Convenience Yields and Asset Pricing Models*

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Abstract

Convenience yields on safe assets affect the constraints that asset price data impose on asset pricing models. We show that high Sharpe ratios imply either a volatile stochastic discount factor or a high convenience yield, which generalizes the Hansen-Jagannathan bound. We apply this insight to the Treasury market by incorporating convenience yields into affine term structure models. The solutions link yields across the entire term structure to SDF and convenience yield dynamics. We show that introducing convenience yields rules in more economically sensible models. This improvement becomes more pronounced as we adopt more realistic convenience yield measures.

Keywords: Convenience Yields, Hansen-Jagannathan Bound, Bond Risk Premium

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Asset price data impose strong constraints on the properties of asset pricing models. For example, Hansen and Jagannathan (1991) show that the Sharpe ratio of any risky asset puts a bound on the volatility of the stochastic discount factor (SDF). Alvarez and Jermann (2005) show that a risky asset's average excess return also informs the size of permanent component of the SDF. In these calculations, the excess return, defined as the return differential between a risky asset and the benchmark risk-free bond, is a key input. The magnitude of this excess return sheds light on the level of risk premium that investors require to compensate their risk exposures.

A recent literature shows that safe assets carry varying degrees of convenience yields, which can be measured by the reduction in yields that investors are willing to give up to hold these safe assets (Krishnamurthy and Vissing-Jorgensen, 2012; Nagel, 2016). These convenience yields reflect non-pecuniary benefits such as liquidity and pleageability that investors value. As such, attributing the return differential entirely to compensation for risk is no longer appropriate. In this paper, we revist the constraints that asset price data impose on asset pricing models. We derive a general result that an asset's excess return needs to be adjusted for convenience yields before it can be related to the SDF volatility (Hansen and Jagannathan, 1991) and empirically demonstrate the significance of this adjustment for interest rate models.

This required adjustment generalizes the Hansen-Jagannathan bound to a *two-dimensional* trade-off: observing a high Sharpe ratio implies either a volatile SDF or a high convenience yield. We analytically characterize this generalization, which expresses each risky asset's risk-return trade-off as a linear constraint on the joint distribution of the SDF volatility and the convenience yield. In particular, when an asset's return volatility is low, this trade-off becomes "steeper" in the sense that a small convenience yield can rationalize a high Sharpe ratio without requiring a volatile SDF. In comparison, the adjustment term approaches zero as the asset return volatility approaches infinity. Therefore, this generalized Hansen-Jagannathan bound is most useful for studying financial assets or portfolios that have low return volatility or can be regarded as "near-arbitrage" trades. To the extent that trading strategies with high Sharpe ratios and high return volatilities exist, high SDF volatility is still required to be consistent with the generalized Hansen-Jagannathan bound.

Having derived this general trade-off formula, we demonstrate its significance in the context of the Treasury bond market. As Treasury bond returns are not volatile, our adjustment term becomes quantitatively significant. We show that a small convenience yield is capable of explaining the high Sharpe ratio of the short-term debt while relaxing the bound on the bond market's SDF volatility. Specifically, Treasury debt has a downward-sloping term structure of Sharpe ratios, with the short-term debt having a Sharpe ratio exceeding that of equity markets (Hansen and Jagannathan, 1991; Duffee, 2011; Van Binsbergen and Koijen, 2017). This high Sharpe ratio is very surprising since the short-term Treasury debt is usually regarded as a safe asset and should therefore earn a low risk premium. We show that, once we incorporate the convenience yield on the benchmark risk-free rate based on the 3-month Treasury, the term structure of Treasury Sharpe ratios becomes flat, implying that Treasury debt has similarly low risk prices across the entire term structure.

This insight has important implications for how we model the term structure of bond yields, which is a key object in asset pricing. We consider a standard class of dynamic, affine term structure models. To properly embed convenience yields in this setting, we present a simple economic setting which microfounds the convenience yields from bond-in-utility preferences. We derive closed-form expressions for the term structure of bond yields, which remain affine after we introduce the convenience yields. Importantly, we show that the entire term structure of bond yields needs to be adjusted for convenience yields, even when only short-term bonds provide convenience benefits. This adjustment across the entire term structure is the result of investors expecting long-term bonds to become more convenient as they mature.

Having generalized the affine term structure model to incorporate convenience yields, we then turn to the empirical implementation. We minimize our objective function, which combines the distance between model-implied and actual yields and the no-arbitrage constraints arising from including the yield spread as a state variable. To make the estimates comparable across measures of convenience yields, we search in the family of affine models with economically sensible SDF volatility. We use three different measures of convenience yields: a constant convenience yield on the 3-month Treasury, a time-varying convenience yield on the 3-month Treasury, and time-varying convenience yields across the entire term structure.

We find that incorporating each variation of convenience yields substantially improves the fit of the affine term structure models. Compared to the model with constant convenience yields, the model with time-varying convenience yields requires a much smaller, and empirically more plausible magnitude for the convenience yields to improve the model fit—which is consistent with our prior that the time-varying convenience yields are more realistic. These results suggest that convenience yields are an important factor in the bond market. Furthermore, our findigs are robust to different specifications of the affine SDF: while additional state variables could improve the bond yields' fit and predictability, in the absence of convenience yields, they do not help escape the tension between a high Sharpe ratio on the short-term bonds and an economically sensible SDF volatility.

The results from this particular term structure model illustrate our general trade-off between the conditional Sharpe ratio and the convenience yield. In particular, Duffee (2011) shows that the conditional Sharpe ratio implied by a flexible four- or five- factor affine term structure model (without convenience yields) is implausibly high. This issue is interpreted as a result of overfitting due to the flexibility permitted by the four or five factors, and the proposed solution is to impose a good deal bound on the maximum Sharpe ratio in model estimation. We offer a different interpretation of this result by showing that, once we introduce a realistic amount of convenience yield to the risk-free rate, the model can generate a good fit to bond yields while maintaining plausible Sharpe ratios. In comparison, if the convenience yield is zero or too small, imposing a good deal bound on the maximum Sharpe ratio leads to a poor fit of the model; in other words, for models without convenience yields to have a reasonable fit, a much greater maximum Sharpe ratio is required. We also find that, when we impose a convenience yield that is higher than the conventional estimates, the model fit deteriorates again.

Finally, to consider the entire term structure of convenience yields, we use the Treasury-Refcorp bond spreads as proxies (Longstaff, 2004; Joslin, Li, and Song, 2021). This measurement implies a flat term structure of the average convenience yields. Thus, incorporating these convenience yield proxies does not change the unconditional return differentials earned by bonds with different maturities. Yet, incorporating these convenience yields still improves the model fit. We find that this result is driven by these convenience yields improving the *conditional* risk-return trade-off. Specifically, while short-term and long-term Treasurys have similar convenience yield levels on average, short-term Treasurys have higher convenience yields in recessions while long-term Treasurys have higher convenience yields in expansions. As such, the cyclical properties of these convenience yields lower the required SDF volatility in recessions, which is exactly when it needs to be high.

In summary, we develop a generalized Hansen and Jagannathan (1991) bound that maps asset price data to restrictions on the SDF volatility in the presence of convenience yields. This result improves our understanding of the term structure of Treasury Sharpe ratios, and highlights the importance of incorporating convenience yields when considering data-based restrictions on asset pricing models.

Related Literature This paper relates to three areas in the asset pricing literature. First, this paper is related to no-arbitrage asset pricing, which uses asset price and return moments to inform the dynamics of SDFs (Hansen and Jagannathan, 1991; Cochrane and Saa-Requejo, 2000; Alvarez and Jermann, 2005; Sandulescu, Trojani, and Vedolin, 2021). Our paper generalizes the result in Hansen and Jagannathan (1991) to a two-dimensional trade-off between SDF volatility and convenience yields based on the special demand for safe and liquid assets.

Second, we use insights about bond convenience yields to improve our understanding of asset pricing models. Our mechanism is consistent with Lenel, Piazzesi, and Schneider (2019), which develops a model that generates convenience yields on short-term Treasuries based on their role in the payment system. Our paper is also related to Grinblatt (2001), which models the liquidity premia on Treasury bonds as the present value of the flow benefit of holding Treasuries. In contrast, our focus is on understanding how introducing convenience yields improves the performance and interpretation of the risk-return trade-off in asset pricing models and affine term structure models.

Our findings also add to the literature on covered interest rate parity (CIP), which finds significant convenience yields on Treasury bonds (Du, Im, and Schreger, 2018; Du and Schreger, 2021; Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2020; Jiang, Krishnamurthy, and Lustig, 2021a; Joslin, Li, and Song, 2021; Diamond and Van Tassel, 2021; Van Binsbergen, Diamond, and Grotteria, 2022; Du, Hébert, and Li, 2022; Choi, Kirpalani, and Perez, 2022). We show that the term structure of Treasury Sharpe ratios is characterized by a downward-sloping term structure of convenience yields and a relatively flat term structure of risk prices. This view is complementary to other explanations of the high Sharpe ratio on the short-term Treasury based on trading frictions (Luttmer, 1996) and idiosyncratic market conditions (Duffee, 1996).[1]

Third, this paper also contributes to a large literature on bond affine term structure models (Dai and Singleton, 2000; Joslin, Singleton, and Zhu, 2011; Joslin, Priebsch, and Singleton, 2014; Cieslak and Povala, 2015; Chernov and Creal, 2018). Duffee (2011) derives conditional Sharpe ratios based on Gaussian term structure models and finds the implied Sharpe ratios to be highly implausible. Augustin, Chernov, Schmid, and Song (2020) studies the term structure of CIP violations in an affine no-arbitrage model, and Payne, Szőke, Hall, and Sargent (2022); Chen, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2022) also seek to embed convenience yields in the term structure of bonds. We formally derive and evaluate affine term structure models in the presence of convenience yields. We show that incorporating convenience yields significantly improves the fit of standard affine term structure models, and also changes their implications for the bonds' risk-return trade-off.

