Fixed Income Derivatives E2025 - Problem Set Week 7

Problem 1

Consider the CIR model where the short rate r has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{1}$$

where $a > 0$ and $b > 0$. We will now denote present time by t and proceed in to find explicit formulas for ZCB prices, spot rates and forward rates in the CIR model.

a) In the following, we will compute ZCB prices, spot rates and forward rates in the CIR model by taking a number of steps.

   i) Show that ZCB prices in the CIR model are of the form $F^{(T)}(t,r) = A(t,T)e^{-B(t,T)r}$ where $A(t,T)$ and $B(t,T)$ solve the following system of ODE's

$$A_t = abAB, \qquad A(T,T) = 1 \tag{2}$$

$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0. \tag{3}$$

   ii) Use the substitution $B = -\frac{2}{\sigma^2 V}V_t$ to transform the ODE for B into the following second order ODE for $V = V(t)$

$$V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0. \tag{4}$$

   iii) Use the conjecture that $V(t)$ is of the form $V(t) = e^{yt}$ to show that all solutions for $V(t)$ can be written as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}, \quad \gamma = \sqrt{a^2 + 2\sigma^2} \tag{5}$$

   where c_1 and c_2 are constants to be found

   iv) Use the boundary condition on $B(T)$ to show that

$$B(t,T) = \frac{2e^{\gamma(T-t)} - 2}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \tag{6}$$

   v) Use the ODE for $A(t,T)$ to show that

$$\ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I, \qquad I = -\gamma\int_t^T \frac{e^{\gamma(T-s)} - 1}{2\gamma + \big(a+\gamma\big)\big(e^{\gamma(T-s)} - 1\big)}ds \tag{7}$$

   vi) Use a substitution of the form $u = e^{\gamma(T-s)}$ to put the integral on the form

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{u-1}{\gamma - a + \big(a+\gamma\big)u}\frac{1}{u}du \tag{8}$$

   vii) Show the following rule for partial fractions

$$\frac{a_0 + a_1 x}{\big(b_0 + b_1 x\big)\big(c_0 + c_1 x\big)} = \frac{y}{b_0 + b_1 x} + \frac{z}{c_0 + c_1 x}, \quad \text{where } y = \frac{a_0 b_1 - a_1 b_0}{c_0 b_1 - c_1 b_0} \text{ and } z = \frac{c_0 a_1 - c_1 a_0}{c_0 b_1 - c_1 b_0} \tag{9}$$

   and use this result to simplify the integral in (8) to

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{2\gamma}{(\gamma - a)[\gamma - a + \big(a+\gamma\big)u]}du - \int_{e^{\gamma(T-t)}}^{1} \frac{1}{(\gamma - a)u}du \tag{10}$$

   viii) Solve the integral in (10) to conclude that

$$A(t,T) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)}\right)^{\frac{2ab}{\sigma^2}} \tag{11}$$

ix) Write down expressions for ZCB prices, spot rates and forward rates in the CIR model.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a CIR model with $a = 2, b = 0.05, \sigma = 0.1, r = 0.025$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Problem 1 - Solution

a) We work towards computing ZCB prices step by step.

i) We know that ZCB prices $P(t,r;T) = F^{(T)}(t,r)$ in the CIR model must satisfy the term structure equation

$$\begin{aligned} F_t^{(T)} + a\big(b-r\big)F_r^{(T)} + \frac{1}{2}\sigma^2 r F_{rr}^{(T)} - rF^{(T)} = 0 \\ F^{(T)}(T,r) = 1 \end{aligned} \tag{12}$$

Inserting the functional form $p(t,r;T) = A(t,T)e^{-B(t,T)r}$ of the ZCB price into (12) gives us that

$$\frac{1}{A}PA_t - rPB_t - a\big(b-r\big)BP + \frac{1}{2}r\sigma^2 B^2 P - rP = 0 \tag{13}$$

For the left hand side to equal 0 for all r, both the term independent of r and the term linear in r must be 0. Also, since $p(T,T;T) = 1$, we must have that $A(T,T) = 1$ and $B(T,T) = 0$. This gives us the following system of ODE's to solve

$$A_t = abAB, \qquad A(T,T) = 1 \tag{14}$$

$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0 \tag{15}$$

ii) The second of these two equations depends only on $B(t,T)$ and must be solved first. This equation is a Ricatti ODE which can be reduced to a second order linear ODE by defining $B(t,T)$ in terms of a new function $V(t,T)$ as follows

$$B = -\frac{2}{\sigma^2 V}V_t \tag{16}$$

Using this definition we get that

$$B_t = -\frac{2}{\sigma^2}\frac{d}{dt}\left(\frac{V_t}{V}\right) = -\frac{2}{\sigma^2}\left(\frac{V_{tt}}{V} - \frac{V_t^2}{V^2}\right) \tag{17}$$

