

Fixed Income Derivatives - Problem Set Week 9

Problem 1

Consider the Hull-White model where the short rate r_t has dynamics

$$dr_t = [\Theta(t) - ar_t]dt + \sigma dW_t \quad (1)$$

- a) Argue that ZCB prices are of the form

$$p(t, T) = e^{A(t, T) - B(t, T)r_t} \quad (2)$$

where

$$\begin{aligned} A(t, T) &= \int_t^T \left[\frac{1}{2}\sigma^2 B^2(s, T) - \Theta(s)B(s, T) \right] ds \\ B(t, T) &= \frac{1}{a} \left[1 - e^{-a(T-t)} \right] \end{aligned} \quad (3)$$

- b) Show that forward rates $f(t, T)$ are of the form

$$f(t, T) = -A_T(t, T) + r_t B_T(t, T) \quad (4)$$

where $A_T(t, T) = \frac{\partial}{\partial T} A(t, T)$ and $B_T(t, T) = \frac{\partial}{\partial T} B(t, T)$.

- c) Argue that the forward rate dynamics can be found from

$$df(t, T) = -\frac{\partial}{\partial T} \left(A_t(t, T)dt - B_t(t, T)r_t dt - B(t, T)dr_t \right) \quad (5)$$

where $A_t(t, T) = \frac{\partial}{\partial t} A(t, T)$ and $B_t(t, T) = \frac{\partial}{\partial t} B(t, T)$.

- d) Show that the forward rate dynamics are

$$df(t, T) = \frac{\sigma^2}{a} e^{-a(T-t)} \left[1 - e^{-a(T-t)} \right] dt + \sigma e^{-a(T-t)} dW_t \quad (6)$$

Now, we will find the forward rate dynamics in a different way. Let us recall that in the Hull-White model, zero coupon bond prices become

$$p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left\{ B(t, T)f^*(0, t) - \frac{\sigma^2}{4a} B^2(t, T)(1 - e^{-2at}) - B(t, T)r_t \right\} \quad (7)$$

- e) Use the above expression to find an expression for forward rates and treat this expression as a function $f(t, T) = g(t, T, r)$.
- f) Show from $g(t, T, r)$ that the forward rate dynamics are of the form

$$df(t, T) = \alpha(t, T)dt + \sigma e^{-a(T-t)} dW_t \quad (8)$$

where $\alpha(t, T)$ is yet to be determined.

- g) Use the HJM drift condition to find $\alpha(t, T)$ and thus show that it is of the same form as in d).

Problem 1 - Solution

- a) Since the drift and the squared diffusion coefficients of the short rate are both affine, the Hull-White model admits an affine term structure with $p(t, T) = e^{A(t, T) - B(t, T)r_t}$ where $A(t, T)$ and $B(t, T)$ satisfy the system of ODE's

$$\begin{aligned} A_t(t, T) &= \Theta(t)B(t, T) - \frac{1}{2}\sigma^2 B^2(t, T), & A(T, T) &= 0 \\ B_t(t, T) &= aB(t, T) - 1, & B(T, T) &= 0 \end{aligned} \quad (9)$$

The equation for $B(t, T)$ is the same as in the Vasicek model and $A(t, T)$ can then be found directly by integration and we have

$$\begin{aligned} A(t, T) &= \int_t^T \left[\frac{1}{2}\sigma^2 B^2(s, T) - \Theta(s)B(s, T) \right] ds \\ B(t, T) &= \frac{1}{a} \left[1 - e^{-a(T-t)} \right] \end{aligned} \quad (10)$$

- b) To use the Hull-White model in practice, we would need to fit the model to observed forward rates $f^*(0, t)$ and thereby find $\Theta(t)$. This involves computing the integral in (10) to find $A(t, T)$ but as we will see next, the forward rate dynamics are independent of $\Theta(t)$ and can be found before fitting the model to data. From the definition of forward rates, we immediately have

$$f(t, T) = -\frac{\partial}{\partial T} p(t, T) = -\frac{\partial}{\partial T} A(t, T) + r_t \frac{\partial}{\partial T} B(t, T) \quad (11)$$

- c) The forward rate dynamics can be found directly the product rule of differentiation and Ito's formula

$$df(t, T) = -\frac{\partial}{\partial T} d(A(t, T) - B(t, T)r_t) = -\frac{\partial}{\partial T} (A_t(t, T)dt - B_t(t, T)r_t dt - B(t, T)dr_t) \quad (12)$$

- d) To find the forward rate dynamics, we need to use $A_t(t, T)$ and $B_t(t, T)$ but they are given from the system of ODE's in (10)

$$\begin{aligned} df(t, T) &= -\frac{\partial}{\partial T} d(\Theta(t)B(t, T)dt - \frac{1}{2}\sigma^2 B^2(t, T)dt - aB(t, T)r_t dt + r_t dt - B(t, T)[\Theta(t) - ar_t]dt - B(t, T)\sigma dW_t) \\ &= \frac{1}{2}\sigma^2 \frac{\partial}{\partial T} B^2(t, T)dt + \sigma \frac{\partial}{\partial T} B(t, T)dW_t = \sigma^2 B(t, T) \frac{\partial}{\partial T} B(t, T)dt + \sigma \frac{\partial}{\partial T} B(t, T)dW_t \\ &= \frac{\sigma^2}{a} e^{-a(T-t)} [1 - e^{-a(T-t)}]dt + \sigma e^{-a(T-t)}dW_t \end{aligned} \quad (13)$$

- e) Using the expression for ZCB prices and the definition of forward rates gives us that

$$f(t, T) = g(t, T, r) = -\frac{\partial}{\partial T} (\log p^*(0, T) - \log p^*(0, t) + B(t, T)f^*(0, t) - \frac{\sigma^2}{4a} B^2(t, T)(1 - e^{-2at}) - B(t, T)r) \quad (14)$$

