Fixed Income Derivatives E2025 - Problem Set Week 5

Problem 1

Suppose that S_t follows a geometric Brownian motion and has dynamics.

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s_0 \end{aligned} \tag{1}$$

where μ and σ are constants. Also assume that there exists a risk-free bank account, B_t, with constant interest rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt \\ B_0 &= b_0 \end{aligned} \tag{2}$$

a) Show that under the risk neutral measure, the dynamics of S_t must be

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ S_0 &= s_0 \end{aligned} \tag{3}$$

by showing that under this measure $\frac{S_t}{B_t}$ is a martingale.

b) Show that the solution $S(T)$ corresponding to the $\mathbb{Q}$-dynamics of S_t is

$$S(T) = s_0 \exp\Big(\big(r - \frac{1}{2}\sigma^2\big)T + \sigma W(T)\Big) \tag{4}$$

Now define $X_t = \ln(S_t)$ and a European call option with contract function $C\big(S(T)\big) = max\big(S(T) - K, 0\big) = \big(S(T) - K, 0\big)_+$

c) Show that $X(T) \sim N\big(x_0 + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T\big)$ under the risk neutral measure and hence that

$$f_{X(T)}(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left(\frac{-\big[x - x_0 - \big(r - \frac{1}{2}\sigma^2\big)T\big]^2}{2\sigma^2 T}\right) \tag{5}$$

is the probability density function of $X(T)$.

d) Argue that the time 0 price of the call option can be found from

$$\begin{aligned} C &= e^{-rT}\mathrm{E}^{\mathbb{Q}}\big[\max\big(S(T) - K, 0\big)\big] = e^{-rT}\int_{-\infty}^{\infty} \big(e^x - K\big)_+ f_{X(T)}(x)dx = e^{-rT}\int_{\ln K}^{\infty} (e^x - K) f_{X(T)}(x)dx \\ &= e^{-rT}\int_{\ln K}^{\infty} e^x f_{X(T)}(x)dx - Ke^{-rT}\int_{\ln K}^{\infty} f_{X(T)}(x)dx = I_1 + I_2 \end{aligned} \tag{6}$$

e) Show that this results in the Black-Scholes formula

$$C = s_0\Phi(d_1) - e^{-rT}K\Phi(d_2), \quad d_1 = \frac{\ln\big(\frac{s_0}{K}\big) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\big(\frac{s_0}{K}\big) + \big(r - \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{7}$$

where $\Phi()$ is the cumulated distribution function of a standard normal random variable. *Hint: To compute these integrals, you will need to perform a substitution so that you work with the PDF of a standard normal random variable, you will need to complete a square, use that the integral of a density function from $-\infty$ to ∞ equals 1 and you will need to use that for $Z \sim N(0,1)$ we have $P(Z < a) = 1 - P(Z > a)$.*

Problem 1 - Solution

a) Under the risk neutral measure $\mathbb{Q}$, $Z_t = \frac{S_t}{B_t}$ must be a martingale. Let us find the $\mathbb{Q}$-dynamics of this ratio

$$dZ_t = \frac{1}{B_t}\big(rS_tdt + \sigma S_t dW_t\big) - \frac{S_t}{B_t^2}rB_tdt = \sigma S_t dW_t \tag{8}$$

This is a clearly a martingale which stems from the fact that the drift of the stock is rS_t.

b) Apply Ito to $X_t = \ln S_t$ and integrate.

$$\begin{aligned} dX_t &= \frac{1}{S_t}\big(rS_tdt + \sigma S_t dW_t\big) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt = \big(r - \frac{1}{2}\sigma^2\big)dt + \sigma dW_t \\ X_T &= x_0 + \int_0^T \big(r - \frac{1}{2}\sigma^2\big)ds + \int_0^T \sigma dW_t = x_0 + \big(r - \frac{1}{2}\sigma^2\big)T + \sigma W_T \\ S_T &= e^{X_T} = s_0 \exp\Big(\big(r - \frac{1}{2}\sigma^2\big)T + \sigma W_T\Big) \end{aligned} \tag{9}$$

c) From the solution for X_T, we see that it is a constant plus a normal random variable W_T. The mean and variance of X_T are $\mathbb{E}\big[X_T\big] = x_0 + (r - \frac{1}{2}\sigma^2)T$ and $\mathrm{Var}\big[X_T\big] = \sigma^2 T$. thus,

$$X_T \sim N\Big(x_0 + \big(r - \frac{1}{2}\sigma^2\big)T, \sigma^2 T\Big) \tag{10}$$

d) Since the payoff from a European call option can be replicated using a portfolio consisting of the bank account and the underlying asset, the price of a European call option can be computed as an expectation under $\mathbb{Q}$. The second equality follows from the definition of the expectation of a continuous random variable and by expressing S_T in terms of X_T, the third equality uses that the integrand is 0 except when $x \geq \ln K$ and the fourth equality simply splits the integrand in two.

