Fixed Income Derivatives E2025 - Problem Set Week 8

Problem 1

Consider the Ho-Lee model where the short rate r_t has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t \tag{1}$$

a) Argue that ZCB prices are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t} \tag{2}$$

where

$$\begin{aligned} A(t,T) &= \frac{\sigma^2}{2}\frac{(T-t)^3}{3} + \int_t^T \Theta(s)(s-T)ds \\ B(t,T) &= T-t \end{aligned} \tag{3}$$

b) Show that forward rates $f(t,T)$ are of the form

$$f(t,T) = -\frac{\partial}{\partial T}A_T(t,T) + r_t\frac{\partial}{\partial T}B_T(t,T) \tag{4}$$

where $A_T(t,T) = \frac{\partial}{\partial T}A(t,T)$ and $B_T(t,T) = \frac{\partial}{\partial T}B(t,T)$.

c) Argue that the forward rate dynamics can be found from

$$df(t,T) = -\frac{\partial}{\partial T}\Big(A_t(t,T)dt - B_t(t,T)r_t dt - B(t,T)dr_t\Big) \tag{5}$$

where $A_t(t,T) = \frac{\partial}{\partial t}A(t,T)$ and $B_t(t,T) = \frac{\partial}{\partial t}B(t,T)$.

d) Show that the forward rate dynamics are

$$df(t,T) = \sigma^2\big(T-t\big)dt + \sigma dW_t \tag{6}$$

Now, we will find the forward rate dynamics in a different way. Let us recall that in the Ho-Lee model, zero coupon bond prices become

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\Big\{\big(T-t\big)f^*(0,t) - \frac{\sigma^2}{2}t\big(T-t\big)^2 - \big(T-t\big)r\Big\} \tag{7}$$

e) Use the above expression in (7) to find an expression for forward rates and treat this expression as a function $f(t,T) = g(t,T,r)$.

f) Show from $g(t,T,r)$ that the forward rate dynamics are of the form

$$df(t,T) = \alpha(t,T)dt + \sigma dW_t \tag{8}$$

where $\alpha(t,T)$ is yet to be determined.

g) Use the HJM drift condition to find $\alpha(t,T)$ and thus show that it is of the same form as in d).

Problem 2

Consider the Ho-Lee model where the short rate has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t. \tag{9}$$

Recall that the dynamics of forward rates in the Ho-Lee model are of the form

$$df(t,T) = \sigma^2\big(T-t\big)dt + \sigma dW_t \tag{10}$$

a) Use a result from the chapter 'Bonds and Interest Rates' in Bjørk also given in the lecture slides set IV that allows us to find the dynamics of zero coupon bond prices from the dynamics of forward rates to find the dynamics of zero coupon bond prices under the risk neutral measure $\mathbb{Q}$.

We also know that zero coupon bond prices in the Ho-Lee model are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\Big\{(T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r\Big\}. \tag{11}$$

b) Consider zero coupon bond prices as a function $p(t,T) = g(t,T,r)$ written as

$$p(t,T) = \exp\Big\{\ln p^*(0,T) - \ln p^*(0,t) + (T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r\Big\}. \tag{12}$$

and find the dynamics of zero coupon bond prices directly to confirm your findings in a).

c) In a) and b), you found the dynamics of zero coupon bond prices under the risk-neutral measure using two different approaches but should arrive at the same result. You will have found that zero coupon bond prices follow a familiar process but which? From the drift of zero coupon bond prices, you will also be able to say something about the behavior of these prices. Relate your findings to the fact that we are working under the risk-neutral measure and that we assumed the absence of arbitrage.

d) Find the distribution of forward rates at a future point in time denoted $f(s,T)$, where $t < s < T$ in this model and argue that they are Gaussian.

e) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time t price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.

Problem 3

In this problem, we will study a version of the Nelson-Siegel function often used when modeling the term structure of spot or forward rates. In particular, we will assume that instantaneous forward rates are given by

$$f(t,T) = f_\infty + a_0 e^{-b_0(T-t)} + a_1(T-t)e^{-b_1(T-t)} + a_2(T-t)e^{-b_2(T-t)} + \ldots = f_\infty + \sum_{k=0}^{K} a_k (T-t)^k e^{-b_k(T-t)} \tag{13}$$

where $b_k > 0$ and K is a strictly positive integer. As usual, t denotes present time and T is some time in the future

a) Show that we have the following relationship between forward rates $f(t,T)$ and zero coupon bond prices $p(t,T)$

$$p(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big) \tag{14}$$

b) Show that when forward rates are defined as in (13), ZCB prices are given by

$$\begin{aligned} p(t,T) &= \exp\Big(-\int_t^T f(t,s)ds\Big) = \exp\Big(-f_\infty(T-t) - \sum_{k=0}^{K} a_k \int_t^T (s-t)^k e^{-b_k(s-t)}ds\Big) \\ &= \exp\Big(-f_\infty(T-t) - \sum_{k=0}^{K} a_k I_k\Big), \quad I_k = \int_t^T (s-t)^k e^{-b_k(s-t)}ds \end{aligned} \tag{15}$$

c) Show that I_k can be written as

$$I_k = b_k^{-k-1} \int_0^{b_k(T-t)} u^k e^{-u} du \tag{16}$$

An integral of the form

$$\Gamma(a,b) = \int_0^b x^{a-1}e^{-x}dx \tag{17}$$

for $a > 0$ and $b > 0$ is called an *Incomplete Gamma Function*. The incomplete gamma function is not defined for a a non-positive integer and has to be evaluated numerically for a general a. However, when a is a positive integer, $a \in \mathbb{Z}^+$, the situation is much simpler and we will now explore this case.

