Risk Analytics

Monitoring and reporting risk

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Talking about risk

Risk is notoriously hard to quantify:

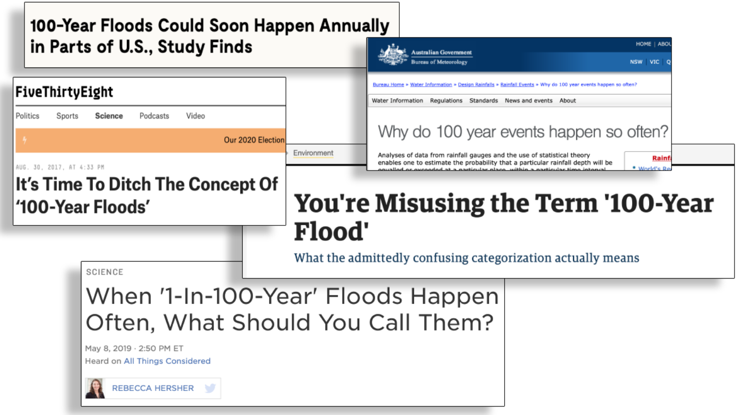


Figure: Even an event with 1% chance can happen regularly (!).

Risk management

- Many management decisions are taken under risk.
- Understanding and communicating risk effectively is important.
- It is also important to check if your risk model is well-calibrated.

- In this module:
  1. Modelling Risks
  2. Risk measures
  3. Uncertainty
  4. Backtesting

Modelling Risks

- Risk = possibility of loss or of an unfavourable outcome happening associated with an action.
- To model risk we use language of probability theory. Risks are represented by random variables mapping unforeseen future states of the world.
- The risks which interest us include:
  - number of patients in the hospital
  - stocks and bonds rate
  - sea-level
  - heatwave
  - wildfire
  - a demand too big to be met
  - ...

Variable of interest or risk factor

- Denote X_t the variable value at time t or the risk factor at time t. We assume this random variable is observable at time t.
- Suppose we look at risk from perspective of time t and we consider the time period $[t, t+1]$. The value X_{t+1} at the end of the time period is unknown to us.

Conditional or Unconditional Distribution?

- This issue is related to the time series properties of $(X_t)_{t\in\mathbb{N}}$, the series of risk factor changes. If we assume that $X_t, X_{t-1}, \ldots$ are iid random variables, the issue does not arise. But, if we assume that they form a strictly stationary time series then we must differentiate between conditional and unconditional.
- Many standard accounts of risk management fail to make the distinction between the two.
- If we cannot assume stationarity of the time series at risk for at least some window of time extending from the present back into intermediate past, then any statistical analysis is difficult.

The Conditional Problem

- Let $\mathcal{H}_t$ represent the history of the risk factor up to the present.
- In the conditional problem we are interested in the distribution of X_{t+1} given $\mathcal{H}_t$, i.e. the conditional (or predictive) distribution of the risk factor for the next time interval given the history of risk factor developments up to time t.
- This problem forces us to model the dynamics of the risk factor time series

The Unconditional Problem

- In the unconditional problem we are interested in the distribution of X_{t+1} when X is a generic variable of risk factor change with the same distribution F_X as $X_t, X_{t-1}, \ldots$.
- When we neglect the modelling of dynamics we inevitably take this view. Particularly when the time interval is large, it may make sense to do this.

More Formally

Conditional loss distribution: $F_{[X_{t+1} | \mathcal{H}_t]}$
Unconditional loss distribution: F_X

Risk measure: Value-at-Risk

- In our usual context, X is an indicator with higher values being worse.
- The Value-at-Risk of X at level α, denoted $\mathrm{VaR}_\alpha(X)$, is the α-quantile of F_X:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\},$$

  with α typically high (close to 1).

Interpretation 1: With probability α, the variable X will be smaller than $\mathrm{VaR}_\alpha(X)$.
Interpretation 2: $\mathrm{VaR}_\alpha(X)$ is the value that the variable can exceed with low probability $1 - \alpha$.

