ECON880, Computational Economics
Fall 2024, University of Wisconsin
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Problem Set #3 - Due Date 9/27/24

This problem set is designed to have you solve two simplified versions of the general equilibrium model of firm dynamics in Hopenhayn and Rogerson (1993, JPE, hereafter H-R). The key difference here (which makes it so much simpler) is that there are no firing costs (hence no need to keep track of a firm's past employment as a state variable). The specific tasks you need to complete are listed in **section 3** of this document.

1 Standard H-R Setup

1.1 Environment

There is a unit measure of identical households and a continuum of firms (mass not necessarily 1) which produce a homogeneous final product that sells at price p_t (the numeraire is units of labor). Any given firm's production function is given by $q_t = s_t n_t^\theta$ where $s_t \in \mathbb{R}_+$ is a productivity shock which follows a first order Markov process,iid across firms with conditional distribution $F(s' = s_{t+1}|s = s_t)$. Since the data on the size distribution of firms in Table 2 of H-R lists firm size (number of workers employed) in 4 bins (1-19, 20-99, 100-499, 500+), we will include a very low productivity state (which should induce exit) and consider a 5 state markov process which is persistent. Each period that the firm stays in the market, it bears a fixed cost c_f valued at p_t. The timing of an incumbent firm's decisions is:

1. enter period t in state s_{t-1}.
2. decide whether to exit. If the firm exits, it avoids c_f but receives profits of zero in all future periods.
3. If firm doesn't exit, it pays costs c_f,receives this period's shock s_t from $F(s_t|s_{t-1})$, chooses labor demand $n_t = N^d(s_t; p_t)$ and pays out its profits.

The timing for a potential entrant is:

1. decide whether to pay entry cost c_e

2. if an entrant decides to pay c_e (i.e. it enters), it is the same position as an incumbent in stage 3 except its draw s_t is from $\nu(s_t)$ (which is iid across entrants).

Household preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \left[\ln(C_t) - AN_t\right].$$

Profits of firms are distributed equally among all households.

1.2 Equilibrium

Since the only uncertainty is idiosyncratic, we will focus on a stationary equilibrium where $p_t = p$. There are two decisions of an incumbent firm: optimal employment $n^* = N^d(s_t; p)$ and optimal exit next period $x' = X(s_t; p) \in \{0, 1\}$ with convention that exit equals 1. The labor choice problem is static and n^* is obtained by solving the firm's profit maximization problem:

$$\pi(s; p) = \max_{n \geq 0} \; psn^{\theta} - n - pc_f \tag{1}$$

Having obtained n^*, the dynamic programming problem starting in stage 3 is:

$$W(s; p) = \max_{x' \in \{0,1\}} \left\{ \pi(s; p) + \beta(1 - x') \int_{s'} W(s'; p) dF(ds'|s) \right\} \tag{2}$$

where exit in next period's stage 2 is the only dynamic choice for the firm.

Conditional upon incurring the cost c_e, an entrant solves the same problem of an incumbent in stage 3 except in its draw s_t is from $\nu(s_t)$. Hence (2) is also the dynamic programming problem of an entrant conditional on entering. Furthermore, free entry requires

$$\int W(s; p)\nu(ds) \leq pc_e \tag{3}$$

with equality if the mass of new entrants in period t, denoted M, is strictly positive.

The distribution of firms at the beginning of stage 3 of period t is denoted $\mu(s; p)$. For any set $S_0 \in S$, the law of motion for the distribution of firms is:

$$\begin{aligned} \mu'(S_0) \quad = \quad & \int_{s' \in S_0} \left\{ \int_{s \in S} \left[1 - X(s; p)\right] dF(s'|s) d\mu(s) \right\} ds' \\ & + \int_{s' \in S_0} \left\{ \int_{s \in S} \left[1 - X(s; p)\right] dF(s'|s) M\nu(ds) \right\} ds'. \end{aligned} \tag{4}$$

Note that the exit decision in (4) accounts for stage 2 of next period. Defining the operator T^*, (4) can be written as[1]

$$\mu' = T^*(\mu, M; p). \tag{5}$$

In a steady state equilibrium where interest rates satisfy $\beta(1+r) = 1$ and all households own the same diversified portfolio of firms, the household's problem simplifies to a static labor/leisure choice since there is no desire to save when there is no uncertainty at the individual household or aggregate level:

$$\begin{aligned} \max_{C,N^s} \quad & u(C) - AN^s \\ s.t.: \quad & pC \leq N^s + \Pi \end{aligned} \tag{6}$$

where

$$\Pi(\mu, M; p) \;=\; \int \pi(s;p)d\mu(s;p) + M \int \pi(s;p)d\nu(s). \tag{7}$$

The solution to (6) implies a decision rule $N^s\left[p, \Pi(\mu, M; p)\right]$.