- f) Applying Ito's formula and simply denoting the drift term by $\alpha(t, T)$ gives us that

$$\begin{aligned} df(t, T) = dg(t, T, r) &= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial r} [\Theta(t) - ar_t]dt + \frac{\partial g}{\partial r} \sigma dW_t + \frac{1}{2} \frac{\partial^2 g}{\partial r^2} \sigma^2 dt = \alpha(t, T)dt + \frac{\partial g}{\partial r} \sigma dW_t \\ &= \alpha(t, T)dt + \frac{\partial}{\partial T} \frac{1}{a} [1 - e^{-a(T-t)}] \sigma dW_t = \alpha(t, T)dt + \sigma e^{-a(T-t)}dW_t \end{aligned} \quad (15)$$

- g) Now, we know that the drift of the forward rates must satisfy the HJM drift condition and that can be used to find $\alpha(t, T)$. The HJM drift condition tells us that the drift coefficient of forward rates depends on the diffusion coefficient $\sigma(t, T)$ as follows

$$\begin{aligned} \alpha(t, T) &= \sigma(t, T) \int_t^T \sigma(s, s)ds = \sigma e^{-a(T-t)} \int_t^T \sigma e^{-a(T-s)}ds = \frac{\sigma^2}{a} e^{-a(T-t)} e^{-aT} \left[e^{as} \right]_t^T \\ &= \frac{\sigma^2}{a} e^{-a(T-t)} [1 - e^{-a(T-t)}] \end{aligned} \quad (16)$$

And indeed, we rediscover the dynamics of forward rates in the Hull-White model just as in d).

Problem 2

Take as given an HJM model under the risk neutral measure \mathbb{Q} where

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t \quad (17)$$

- a) Show that all forward rates and also the short rate are normally distributed.
b) Show that zero coupon bond prices are log-normally distributed.

Problem 2 - Solution

- a) Since the drift- and diffusion coefficients of the SDE satisfied by $f(t, T)$ has deterministic coefficients, the solution for $f(t, T)|\mathcal{F}_0$ for $0 < t \leq T$ can be found directly by integration

$$\begin{aligned} f(t, T) - f(0, T) &= \int_0^t df(s, T) = \int_0^t \alpha(s, T)ds + \int_0^t \sigma(s, T)dW_s \Rightarrow \\ f(t, T) &= f(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \sigma(s, T)dW_s \end{aligned} \quad (18)$$

The first of the two integrals in (18) is a regular \mathcal{F}_0 -adapted Riemann-Stieltjes time integral that, in principle at least, can be computed. The second integral of (18) is an Ito integral with a deterministic coefficient and hence, this integral is Gaussian with mean zero and a variance that can be computed using Ito isometry. We thus have that

$$f(t, T) | \mathcal{F}_0 \sim N\left(f(0, T) + \int_0^t \alpha(s, T) ds, \int_0^t \sigma^2(s, T) ds\right) \quad (19)$$

- b) To show that zero coupon bond prices, $p(t, T)$, follow a log-normal distribution, we first recall the link between $p(t, T)$ and $f(t, T)$

$$\begin{aligned} f(t, T) &= -\frac{\partial}{\partial T} \log p(t, T) \Leftrightarrow p(t, T) = \exp\left\{-\int_t^T f(t, s) ds\right\} \Rightarrow \\ \log p(t, T) &= -\int_t^T \left[f(t, T) + \int_t^s \alpha(u, T) du + \int_t^s \sigma(u, T) dW_u\right] ds \\ &= -(T-t)f(t, T) - \int_t^T \int_t^s \alpha(u, T) du ds - \int_t^T \int_t^s \sigma(u, T) dW_u ds \end{aligned} \quad (20)$$

In order to proceed, we will need to change the order of integration and for that we will need a version of Fubini's theorem that we strictly speaking have not proven. In particular, have not proven such a theorem when integration with respect to Brownian motion is involved. However, under suitable conditions on $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$, we proceed to change the order of integration to get that

$$\log p(t, T) = -(T-t)f(t, T) - \int_t^T \int_t^u \alpha(s, T) ds du - \int_t^T \int_t^u \sigma(s, T) ds dW_u = -(T-t)f(t, T) + I_1 + I_2 \quad (21)$$

Now, $\sigma(s, T)$ is a deterministic function and so

$$g(u) = \int_t^u \sigma(s, T) ds \quad (22)$$

is also a deterministic function of u and we can find the distribution of I_2 using what we know about the distribution of an Ito integral with a deterministic integrand

$$I_2 = -\int_t^T g(u) dW_u \Rightarrow I_2 \sim N\left(0, \int_t^T \left(\int_t^u \sigma(s, T) ds\right)^2 du\right). \quad (23)$$

We can therefore conclude that

$$\log p(t, T) \sim N\left(- (T-t)f(t, T) - \int_t^T \int_t^u \alpha(s, T) ds du, \int_t^T \left(\int_t^u \sigma(s, T) ds\right)^2 du\right). \quad (24)$$

Problem 3

Take as given an HJM model under the risk neutral measure \mathbb{Q} of the form

$$df(t, T) = \alpha(t, T) dt + \sigma_1(T-t) dW_{1t} + \sigma_2 e^{-a(T-t)} dW_{2t} \quad (25)$$

- a) Use the HJM drift condition to find $\alpha(t, T)$.
- b) Find the dynamics of zero coupon bond prices under \mathbb{Q} .
- c) Find the distribution of forward rates in this model and argue that they are Gaussian.
- d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time $t = 0$ price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.