Homework 02-02

2U

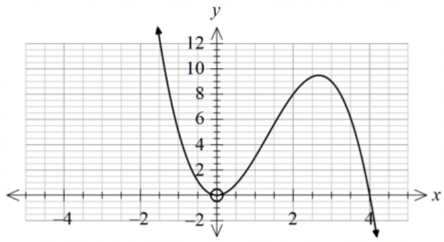
1. Integration

If you have not completed the holiday classes, read and complete the Integration notes and questions previously sent.

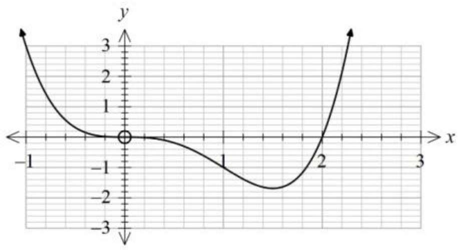
2. Curve sketching using the derivative

Complete the following questions.

The adjacent graph shows the sketch of the curve $y = 4x^2 - x^3$. The coordinates of the local maximum are?



From the graph, $y = x^4 - 2x^3$, the curve increases in the domain?



For each question, determine the domain and range, the x and y-intercepts, and the turning points (and their nature). Use these points to help you then sketch the curve.

For the function $y = x^4 - 2x^3$

(i) Find the coordinates of the points where the curve crosses the axes

(ii) Find the coordinates of the stationary points and determine their nature

~~(iii) Find the coordinates of the points of inflection~~

(iv) Sketch the graph of $f(x)$ clearly indicating the intercepts, stationary points, ~~and points of inflection.~~

Consider the curve $y = x^4 - 8x^2 + 16$

(i) Show that $\dfrac{dy}{dx} = 4x(x-2)(x+2)$

(ii) Find the stationary points on the curve and determine their nature

(iii) Sketch the curve showing all intercepts on the axes and stationary points

The gradient function of a curve is given by $f'(x) = 3(x+1)(x-3)$ and the curve $y = f(x)$ passes through the point $(0, 12)$

i) Find the equation of the curve $y = f(x)$.

ii) Sketch the curve $y = f(x)$, clearly labeling turning points and the y intercept.

3U

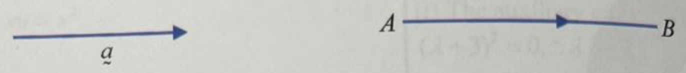
Complete the below questions and read the notes! The starred questions are for those feeling confident.

We measure distance, area, time, etc. with scalar real numbers. However, there are things that need both the magnitude (which use real numbers) and direction to describe them, e.g. force (of a push or a pull) or velocity (of a storm or a car). Such quantities are called vectors.

8.1 Two-dimensional vectors

8.1.1 Definitions

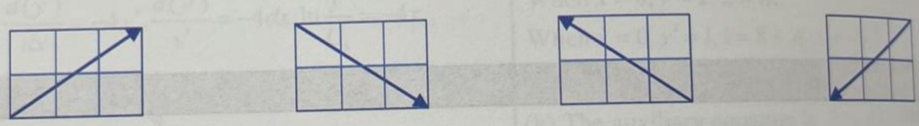
A vector is a directed line segment, drawn with an arrow placed either on its middle or at its head. The arrow indicates the direction of the vector. A vector may be labelled by a small letter above a tilde or by the two capital letters representing the points at each end of the line segment placed under an arrow. The diagram below shows vector $\underset{\sim}{a}$ and vector $\overrightarrow{AB}$.[1]



A vector has direction. Vector $\overrightarrow{AB}$ means it goes from A (called the tail) to B (called the head). Vector $\overrightarrow{AB}$ is opposite to vector $\overrightarrow{BA}$, i.e. $\overrightarrow{AB} = -\overrightarrow{BA}$.

A vector in a two-dimensional space can be represented by a 2×1 matrix, whose top row represents the horizontal directed line segment and bottom row represents the vertical directed line segment.

Below are the vectors $\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}-3\\2\end{pmatrix}$ and $\begin{pmatrix}-3\\-2\end{pmatrix}$ respectively from left to right.