This paper is organized as follows. In Section (1), we generalize the Hansen-Jagannathan bound to a two-dimensional trade off and show that this adjustment is economically significant when applied to the Treasury market. In Section (2), we present a theoretical model which incorporates convenience yields in the term structure of Treasury yields and specify the details of empirical model. Section (3) presents our estimation of and results from the model. Section (4)

[1] In the broader asset pricing literature, high Sharpe ratios on zero-beta assets have been noted in the context of risky assets and equity pricing in particular (Black, 1972; Frazzini and Pedersen, 2014; Hébert, Kurlat, and Wang, 2023). Our results are more relevant to bond markets and other near-arbitrage spreads, as our economic mechanism is stronger when the asset's return volatility is lower.

concludes.

1 Risk-Return Trade-Off with Convenience Yields

Any asset pricing model has strong implications for the trade-off between asset risks and returns. When we confront models with asset price data, we evaluate whether the assets' observed risks and returns are consistent with the trade-off described by the given model. In this section, we consider a general, one-period setting. We study a dynamic setting in the next section when we consider specific term structure models.

We assume the existence of a stochastic discount factor (SDF) m, but do not require the SDF to be unique. We interpret the SDF m as the marginal utility growth of the marginal investor. Then, this investor's Euler equation for an arbitrary risky asset with stochastic return r is

$$\mathbb{E}[mr] = 1.$$

By straight-forward algebra,

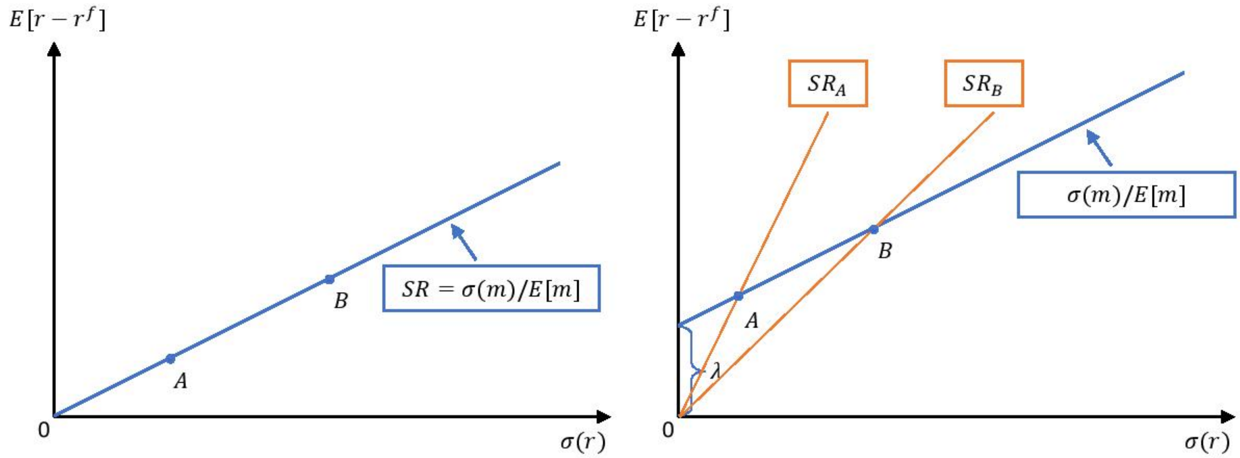
$$\frac{\sigma(m)}{\mathbb{E}[m]} corr(-m, r) = \frac{\mathbb{E}[r - 1/\mathbb{E}[m]]}{\sigma(r)}. \tag{1}$$

Traditionally, we use the Euler equation for the risk-free rate r^f, i.e., $\mathbb{E}[mr^f] = 1$, to measure $1/\mathbb{E}[m] = r^f$. Then, the right-hand side of Eq. (1) measures the Sharpe ratio of the risky asset:

$$\frac{\sigma(m)}{\mathbb{E}[m]} corr(-m, r) = \frac{\mathbb{E}[r] - r^f}{\sigma(r)},$$

which relates the expected excess return on this asset per unit of volatility of the return to the volatility of the SDF m.

Figure (1)(a) illustrates this standard relation. Each asset can be represented by a point in this figure, where each point's coordinates are its expected excess return and return volatility. When



Panel (a) Standard Trade-Off *Panel (b) Trade-Off with Convenience Yield*

Figure 1: Illustrations of Risk-Return Trade-Offs in Different Scenarios

the asset return is perfectly negatively correlated with the SDF, i.e., $corr(-m, r) = 1$, it lies on the blue line whose slope is exactly $\sigma(m)/\mathbb{E}[m]$. Let A and B denote two such assets. Then, these two assets have the same Sharpe ratio of $\sigma(m)/\mathbb{E}[m]$. If the asset B has a higher expected excess return, its return volatility has to be proportionally greater. Other assets whose returns are not perfectly correlated with the SDF lie to the right of the blue line, and therefore have lower Sharpe ratios.

Now, suppose the risk-free rate carries a convenience yield. Its Euler equation becomes

$$\mathbb{E}[mr^f] = \exp(-\lambda) < 1,$$

where the convenience yield λ captures the reduction in the log return of the risk-free asset that investors are willing to accept in order to hold this asset. This convenience yield captures the premium that investors pay for the safety and for the liquidity service provided by the risk-free asset (Krishnamurthy and Vissing-Jorgensen, 2012). Then, Eq. (1) becomes

$$\frac{\sigma(m)}{\mathbb{E}[m]} corr(-m, r) = \frac{\mathbb{E}[r - r^f \exp(\lambda)]}{\sigma(r)} \approx \frac{\mathbb{E}[r] - r^f}{\sigma(r)} - \frac{\lambda}{\sigma(r)} \tag{2}$$

in the presence of the convenience yield. This expression implies that we need to adjust the Sharpe ratio $(\mathbb{E}[r] - r^f)/\sigma(r)$ by the ratio between the convenience yield λ and the standard deviation of asset return, $\sigma(r)$, before relating it to the SDF volatility. This adjustment term is greater when the asset return is less volatile.

Figure (1)(b) illustrates this case. Again let A and B denote two risky assets whose returns are perfectly correlated with the SDF. Since the risk-free rate now has a convenience yield λ, these assets' risk-return trade-offs are now shifted up. As a result, their Sharpe ratios are higher than $\sigma(m)/\mathbb{E}[m]$. Graphically, the Sharpe ratios are equal to the slopes of the red lines that connect $(0,0)$ and the points A and B respectively, and the scaled SDF volatility $\sigma(m)/\mathbb{E}[m]$ is equal to the slope of the blue line. The presence of the convenience yield λ makes the orange lines steeper than the blue line, and this effect is stronger for assets with lower return volatilities $\sigma(r)$. On the other hand, if an asset has a very high return volatility, the adjustment term $\lambda/\sigma(r)$ approaches 0, and the Sharpe ratio of this asset converges to $\sigma(m)/\mathbb{E}[m]$ from above.

1.1 A Generalized Hansen-Jagannathan Bound

This result generalizes the Hansen-Jagannathan bound to two dimensions in the presence of convenience yields. First, ignoring the convenience yield λ, Eq. (2) implies the standard Hansen-Jagannathan bound:

$$\frac{\sigma(m)}{\mathbb{E}[m]} \geq \left| \frac{\mathbb{E}[r] - r^f}{\sigma(r)} \right|,$$

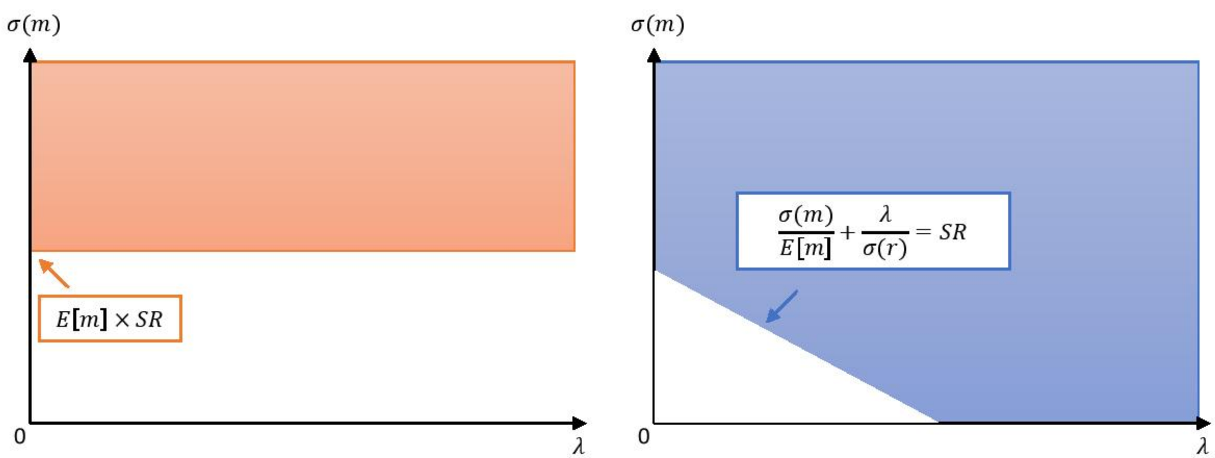
with equality obtained when the asset return is perfectly correlated with the SDF: $corr(-m, r) = 1$. In the absence of the convenience yield, the standard inference problem can be illustrated by Figure (2)(a). If we observe an asset with a given Sharpe ratio $SR = (\mathbb{E}[r] - r^f)/\sigma(r)$, the SDF volatility needs to be higher than the Sharpe ratio times $\mathbb{E}[m] = 1/r^f$. In the space of the SDF volatility $\sigma(m)$ and the convenience yield λ illustrated in this figure, the admissible area is colored in orange.