A **stationary competitive equilibrium** is a list $\{p^*, \mu^*, M^*\}$ such that: (i) the labor market clears $L^d(\mu^*, M^*; p^*) = N^s\left[p^*, \Pi(\mu^*, M^*; p^*)\right]$ where $L^d(\mu, M; p) = \int N^d(s;p)d\mu(s) + M\int N^d(s;p)d\nu(s)$ (ii) there is an invariant distribution over firms $\mu^* = T(\mu^*, M^*; p^*)$; and (iii) $\int W^e(s;p^*)\nu(ds) \leq p^* c_e$ with equality if $M^* > 0$.

1.3 Algorithm

. Basically, 2 "Do Loops"

1. Iterate over p_i until the entry condition is satisfied at p^* :

   (a) For each p_i, calculate $W_i(s;p_i)$.

   (b) Let $EC(p_i) \equiv \left[\int W(s;p_i)\nu(ds)\right]/p_i - c_e$. See first figure in the appendix of this problem set. If $EC(p_i) > 0$, then set $p_{i+1} < p_i$, otherwise set $p_{i+1} > p_i$. When $EC(p_{i+1}) \approx 0$, $p_{i+1} = p^*$.

2. Iterate over (μ_i, M_i) until the labor market clearing condition is satisfied at (μ^*, M^*).

   (a)

[1] The operator T^* is linearly homogeneous in μ and M jointly.

(b) Let $LMC(\mu_i, M_i) = L^d(\mu_i^{ss}(M_i), M_i; p^*) - N^s\left[p^*, \Pi(\mu_i^{ss}(M_i), M_i; p^*)\right]$. See second figure in appendix. If $LMC(\mu_i, M_i) > 0$, then set $M_{i+1} < M_i$, otherwise set $M_{i+1} > M_i$. When $LMC(\mu_{i+1}, M_{i+1}) \approx 0$, then $(\mu_{i+1}, M_{i+1}) = (\mu^*, M^*)$.

Since H-R prove linear homogeneity of (4) in (μ, M), you needn't add a loop prior to (2a) applying the T^* operator until convergence.

1.4 Calibration

Two parameters can be set independent of the firm distribution data: $\{\beta = 0.8, \theta = 0.64\}$. The distribution of shocks $\{F, \nu\}$, costs $\{c_f, c_e\}$, and the employment to population ratio (which pins down A) need to be set to match the firm distribution data. For this assignment, let $s \in \{3.98e^{-4}, 3.58, 6.82, 12.18, 18.79\}$. This grid of shocks gives employment levels of $\{1.3e^{-9}, 10, 60, 300, 1000\}$ which except for 0, are in the bins of firm size in Table 1.B. Let

$$F(s'|s) = \begin{bmatrix} 0.6598 & 0.2600 & 0.0416 & 0.0331 & 0.0055 \\ 0.1997 & 0.7201 & 0.0420 & 0.0326 & 0.0056 \\ 0.2000 & 0.2000 & 0.5555 & 0.0344 & 0.0101 \\ 0.2000 & 0.2000 & 0.2502 & 0.3397 & 0.0101 \\ 0.2000 & 0.2000 & 0.2500 & 0.3400 & 0.0100 \end{bmatrix}$$

This transition matrix gives an invariant distribution, which we take to be the entrant distribution $\nu(s)$,given by:

$$v(s) = \{0.37, 0.4631, 0.1102, 0.0504, 0.0063\}.$$

The way we calibrated $F(s'|s)$ was to change parameters until the invariant distribution matched the exit rate 37% in Table 1 and then take $(1 - 0.37)*$Table 1 bin to arrive at $v(s), s \in \{s_2, s_3, s_4, s_5\}$. For example, for the 500+ bin, the share of Total Firms is 0.01,so $v(s_5) = (1 - 0.37) * 0.01 = 0.0063$. Let $A = 1/200$, $c_f = 10$, and $c_e = 5$.