Two vectors are equal if they have equal length and both head in the same direction. Therefore, vector $\begin{pmatrix}3\\2\end{pmatrix}$, for example, can be placed anywhere in the two-dimensional space. Such a vector is called a free vector. However, the one that has its tail at the origin is called the radius vector (or position vector, as it is used to show the position of a point in polar (r, θ) coordinates).

Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if there is a scalar $k, k \in R, k \neq 0$, such that $\underset{\sim}{a} = k\underset{\sim}{b}$. If $k > 0$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are in the same direction. If $k < 0$, they are in opposite directions. If two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, they are said to be linearly dependent. If they are not parallel, they are linearly independent.

A unit vector is a vector that has length of 1 unit. It is labelled as $\hat{\underset{\sim}{u}}$. In two-dimensional space, the two special unit vectors are the radius unit vectors along the x-axis and the y-axis. They are labelled as $\underset{\sim}{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\underset{\sim}{j} = \begin{pmatrix}0\\1\end{pmatrix}$. Any vector $\begin{pmatrix}x\\y\end{pmatrix}$ can be written as $x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$, i.e. $x\underset{\sim}{i} + y\underset{\sim}{j}$.

Any non-zero vector can be converted into a unit vector by dividing by its length, $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

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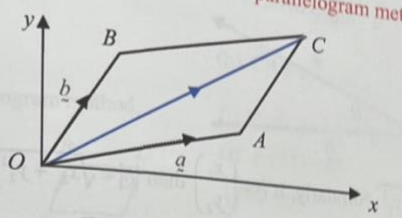
[1] Some textbooks write vector $\underset{\sim}{a}$ in bold, $\boldsymbol{a}$, without the tilde.

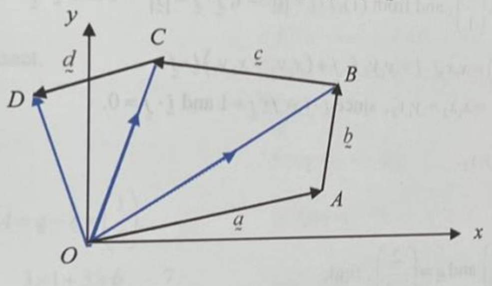
8.1.2 Addition of vectors

To add two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, draw both vectors with their tails at the origin O, let $\overrightarrow{OA}=\underset{\sim}{a}$ and $\overrightarrow{OB}=\underset{\sim}{b}$. By completing the parallelogram $OACB$, vector $\overrightarrow{OC}$ is the sum of the other two vectors, $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}=\underset{\sim}{a}+\underset{\sim}{b}$, as shown. This is known as the parallelogram method.



However, since vector $\overrightarrow{OB}=$ vector $\overrightarrow{AC}$, as two vectors are equal if they have equal lengths and both head in the same direction, we can add two or more vectors by the 'head to tail' method: Draw the second vector such that its tail is at the head of the first vector, and draw the third vector such that its tail is at the head of the second vector, and so on. The sum of these vectors is the vector drawn to complete the polygon, hence, this method is also called the polygon rule.

In the diagram below, $\overrightarrow{OB}=\underset{\sim}{a}+\underset{\sim}{b}, \overrightarrow{OC}=\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}, \overrightarrow{OD}=\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}$.

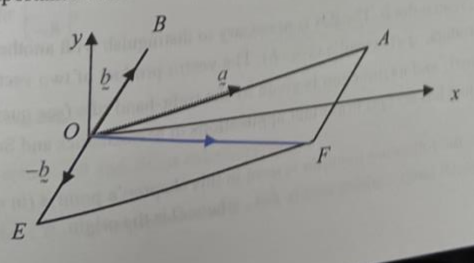


8.1.3 Subtraction of vectors

To draw vector $\underset{\sim}{a}-\underset{\sim}{b}$, we simply change vector $\underset{\sim}{b}$ to vector $-\underset{\sim}{b}$ then add vector $\underset{\sim}{a}$ with vector $-\underset{\sim}{b}$. The result of $\underset{\sim}{a}+\left(-\underset{\sim}{b}\right)$ is $\underset{\sim}{a}-\underset{\sim}{b}$.

