Fixed Income Derivatives E2025 - Problem Set Week 4

Problem 1

Let W_t be a Bownian motion, assume $s < t < u < v$ and solve the problems below. In doing so, you will need to use that W_t is Markov and has stationary independent increments. That is, for $0 < s < t$ we know that $W_t - W_s | \mathcal{F}_s = W_t - W_s | W_s = w_s \sim N\big(0, t-s\big)$.

a) Find the conditional distribution of W_t given $\mathcal{F}_s$.

c) Find $\mathbb{E}\big[W_s W_t\big]$, $\text{Cov}\big[W_s, W_t\big]$ and $\text{Cor}\big[X_t, Z_t\big]$.

c) Show that $W_t^2 - t$ is a Martingale.

d) Find $\mathbb{E}\big[W_s W_t W_u\big]$.

e) Find $\mathbb{E}\big[W_s W_t W_u W_v\big]$.

Problem 1 - Solution

a) $\big(W_t | W_s = w_s\big) \sim N(w_s, t-s)$.

b) We of course have that $\text{Var}\big[W_s\big] = s$ and $\text{Var}\big[W_t\big] = t$ but also

$$\begin{aligned} \mathbb{E}\big[W_s W_t\big] &= \mathbb{E}\big[\mathbb{E}\big[W_s W_t \big| \mathcal{F}_s\big]\big] = \mathbb{E}\big[\mathbb{E}\big[W_s W_t \big| \mathcal{F}_s\big]\big] = \mathbb{E}\big[W_s^2\big] = s \\ \text{Cov}\big[W_s, W_t\big] &= \mathbb{E}\big[W_s W_t\big] - \mathbb{E}\big[W_s\big]\mathbb{E}\big[W_t\big] = s \\ \text{Cor}\big[X_t, Z_t\big] &= \frac{s}{\sqrt{s}\sqrt{t}} = \sqrt{\frac{s}{t}} \end{aligned} \tag{1}$$

c) To show that $W_t^2 - t$ is a Martingale, we need to show that $\mathbb{E}\big[W_t^2 - t \big| \mathcal{F}_s\big] = W_s^2 - s$.

$$\begin{aligned} \mathbb{E}\big[W_t^2 - t \big| \mathcal{F}_s\big] &= \mathbb{E}\big[(W_t - W_s + W_s)^2 - t \big| \mathcal{F}_s\big] = \mathbb{E}\big[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t \big| \mathcal{F}_s\big] \\ &= \mathbb{E}\big[(W_t - W_s)^2 + 2W_t W_s - W_s^2 - t \big| \mathcal{F}_s\big] = (t-s) + 2W_s^2 - W_s^2 - t = W_s^2 - s \end{aligned} \tag{2}$$

d) Let us find $\mathbb{E}\big[W_s W_t W_u\big]$ by conditioning

$$\begin{aligned} \mathbb{E}\big[W_s W_t W_u\big] &= \mathbb{E}\big[\mathbb{E}\big[W_s W_t W_u \big| \mathcal{F}_t\big]\big] = \mathbb{E}\big[W_s W_t \mathbb{E}\big[W_u \big| \mathcal{F}_t\big]\big] = \mathbb{E}\big[W_s W_t^2\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[W_s W_t^2 \big| \mathcal{F}_s\big]\big] = \mathbb{E}\big[W_s \mathbb{E}\big[W_t^2 \big| \mathcal{F}_s\big]\big] = \mathbb{E}\big[W_s(t - s + W_s^2)\big] = (t-s)\mathbb{E}\big[W_s\big] + \mathbb{E}\big[W_s^3\big] = 0 \end{aligned}$$