e) Let us first compute I_1. Set $\ln K = k$ and $\mu_0 = x_0 + \big(r - \frac{1}{2}\sigma^2\big)T$.

$$\begin{aligned} I_1 &= e^{-rT}\int_{\ln K}^{\infty} e^x f_{X_T}(x) = e^{-rT}\int_k^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left(\frac{-\big(x-\mu_0\big)^2}{2\sigma^2 T}\right)dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}\sigma\sqrt{T}}\int_k^{\infty} \exp\left(-\frac{x^2 - 2x\mu_0 + \mu_0^2 - 2x\sigma^2 T}{2\sigma^2 T}\right)dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}\sigma\sqrt{T}}\int_k^{\infty} \exp\left(-\frac{\big(x - [\mu_0 + \sigma^2 T]\big)^2 - 2\mu_0\sigma^2 T - \sigma^4 T^2}{2\sigma^2 T}\right)dx \\ &= e^{-rT + \mu_0 + \frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}\int_k^{\infty} \exp\left(-\frac{\big(x - [\mu_0 + \sigma^2 T]\big)^2}{2\sigma^2 T}\right)dx \end{aligned} \tag{11}$$

Set $z = \frac{x - \mu_0 - \sigma^2 T}{\sigma\sqrt{T}} = \frac{x - x_0 - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $z^* = \frac{k - x_0 - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -d_1$, note that $dx = \sigma\sqrt{T}dz$ and substitute.

$$I_1 = e^{x_0}\int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = S_0 \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = S_0 \Phi(d_1) \tag{12}$$

Let us then compute I_2. Again, set $\ln K = k$ and $\mu_0 = x_0 + \big(r - \frac{1}{2}\sigma^2\big)T$.

$$I_2 = -Ke^{-rT}\int_{\ln K}^{\infty} f_{X_T}(x) = -Ke^{-rT}\int_k^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left(\frac{-\big(x-\mu_0\big)^2}{2\sigma^2 T}\right)dx$$

Set $z = \frac{x - \mu_0}{\sigma\sqrt{T}} = \frac{x - x_0 - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $z^* = \frac{k - x_0 - (r - \frac{1}{2}\sigma^2)}{\sigma\sqrt{T}} = -d_2$, note that $dx = \sigma\sqrt{T}dz$ and substitute.

$$I_2 = -Ke^{-rT}\int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -Ke^{-rT}\int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -Ke^{-rT}\Phi(d_2) \tag{13}$$

Problem 2

In this problem, we will consider the Vasicek model for the short rate r_t with dynamics given by

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{14}$$

We know that the distribution of $r_T|r_t$ is given by

$$r_T \sim N\Big(e^{-aT}r(0) + \frac{b}{a}\big(1 - e^{-aT}\big), \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big]\Big) \tag{15}$$

and that the stationary distribution is

$$r_\infty \sim N\Big(\frac{b}{a}, \frac{\sigma^2}{2a}\Big) \tag{16}$$

a) Write a function in Python that takes T, r_0, a, b and σ and a confidence level α as inputs and returns the lower-, upper- or two-sided confidence bounds of r_T.

b) Plot the two-sided confidence bounds for $\alpha = 0.05$ and appropriately many choices of $T < 10$ setting $r_0 = 0.04$, $a = 2$, $b = 0.1$, $\sigma = 0.02$. Also include the two-sided confidence bounds under the stationary distribution in your plot.

c) For combinations of $a \in [1, 2, 4, 8]$ and b such that $\frac{b}{a} = 0.05$ (That is, as you are changing a also change b to keep $\frac{b}{a} = 0.05$), and $\sigma \in [0.01, 0.02, 0.03, 0.04]$, redo the plot from b) for a sufficiently large T. For each of the combinations of parameters, assess how large T must be for r_T to have settled to it's stationary distribution. How does the rate at which r_T settles to it's stationary distribution depend on a and σ?

In the following, you will simulate the short rate on a grid of mesh δ that runs from initial time $t_0 = 0$ to some terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T\big]$ and $\delta = \frac{T}{M}$. In the following, please consider the following three schemes

i) An Euler scheme

ii) A Milstein scheme

iii) An exact scheme

d) Derive the difference equation for the short rate for each of the three schemes in terms of a standard normal variable denoted Z_m drawn in each of the steps. Are some of the schemes equivalent? Which of the three schemes do you expect to be more accurate?