Refer to the diagram below, $\overrightarrow{OE}$ represents vector $-\underset{\sim}{b}$ and $OAFE$ is a parallelogram. By the parallelogram method, $\overrightarrow{OF}$ represents vector $\underset{\sim}{a}-\underset{\sim}{b}$.

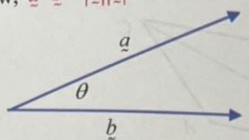
Since $OBAF$ is also a parallelogram, vector $\overrightarrow{OF}=$ vector $\overrightarrow{BA}$, $\therefore$ the value of $\underset{\sim}{a}-\underset{\sim}{b}$ can be found from vector $\overrightarrow{BA}$. This is an important result: $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}$.



8.1.4 Scalar product of vectors

The scalar product or the dot product of two vectors is defined as the product of their magnitudes and the cosine of the angle between their directions.
It is so named because the result is a scalar real number[2].

Refer to the diagram below, $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$.



If $\underset{\sim}{a}=\begin{pmatrix}x_1\\y_1\end{pmatrix}$ then $|\underset{\sim}{a}|=\sqrt{x_1^2+y_1^2}$. Similarly, if $\underset{\sim}{b}=\begin{pmatrix}x_2\\y_2\end{pmatrix}$ then $|\underset{\sim}{b}|=\sqrt{x_2^2+y_2^2}$.

First, notice that if two vectors are parallel, $\underset{\sim}{a}\cdot\underset{\sim}{b}=\pm|\underset{\sim}{a}||\underset{\sim}{b}|$ (because $\cos 0°=1$ and $\cos 180°=-1$) and if they are orthogonal (i.e. perpendicular), $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$ (because $\cos 90°=0$). (1)

Now, $\begin{pmatrix}x_1\\y_1\end{pmatrix}\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix}$ can be calculated by multiplying $(x_1\underset{\sim}{i}+y_1\underset{\sim}{j})\cdot(x_2\underset{\sim}{i}+y_2\underset{\sim}{j})$, using the distributive law,

where $\underset{\sim}{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\underset{\sim}{j}=\begin{pmatrix}0\\1\end{pmatrix}$, and from (1), $\underset{\sim}{i}\cdot\underset{\sim}{i}=|\underset{\sim}{i}|^2=1$, $\underset{\sim}{j}\cdot\underset{\sim}{j}=|\underset{\sim}{j}|^2=1$ and $\underset{\sim}{i}\cdot\underset{\sim}{j}=0$.

$$\begin{aligned}(x_1\underset{\sim}{i}+y_1\underset{\sim}{j})\cdot(x_2\underset{\sim}{i}+y_2\underset{\sim}{j})&=x_1x_2\underset{\sim}{i}\cdot\underset{\sim}{i}+y_1y_2\underset{\sim}{j}\cdot\underset{\sim}{j}+(x_1y_2+x_2y_1)\underset{\sim}{i}\cdot\underset{\sim}{j}\\&=x_1x_2+y_1y_2,\ \text{since } \underset{\sim}{i}\cdot\underset{\sim}{i}=\underset{\sim}{j}\cdot\underset{\sim}{j}=1 \text{ and } \underset{\sim}{i}\cdot\underset{\sim}{j}=0.\end{aligned}$$

$$\therefore \begin{pmatrix}x_1\\y_1\end{pmatrix}\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix}=x_1x_2+y_1y_2.$$

Example 8.1

Given $\underset{\sim}{a}=\begin{pmatrix}3\\4\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}0\\1\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}2\\-2\end{pmatrix}$, find

(a) $\underset{\sim}{a}+\underset{\sim}{b}$. (b) $\underset{\sim}{a}+\underset{\sim}{b}-\underset{\sim}{c}$. (c) $\underset{\sim}{a}\cdot\underset{\sim}{b}$. (d) $\underset{\sim}{a}\cdot\underset{\sim}{c}$.
(e) the angle γ between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. (f) the angle β between vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

Given O, A, B and C are points on the Cartesian plane with radius vectors $\underset{\sim}{0}$, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.[3] Find

(g) $\overrightarrow{BA}$. (h) $\overrightarrow{CA}$. (i) point D if $OADB$ is a parallelogram.
(j) point E if $OACE$ is a parallelogram. (k) $\angle BAC$. (l) $\angle ABC$.
(m) point F if $\overrightarrow{BA} \parallel \overrightarrow{CF}$ and $CF=2BA$. (n) point G if $\overrightarrow{BA}\perp\overrightarrow{CG}$ and $CG=BA$.

