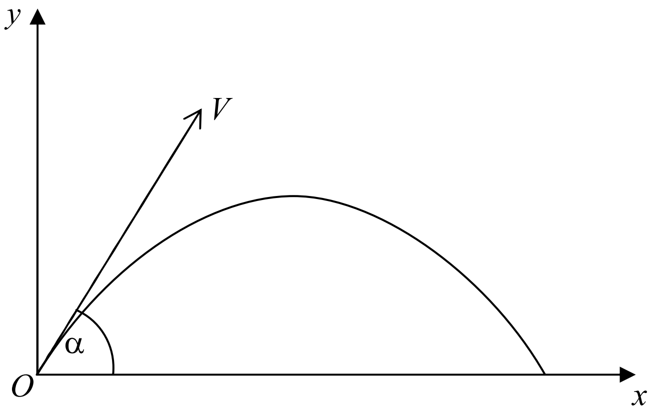
Projectile Motion

Mathematics Extension 1 (Waverley College)

Projectile Motion HSC Questions

Question 2s

1. ’90 2c



The path of a projectile fired from the origin O is given by $x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$ where V is the initial speed in metres per second, and α is the angle of projection as in the diagram, and t is the time in seconds.

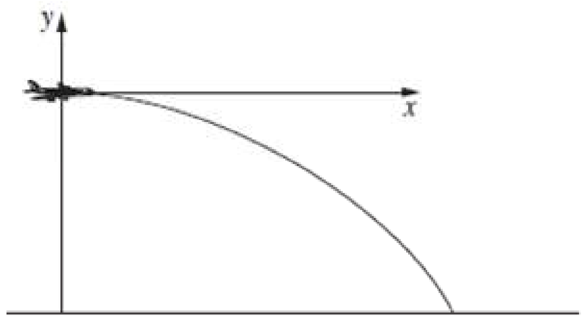
(i) Find the maximum height reached by the projectile in terms of V and α.

(ii) Find the range in terms of V and α.

(iii) Prove that the range is maximum when $\alpha = 45°$.

Question 4s

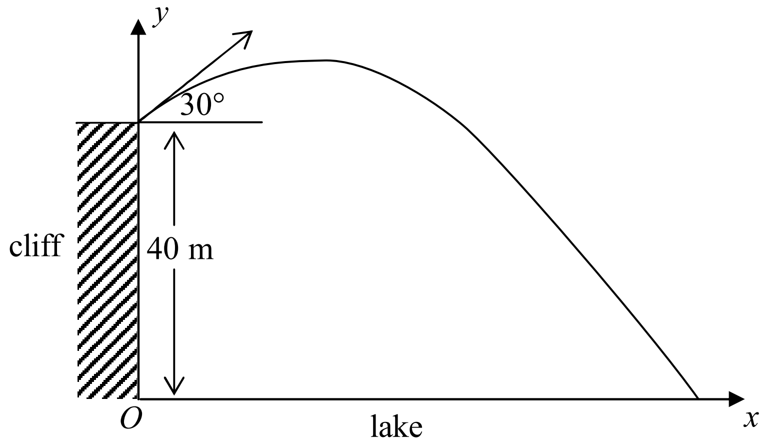
***2.** ’01 4b An aircraft flying horizontally at V ms^{-1} releases a bomb that hits the ground 4000 m away, measured horizontally. The bomb hits the ground at an angle of 45° to the vertical.



Assume that, t seconds after release, the position of the bomb is given by $x = Vt$, $y = -5t^2$.

Find the speed V of the aircraft.

3. ’87 4ii A pebble is projected from the top of a vertical cliff with velocity $20\ \text{ms}^{-1}$ at an angle of elevation of $30°$. The cliff is 40 metres high and overlooks a lake.



(a) Take the origin O to be the point at the base of the cliff immediately below the point of projection. Derive expressions for the horizontal component $x(t)$ and vertical component $y(t)$ of the pebble’s displacement from O after t seconds.

(b) Calculate the time which elapses before the pebble hits the lake and the distance of the point of impact from the face of the cliff. (Assume $g = 10$.)

Question 5s

***4.** '85 5(i) Firefighters are forced to stay 60 metres away from a dangerous fire burning in a low open tank on horizontal ground. They have two pumps. One, which can eject water in any direction at $30\ \text{ms}^{-1}$, is on the ground, while the other which can eject water at $40\ \text{ms}^{-1}$ but only horizontally, is on a vertical stand 5 metres high.

Can both pumps reach the fire? Justify your answer. (Assume $g = 10$)

***5.** '83 5 A steady wind is blowing with speed 36 km/h. From clouds moving horizontally with the wind, heavy raindrops fall to the ground 200 metres below.

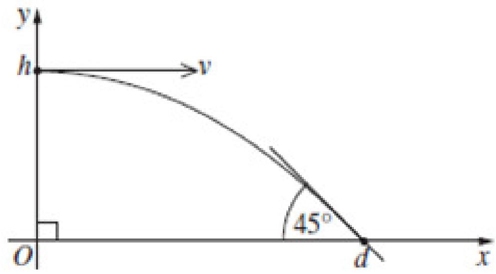
(a) Find the time taken for a drop to reach the ground.

(b) Find the speed and angle at which a drop hits the ground.

(c) At what angle does the drop hit the ground when the wind speed is doubled?

Question 6s

6. '11 6b The diagram shows the trajectory of a ball thrown horizontally, at speed v ms^{-1}, from the top of a tower h metres above ground level.



