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Analytical solution of a spatially variable coefficient advection–diffusion equation in up to three dimensions

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Abstract

Analytical solutions are provided for the two- and three-dimensional advection–diffusion equation with spatially variable velocity and diffusion coefficients. We assume that the velocity component is proportional to the distance and that the diffusion coefficient is proportional to the square of the corresponding velocity component. There is a simple transformation which reduces the spatially variable equation to a constant coefficient problem for which there are available a large number of known analytical solutions for general initial and boundary conditions. These solutions are also solutions to the spatially variable advection–diffusion equation. The special form of the spatial coefficients has practical relevance and for divergent free flow represent *corner* or *straining* flow. Unlike many other analytical solutions, we use the transformation to obtain solutions of the spatially variable coefficient advection–diffusion equation in two and three dimensions. The analytical solutions, which are simple to evaluate, can be used to validate numerical models for solving the advection–diffusion equation with spatially variable coefficients. For numerical schemes which cannot handle flow stagnation points, we provide analytical solution to the spatially variable coefficient advection–diffusion equation for two-dimensional corner flow which contains an impermeable flow boundary. The impermeable flow boundary coincides with a streamline along which the fluid velocity is finite but the concentration vanishes. This example is useful for validating numerical schemes designed to predict transport around a curved boundary.

Keywords: Advection–diffusion equation; Analytical solutions; Variable coefficients; Two-, three-dimensional problems

1. Introduction

The transport of pollutant occurs in a large variety of environmental, agricultural and industrial processes. Accurate prediction of the transport of these pollutants is crucial to the effective management of these processes. The transport of these pollutants can be adequately described by the advection–diffusion equation. Due to the complexity of many of these problems, the advection–diffusion equation is usually solved numerically. Numerical models however, represent an approximation to the equations describing the transport process. Therefore, it is important that during the development of these models their accuracy is evaluated. There are several approaches that could be used to assess the accuracy of a numerical scheme. These

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include; (i) comparing numerical schemes, (ii) successive grid refinement and (iii) using analytical solutions. Only the latter approach will provide a reliable assessment of the accuracy of a numerical scheme.

There are numerous analytical solutions to the linear advection–diffusion equation with uniform flow and constant coefficients (see, for example Refs. [1,2]). Although practical problems generally involve non-uniform velocity fields and variable diffusion coefficients, it is frequently the case that numerical schemes are only tested against known analytical solutions to the constant coefficient advection–diffusion equation (see, for example Refs. [3–5]). A reason for this is that there are very few analytical solutions to the advection–diffusion equation with variable coefficients. There are a few notable exceptions; Warrick et al. [6], Barry and Sposito [7] and Basha and El-Habel [8] provided analytical solutions to the one-dimensional advection–diffusion equation with arbitrary time-dependent diffusion and velocity coefficients. Yates [9] proved analytical solutions to the one-dimensional advection–diffusion equation with the diffusion coefficient as an asymptotic function of distance. Philip [10] provided analytical solutions for radial flow in two and three dimensions to the advection–diffusion equation with the diffusion coefficient proportional to some power of the Péclet number. Aral and Liao [11] derived analytical solutions to the two-dimensional advection–diffusion equations with constant, linear, asymptotic and exponentially time-dependent diffusion coefficients. Many of these analytical solutions are restricted to problems with specific boundary conditions; some have limited practical relevance or are complicated to evaluate.

We derive analytical solutions to the advection–diffusion equation with a particular form of spatially variable diffusion and velocity coefficients. Each fluid velocity component is taken to be a linear function of distance and the diffusion coefficient proportional to the square of the velocity component, and therefore proportional to the square of the distance. The simple expressions for the spatial variation of the coefficients give rise to self-similar analytical solutions to these equations. The spatially variable coefficient equations reduce to constant coefficient equations through a simple transformation. Many of the known analytical solutions to the constant coefficient equations can be used to obtain analytical solutions to our spatially variable coefficient equation. Our analytical solutions differ from others in the literature because they are simple to evaluate and the transformation applies to problems of any dimension.

Analytical solutions are given for an instantaneous unit slug and a steady release of contaminant in the flow. These analytical solutions are extremely useful for testing numerical schemes developed to solve the advection–diffusion equation applied to problems containing non-uniform flow field and spatially variable diffusion (see, for example Refs. [12,13]).

We provide analytical solutions to the advection–diffusion equation for two-dimensional corner flow with an impermeable flow boundary located along a streamline. Along the impermeable boundary the flow velocity is finite and the concentration of contaminant vanishes. Analytical solutions to this problem are useful for testing the irregular boundary capability of a numerical scheme and to test numerical schemes which cannot accommodate flow stagnation points.