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[2] For scalar product $\underset{\sim}{a}\cdot\underset{\sim}{b}$, read a dot b. The dot is necessary to distinguish with another type of product, called vector product or cross product, $\underset{\sim}{a}\times\underset{\sim}{b}$ (read a cross b). The vector product of two vectors is a vector with magnitude equals $|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$, and its direction is given by the right-hand rule (see question 1, Challenge Problems 8). Vector product has several important applications in Mathematics and Science, but it is not in our syllabus.

[3] Unless stated otherwise, the following notation is used in this chapter: a point A (in capital letter) is represented by a radius vector $\underset{\sim}{a}$ (in small letter), which also is $\overrightarrow{OA}$, where O is the origin.

(a) $\underset{\sim}{a}+\underset{\sim}{b}=\begin{pmatrix}3+0\\4+1\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$.

(c) $\underset{\sim}{a}\cdot\underset{\sim}{b}=3\times0+4\times1=4$.

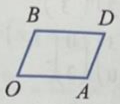
(e) $\cos\gamma=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\dfrac{4}{5\times1}=\dfrac{4}{5},\therefore\gamma\approx37°$.

(g) $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}=\begin{pmatrix}3\\3\end{pmatrix}$.

(i) Method 1: Using parallelogram method,

$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{OB}$.

$\underset{\sim}{d}=\underset{\sim}{a}+\underset{\sim}{b}=\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$.



$\therefore D(3,5)$.

Method 2: Opposite sides of a parallelogram are equal,

$\overrightarrow{OB}=\overrightarrow{AD}$.

$\underset{\sim}{b}=\underset{\sim}{d}-\underset{\sim}{a}$.

$\therefore\underset{\sim}{d}=\underset{\sim}{a}+\underset{\sim}{b}$, as above.

Method 3: Diagonals bisect.

$\dfrac{\underset{\sim}{0}+\underset{\sim}{d}}{2}=\dfrac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$.

$\therefore\underset{\sim}{d}=\underset{\sim}{a}+\underset{\sim}{b}$, as above.

(k) $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}=\begin{pmatrix}3\\3\end{pmatrix},\overrightarrow{CA}=\underset{\sim}{a}-\underset{\sim}{c}=\begin{pmatrix}1\\6\end{pmatrix}$.

$\cos\angle BAC=\dfrac{\overrightarrow{BA}\cdot\overrightarrow{CA}}{|\overrightarrow{BA}||\overrightarrow{CA}|}=\dfrac{3\times1+3\times6}{3\sqrt{2}\sqrt{37}}=\dfrac{7}{\sqrt{74}}$,

$\therefore\angle BAC\approx35°32'$.

(m) $\overrightarrow{BA}\,//\,\overrightarrow{CF}$ and $2BA=CF\Leftrightarrow\overrightarrow{CF}=\pm2\overrightarrow{BA}$,

$\underset{\sim}{f}-\underset{\sim}{c}=\pm2(\underset{\sim}{a}-\underset{\sim}{b})$.

$\underset{\sim}{f}=\underset{\sim}{c}\pm2(\underset{\sim}{a}-\underset{\sim}{b})$

$=\begin{pmatrix}2\\-2\end{pmatrix}\pm2\begin{pmatrix}3\\3\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$ or $\begin{pmatrix}-4\\-8\end{pmatrix}$.

$\therefore F(8,4)$ or $(-4,-8)$.

(b) $\underset{\sim}{a}+\underset{\sim}{b}-\underset{\sim}{c}=\begin{pmatrix}3+0-2\\4+1+2\end{pmatrix}=\begin{pmatrix}1\\7\end{pmatrix}$.

(d) $\underset{\sim}{a}\cdot\underset{\sim}{c}=3\times2+4\times(-2)=-2$.