The ball strikes the ground at an angle of 45°, d metres from the base of the tower, as shown in the diagram. The equations describing the trajectory of the ball are

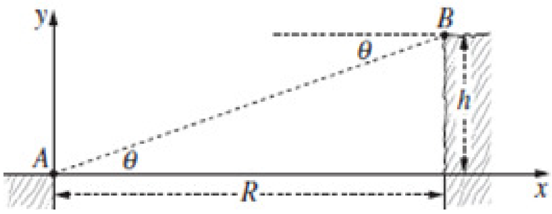
$x = vt$ and $y = h - \frac{1}{2}gt^2$, (Do NOT prove this.)

where g is the acceleration due to gravity, and t is time in seconds.

(i) Prove that the ball strikes the ground at time $t = \sqrt{\frac{2h}{g}}$ seconds.

(ii) Hence, or otherwise, show that $d = 2h$.

***7.** '09 6a Two points, A and B, are on cliff tops on either side of a deep valley. Let h and R be the vertical and horizontal distances between A and B as shown in the diagram. The angle of elevation of B from A is θ, so that $\theta = \tan^{-1}\left(\frac{h}{R}\right)$.



At time $t = 0$, projectiles are fired simultaneously from A and B. The projectile from A is aimed at B, and has initial speed U at an angle θ above the horizontal. The projectile from B is aimed at A and has initial speed V at an angle θ below the horizontal. The equations for the motion of the projectile from A are

$$x_1 = Ut\cos\theta \text{ and } y_1 = Ut\sin\theta - \frac{1}{2}gt^2$$

and the equations for the motion of the projectile from B are

$$x_2 = R - Vt\cos\theta \text{ and } y_2 = h - Vt\sin\theta - \frac{1}{2}gt^2$$

(Do NOT prove these equations.)

(i) Let T be the time at which $x_1 = x_2$. Show that $T = \frac{R}{(U+V)\cos\theta}$.

(ii) Show that the projectiles collide.

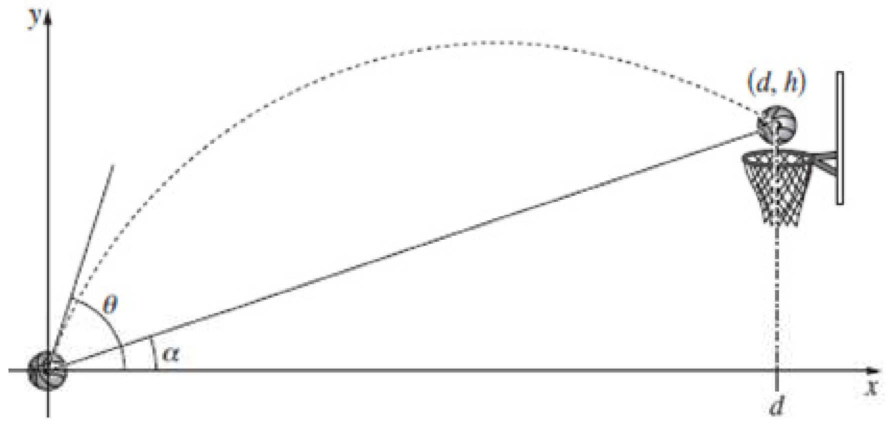
(iii) If the projectiles collide on the line $x = \lambda R$, where $0 < \lambda < 1$, show that $V = \left(\frac{1}{\lambda} - 1\right)U$.

***8.** '10 Q6 (a) (i) Show that $\cos(A-B)=\cos A\cos B(1+\tan A\tan B)$.

(ii) Suppose that $0<B<\frac{\pi}{2}$ and $B<A<\pi$.

Deduce that if $\tan A\tan B=-1$, then $A-B=\frac{\pi}{2}$.

(b) A basketball player throws a ball with an initial velocity v ms^{-1} at an angle θ to the horizontal. At the time the ball is released its centre is at (0, 0), and the player is aiming for the point (d, h) as shown on the diagram. The line joining (0, 0) and (d, h) makes an angle α with the horizontal, where $0<\alpha<\theta<\frac{\pi}{2}$.



Assume that at time t seconds after the ball is thrown its centre is at the point (x, y), where $x=vt\cos\theta$

$$y=vt\sin\theta-5t^2$$

(You are NOT required to prove these equations.)

(i) If the centre of the ball passes through (d, h) show that

$$v^2=\frac{5d}{\cos\theta\sin\theta-\cos^2\theta\tan\alpha}.$$

(ii) (1) What happens to v as $\theta\to\alpha$?

(2) What happens to v as $\theta\to\frac{\pi}{2}$?

(iii) For a fixed value of α, let $F(\theta)=\cos\theta\sin\theta-\cos^2\theta\tan\alpha$.
Show that $F'(\theta)=0$ when $\tan 2\theta\tan\alpha=-1$.

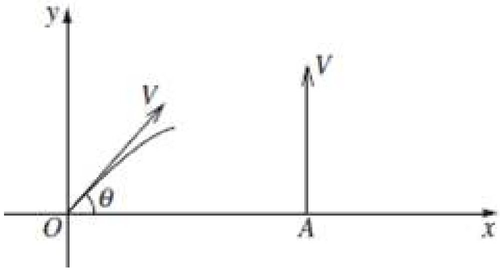
(iv) Using part (a) (ii) or otherwise show that $F'(\theta)=0$ when $\theta=\frac{\alpha}{2}+\frac{\pi}{4}$.

(v) Explain why v^2 is a minimum when $\theta=\frac{\alpha}{2}+\frac{\pi}{4}$.

***9.** '06 6a Two particles are fired simultaneously from the ground at time $t = 0$.

Particle 1 is projected from the origin at an angle θ, $0 < \theta < \frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from the point A, at a distance a to the right of the origin, also with an initial velocity of V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

and Particle 2 has equations of motion:

$$x = a$$

$$y = Vt - \frac{1}{2}gt^2$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t.

