Fixed Income Derivatives E2025 - Problem Set Week 7

Problem 1

Consider the CIR model where the short rate r has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{1}$$

where $a > 0$ and $b > 0$. We will now denote present time by t and proceed in to find explicit formulas for ZCB prices, spot rates and forward rates in the CIR model.

a) In the following, we will compute ZCB prices, spot rates and forward rates in the CIR model by taking a number of steps.

i) Show that ZCB prices in the CIR model are of the form $F^{(T)}(t,r) = A(t,T)e^{-B(t,T)r}$ where $A(t,T)$ and $B(t,T)$ solve the following system of ODE's

$$A_t = abAB, \qquad A(T,T) = 1 \tag{2}$$

$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0. \tag{3}$$

ii) Use the substitution $B = -\frac{2}{\sigma^2 V}V_t$ to transform the ODE for B into the following second order ODE for $V = V(t)$

$$V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0. \tag{4}$$

iii) Use the conjecture that $V(t)$ is of the form $V(t) = e^{yt}$ to show that all solutions for $V(t)$ can be written as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}, \quad \gamma = \sqrt{a^2 + 2\sigma^2} \tag{5}$$

where c_1 and c_2 are constants to be found

iv) Use the boundary condition on $B(T)$ to show that

$$B(t,T) = \frac{2e^{\gamma(T-t)} - 2}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \tag{6}$$

v) Use the ODE for $A(t,T)$ to show that

$$\ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I, \qquad I = -\gamma\int_t^T \frac{e^{\gamma(T-s)} - 1}{2\gamma + \big(a+\gamma\big)\big(e^{\gamma(T-s)} - 1\big)}ds \tag{7}$$

vi) Use a substitution of the form $u = e^{\gamma(T-s)}$ to put the integral on the form

$$I = \int_{e^{\gamma(T-t)}}^1 \frac{u-1}{\gamma - a + \big(a+\gamma\big)u}\frac{1}{u}du \tag{8}$$

vii) Show the following rule for partial fractions

$$\frac{a_0 + a_1 x}{\big(b_0 + b_1 x\big)\big(c_0 + c_1 x\big)} = \frac{y}{b_0 + b_1 x} + \frac{z}{c_0 + c_1 x}, \quad \text{where } y = \frac{a_0 b_1 - a_1 b_0}{c_0 b_1 - c_1 b_0} \text{ and } z = \frac{c_0 a_1 - c_1 a_0}{c_0 b_1 - c_1 b_0} \tag{9}$$

and use this result to simplify the integral in (8) to

$$I = \int_{e^{\gamma(T-t)}}^1 \frac{2\gamma}{(\gamma - a)[\gamma - a + \big(a+\gamma\big)u]}du - \int_{e^{\gamma(T-t)}}^1 \frac{1}{(\gamma - a)u}du \tag{10}$$

viii) Solve the integral in (10) to conclude that

$$A(t,T) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)}\right)^{\frac{2ab}{\sigma^2}} \tag{11}$$

ix) Write down expressions for ZCB prices, spot rates and forward rates in the CIR model.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a CIR model with $a = 2, b = 0.05, \sigma = 0.1, r = 0.025$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Problem 2

Consider the CIR model where the short rate has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{12}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. Present time is denoted by t, the short rate at time t is denoted by r and the price of a zero coupon bond with maturity T is denoted $p(t, T)$. In this problem, we will first generate zero coupon bond prices using the CIR model with known parameters and then seek to recover these parameters by fitting a CIR model to the zero coupon bond prices we generated.

a) Generate ZCB prices for times to maturity $\tau = T - t = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.045$ and parameters $a = 1.5$, $b = 0.06$, $\sigma = 0.08$. Denote these 'empirical' prices by $p^*(t, T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a CIR model to the prices $p^*(t, T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.05, 1.8, 0.08, 0.08$. Plot the fitted values $\hat{p}(t, T)$ and the empirical values $p^*(t, T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.08$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b and σ. You will need to use that method 'trust-constr' also described in the documentation.

e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method.

Problem 3

In this problem, we will consider the CIR model for the short rate r_t with dynamics given by

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r, \end{aligned} \tag{13}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. We know that $r_T|r_0$ is equal in distribution to

$$\frac{\sigma^2}{4a}\Big[1 - e^{-aT}\Big]Y \tag{14}$$

where Y follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-aT}}{\sigma^2\big[1 - e^{-aT}\big]}r_0 \tag{15}$$

The stationary distribution of the short rate is a gamma where

$$r_\infty \sim \text{Gamma}\big(\alpha, \beta\big), \quad \alpha = \frac{2ab}{\sigma^2}, \quad \beta = \frac{\sigma^2}{2a}, \quad f_{r_\infty}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, \; x > 0$$

a) Write a function in Python that takes T, r_0, a, b and σ and a confidence level α as inputs and returns the lower-, upper- or two-sided confidence bounds of r_T.

b) Plot the two-sided confidence bounds for $\alpha = 0.05$ and appropriately many choices of $T < 10$ setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$. Also include the two-sided confidence bounds under the stationary distribution in your plot.

c) For combinations of $a \in [1, 2, 4, 8]$, b fixed at 0.05, $\sigma \in [0.05, 0.1, 0.15, 0.2]$, redo the plot from b) for a sufficiently large T. For each of the combinations of parameters, assess how large T must be for r_T to have settled to it's stationary distribution. How does the rate at which r_T settles to it's stationary distribution depend on a and σ?

In the following, you will simulate the short rate on a grid of mesh δ that runs from initial time $t_0 = 0$ to some terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M - 1, M$, the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T - \delta, T\big]$ and $\delta = \frac{T}{M}$. In the following, please consider the following three schemes

i) An Euler scheme

ii) A Milstein scheme

iii) An exact scheme

d) Derive the difference equation for the short rate for each of the three schemes (if it is possible!) in terms of a standard normal variable denoted Z_m drawn in each of the steps. Are some of the schemes equivalent? Which of the three schemes do you expect to be more accurate?

e) Write a python function that take as inputs T, M, r_0, a, b, σ, and "*scheme*", and returns a simulated trajectory of the short rate. Plot a single trajectory setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$ for each of the three schemes and include the lower, upper and two-sided confidence bounds in your plot for a choice of $\alpha = 0.1$.

f) Now set $T = 3$, repeat the simulations N times and denote the value of the short rate at $T = 3$ in the n'th simulation by r_{3n}, $n = 1, 2, ..., N$. Construct at least 50 but ideally more equally spaced bins to cover the range of r_{3n} from the smallest to the largest value. Sort your simulated values into these bins and use the proportion in each bin to construct an empirical probability mass function. Plot the empirical mass function for your favorite scheme with $N = 1000$ and $M = 1000$, and also plot the theoretical mass function in the same figure.

g) Finally, we will investigate how the difference between the empirical and theoretical PMF's depend on M and N. For a choice of 100 bins and combinations of values of M in [2000,4000,6000,8000,10000] and N in [2000,4000,6000,8000,10000], compute the total square difference between the theoretical probabilities and empirical frequencies across the 100 bins. Report these total squared differences for all combinations of M and N, and for all three schemes. Compare the accuracy of the three schemes and try to assess how large M and N need to be in each of the three cases to arrive at a reasonable accuracy.