(f) $\cos\beta=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|}=\dfrac{-2}{5\times2\sqrt{2}}=\dfrac{-1}{5\sqrt{2}},\therefore\beta\approx98°$.[4]

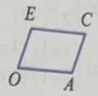
(h) $\overrightarrow{CA}=\underset{\sim}{a}-\underset{\sim}{c}=\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}2\\-2\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}$.

(j) Method 1: Using parallelogram method,

$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OE}$.

$\therefore\overrightarrow{OE}=\overrightarrow{OC}-\overrightarrow{OA}$.

$\therefore\underset{\sim}{e}=\underset{\sim}{c}-\underset{\sim}{a}=\begin{pmatrix}2\\-2\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-1\\-6\end{pmatrix}$.



$\therefore E(-1,-6)$.

Method 2: Opposite sides of a parallelogram are equal,

$\overrightarrow{OE}=\overrightarrow{AC}$.

$\therefore\underset{\sim}{e}=\underset{\sim}{c}-\underset{\sim}{a}$, as above.

Method 3: Diagonals bisect.

$\dfrac{\underset{\sim}{0}+\underset{\sim}{c}}{2}=\dfrac{\underset{\sim}{a}+\underset{\sim}{e}}{2}$.

$\therefore\underset{\sim}{e}=\underset{\sim}{c}-\underset{\sim}{a}$, as above.

(l) $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}=\begin{pmatrix}3\\3\end{pmatrix},\overrightarrow{BC}=\underset{\sim}{c}-\underset{\sim}{b}=\begin{pmatrix}2\\-3\end{pmatrix}$.

$\cos\angle ABC=\dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}=\dfrac{3\times2+3\times(-3)}{3\sqrt{2}\sqrt{13}}=\dfrac{-1}{\sqrt{26}}$,

$\therefore\angle ABC\approx101°19'$.

(n) $\overrightarrow{BA}\perp\overrightarrow{CG}\Leftrightarrow\overrightarrow{BA}\cdot\overrightarrow{CG}=0$.

Let G be $\begin{pmatrix}x\\y\end{pmatrix},\therefore\begin{pmatrix}3\\3\end{pmatrix}\cdot\begin{pmatrix}x-2\\y+2\end{pmatrix}=0$.

$3x-6+3y+6=0,\therefore y=-x$.

$CG=BA\Leftrightarrow(x-2)^2+(-x+2)^2=3^2+3^2$.

$x-2=\pm3,\therefore x=5$ or -1.

$\therefore G(5,-5)$ or $(-1,1)$.

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[4] The angle ABC between vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is calculated from $\cos\theta=\dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$. The numerator is $\overrightarrow{BA}\cdot\overrightarrow{BC}$ or $\overrightarrow{AB}\cdot\overrightarrow{CB}$, not $\overrightarrow{AB}\cdot\overrightarrow{BC}$ nor $\overrightarrow{BA}\cdot\overrightarrow{CB}$. The angle between 2 vectors is not always acute.

Exercise 8.1

1 (a) Prove this theorem by contrapositive: If two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent, then $k\underset{\sim}{a}+\ell\underset{\sim}{b}=0$ only when $k=\ell=0$.

(b) If $(3x+2y)\underset{\sim}{a}+(x-y)\underset{\sim}{b}=\underset{\sim}{a}+12\underset{\sim}{b}$, find the values of x and y if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent.

2 State whether the following pairs of vectors are parallel, orthogonal or neither.

(a) $\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}-3\\-2\end{pmatrix}$.
(b) $\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix}$.
(c) $\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}-2\\3\end{pmatrix}$.
(d) $\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}6\\4\end{pmatrix}$.

(e) $\begin{pmatrix}x+1\\x-1\end{pmatrix},\begin{pmatrix}1-x\\1+x\end{pmatrix}$.
(f) $\begin{pmatrix}q\\p\end{pmatrix},\begin{pmatrix}p\\q\end{pmatrix}, p,q\neq 0$.
(g) $\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix}$.
(h) $\begin{pmatrix}\sin x\\1+\cos x\end{pmatrix},\begin{pmatrix}\sin x\\1-\cos x\end{pmatrix}$

3 Given $\underset{\sim}{a}=p\underset{\sim}{i}+8\underset{\sim}{j}, \underset{\sim}{b}=q\underset{\sim}{i}+4\underset{\sim}{j}$, find the relations of p and q when

(a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel.
(b) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal.