2 Adding Random Disturbances to Action Values

2.1 Environment

In this version of the model, we keep everything the same as in the first version except that now we introduce action-specific stochastic disturbances to the values associated with the discrete exit choice. The idea of this shock stems from McFadden (1973) and Rust (1987) that two agents with identical state variables (i.e. observables) may take different actions.

Recall that in the standard version, we may write the action value of staying $x' = 0$ and exiting $x' = 1$ respectively as follows:

$$V^{(x'=0)}(s;p) = psn^{*\theta} - n^* - pc_f + \beta \int_{s'} W(s';p)dF(ds'|s) \tag{8}$$

$$V^{(x'=1)}(s;p) = psn^{*\theta} - n^* - pc_f \tag{9}$$

Then the dynamic discrete choice problem in the standard H-R model in the previous section (i.e (2) can be re-written equivalently as follows:

$$W(s;p) = \max_{x' \in \{0,1\}} V^{(x')}(s;p) \tag{10}$$

In this section, the additional component we add to (10) is an action-specific disturbance $\epsilon^{x'}$ to the fundamental action values $V^{(x')}(s;p)$. The dynamic discrete choice problem with this new feature is given by:

$$W(\epsilon, s;p) = \max_{x' \in \{0,1\}} V^{(x')}(s;p) + \epsilon^{x'} \tag{11}$$

where $\epsilon^{x'}$ is an iid shock across actions, time, and agents drawn from some distribution $G(\epsilon)$. Due to $\epsilon^{x'}$, $V^{(x')}(s;p)$ is also different from (8) and (9) above in the following way:

$$V^{(x'=0)}(s;p) = psn^{*\theta} - n^* - pc_f + \beta \int_{s'} U(s';p)dF(ds'|s) \tag{12}$$

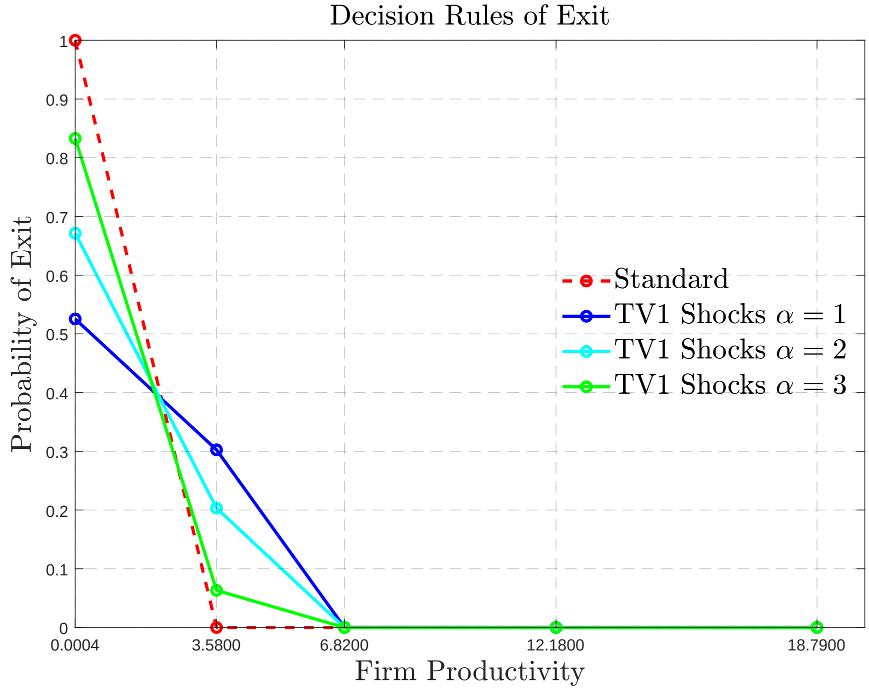
$$V^{(x'=1)}(s;p) = psn^{*\theta} - n^* - pc_f \tag{13}$$

where $U(s';p)$ is the *ex-ante* value function, which is given by integrating $W(\epsilon, s;p)$ over $\epsilon^{x'}$:

$$U(s;p) = \int W(\epsilon, s;p)dG(\epsilon) \tag{14}$$

Our object here is to compute the probability distribution of the exit decision rule for all states $(s;p)$. This could be extremely demanding computationally for general distributions $G(\epsilon)$ but the magic lies in extreme value distributions. Specifically, we will assume that $\epsilon^{x'}$ is drawn from a Type One Extreme Value (henceforth TV1) distribution. With this distribution, there is a closed form solution for the exit probability distribution (i.e. exit