Inserting (17) into (15) gives us that

$$-\frac{2}{\sigma^2}\left(\frac{V_{tt}}{V} - \frac{V_t^2}{V^2}\right) = -1 - \frac{2a}{\sigma^2 V}V_t + \frac{2}{\sigma^2}\frac{V_t^2}{V^2} \quad \Rightarrow \quad V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0 \tag{18}$$

iii) To solve the second order ODE in (18), we conjecture that the solution is of the form $V(t) = e^{yt}$, where y is some function of the parameters of the model. Inserting the conjecture into (18) gives us that

$$y^2e^{yt} - aye^{yt} - \frac{\sigma^2}{2}e^{yt} = 0 \quad \Rightarrow \quad y^2 - ay - \frac{\sigma^2}{2} = 0 \quad \Rightarrow \quad y = \frac{a \pm \sqrt{a^2 + 2\sigma^2}}{2} = \frac{a \pm \gamma}{2}. \tag{19}$$

We now have two potential solutions to the ODE in (18) given by

$$V_1(t) = e^{\left(\frac{a+\gamma}{2}\right)t} \quad \text{and} \quad V_2(t) = e^{\left(\frac{a-\gamma}{2}\right)t} \tag{20}$$

However, any linear combination of the two solutions $V_1(t)$ and $V_2(t)$ will also solve (18) and we can write all solutions of (18) as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t} \tag{21}$$

where c_1 and c_2 are constants to be found.

iv) Inserting (21) into (16) gives us that

$$\begin{aligned} B(t) &= -\frac{1}{\sigma^2}\frac{c_1(a+\gamma)e^{\left(\frac{a+\gamma}{2}\right)t} + c_2(a-\gamma)e^{\left(\frac{a-\gamma}{2}\right)t}}{c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}} \\ &= -\frac{1}{\sigma^2}\frac{c_1(a+\gamma)e^{\frac{\gamma}{2}t} + c_2(a-\gamma)e^{-\frac{\gamma}{2}t}}{c_1 e^{\frac{\gamma}{2}t} + c_2 e^{-\frac{\gamma}{2}t}} = -\frac{1}{\sigma^2}\frac{c_3(a+\gamma)e^{\frac{\gamma}{2}t} + (a-\gamma)e^{-\frac{\gamma}{2}t}}{c_3 e^{\frac{\gamma}{2}t} + e^{-\frac{\gamma}{2}t}} \end{aligned} \tag{22}$$

where $c_3 = \frac{c_1}{c_2}$ is just another constant. The solution for $B(t,T)$ must satisfy the boundary condition that $B(t=T)=0$ which will allow us to find c_3

$$B(T) = 0 \quad \Rightarrow \quad c_3(a+\gamma)e^{\frac{\gamma}{2}T} + (a-\gamma)e^{\frac{-\gamma}{2}T} = 0 \quad \Rightarrow \quad c_3 = -\frac{(a-\gamma)}{(a+\gamma)}e^{-\gamma T} \tag{23}$$

Inserting c_3 from (23) into the general solution for B from (22) gives us

$$\begin{aligned} B(t,T) &= \frac{1}{\sigma^2}\frac{(a-\gamma)e^{-\gamma T+\frac{\gamma}{2}t} - (a-\gamma)e^{-\frac{\gamma}{2}t}}{-\frac{(a-\gamma)}{(a+\gamma)}e^{-\gamma T+\frac{\gamma}{2}t} + e^{-\frac{\gamma}{2}t}} = \ldots \quad \Rightarrow \\ B(t,T) &= \frac{2e^{\gamma(T-t)} - 2}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \end{aligned} \tag{24}$$

We have now found the solution for $B(t,T)$ in (15).

v) We proceed to find $A(t,T)$ from (14)

$$\begin{aligned} &A_t = abAB \quad \Rightarrow \quad \frac{1}{A}\frac{dA}{dt} = abB \quad \Rightarrow \quad \int_t^T \frac{1}{A(s,T)}dA(s,T) = ab\int_t^T B(s,T)ds \quad \Rightarrow \\ &\big[\ln A(s,T)\big]_t^T = ab\int_t^T B(s,T)ds \quad \Rightarrow \quad \ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I \end{aligned} \tag{25}$$

where I is an integral that we must evaluate

$$I = -\gamma\int_t^T \frac{e^{\gamma(T-s)} - 1}{2\gamma + (a+\gamma)\big(e^{\gamma(T-s)} - 1\big)}ds \tag{26}$$

vi) To evaluate this integral, we perform the substitution

$$\begin{aligned} &u = e^{\gamma(T-s)} \quad \Rightarrow \quad du = -\gamma e^{\gamma(T-s)}ds \quad \Rightarrow \quad ds = -\frac{1}{\gamma}e^{-\gamma(T-s)} = -\frac{1}{u\gamma}du \\ &s = t \quad \Rightarrow \quad u = e^{\gamma(T-t)}, \quad s = T \quad \Rightarrow \quad u = 1 \end{aligned} \tag{27}$$

and we get that

$$I = \int_{e^{\gamma(T-t)}}^1 \frac{u-1}{2\gamma + (a+\gamma)(u-1)}\frac{1}{u}du = \int_{e^{\gamma(T-t)}}^1 \frac{u-1}{\gamma - a + \big(a+\gamma\big)u}\frac{1}{u}du \tag{28}$$

vii) In order to evaluate this integral, we will need a trick involving partial fractions.