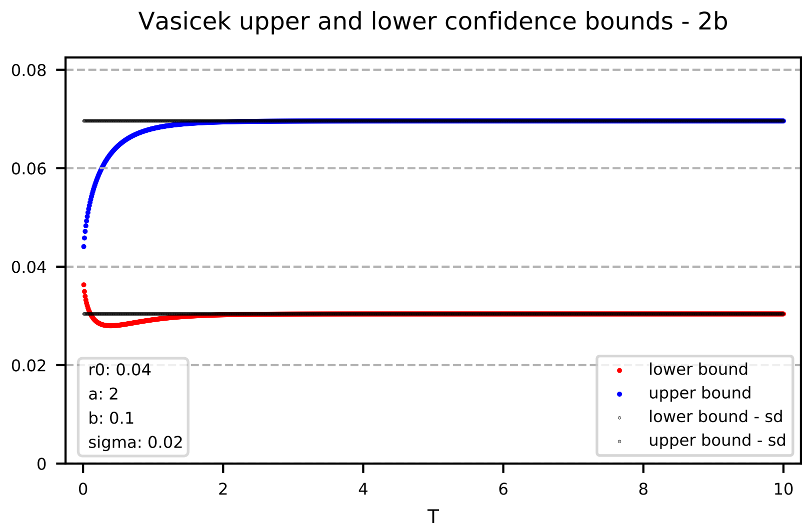
e) Write a python function that take as inputs T, M, r_0, a, b, σ, and "*scheme*", and returns a simulated trajectory of the short rate. Plot a single trajectory for each of the three schemes setting $r_0 = 0.04$, $a = 2$, $b = 0.1$, $\sigma = 0.02$ up to time $T = 10$ and for $M = 10,000$. Include the lower, upper and two-sided confidence bounds in your plot for a choice of $\alpha = 0.1$.

f) Now set $T = 3$, repeat the simulations N times and denote the value of the short rate at $T = 3$ in the n'th simulation by r_{3n}, $n = 1, 2, ..., N$. Construct at least 50 but ideally more equally spaced bins to cover the range of r_{3n} from the smallest to the largest value. Sort your simulated values into these bins and use the proportion in each bin to construct an empirical probability mass function. Plot the empirical mass function for your favorite scheme with $N = 1000$ and $M = 1000$, Also plot the theoretical mass function in the same figure.

g) Finally, we will investigate how the difference between the empirical and theoretical PMF's depend on M and N. For a choice of 100 bins and combinations of values of M in [2000,4000,6000,8000,10000] and N in [2000,4000,6000,8000,10000], compute the total square difference between the theoretical probabilities and empirical frequencies across the 100 bins. Report the total squared differences for all combinations of M and N, and for all three schemes. Compare the accuracy of the three schemes and try to assess how large M and N need to be in each of the three cases to arrive at a reasonable accuracy.

Problem 2 - Solution

b) The two-sided confidence bounds for the short rate in the Vasicek model look as follows



c) Plotting the confidence intervals for a range of choices of a, it is quite clear that the smaller a is, the slower the convergence to the stationary distribution. Also, it seems that though changing σ has an effect on the size of the confidence interval, it does not seem that changing σ has much effect on the rate of convergence. Recall that the distributions of the short rate for finite T and as T tends to infinity are

$$r_T|\mathcal{F}_t \sim N\left(r_t e^{-a(T-t)} + \frac{b}{a}\left[1 - e^{-a(T-t)}\right], \frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]\right), \quad r_\infty \sim N\left(\frac{b}{a}, \frac{\sigma^2}{2a}\right)$$

To more precisely say how the parameters affect convergence to the stationary distribution, we can for example find the value of T for which the difference between the upper bounds have fallen to a certain size, x say, for a given level of α, here $\alpha = 0.05$, by solving the following equation with respect to some T^*

$$x = r_t e^{-a(T^*-t)} + \frac{b}{a}\left[1 - e^{-a(T^*-t)}\right] + \sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]} z_{0.975} - \frac{b}{a} - \sqrt{\frac{\sigma^2}{2a}} z_{0.975} \tag{17}$$

Setting $x = 0.0001$ gives us the following values of T^* for $a \in [1, 2, 4, 8]$ and $\sigma \in [0.01, 0.02, 0.03, 0.04]$ where a varies by row and σ by column.

$$\mathbf{T}^* = \begin{bmatrix} 18.88602161 & 19.23595384 & 19.51892121 & 19.75760538 \\ 9.38254066 & 9.52063898 & 9.63662886 & 9.7369882 \\ 2.3254584 & 2.34563516 & 2.36372909 & 2.38015974 \\ 1.15953625 & 1.16705114 & 1.17396578 & 1.18037635 \end{bmatrix} \tag{18}$$