e) Let us find $\mathbb{E}\big[W_s W_t W_u W_v\big]$ once again by conditioning

$$\begin{aligned} \mathbb{E}\big[W_s W_t W_u W_v\big] &= \mathbb{E}\big[\mathbb{E}\big[W_s W_t W_u W_v \big| \mathcal{F}_u\big]\big] = \mathbb{E}\big[W_s W_t W_u \mathbb{E}\big[W_v \big| \mathcal{F}_u\big]\big] = \mathbb{E}\big[W_s W_t W_u^2\big] \\ &= \mathbb{E}\big[W_s W_t \mathbb{E}\big[W_u^2 \big| \mathcal{F}_t\big]\big] = \mathbb{E}\big[W_s W_t \big(u - t + W_t^2\big)\big] = s(u-t) + \mathbb{E}\big[W_s \mathbb{E}\big[W_t^3 \big| \mathcal{F}_s\big]\big] \\ &= s(u-t) + \mathbb{E}\big[W_s\big(W_s^3 + 3W_s(t-s)\big)\big] = s(u-t) + 3s^2 + 3s(t-s) = s\big(2t+u\big) \end{aligned} \tag{3}$$

Problem 2

Let X_t and Y_t be independent Brownian motions for $t \geq 0$. Define $Z_t = \rho X_t + \sqrt{1-\rho^2}Y_t$.

a) Show that Z_t is a Brownian motion

b) Find $\text{Cor}\big[X_t, Z_t\big]$.

c) Find $\mathbb{E}\big[Z_t \big| X_t = x\big]$ and $\text{Var}\big[Z_t \big| X_t = x\big]$.

Let $W_t^{(1)}, W_t^{(2)}, ..., W_t^{(N)}$ be independent Brownian motions and let $\mathbf{\Sigma}$ be an $M \times N$-dimensional matrix where row i, $\mathbf{\Sigma}_{i\cdot}$, satisfies $\|\mathbf{\Sigma}_{i\cdot}\| = \Sigma_{i1}^2 + \Sigma_{i2}^2 + ... + \Sigma_{iN}^2 = 1$. Define the M-dimensional vector $\mathbf{Y}_t = \mathbf{\Sigma}\mathbf{W}_t$

d) Find the covariance matrix of the random vector $\mathbf{Y}_t$. Show that the covariance matrix is positive definite?

e) What is the correlation matrix of $\mathbf{Y}_t$?

f) What is the distribution of $Y_t^{(i)}$ and what is the joint distribution of $\mathbf{Y}_t$?

g) Is $\mathbf{Y}_t$ a multivariate Brownian motion?

Problem 2 - Solution

a) X_t and Y_t have stationary independent increments and therefore, Z_t has stationary independent increments. $Z_t - Z_s$ is the sum of two independent Gaussian random variables and is also Gaussian.

$$\begin{aligned}\mathbb{E}\big[Z_t - Z_s\big|\mathcal{F}_s\big] &= \rho\mathbb{E}\big[X_t - Z_s\big|\mathcal{F}_s\big] + \sqrt{1-\rho^2}\mathbb{E}\big[Y_t - Y_s\big|\mathcal{F}_s\big] = \rho\cdot 0 + \sqrt{1-\rho^2}\cdot 0\\ \mathrm{Var}\big[Z_t - Z_s\big|\mathcal{F}_s\big] &= \mathbb{E}\big[\big(Z_t - Z_s\big)^2\big|\mathcal{F}_s\big] = \rho^2\mathbb{E}\big[\big(X_t - X_s\big)^2\big|\mathcal{F}_s\big] + \big(1-\rho^2\big)\mathbb{E}\big[\big(Y_t - Y_s\big)^2\big|\mathcal{F}_s\big]\\ &= \rho^2\big(t-s\big) + \big(1-\rho^2\big)^2\big(t-s\big) = t - s \end{aligned} \tag{4}$$

In conclusion, Z_t is a continuous stochastic process that has stationary independent increments each of which are Gaussian with mean 0 and variance proportional to time. Thus, Z_t is a Brownian motion.