In the following sections several examples of analytical solutions to two- and three-dimensional problems are provided. In two and three dimensions we have assumed that the divergence of the velocity is zero. Therefore, the two- and three-dimensional problems considered represent *corner* or *straining flow*. The approach described in this paper is also applicable to problems with non-zero divergence of the flow. In Section 2 we establish the functional form of the spatial variable coefficients required to reduce the spatially variable coefficient advection–diffusion equation into a constant coefficient problem. Two-dimensional problems are easier to visualize than three-dimensional problems. Therefore, in Section 3 we derive in detail, the analytical solution for an instantaneous unit line release in two-dimensional corner flow and examine the properties of this analytical solution. We also derive an analytical solution for the instantaneous unit line release of contaminant in two-dimensional corner flow with an impermeable boundary located along a streamline. Analytical solutions are also provided for a steady unit line source of contaminant in two-dimensional corner flow with and without an impermeable boundary. In Section 4 we provide analytical solutions to three-dimensional corner flows for an instantaneous unit point release and a steady unit point source. Some one-dimensional solutions have been published elsewhere [13,14].

2. Required form of the spatial coefficients

The three-dimensional advection–diffusion equation written in conservative form is given by

$$\frac{\partial C}{\partial t}+\frac{\partial (uC)}{\partial x}+\frac{\partial (vC)}{\partial y}+\frac{\partial (wC)}{\partial z} = \frac{\partial}{\partial x}\left(D_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(D_z\frac{\partial C}{\partial z}\right)-\lambda C, \tag{1}$$

in which $C(x,y,z,t)$ is the contaminant concentration, $\mathbf{u}=(u(\mathbf{x}),v(\mathbf{x}),w(\mathbf{x}))$ is the vector of fluid velocity in the Cartesian coordinates, $\mathbf{x}=x,y$ and z respectively, $D_x(\mathbf{x})$, $D_y(\mathbf{x})$ and $D_z(\mathbf{x})$ are the diffusion coefficients in each coordinate direction, t is the time and λ is a coefficient governing the rate of decay of C.

Expanding Eq. (1) we obtain

$$\frac{\partial C}{\partial t}+\left(u-\frac{\partial D_x}{\partial x}\right)\frac{\partial C}{\partial x}+\left(v-\frac{\partial D_y}{\partial y}\right)\frac{\partial C}{\partial y}+\left(w-\frac{\partial D_z}{\partial z}\right)\frac{\partial C}{\partial z}+C\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\lambda\right) = D_x\frac{\partial^2 C}{\partial x^2}+D_y\frac{\partial^2 C}{\partial y^2}+D_z\frac{\partial^2 C}{\partial z^2}. \tag{2}$$

Because the coefficients in Eq. (1) depend on space, the diffusion terms contribute to the advection term, and the advection terms contribute to the decay term. The latter contribution is zero if the divergence, $\nabla\cdot\mathbf{u}=0$.

Our aim is to choose particular forms of $u(x), v(y), w(z), D_x(x)$, $D_y(y)$ and $D_z(z)$ so that Eq. (2) reduces to a constant coefficient equation. Hildebrand [15] shows that if a linear ordinary differential equation with variable coefficients can be written as an *equidimensional linear differential equation* then with the use of a simple transformation it reduces to a linear differential equation with constant coefficients. An equidimensional linear differential equation is an ordinary differential equation in the variable x with coefficients that are powers of x, which has the additional property that it remains unchanged when x is replaced by αx, where α is a non-zero constant. This approach of transforming a variable coefficient equation into a constant coefficient equation is also valid for partial differential equations such as Eq. (2).

The transformation consists of introducing the new variables;

$$x=\exp(u_0X),\quad y=\exp(v_0Y),\quad z=\exp(w_0Z), \tag{3}$$

in which $u_0, v_0,$ and w_0 are constants. Substituting into Eq. (2) then

$$
\begin{aligned}
&\frac{\partial C}{\partial t} + \left(\frac{u}{u_0 x} + \frac{D_x}{u_0 x^2} - \frac{1}{u_0^2 x^2}\frac{\partial D_x}{\partial X}\right)\frac{\partial C}{\partial X} + \left(\frac{v}{v_0 y} + \frac{D_y}{v_0 y^2} - \frac{1}{v_0^2 y^2}\frac{\partial D_y}{\partial Y}\right)\frac{\partial C}{\partial Y} \\
&\quad + \left(\frac{w}{w_0 z} + \frac{D_z}{w_0 z^2} - \frac{1}{w_0^2 z^2}\frac{\partial D_z}{\partial Z}\right)\frac{\partial C}{\partial Z} + C\left(\frac{1}{u_0 x}\frac{\partial u}{\partial X} + \frac{1}{v_0 y}\frac{\partial v}{\partial Y} + \frac{1}{w_0 z}\frac{\partial w}{\partial Z}\right) \\
&\quad = \frac{D_x}{u_0^2 x^2}\frac{\partial^2 C}{\partial X^2} + \frac{D_y}{v_0^2 y^2}\frac{\partial^2 C}{\partial Y^2} + \frac{D_z}{w_0^2 z^2}\frac{\partial^2 C}{\partial Z^2}.
\end{aligned}
\tag{4}
$$