Return levels vs Value-at-Risk

- Value-at-Risk is typically used in finance and insurance.
- In other fields, the notion of return level is preferred.
- The n-period return level is $R_n = F^{-1}(1 - \frac{1}{n})$, the $(1 - 1/n)$-quantile. The name refers to the following idea:

$$\begin{aligned} F(y_n) = 1 - 1/n &\Longleftrightarrow \mathbb{P}(Y \leq y_n) = 1 - 1/n \\ &\Longleftrightarrow \mathbb{P}(Y > y_n) = 1/n, \end{aligned}$$

  so that, on average, among n draws of Y, one exceeds the value y_n.
- But this doesn't mean only one excess happens!
  - Under independence, the number of excesses among n is Binomial$(n, 1/n)$, so the probability of having at least two excesses among $n = 100$ is 26%.

Risk measure: Expected Shortfall

- Value-at-Risk gives an upper bound on risk.
- It doesn't quantify what happens if $X > x$, only the probability of this happening.
- But what if in fact $X > x$?

The Expected Shortfall at level α is the average value of X, when it is higher than $\mathrm{VaR}_\alpha(X)$:

$$\mathrm{ES}_\alpha(X) = \mathbb{E}\left(X \mid X > \mathrm{VaR}_\alpha(X)\right)$$

Interpretation: If I have an extreme event, how bad does it get?

For a continuous distribution,

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_\beta(X) \, \mathrm{d}\beta$$

VaR and ES in Visual Terms

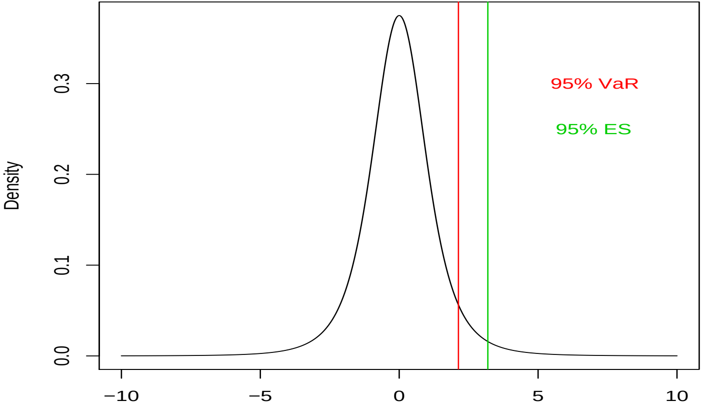


Figure: Risk measures at the 95% level. The area right of the red line has probability 0.05. $\mathrm{VaR}_\alpha(X) \simeq 2$, $\mathrm{ES}_\alpha(X) \simeq 3.5$.

Ratio of risk measures

The expected shortfall can be much larger than the Value-at-Risk.

Examples

$$X \sim N(\mu, \sigma^2) \quad \Longrightarrow \quad \lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = 1$$

$$X \sim t_\nu \quad \Longrightarrow \quad \lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = \frac{\nu}{\nu - 1} > 1$$

Return Levels: block maxima approach

We address the following question:

- What is the k–period return level $R_{n,k}$?

In this question we define and estimate a rare stress or scenario loss. $R_{n,k}$, the k n–block return level, is defined by

$$\mathbb{P}\left(M_n > R_{n,k}\right) = 1/k\,;$$

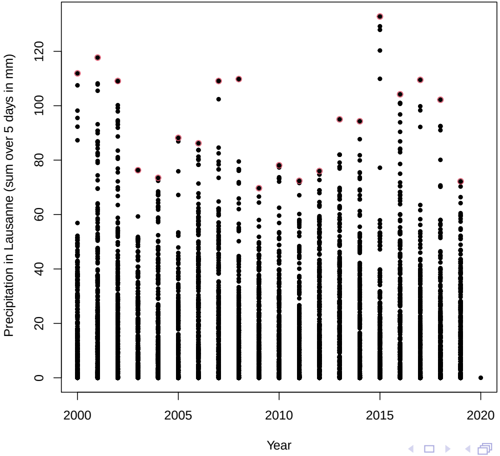
i.e. it is that level which is exceeded in one out of every k n–blocks, on average.
We use the approximation by the $(1-1/k)$-quantile of a GEV

