Fixed Income Derivatives E2025 - Problem Set Week 11

Problem 1

In this problem, we will fit the market using a Hull-White Extended Vasicek model, use Jamshidian decomposition to compute prices of interest rate swaptions on an underlying 5Y receiver swap paying an annual fixed coupon rate against 6M EURIBOR and investigate what the Black implied volatility surface looks like for these swaptions. Assume that the 6M EURIBOR has just been announced and that you have the following market data available for FRA's and interest rate swaps exchanging a fixed annual simple rate against 6M EURIBOR.

Table 1: Euribor fixing, FRA and swap market data

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.04098	1X7	0.04178	2Y	0.04674
		2X8	0.04255	3Y	0.04890
		3X9	0.04327	4Y	0.05041
		4X10	0.04396	5Y	0.05150
		5X11	0.04461	7Y	0.05291
		6X12	0.04523	10Y	0.05406
		7X13	0.04581	15Y	0.05496
		8X14	0.04637	20Y	0.05540
		9X15	0.0469	30Y	0.05580

a) Fit a term structure of continuously compounded spot rates to the market data in Table 1 and plot the term structures of continuously compounded spot rates, forward rates and the original market data in a plot.

b) Fit a Hull-White Extended Vasicek Model to the term structures from a) in which you set $a = 0.5$ and $\sigma = 0.025$. Then plot f^*, f^*_T and $\Theta(t)$ corresponding to your fit.

c) Compute forward 5Y par swap rates for 5Y interest rates swap beginning at times $T_n \in [0.5, 1, 2, 3, 5]$ and report these in a table.

d) For the following choices of strike offsets in bps from the 5Y forward par swap rates you found in c): $[-100, -75, -50, -25, 0, 25, 50, 75, 100]$, compute swaption prices per unit of principal (that is in bps) for all exercise times $T_n \in [0.5, 1, 2, 3, 5]$ and report these values in a table. In a second table, report the value of r^* used when computing each of the swaption prices and in a third table, report the strike of each of the swaptions in terms of a forward par swap rate and not just the offset from the ATM value of the underlying par swap rate.

e) Give an interpretation of r^* and relate this value to the strikes in terms of the forward par swap rates you found in d).

f) Find Black implied volatilities corresponding to each of the swaption prices in d) and plot these implied volatilities as a function of the strike offset for each swaption exercise time.

g) Interpret the plot of implied volatilities from e). Are swaption prices consistent with an assumption that 5Y forward par swap rates follow a Geometric Brownian motion? If not, what can then be said about the distribution of the underlying forward par swap rates in the Hull-White Extended Vasicek Model? How do you think the shape if the implied volatility surface depends on the parameters a and σ?

Problem 1 - Solution

a) The term structures of spot rates, forward rates and the original swap market data is given in Figure 1 below.

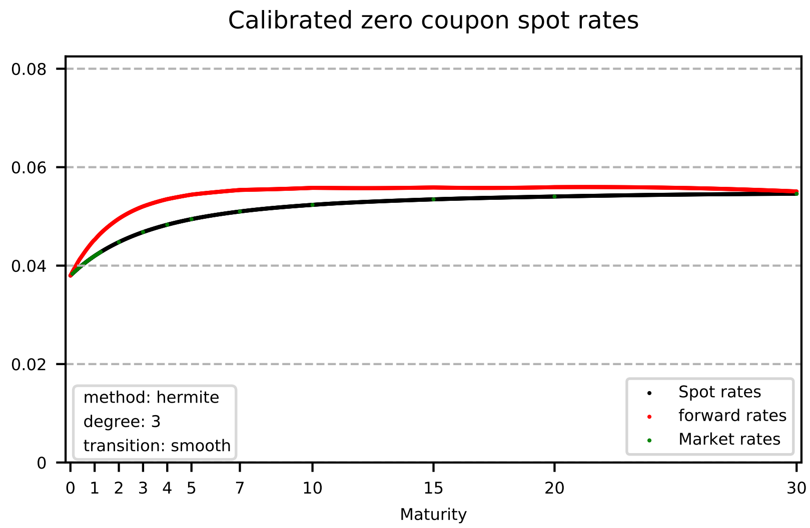

Figure 1: The Term Structure of Interest Rates

b) The plot of f^*, f^*_T and $\Theta(t)$ corresponding to our fit of the HWEV model is given in Figure 2 below.