Eq. (4) provides some indication of suitable spatial forms of the velocity and diffusion coefficients that are required to reduce Eq. (4) into a constant coefficient equation. One suitable choice is

$$
u = u_0 x, \quad v = v_0 y, \quad w = w_0 z, \quad D_x = D_0 u_0^2 x^2, \quad D_y = D_0 v_0^2 y^2, \quad D_z = D_0 w_0^2 z^2 \tag{5}
$$

in which D_0 is a constant. In each direction, the velocity component has a linear dependence on distance and the corresponding diffusion coefficient has a quadratic dependence on distance. Physically, this model has the disadvantage that the diffusion coefficients are zero where the velocity components are zero. In a practical problem, although the flow may be zero in some region, these regions are still affected by the action of diffusion.

Substituting Eq. (5) into Eq. (4) gives

$$
\begin{aligned}
&\frac{\partial C}{\partial t} + (1 - D_0 u_0)\frac{\partial C}{\partial X} + (1 - D_0 v_0)\frac{\partial C}{\partial Y} + (1 - D_0 w_0)\frac{\partial C}{\partial Z} \\
&\quad = D_0\left(\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} + \frac{\partial^2 C}{\partial Z^2}\right) - C(u_0 + v_0 + w_0 + \lambda),
\end{aligned}
\tag{6}
$$

which is a constant coefficient advection–diffusion equation. Provided that the initial and boundary conditions for Eq. (6) are transformed using Eq. (3) then we have available a large class of analytical solutions to Eq. (6) (see, for example, Ref. [1]).

This approach is not only restricted to the advection–diffusion equation. It is applicable to any order partial differential equation, such as the linearized Korteweg–de Vries equation and the advection equation. Zoppou and Knight [14] used this approach to obtain analytical solutions to the one-dimensional advection equation in a spatially variable flow field.

Although a particular form of the spatial variation of the velocity and diffusion coefficients is required, the assumed relationship can be interpreted physically. For flow in a tube of circular cross section, Taylor [16] showed that the diffusion was approximately proportional to the square of the flow velocity. Wooding [17] derived a corresponding dependence of the diffusion on the square of the velocity for flow between parallel plates. Field experimental studies have suggested that the diffusion coefficient is not constant but an increasing function of distance during the transport of solutes in subsurface water [18–21]. There seems to be evidence in the literature to suggest that in some transport processes the velocity and diffusion coefficients are not constant but functions of time or space. The particular forms of the spatially variable coefficients considered here are consistent with this observation. Another practical problem that can be described approximately by spatially variable coefficient advection–diffusion equation is the transport of pollutant in an open channel where the flow in the channel is augmented by steady, unpolluted lateral inflow distributed along the whole length of the channel, such as steady inflow of ground water. There are similar problems in two and three dimensions.

3. Two-dimensional solutions

The analytical solution of the two-dimensional spatially variable coefficient advection–diffusion equation provides a greater scope to explore the properties of the analytical solution than its three-dimensional counterparts. Therefore, we provide a more detailed examination of the two-dimensional problem than the three-dimensional problem.

We take the decay coefficient to be zero from now on and analytical solutions are presented for the following advection–diffusion equation

$$\frac{\partial C}{\partial t}+\frac{\partial (uC)}{\partial x}+\frac{\partial (vC)}{\partial y}=\frac{\partial}{\partial x}\left(D_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}\right) \tag{7}$$

for a line source of unit strength and an instantaneous release of contaminant of unit mass in two-dimensional corner flow. Corner or straining flow occurs when the divergence, $\nabla\cdot\mathbf{u}=0$. For two-dimensional corner flow, the stream function in the first quadrant are given by $\psi(x,y)=-u_0xy$, and the velocity potential is given by $\phi(x,y)=u_0(y^2+x^2)/2$. The streamlines in the positive quadrant are shown in Fig. 1 where $u_0=1$.