Figure 1: Decision Rules Comparison



decision rule) given by:

$$\sigma^{(x')}(s;p) = \frac{exp\{\alpha V^{(x')}(s;p)\}}{\sum_{x'} exp\{\alpha V^{(x')}(s;p)\}} \tag{15}$$

where α parameterizes the variance of the distribution. Specifically, a higher α implies a lower variance of the shock (i.e. higher α is closer to the standard macro model where there are no shocks). Based on (15) above, the action that has the highest probability, which is termed the *modal action* in this framework, is the one that delivers the highest fundamental action value $V^{(x')}(s;p)$. This means that individual state $(s;p)$ still plays a crucial role in shaping the decision rules; however there is no longer a one-to-one relation. Intuitively, a smaller variance of $\epsilon^{x'}$ indicates higher probability assigned to the modal action. Hence, as the variance of $\epsilon^{x'}$ approaches 0 in the limit, $\sigma^{(x')}(s;p)$ approaches a degenerate distribution that has all mass on the modal action, which is equivalent to the decision rules in the version of the model without any $\epsilon^{x'}$ shocks. The parameter that governs variance of $\epsilon^{x'}$ is α: the bigger α is, the smaller variance of $\epsilon^{x'}$ will be. Figure 1 below shows that as we increase α, the decision rules become closer to the deterministic cut-off rule in the standard H-R model.

There is an important point illustrated in Figure 1. In the deterministic case, *all* firms with productivity $s = 4e^{-4}$ exit and *all* firms with higher productivity do not exit. Since productivity (which implies a number of workers) is observable, this implies that *all* firms with a small number of workers exit. In the data, not all firms with a small number of workers exit. In Figure 1, only half of the small employment firms exit while not all firms with higher numbers of workers choose to stay (i.e. not exit). The extreme value shocks break that perfect correlation between size and exit as in the data.

We see from (15) that in order to compute $\sigma^{(x')}(s;p)$, we need to find $V^{(x')}(s;p)$ first. The iterative algorithm for computing $V^{(x')}(s;p)$ for $x' \in \{0,1\}$ is as follows.

Algorithm

1. Set a tolerance level ε and take an initial guess of the ex-ante value function $U_0(s;p)$;

2. Compute $V^{(x')}(s;p)$ as in (12) and (13) using $U_0(s;p)$;

3. Compute the new ex-ante value function using the below formula:

$$U_1(s;p) = \frac{\gamma_E}{\alpha} + \frac{1}{\alpha} ln\Bigg(\sum_{x'} exp\{\alpha V^{(x')}(s;p)\} \Bigg)$$

   where γ_E is the Euler constant.

4. If $\|U_1(s;p) - U_0(s;p)\|_\infty < \varepsilon$, done; otherwise set $U_0(s;p) = U_1(s;p)$ and go back to 2.

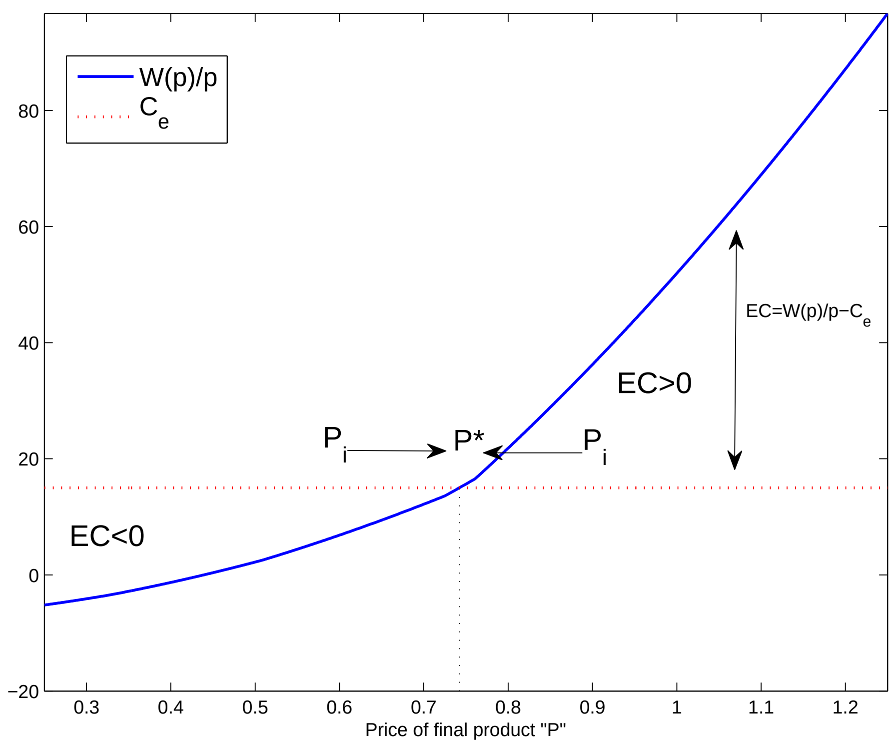
Having obtained $\sigma^{(x')}(s;p)$, the rest of the model can be solved in the same way as in the standard version. Since $U(s;p)$ and $\sigma^{(x')}(s;p)$ have closed form solutions, computation of this type of model takes much less time than the standard H-R setup. In addition, since the probabilistic decision rule $\sigma^{(x')}(s;p)$ is a much smoother function than the deterministic cut-off rule, the burden of seeking convergence of equilibrium objects (e.g. aggregate price) is simpler.