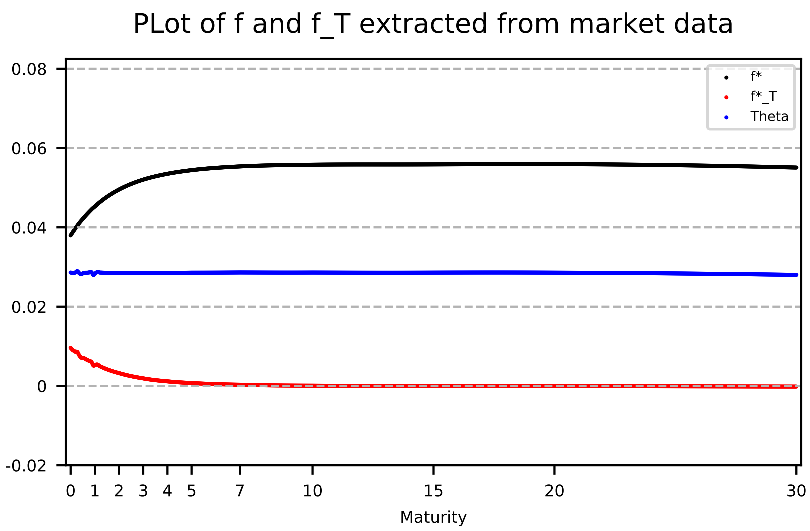

Figure 2: Fitted Values from the HWEV Fit

c) The forward 5Y par swap rates for 5Y interest rates swap beginning at times $T_n \in [0.5, 1, 2, 3, 5]$ are as given in Table 2.

Table 2: Forward 5Y Par Swap Rates

T_n	0.5	1	2	3	5
$F_{T_n}^{T_n+5}$	0.05195468	0.05316186	0.05482378	0.05581652	0.05677708

d) Swaption strikes, r^* and swaption prices are shown in the three tables below.

Table 3: Payer Swaption Strikes

T_n \ offset	-100	-75	-50	-25	0	25	50	75	100
0.5	0.04195	0.04445	0.04695	0.04945	0.05195	0.05445	0.05695	0.05945	0.06195
1	0.04316	0.04566	0.04816	0.05066	0.05316	0.05566	0.05816	0.06066	0.06316
2	0.04482	0.04732	0.04982	0.05232	0.05482	0.05732	0.05982	0.06232	0.06482
3	0.04582	0.04832	0.05082	0.05332	0.05582	0.05832	0.06082	0.06332	0.06582
5	0.04678	0.04928	0.05178	0.05428	0.05678	0.05928	0.06178	0.06428	0.06678

Table 4: Payer Swaption r^*

T_n \ offset	-100	-75	-50	-25	0	25	50	75	100
0.5	-0.00592	0.00509	0.01594	0.02662	0.03715	0.04753	0.05776	0.06785	0.07780
1	0.00255	0.01348	0.02425	0.03486	0.04531	0.05562	0.06578	0.0758	0.08568
2	0.01410	0.02492	0.03559	0.04609	0.05645	0.06666	0.07672	0.08665	0.09644
3	0.02091	0.03167	0.04227	0.05272	0.06302	0.07317	0.08318	0.09305	0.10279
5	0.02746	0.03816	0.04870	0.05909	0.06933	0.07942	0.08938	0.09920	0.10889

Table 5: Payer Swaption Prices

T_n \ offset	-100	-75	-50	-25	0	25	50	75	100
0.5	429.79	332.65	243.93	167.57	106.66	62.28	33.07	15.86	6.84
1	428.90	339.96	259.48	189.67	132.11	87.28	54.47	32.01	17.66
2	417.58	337.12	264.37	200.72	147.11	103.80	70.33	45.67	28.38
3	399.55	324.48	256.55	196.95	146.43	105.20	72.87	48.59	31.14
5	359.84	293.08	232.67	179.58	134.44	97.41	68.18	46.03	29.93

e) The value r^* is the value that the short rate must assume at time of exercise for the underlying par swap rate to be exactly equal to its strike. Recall that the explicit formula we derived for the price of a swaption in the HWEV model uses Jamshidian composition and the fact that in a short rate model, the price of a swaption can be computed as the sum of caplet prices each of which have a cap rate equal to r^*.

f) Black implied volatilities corresponding to the swaption prices from the previous question are shown in Figure 3 below.

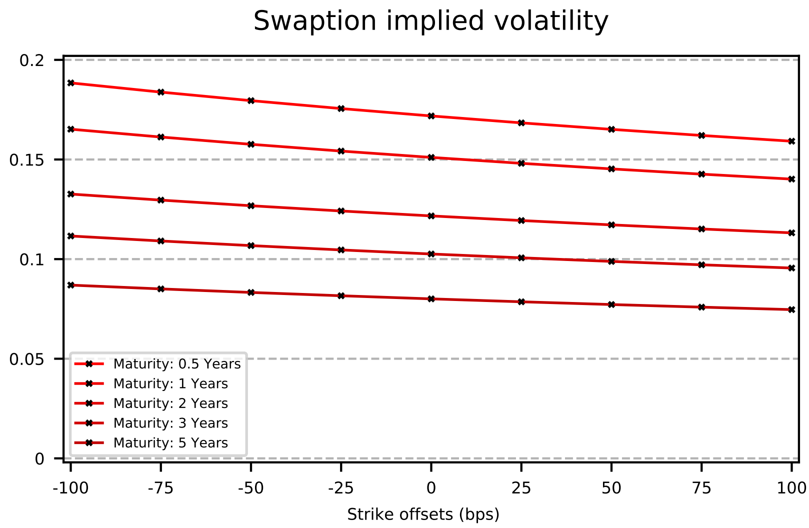

Figure 3: Swaption implied volatilities

g) We can see that swaption implied volatilities are almost constant as a function of strike and hence, the underlying par swap rate follows a distribution that is close to the log-normal distribution in the HWEV model. We also see that implied volatilities are higher for shorter time of exercise of the swaption and hence, the distributions of the underlying swap rates, though all roughly log-normal, are not log-normal with the same parameter values. The reason why the HWEV model assigns relatively higher implied volatilities and thus prices to swaps with an exercise in the immediate future is that there is mean-reversion in this model governed by the parameter a. If we were to increase σ, implied volatilities would shift upwards which is a simply because options are generally more valuable of the variation in the underlying is higher.

