CISC260 Machine Organization and Assembly Language

Boolean logic, Gates, NAND universality

Any function can be implemented in Boolean logic.

- Function is a mapping from input variables **I** to output value **O**.

F: I → O,

where I ∈ {0,1}^{N}, O ∈{0, 1}^{M}.

Inputs ABC	Output XY
000	00
001	01
010	01
011	10
100	01
101	10
110	10
111	11

A 1
B 1
+) C 1

X Y
carry sum

- Boolean logic
  - Boolean variables
  - Boolean operators

    AND (&), OR (|), NOT(~)

Y = AND (A, B) = A & B

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

Y = OR (A, B) = A | B

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

Y = NOT(A) = ~A

A	Y
0	1
1	0

Boolean expressions: *made of Boolean variables and Boolean operators much like arithmetic expressions.*

Boolean variables: take values {0, 1} or {F, T}.

Boolean operators:

- (& : .)
- (| : + module 2)
- (~ : . (-1))

E.g., Y=A&B|C&~B

Precedence(high to low): ~, &, |

Meaning of a Boolean Expression

- Each such expression implies a truth table → performs a particular function
- *E.g.* Y=A&B|C&~B $\Rightarrow$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Canonical representation

every Boolean function (defined in a truth can be expressed using at least one Boolean expression called the canonical representation.

Start with a truth table

Procedure:

- And together all literals (negate if 0) in each row (conjunct)
- Or together rows that have true output
- Repeat for each output bit of the function.

The Boolean expression thus obtained is called the “sum-of-product” canonical form.

权威的，牧师的，依教规的

Conclusion: Every Boolean function, no matter how complex, can be expressed using three Boolean operators: AND, OR and NOT.

Boolean Logic Example

define x first: x could be—1. ~A&B&C | A&~B&~C | A&B&~C

define y: y could be — ~A&~B&~C | A&B&~C

Conjuncts:

- ~A&~B&~C
- ~A&~B&C
- ~A&B&~C
- ~A&B&C
- A&~B&~C
- A&~B&C
- A&B&~C
- A&B&C

Inputs ABC	Output XY
000	01
001	00
010	00
011	10
100	10
101	00
110	11
111	00

Boolean Logic Example

Conjuncts:

- ~A&~B&~C
- ~A&~B&C
- ~A&B&~C
- ~A&B&C
- A&~B&~C
- A&~B&C
- A&B&~C
- A&B&C

Inputs ABC	Output XY
000	01
001	00
010	00
011	10
100	10
101	00
110	11
111	00

X=~A&B&C
| A&~B&~C
| A&B&~C

Boolean Logic Example

- Conjuncts:
  - ~A&~B&~C
  - ~A&~B&C
  - ~A&B&~C
  - ~A&B&C
  - A&~B&~C
  - A&~B&C
  - A&B&~C
  - A&B&C

Inputs ABC	Output XY
000	01
001	00
010	00
011	10
100	10
101	00
110	11
111	00

X=~A&B&C
| A&~B&~C
| A&B&~C

Y=~A&~B&~C
| A&B&~C

Boolean Logic Example

Inputs ABC	Output XY
000	01
001	00
010	00
011	10
100	10
101	00
110	11
111	00

X=~A&B&C | A&~B&~C | A&B&~C

Y=~A&~B&~C | A&B&~C

Alternative Procedure:

OR together all literals (negate if true) in each row
AND together rows that have false output
Repeat for each output bit of the function.

The semantic/syntax needs to be consistent

The Boolean expression thus obtained has the so-called the "product-of-sum" form.

This procedure works for the same reason that the canonical procedure works.

Just look at the truth tables for AND, OR. They are "identical" in the sense that AND to 1 is just like OR to 0, vice versa.
Namely, AND has value 1 iff all input values are 1, and OR has value 0 iff all input values are 0.

Therefore, this alternative procedure is to define the output from the zero's perspective.

NB: equivalence of sum-of-product and product-of-sum can also be proved using DeMorgan's Law. (though the logic could be the other way around, or circular.)

The number of Boolean functions that can be ~~defined~~ over n Boolean variables is 2^{2^n}