d) Show that

$$\Gamma(1,b) = 1 - e^{-b} \tag{18}$$

e) Use integration by parts to show the following recursive relationship

$$\Gamma(a+1,b) = a\Gamma(a,b) - b^a e^{-b} \tag{19}$$

f) Using the recursive relationship between incomplete gamma functions, show that

$$\Gamma(a+1,b) = a! - e^{-b}\sum_{k=0}^{a} b^{a-k}\frac{a!}{(a-k)!} \tag{20}$$

for $a \in \mathbb{Z}^+$ and $b > 0$.

g) Finally, show that if forward rates are given by (13) then ZCB prices are given by

$$p(t,T) = \exp\Big(- f_\infty(T-t) - \sum_{k=0}^{K} a_k I_k\Big) = \exp\Big(- f_\infty(T-t) - \sum_{k=0}^{K} a_k b_k^{-k-1}\Gamma\big(k+1, b_k(T-t)\big)\Big) \tag{21}$$

Problem 4

In this problem, we will consider the Ho-Lee model in which the short rate under the risk neutral measure $\mathbb{Q}$ has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t. \tag{22}$$

Our objective will be to fit the Ho-Lee model to observed forward rates extracted from the market. So, assume that we observe the forward rates given in the vector *f_star* in the file *homework_8.py* for the maturities in the vector T from that same file and denote these observed forward rates by f^*. Also assume that $\sigma = 0.03$. To estimate $\Theta(t)$ in the Ho-Lee model, we will fit a Nelson-Siegel type function $f(t,T)$ to the observed prices f^*

$$f(t,T) = f_\infty + a_0 e^{-b_0(T-t)} + a_1(T-t)e^{-b_1(T-t)} + a_2(T-t)e^{-b_2(T-t)} + ... = f_\infty + \sum_{k=0}^{K} a_k(T-t)^k e^{-b_k(T-t)} \tag{23}$$

where $b_k > 0$ and K governs the number of terms included in the fit.

a) Set present time to $t = 0$ and plot the forward rates, spot rates and zero coupon bond prices generated by the function $f(T) = f(t = 0, T)$ for all maturities in the $T = [0, 0.1, ..., 9.9, 10]$ and parameters $[f_\infty, a_0, a_1, b_0, b_1] = [0.05, -0.02, 0.01, 0.5, 0.4]$. Explain the role each parameter plays in the shape of the spot and forward rate curves.

b) Plot the observed forward rates $f^*(T)$ in a separate plot and try to guess how many terms K will at least need to be included in the fit. Also, based on the plot come up with a set of plausible parameter values so that $f(T)$ with your choice of parameters is likely to fit $f^*(T)$ for $K = 1$.

c) Fit the function $f(T)$ to the observed values in $f^*(T)$ using 'scipy.optimize' and the 'nelder-mead' method. Your objective function should compute the total squared error between the fitted and observed values and hence, you should solve the following minimization problem

$$\min \sum_{m=0}^{M} \Big(f^*(T_m) - f(f_\infty, \mathbf{a}, \mathbf{b}; T_m) \Big)^2 \quad \text{wrt. } f_\infty,\ \mathbf{a},\ \mathbf{b} \tag{24}$$

Do your fit recursively for increasing values of K starting with $K = 1$ and try to go up to no more than $K = 4$. Plot the fitted values $\hat{f}(T)$ versus the observed values $f^*(T)$ for the best fit you achieve.

d) Given your choice of K and preferred parameter estimates, find the function $\Theta(t)$ in the drift of the Ho-Lee model using that

$$\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t \tag{25}$$

where $\frac{\partial f^*(0,t)}{\partial T}$ denotes derivative in the second argument of f^* evaluated at $(0,t)$. Plot the $\Theta(t)$ for all your values of t up to $t = 10$.

e) Now try to fit the function $f(T)$ using the method 'Newton-CG'. To do this, you will need to supply the algorithm with a function that returns the Jacobian (a vector of first-order derivatives of the objective function wrt. the parameters) and the Hessian (a matrix of second-order derivatives of the objective function wrt. the parameters). For example, the derivative of your objective function with respect to a_0 will be

$$\sum_{m=0}^{M} 2 \cdot \Big(f^*(T_m) - f(f_\infty, \mathbf{a}, \mathbf{b}; T_m) \Big) \cdot \frac{\partial f}{\partial a_0} \tag{26}$$

Report the parameter estimates you find using this method and plot the both the empirical and fitted values.