Assuming the required form for the spatial variation of the velocity and diffusion coefficients

$$u=u_0x,\quad v=-u_0y,\quad D_x=D_0u_0^2x^2,\quad D_y=D_0u_0^2y^2,\ \ x\geqslant 0, y\geqslant 0,$$

then Eq. (7) becomes

$$\frac{\partial C}{\partial t}+\frac{\partial (u_0xC)}{\partial x}+\frac{\partial (-u_0yC)}{\partial y}=\frac{\partial}{\partial x}\left(D_0u_0^2x^2\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_0u_0^2y^2\frac{\partial C}{\partial y}\right). \tag{8}$$

This is in the form of an equidimensional equation. Substituting the following transformations

$$x=\exp(u_0X),\quad y=\exp(-u_0Y),\quad -\infty\leqslant X\leqslant\infty,\quad -\infty\leqslant Y\leqslant\infty, \tag{9}$$

which are consistent with divergence free flow, into Eq. (8) yields

$$\frac{\partial C}{\partial t}+(1-D_0u_0)\frac{\partial C}{\partial X}+(1+D_0u_0)\frac{\partial C}{\partial Y}=D_0\left(\frac{\partial^2 C}{\partial X^2}+\frac{\partial^2 C}{\partial Y^2}\right). \tag{10}$$

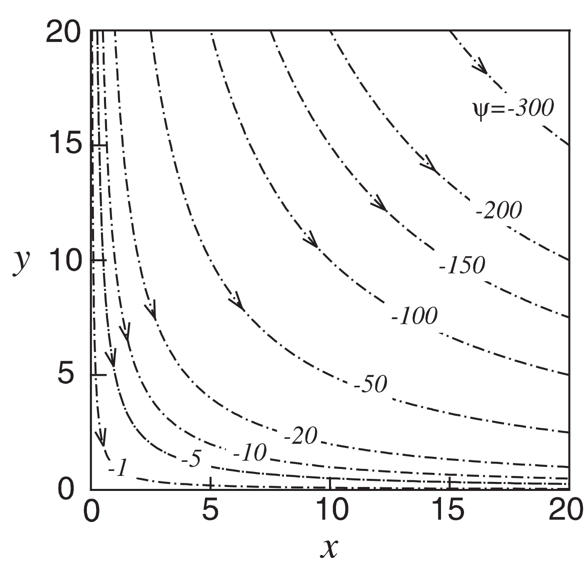

Fig. 1. Contours of streamlines in the two-dimensional corner flow with $u_0=1$.

3.1. Instantaneous release

We seek a solution to this equation, for an instantaneous release at (X_0, Y_0) of the form

$$C(X,Y,t) = g_1(X,X_0,t)\ g_2(Y,Y_0,t), \tag{11}$$

where g_1 and g_2 are solutions to the one-dimensional constant coefficient advection–diffusion equation in the transformed space and are given by

$$g_1(X,X_0,t) = \frac{A_1}{2\sqrt{D_0 u_0^2 \pi t}} \exp\left(\frac{-[X - X_0 - (1 - D_0 u_0)t]^2}{4D_0 t}\right)$$

and

$$g_2(Y,Y_0,t) = \frac{A_2}{2\sqrt{D_0 u_0^2 \pi t}} \exp\left(\frac{-[Y - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 t}\right).$$

The coefficients A_1 and A_2 must be determined. For a source of unit mass

$$\int_0^\infty \int_0^\infty C(x,y,t)\ \mathrm{d}x\ \mathrm{d}y = 1, \quad \forall t,$$

therefore we must have

$$\int_0^\infty \int_0^\infty C(x,y,t)\ \mathrm{d}x\ \mathrm{d}y = \int_{-\infty}^\infty \int_{-\infty}^\infty - u_0^2 \exp(u_0(X - Y)) C(X,Y,t)\ \mathrm{d}X\ \mathrm{d}Y = 1$$

or

$$\left[\int_{-\infty}^\infty u_0 \exp(u_0 X) g_1(X,X_0,t)\ \mathrm{d}X\right] \left[\int_{-\infty}^\infty - u_0 \exp(-u_0 Y) g_2(Y,Y_0,t)\ \mathrm{d}Y\right] = 1.$$

Solving, then

$$A_1 \exp(u_0(X_0 + t)) A_2 \exp(-u_0(Y_0 + t)) = A_1 A_2 x_0 y_0 = 1.$$

An obvious choice for A_1 and A_2 is $A_1 = 1/x_0$ and $A_2 = 1/y_0$. Therefore,

$$g_1(X,X_0,t) = \frac{1}{2x_0\sqrt{D_0 u_0^2 \pi t}} \exp\left(\frac{-[X - X_0 - (1 - D_0 u_0)t]^2}{4D_0 t}\right), \tag{12}$$

$$g_2(Y,Y_0,t) = \frac{1}{2y_0\sqrt{D_0 u_0^2 \pi t}} \exp\left(\frac{-[Y - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 t}\right), \tag{13}$$

and Eq. (11) becomes

$$C(X,Y,t) = \frac{1}{4D_0 u_0^2 \pi t x_0 y_0} \exp\left(\frac{-[X - X_0 - (1 - D_0 u_0)t]^2 - [Y - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 t}\right). \tag{14}$$