2 variables, each variable could have 2 values-0 and 1

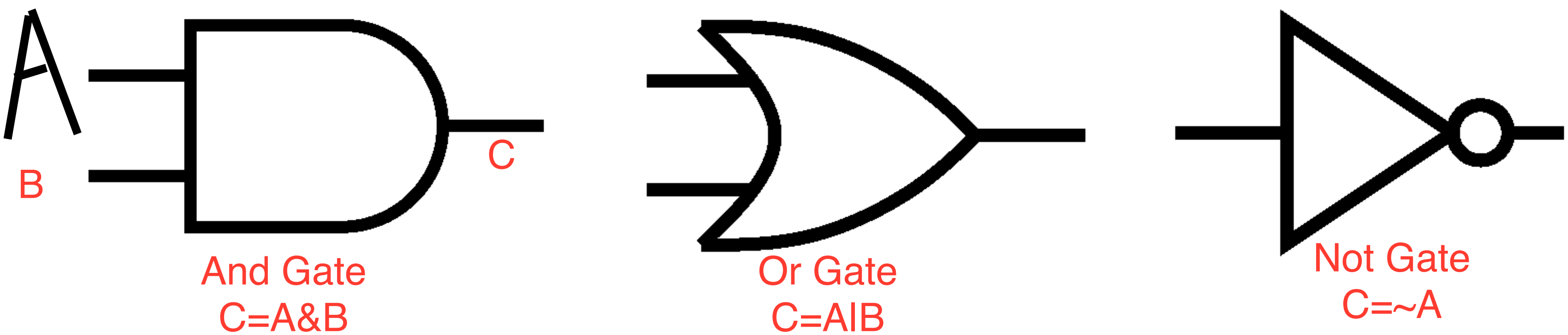
the function you can define grows exponentially

All Boolean functions of 2 variables

Function	x	**0**	**0**	**1**	**1**
	y	**0**	**1**	**0**	**1**
Constant 0	0	0	0	0	0
And	$x \cdot y$	0	0	0	1
x And Not y	$x \cdot \overline{y}$	0	0	1	0
x	x	0	0	1	1
Not x And y	$\overline{x} \cdot y$	0	1	0	0
y	y	0	1	0	1
Xor	$x \cdot \overline{y} + \overline{x} \cdot y$	0	1	1	0
Or	$x + y$	0	1	1	1
Nor	$\overline{x + y}$	1	0	0	0
Equivalence	$x \cdot y + \overline{x} \cdot \overline{y}$	1	0	0	1
Not y	$\overline{y}$	1	0	1	0
If y then x	$x + \overline{y}$	1	0	1	1
Not x	$\overline{x}$	1	1	0	0
If x then y	$\overline{x} + y$	1	1	0	1
Nand	$\overline{x \cdot y}$	1	1	1	0
Constant 1	1	1	1	1	1

Gates implement Boolean logic

- Physical, Boolean Operator
  - A physical entity that implements a small truth table
  - *E.g.* AND, OR, NOT



How to wire the gates according to a Boolean expression?

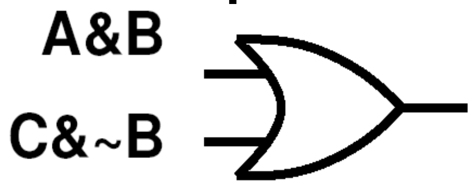
- Can implement any Boolean expression with a collection of gates

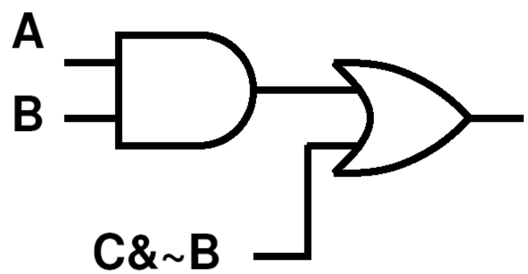
Lowest precedence logical operator

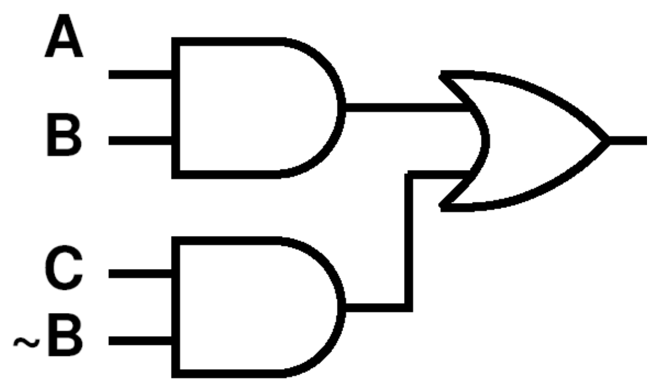
1. Find outer-most operator
2. Replace with gate
3. Work recursively on input functions

