Risk Analytics

Time Series

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Basics of Time Series Analysis

Definition A time series is a collection of random elements indexed by an index set.

Mathematically, we write such a collection $(X_t)_{t \in I}$, where I is an index set.

Usually, the random elements are random variables with values in $\mathbb{R}$ or $\mathbb{R}^d$.

If the index set $I \subset \mathbb{R}$ (real numbers), it is a continuous time series.
If the index set $I \subset \mathbb{N}$ or $\mathbb{Z}$ (integers), it is a discrete time series.

In this course, we will work with real-valued discrete time series.

(In what follows, we take $I = \mathbb{Z}$ for ease of presentation.)

Moments of a Random Variable

The moments of a random variable X are indicators of the shape of its distribution.

- First moment is the expectation or the mean of the variable:

$$\mu = \mathbb{E}[X]$$

- Second moment is the variance of the variable:

$$\sigma^2 = \mathbb{E}[(X - \mu)^2]$$

- Third moment is the skewness of the variable:

$$\gamma = \mathbb{E}\left[\left(\frac{X - \mu}{\sigma}\right)^3\right]$$

Stationarity of a Time Series

Definition

A time series $(X_t)_{t \in \mathbb{Z}}$ is stationary if

$$(X_{t_1}, \ldots, X_{t_n}) \stackrel{\mathrm{d}}{=} (X_{t_1+h}, \ldots, X_{t_n+h})$$

for all $t_1, \ldots, t_n, h \in \mathbb{Z}$.

In particular this means that X_t has the same distribution for all $t \in \mathbb{Z}$, and this distribution is known as the stationary distribution (or marginal distribution).

Unlike an independent series, stationarity does not forbid X_t being dependent on previous values, although X_{t+h} must have the same dependence on its previous values as X_t.

A stationary time series cannot have trend, seasonality or other deterministic cycles.

How to make a time-series stationary ?

Motivation

- Stationary time-series are easier to model.
- Many theoretical results are only valid under stationarity conditions.

Tranformation techniques

- **Differencing** $\tilde{X}_t = X_t - X_{t-1}$
  - Helps to stabilize the mean.
  - Eliminates trends.
  - Can also remove seasonality, if differences are taken appropriately (e.g. differencing observations 1 year apart to remove a yearly seasonality).
- **Logarithm** $\tilde{X}_t = \log(X_t)$
  - Helps to stabilize the variance.
  - Removes multiplicative dependency.

Autocovariance

- Let $(X_t)_{t\in\mathbb{Z}}$ be a time series.
- The autocovariance function of (X_t) is defined by

$$\gamma(t,s) = \mathrm{cov}(X_t, X_s) = \mathbb{E}\left[(X_t - \mathbb{E}(X_t))(X_s - \mathbb{E}(X_s))\right].$$

- If (X_t) is a stationary time series, then

$$\gamma(t,s) = \gamma(t+h, s+h), \quad \text{for all } t,s,h \in \mathbb{Z}$$

- This implies that covariance only depends on the separation in time of the observations $|t-s|$, also known as the lag.
- A time series for which the first two moments are constant over time (and finite) and for which this condition holds, is known as covariance stationary, or second-order stationary.

The Autocorrelation Function

- Rewrite the autocovariance function of a stationary time series as

$$\gamma(h) = \gamma(h, 0) = \mathrm{cov}(X_h, X_0), \quad h \in \mathbb{Z}.$$

- Note that $\gamma(0) = \mathrm{var}(X_0) = \mathrm{var}(X_t)$, $t \in \mathbb{Z}$.
- The autocorrelation function is given by

$$\rho(h) = \gamma(h)/\gamma(0), \quad h \in \mathbb{Z}.$$

  Observe that $\rho(0) = 1$.
- We refer to $\rho(h), h = 1, 2, 3, \ldots$ as autocorrelations or serial correlations.

Time Domain

- If we study dependence structure of a time series by analysing the autocorrelations we analyse (X_t) in the time domain.
- An important instrument in the time domain is the correlogram, which gives empirical estimates of serial correlations.

