Fixed Income Derivatives E2025 - Problem Set Week 10

Problem 1

Imagine that todays date is January 1. 2017 and the 3M LIBOR fixing has just been announced. You are working for a financial institution and have been approached by a client who needs to pay 3M LIBOR on a loan of 1 British pound on April 1. 2017, July 1. 2017, October 1. 2017, January 1. 2018, April 1. 2018 July 1. 2018, October 1. 2018 and finally on January 1. 2019. Each of these payments equal the fixing announced exactly 3 months prior. The client wishes to protect himself from 3M LIBOR exceeding 0.045 and your job will be to quote a risk-neutral price on an interest rate cap with a strike of $R = 0.045$. As always, assume that the year has 360 days, that each month has 30 days and that there is no credit risk. Your trusted quant coworker has extracted ZCB prices for you and they are

Table 1: Zero Coupon Bond Prices

T_i	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$p(0,T_i)$	0.99228605	0.98417493	0.97571048	0.96693366	0.95788267	0.94859299	0.93909755	0.92942683

Also, you can see from market data that Black Implied Volatilities, σ_i, for caplets on 3M with maturities T_i and resettlement date one quarter earlier with a strike of 0.045 are

Table 2: Caplet Implied Volatilities

T_i	0.50	0.75	1.00	1.25	1.50	1.75	2.00
σ_i	0.15	0.19	0.23	0.27	0.3	0.33	0.36

a) Find the spot rates, the continuously compounded 3M forward rates and the 3M forward Libor rates corresponding to the ZCB prices in Table 1 and plot these term structures.

b) Explain why there is no caplet available for a maturity of $T = 0.25$ and resettlement at $T = 0$, and compute the caplet prices corresponding to the remaining maturities.

c) Compute the price of the cap that would prevent the client from ever paying more than 0.045 on his floating rate obligation.

d) The client would like to not pay for this cap upfront but rather pay a fixed premium at the dates where he pays regular interest on his floating rate obligation. Find the premium that the client would have to pay to your bank quarterly starting at April 1. 2017 and ending January 1. 2019 should he enter into the interest rate cap.

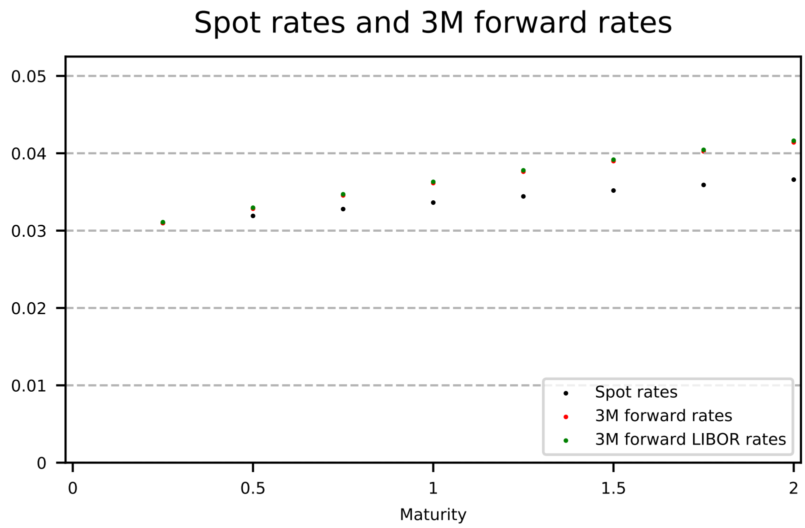
e) Alternatively, you can offer the client to pay a fixed rate by offering him an interest rate swap. What would be the fixed rate that the client would need to pay instead of 3M LIBOR?

f) From the viewpoint of the client, discuss the pros and cons of these different two methods of eliminating interest rate risk.

g) Also discuss the pros and cons, if any, seen from the viewpoint of your financial institution. Do any these trades involve risk for the bank? How can the bank minimize/eliminate the risk from offering the client any of the two different solutions.