4 (a) Determine the condition for three points A, B and C to be collinear.
(b) Are $A(5,0)$, $B(-3,4)$ and $C(1,2)$ collinear?

5 If $\underset{\sim}{a}=3\underset{\sim}{i}+4\underset{\sim}{j}$, find the following.

(a) $5\underset{\sim}{a}$.
(b) $\underset{\sim}{a}\cdot\underset{\sim}{i}$.
(c) $\underset{\sim}{a}\cdot\underset{\sim}{j}$.
(d) $\hat{\underset{\sim}{u}}$.

6 Given $\underset{\sim}{u}=3\underset{\sim}{i}+\underset{\sim}{j}, \underset{\sim}{v}=-\underset{\sim}{i}+4\underset{\sim}{j}$ and $\underset{\sim}{w}=-5\underset{\sim}{i}-3\underset{\sim}{j}$, calculate and illustrate the following sums on the Cartesian plane.

(a) $\underset{\sim}{u}+\underset{\sim}{v}$.
(b) $\underset{\sim}{u}-\underset{\sim}{v}$.
(c) $\underset{\sim}{u}+\underset{\sim}{v}+\underset{\sim}{w}$.
(d) $\underset{\sim}{u}+\underset{\sim}{v}-\underset{\sim}{w}$.

7 Simplify, giving reasons for your answers.

(a) $\overrightarrow{AB}+\overrightarrow{BC}$.
(b) $\overrightarrow{DE}+\overrightarrow{FD}$.
(c) $\overrightarrow{AB}-\overrightarrow{CA}+\overrightarrow{BC}$.
(d) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$.

8 Simplify

(a) $(\underset{\sim}{i}+3\underset{\sim}{j})+(4\underset{\sim}{i}+\underset{\sim}{j})$.
(b) $(3\underset{\sim}{i}+4\underset{\sim}{j})-(\underset{\sim}{i}-3\underset{\sim}{j})$.
(c) $(3\underset{\sim}{i}-2\underset{\sim}{j})\cdot(\underset{\sim}{i}-4\underset{\sim}{j})$.
(d) $(-3\underset{\sim}{i}+\underset{\sim}{j})\cdot(-4\underset{\sim}{i}-$

9 Given O, A, B and C are points on the Cartesian plane with position vectors $\underset{\sim}{0}, \underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$, where $\underset{\sim}{a}=3\underset{\sim}{i}+4\underset{\sim}{j}, \underset{\sim}{b}=2\underset{\sim}{i}+\underset{\sim}{j}$ and $\underset{\sim}{c}=2\underset{\sim}{i}-3\underset{\sim}{j}$, find

(a) $\angle AOB$.
(b) $\angle AOC$.
(c) $\angle ABC$.
(d) $\angle CAB$.
(e) point D if $OADB$ is a parallelogram.
(f) point E if $OBAE$ is a parallelogram.
(g) point F if $ABFC$ is a parallelogram, using at least 3 different methods.
(h) point H if $ABCH$ is a parallelogram, using at least 3 different methods.
(i) point P if $\overrightarrow{AB}\ //\ \overrightarrow{PC}$ and $PC=3AB$.
(j) point Q if $\overrightarrow{CB}\perp\overrightarrow{AQ}$ and $CB=AQ$.

8.2 Geometrical applications

8.2.1 Vector equation of a line

Since $\underset{\sim}{p}$ represents the radius vector (also called position vector) $\overrightarrow{OP}$, $\overrightarrow{AB}=\underset{\sim}{b}-\underset{\sim}{a}$, $\underset{\sim}{a}\ //\ \underset{\sim}{b}$ iff $\underset{\sim}{a}=\lambda\underset{\sim}{b}$, where $\lambda\in R$ and $\underset{\sim}{a}\perp\underset{\sim}{b}$ iff $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$, we can use them to state the following as vector equations:

The vector equation of the line through point A (radius vector $\underset{\sim}{a}$) and parallel to vector $\underset{\sim}{d}$ (hence, $\underset{\sim}{d}$ is called the direction vector) is $\underset{\sim}{p}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$.