Correlogram

- Given time series data $X_1, \ldots, X_n$ we calculate the sample autocovariances

$$\widehat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_t - \overline{X})(X_{t+h} - \overline{X}) \quad \text{where} \quad \overline{X} = \sum_{t=1}^{n} X_t / n.$$

and the sample autocorrelations

$$\widehat{\rho}(h) = \widehat{\gamma}(h) / \widehat{\gamma}(0), \; h = 0, 1, \ldots .$$

- The correlogram (ACF) is the plot

$$\{(h, \widehat{\rho}(h)), \; h = 0, 1, 2, \ldots\}.$$

- For many standard underlying processes, it can be shown that the $\widehat{\rho}(h)$ are consistent, and asymptotically normal estimators of the autocorrelations $\rho(h)$.

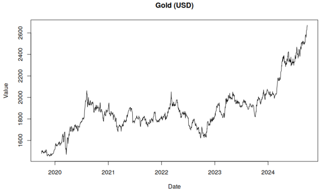
Correlogram (variant)

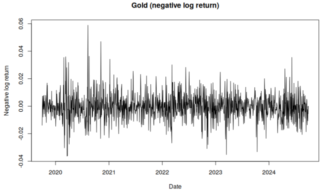
- A variant to the AutoCorrelation Function (ACF) is the Partial AutoCorrelation Function (PACF).
- The partial autocorrelation π_k at lag k is the correlation that results after removing the effect of any correlations due to the terms at shorter lags.
- Formally, it is calculated recursively as

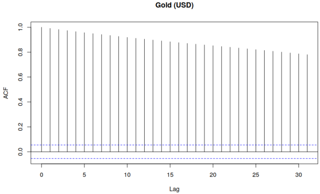
$$\pi_n = \pi_{n,n} = \frac{\rho(n) - \sum_{k=1}^{n-1} \pi_{n-1,k}\, \rho(n-k)}{1 - \sum_{k=1}^{n-1} \pi_{n-1,k}\, \rho(k)}$$

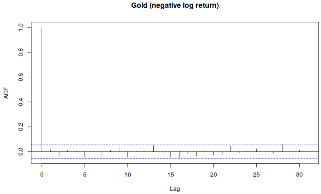
where $\pi_{n,k} = \pi_{n-1,k} - \pi_n\, \pi_{n-1,n-k}$ for $1 \leq k \leq n-1$.

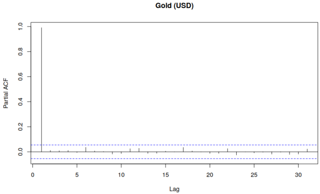
Example: Gold rate

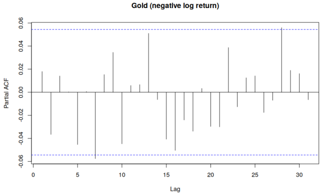












White Noise Processes

- Processes with no appreciable dependence structure in the time domain.
- A white noise process is a covariance stationary time series process whose autocorrelation function is given by

$$\rho(0) = 1, \quad \rho(h) = 0, \; h \neq 0.$$

  That is, a process showing no serial correlation.
- A strict white noise process is a process of independent and identically distributed (i.i.d.) random variables.

Classical ARMA Processes

- Classical ARMA (AutoRegressive Moving Average) processes are constructed from white noise.
- Let $(\varepsilon_t)_{t \in \mathbb{Z}}$ be a white noise process with mean zero and finite variance σ_ε^2.
- The (ε_t) form the innovations that drive the ARMA process.

Moving Average Process

- These are defined as linear sums of the noise (ε_t).
- (X_t) follows a MA(q) process if

$$X_t = \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t.$$

Autoregressive Process

- These are defined by stochastic difference equations, or recurrence relations.
- $(X_t)_{t\in\mathbb{Z}}$ follows a AR(p) process if for every t

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

  where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise.
- In order for these equations to define a covariance stationary causal process (depending only on past innovations) the coefficients ϕ_j must obey certain conditions.