Now, considering the convenience yield, Eq. (2) generalizes to

$$\frac{\sigma(m)}{\mathbb{E}[m]} + \frac{\lambda}{\sigma(r)} \geq \left| \frac{\mathbb{E}[r] - r^f}{\sigma(r)} \right|, \tag{3}$$

which represents a trade-off between the SDF volatility $\sigma(m)$ and the convenience yield λ. Figure (2)(b) illustrates this case: when we observe an asset with a high Sharpe ratio, it indicates either the presence of a volatile SDF or the presence of a high convenience yield. The minimal possible SDF volatility for a given level of convenience yield is described by Eq. (3) with equality, which is graphically represented by the line that separates the rejected area, represented by the white triangle. As this line is downward sloping, if the convenience yield is higher, the bound on the minimal SDF volatility is going to be lower.

Moreover, if the asset's return volatility is lower, the rejected area will be smaller. In other words, high Sharpe ratios implied by low-volatility assets may be consistent with a low SDF volatility and a small convenience yield.



Panel (a) Standard Bound *Panel (b) Bound with Convenience Yield*

Figure 2: Hansen-Jagannathan Bounds

1.2 Convenience Yield-Adjusted Sharpe Ratios on U.S. Treasurys

Having derived the theoretical implications of convenience yields on the risk-return trade-off, we now consider how these insights apply to the U.S. Treasury market. We obtain the Treasury yield data from Gürkaynak, Sack, and Wright (2007), and we construct a proxy for the convenience yield on the 3-month Treasury based on a scaled 3-month CD-Treasury spread (Nagel, 2016). Our main sample is from 1971Q3 to 2020Q4. Details of data sources and construction are in Appendix A. The time series of our convenience yield measure (3-month scaled CD rates minus Treasury yields) is presented in Appendix Figure (A1). There is substantial variation in the convenience yield, with the average value being 30 basis points annually.

For this application, we assume that the convenience yields λ are specific to the 3-month Treasury, and that Treasury bonds with longer maturities do not enjoy convenience yields. Table (1) reports the average excess returns, standard deviations, and Sharpe ratios for Treasury bond portfolios of varying maturities. All values reported are annualized from the quarterly data. The first three rows are without adjusting for convenience yields while the last two rows adjust for convenience yields, where convenience yields are measured using the CD-Treasury spread. Average excess returns are increasing in maturity, with Treasurys with maturities less than 6 months having average excess returns over 3-month Treasury of 32 basis points. Bonds with maturities of 5-10 years have average excess returns of 2.68% per year. Volatilities of these returns are increasing in maturity and do so more quickly than the average excess returns. As a result, Sharpe ratios on these Treasurys are decreasing in maturity. Of particular note is that the Sharpe ratio on short (less than 6 month) maturity debt is very high — 0.82. Van Binsbergen and Koijen (2017) cast this pattern in a broader context and show the downward-sloping expected return across the maturity spectrum is a general pattern across asset classes.

In the bottom two rows of Table (1) we subtract convenience yields from the excess returns of the bonds, as in Equation (2). The average annual convenience yield is 29 basis points over our sample. As a result, the convenience yield-adjusted average excess returns are lower by 29 basis points across all maturity portfolios. This convenience yield also affects the Sharpe ratios on the

Table 1: Treasury Sharpe Ratios across Maturities

The (convenience yield-) Adjusted Avg Excess Return is $(\mathbb{E}[r] - r^f - \lambda)$ and the Adjusted Sharpe Ratio is $(\mathbb{E}[r] - r^f - \lambda)/\sigma(r - r^f)$. All values are annualized. Data are from CRSP and Bloomberg.

	< 6 months	6-12 months	1-2 years	2-3 years	3-4 years	4-5 years	5-10 years
Avg Excess Return (%)	0.32	0.66	1.06	1.56	1.99	2.16	2.68
Std Dev (%)	0.39	1.26	2.37	3.66	4.59	5.48	6.93
Sharpe Ratio	0.82	0.52	0.45	0.43	0.43	0.39	0.39
Adjusted Avg Excess Return (%)	0.03	0.36	0.77	1.27	1.70	1.87	2.39
Adjusted Sharpe Ratio	0.07	0.29	0.32	0.35	0.37	0.34	0.34

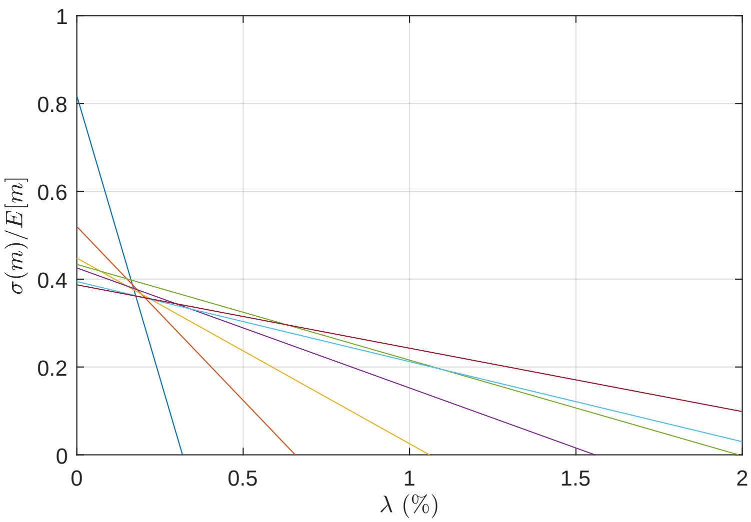


Figure 3: Hansen-Jagannathan Bounds: Application to Treasury Debt

bond portfolios, with the Sharpe ratio of short maturity bonds being impacted the most due to their relatively low volatility. For example, the adjusted Sharpe ratio of Treasurys with maturity less than 6 months goes from 0.82 to 0.07, whereas it goes from 0.39 to 0.34 for bonds with maturities of 5-10 years. It is also notable that for all bonds with maturities beyond 6 months, their adjusted Sharpe ratios are all very close to 0.30 per annum, implying similarly low prices of risk. As a result, these adjusted Sharpe ratios imply a bound on the SDF volatility of 0.37 (corresponding to 3-4 year Treasurys) instead of 0.82 without accounting for convenience yields.

To understand how different levels of convenience yields impact the Hansen-Jagannathan bound in Treasure markets, we can apply our generalized bound to the returns in Table (1). Each bond portfolio imposes one inequality that governs the joint distribution of the scaled SDF volatility $\sigma(m)/\mathbb{E}[m]$ and the convenience yield λ. Figure (3) provides a graphical illustration of these inequalities for varying possible average convenience yields. Each line is the inequality for one of the bond portfolios with a varying level of convenience yield. The set of admissible values for the pair of SDF volatility and convenience yield lies to the right of each line. This figure shows that, when the convenience yield λ is 0, the scaled SDF volatility $\sigma(m)/\mathbb{E}[m]$ has to be above 0.82 according to the standard Hansen-Jagannathan bound. This bound on the scaled SDF volatility declines as we increase the convenience yield λ. For example, if we are willing to consider a convenience yield of $\lambda = 30$ basis points per annum, then, the bound on the scaled SDF volatility $\sigma(m)/\mathbb{E}[m]$ declines to about 0.35.

2 Implications for Dynamic Asset Pricing Models

Having derived the risk-return trade-off with convenience yields in a general one-period setting, we now apply this insight in a dynamic affine term structure models. This family of models have very broad applications from pricing the bond market to evaluating other cash flow patterns (Duffie and Kan, 1996; Dai and Singleton, 2000; Ang and Piazzesi, 2003; Lustig, Van Nieuwerburgh, and Verdelhan, 2013; Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2019; Chernov and Creal, 2023). Yet, as the short-term Treasury earns a high Sharpe ratio, these models fail to generate an SDF with a reasonable volatility that fits the bond data. Duffee (2011) finds that the implied SDF volatility in estimated affine term structure models is astronomically high. A standard solution is to impose good-deal bonds in estimation (Cochrane and Saa-Requejo, 2000). In this section, we re-evaluate this result in a model that incorporates convenience yields and show that incorporating a convenience yield can substantially improve the fit of a standard affine term structure model.

To understand how we should adjust the term structure model in the presence of convenience yields, we begin with a simple economic setting which microfounds the convenience yields from bond-in-utility preferences (Valchev, 2020; Jiang, Krishnamurthy, Lustig, and Sun, 2021b). In this setting, there is a representative investor who holds the bond market and only the one-period bond generates non-pecuniary utility. Nonetheless, the convenience yield on the one-period bond affects yields across the entire term-structure.

With the insight on how to properly adjust yields across the term-structure, we then turn to the empirical specification of our affine term structure model which we use, along with our estimation methodology. In the next section we present results using three different measures of convenience yields.