Expanding and recalling that $x = \exp(u_0 X)$ and $y = \exp(-u_0 Y)$, then for a unit instantaneous release at (x_0, y_0), the evolution in time of the concentration profile is given by

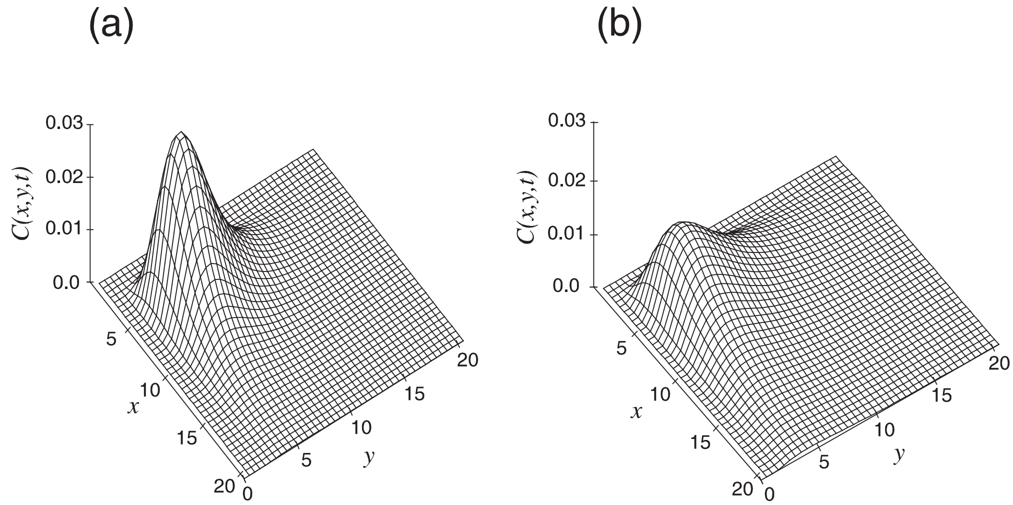

Fig. 2. Concentration profile, $C(x,y,t)$ for an instantaneous unit line release located at $x_0 = 5, y_0 = 5$, with $u_0 = 1$, $D_0 = 2$ and $t = 0.05$ in (a) and $t = 0.1$ in (b).

$$C(x,y,t) = \frac{1}{4\pi D_0 u_0^2 t\sqrt{xyx_0y_0}}\left(\frac{xy_0}{x_0y}\right)^{1/(2u_0D_0)} \exp\left(\frac{-\rho^2 - 2(1+D_0^2u_0^2)t^2}{4D_0t}\right), \tag{15}$$

where $\rho = (1/u_0)\sqrt{\ln^2(x/x_0) + \ln^2(y/y_0)}$.

The evolution with time of an instantaneous line source in two-dimensional corner flow is shown in Fig. 2. The source is located at $x_0 = 5$ and $y_0 = 5$, the velocity coefficient, $u_0 = 1$ and the diffusion coefficient, $D_0 = 2$. The profiles predicted by Eq. (15) at $t = 0.05$ and $t = 0.1$ are shown in Eq. (2)(a) and (b) respectively. With increasing time, the maximum concentration attenuates and the tail of the plume is swept further downstream.

We will now examine some of the properties of this analytical solution. When $D_0 = 0$, Eq. (10) reduces to a constant coefficient advection equation. A particle initially at (x_0, y_0) evolves according to

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 1 \quad \text{and} \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = 1,$$

so that $X - X_0 = t$ and $Y - Y_0 = t$. Since $x = \exp(u_0X)$ and $y = \exp(-u_0Y)$ then the position of the particle is given by

$$\mathbf{x}_\mathrm{p}(t) = (x_\mathrm{p}, y_\mathrm{p}) = (x_0\exp(u_0t),\ y_0\exp(-u_0t)).$$

Recalling that the divergence of the velocity is zero, then the path of the maximum, centroid of the concentration plume and the fluid particle will always lie on the curve

$$x_\mathrm{p}y_\mathrm{p} = x_0y_0, \tag{16}$$

which is a streamline. For $D_0 > 0$ the concentration plume spreads, with the maximum concentration located at

$$\mathbf{x}_\mathrm{m}(t) = (x_\mathrm{p}\exp(-D_0u_0^2t),\ y_\mathrm{p}\exp(-D_0u_0^2t)), \tag{17}$$

which was obtained by differentiating Eqs. (12) and (13). The path of the maximum concentration is given by

$$x_m y_m = x_0 y_0 \exp(-2D_0 u_0^2 t). \tag{18}$$

Comparing Eq. (16) with Eq. (18) shows that the effect of the velocity dependent diffusion coefficient is to move the maximum concentration below and away from the surface given by Eq. (16). From Eq. (17), when $D_0 = 1/u_0$, $\mathbf{x}_m(t) = (x_0, y_0 \exp(-2u_0 t))$ the path of the maximum is a straight line. When $D_0 > 1/u_0$, then $x_m(t) < x_0$ and when $D_0 < 1/u_0$, then $x_m(t) > x_0$.