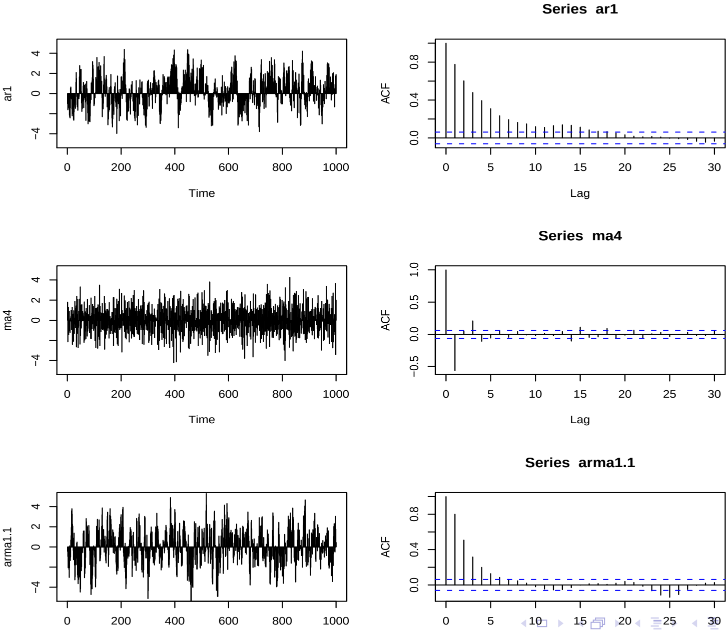
ARMA Process

- Autoregressive and moving average features can be combined to form ARMA processes.
- $(X_t)_{t\in\mathbb{Z}}$ follows an ARMA(p,q) process if for every t

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t, \qquad (1)$$

  where $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a white noise.
- Again, there are conditions on the coefficients ϕ_j and θ_i for these equations to define a covariance stationary causal process.
- The autocorrelation functions of ARMA processes show a number of typical patterns, including exponential decay and damped sinusoidal decay.

ARMA Examples



Example: AR(1) Process

Consider an AR(1) process:

$$X_t = \varepsilon_t + \phi X_{t-1} \tag{2}$$

We can calculate recursively that

$$\begin{aligned} X_t &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2 X_{t-2} \\ &= \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \phi^3 X_{t-3} \\ &= \sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i} \end{aligned} \tag{3}$$

Thus this process is an infinite order MA process (or MA(∞)). The condition for a stationary solution to (2) is that the sum (3) should converge. We require $|\phi| < 1$.

Example: AR(1) Process, Continued

- The autocovariances of this process can be calculated:

$$
\begin{aligned}
\gamma(s) &= \mathrm{cov}(X_t, X_{t+s}) \\
&= \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i}\right)\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t+s-j}\right)\right] \\
&= \frac{\phi^{|s|}\sigma_\varepsilon^2}{1-\phi^2}
\end{aligned}
$$

- Therefore $\rho(s) = \gamma(s)/\gamma(0) = \phi^{|s|}, \quad s \in \mathbb{Z}$.
- The serial correlations decay exponentially with possibly alternating sign.

Stationarity conditions

AR(p) process $$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t$$

- For an AR(1) model: $-1 < \phi_1 < 1$.
- For an AR(2) model: $-1 < \phi_2 < 1$, $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$.
- When $p \geq 3$, the conditions are much more complicated.

MA(q) process $$X_t = \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

- For an MA(1) model: $-1 < \theta_1 < 1$.
- For an MA(2) model: $-1 < \theta_2 < 1, \quad \theta_2 + \theta_1 > -1,$ $\theta_1 - \theta_2 < 1$.
- When $q \geq 3$, the conditions are much more complicated.

ARMA(p, q) process $$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

- Conditions on ϕ_j and θ_i are also quite complicated and are usually taken care of by the software function.

ARIMA process

- ARIMA stands for AutoRegressive Integrated Moving Average.
- An ARIMA(p, d, q) process is an ARMA(p, q) process over the time series differenced d times.
- The I stands for integrated (i.e. opposite of differenced)

- ARIMA processes allow to model a very wide variety of time series patterns. They provide statistical models that fit the historical data well, taking into account the time based variation, and therefore they allow to produce meaningful and relevant forecasts.

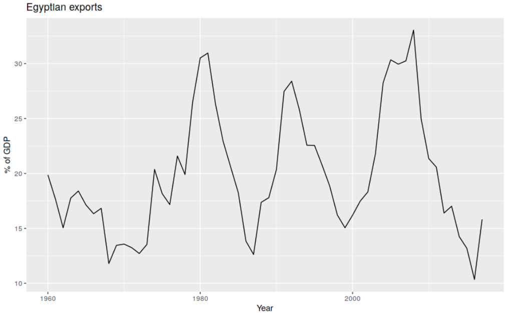
How to choose p, q and d ?