Proof

If $\overrightarrow{AP}$ is parallel to vector $\underset{\sim}{d}$, then $\overrightarrow{AP}=\lambda\underset{\sim}{d}$, where $\lambda\in R$.

But $\overrightarrow{AP}=\underset{\sim}{p}-\underset{\sim}{a}, \therefore \underset{\sim}{p}-\underset{\sim}{a}=\lambda\underset{\sim}{d}, \therefore \underset{\sim}{p}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$.

In $\underset{\sim}{p}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$, λ is called a parameter.

If $\underset{\sim}{p}=\begin{pmatrix}x\\y\end{pmatrix}$, $\underset{\sim}{a}=\begin{pmatrix}a_x\\a_y\end{pmatrix}$ and $\underset{\sim}{d}=\begin{pmatrix}d_x\\d_y\end{pmatrix}$, then $\begin{cases}x=a_x+\lambda d_x\\y=a_y+\lambda d_y\end{cases}$.

The parametric equation is $\underset{\sim}{p}=\left(a_x\underset{\sim}{i}+a_y\underset{\sim}{j}\right)+\lambda\left(d_x\underset{\sim}{i}+d_y\underset{\sim}{j}\right)$.

The symmetric equation is $\dfrac{x-a_x}{d_x}=\dfrac{y-a_y}{d_y}$ (because they equal λ).

Example 8.2

(a) Find the equation of the line through $(-2,3)$ and parallel to vector $\begin{pmatrix}1\\2\end{pmatrix}$, giving your answer in

(i) parametric form (ii) symmetric form (iii) either tangent form or general form.

(b) Find the direction vector of the line $\dfrac{x-2}{3}=\dfrac{y-4}{5}$.

(a) In parametric form, $\underset{\sim}{p}=\left(-2\underset{\sim}{i}+3\underset{\sim}{j}\right)+\lambda\left(\underset{\sim}{i}+2\underset{\sim}{j}\right)$.

Let $\underset{\sim}{p}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\end{pmatrix}\therefore \begin{cases}x=-2+\lambda\\y=3+2\lambda\end{cases}$

In symmetric form, $x+2=\dfrac{y-3}{2}$.

The above equation can be simplified to $y=2x+7$, which is the familiar tangent form.

(b) $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}+\lambda\begin{pmatrix}3\\5\end{pmatrix}$.

In parametric form, $\underset{\sim}{p}=\left(2\underset{\sim}{i}+4\underset{\sim}{j}\right)+\lambda\left(3\underset{\sim}{i}+5\underset{\sim}{j}\right)$.

$\therefore$ The direction vector is $3\underset{\sim}{i}+5\underset{\sim}{j}$.

Exercise 8.2

1 Find the equations of the following lines in (i) parametric, (ii) symmetric and (iii) either tangent or general forms.

(a) through (2,1) and parallel to $\begin{pmatrix}-1\\3\end{pmatrix}$.

(b) through $(-1,-3)$ and perpendicular to $\begin{pmatrix}1\\2\end{pmatrix}$.

(c) through (1,2) and perpendicular to $\begin{pmatrix}-2\\3\end{pmatrix}$.

(d) through $A(5,0)$ and $B(3,4)$.

2 Find the direction vectors and the normal (= orthogonal) vectors of the following lines.

(a) $\dfrac{x+2}{3}=\dfrac{3-y}{4}$. (b) $\dfrac{4x+2}{3}=\dfrac{3y-1}{2}$. (c) $\dfrac{3x+2}{3}=10-5y$. (d) $4x+2=\dfrac{2-3y}{5}$.

3 $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the radius vectors of 2 points A and B and $\underset{\sim}{d}$ is another vector.

(a) Why is $\left|\underset{\sim}{p}-\underset{\sim}{a}\right|=R$, $R>0$, the equation of a circle? What is $\left|\underset{\sim}{p}+\underset{\sim}{a}\right|=R$?

(b) Why is $\left(\underset{\sim}{p}-\underset{\sim}{a}\right).\underset{\sim}{d}=0$ the equation of a line? What is it if $\underset{\sim}{a}$ and $\underset{\sim}{d}$ are 3-D vectors?