Calculating the spatial moments of the concentration plume, it can be shown that the solution of the centroid of the concentration plume is given by

$$\mathbf{x}_c(t) = \left(x_p \exp(2D_0 u_0^2 t), y_p \exp(2D_0 u_0^2 t)\right),$$

and it lies on the curve

$$x_c y_c = x_0 y_0 \exp(4D_0 u_0^2 t). \tag{19}$$

Therefore, the centroid of the concentration plume is always above the surface given by Eq. (16). When $D_0 = 1/(2u_0)$, $\mathbf{x}_c(t) = (x_0 \exp(2u_0 t), y_0)$, the path of the maximum is a straight line. When $D_0 < 1/(2u_0)$, then $y_m(t) < y_0$ and when $D_0 > 1/(2u_0)$, then $x_c(t) > y_0$.

The trajectory of the maximum and centroid of a concentration plume produced from an instantaneous release at $x_0 = 5$ and $y_0 = 5$ with $u_0 = 1$ and $D_0 = 0.2, 1$ and 2 are shown in Fig. 3. The paths of the maximum and centroid of the concentration plume always diverge from each other and from the streamline $\psi = x_0 y_0 = 25$. This is in contrast to the behaviour of a plume in a uniform velocity field and when the diffusion coefficient is constant, where the three paths coincide. For large diffusion coefficients, $D_0 > 1$ the maximum is moving against the direction of the flow towards the stagnant point $\mathbf{x} = (0, 0)$. For moderate values of the diffusion coefficient, $D_0 > 0.5$ the centroid also moves against the flow. These observations seem counterintuitive, but can be readily explained. As the plume evolves, the material near the maximum concentration is being eroded and swept downstream. This has the effect of decreasing the maximum concentration and making the position of the maximum concentration move against the direction of the flow. The effect of the quadratic dependence of the diffusion coefficient with distance is that as material is swept downstream by the flow, it is also subject to increasing diffusion, which plays a

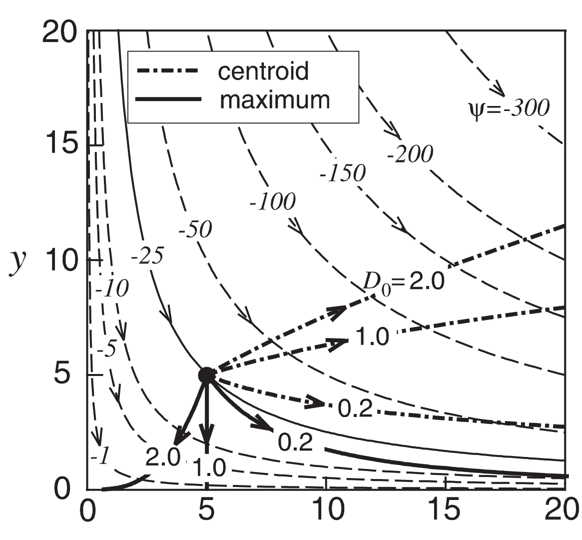

Fig. 3. Streamlines, path of the maximum and centroid of the concentration plume from an instantaneous unit line release located at $x_0 = 5, y_0 = 5$ in two-dimensional corner flow with $u_0 = 1, D_0 = 0.2, 1.0$ and 2.0.