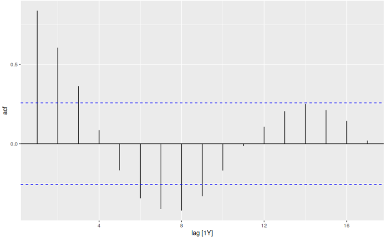
- Each partial autocorrelation can be estimated as the last coefficient in an autoregressive model. Specifically, π_k, the k^{th} partial autocorrelation coefficient, is equal to the estimate of ϕ_k in an AR(k) model.
- The data may follow an ARIMA(p,d,0) model if the ACF and PACF plots of the differenced data show the following patterns:
  - the ACF is exponentially decaying or sinusoidal;
  - there is a significant spike at lag p in the PACF, but none beyond lag p.
- The data may follow an ARIMA(0,d,q) model if the ACF and PACF plots of the differenced data show the following patterns:
  - the PACF is exponentially decaying or sinusoidal;
  - there is a significant spike at lag q in the ACF, but none beyond lag q.
- The time series should be differenced as many times as needed so that it becomes stationary. Thus d is the number of differencing needed.

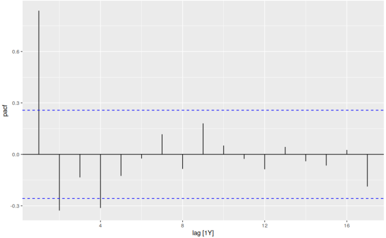
Estimating the parameters

- We need to estimate the parameters of the model:
  - autoregressive coefficients ϕ_j
  - moving average coefficients θ_i
  - variance σ_ε of the innovation process (ε_i)
- Let $\mathbf{\Phi} = (\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma_\varepsilon)$. The goal is to optimize a function $Q(\mathbf{\Phi})$ subject to some linear and non-linear constraints.
- The following methods can be used:
  - Least squares method:
    minimize the sum of squared residuals
  - Maximum likelihood estimation:
    maximize the (log-)likelihood function
  - Generalized method of moments:
    minimize the distance between the theoretical moments and 0
- High quality software programs, such as R are available to do it.

ARIMA example







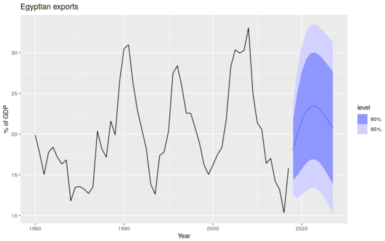
ARIMA example (cont'd)

- Model 1: ARIMA(4,0,0)

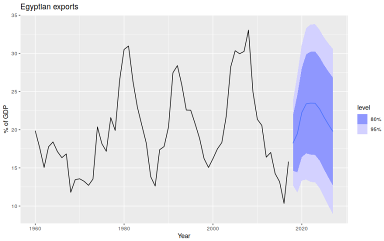
$$X_t = 0.9861X_{t-1} - 0.1715X_{t-2} + 0.1807X_{t-3} - 0.3283X_{t-4} + 6.6922$$

- Model 2: ARIMA(2,0,1)

$$X_t = 1.6764X_{t-1} - 0.8034X_{t-2} - 0.6896\varepsilon_{t-1} + 2.5623$$



ARIMA(4,0,0)
log likelihood = -140.53



ARIMA(2,0,1)
log likelihood = -141.57

The Conditional Mean

- All ARMA processes can be written in the form

$$X_t = \mu_t + \varepsilon_t, \quad t \in \mathbb{Z},$$

  where, for example, $\mu_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ in the general ARMA process (1).
- Let $\mathcal{F}_t$ denote the history (or information content) of the process up to and including time t — in mathematical language the filtration generated by $(X_s)_{s \leq t}$.
- Then $\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \mathbb{E}(\mu_t + \varepsilon_t \mid \mathcal{F}_{t-1}) = \mu_t$, since $\mathbb{E}(\varepsilon_t \mid \mathcal{F}_{t-1}) = 0$.
- ARMA models are models for the conditional mean, the expected value of tomorrow's obervation, given the history until today.
- ARCH and GARCH are models for the conditional standard deviation.