2.1 Model Set-Up

Let $q_t(\tau)$ denote the book value of bond holdings and $p_t(\tau)$ denote the bond price of maturity τ. All bonds are default-free. The representative investor's utility is derived over consumption and holding the one-period bond:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\delta^t(u(c_t)+v(q_t(1)p_t(1);\theta_t))\right],$$

where θ_t is a time-varying demand shifter for one-period bonds. We assume that the utility is increasing in the consumption and the holding in the U.S. bonds, i.e. $u'(c)>0$ and $v'(q;\theta_t)>0$. In this way, the one-period bond carries a convenience yield, which captures its non-pecuniary benefits to the investor. We assume that the marginal utility for holding U.S. bonds is decreasing in quantity, i.e., $v''(q;\theta_t)<0$, so that the convenience yield is decreasing in the quantity held.

The representative investor receives an endowment of y_t in period t. The Lagrangian is

$$\mathbb{E}_0\left[\sum_{t=1}^{\infty}\delta^t(u(c_t)+v(q_t(1)p_t(1);\theta_t))+\sum_{t=1}^{\infty}\zeta_t\left(y_t+\sum_{j=1}^{H}q_{t-1}(j)p_t(j-1)-c_t-\sum_{j=1}^{H}q_t(j)p_t(j)\right)\right].$$

As we show in Appendix B.1, the Euler equation for holding one-period bond can be expressed as

$$1-\frac{v'(q_t(1);\theta_t)}{u'(c_t)}=\mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\frac{1}{p_t(1)}\right].$$

We note that this Euler equation deviates from the standard Euler equation because the investors derive utility directly from holding the bond. In other words, the investors are willing to accept a lower expected return on the bond in exchange for the non-pecuniary benefits.

Following the affine term structure literature, we define the pricing kernel as

$$m_{t,t+k}=\delta^k\frac{u'(c_{t+k})}{u'(c_t)}.$$

We also use λ_t to denote the log convenience yield on the 1-period bond:

$$\exp(-\lambda_t)=1-\frac{v'(q_t(1);\theta_t)}{u'(c_t)}; \tag{4}$$

then, the Euler equation for the one-period bond becomes

$$\exp(-\lambda_t)=\mathbb{E}_t\left[m_{t,t+1}\frac{1}{p_t(1)}\right].$$

Similarly, the Euler equation for holding two-period bond can be expressed as

$$1-\mathbb{E}_t\left[\delta\frac{p_{t+1}(1)}{p_t(2)}\frac{v'(q_{t+1}(1);\theta_{t+1})}{u'(c_t)}\right]=\mathbb{E}_t\left[m_{t,t+2}\frac{1}{p_t(2)}\right],$$

which also deviates from the standard Euler equation because, even though the investors do not derive utility directly from holding this two-period bond right away, they derive utility from holding it in the next period. In fact, using the definition of the convenience yield in Eq. (4), we can express the Euler equation for the two-period bond as

$$1-\mathbb{E}_t\left[m_{t,t+1}\frac{p_{t+1}(1)}{p_t(2)}\left(1-\exp(-\lambda_{t+1})\right)\right]=\mathbb{E}_t\left[m_{t,t+2}\frac{1}{p_t(2)}\right],$$

where the deviation of left-hand side from 1 in standard Euler equations reflects the expected convenience yield on this bond in the next period after proper discounting.

2.2 State Space and Bond Pricing

Following the affine term structure literature, we directly impose exogenous processes on this pricing kernel and on the convenience yield. We calibrate one period to be one quarter. We use z_t to denote the $N \times 1$ vector of state variables, which include log inflation π_t, log GDP growth x_t, log 3-month short rate $i_t(1)$, the yield spread $yspr_t$ between the log 5-year nominal yield and the log 3-month nominal yield, and the convenience yield λ_t:

$$z_t = [\pi_t, x_t, i_t(1), yspr_t, \lambda_t]'.$$

For notational convenience, these state variables are demeaned by their sample averages. Let $e^{i1} = [0, 0, 1, 0, 0]'$, $e^{yspr} = [0, 0, 0, 1, 0]'$, and $e^{\lambda} = [0, 0, 0, 0, 1]'$ denote the vector that selects the short rate, the yield spread, and the convenience yield, respectively.

To incorporate the convenience yield while maintaining affine solutions, the 3-month short rate $i_t(1)$ is the risk-free rate without convenience yield. Let $y_t(1)$ denote the log 3-month Treasury yield that contains the convenience yield. Then, we can obtain $i_t(1)$ from

$$i_t(1) = y_t(1) + \lambda_t.$$

Moreover, the yield spread $yspr_t$ is defined as the spread between the log 5-year (i.e., 20-quarter) Treasury yield and log 3-month risk-free rate without convenience yield:

$$yspr_t = y_t(20) - i_t(1),$$

so that the sum of the short rate $i_t(1)$ and the yield spread $yspr_t$ equals the log 5-year Treasury yield $y_t(20)$.

We assume that the state variables evolve according to a Gaussian first-order VAR:

$$z_t = \Psi z_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_t, \tag{5}$$

where the shocks $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I)$.

We specify an exponentially affine SDF:

$$m_{t+1} = -i_t(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1}, \tag{6}$$

where $i_t(1)$ is the nominal risk-free rate that carries no convenience yield, and the market price of risk Λ_t is an $N \times 1$ vector of the affine form:

$$\Lambda_t = \Lambda_0 + \Lambda_1 z_t.$$

The $N \times 1$ vector Λ_0 collects the average prices of risk, while the $N \times N$ matrix Λ_1 governs the time variation in risk premia. Asset pricing in this model amounts to estimating the market prices of risk in Λ_0 and Λ_1.

With this set-up, we derive the following affine bond pricing formula:

Proposition 1. *The log Treasury yields are affine in the state vector:*

$$y_t(h) = -\frac{A(h)}{h} - \frac{B(h)'}{h} z_t. \tag{7}$$

Equivalently, the log Treasury prices are

$$\log p_t(h) = A(h) + B(h)' z_t,$$

where $A(0) = 0$ *and* $B(0) = \boldsymbol{0}$*,*

$$A(1) = -i_0(1) + \lambda_0,$$

$$B(1)' = -(e^{i1})' + (e^{\lambda})',$$

and

$$A(h+1) = -i_0(1) + A(h) + \frac{1}{2}(B(h))'\Sigma B(h) - B(h)'\Sigma^{\frac{1}{2}}\Lambda_0, \tag{8}$$

$$(B(h+1))' = B(h)'\Psi - (e^{i1})' - B(h)'\Sigma^{\frac{1}{2}}\Lambda_1, \tag{9}$$

for $h \geq 2$*.*

The proof is in Appendix B.2. This formula embeds the bond convenience yield in the pricing parameters $A(1)$ and $B(1)$ for the one-period Treasury bond, which imply $y_t(1) = -A(1) - B(1)'z_t = i_t(1) - \lambda_t$. In comparison, the textbook affine bond pricing formula is $y_t(1) = i_t(1)$, which directly maps the observed one-period Treasury yield as the one-period risk-free rate.

Moreover, while the pricing parameters $A(h)$ and $B(h)$ for the h-period bond look identical to those in the standard affine bond pricing formula, the convenience yield is also embedded in the longer-term bonds. For example, $(B(2))' = (e^{\lambda})'\Psi - (e^{i1})'(I+\Psi) - B(h)'\Sigma^{\frac{1}{2}}\Lambda_1$, which exposes the 2-period bond price to the convenience yield $\lambda_t(1)$ via the $e^{\lambda}\Psi$ term. In other words, as the bonds are priced recursively, the price of a longer-term bond today depends on the expectation of the price of a shorter-term bond in the future, which incorporates the investors' expected convenience yield in the future.

2.3 Convenience Yield Measures

We consider three measures for convenience yields. First, we assume that the convenience yield on the three-month bond is a constant:

$$\lambda_t(1) = c_1.$$

Second, we assume the convenience yield on the three-month bond is time-varying. To measure it we use the yield spread between 3-month Treasury bond and scaled 3-month CD as a proxy, as in the previous section. Because the 3-month CD could carry additional default risk, we scale this yield spread by a constant multiplier c_2:

$$\lambda_t(1) = c_2(y_t^{CD}(1) - y_t(1)).$$

Third, recent work points to the existence of the term structure of convenience yields beyond the very short durations. Therefore, we use the spread between Resolution Funding Corporation (Refcorp) strips and Treasury strips as the proxy for the tenor-specific convenience yield $\lambda_t(\tau)$ (Longstaff, 2004; Joslin, Li, and Song, 2021). Furthermore, we scale this spread by a constant multiplier across maturities to study the impact of the differing magnitudes of convenience yields:

$$\lambda_t(\tau) = c_3(y_t^{Refcorp}(\tau) - y_t(\tau)).$$

We do not take a stance on the constants c_1, c_2, and c_3. Instead, we vary them across different specifications of our model and examine their fit.

3 Estimation and Results

3.1 Moment Conditions and Parametric Restrictions

We match bond yields of maturities 1, 4, 8, 12, 16, 20, 28, 40 quarters in each quarter t. Moreover, since both the nominal short rate ($i_t(1)$) and the slope of the term structure ($yspr_t = y_t(20) - i_t(1)$) are included in the VAR, internal consistency requires the model to price these bonds closely. The nominal short rate is matched automatically — it does not identify any market price of risk parameters. Then, matching the slope of the yield curve generates $N+1$ parameter restrictions:

$$-A(20)/20 = i_0(1) + yspr_0 = y_0(20), \tag{10}$$

$$-B(20)/20 = e^{i1} + e^{yspr}. \tag{11}$$

Our moment conditions include the no-arbitrage restrictions represented by Eq. (10) and (11), as well as the pricing error between the actual bond yields and the bond yields implied by Eq. (7), for the tenors considered above. We estimate the risk prices Λ_0 and Λ_1 by minimizing the sum of the squared moment conditions.