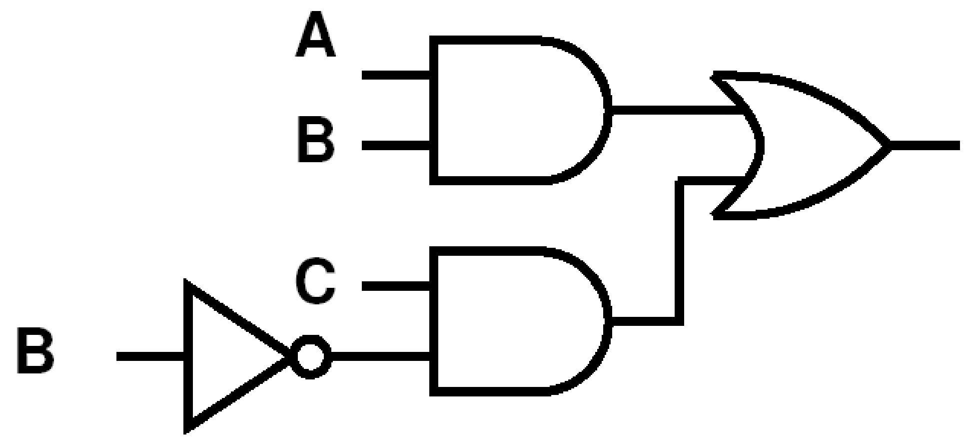
Example

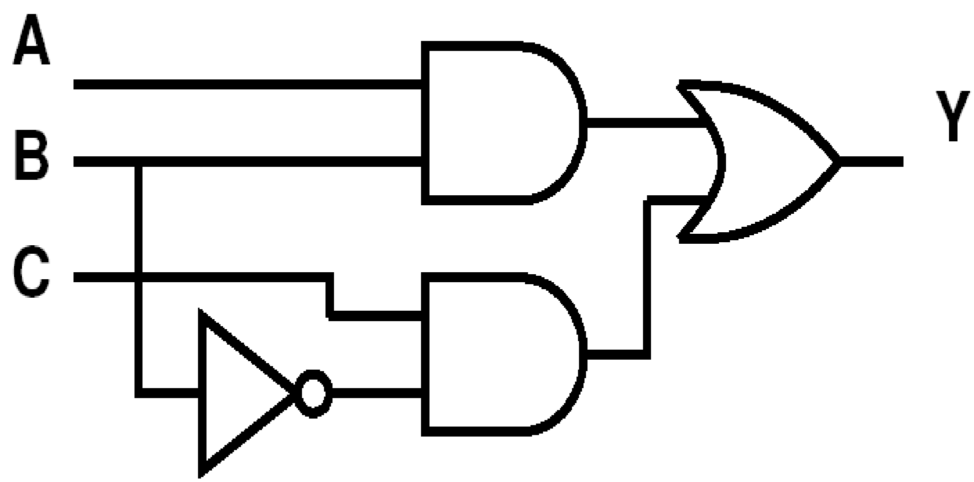
- Y=A&B|C&~B











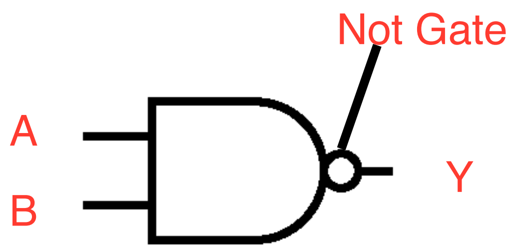
Any Function in Gates

- Combining: we can
  - Express any function with Boolean expressions
  - Implement any Boolean expression with gates

→ Can implement any function with gates

NAND gate universality

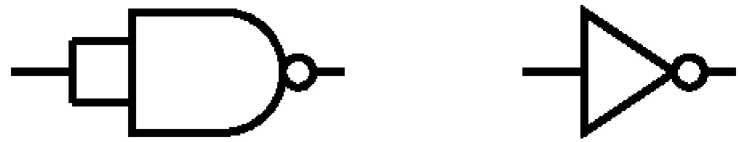
- NAND gate AND gate followed by a nagation



A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

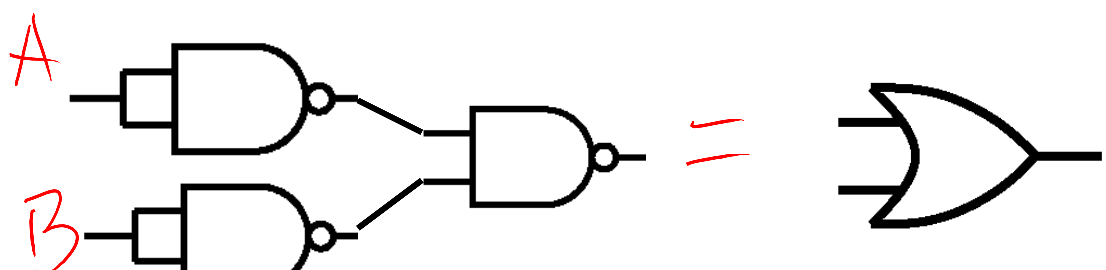
- Universality: Any Boolean expression can be implemented using NAND gates only.