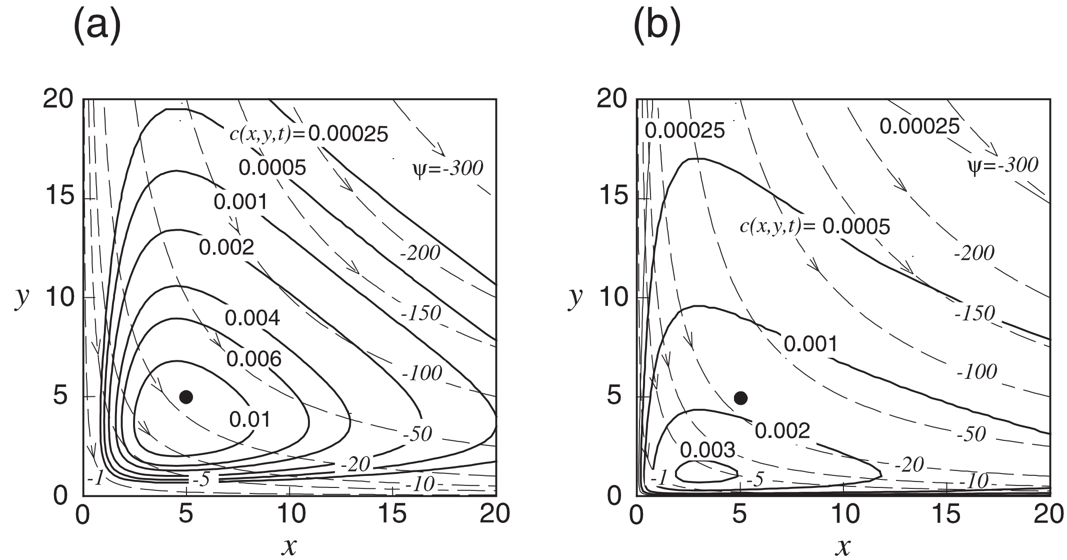

Fig. 4. Contours of the concentration profile, $C(x, y, t)$ for an instantaneous unit line release located at $x_0 = 5, y_0 = 5$ in two-dimensional corner flow with $u_0 = 1$, $D_0 = 2$ and $t = 0.1$ in (a) and $t = 0.5$ in (b).

more dominant role in the transport process. As more and more material is dispersed throughout the domain, the bulk of the contaminant, which is characterised by the centroid, moves against the flow. This is illustrated using the following example, where the same parameters used to produce Fig. 2 were used to produce the contours shown in Fig. 4.

The concentration contours at $t = 0.1$ and $t = 0.5$ are shown in Fig. 4(a) and (b) respectively. It is obvious that the concentration profile has attenuated. In Fig. 4(a) most of the plume is contained within the domain. However, in Fig. 4(b) material is being swept downstream outside the domain. The source of this material is the region surrounding the maximum concentration. The erosion of this material gives the impression that the maximum is travelling against the direction of the flow, towards the stagnation point. As the contamination plume evolves with time, the diffusion process dominates the transport process. In this case the diffusion coefficient is sufficiently large to disperse the contaminant throughout the domain. This spreading gives the impression that the bulk of the plume is travelling upstream, negative y-direction.

3.2. Steady source

Since Eq. (10) is linear, then Eq. (15) is a fundamental solution to the governing equations. This analytical solution for a unit instantaneous release can be integrated to provide solutions to problems involving sources with arbitrary shapes. For example, a steady point source may be treated as a series of instantaneous point sources [1,22]. An analytical solution for a steady line source located at $\mathbf{x} = (x_0, y_0)$ can be obtained by integrating Eq. (14) with respect to time and substituting Eq. (9), to give

$$C(x, y, t) = \frac{1}{2\pi D_0 u_0^2 \sqrt{x y x_0 y_0}} \left(\frac{x y_0}{x_0 y}\right)^{1/(2u_0 D_0)} K_0\left(\frac{\rho\sqrt{1 + D_0^2 u_0^2}}{\sqrt{2} D_0}\right), \tag{20}$$

in which $K_0(r)$ is the modified Bessel function of the second kind of order zero and argument r. The concentration plume in the two-dimensional corner flow for a steady unit line source located at $x_0 = 5$, $y_0 = 5$, with $u_0 = 1$ is shown in Fig. 5 where $D_0 = 1$ in (a) and $D_0 = 0.1$ in (b).