We impose two more constraints to ensure that the model is well-behaved. First, we calculate the model-implied bond yield in the infinite horizon defined as

$$y_0(\infty) = i_0(1) - \frac{1}{2}(B(\infty))'\Sigma B(\infty) + B(\infty)'\Sigma^{\frac{1}{2}}\Lambda_0, \tag{12}$$

where the risk loading $(B(\infty))' = -(e^{i1})'(I - (\Psi - \Sigma^{\frac{1}{2}}\Lambda_1))^{-1}$ is obtained by setting $B(h+1) = B(h)$ for $h \to \infty$ in the iteration formula (9). We require that the infinite-horizon bond yield $y_0(\infty)$ is at least 1% per annum, to rule out the possibility that the model generates too low an interest rate beyond the longest horizon that has observed yield data (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009).

Second, we impose a good deal bound following Cochrane and Saa-Requejo (2000) and Duffee (2011). In particular, the latter paper finds that affine models generate implausibly high SDF volatility in the absence of the good deal bound. By the Hansen-Jagannathan bound, the log Sharpe ratio of any asset in our model is bounded by the volatility of the SDF given in Eq. (6):

$$\theta_t = \sqrt{var_t(m_{t+1})} = \sqrt{\Lambda_t' \Lambda_t}.$$

Let θ_t denote the maximal log Sharpe ratio evaluated at the values of state variables at time t. We require that the model's average maximum log Sharpe ratio is bounded by a constant b:

$$mean(\theta_t) < b, \tag{13}$$

and we calibrate this bound b to 0.35 per quarter, which is based on the adjusted Sharpe ratios of the bonds in Table (1). This bound implies a maximum annual log Sharpe ratio of 0.7, which is also a similar to the magnitude of the U.S. equity market's Sharpe ratio.

In our empirical implementation, we also run the estimation without imposing the good deal bound. We find that, in the presence of the good deal bound, the models with convenience yields generate better fits with the bond data; in the absence of the good deal bound, the models with convenience yields instead generate much smaller maximal Sharpe ratios. This duality result suggests that evaluating the model fit while imposing the good deal bound offers a useful comparison across models, a point we will expand further in the next subsection.

Finally, we only allow the first four state variables $(\pi_t, x_t, i_t(1), yspr_t)$ to carry priced risks, which amounts to assuming that $\Lambda_0(5) = 0$, $\Lambda_1(5, i) = \Lambda_1(i, 5) = 0$ for all i. We impose this restriction in order to have a direct comparison with the benchmark affine model that contains no convenience yields. In fact, our model developed in this section offers two channels through which convenience yields can affect the bond prices. First, the convenience yield can affect the bond prices directly by imposing a wedge in the Euler equation (4). Second, the convenience yield can also affect the pricing kernel as its market price of risk Λ_t loads on the state vector. The second channel can be present in standard affine models if we introduce the convenience yield as an extra state variable. For this paper, we mute this second channel to focus on how the Euler equation wedge affects the bond prices.

3.2 Estimation Result with a Constant Convenience Yield

We first report the estimation results in the model with a constant convenience yield $\lambda_t = c_1$ in Figure (4). Panel (a) reports the average maximal Sharpe ratio $mean(\theta_t)$ of the model. For all convenience yield levels we consider, the good deal bound is binding: the maximal Sharpe ratio is at our bound of 0.7 per annum.

To measure the model's fits, we define the root mean squared error for tenor h as the square root of the average squared bond pricing error:

$$rmse(h) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(-\frac{A(h)}{h} - \frac{B(h)'}{h}z_t - y_t(h)\right)^2}.$$

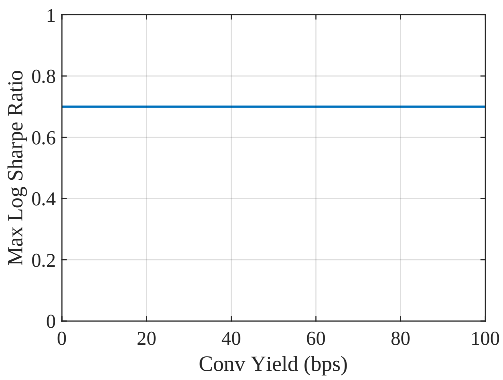
Panel (b) reports the root mean squared error averaged across the tenors used in our moment conditions, and Panel (c) reports the root mean squared errors in six selected tenors. We note that these pricing errors exhibit a U-shape pattern as we vary the convenience yield level: when the convenience yield is 0, we have fairly large pricing errors across the tenors we considered. The mean pricing error declines as we raise the convenience yield level, and reaches its minimum when the convenience yield is 60 basis points per annum. When we further increase the convenience yield level beyond 60 basis points, the fit begins to worsen.

This improvement in the model fit is consistent with the intuition we derived from the general trade-off (3) between SDF volatility and convenience yield in the previous section:

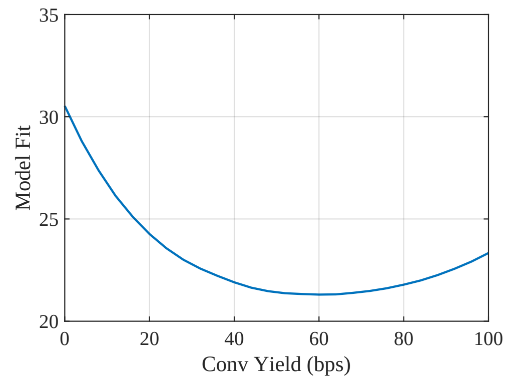
$$\frac{\sigma(m)}{\mathbb{E}[m]} + \frac{\lambda}{\sigma(r)} \geq \left|\frac{\mathbb{E}[r] - r^f}{\sigma(r)}\right|.$$

To see this point, note that we estimate our term structure model with a binding good deal bound. Under this constraint, a model with zero convenience yield has a poor fit since its scaled SDF volatility $\frac{\sigma(m)}{\mathbb{E}[m]}$ is too low relative to the observed Sharpe ratios of the bonds $\left|\frac{\mathbb{E}[r]-r^f}{\sigma(r)}\right|$. In compar-

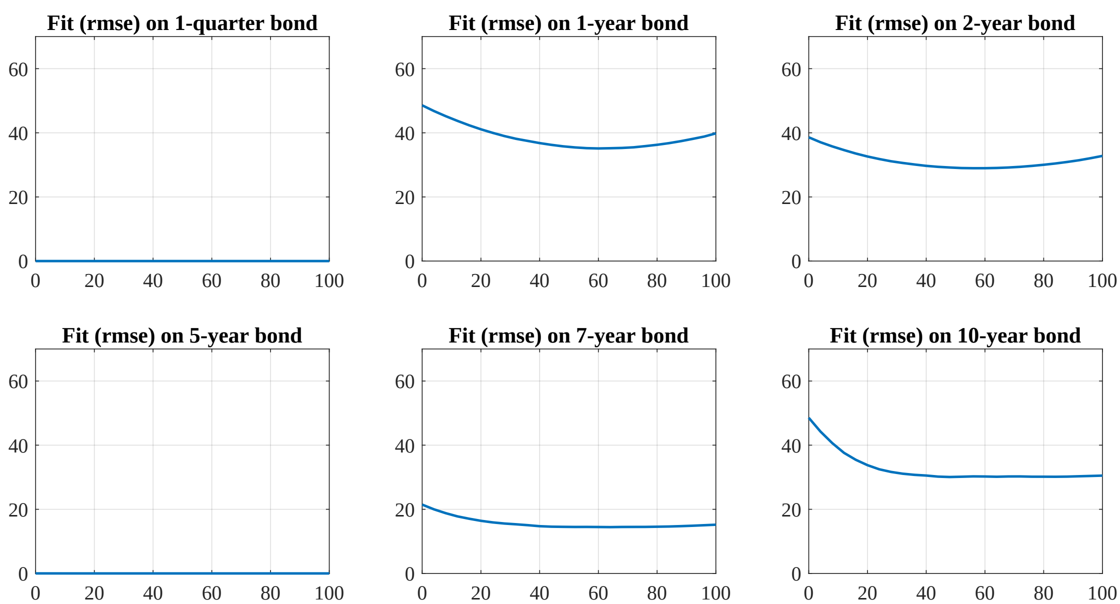
Figure 4: Estimation Results of the Model with Constant Convenience Yield



Panel (a) Max Log Sharpe Ratio



Panel (b) Mean Bond Pricing Error



Panel (c) Bond Pricing Error by Tenor

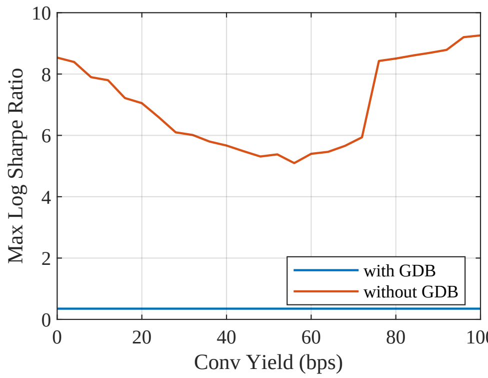
Notes: This figure plots the SDF's maximum quarterly log Sharpe ratio and model fit as we vary the convenience yield on the 3-month Treasury bill. The x axis is the constant annualized convenience yield adjustment on the 3-month Treasury bill. The yields and the convenience yield are annualized, in basis points. Data are 1971Q3–2020Q4.

ison, when we introduce a positive convenience yield, we relax the tight link between the SDF volatility and the asset's Sharpe ratio, which allows the model to fit the yield data better.