Problem 1 - Solution

a) The term structures of spot rates and 3M forward rates are shown in the plot below.



b) A maturity $T = 0.25$ caplet would correspond to a European option on $L(0, 0.25)$ but this is simply the Libor rate that was just announced and hence, there is no option to be had.

c) Caplet prices can be found from the implied volatilities in Table 2 and the price of the cap is the sum of the caplet prices. The price of the cap is 0.0045350046 GBP per 1 unit of notional which corresponds to 45.35 basispoints.

d) The total price of the cap corresponds to an annual premium of 23.59 basispoints and a quarterly premium of 5.89 basispoints.

e) The par swap rate of a 2Y interest rate swap is $R_{swap} = 0.036689$.

f) Using an interest rate cap to protect against interest rate increases comes with a cost but on the other hand, choosing this option implies that the client would still benefit if future Libor rates turn out to be relatively small. Entering into an interest rate swap eliminates all future uncertainty as it locks in all future interest rate payments at the fixed 2Y par swap rate. Now, entering into the 2Y swap does not come with an upfront cost, but choosing this solution also eliminates any upside that would arise if future Libor rates turn out to be lower than the 3M forward Libor rates observed at time $t = 0$.

g) The financial institution only acts to facilitate whatever solution the client chooses and is, in the absence of credit risk at least, not exposed to any risk. The financial institution can however charge the client a fee for its services.

Problem 2

In this problem, we will write a function which computes the Black implied volatility for a caplet with a given price. That is, we will write a function that takes as input: caplet price C, time to maturity T, strike R, tenor α, ZCB price $p(0,T)$ and LIBOR rate $L(0;T-\alpha,T)$ and returns the Black implied volatility of the caplet. We will do this in two different ways.

a) Write a function to compute Black implied volatility of a caplet that uses the scipy.minimize and the 'nelder-mead' method. Apply the function you have written to the data from Problem 1 and make sure that the function works properly by checking that the function recovers the correct implied volatilities from the table above.

b) Now write a function to compute the implied volatility that only uses the vega of a caplet and no numerical optimization. Is the method you are using guaranteed to converge?. Could you improve the method by also including higher order derivatives of the caplet price with respect to σ.

Problem 3

Suppose present time is set to $t = 0$ and you have market data for zero coupon bond prices and 6M LIBOR caplet prices C_i corresponding to the maturity dates T_i, a strike of $R = 0.035$ and a principal of one EUR seen in the table below. Also note that the resettlement (LIBOR fixing announcement) dates are all one period prior. That, is the LIBOR rate serving as the underlying for the $T_i = 1$ caplet is announced at time $T_{i-1} = 0.5$.

Table 3: ZCB and caplet prices

T_i	0.50	1.0	1.5	2.0	2.50	3.0	3.5	4.0
$C_i(0)$	-	0.00062138	0.00193406	0.00329997	0.00462232	0.00588119	0.00707032	0.00818548
$p_i(0)$	0.98530023	0.96939649	0.95255338	0.93499966	0.91693156	0.89851614	0.87989458	0.86118526

a) Find spot-rates, 6M continuously compounded forward rates and 6M forward LIBOR rates. Plot these in an appropriate figure.

b) Find the implied volatilities $\bar{\sigma}_i$ corresponding to these caplet prices and plot the term structure of Black implied volatilities. (You should get numbers ranging from 0.17 for $T = 1$ to 0.48 for $T = 4$.)

In the following you are to construct a LIBOR market model based on the Black implied volatilities you have just computed. That is, you will assume a model where LIBOR rates have dynamics of the form

$$dL_i(t) = L_i(t)\sigma_i(t)dW_i(t), \quad 0 \le t < T_{i-1} \tag{1}$$

and your job will be to determine the coefficients σ_i that are consistent with the term structure of Black implied volatilities you have just found. To do this, assume that for the caplet with maturity T_i and resettlement date T_{i-1}, $\sigma_i(t)$ is piecewise constant and of the following form

$$\sigma_i^2(t) = \begin{cases} \beta_1^2, & 0 \le t < 0.5 \\ \beta_2^2, & 0.5 \le t < 1 \\ \vdots & \\ \beta_{i-1}^2, & T_{i-2} \le t < T_{i-1} \end{cases} \tag{2}$$

c) For $i = 2, ...8$ denote $T_2 = 1, ..., T_8 = 4$ and note that there is the following relationship between the Black implied volatilities $\bar{\sigma}_i$ and the diffusion coefficients in the dynamics of LIBOR rates $\sigma_i(t)$