(c) Why is $\left|\underset{\sim}{p}-\underset{\sim}{a}\right|+\left|\underset{\sim}{p}-\underset{\sim}{b}\right|=k, k>\left|\underset{\sim}{a}-\underset{\sim}{b}\right|$, the equation of an ellipse? What is it if $k=\left|\underset{\sim}{a}-\underset{\sim}{b}\right|$?

4 Prove the following by using the methods given in the hints.

(a) $EFGH$, $E(1,1)$, $F(2,6)$, $G(-1,7)$, $H(-2,2)$, is a parallelogram. Hint: A quadrilateral that has a pair of sides both parallel and equal is a parallelogram).

(b) $PQRS$, $P(1,3)$, $Q(2,7)$, $R(-2,6)$, $S(-3,2)$, is a rhombus. Hint: A parallelogram that has a pair of adjacent sides equal is a rhombus.

(c) $ABCD$, $A(2,1)$, $B(-1,3)$, $C(1,6)$, $D(4,4)$, is a square. Hint: A rhombus that has a right angle is a square.

~~**5** Redo question 4 by using scalar product only.~~

6 (a) Point P divides the line segment AB internally the ratio of $m{:}n$, prove that $\underset{\sim}{p}=\dfrac{m\underset{\sim}{b}+n\underset{\sim}{a}}{m+n}$.

(b) P is the midpoint of AB, prove that $\underset{\sim}{p}=\dfrac{\underset{\sim}{a}+\underset{\sim}{b}}{2}$.

(c) P, Q and R are the midpoints of the sides of ΔABC, show that $\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}=\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}$.

7 (a) $OABC$ is a rhombus, where O is the origin, $A(3,7)$ and $C(7,3)$. Find the lengths of the diagonals, hence, the area of the rhombus.

(b) Explain why $AB // CD$, where $A(0,-2)$, $B(5,-1)$, $C(8,-3)$ and $D(-2,-5)$. Is $AD \perp BC$?

8 Prove the cosine rule that for any ΔOAB, $\left|\underset{\sim}{a}-\underset{\sim}{b}\right|^2=\left|\underset{\sim}{a}\right|^2+\left|\underset{\sim}{b}\right|^2-2\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos\angle AOB$.

9 Prove that the medians of a triangle meet at a point that divides each median in the ratio 2:1.

10 Describe the following curves both geometrically and algebraically, where possible. Sketch the curves, showing direction vectors for straight lines.

(a) $\left|\underset{\sim}{p}\right|=2$. (b) $\left|\underset{\sim}{p}-\underset{\sim}{i}\right|=2$. (c) $\underset{\sim}{p}=(3+\lambda)\underset{\sim}{i}+2\underset{\sim}{j}$. (d) $\underset{\sim}{p}=(2+\lambda)\underset{\sim}{i}+\lambda\underset{\sim}{j}$.

(e) $\underset{\sim}{p}\cdot\underset{\sim}{i}=0$. (f) $\underset{\sim}{p}\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}\right)=0$. (g) $\left(\underset{\sim}{p}+\underset{\sim}{j}\right)\cdot\left(2\underset{\sim}{i}-\underset{\sim}{j}\right)=0$ (h) $\left(\underset{\sim}{p}-\underset{\sim}{i}\right)\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}\right)=0$

(i) $\underset{\sim}{p}\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}\right)=1$ (j) $\underset{\sim}{p}\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}\right)=\underset{\sim}{i}\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}\right)$ (k) $\left|\underset{\sim}{p}-3\underset{\sim}{i}\right|+\left|\underset{\sim}{p}+\underset{\sim}{i}\right|=8$. (l) $\left|\underset{\sim}{p}-\underset{\sim}{i}\right|+\left|\underset{\sim}{p}+\underset{\sim}{i}\right|=2$.

8.3 Projectile motion

8.3.1 The parametric equation of motion.

In this section we consider the two-dimensional motion of a particle projected into the air but subject to the gravity only, i.e. we assume that air resistance takes no effect in the motion of the particle. Consider a particle that is projected from a point O with initial velocity U that makes an angle of projection α with the horizontal.