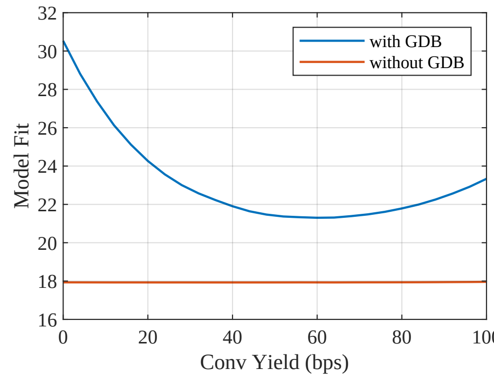
We also note that the improvement in the model fit is driven by reducing the pricing errors across all horizons, which is surprising because the formula in Proposition 1 suggests that, for a given model, the convenience yield adjustment affects the short-term bond yields much more than the long-term bond yields. The improvement from the long-term bond yields has to come from different prices of risks Λ_0 and Λ_1 that are consistent with the short bonds' pricing dynamics after the convenience yield adjustment. In other words, incorporating the convenience yields on the short-term Treasury also changes the pricing dynamics of the long-term Treasury under the best-fitting parameter set.

Finally, these results show that, in the class of affine models whose maximal Sharpe ratio is 0.7 per annum or lower, a convenience yield on the risk-free rate around 60 basis points per annum greatly improves the fit of the model. This insight is general to the class of affine models that allow higher maximal Sharpe ratios. In Figure (5), we report the models' maximal Sharpe ratios and fit when we estimate the models with and without imposing the good deal bound Eq. (13). The blue line reports the results from the models estimated with the good deal bound, which replicates the results we see in Figure (4). The red line reports the results from the models estimated without the good deal bound. This figure uncovers a "duality" result: in the absence of the good deal bound, the model that incorporates the convenience yield generates a much smaller maximal Sharpe ratio conditional on the same model fit. In the presence of the good deal bound, while the maximal Sharpe ratio is similar across different models, the model that incorporates the convenience yield generates a much better model fit. Given this symmetry between the results estimated with and without the good deal bound, we focus on the model fit conditional on imposing the good deal bound as the sufficient statistics to compare the models.

Figure 5: Comparing Models Estimated with and without Good Deal Bound



Panel (a) Max Log Sharpe Ratio



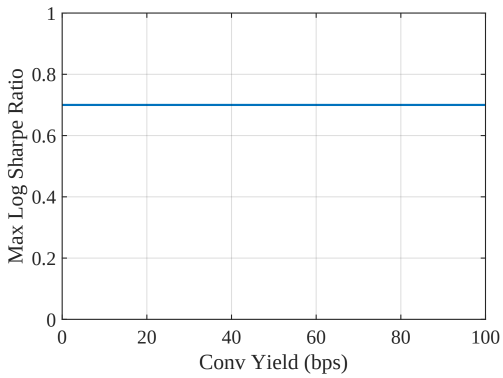
Panel (b) Mean Bond Pricing Error

Notes: This figure plots the SDF's maximum quarterly log Sharpe ratio and model fit as we vary the convenience yield on the 3-month Treasury bill. The blue line reports the results from the models estimated with the good deal bound Eq. (13), and the red line reports the results from the models estimated without the good deal bound. The x axis is the constant annualized convenience yield adjustment on the 3-month Treasury bill. The yields and the convenience yield are annualized, in basis points. Data are 1971Q3–2020Q4.

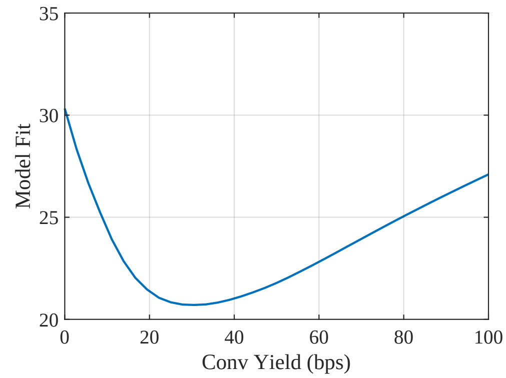
3.3 Estimation Result with a Time-Varying Convenience Yield

Next, we report the estimation results in the model with a time-varying convenience yield $\lambda_t(1) = c_2(y_t^{\$,CD}(1) - y_t^{\$,Treasury}(1))$ in Figure (6). Each point in this figure represents a different instance of the model, which has a different value for the constant multiplier c_2 in front of the 3-month Treasury bill-CD spread. We report the mean convenience yield in the horizontal axis, which is increasing in c_2. We obtain similar results: the model has a high mean pricing error if the convenience yield is zero, and it improves as we increase the convenience yield. We obtain the best model fit when the mean convenience yield is about 30 basis points per annum, which is close to the mean 3-month CD-Treasury spread of about 30 basis points per annum. In this case, the mean pricing error is around 20 basis points, which is similar to that obtained in the best case of the constant convenience yield model in Figure (4). In other words, this model with a time-varying convenience yield allows us to obtain a good model fit with a lower, and more realistic, magnitude of the convenience yield.

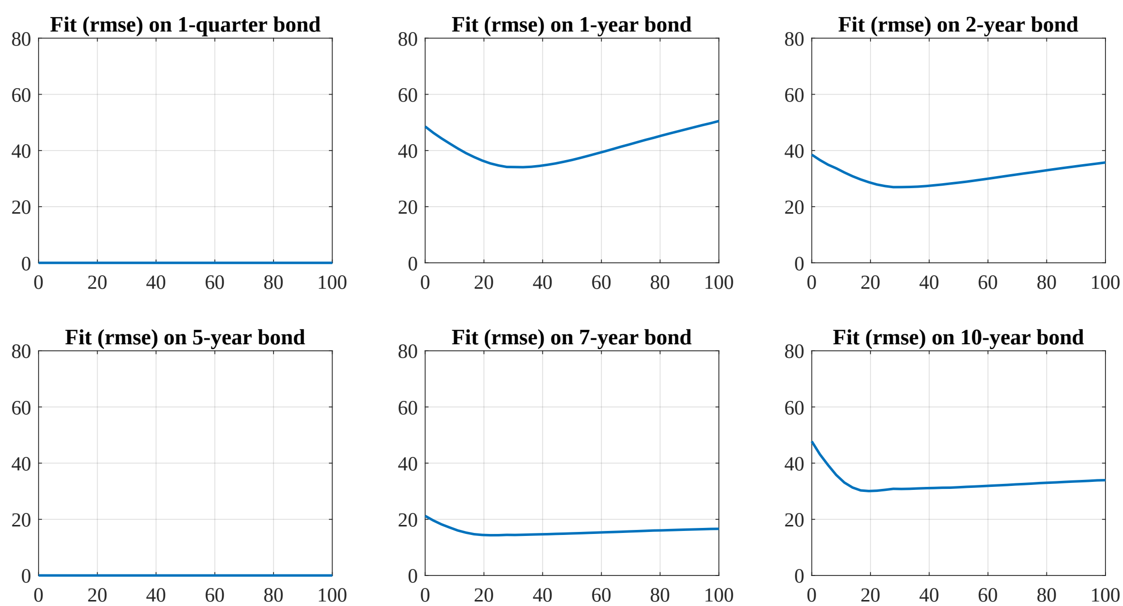
Figure 6: Estimation Results of the Model with Time-Varying Convenience Yield



Panel (a) Max Log Sharpe Ratio



Panel (b) Mean Bond Pricing Error

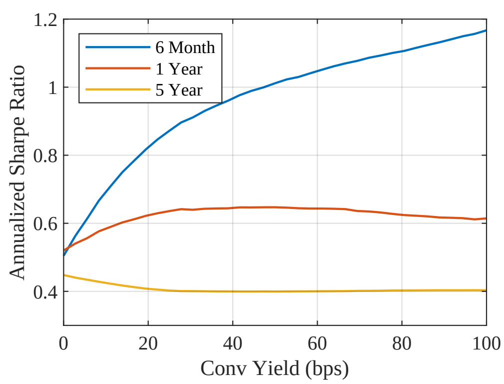


Panel (c) Bond Pricing Error by Tenor

Notes: This figure plots the SDF's maximum quarterly log Sharpe ratio and model fit as we vary the convenience yield on the 3-month Treasury bill. The x axis is the average annualized convenience yield adjustment on the 3-month Treasury bill, which is based on a constant times the CD-Treasury bill spread. The yields and the convenience yield are annualized, in basis points. Data are 1971Q3–2020Q4.

Besides the model fit, we also inspect if the models with different magnitudes of convenience yields have different implications for the bonds' Sharpe ratios. In Figure (7), we report the model-implied Sharpe ratios of the 6-month, 1-year, and 5-year bonds. In the presence of a binding good deal bound, a model with a zero or low convenience yield compresses bond Sharpe ratios below the maximal Sharpe ratio implied by the good deal bound. Specifically, the model-implied 6-month bond's Sharpe ratio is significantly lower than its empirical counterpart. As we increase the convenience yield in the model, we disentangle the link between the SDF volatility and the bond's Sharpe ratio. This allows us to obtain higher and more realistic Sharpe ratios, especially for the short-term bonds that have low return volatilities.