Modelling Return Series with ARCH/GARCH

- Let $(Z_t)_{t \in \mathbb{Z}}$ follow a strict white noise process with mean zero and variance one.
- ARCH and GARCH processes $(X_t)_{t \in \mathbb{Z}}$ take general form

$$X_t = \sigma_t Z_t, \quad t \in \mathbb{Z}, \tag{4}$$

  where σ_t, the (random) volatility, is a function of the history up to time $t-1$ represented by $\mathcal{F}_{t-1}$.
- Z_t is assumed independent of $\mathcal{F}_{t-1}$.
- Mathematically, σ_t is $\mathcal{F}_{t-1}$-measurable, where $\mathcal{F}_{t-1}$ is the filtration generated by $(X_s)_{s \leq t-1}$, and therefore

$$\mathrm{var}(X_t \mid \mathcal{F}_{t-1}) = \mathrm{var}(\sigma_t Z_t \mid \mathcal{F}_{t-1}) = \sigma_t^2 \mathrm{var}(Z_t \mid \mathcal{F}_{t-1}) = \sigma_t^2,$$

  using the measurability of σ_t w.r.t. to $\mathcal{F}_{t-1}$.
- Hence, volatility is the conditional standard deviation of the process.

ARCH and GARCH Processes

ARCH = AutoRegressive Conditional Heteroskedasticity

- (X_t) follows an ARCH(p) process if, for all t,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2, \quad \alpha_j > 0.$$

  Intuition: volatility is influenced by large observations in recent past.

- (X_t) follows a GARCH(p,q) process (generalised ARCH) if, for all t,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2, \quad \alpha_j, \beta_k > 0. \tag{5}$$

  Intuition: more persistence is built into the volatility.

Stationarity and Autocorrelations

- The condition for the GARCH equations to define a covariance stationary process with finite variance is that

$$\sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1.$$

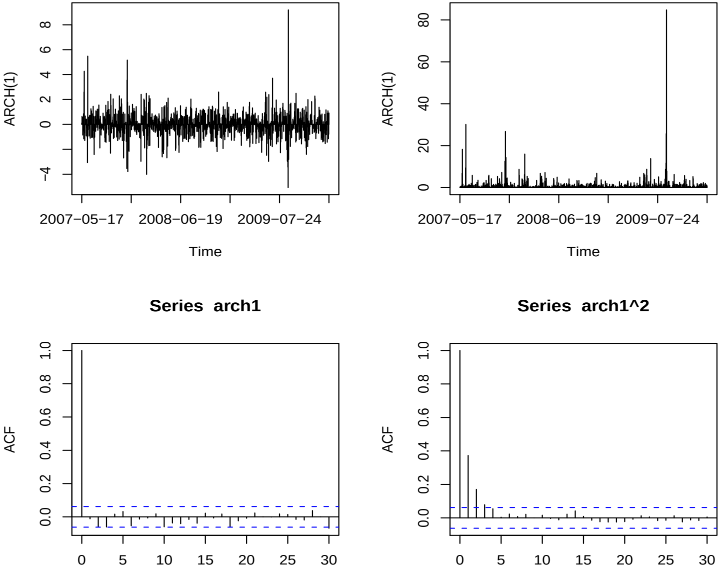
- ARCH and GARCH are technically white noise processes since

$$\begin{aligned} \gamma(h) &= \operatorname{cov}(X_t, X_{t+h}) \\ &= \mathbb{E}(\sigma_{t+h} Z_{t+h} \sigma_t Z_t) - \mathbb{E}(\sigma_{t+h} Z_{t+h}) \mathbb{E}(\sigma_t Z_t) \\ &= \mathbb{E}(Z_{t+h}) \mathbb{E}(\sigma_{t+h} \sigma_t Z_t) = 0. \end{aligned}$$