$$\frac{a_0 + a_1 x}{\big(b_0 + b_1 x\big)\big(c_0 + c_1 x\big)} = \frac{y}{b_0 + b_1 x} + \frac{z}{c_0 + c_1 x}, \quad \text{where } y = \frac{a_0 b_1 - a_1 b_0}{c_0 b_1 - c_1 b_0} \text{ and } z = \frac{c_0 a_1 - c_1 a_0}{c_0 b_1 - c_1 b_0} \tag{29}$$

which can be shown by direct computation

viii) Using this little trick gives us that

$$
\begin{aligned}
I &= \int_{e^{\gamma(T-t)}}^{1} \frac{2\gamma}{(\gamma-a)[\gamma-a+(a+\gamma)u]}du - \int_{e^{\gamma(T-t)}}^{1} \frac{1}{(\gamma-a)u}du \\
&= \frac{2\gamma}{\gamma-a}\int_{e^{\gamma(T-t)}}^{1} \frac{1}{[\gamma-a+(a+\gamma)u]}du + \frac{1}{a-\gamma}\int_{e^{\gamma(T-t)}}^{1} \frac{1}{u}du \\
&= \frac{2\gamma}{(\gamma-a)(a+\gamma)}\Big[\ln\big(\gamma-a+(a+\gamma)u\big)\Big]_{e^{\gamma(T-t)}}^{1} + \frac{1}{a-\gamma}\Big[\ln\big(u\big)\Big]_{e^{\gamma(T-t)}}^{1} \\
&= \frac{2\gamma}{(\gamma-a)(a+\gamma)}\Big[\ln\big(2\gamma\big) - \ln\big(\gamma-a+(a+\gamma)e^{\gamma(T-t)}\big) + \frac{a+\gamma}{2\gamma}\ln\big(e^{\gamma(T-t)}\big)\Big] \\
&= \frac{2\gamma}{\gamma^2-a^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right) = \frac{\gamma}{\sigma^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right)
\end{aligned}
\tag{30}
$$

Substituting I from (30) back into (25) gives us that

$$
\begin{aligned}
\ln A(t,T) =& \frac{2ab}{\gamma}I = \frac{2ab}{\gamma}\frac{\gamma}{\sigma^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right) \quad \Rightarrow \\
A(t,T) =& \left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big)}\right)^{\frac{2ab}{\sigma^2}}
\end{aligned}
\tag{31}
$$

ix) Zero coupon bond prices in the CIR model are in other words given by

$$
P(t,T) = \left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big)}\right)^{\frac{2ab}{\sigma^2}} \cdot \exp\left(\frac{-2\big(e^{\gamma(T-t)}-1\big)}{2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big)}r_t\right). \tag{32}
$$

Spot rates and forward rates can most easily be expressed by setting be found by setting $A = \big(\frac{N}{D}\big)^c$ and $B = \frac{M}{D}$ so that $B_T = \frac{M_T D - M D_T}{D^2}$. We then have that spot rates $R(t,T)$ are given by

$$
R(t,T) = -\frac{\ln P(t,T)}{T-t} = -\ln A - Br_t = -c\big(\ln N - \ln D\big) - \frac{M}{D}r_t. \tag{33}
$$

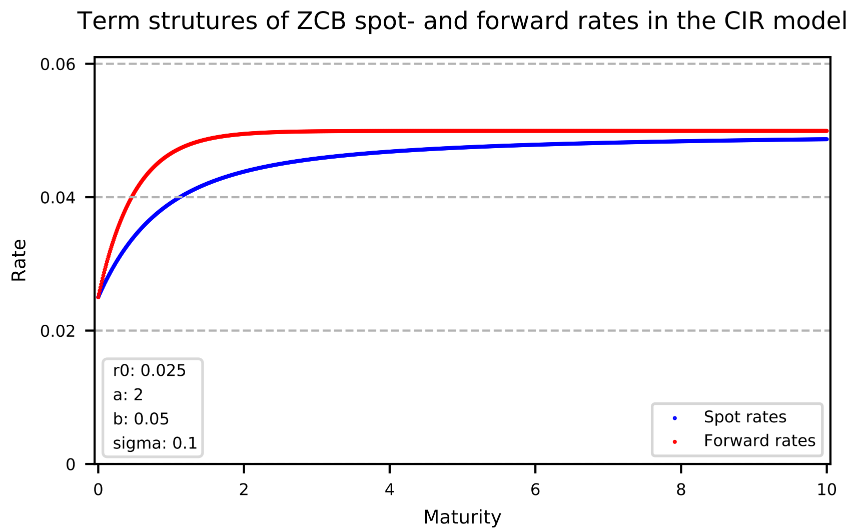
Forward rates are given by

$$
f(t,T) = -\frac{\partial log\, P(t,T)}{\partial T} = -\frac{\partial}{\partial T}\Big(\log\, A + rB_T\Big) = -c\Big(\frac{N_T}{N} - \frac{D_T}{D}\Big) + \frac{M_T D - M D_T}{D^2}r_t \tag{34}
$$