3 Tasks

1. Solve both versions of the model presented above. Use the parameter values in the calibration section of section 1 for both versions.

2. Compute the following model moments and fill in the table. Are they any different across model specifications? If yes, try to explain intuitively what drives the differences.

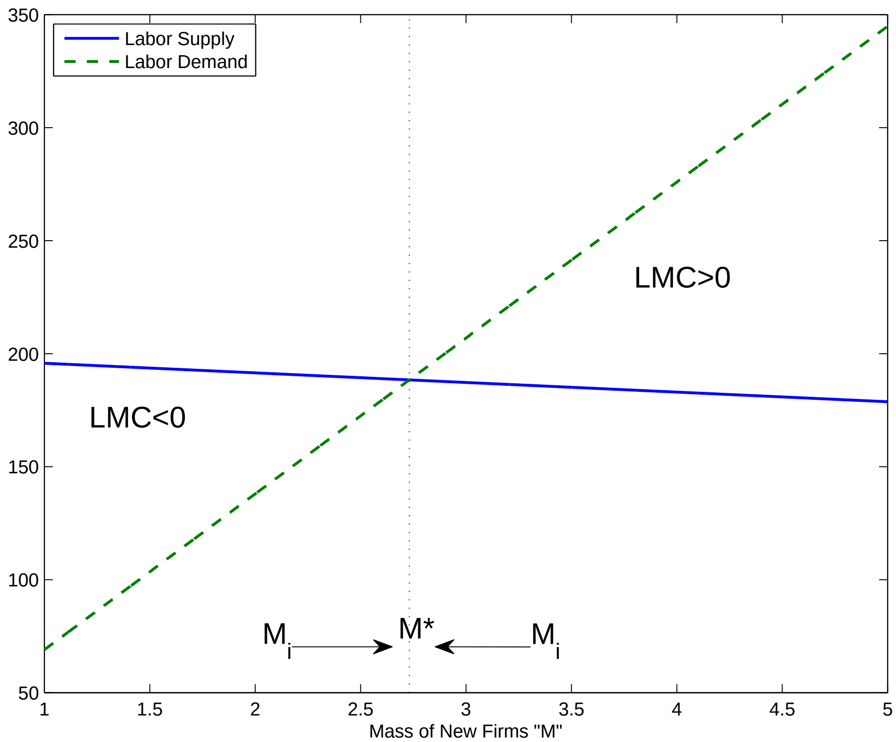
Variable	Standard	TV1 Shock $\alpha = 1$	TV1 Shock $\alpha = 2$
Price Level			
Mass of Incumbents			
Mass of Entrants			
Mass of Exits			
Aggregate Labor			
Labor of Incumbents			
Labor of Entrants			
Fraction of Labor in Entrants			



3. Plot the decision rules of exit in all model specifications you have solved. Are they any different? If yes, try to explain intuitively what drives the differences.

4. How does the exit decision rule change if c_f rises from 10 to 15?

4 Appendix



350
300
250
200
150
100
50
1
1.5
2
2.5
3
3.5
4
4.5
5
Labor Supply
Labor Demand
LMC>0
LMC<0
M*
M
i
M
i
Mass of New Firms "M"