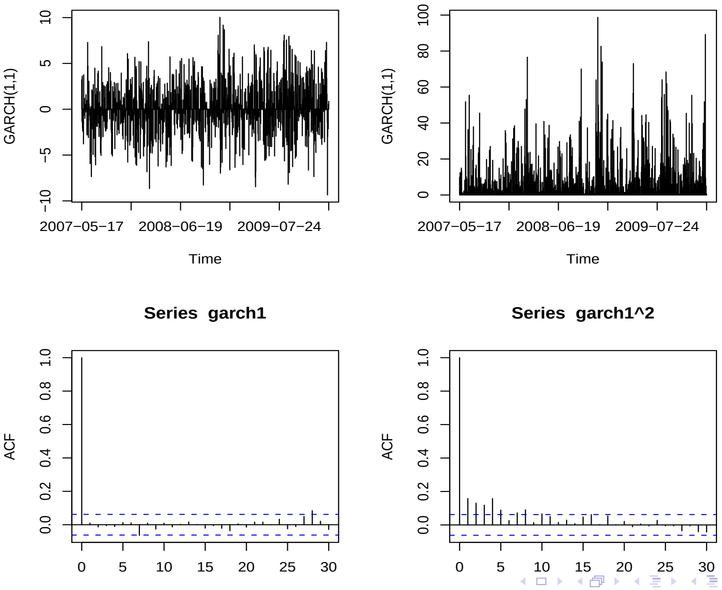
Absolute and Squared GARCH Processes

- Although (X_t) is an uncorrelated process, it can be shown that the processes (X_t^2) and $(|X_i|)$ possess profound serial dependence.
- In fact (X_t^2) can be shown to have a kind of ARMA-like structure.
- A GARCH(1,1) model is like an ARMA(1,1) model for (X_t^2).

GARCH Simulated Example I



GARCH Simulated Example II



Hybrid ARMA/GARCH Processes

- Although changes in volatility are the most obvious feature of financial return series, there is sometimes some evidence of serial correlation at small lags. This can be modelled by

$$\begin{aligned} X_t &= \mu_t + \varepsilon_t, \\ \varepsilon_t &= \sigma_t Z_t, \end{aligned} \tag{6}$$

  where μ_t follows an ARMA specification, σ_t follows a GARCH specification, and (Z_t) is a zero mean, variance 1 strict white noise.
- μ_t and σ_t are respectively the conditional mean and standard deviation of X_t given history to time $t-1$; they satisfy

$$\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = \mu_t, \quad \mathrm{var}(X_t \mid \mathcal{F}_{t-1}) = \sigma_t^2.$$

A Simple Effective Model: AR(1)+GARCH(1,1)

- The following model often suffices in practice:

$$\begin{aligned} \mu_t &= c + \phi(X_{t-1} - c)\,, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \left(X_{t-1} - \mu_{t-1}\right)^2 + \beta\sigma_{t-1}^2, \end{aligned} \tag{7}$$

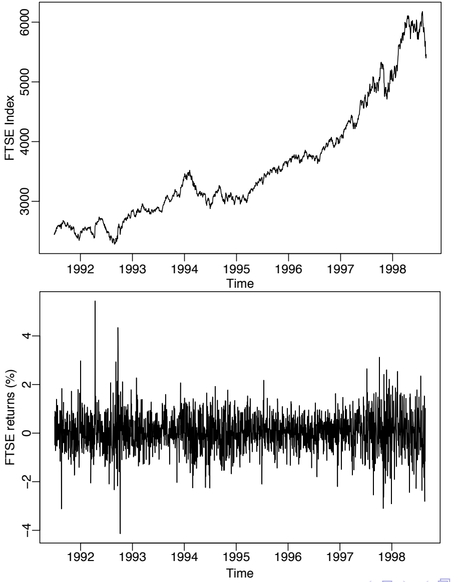
  with α_0, α_1, $\beta > 0$, $\alpha_1 + \beta < 1$ and $|\phi| < 1$ for a stationary model with finite variance.
- This model is a reasonable fit for many daily financial return series, particularly under the assumption that the driving innovations are heavier-tailed than normal.

Fitting GARCH Models to Financial Data

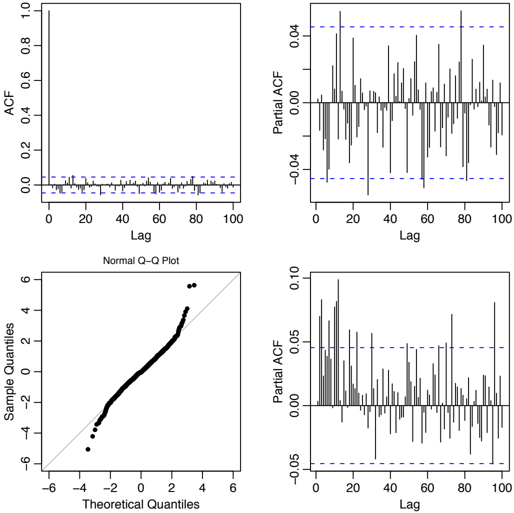
- There are a number of possible fitting methods, but the most common is maximum likelihood (ML), which is also a standard method of fitting ARMA processes to data.
- **Possibilities:**
  - Assume (Z_t) are standard iid normal innovations and estimate GARCH parameters (α_j and β_k) by ML.
  - Assume (Z_t) are (scaled) Student t_ν innovations and estimate GARCH parameters plus ν by ML.
  - Make no distributional assumptions and estimate GARCH parameters by quasi maximum likelihood (QML). (Effectively uses Gaussian ML but calculates standard errors differently.)

