Kurt Godel's Incompleteness Theorems were remarkable milestones in mathematics. His theorems indicate that mathematics has limitations; i.e., he showed that there are true statements that cannot be proved within some system of mathematics.

Godel has two Incompleteness Theorems. The First Incompleteness Theorem states that for any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F. The Second Incompleteness Theorem states that for any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself.

If we're dealing with a system like the natural numbers, the First Theorem says that the natural number system has a true sentence which cannot be proved, whereas the second theorem says that the consistency of the natural number system cannot be proved by using its own system of proofs. Note, we refer to the natural number system as arithmetic.

To exemplify incompleteness in terms of Godel's theorems, let's imagine a computer that carries out a set of instructions and outputs an answer according to those instructions. This computer's input and output consist of integers and utilize the following operations: addition, subtraction, multiplication, and division. If we wanted the computer to determine if a number, M, is prime, then we would tell it to divide M by every integer between 1 and $M - 1$. Godel's theorem states that there are questions which involve only the arithmetic of integers that the computer cannot determine and, therefore, there are statements that the computer can't declare if it is true or false. At this, we arrive at our example of incompleteness.

Another example of an incomplete system is arithmetic; that is, arithmetic is negation incomplete. How so? We start by assuming that $N = (\mathbb{N}, +, \times)$ to be a consistent. Also, we write $N \vdash X$ to mean "X is derivable in N." Now, we may use one of Godel's Incompleteness Theorems to validate this claim, namely, his First Incompleteness Theorem. Given Godel's First Incompleteness Theorem, there exists a sentence X in N such that $N \nvdash X$ and $N \nvdash \neg X$. To prove this theorem, we need the typical laws of logic, as well as make use of a provability predicate, which helps us determine whether or not a sentence can be proven.

Moreover, Godel designed a way to talk about mathematics using mathematics. He created Godel numbering, and what this number does is assigns a unique number (or code) to a particular statement. Thus, every statement about mathematics can be turned into a number. The practicality of this is that we can now discuss proofs in mathematics using Godel numbers; that is, we can talk about proof mathematically. The Godel numbers can be used to prove Godel's Incompleteness Theorems.

Godel's Incompleteness theorem also says that there exists a true sentence in N which is not provable in N. This is important because it shows that no axiomatic system for arithmetic may be complete. Revisiting his second Incomplete theorem, Godel shows that the consistency of arithmetic cannot be mechanically constructed from arithmetic. Hence, we may show that a statement like $N \nvdash P(0=1)$ is correct, where, for example, $P(0=1)$ means $0=1$ is provable, which is of course inconsistent in N.

We shall finally discuss the crux of Godel's proof of the Incompleteness Theorems. Godel's proof for the First Incompleteness Theorem consists of multiple parts. For the First Theorem, it requires the formal system to be ω-consistent or 1-consistent, there must be representability, there should occur arithmetization of the formal language, and it requires diagonalization. We will unpack how each of these are implemented. As for the second Incompleteness theorem, we require the provability predicate and/or use several $\sum_0^1$-formulas.

For the First Incompleteness Theorem, the reason why we want to consider a formal system F to be ω-consistent or 1-consistent is to make use of the assumption that the natural numbers satisfy the axioms of that particular formal system F. Note, a formal system is ω-consistent if it is not the case that for some formula A(x), both $F \vdash \neg A(n)$ for all n and $F \vdash \exists x A(x)$. We also want representability, which explains whether a set is recursive or recursively enumerable. This is useful because we can determine whether or not that set is decidable within a formal system F. Another part of his proof deals with arithmetization, or Godel numbering. As explained above, Godel numbering is practical because synactical properties, relations, and operations are shown in arithmetic; and we're using mathematics to describe mathematics. Then, we need the concept of diagonalization, which is very necessary because it uses a computable function to talk about decidable sets. In fact, the diagonalization lemma states that by letting $A(x)$ be an arbitrary formula of the language F with only one free variable, it can be shown that a sentence D can be mechanically constructed such that $F \vdash D \leftrightarrow A([D])$, where $[D]$ is the Godel number of D. Using all of these components, we use the negated provability predicate $\neg Prov_F(x)$ to claim that it gives a sentence G_F such that $F \vdash G_F \leftrightarrow \neg Prov_F([G_F])$. Thus, we see that G_F is true $\iff$ it is not provable in F.

As for his second Incompleteness theorem, we shall prove this using some statement like $0=1$, denote it by $\perp$, and define the consistency of the system to be $\neg Prov_F([\perp])$ (or abbreviate it by $Cons(F)$. Then, we must show that $F \nvdash Cons(F)$. For the second theorem, we prove it using a set of derivable conditions to show that $F \vdash G_F \leftrightarrow Cons(F)$, which yields that $Cons(F)$ is unprovable by the First Incompleteness Theorem, and so the consistency of a formal system cannot be proved using the mechanical construction of arithmetic. An alternative way of proving his Second Incompleteness Theorem is by using Feferman's alternative approach, which makes use of $\sum_0^1$-formulas and representability to logically deduce that $Cons(F)$ is not provable in F.

Given Godel's Incompleteness theorem(s), we now understand that if you could prove that something does not have a proof, then oddly, it would prove that it must be true. But what if that same thing were false? If it were false, then that means there's a way to mechanically construct it to be false; that is, if it is false, then it has to be provably false.