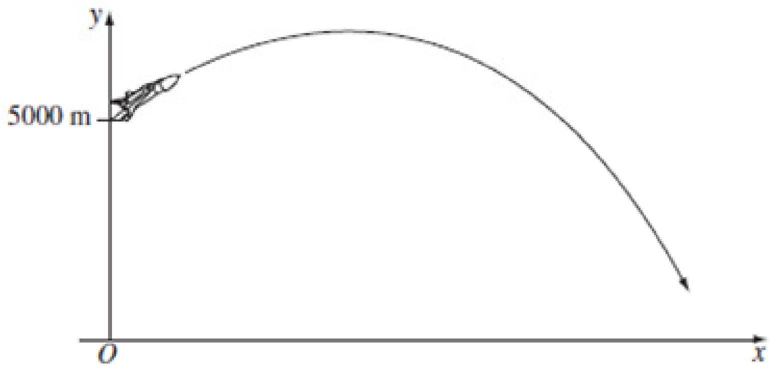
(i) Show that, while both particles are in flight, $L^2 = 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2$.

(ii) An observer notices that the distance between the particles in flight first decreases, then increases. Show that the distance between the particles in flight is smallest when $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$, and that this smallest distance is $a\sqrt{\dfrac{1-\sin\theta}{2}}$.

(iii) Show that the smallest distance between the two particles in flight occurs while Particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$.

***10.** '05 6b

An experimental rocket is at a height of 5000 m, ascending with a velocity of $200\sqrt{2}$ ms^{-1} at an angle of 45° to the horizontal, when its engine stops.



After this time, the equations of motion of the rocket are:

$$x = 200t$$
$$y = -4.9t^2 + 200t + 5000$$

where t is measured in seconds after the engine stops. (Do NOT show this.)

(i) What is the maximum height the rocket will reach, and when will it reach this height?

(ii) The pilot can only operate the ejection seat when the rocket is descending at an angle between 45° and 60° to the horizontal. What are the earliest and latest times that the pilot can operate the ejection seat?

(iii) For the parachute to open safely, the pilot must eject when the speed of the rocket is no more than 350 ms^{-1}. What is the latest time at which the pilot can eject safely?

11. '04 6b

A fire hose is at ground level on a horizontal plane. Water is projected from the hose. The angle of projection, θ, is allowed to vary. The speed of the water as it leaves the hose, v metres per second, remains constant. You may assume that if the origin is taken to be the point of projection, the path of the water is given by the parametric equations

$$x = vt\cos\theta$$
$$y = vt\sin\theta - \frac{1}{2}gt^2$$

where g ms^{-2} is the acceleration due to gravity. (Do NOT prove this.)

(i) Show that the water returns to ground level at a distance $\dfrac{v^2 \sin 2\theta}{g}$ metres from the point of projection.

This fire hose is now aimed at a 20 metre high thin wall from a point of projection at ground level 40 metres from the base of the wall. It is known that when the angle θ is 15°, the water just reaches the base of the wall.

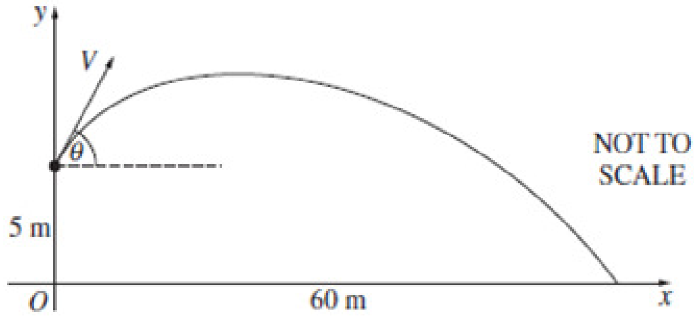
(ii) Show that $v^2 = 80g$.

(iii) Show that the cartesian equation of the path of the water is given by
$$y = x\tan\theta - \frac{x^2\sec^2\theta}{160}.$$

(iv) Show that the water just clears the top of the wall if $\tan^2\theta - 4\tan\theta + 3 = 0$.

(v) Find all values of θ for which the water hits the front of the wall.

12. '02 6a An angler casts a fishing line so that the sinker is projected with a speed V ms^{-1} from a point 5 metres above a flat sea. The angle of projection to the horizontal is θ, as shown.



Assume that the equations of motion of the sinker are $\ddot{x} = 0$ and $\ddot{y} = -10$, referred to the coordinate axes shown.

(i) Let (x, y) be the position of the sinker at time t seconds after the cast, and before the sinker hits the water. It is known that $x = Vt\cos\theta$.

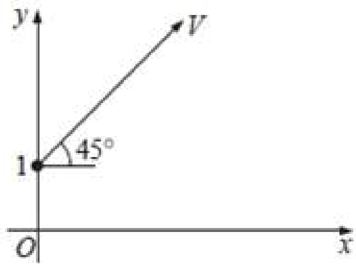
Show that $y = Vt\sin\theta - 5t^2 + 5$.

(ii) Suppose the sinker hits the sea 60 metres away as shown in the diagram.

Find the value of V if $\theta = \tan^{-1}\dfrac{3}{4}$.

(iii) For the cast described in part (ii), find the maximum height above sea level that the sinker achieved.

***13.** '98 6a A particle is projected from the point $(0, 1)$ at an angle of 45° with a velocity of V m/s.



The equations of motion of the particle are $\ddot{x} = 0$ and $\ddot{y} = -g$.

(i) Using calculus, derive the expressions for the position of the particle at time t.