From these results we see that indeed the distribution of r_T settles to it's stationary distribution much faster when a is large and also that the rate of convergence is only slightly slower when σ is larger.

d) Since the diffusion coefficient of the dynamics of the short rate is independent of r_t, the Milstein and Euler schemes are the same. The exact scheme can be found by using the distribution of $r_{T=t+\delta}|r_t$ from the above. The difference equations become

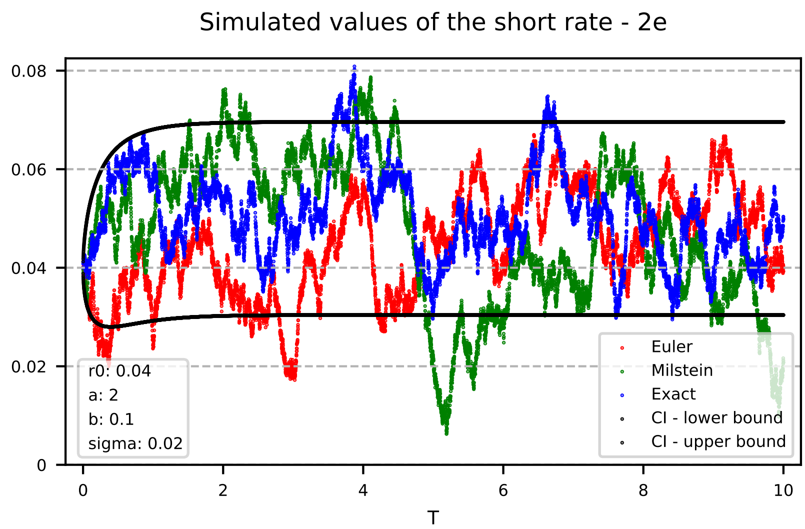
i) Euler: $$r_{m+1} = r_m + (b - ar_m)\delta + \sigma\sqrt{\delta}Z_{m+1}$$

ii) Milstein scheme: $$r_{m+1} = r_m + (b - ar_m)\delta + \sigma\sqrt{\delta}Z_{m+1}$$

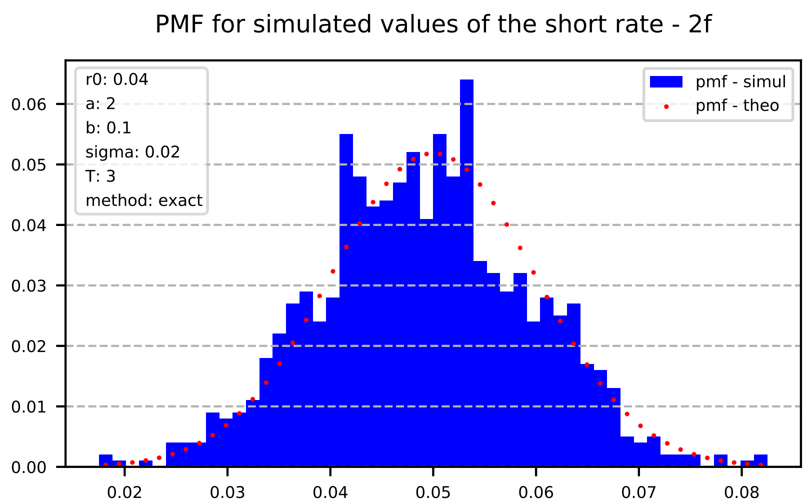
iii) Exact scheme: $$r_{m+1} = r_m e^{-a\delta} + \frac{b}{a}\left[1 - e^{-a\delta}\right] + \sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2a\delta}\right]}Z_{m+1}$$

and needless to say, the scheme based on the solution to the SDE satisfied by the short rate is expected to be most accurate but also, we would expect the difference to be small for large M.