b) We note that $\mathbb{E}\big[X_t\big] = 0$, $\mathbb{E}\big[Z_t\big] = 0$, $\mathrm{Var}\big[X_t\big] = t$ and $\mathrm{Var}\big[X_t\big] = t$

$$\begin{aligned}\mathrm{Cov}\big[Z_tX_t\big] &= \mathbb{E}\big[Z_tX_t\big] = \mathbb{E}\big[\big(\rho X_t + \sqrt{1-\rho^2}Y_t\big)X_t\big] = \rho\,\mathbb{E}\big[X_t^2\big] + \sqrt{1-\rho^2}\mathbb{E}\big[X_tY_t\big] = \rho t\\ \mathrm{Cor}\big[Z_tX_t\big] &= \frac{\mathrm{Cov}\big[Z_tX_t\big]}{\sqrt{\mathrm{Var}\big[X_t\big]}\sqrt{\mathrm{Var}\big[X_t\big]}} = \rho \end{aligned} \tag{5}$$

c) The conditional mean and variance of Z_t given $X_t = x$ can be found from

$$\begin{aligned}\mathbb{E}\big[Z_t\big|X_t = x\big] &= \mathbb{E}\big[\rho X_t\big|X_t = x\big] + \sqrt{1-\rho^2}\mathbb{E}\big[Y_t\big|X_t = x\big] = \rho\, x\\ \mathrm{Var}\big[Z_t\big|X_t = x\big] &= \mathbb{E}\big[\big(\rho X_t + \sqrt{1-\rho^2}Y_t - \rho\, x\big)^2\big|X_t = x\big]\\ &= \mathbb{E}\big[\big(\rho X_t - \rho\, x\big)^2\big|X_t = x\big] + 2\mathbb{E}\big[\big(\rho X_t - \rho\, x\big)\big|X_t = x\big]\mathbb{E}\big[\sqrt{1-\rho^2}Y_t\big|X_t = x\big] + \big(1-\rho^2\big)\mathbb{E}\big[Y_t^2\big|X_t = x\big]\\ &= \big(1-\rho^2\big)t \end{aligned} \tag{6}$$

d) Let us find the moments of the vector $\mathbf{Y}_t$. Denote dot product by $\cdot$, use Y_i for the ith entry of $\mathbf{Y}_t$, use Σ_{ij} for the entry in the ith row and jth column of $\mathbf{\Sigma}$ and finally use $\mathbf{\Sigma}_{i\cdot}$ to denote the i'th row of $\mathbf{\Sigma}$. Then

$$\begin{aligned}\mathbb{E}\big[Y_i\big] &= \mathbb{E}\Big[\sum_{n=1}^{N}\Sigma_{in}W_t^{(n)}\Big] = 0\\ \mathrm{Var}\big[Y_i\big] &= \mathbb{E}\big[Y_i^2\big] = \mathbb{E}\Big[\sum_{m=1}^{M}\sum_{n=1}^{N}\Sigma_{im}\Sigma_{in}W_t^{(m)}W_t^{(n)}\Big] = t\sum_{n=1}^{N}\Sigma_{in}^2 = t\mathbf{\Sigma}_{i\cdot}\mathbf{\Sigma}_{i\cdot} = t\,\|\mathbf{\Sigma}_{i\cdot}\| = t\\ \mathrm{Cov}\big[Y_iY_j\big] &= \mathbb{E}\big[Y_iY_j\big] = \mathbb{E}\Big[\sum_{m=1}^{M}\sum_{n=1}^{N}\Sigma_{im}\Sigma_{jn}W_t^{(m)}W_t^{(n)}\Big] = t\sum_{n=1}^{N}\Sigma_{in}\Sigma_{jn} = t\,\mathbf{\Sigma}_{i\cdot}\mathbf{\Sigma}_{j\cdot} \end{aligned} \tag{7}$$

The covariance matrix of $\mathbf{Y}_t$ can therefore be written as

$$\mathrm{Cov}\big[\mathbf{Y}_t\big] = \mathrm{Cov}\big[\mathbf{\Sigma}\mathbf{W}_t\big] = \mathbf{\Sigma}\mathbf{\Omega}\mathbf{\Sigma}', \qquad \mathbf{\Omega} = \begin{bmatrix} t & 0 & \cdots & 0\\ 0 & t & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & t \end{bmatrix} \tag{8}$$