Figure 7: Implications for Bond Sharpe Ratios



Notes: This figure plots the bond Sharpe ratios as we vary the convenience yield on the 3-month Treasury bill. The x axis is the average annualized convenience yield adjustment on the 3-month Treasury bill, which is based on a constant times the CD-Treasury bill spread. The yields, the Sharpe ratios, and the convenience yield are annualized. Data are 1971Q3–2020Q4.

3.4 Estimation Result with a Term Structure of Convenience Yields

Finally, we use the spread between Refcorp strips and Treasury strips as the proxy for the tenor-specific convenience yield $\lambda_t(\tau)$[2]. Due to data availability, the sample is from 1991Q2 to 2020Q1,

[2]We thank Joslin, Li, and Song (2021) for sharing the data with us.

and only includes 3-month, 1-year, 2-year, 3-year, 4-year, 5-year, and 10-year bonds. Interestingly, in this data, the Refcorp-Treasury spread has a relatively flat term structure: the average 3-month convenience yield is 7 basis points whereas the average 1-year convenience yield is 6 basis points and the average 10-year convenience yield is 7 basis points. However, Joslin, Li, and Song (2021) document that this term structure is downward-sloping in recessions but upward-sloping in booms. In other words, the short-term Treasury debt's convenience yield tends to increase in recessions.

Using this term-structure of Refcorp-Treasury spreads, we consider a more flexible specification of Treasury convenience yields $\lambda_t(\tau)$ across the entire term structure. Then, we back out the risk-free rates without convenience yields $i_t(\tau)$ as

$$i_t(\tau) = y_t(\tau) + \lambda_t(\tau).$$

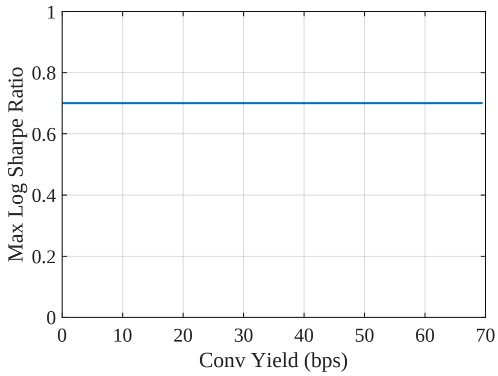
It is important to note that these convenience yields $\lambda_t(\tau)$ reflect the cumulative effect of the convenience effect for the next τ periods. It is different from the wedge $w_t(\tau)$ in a one-period Euler equation:

$$\exp(-w_t(\tau)) = \mathbb{E}_t \left[m_{t,t+1} \frac{p_{t+1}(\tau - 1)}{p_t(\tau)} \right],$$

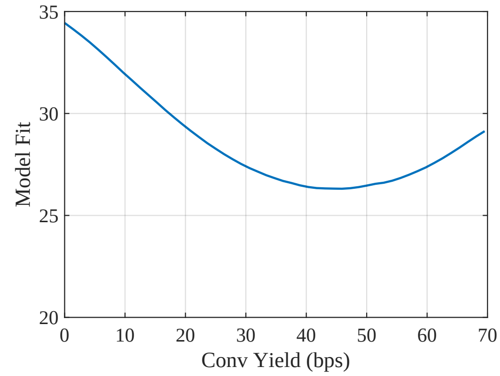
which is required for incorporating the term structure of convenience yields in the affine bond pricing model. As a result, we adopt a more convenient estimation strategy by directly estimating the affine term structure model using the risk-free rates $i_t(\tau)$ after stripping out the convenience yields, which amounts to repeating our estimation algorithm using these adjusted risk-free rates while assuming the convenience yields are zero.

We report our results in Figure (8). Even though this convenience yield adjustment barely changes the unconditional return difference between 3-month bonds and longer-term bonds, the conditional variation in the convenience yields at the business cycle frequency is helpful for improving the fit of the model. In particular, the 10-year yield has a much better fit once we account for a convenience yield above 40 basis points per annum.

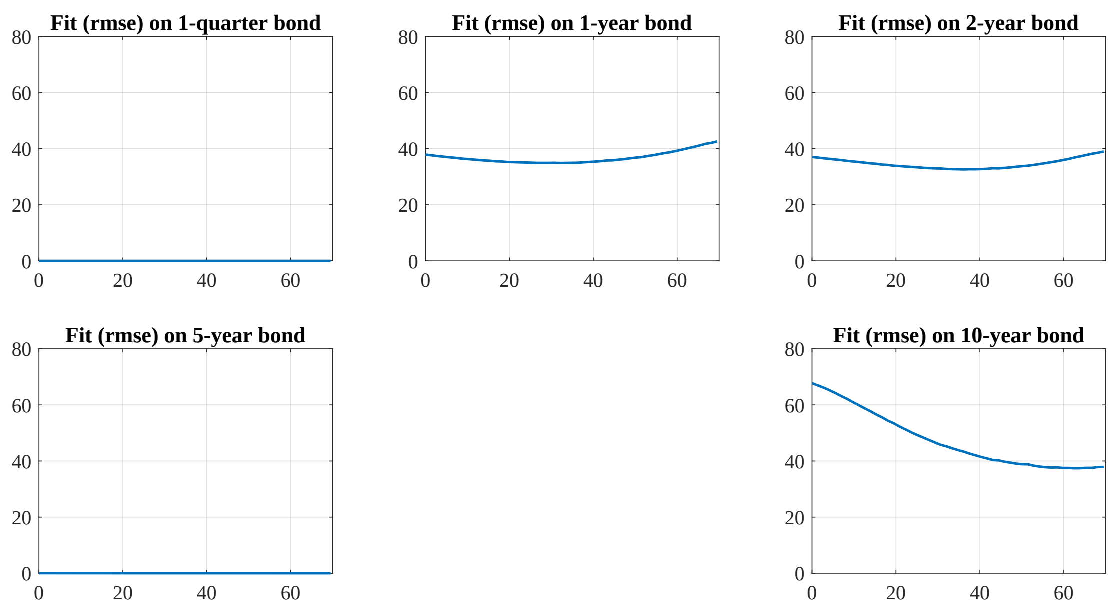
Figure 8: Estimation Results of the Model with Time-Varying Convenience Yield for the Entire Term Structure



Panel (a) Max Log Sharpe Ratio



Panel (b) Mean Bond Pricing Error



Panel (c) Bond Pricing Error by Tenor

Notes: This figure plots the SDF's maximum quarterly log Sharpe ratio and model fit as we vary the convenience yield on the 3-month Treasury bill. The x axis is the average annualized convenience yield adjustment on the 3-month Treasury bill, which is based on a constant times the Refcorp-Treasury spread. In this model, we also adjust other tenors by the same constant times the Refcorp-Treasury spreads for these tenors. The yields and the convenience yield are annualized, in basis points. Data are 1991Q2–2020Q1.

To understand this result, we note that, after all, we are working with a conditional asset pricing model, which implies the following Euler equation:

$$\mathbb{E}_t[m_{t+1}r_{t+1}(k)] = \exp(-\lambda_t(k)).$$

The high Sharpe ratio of short-term Treasury debt is concentrated during U.S. recessions. If we only use the subsample of NBER U.S. recessions, the Treasury debt below 6 months has a Sharpe ratio of 1.47 per annum, whereas its Sharpe ratio is 0.97 in the subsample of NBER U.S. expansions.[3] Therefore, in the absence of bond convenience yields, the issue with the affine model is that this conditionally high Sharpe ratio U.S. recessions requires a conditionally high SDF volatility, which also raises the unconditional SDF volatility.

While the convenience yield estimates based on the Treasury-Refcorp spread barely change the unconditional return difference between 3-month bonds and longer-maturity bonds, they relax the link from the conditionally high Sharpe ratio during recessions to the conditional SDF volatility according to our two-dimensional trade-off. As such, they lower the required SDF volatility exactly when it needs to be high, which allows us to obtain a much better model fit subject to the good-deal bound.

4 Conclusion

In this paper, we revisit the risk-return trade-off that characterizes the Hansen-Jagannathan bound in the presence of convenience yields. We show that the Hansen-Jagannathan bound needs to be extended into a two-dimensional trade-off: the presence of a high-Sharpe-ratio asset can be rationalized by either a higher SDF volatility or a high bond convenience yield level. When the asset return has a low volatility, a small convenience yield can "replace" a very high SDF volatility to explain the high Sharpe ratio.

We illustrate the economic significance of this trade-off in the context of the Treasury bond

[3]These Sharpe ratios, though annualized, are based on monthly returns to match the recession periods more precisely.

market. Consistent with our general characterization, we show that introducing a realistic convenience yield changes the trade-off between model fit and the SDF volatility. Specifically, both a time-varying convenience yield on the 3-month bond alone and convenience yields on the entire term structure help improve the model fit while maintaining a reasonable SDF volatility. Our model remains simple and tractable after incorporating convenience yields, and can be used to model asset pricing dynamics involving bond markets, such as Treasury bonds, corporate bonds, currency forwards, and international CIP deviations.

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Appendix

A Data

Data on zero-coupon yields on Treasurys are from the U.S. Treasury website. We collect data on the returns to U.S. Treasurys from CRSP. Returns are compounded to the quarterly frequency. We use the yield on 3-month Treasury for the risk-free rate. For estimation of our term structure models we collect data on the U.S consumer price index and GDP from FRED. Our primary merged quarterly sample runs from 1971Q3 to 2020Q4.