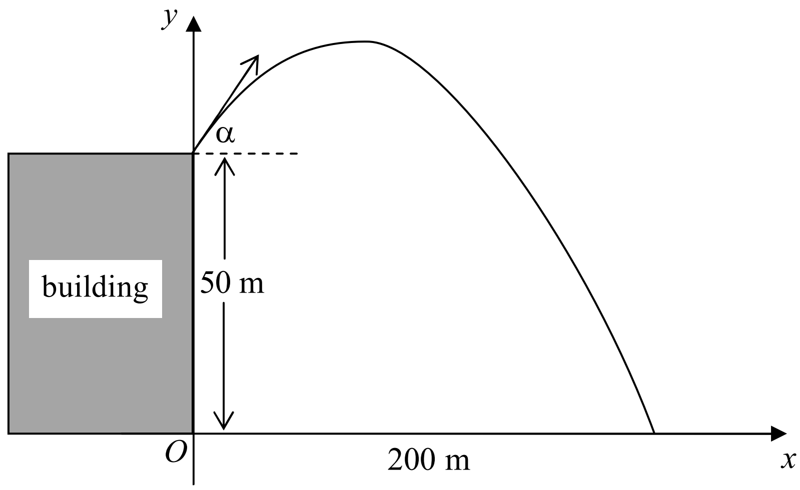
Hence show that the path of the particle is given by $y = 1 + x - g\dfrac{x^2}{V^2}$.

A volleyball player serves a ball with initial speed V metres per second and angle of projection 45°. At that moment the bottom of the ball is 1 metre above the ground and its horizontal distance from the net is 9·3 metres. The ball just clears the net, which is 2·3 metres high.

(ii) Show that the initial speed of the ball is approximately 10·3 metres per second. (Take $g = 9.8\ \text{ms}^{-2}$.)

(iii) What is the horizontal distance from the net to the point where the ball lands?

14. '91 6b The diagram shows the path of a projectile launched at an angle of elevation α, with an initial speed of 40 m/s, from the top of a 50 metre high building. The acceleration due to gravity is assumed to be 10 m/s².

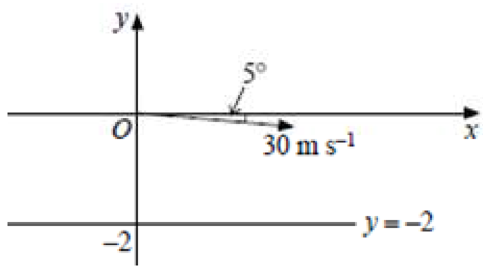


(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, show that $x = 40t\cos\alpha$ and $y = -5t^2 + 40t\sin\alpha + 50$ where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after launching.

(ii) The projectile lands on the ground 200 metres from the base of the building. Find the two possible values for α. Give your answers to the nearest degree.

Question 7s

15. '99 7a A cricket ball leaves the bowler's hand 2 metres above the ground with a velocity of 30 ms^{-1} at an angle of 5° below the horizontal.



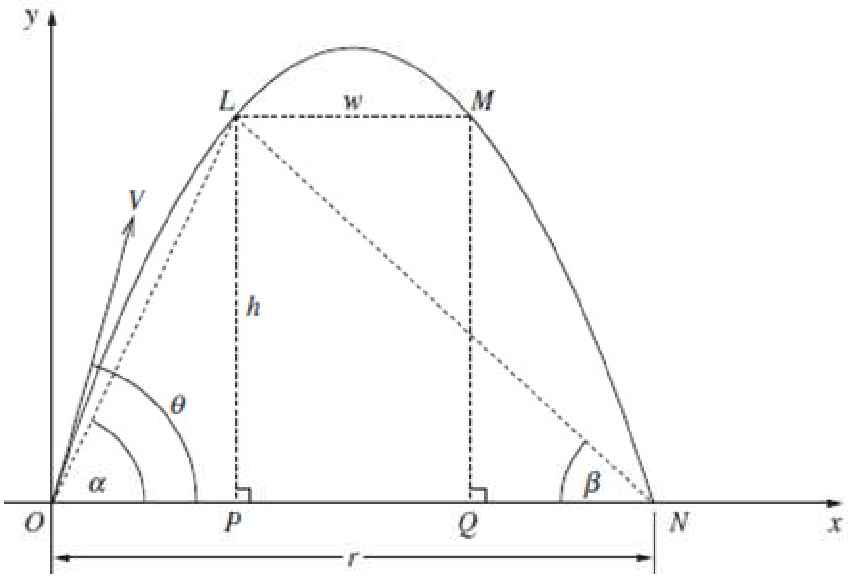
The equations of motion for the ball are $\ddot{x} = 0$ and $\ddot{y} = -10$.
Take the origin to be the point where the ball leaves the bowler's hand.

(i) Using calculus, prove that the coordinates of the ball at time t are given by $x = 30t\cos(5°)$, and $y = -30t\sin(5°) - 5t^2$.

(ii) Find the time at which the ball strikes the ground.

(iii) Calculate the angle at which the ball strikes the ground.

***16.** '08 Q7 A projectile is fired from O with velocity V at an angle of inclination θ across level ground. The projectile passes through the points L and M, which are both h metres above the ground, at times t_1 and t_2 respectively. The projectile returns to the ground at N.



The equations of motion of the projectile are

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{1}{2}gt^2. \qquad \text{(Do NOT prove this.)}$$

(a) Show that $t_1 + t_2 = \dfrac{2V}{g}\sin\theta$ AND $t_1 t_2 = \dfrac{2h}{g}$.

Let $\angle LON = \alpha$ and $\angle LNO = \beta$. It can be shown that

$$\tan\alpha = \frac{h}{Vt_1\cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2\cos\theta}. \qquad \text{(Do NOT prove this.)}$$

(b) Show that $\tan\alpha + \tan\beta = \tan\theta$.