Example: FTSE returns

The Financial Times Stock Exchange Index, 1991–1998

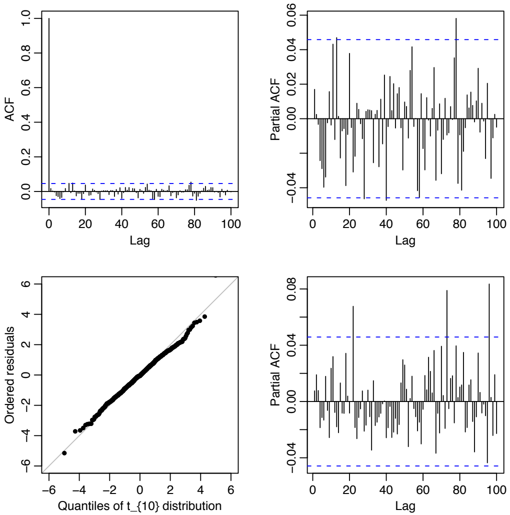


FTSE data: Fit of AR(1)-ARCH(1) model



Residuals show signs of structure, including non-normality, so try fitting an AR(1)-GARCH(1,1) with scaled t_ν residuals.

FTSE data: Fit of AR(1)-GARCH(1,1), t_ν model



Residuals show some possible signs of structure, but overall fit is much better, and the data seem to be white noise (next).

FTSE data: Fitted model

- The model is fitted by maximum likelihood and gives

$$
\begin{aligned}
Y_t - 0.051_{0.018} &= 0.070_{0.024}(Y_{t-1} - \mu) + \sigma_t \varepsilon_t, \\
\sigma_t^2 &= 0.006_{0.004} + 0.036_{0.011}(Y_{t-1} - \mu)^2 + 0.955_{0.016}\sigma_{t-1}^2, \\
\varepsilon_t &\overset{\text{iid}}{\sim} \sqrt{7.7/9.7}\, t_{9.7_{1.9}}
\end{aligned}
$$

- Comments:
  - the mean and autocorrelation are significantly positive but small, so not very useful for prediction;
  - dependence between successive σ_t^2 accounts for most of the changes in variance, with some being added by the $(Y_{t-1} - \mu)^2$;
  - the usual t_ν density has variance $\nu/(\nu - 2)$, but here is scaled to have unit variance.
- More about the innovation distribution:
  - other heavy-tailed innovation distributions could be used;
  - often the innovations are asymmetric.

Some extensions

- (G)ARCH models have the following weaknesses:
  - σ_t reacts identically to positive and negative shocks
  - Strong restrictions on the parameters are needed for stationarity and finite variance
  - They give no insight into why a series behaves as it does
  - They react slowly to large shocks in the returns, so may over-predict volatility
- They have been extended to integrated GARCH (IGARCH) models, for example

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) Y_{t-1}^2,$$

  which is analogous to an ARIMA model, in that past volatility shocks persist
- The exponential GARCH (EGARCH) model allows the variance to depend on the sign of the series, for example giving

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha} \exp\left(\alpha_* + (\gamma + \theta \mathrm{sign}(Y_{t-1})) \frac{|Y_{t-1}|}{\sigma_{t-1}}\right)$$

  for suitable constants α, α_*, γ, θ; we expect that $\theta < 0$ if negative shocks have higher impacts