$$R_{n,k} \approx H^{-1}_{\xi,\mu,\sigma}(1-1/k) \approx \mu + \sigma\left(\left(-\log(1-1/k)\right)^{-\xi} - 1\right)/\xi\,.$$

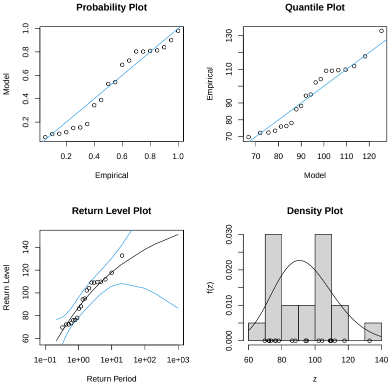
We wish to estimate this functional of the unknown parameters of our GEV model for maxima of n–blocks.

Return Levels: block maxima approach
Example: Rainfall yearly maxima

- What is the 10–period return level $R_{365,10}$? i.e. what level is exceeded in one out of every 10 years, on average?



Return Levels: block maxima approach
Example: Rainfall yearly maxima



$$\hat{R}_{365,10} \approx \hat{H}^{-1}_{\hat{\xi},\hat{\mu},\hat{\sigma}}(1-1/k) \approx 87.2 + 16.5\frac{(-\log(1-1/10))^{0.18}-1}{-0.18}$$

≈ 117 mm is the estimated value of daily rainfall (sum over the last 5 days) that can be exceeded once every 10 years.

Quantifying uncertainty

- Whenever reporting or assessing a measure of risk, one should always take into account uncertainty.
  - EVT model is an approximation for finite data.
  - Parameters are estimated with some uncertainty (e.g. profile likelihood).
- More flexible: simulation from the fitted model (accounts for uncertainty in formulas).
- One can even simulate from estimator distribution (accounts for parameter uncertainty).

Quantifying uncertainty: simulation example

- Concrete example
- $X_i \sim \mathrm{GEV}(\hat{\mu}, \hat{\sigma}, \hat{\xi})$, with the usual maximum likelihood distributions

$$\hat{\mu} \sim N(397, 5), \quad \hat{\sigma} \sim N(48, 2.9), \quad \hat{\xi} \sim N(-0.25, 0.03)$$

- Algorithm to estimate $\mathrm{VaR}_\alpha(X_i)$:
  1. For $b = 1, \ldots, B$
     1.1 Draw parameters μ_b, σ_b, ξ_b from the above distributions
     1.2 Set
     $$y_b = \mu_b + \frac{\sigma_b}{\xi_b}\left((-\log(1-\alpha))^{-\xi_b} - 1\right)$$
  2. The $(y_b)_{b=1}^B$ are an approximate distribution for $\mathrm{VaR}_\alpha(X_i)$.
- Parametric, because we assume the ML distributions are accurate. Could do a full bootstrap by re-fitting the model on resampled data in step 1.1.

The POT Model: summary

When the data form a stationary time series then the timing and magnitude of threshold exceedances are both of interest. The POT (peaks-over-thresholds) model is a limit model for threshold exceedances in iid processes. The limit is derived by considering datasets $X_1, \ldots, X_n$ and thresholds u_n that increase with n and letting $n \to \infty$. The limit model says that:

- Exceedances occur according to a homogenous Poisson process in time.
- Excess amounts above the threshold are iid and independent of exceedance times.
- Distribution of excesses is generalised Pareto (GPD).