(c) Show that $\tan\alpha\tan\beta = \dfrac{gh}{2V^2\cos^2\theta}$.

Let $ON = r$ and $LM = w$.

(d) Show that $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$.

Let the gradient of the parabola at L be $\tan\phi$.

(e) Show that $\tan\phi = \tan\alpha - \tan\beta$.

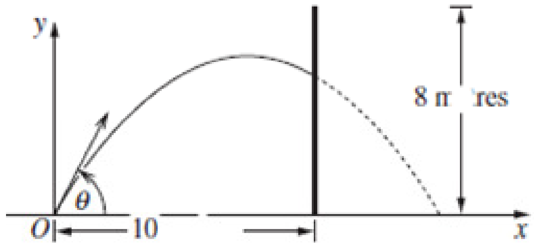
(f) Show that $\dfrac{w}{\tan\phi} = \dfrac{r}{\tan\theta}$.

***17.** '07 7b A small paintball is fired from the origin with initial velocity 14 metres per second towards an eight-metre high barrier. The origin is at ground level, 10 metres from the base of the barrier.

The equations of motion are $x = 14t\cos\theta$

$$y = 14t\sin\theta - 4.9t^2$$

where θ is the angle to the horizontal at which the paintball is fired and t is the time in seconds. (Do NOT prove these equations of motion.)



(i) Show that the equation of trajectory of the paintball is $y = mx - \left(\dfrac{1+m^2}{40}\right)x^2$, where $m = \tan\theta$.

(ii) Show that the paintball hits the barrier at height h metres when $m = 2 \pm \sqrt{3-0.4h}$. Hence determine the maximum value of h.

(iii) There is a large hole in the barrier. The bottom of the hole is 3.9 metres above the ground and the top of the hole is 5.9 metres above the ground. The paintball passes through the hole if m is in one of two intervals. One interval is $2.8 \le m \le 3.2$. Find the other interval.

(iv) Show that, if the paintball passes through the hole, the range is $\dfrac{40m}{1+m^2}$ metres.

Hence find the widths of the two intervals in which the paintball can land at ground level on the other side of the barrier.

18. '93 7b A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \ne \dfrac{\pi}{2}$. You may assume that $x = Vt\cos\theta$ and $y = -\dfrac{1}{2}gt^2 + Vt\sin\theta$, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

(i) Show that the equation of flight of the projectile can be written as $y = x\tan\theta - \dfrac{1}{4h}x^2\left(1+\tan^2\theta\right)$ where $\dfrac{V^2}{2g} = h$.

(ii) Show that the point (X, Y), where $X \ne 0$, can be hit by firing at two different angles θ_1 and θ_2 provided $X^2 < 4h(h-Y)$.

(iii) Show that no point **above** the x axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \dfrac{\pi}{4}$ and $\theta_2 < \dfrac{\pi}{4}$.

***19.** '03 7b

A particle is projected from the origin with velocity v ms^{-1} at an angle α to the horizontal. The position of the particle at time t seconds is given by the parametric equations

$$x = vt\cos\alpha$$

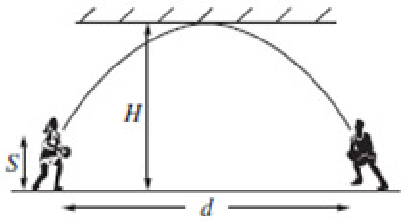
$$y = vt\sin\alpha - \frac{1}{2}gt^2$$

where g ms^{-2} is the acceleration due to gravity. (You are NOT required to derive these.)

(i) Show that the maximum height reached, h metres, is given by $h = \dfrac{v^2\sin^2\alpha}{2g}$.

(ii) Show that it returns to the initial height at $x = \dfrac{v^2}{g}\sin 2\alpha$.

(iii) Chris and Sandy are tossing a ball to each other in a long hallway. The ceiling height is H metres and the ball is thrown and caught at shoulder height, which is S metres for both Chris and Sandy.



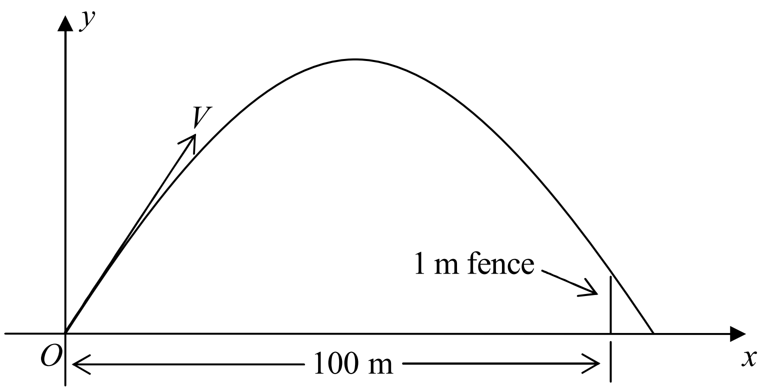
The ball is thrown with a velocity v ms^{-1}. Show that the maximum separation, d metres, that Chris and Sandy can have and still catch the ball is given by

$$d = 4\times\sqrt{(H-S)\left(\frac{v^2}{2g}\right)-(H-S)^2}, \qquad \text{if } v^2 \geq 4g(H-S), \text{ and}$$

$$d = \frac{v^2}{g}, \qquad \text{if } v^2 \leq 4g(H-S).$$

20. '89 7a

A "six" is scored in a cricket game when the ball is hit over the boundary fence on the full as in the diagram. A ball is hit from O with velocity $V = 32$ ms^{-1} at an angle θ to the horizontal and towards the 1 metre high boundary fence 100 metres away.