We construct a proxy for the convenience yield on 3-month Treasury. To measure the convenience yield we need an equivalent maturity risk-free rate that does not contain the liquidity/convenience premium of Treasurys. The most natural measure of convenience yields would be the difference between 3-month GC Repo rates (Nagel, 2016) and 3-month Treasury yield. That said, repo rates are only available starting in 1991 and in order to obtain good estimates of average excess returns we need a long sample. Given this, we collect data on 3-month CD rates which are available beginning in the 1960s. Since CD rates contain a credit risk component, their level is higher than that of repo rates. To account for this we scale the CD rates by the average ratio of repo rates to CD rates over the period for which they are both available. The time series of our convenience yield measure (3-month scaled CD rates minus Treasury yields) is presented in Figure (A1). There is substantial variation in the convenience yield, with the average value being 30 basis points per annum.

We also study the impact of convenience yields across the entire term structure of Treasurys. To do so, we use the spread between Resolution Funding Corporation (Refcorp) strips and Treasury strips as the proxy for the tenor-specific convenience yield $\lambda_t(\tau)$ (Longstaff, 2004; Joslin, Li, and Song, 2021)[4]. Due to data limitations, this sample is from 1991Q2 to 2020Q1, and only includes 3-month, 1-year, 2-year, 3-year, 4-year, 5-year, and 10-year bonds.

[4] We thank Joslin, Li, and Song (2021) for sharing the data with us.

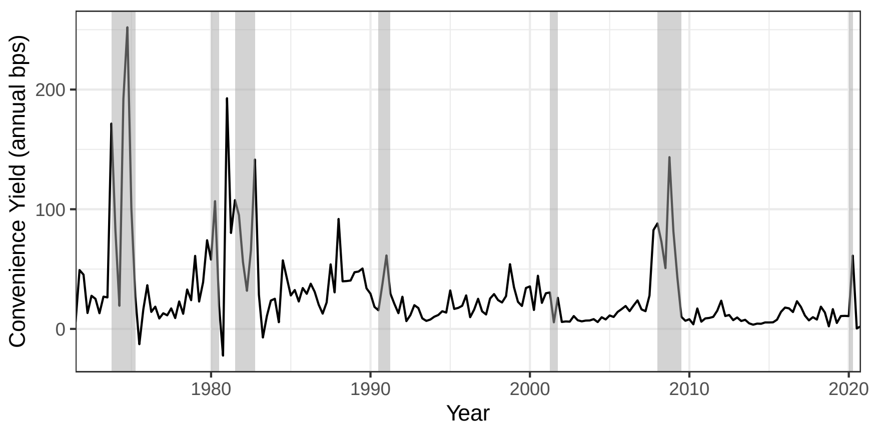

Figure A1: Convenience Yield on Short Term Treasurys over Time

Annual convenience yields (bps) on 3-month U.S. Treasurys. Convenience yields are measured as the difference between 3-month scaled CD rates and yields on 3-month U.S. Treasurys. CD rates are scaled by the ratio of 3-month GC repo rates and 3-month CD rates over the sample where both are available. Data are from CRSP and Bloomberg.

B Derivation

B.1 Model solution

Starting from the Lagrangian, reproduced below,

$$\mathbb{E}_0 \left[\sum_{t=1}^{\infty} \delta^t (u(c_t) + v(q_t(1)p_t(1); \theta_t)) + \sum_{t=1}^{\infty} \zeta_t \left(y_t + \sum_{j=1}^{H} q_{t-1}(j) p_t(j-1) - c_t - \sum_{j=1}^{H} q_t(j) p_t(j) \right) \right].$$

The first-order conditions w.r.t. c_t is

$$\delta^t u'(c_t) - \zeta_t = 0,$$

and the first-order conditions w.r.t. $q_t(1)$ is

$$\mathbb{E}_t \left[\delta^t v'(q_t(1); \theta_t) p_t(1) - \zeta_t p_t(1) + \zeta_{t+1} \right] = 0.$$

These two equations imply the following Euler equation for holding one-period bond:

$$1 - \frac{v'(q_t(1); \theta_t)}{u'(c_t)} = \mathbb{E}_t \left[\delta \frac{u'(c_{t+1})}{u'(c_t)} \frac{1}{p_t(1)} \right].$$

Similarly, the first-order condition w.r.t. $q_t(2)$ is

$$\mathbb{E}_t \left[\delta^{t+1} v'(q_{t+1}(1); \theta_{t+1}) p_{t+1}(1) - \zeta_t p_t(2) + \zeta_{t+2} \right] = 0,$$

which implies the following Euler equation for holding two-period bond:

$$1 - \mathbb{E}_t \left[\delta \frac{v'(q_{t+1}(1); \theta_{t+1})}{u'(c_t)} \frac{p_{t+1}(1)}{p_t(2)} \right] = \mathbb{E}_t \left[\delta^2 \frac{u'(c_{t+2})}{u'(c_t)} \frac{1}{p_t(2)} \right],$$

Let λ_t denote the log convenience yield on the 1-period bond, i.e.,

$$\exp(-\lambda_t) = 1 - \frac{v'(q_t(1);\theta_t)}{u'(c_t)},$$

then, the Euler equations are

$$\begin{aligned} \exp(-\lambda_t) &= \mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\frac{1}{p_t(1)}\right], \\ 1 - \mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\left(1-\exp(-\lambda_{t+1})\right)\frac{p_{t+1}(1)}{p_t(2)}\right] &= \mathbb{E}_t\left[\delta^2\frac{u'(c_{t+2})}{u'(c_t)}\frac{1}{p_t(2)}\right]. \end{aligned}$$

Moreover, we can express the Euler equation for the 2-period bond recursively as

$$\begin{aligned} 1 - \mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\left(1-\exp(-\lambda_{t+1})\right)\frac{p_{t+1}(1)}{p_t(2)}\right] &= \mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\frac{p_{t+1}(1)}{p_t(2)}\mathbb{E}_{t+1}\left[\delta\frac{u'(c_{t+2})}{u'(c_{t+1})}\frac{1}{p_{t+1}(1)}\right]\right] \\ &= \mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\frac{p_{t+1}(1)}{p_t(2)}\exp(-\lambda_{t+1})\right] \end{aligned}$$

Plug in

$$m_{t+1} = \delta\frac{u'(c_{t+1})}{u'(c_t)}.$$

Then, we can express the Euler equations as

$$\begin{aligned} \exp(-\lambda_t) &= \mathbb{E}_t\left[m_{t+1}\frac{1}{p_t(1)}\right], \\ 1 - \mathbb{E}_t\left[m_{t+1}\left(1-\exp(-\lambda_{t+1})\right)\frac{p_{t+1}(1)}{p_t(2)}\right] &= \mathbb{E}_t\left[m_{t+1}\frac{p_{t+1}(1)}{p_t(2)}\exp(-\lambda_{t+1})\right], \end{aligned}$$

which implies

$$1 = \mathbb{E}_t\left[m_{t+1}\frac{p_{t+1}(1)}{p_t(2)}\right].$$

We can also derive this equation from a different first-order condition

$$\mathbb{E}_t\left[-\zeta_t p_t(2) + \zeta_{t+1} p_{t+1}(1)\right] = 0.$$

Similarly, for any $h \geq 2$, we have

$$\begin{aligned}
0 &= \mathbb{E}_t\left[-\delta^t u'(c_t) p_t(h) + \delta^{t+1} u'(c_{t+1}) p_{t+1}(h-1)\right] \\
1 &= \mathbb{E}_t\left[m_{t+1} \frac{p_{t+1}(h-1)}{p_t(h)}\right].
\end{aligned}$$

B.2 Proof of Proposition 1

Proof. We start with the $h = 1$ case:

$$\begin{aligned}
p_t(1) &= \mathbb{E}_t\left[m_{t+1}\right] \exp(\lambda_t) \\
&= \exp(-y_t(1) + \lambda_t)
\end{aligned}$$

so

$$\begin{aligned}
A(1) &= -y_0(1) + \lambda_0 \\
(B(1))' &= -(e^{y1})' + (e^{\lambda})'
\end{aligned}$$

For the $h = 2$ case,

$$\begin{aligned}
p_t(2) &= \mathbb{E}_t\left[m_{t+1} p_{t+1}(1)\right] \\
&= \mathbb{E}_t\left[\exp(-y_t(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A(1) + B(1)'(\Psi z_t + \Sigma^{\frac{1}{2}}\varepsilon_{t+1}))\right] \\
&= \exp(-y_t(1) - \frac{1}{2}\Lambda_t'\Lambda_t + A(1) + B(1)\Psi z_t + \frac{1}{2}(B(1)'\Sigma^{\frac{1}{2}} - \Lambda_t')'(B(1)'\Sigma^{\frac{1}{2}} - \Lambda_t'))
\end{aligned}$$

which implies

$$A(2) = -y_0(1) + A(1) + \frac{1}{2}((B(1)))'\Sigma\,(B(1)) - (B(1))'\,\Sigma^{\frac{1}{2}}\Lambda_0,$$
$$(B(2))' = (B(1))'\,\Psi - (e^{y1})' - (B(1))'\,\Sigma^{\frac{1}{2}}\Lambda_1.$$

Similarly, for $h \geq 2$,

$$p_t(h) = \mathbb{E}_t\left[m_{t+1}p_{t+1}(h-1)\right]$$

which implies

$$A(h+1) = -i_0(1) + A(h) + \frac{1}{2}(B(h))'\Sigma B(h) - B(h)'\Sigma^{\frac{1}{2}}\Lambda_0,$$
$$(B(h+1))' = B(h)'\Psi - (e^{i1})' - B(h)'\Sigma^{\frac{1}{2}}\Lambda_1.$$

$\square$