- NOT gate



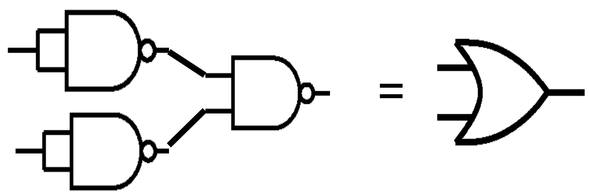
- AND gate

- OR gate



A	B	~(A&B)	A|B	~A	~B	~(~A & ~B)
0	0	1	0	1	1	0
0	1	1	1	1	0	1
1	0	1	1	0	1	1
1	1	0	1	0	0	1

From the truth table, we see A|B = ~ (~A & ~B), which means

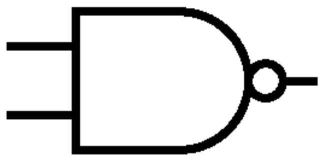


If we let ~A = C and ~B = D, then we have ~(C&D) = (~C)|(~D), which is DeMorgan's Law.

DeMorgan's Law: ~(A&B)=~A|~B

"Not A and B means neither A nor B"

Start with the left hand side, it is a NAND gate with inputs A, B.



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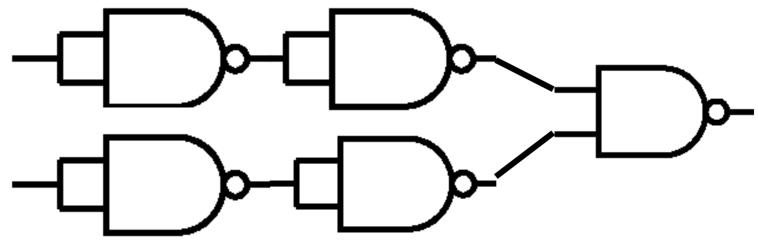

Negate the input twice is still equal to the input itself. That is, ~~A = A. Recognize the OR gate implemented by 3 NAND gates.

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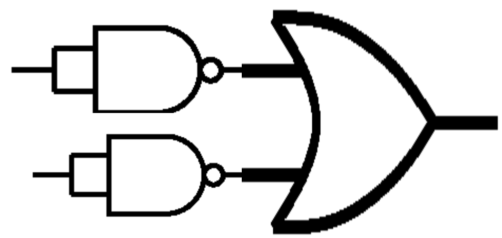

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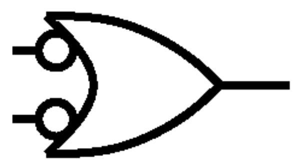


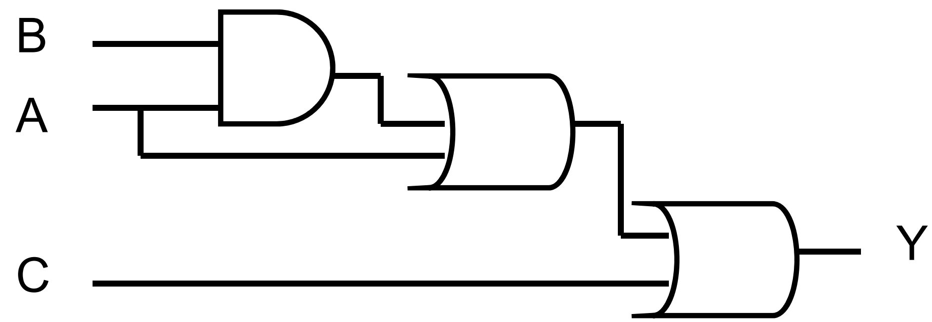
To simplify logic/circuit design

- Identity law: $A + 0 = A$ and $A \cdot 1 = A$.
- Zero and One laws: $A + 1 = 1$ and $A \cdot 0 = 0$.
- Inverse laws: $A + \overline{A} = 1$ and $A \cdot \overline{A} = 0$.
- Commutative laws: $A + B = B + A$ and $A \cdot B = B \cdot A$.
- Associative laws: $A + (B + C) = (A + B) + C$ and $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
- Distributive laws: $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ and $A + (B \cdot C) = (A + B) \cdot (A + C)$.

$\overline{x \cdot y} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x} \cdot \overline{y}$ DeMorgan's Law

Simplifying logic

e.g.,



Y = A&B | A | C

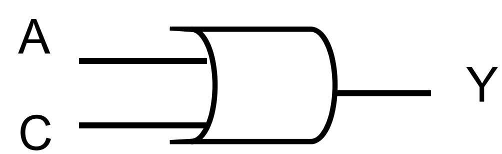
= AB+A +C

= A(B+1) +C   distribution, identity

= A(1) + C   law of 1's

= A + C   identity

= A | C



K-map

- Logic optimization
  - Very difficulty: NP-complete
  - Commercial software

Summary

- Any function on binary input/output can be implemented in Boolean logic
- Boolean logic can be implemented by physical devices – gates.
- Logic gates, as an abstraction, hide the physical details of the devices.
- Only need a small number of primitive gates, actually, only a single gate type NAND2 is enough.