where

$$
\begin{aligned}
c &= \frac{2ab}{\sigma^2}, \\
\gamma &= \sqrt{a^2+2\sigma^2}, \\
N &= 2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}, \\
N_T &= \gamma(a+\gamma)e^{\frac{(T-t)(a+\gamma)}{2}}, \\
D &= 2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big), \\
D_T &= \gamma(\gamma+a)e^{\gamma(T-t)}, \\
M &= 2e^{\gamma(T-t)}-2, \\
M_T &= 2\gamma e^{\gamma(T-t)}.
\end{aligned}
\tag{35}
$$

c) The Term structures of spot- and forward rates for $r_0 = 0.025$, $a = 2$, $b = 0.05$ and $\sigma = 0.1$ look as follows.



d) The long-run stationary mean in the CIR model is simply $b = 0.05$, and we see that the short rate at present time $t = 0$ is only 0.025 and well below the stationary mean resulting in an upward-sloping term structure of zero coupon spot rates.

Problem 2

Consider the CIR model where the short rate has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{36}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. Present time is denoted by t, the short rate at time t is denoted by r and the price of a zero coupon bond with maturity T is denoted $p(t, T)$. In this problem, we will first generate zero coupon bond prices using the CIR model with known parameters and then seek to recover these parameters by fitting a CIR model to the zero coupon bond prices we generated.

a) Generate ZCB prices for times to maturity $\tau = T - t = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.045$ and parameters $a = 1.5$, $b = 0.06$, $\sigma = 0.08$. Denote these 'empirical' prices by $p^*(t, T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a CIR model to the prices $p^*(t, T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.05, 1.8, 0.08, 0.08$. Plot the fitted values $\hat{p}(t, T)$ and the empirical values $p^*(t, T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.08$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b and σ. You will need to use that method 'trust-constr' also described in the documentation.

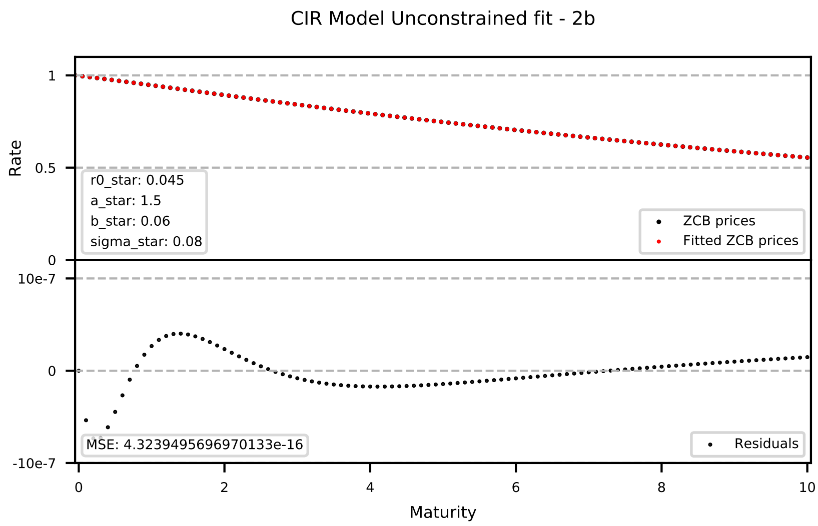
e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method.

Problem 2 - Solution

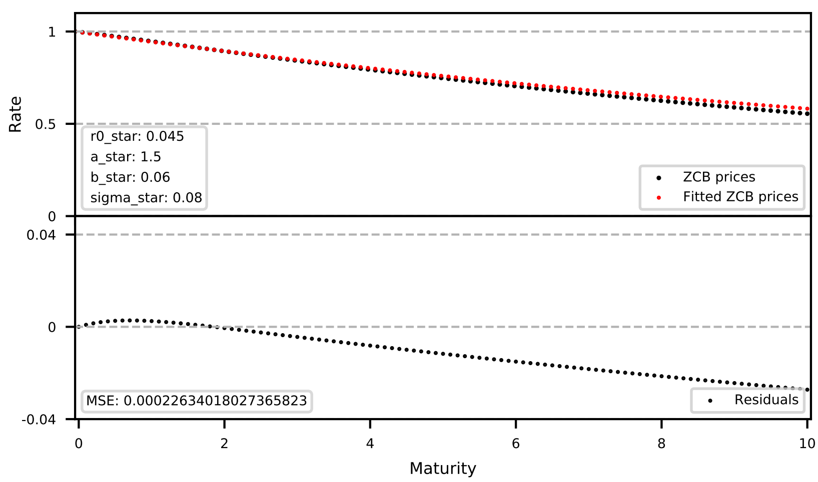
b) The unconstrained fit of the CIR model results in a near perfect fit as illustrated below.



c) In the CIR model as in the Vasicek model, a near perfect fit is obtained as long as the initial parameter values assume reasonable values and in particular met the criteria of the parameters in the model. However, it is also apparent in the CIR model that σ is not well identified reflecting, also in this case, that ZCB prtices are not affected much by σ and that this parameter can be chosen somewhat freely and still, we would be able to achieve a good fit.