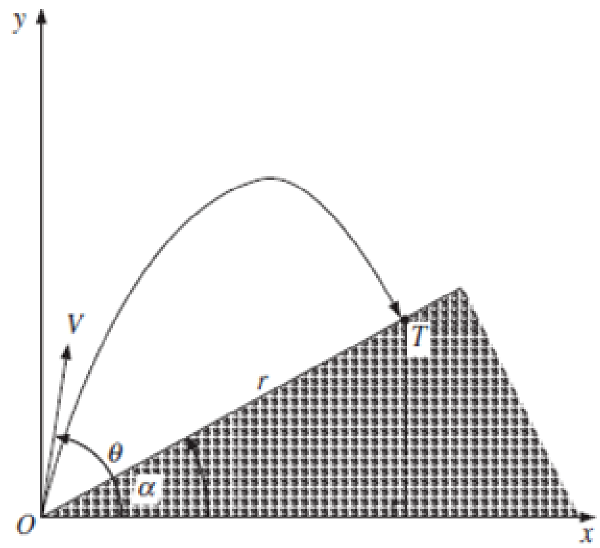
(i) Derive the equations of motion for the ball in flight using axes as in the diagram.
(Air resistance is to be neglected and the acceleration due to gravity is taken as 10 ms^{-2})

(ii) Show that the ball just clears the boundary fence when
$$50\,000\tan^2\theta - 102\,400\tan\theta + 51\,024 = 0.$$

(iii) In what range must θ lie for a "six" to be scored?

(iv) If, during the flight of the ball, its velocity is reduced by piercing an extremely thin "board", show by a sketch how the path is altered.
Without further calculation, discuss qualitatively the effect of air resistance on your answer in (iii).

***21.** ’00 7b The diagram shows an inclined plane that makes an angle of α radians with the horizontal. A projectile is fired from O, at the bottom of the incline, with a speed of V ms^{-1} at an angle of elevation θ to the horizontal, as shown.



With the above axes, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

For simplicity we assume that the unit of length has been chosen so that $\dfrac{2V^2}{g} = 1$.

(i) Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$.

(ii) Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta-\alpha)\cos\theta}{\cos^2\alpha}.$$

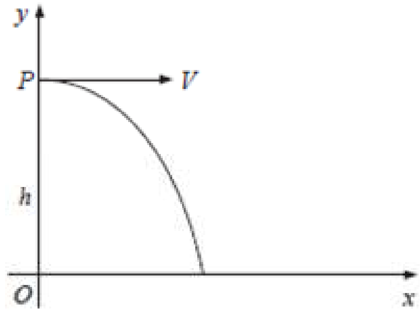
(iii) Hence, or otherwise, deduce that the maximum range, R metres, up the incline is

$$R = \frac{1}{2(1+\sin\alpha)}.$$

(You may assume that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.)

(iv) Consider the trajectory of the projectile for which the maximum range R is achieved. Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

***22.** '97 7a A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second.



The equations of motion of the particle are $\ddot{x}=0$ and $\ddot{y}=-g$.

(i) Using calculus, show that the position of the particle at time t is given by

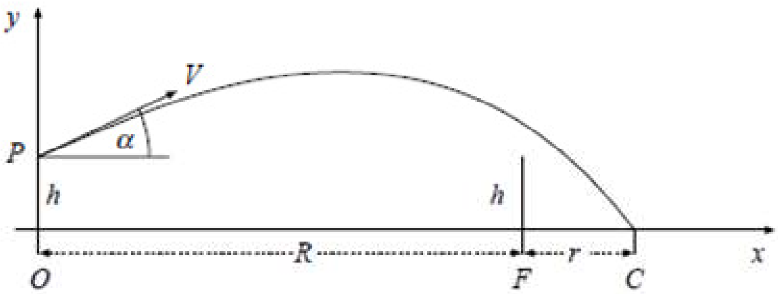
$$x=Vt,\quad y=h-\frac{1}{2}gt^2.$$

A canister containing a life raft is dropped from a plane to a stranded sailor. The plane is travelling at a constant velocity of 216 km/h, at a height of 120 metres above sea level, along a path that passes above the sailor.

(ii) How long will the canister take to hit the water? (Take $g=10$ m/s^2.)

(iii) A current is causing the sailor to drift at a speed of 3·6 km/h in the same direction as the plane is travelling. The canister is dropped from the plane when the horizontal distance from the plane to the sailor is D metres. What values can D take if the canister lands at most 50 metres from the stranded sailor?

23. '95 Q7 A cap C is lying outside a softball field, r metres from the fence F, which is h metres high. The fence is R metres from the point O, and the point P is h metres above O.
Axes are based at O, as shown.



At time $t=0$, a ball is hit from P at a speed V metres per second and at an angle α to the horizontal, towards the cap.

(a) The equations of motion of the ball are $\ddot{x}=0$, $\ddot{y}=-g$.
Using calculus, show that the position of the ball at time t is given by

$$x=Vt\cos\alpha$$

$$y=Vt\sin\alpha-\frac{1}{2}gt^2+h$$

(b) Hence show that the trajectory of the ball is given by $y=h+x\tan\alpha-x^2\dfrac{g}{2V^2\cos^2\alpha}$.

(c) The ball clears the fence. Show that $V^2\geq\dfrac{gR}{2\sin\alpha\cos\alpha}$.

(d) After clearing the fence, the ball hits the cap C. Show that $\tan\alpha\geq\dfrac{Rh}{(R+r)r}$.

(e) Suppose that the ball clears the fence, and that $V\leq 50$, $g=10$, $R=80$, and $h=1$.
What is the closest point to the fence where the ball can land?