Time series dependence

Given two time series data $(X_t)_{t=1}^n$ and $(Y_t)_{t=1}^n$ we calculate the sample cross-covariances

$$\widehat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_t - \overline{X})(Y_{t+h} - \overline{Y}), \quad h = 0, 1, 2, \ldots$$

$$\widehat{\gamma}(-h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h} - \overline{X})(Y_t - \overline{Y}), \quad h \geq 0 \quad \text{(symetrically)}$$

$$\text{where} \quad \overline{X} = \sum_{t=1}^{n}\frac{X_t}{n} \quad \text{and} \quad \overline{Y} = \sum_{t=1}^{n}\frac{Y_t}{n}.$$

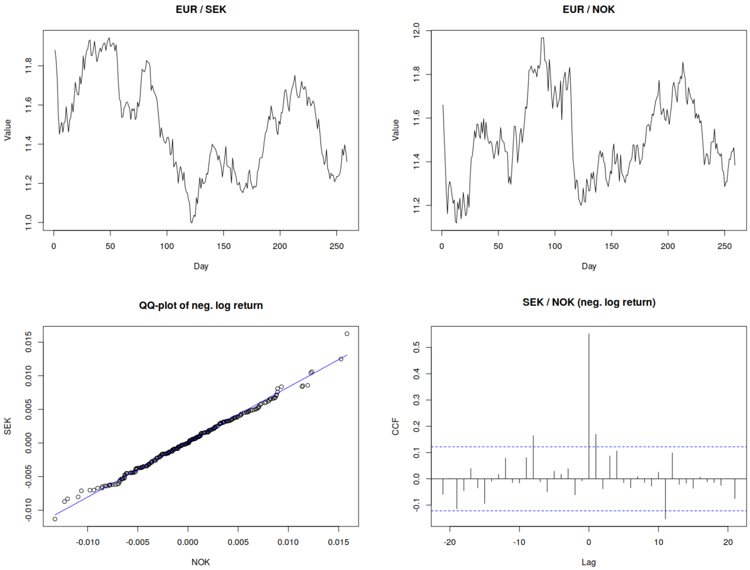
and sample cross-correlations

$$\widehat{\rho}(h) = \widehat{\gamma}(h)/\widehat{\gamma}(0), \quad h \in \mathbb{Z}, \ -n < h < n$$

The cross-correlogram (CCF) is the plot

$$\{(h, \widehat{\rho}(h)), \ h \in \mathbb{Z}, \ -n < h < n\}$$

Example



Exchange rate of SEK and NOK against EUR (June 2023 - June 2024)

Granger Causality Test

The Granger Causality test is used to examine if one time series may be used to forecast another.

- Consider two time series $(X_t)_{t=1}^{n}$ and $(Y_t)_{t=1}^{n}$.
- Knowing the value of a time series (X_t) at a given lag is valuable for forecasting the value of a time series (Y_t) at a later time period is referred to as Granger-causes.
- The Granger Causality Test tests the following H_0 hypothesis:

  Time series (X_t) does not Granger-cause time series (Y_t).
- Alternative Hypothesis (H_1):

  Time series (X_t) Granger-causes time series (Y_t).

Granger Causality Test

Mathematically

$$Y_t = c + \sum_{i=1}^{p} \alpha_i Y_{t-i} + \varepsilon_t \tag{8}$$

$$Y_t = c + \sum_{i=1}^{p} \alpha_i Y_{t-i} + \sum_{j=1}^{q} \beta_j X_{t-j} + \varepsilon_t \tag{9}$$

H_0 : no β_j is significant in the model
The null hypothesis that (X_t) does not Granger-cause (Y_t) is rejected if and only if any lagged values of (X_t) is retained in the regression.

Intuition

A time series (X_t) Granger-causes another time series (Y_t) if predictions of the value of (Y_t) based on its own past values and on the past values of (X_t) are better than predictions of (Y_t) based only on (Y_t) own past values.

Example

```
Granger causality test

Model 1: sek_lr ~ Lags(sek_lr, 1:1) + Lags(nok_lr, 1:1)
Model 2: sek_lr ~ Lags(sek_lr, 1:1)
  Res.Df Df      F   Pr(>F)
1    254
2    255 -1 8.4557 0.003961 **
```

Model

$$\texttt{sek\_lr}_t = \mu + \alpha_1 \, \texttt{sek\_lr}_{t-1} + \beta_1 \, \texttt{nok\_lr}_{t-1}$$

$$\mu \cong 0, \ \alpha_1 = -0.0747_{0.0631}, \ \beta_1 = 0.4513_{0.0420}$$