This matrix is positive definite by construction since, we can write $\mathrm{Cov}\big[\mathbf{Y}_t\big]$ as the matrix square of two matrices

$$\mathrm{Cov}\big[\mathbf{Y}_t\big] = \mathbf{\Sigma}\mathbf{\Omega}\mathbf{\Sigma}' = \mathbf{A}\mathbf{A}' \tag{9}$$

e) The correlation between Y_i and Y_j is thus $\mathrm{Cor}\big[Y_iY_j\big] = \mathbf{\Sigma}_{i\cdot}\mathbf{\Sigma}_{j\cdot}$ and the correlation matrix of $\mathbf{Y}_t$ is

$$\mathrm{Cor}\big[\mathbf{Y}_t\big] = \mathrm{Cor}\big[\mathbf{\Sigma}\mathbf{W}_t\big] = \mathbf{\Sigma}\mathbf{\Sigma}' \tag{10}$$

f) $Y_t^{(i)}$ is the sum of independent Gaussian random variables and is itself Gaussian with mean 0 and variance t for all t. Furthermore, $Y_t^{(i)}$ has continuous trajectories and hence, $Y_t^{(i)}$ is a Brownian motion. Now since $Y_t^{(i)}$ is a sum of independent normal random variables for all i, it follows that any linear combination $\mathbf{b}'\mathbf{Y}_t$, $\mathbf{b} \in \mathbb{R}^n$ is also a Gaussian random variable. Hence, the joint distribution of the processes collected in $\mathbf{Y}_t^{(i)}$ is multivariate normal

$$\mathbf{Y}_t \sim N\big(\mathbf{0}, \mathbf{\Sigma}\mathbf{\Omega}\mathbf{\Sigma}'\big). \tag{11}$$

g) We can therefore define $\mathbf{Y}_t$ as a multivariate Brownian motion with correlation matrix $\mathbf{\Sigma}\mathbf{\Sigma}'$ and we have just seen a very straight-forward method to construct a multivariate Brownian motion with a specific correlation matrix.

Problem 3

Consider a stochastic process r_t for $t \geq 0$ with dynamics

$$dr_t = (b - ar_t)dt + \sigma dW_t, \quad b > 0$$

a) Show that the solution $r(T)$ corresponding to these dynamics are

$$r_T = e^{-aT}r_0 + \frac{b}{a}\big(1 - e^{-aT}\big) + \sigma\int_0^t e^{-a(T-t)}dW_t$$

by performing the following steps

i) Apply Ito's formula to $f(t,r) = e^{at}r$.

ii) Simplify to get an expression for $d\big(e^{at}r\big)$ that does not depend on r_t.

iii) Integrate from 0 to T and solve the time-integral.

b) Use Ito isometry to show that $r_T|r_t \sim N\Big(e^{-aT}r_0 + \frac{b}{a}\big(1 - e^{-aT}\big), \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big]\Big)$.

c) Find the limiting distribution of r_T as $T \nearrow \infty$.

d) If you had to guess, what is your best guess of the r in the long run? How does the limiting distribution of r_T depend on r_0 and what is the implication?

Problem 3 - Solution

a) The SDE in (12) can be solved explicitly by applying *Ito* to $f(t,r_t) = e^{at}r_t$

$$df(t,r_t) = ae^{at}r_tdt + e^{at}(b - ar_t)dt + e^{at}dW_t = be^{at}dt + \sigma e^{at}dW_t \tag{12}$$