24. '92 7

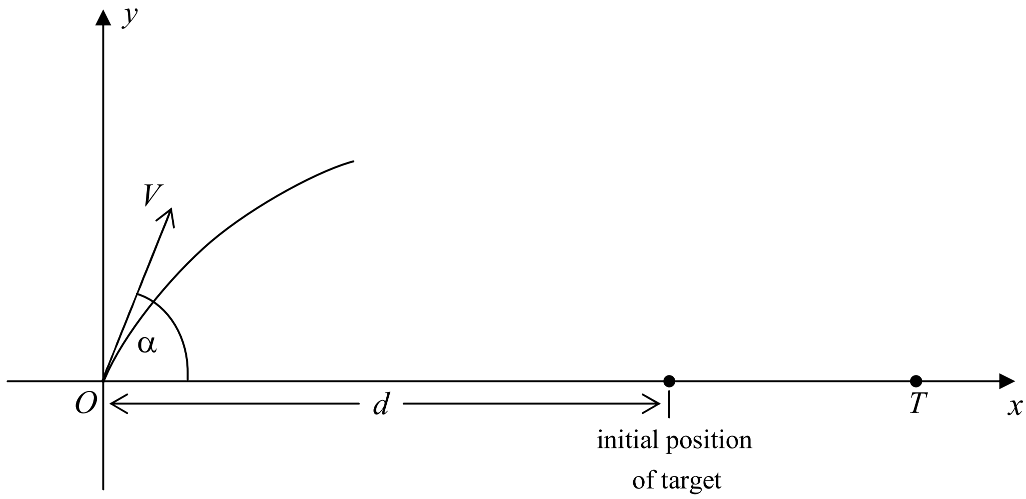
(a) Consider the function $y = f(\theta)$, where $f(\theta) = \cos\theta - \dfrac{1}{4\sqrt{3}\sin\theta}$.

(i) Verify that $f'\left(\dfrac{\pi}{6}\right) = 0$.

(ii) Sketch the curve $y = f(\theta)$ for $0 < \theta \le \dfrac{\pi}{2}$ given that $f''(\theta) < 0$.

On your sketch, write the coordinates of the turning point in exact form and label the asymptote.

(b) A projectile, of initial speed V m/s, is fired at an angle of elevation α from the origin O towards a target T, which is moving away from O along the x axis.



You may assume that the projectile's trajectory is defined by the equations

$$x = Vt\cos\alpha \quad \text{and} \quad y = -\frac{1}{2}gt^2 + Vt\sin\alpha$$

where x and y are the horizontal and vertical displacements of the projectile in metres at time t seconds after firing, and where g is the acceleration due to gravity.

(i) Show that the projectile is above the x axis for a total of $\dfrac{2V\sin\alpha}{g}$ seconds.

(ii) Show that the horizontal range of the projectile is $\dfrac{2V^2\sin\alpha\cos\alpha}{g}$ metres.

(iii) At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α.

Show that $u = V\cos\alpha - \dfrac{gd}{2V\sin\alpha}$.

In parts (iv) and (v), assume that $gd = \dfrac{V^2}{2\sqrt{3}}$.

(iv) By using (iii) and the graph of part (a), show that if $u > \dfrac{V}{\sqrt{3}}$ the target cannot be hit by the projectile, no matter at what angle of elevation α the projectile is fired.

(v) Suppose $u < \dfrac{V}{\sqrt{3}}$. Show that the target can be hit when it is at precisely two distances from O.

New Format HSC and Older Formats

25. '82 10 A particle P is projected from a point O on a horizontal plane with initial velocity V ms^{-1} in a direction inclined at an angle α upwards from the horizontal. At time t seconds after the instant of projection its horizontal and vertical distances from O are x metres and y metres respectively. Air resistance may be neglected.

(i) Write down expressions for x and y as functions of t.

(ii) Show that the time of flight T seconds and the range R metres are given by $T = \left(\frac{2V}{g}\right)\sin\alpha$, $R = \left(\frac{V^2}{g}\right)\sin 2\alpha$, and derive similar expressions for the maximum height reached by P.

(iii) A ball is thrown from a height 1 metre from the ground and is caught without bouncing 2 seconds later, 50 metres away, also at a height of 1 metre. Assuming no air resistance and that g has the approximate value 10 ms^{-2}, find

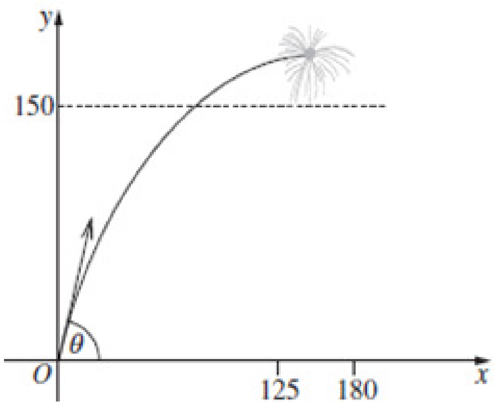
(a) the velocity and angle of projection of the ball

(b) the maximum height of the ball above the ground during its flight.

26. '12 14b A firework is fired from O, on level ground, with velocity 70 metres per second at an angle of inclination θ. The equations of motion of the firework are

$x = 70t\cos\theta$ and $y = 70t\sin\theta - 4.9t^2$. (Do NOT prove this.)

The firework explodes when it reaches its maximum height.



(i) Show that the firework explodes at a height of $250\sin^2\theta$ metres.

(ii) Show that the firework explodes at a horizontal distance of $250\sin 2\theta$ metres from O.