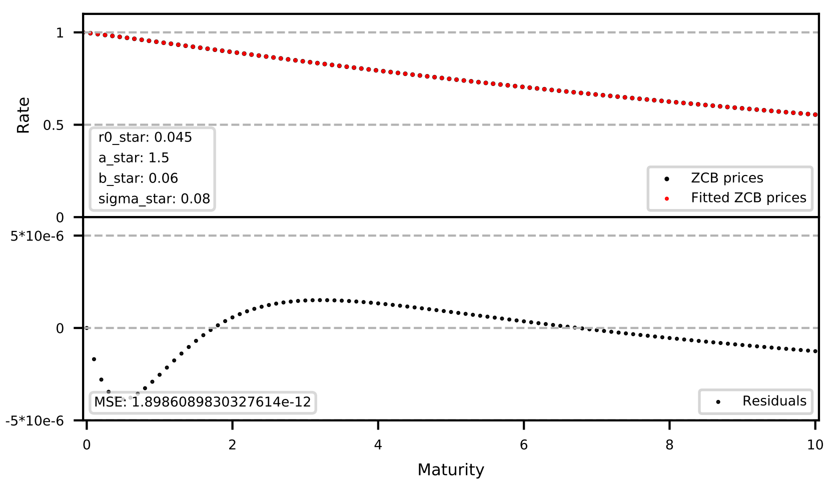
d) Forcing b to taking a values that is not equal to the true value parameter value results, also in this case, results in a poor fit as shown below.

CIR when b is set to 0.06 - 2d



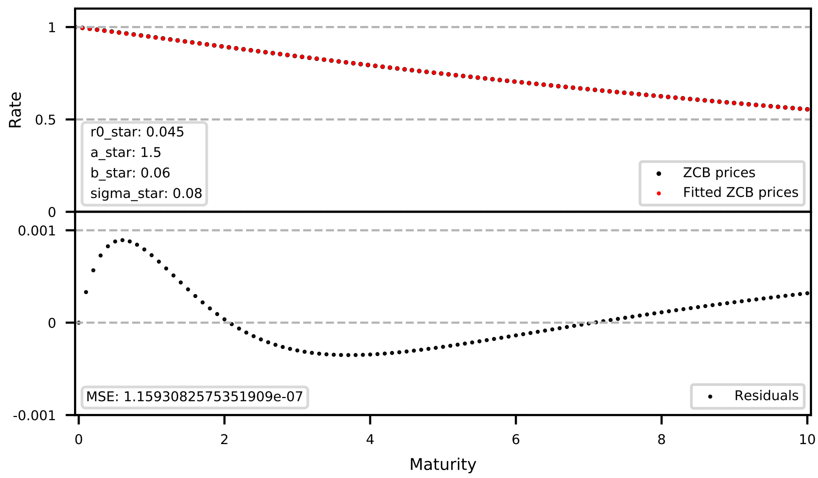
e) Imposing bounds such that the true values of the parameters sit inside the bounds, we recover the true values of the parameters and get a perfect fit once again.

CIR model bounded non-binding fit - 2e



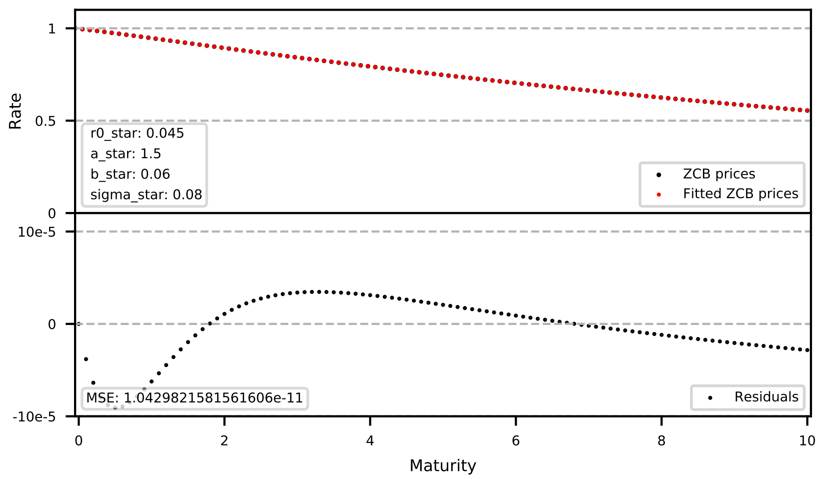
f) Imposing bounds such that the true parameter values are outside the bounds reduces the quality of the fit, but also in the case of these bounds the algorithm is able to compensate and produce a fairly solid fit of ZCB prices.

CIR model bounded binding fit - 2f



g) Imposing the Feller condition on the parameters of this version of the CIR model, we are able to get a perfect fit because the Feller condition is met for the parameters used to generate the ZCB prices we are fitting.