GPD Model: Calculation of VaR and Expected Shortfall

Let $X \sim F$ with $F \in \mathrm{MDA}(\mathrm{H}_\xi)$ and consider the excess $W = X - u$ where u is a high threshold. We have $W \sim \mathrm{GPD}(\xi, \beta)$.

- For $\alpha \geq F(u)$, the Value at Risk is:

$$\mathrm{VaR}_\alpha(X) = u + \frac{\beta}{\xi}\left(\left(\frac{1-\alpha}{\bar{F}(u)}\right)^{-\xi} - 1\right).$$

  where $\bar{F}(u) = \mathbb{P}(X > u)$ can be approximated by N_u/n.

- Assuming that $\xi < 1$ the associated expected shortfall can be calculated easily. We obtain

$$\mathrm{ES}_\alpha(X) = \frac{\mathrm{VaR}_\alpha(X)}{1-\xi} + \frac{\beta - \xi u}{1-\xi}.$$

Ratios of Risk Measures

It is interesting to look at how the ratio of the two risk measures behaves for large values of the quantile probability α. We can show that

$$\lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = \begin{cases} (1-\xi)^{-1}, & \xi \geq 0, \\ 1, & \xi < 0, \end{cases}$$

so that the shape parameter ξ of the GPD effectively determines the ratio when we go far enough out into the tail.

Estimating Tails and Risk Measures

- Recall that for $x > u$ in the GPD model,

$$\bar{F}(x) = \bar{F}(u) \left(1 + \xi \frac{x - u}{\beta} \right)^{-1/\xi}$$

- Tail probabilities, VaRs and ESs are all given by formulas of the form $g(\xi, \beta, \bar{F}(u))$. We estimate these quantities by replacing ξ and β by their estimates and replacing $\bar{F}(u)$ by the simple empirical estimator N_u/n (fraction of excesses).
- For tail probabilities we use the estimator

$$\hat{\bar{F}}(x) = \frac{N_u}{n} \left(1 + \hat{\xi} \frac{x - u}{\hat{\beta}} \right)_+^{-1/\hat{\xi}},$$

  which is valid for $x \geq u$.
- Asymmetric confidence intervals can be constructed using profile likelihood method.
  - A high u reduces bias in estimating excess function.
  - A low u reduces variance in estimating excess function and $F(u)$.

Purposes of Risk Measurement

Risk measures are used for the following purposes:

- Determination of risk capital. Risk measure gives amount of capital needed as a buffer against (unexpected) future losses to satisfy a regulator. *Basel committee. Solvency*.
- Management tool. Risk measures are used in internal limit systems. *High demands*.
- Protect against climate/meteorological extremes. *Floods, droughts, heatwaves*.
- Protect against technological problems. *Cyber attacks*.

Backtesting (I)

Suppose we observe a time series of T iid data $x_1, \dots, x_T$ on which we fit a model and want to backtest the VaR/risk measure.

- We split the data into two parts: one $(x_1, \dots, x_m)$ is the historical data we will use to estimate the VaR and $x_{m+1}, \dots, x_T$ is a window of $n = T - m$ observations on which we want to backtest the VaR_α.
- At each time $t = 1, \dots, n$, we use a window of time of size w of the historical data $x_{m+t-w}, \dots, x_{m+t-1}$ to estimate the VaR at time $m + t$.
- We compare the estimated VaR_α with the observed value at time $m + t$, that is x_{m+t}.

Backtesting (II)

- A violation occurs whenever $x_{m+t} > \widehat{\mathrm{VaR}}_\alpha(x_{m+t})$.
- If the model is correct, for a level α, a violation occurs according to a Bernoulli law with probability $1-\alpha$.
- Considering that the violations are independent, the total number of violations over the n tested data follows a Binomial distribution with size n and probability $1-\alpha$.
- We can perform a Binomial test to formally test the null hypothesis that the observed number of violations (number of times the observation is larger than the estimated VaR) is equal to the expected number of violations, that is $n(1-\alpha)$.