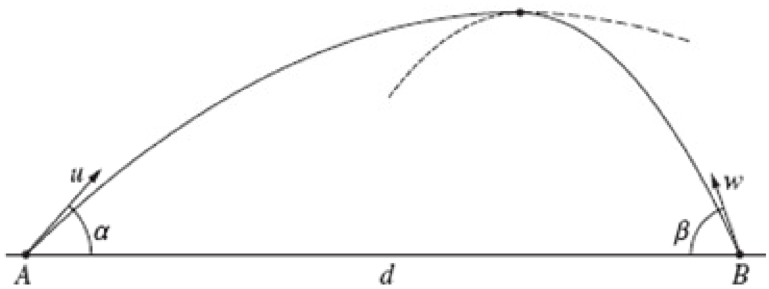
(iii) For best viewing, the firework must explode at a horizontal distance between 125 m and 180 m from O, and at least 150 m above the ground.

For what values of θ will this occur?

27. '13 13c Points A and B are located d metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity u ms^{-1} at angle α to the horizontal.

At the same time, another projectile is fired from B towards A with initial velocity w ms^{-1} at angle β to the horizontal, as shown on the diagram.

The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity V ms^{-1} at angle θ to the horizontal are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{g}{2}t^2. \qquad \text{(Do NOT prove this.)}$$

(i) How long does the projectile fired from A take to reach its maximum height?

(ii) Show that $u\sin\alpha = w\sin\beta$.

(iii) Show that $d = \dfrac{uw}{g}\sin(\alpha+\beta)$.

Extension 2

***28.** '84 Q6 Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α, but with different velocities U, V metres per second, $U < V$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(a) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. Hence express the horizontal distance from P to the foot of the wall in terms of h, α.

(b) Show that $V(\tan\alpha + \tan\beta) = 2U\tan\alpha$, and deduce that, if $\beta = \frac{1}{2}\alpha$, then $U < \frac{3}{4}V$.

***29.** '79 Q3 Prove that the range on a horizontal plane of a particle projected upwards at an angle α to the plane with velocity V metres per second is $\dfrac{V^2 \sin 2\alpha}{g}$ metres, where g metres per second per second is the acceleration due to gravity.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second. The initial direction of the spray varies continuously between angles of $15°$ and $60°$ to the horizontal.

Prove that, from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O and with internal and external radii $\dfrac{V^2}{2g}$ metres and $\dfrac{V^2}{g}$ metres respectively.

Deduce that by locating the sprinkler appropriately relative to a rectangular garden bed of size 6 metres by 3 metres, the entire bed may be watered provided that $\dfrac{V^2}{2g} \geq 1+\sqrt{7}$.

***30.** '74 Q10 A large vertical wall stands on horizontal ground. The nozzle of a water hose is positioned at a point C on the ground at a distance c from the wall and the water jet can be pointed in any direction from C. Also the water issues from the nozzle with speed V. (Air resistance may be neglected and the constant g denotes the acceleration due to gravity.)

(i) Prove that the jet can reach the wall above ground level if and only if $V > \sqrt{gc}$.

(ii) If $V = 2\sqrt{gc}$ prove that the portion of the wall that can be reached by the jet is a parabolic segment of height $\dfrac{15c}{8}$ and area $\dfrac{5\sqrt{15}\,c^2}{2}$.

31. '69 Q10 A gun fires a shot from O with initial speed V at an angle α with the horizontal.

If the acceleration due to gravity is constant $(= g)$ prove that the shot describes a parabola of focal length $\dfrac{V^2 \cos^2 \alpha}{2g}$.

If the initial speed V is fixed but the direction of firing can be varied prove that the region of vulnerability (ie. the set of points that can be hit) consists of points within and on the paraboloid whose equation (referred to a Cartesian x, y, z-frame with origin at O and z-axis vertically upwards) is $x^2 + y^2 + \dfrac{2V^2}{g} z = \dfrac{V^4}{g^2}$.

Answers

1. (i) $\dfrac{V^2 \sin^2 \alpha}{20}$ (ii) $\dfrac{V^2 \sin 2\alpha}{10}$

2. 200 m/s

3. (a) $x = 10t\sqrt{3}$, $y = -5t^2 + 10t + 40$ (b) 4 seconds, $40\sqrt{3}$ metres

4. First pump can reach the fire – its max range is 90 m.
Second pump has a range of 40 m and cannot reach the fire.

5. (a) $2\sqrt{10}$ s (b) 64 m/s at $81°01'$ (c) $72°27'$

8. (b) (ii) (1) $v \to \infty$ (2) $v \to \infty$

10. (i) 7041 m; $t = 20.4$ (ii) 40.8 s; 55.7 s (iii) $t = 49.7$

11. (v) $15° < \theta < 45°$ or $72° < \theta < 75°$

12. (ii) $V = \dfrac{15\sqrt{10}}{2}$ (iii) 15.125 m

13. (i) $x = \dfrac{V}{\sqrt{2}}t$; $y = 1 + \dfrac{V}{\sqrt{2}}t - \dfrac{1}{2}gt^2$ (iii) 2.4 m

14. (ii) 31°, 45°

15. (ii) 0.423 s (iii) 13°

17. (ii) 7.5 m (iii) $0.8 \le m \le 1.2$ (iv) 0.49 m and 1.28 m

20. (i) $x = 32t\cos\theta$, $y = -5t^2 + 32t\sin\theta$ (iii) $40°35' \le \theta \le 50°0'$
(iv) the range of possible angles will be reduced

22. (ii) $\sqrt{24}$ s (iii) $239.1 \le D \le 339.1$

23. (e) 80.16 m from O

24. (a) (ii)

25. (i) (ii) (iii) (a) (b)

26. (iii) $67° \le \theta \le 75°$

27. (i) $\dfrac{u\sin\alpha}{g}$

28. (a)