CIR model with non-linear constraint - 2g



Problem 3

In this problem, we will consider the CIR model for the short rate r_t with dynamics given by

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r, \end{aligned} \tag{37}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. We know that $r_T | r_0$ is equal in distribution to

$$\frac{\sigma^2}{4a}\left[1 - e^{-aT}\right]Y \tag{38}$$

where Y follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-aT}}{\sigma^2\left[1 - e^{-aT}\right]}r_0 \tag{39}$$

The stationary distribution of the short rate is a gamma where

$$r_\infty \sim \text{Gamma}(\alpha, \beta), \quad \alpha = \frac{2ab}{\sigma^2}, \quad \beta = \frac{\sigma^2}{2a}, \quad f_{r_\infty}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, \, x > 0$$

a) Write a function in Python that takes T, r_0, a, b and σ and a confidence level α as inputs and returns the lower-, upper- or two-sided confidence bounds of r_T.

b) Plot the two-sided confidence bounds for $\alpha = 0.05$ and appropriately many choices of $T < 10$ setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$. Also include the two-sided confidence bounds under the stationary distribution in your plot.

c) For combinations of $a \in [1, 2, 4, 8]$, b fixed at 0.05, $\sigma \in [0.05, 0.1, 0.15, 0.2]$, redo the plot from b) for a sufficiently large T. For each of the combinations of parameters, assess how large T must be for r_T to have settled to it's stationary distribution. How does the rate at which r_T settles to it's stationary distribution depend on a and σ?

In the following, you will simulate the short rate on a grid of mesh δ that runs from initial time $t_0 = 0$ to some terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $\left[t_0, t_1, ..., t_{M-1}, t_M\right] = \left[0, \delta, 2\delta, ..., T-\delta, T\right]$ and $\delta = \frac{T}{M}$. In the following, please consider the following three schemes

i) An Euler scheme

ii) A Milstein scheme

iii) An exact scheme

d) Derive the difference equation for the short rate for each of the three schemes (if it is possible!) in terms of a standard normal variable denoted Z_m drawn in each of the steps. Are some of the schemes equivalent? Which of the three schemes do you expect to be more accurate?

e) Write a python function that take as inputs T, M, r_0, a, b, σ, and "*scheme*", and returns a simulated trajectory of the short rate. Plot a single trajectory setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$ for each of the three schemes and include the lower, upper and two-sided confidence bounds in your plot for a choice of $\alpha = 0.1$.

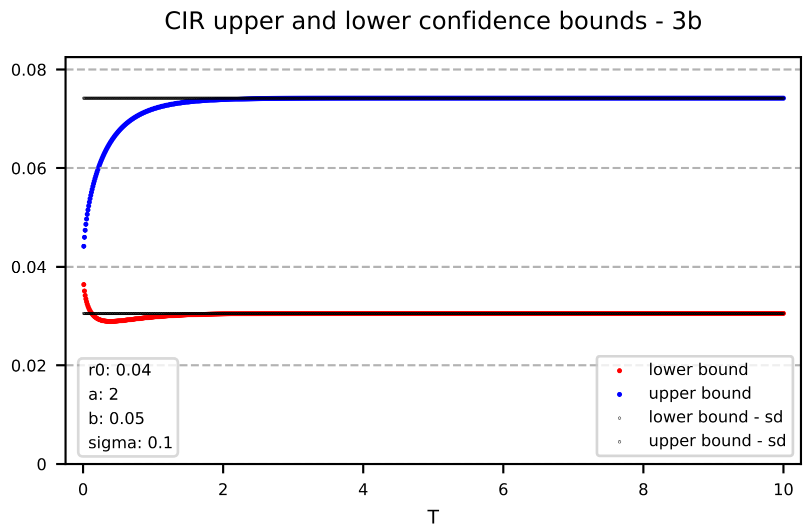
f) Now set $T = 3$, repeat the simulations N times and denote the value of the short rate at $T = 3$ in the n'th simulation by r_{3n}, $n = 1, 2, ..., N$. Construct at least 50 but ideally more equally spaced bins to cover the range of r_{3n} from the smallest to the largest value. Sort your simulated values into these bins and use the proportion in each bin to construct an empirical probability mass function. Plot the empirical mass function for your favorite scheme with $N = 1000$ and $M = 1000$, and also plot the theoretical mass function in the same figure.

g) Finally, we will investigate how the difference between the empirical and theoretical PMF's depend on M and N. For a choice of 100 bins and combinations of values of M in [2000,4000,6000,8000,10000] and N in [2000,4000,6000,8000,10000], compute the total square difference between the theoretical probabilities and empirical frequencies across the 100 bins. Report these total squared differences for all combinations of M and N, and for all three schemes. Compare the accuracy of the three schemes and try to assess how large M and N need to be in each of the three cases to arrive at a reasonable accuracy.