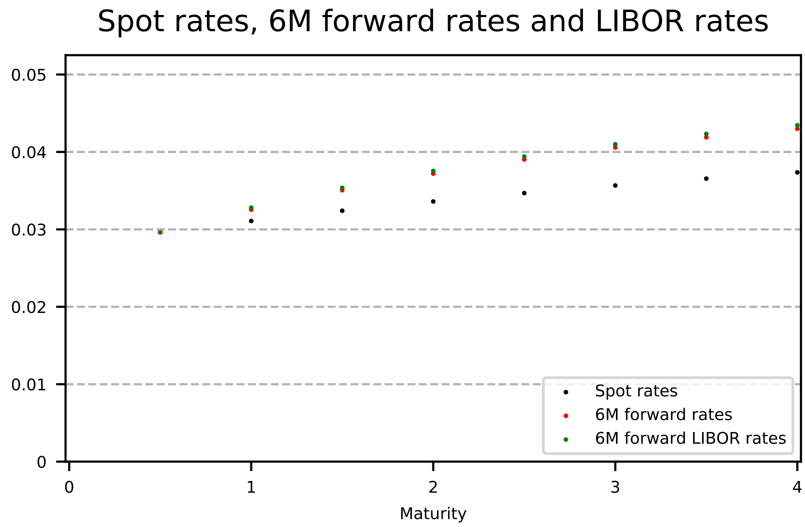
$$\bar{\sigma}_i^2 = \frac{1}{T_i}\int_{t=0}^{T_{i-1}} \sigma_i^2(s)ds \tag{3}$$

Use this relation to find a linear system of equations relating the $\bar{\sigma}_i$ and the β_{i-j} for $i = 2, ..., 8$

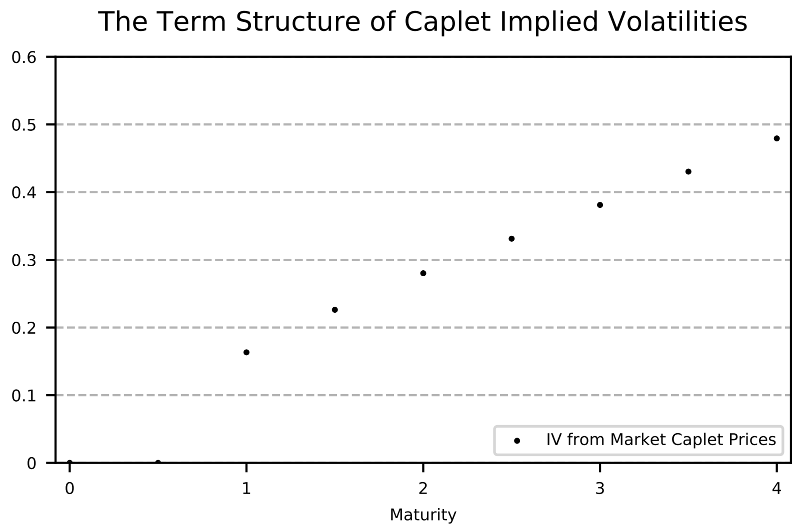
d) Solve the linear system of equations to find the coefficients $\beta_1, ..., \beta_7$.

Problem 3 - Solution

a) The term structures of spot rates, 6M continuously compounded forward rates and 6M forward Libor rates are plotted below



b) The term structure of caplet implied volatilities is shown below.



c) The system of equations linking $\bar{\sigma}_i^2$ and the β_i coefficients looks as follows

$$\bar{\sigma}_1^2 = \frac{1}{1}\int_0^{0.5} \beta_1^2 ds = \frac{1}{1}\left[\frac{1}{2}\beta_1^2\right]$$

$$\bar{\sigma}_2^2 = \frac{1}{1.5}\left[\int_0^{0.5} \beta_1^2 ds + \int_{0.5}^{1} \beta_2^2 ds\right] = \frac{1}{1.5}\left[\frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2\right]$$

$$\vdots$$

$$\bar{\sigma}_7^2 = \frac{1}{4}\left[\frac{1}{2}\int_0^{0.5} \beta_1^2 ds + \frac{1}{2}\int_{0.5}^{1} \beta_2^2 ds + ... + \frac{1}{2}\int_{3}^{3.5} \beta_7^2 ds\right] = \frac{1}{4}\left[\frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2 + ... + \frac{1}{2}\beta_7^2\right] \tag{4}$$

d) Solving the linear system of equations gives us

β_1	β_2	β_3	β_4	β_5	β_6	β_7
0.23093713	0.3164986	0.40060412	0.48433441	0.56808164	0.6520088	0.73617678