Problem 2

In this problem, we will assume the market data given used in Problem 1 and given in Table 1, and we will use the Hull-White Extended Vasicek Model we fitted in Problem 1b. In addition, we will consider a 10Y fixed rate bullet bond paying an annual simple coupon rate, $R_c = 0.06$, of 6% semi-annually. This bond has just been issued and for simplicity, we will assume that it has a principal of 1.

a) Compute the price of the 10Y fixed rate bullet bond at present time $t = 0$.

b) Compute the DV01 of the fixed rate bond for a 1 bps increase in each of the 2Y, 4Y, 6Y, 8Y and 10Y continuously compounded spot rates. Also compute the DV01 if the entire continuously compounded spot rate curve were to increase by one bps.

c) Find an expression for the return to the 10Y fixed rate bond in exactly one year from now right after the second coupon has been paid. In doing so, you can assume that the known coupon paid in six months is invested at present time $t = 0$ to yield a payoff in one year using forward zero coupon bonds.

d) Using simulation and at least $M = 1000$ steps, compute an estimate of the $\alpha = 0.95$ Value-at-Risk in exactly one year with the assumptions from above. Report your answer both as a monetary value and as a loss in percent.

e) Using what we know about the relationship between the value of a fixed rate bond and the short rate in a Hull-White Extended Vasicek Model, compute the $\alpha = 0.95$ VaR from d) explicitly both in monetary units and in terms of a loss in percent.

f) Give an interpretation of the VaR numbers in d) and e) and explain why it can be very problematic to compute VaR, especially for α close to 1. using simulation. Also explain in which cases you are forced to use simulation, both in the single asset and portfolio case.

Problem 2 - Solution

a) The price of the 10Y fixed rate bullet bond at present time is 1.0586172 bps.

b) The DV01 of the fixed rate bond for a 1 bps increase in each of the 2Y, 4Y, 6Y, 8Y and 10Y continuously compounded spot rates become as shown in Table 6 below.

Table 6: DV01 of the Fixed Rate Bond

T	2	4	6	8	10
DV01(bps)	-0.05485474	-0.09889372	-0.13305938	-0.15883792	-6.09935031

The DV01 if the entire continuously compounded spot rate curve were to increase by one bps is -8.162293459 bps.

c) Let us denote the principal of the fixed rate bond by K, the price of the fixed rate coupon bond at time t by $\Pi(t)$, then the total discretely compounded return, $y(1)$, to the fixed rate bond at time $t = 1$ will be

$$y(1) = \frac{\frac{\alpha R_c K}{p(0,0.5,1)} + \alpha R_c K + \Pi(1) - \Pi(0)}{\Pi(0)} \tag{1}$$

d) The estimate of the 95% Var in monetary units became 0.0155 per unit of principal corresponding to a loss of $= 0.0146\%$.

e) The value of the fixed rate bond in one number will depend on the term structure of interest rates in one year, but the entire term structure is governed by the short rate in one year, so the time $t = 1$ value of the fixed rate bond is entirely determined by r_1. We also know that the future price of the bond, $\Pi(1)$, is a strictly decreasing function of r_1. Thus, we can compute the 95% VaR by finding the upper 95% confidence bound on r_1 denoted $\overline{r_1}$ and then compute $\Pi(1, \overline{r_1})$. Doing so, we get that $\overline{r_1} = 0.07874687$ and that the VaR in monetary units becomes 0.01643865 corresponding to a loss of 0.01552842.

f) The exact VaR numbers reveal that there is a 5% risk that we will loose 1.643865 cent per unit of principal corresponding to 1.552842 percent or more on a position in the 10 year fixed rate bond. We also estimated the VaR using simulation and were able to get an estimate that was reasonably close to the 'exact' value. The VaR is by definition a percentile in the distribution of the loss to the fixed rate bond, where the loss is simply the negative of the gain/return. By simulating the short rate, we were effectively able to draw from the loss distribution of the fixed rate bond and could then find the 95% percentile and use that as the estimate of the VaR. Simulating is generally something we only do when there are no other options and especially when computing the VaR. Since the VaR is nothing more than the tail of a distribution, even more simulations are needed because observations in the tail are of course more rare. We were however also able to compute an exact value of the VaR, but this is only possible because we know that the price of the fixed rate bond is a monotone function of the short rate and because we know the distribution of the short rate in the HWEV model.