Problem 3

Solution

b) The two-sided confidence bounds for the short rate in the CIR model look as follows



c) Plotting the confidence bounds for different values of a and σ, we can conclude that increasing a implies that the distribution of r_T settles to it's stationary distribution much faster. Decreasing σ also some effect but the effect is much less pronounced. These conclusions are similar to that of the Vasicek model reflecting that a and σ have similar interpretations in those two models.

d) The SDE in (37) for the short rate in the CIR model can not be solved explicitly and the closest we can get is to write

$$r_T = r_t e^{-a(T-t)} + b\Big[1 - e^{-a(T-t)}\Big] + \sigma \int_t^T e^{-a(T-u)} \sqrt{r_u} dW_u. \tag{40}$$

It is therefore not possible to write an exact difference equation for r_{m+1} in terms of r_m and a standard normal random variable. What we can however do is to use that $r_{m+1}|r_m$ is equal in distribution to a random variable X_{m+1} where

$$X_{m+1} = \frac{\sigma^2}{4a}\Big[1 - e^{-a\delta}\Big] Y_{m+1} \tag{41}$$

and Y_{m+1} follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-a\delta}}{\sigma^2\big(1 - e^{-a\delta}\big)} r_m. \tag{42}$$

An exact approach would thus involve drawing Y_{m+1} from the appropriate non-central chi-squared distribution and using (41) to generate the increments to the short rate. The three schemes therefore become

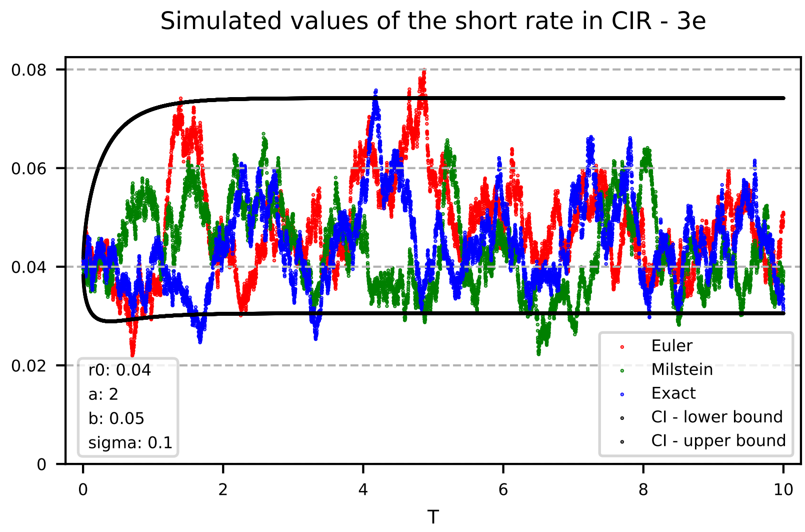
i) Euler: $r_{m+1} = r_m + a(b - r_m)\delta + \sigma\sqrt{r_m}\sqrt{\delta} Z_{m+1}$

ii) Milstein scheme: $r_{m+1} = r_m + a(b - r_m)\delta + \sigma\sqrt{r_m}\sqrt{\delta} Z_{m+1} + \frac{1}{2}\sigma^2\big(Z_{m+1}^2 - 1\big)$

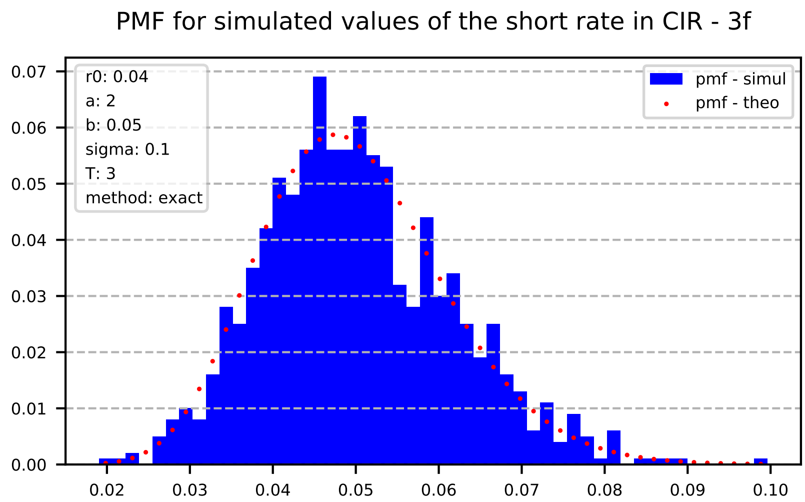
iii) Exact scheme: $r_{m+1} = X_{m+1}$

We would expect the third scheme to be the most accurate in terms of reproducing the true distribution of the short rate but for large values of M, the difference is likely small.

e) A plot of the simulated values of the short rate in the CIR model using each of the three schemes could look as follows.