Integrating from t to T gives us that

$$\int_t^T df(u,r_u) = e^{aT}r_T - e^{at}r_t = \int_t^T be^{au}du + \sigma\int_t^T e^{au}dW_u \quad \Rightarrow$$
$$r_T = r_te^{-a(T-t)} + be^{-aT}\int_t^T e^{au}du + \sigma\int_t^T e^{-a(T-u)}dW_u$$

and hence, the solution becomes

$$r_T = r_te^{-a(T-t)} + \frac{b}{a}\Big[1 - e^{-a(T-t)}\Big] + \sigma\int_t^T e^{-a(T-u)}dW_u. \tag{13}$$

b) We know that from Problem 5) below that an Ito integral with a deterministic integrand will follow a Gaussian distribution. We also know that the expected value of an Ito integral is 0, so we have

$$\mathbb{E}\big[r_T \big| r_t\big] = e^{-aT} r_0 + \frac{b}{a}\big(1 - e^{-aT}\big) \tag{14}$$

Computing the variance of $r_T|r_t$ gives us that

$$\text{Var}\big[r_T\big|r_t\big] = \mathbb{E}\Big[\Big(\sigma \int_t^T e^{-a(T-u)} dW_u\Big)^2 \Big| r_t\Big] \overset{\text{Ito isometry}}{=} \sigma^2 \int_t^T e^{-2a(T-u)} du = \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big] \tag{15}$$

and we conclude that

$$r_T|r_t \sim N\Big(e^{-aT} r_0 + \frac{b}{a}\big(1 - e^{-aT}\big), \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big]\Big) \tag{16}$$

c) Sending $T \nearrow \infty$ to find the limiting distribution gives us that

$$r_\infty \sim N\Big(\frac{b}{a}, \frac{\sigma^2}{2a}\Big) \tag{17}$$

The fact that the limiting distribution exists and is well-defined allows us to conclude that the short rate in this model settles to a stationary distribution.

d) The long run mean under the stationary distribution is $\frac{b}{a}$ and that would be our best long-run guess for the short rate. The limiting distribution does not depend on r_0 implying that this process forgets its origin. The mean of $r_T|r_t$ for T finite is a weighted average of the initial value r_t and the long-run mean $\frac{b}{a}$ where the weight on r_t decays exponentially fast and at the rate a. Likewise, $\text{Var}\big[r_T|r_t\big]$ also decays exponentially fast at a rate of $2a$ to the long-run variance. It is therefore quite clear that the parameter a governs the rate at which the distribution of the short rate settles to it's stationary distribution.

Problem 4

Suppose that the stochastic process S_t follows a Geometric Brownian motion and has dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s_0 \end{aligned}$$

a) Show that the solution $S(T)$ corresponding to these dynamics is $S(T) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$.

b) Find $\mathbb{E}[S(T)]$ in terms of s_0, μ and σ.

c) Find the dynamics of $Z_t = S_t^m$ and show that Z_t also follows a geometric Brownian motion.

d) Use these results to find $\mathbb{E}\big[S^m(T)\big]$.

Problem 4 - Solution

a) Applying Ito to $X_t = \ln X_t$ and integrating from 0 to T will, as in Problem 2, give us that

$$S(T) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \tag{18}$$

b) To find $\mathbb{E}\big[S(T)\big]$, we need to use that if

$$X \sim N\big(\mu, \sigma^2\big) \quad \Rightarrow \quad \mathbb{E}\big[e^{\omega X}\big] = e^{\omega\mu + \frac{1}{2}\omega^2\sigma^2} \tag{19}$$

We then get that

$$\mathbb{E}\big[S(T)\big] = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}\big] = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T}\mathbb{E}\Big[e^{\sigma W(T)}\Big] = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T} e^{\frac{1}{2}\sigma^2 T} = s_0 e^{\mu T} \tag{20}$$

c) We find the dynamics of $Z_t = S_t^m$ using Ito's formula

$$dZ_t = m S_t^{m-1}\big(\mu S_t dt + \sigma S_t dW_t\big) + \frac{1}{2} m(m-1) S_t^{m-2} \sigma^2 S_t^2 dt = \big(m\mu + \frac{1}{2} m(m-1)\sigma^2\big) Z_t dt + m\sigma Z_t dW_t \tag{21}$$

d) From the dynamics of Z_t, we can see that Z_t follows a Geometric Brownian motion and hence

$$\mathbb{E}\big[S_t^m\big] = \mathbb{E}\big[Z_t\big] = e^{m\mu + \frac{1}{2}m(m-1)\sigma^2} \tag{22}$$