e) The simulated trajectories could look something like

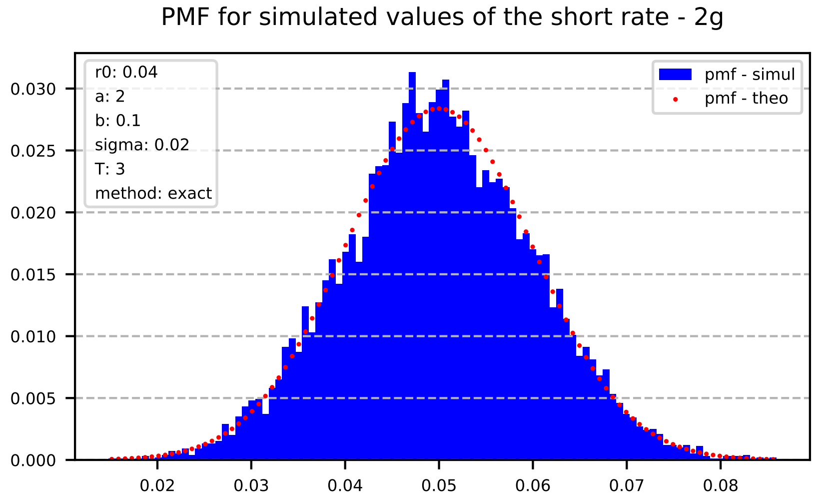


f) Simulating the short rate for $T = 3$, $M = 1000$, repeating the simulation $n = 1000$ times and placing the data in 50 equally spaced bins allows us create a plot of the empirical and theoretical probability mass function. Such a plot may look as follows



Looking at the plot from, we see that there is, not surprisingly, a difference between the simulated and theoretical probability mass function reflecting that the distribution of the simulated values of the short rate after 3 years differs from the theoretical distribution.

g) Setting $N = 10,000$ and $M = 10,000$ and drawing a plot of the empirical- and theoretical probability mass function with 100 bvins results in a plot like the one below.



As one would expect, the two PMF's align much more closely now compared to the plot in 2f. The average of r_3 from the simulation was in this instance 0.049924927 compared to the theoretical mean of 0.049975212 and the sample standard deviation of r_3 based on the simulation came out to 0.010020180 which should be compared to a theoretical standard deviation of 0.009999969. The first two moments are in other words fairly accurately estimated from the simulated values of the short rate. Below the Sum of Squared Errors (SSE) and the run-times are reported for both the Euler scheme and the exact scheme where M increases by the row and N increases by the column. The SSE for the Euler scheme reveals that there might be a small improvement in accuracy when increasing M but the pattern is not clear, however increasing N certainly reduces the SSE. Similarly for the exact scheme, it is not clear that the SSE falls as M increases but certainly, the SSE falls as N increases. Overall, it does not seem that the exact scheme is significantly better than the Euler Scheme in terms of the SSE, but the exact scheme is roughly four times slower. We might have expected that the exact scheme would perform better than the Euler scheme but the fact that this does not seem to be the case simply reflects that the choices of M were sufficient to guarantee that the simulated values of r_3 come from the correct distribution. As for the reduction of SSE as a function of N, we did see a significant improvement in SSE as function of N and in fact, it can be shown that doubling the N roughly reduces the expected SSE by a factor of 2.

Euler scheme:

$$SSE = \begin{bmatrix} 3.03637320e-04 & 1.24607517e-04 & 1.00344431e-04 & 1.26920101e-04 \\ 5.82437990e-04 & 1.99265794e-04 & 9.47600231e-05 & 1.07911005e-04 \\ 4.45664985e-04 & 1.87112542e-04 & 1.32142879e-04 & 1.23209548e-04 \\ 2.19063747e-04 & 1.81049095e-04 & 1.68982400e-04 & 8.13104985e-05 \end{bmatrix} \tag{19}$$

$$T_{comp} = \begin{bmatrix} 6.35144615 & 12.56103039 & 19.45837235 & 25.64430261 \\ 13.00798297 & 29.47159219 & 41.55831909 & 55.29717541 \\ 20.53233314 & 41.67048883 & 62.11731696 & 82.93085575 \\ 27.78885531 & 55.15834522 & 82.14740467 & 104.37330103 \end{bmatrix} \tag{20}$$

Exact scheme:

$$SSE = \begin{bmatrix} 3.29639937e-04 & 2.85697148e-04 & 2.36012075e-04 & 1.11909980e-04 \\ 4.34806626e-04 & 2.14826032e-04 & 1.61631886e-04 & 1.46186736e-04 \\ 5.62213974e-04 & 3.24304573e-04 & 1.48412118e-04 & 7.87769575e-05 \\ 5.35065328e-04 & 3.24571086e-04 & 1.56468608e-04 & 1.06224244e-04 \end{bmatrix} \tag{21}$$

$$T_{comp} = \begin{bmatrix} 25.12175274 & 51.00440907 & 77.16424203 & 102.73119712 \\ 50.25497413 & 103.24376798 & 152.66686845 & 201.46178961 \\ 74.97726488 & 148.74038649 & 224.7333796 & 303.31468487 \\ 103.79146314 & 206.60571575 & 305.91663241 & 407.46576643 \end{bmatrix} \tag{22}$$