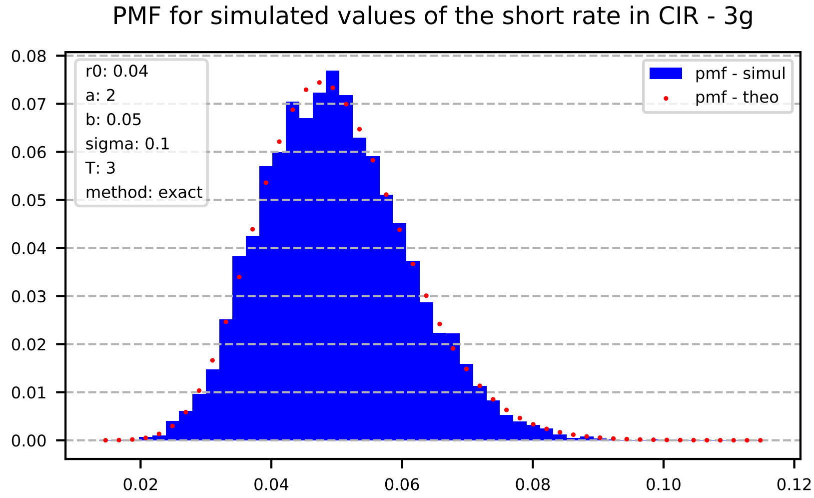


f) Simulating the short rate for $T = 3$, $M = 1000$, repeating the simulation $n = 1000$ times and placing the data in 50 equally spaced bins allows us create a plot of the empirical and theoretical probability mass function. Such a plot may look as follows



Also in the case of the CIR model, there is some difference between the empirical and theoretical PMF's, at least for the choices of M, N and number of bins used here.

g) Setting $M = 10,000$, simulating $N = 10,000$ times and setting the number of bins to 100 resulted in the following plot of the theoretical and empirical PMF's.



Not surprisingly, the empirical and theoretical PMF's match much better as M and N have been increased by a factor of 10. We also compute the SSE and run-time for the three different schemes for $M \in [2000, 4000, 6000, 8000]$ and $N \in [2000, 4000, 6000, 8000]$ and the results are shown below where again M increases by the row and M increases by the column. The conclusions regarding the SSE remain the same in the SSE namely that increasing M does not reduce the SSE reflecting that setting $M = 2000$ suffices for all of the schemes to produce a randome variable r_3 with the correct distribution. Reducing N does indeed reduce the SSE and also in the case of the CIR, it can be shown that doubling N will cut the SSE in half. As far as computational efforts goes, we see that the Milstein scheme is only slightly slower than the Euler scheme but the exact scheme is by far the slowest. This stems from the fact that it is much slower to sample from the non-central chi-squared distribution than from the standard normal.

Euler scheme:

$$SSE = \begin{bmatrix} 8.59610011e-06 & 6.36788751e-06 & 2.58475602e-06 & 2.79341057e-06 \\ 8.24710435e-06 & 5.28591187e-06 & 4.96186105e-06 & 2.78465983e-06 \\ 5.55421598e-06 & 4.35602278e-06 & 3.38883459e-06 & 3.04578379e-06 \\ 5.88308273e-06 & 7.96564165e-06 & 2.54329266e-06 & 3.05976957e-06 \end{bmatrix} \tag{43}$$

$$T_{comp} = \begin{bmatrix} 15.11335707 & 32.33399987 & 44.82680082 & 57.02715254 \\ 29.58613658 & 58.41017938 & 84.59104967 & 110.31846738 \\ 40.74923539 & 84.18940091 & 124.07491994 & 178.17956495 \\ 57.11244535 & 110.88269496 & 180.63092899 & 243.30869412 \end{bmatrix} \tag{44}$$

Milstein scheme:

$$SSE = \begin{bmatrix} 1.15393197e-05 & 4.79633224e-06 & 3.68116930e-06 & 1.61109868e-06 \\ 1.04495999e-05 & 2.40852156e-06 & 2.52195316e-06 & 2.62205451e-06 \\ 1.91154642e-05 & 3.26320192e-06 & 3.69727708e-06 & 1.44500955e-06 \\ 8.46220147e-06 & 5.38589912e-06 & 2.36558232e-06 & 2.31786703e-06 \end{bmatrix} \tag{45}$$

$$T_{comp} = \begin{bmatrix} 19.50208116 & 38.59039235 & 58.6373992 & 77.90965629 \\ 38.83999777 & 80.64415622 & 115.91019917 & 153.11310601 \\ 57.32057571 & 114.65791154 & 175.48505974 & 232.12298465 \\ 78.78927541 & 154.25125146 & 233.44477272 & 311.38987708 \end{bmatrix} \tag{46}$$

Exact scheme:

$$SSE = \begin{bmatrix} 1.80995454e-05 & 5.75707717e-06 & 3.25716517e-06 & 2.03648461e-06 \\ 4.79698650e-06 & 4.90022748e-06 & 1.92974510e-06 & 2.09689486e-06 \\ 9.04558118e-06 & 3.37096677e-06 & 1.90225161e-06 & 2.22526612e-06 \\ 8.08880555e-06 & 5.32827308e-06 & 2.74636634e-06 & 2.33589853e-06 \end{bmatrix} \tag{47}$$

$$T_{comp} = \begin{bmatrix} 191.87936044 & 377.43501616 & 565.61707711 & 761.02315211 \\ 191.87936044 & 377.43501616 & 565.61707711 & 761.02315211 \\ 554.14963913 & 1117.08205152 & 1690.55293155 & 2236.28040481 \\ 746.93022609 & 1483.21049786 & 2221.9847548 & 2955.58541226 \end{bmatrix} \tag{48}$$