Problem 5

Let $\sigma(t)$ be a given *deterministic* function of time and define the process X_t by

$$X(t) = \int_0^t \sigma(s) dW_s$$

Also define $Z(t) = e^{i\omega X(t)}$ where i is the complex unit and thus a constant and ω is also a constant.

a) Find the dynamics of X_t.

b) Find the dynamics of Z_t and show that Z_t has dynamics

$$\begin{aligned} dZ_t &= -\frac{1}{2}\omega^2\sigma^2(t)Z(t)dt + i\omega\sigma(t)Z_t dW_t \\ Z_0 &= 1 \end{aligned}$$

c) Integrate dZ_t and take expectations to find an expression for $\mathrm{E}\big[Z(t)\big]$.

d) Define $m(t) = \mathrm{E}\big[Z(t)\big]$ and show that $m(t)$ satisfies the ODE.

$$\begin{aligned} m'(t) &= -\frac{1}{2}\omega^2\sigma^2(t)m(t) \\ m(0) &= 1 \end{aligned}$$

e) Argue that $\mathrm{E}\big[e^{i\omega X(t)}\big] = \exp\Big(-\frac{1}{2}\omega^2\int_0^t \sigma^2(s)ds\Big)$ and why we can say that $X(t) \sim N\Big(0, \int_0^t \sigma^2(s)ds\Big)$.

Problem 5 - Solution

a) The dynamics of X_t can be found directly as $dX_t = \sigma_t dt$.

b) Applying to $Z(t) = e^{i\omega X(t)}$ gives us that

$$dZ_t = d\big(e^{i\omega X(t)}\big) = i\omega e^{i\omega X(t)}dX_t - \frac{1}{2}\omega^2\sigma^2(t)e^{i\omega X(t)}\big(dX_t\big)^2 = -\frac{1}{2}\omega^2\sigma^2(t)Z(t)dt + i\omega\sigma(t)Z_t dW_t \tag{23}$$

c) Integrating and taking expectations will now give us that

$$\begin{aligned} Z(t) &= Z(0) - \frac{1}{2}\omega^2\int_0^t \sigma^2(s)Z(s)ds + i\omega\int_0^t \sigma(s)Z_s dW_s \quad \Rightarrow \\ \mathbb{E}\big[Z(t)\big] &= 1 - \frac{1}{2}\omega^2\int_0^t \sigma^2(s)\mathbb{E}\big[Z(s)\big]ds \end{aligned} \tag{24}$$

d) Setting $m(t) = \mathbb{E}\big[Z(t)\big]$ we have directly from (24) that $m(t)$ satisfies the ODE

$$\begin{aligned} m'(t) &= -\frac{1}{2}\omega^2\sigma^2(t)m(t) \\ m(0) &= 1 \end{aligned} \tag{25}$$

e) The solution to the ODE for $m(t)$ is

$$m(t) = \exp\Big(-\frac{1}{2}\omega^2\int_0^t \sigma^2(s)ds\Big) \tag{26}$$

Now, we can put everything together to get that

$$\hat{f}_{X(t)}(\omega) = \mathrm{E}\big[e^{i\omega X(t)}\big] = \mathbb{E}\big[Z(t)\big] = \exp\Big(-\frac{1}{2}\omega^2\int_0^t \sigma^2(s)ds\Big) \tag{27}$$

The function $\hat{f}_{X(t)}(\omega)$ is the characteristic function of a Gaussian random variable with mean 0 and variance $\int_0^t \sigma^2(s)ds$ and we conclude that

$$X(t) \sim N\Big(0, \int_0^t \sigma^2(s)ds\Big) \tag{28}$$