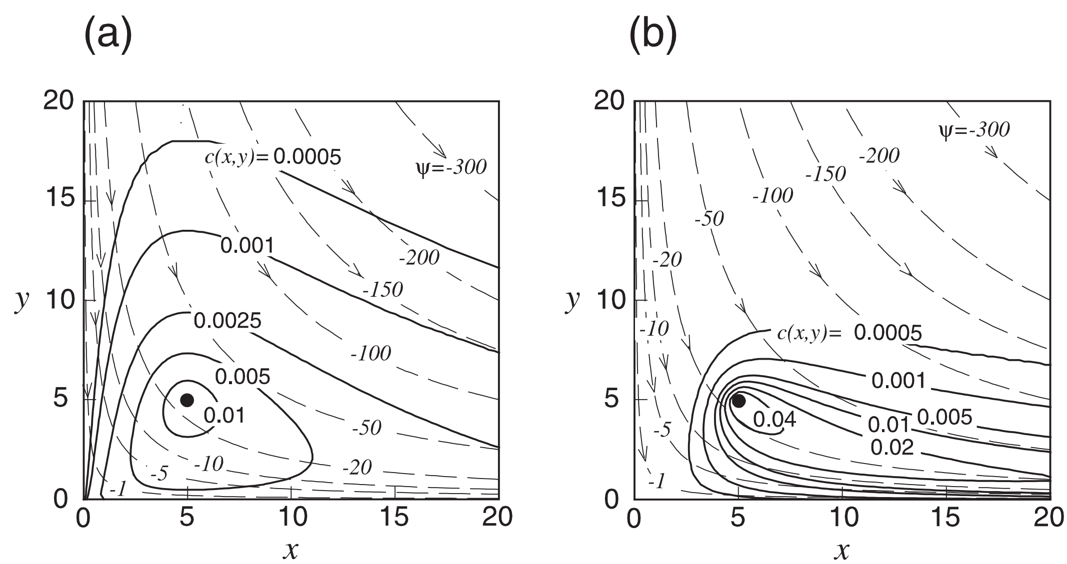

Fig. 5. Contours of the concentration profile, $C(x,y)$ for a steady line source located at $x_0 = 5, y_0 = 5$ in two-dimensional corner flow with $u_0 = 1$ and $D_0 = 1$ in (a) and $D_0 = 0.1$ in (b).

Comparing the results in this figure suggests that when $D_0 = 1$, diffusion has a significant influence in the transport process. However, when $D_0 = 0.1$, the transport of material is dominated by the flow.

The only other solution of this type known to the authors is that due to Maas [23], who provided a solution to the steady line release of contaminant in two-dimensional straining flow with constant diffusion coefficient.

3.3. Conjugate solutions

The spatial form of the velocity given by Eq. (5) implies that $u(x=0) = 0$ and $v(y=0) = 0$. Some numerical schemes become unstable or fail when the Courant number, $Cr_x = u\Delta t/\Delta x = 0$, or $Cr_y = v\Delta t/\Delta y = 0$ where Δx and Δy are the computational grid spacing in the x- and y-directions respectively. Under these circumstances the above analytical solutions are only useful if the numerical scheme avoids the region $x = 0$ and $y = 0$. Accordingly, we provide analytical solutions to a problem where along a hyperbola, $a^2 = xy$ which corresponds to the streamline $\psi = a^2$, the velocity is finite but the concentration vanishes. This represents the transport of pollutant, originally located on the streamline $x_0 y_0 > a^2$ with a boundary, located along the streamline $a^2 = xy$ (see Fig. 6) which is impermeable to fluid flow and along which zero concentration is maintained. To satisfy these properties, a *conjugate* pollutant sink is required, its location is (x_1, y_1) with $x_1 y_1 < a^2$ and strength to be determined.

3.3.1. Instantaneous release

Take (x_2, y_2) to be a point on the hyperbola $xy = a^2$. To find a solution which is zero on the hyperbola we assume a solution of the form

$$C(x,y,t) = C_0(x,y,x_0,y_0,t) - AC_0(x,y,x_1,y_1,t), \tag{21}$$

where $C_0(x,y,x_n,y_n,t)$ is the instantaneous unit release of contaminant at $x = x_n$ and $y = y_n$ and $A > 0$ is the strength of the conjugate sink. Since

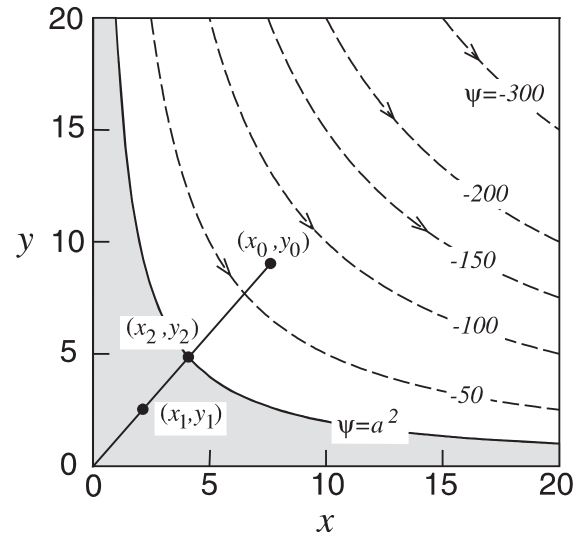

Fig. 6. Location of the source and conjugate sink in two-dimensional corner flow with the impermeable flow boundary located at $\psi = a^2 = 25$.

$$C(x = x_2, y = y_2, t) = 0, \quad \forall t$$

then from Eq. (21) and using Eq. (14)

$$\frac{1}{x_0 y_0} \exp\left(\frac{-[X_2 - X_0 - (1 - D_0 u_0)t]^2 - [Y_2 - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 t}\right)$$
$$= \frac{A}{x_1 y_1} \exp\left(\frac{-[X_2 - X_1 - (1 - D_0 u_0)t]^2 - [Y_2 - Y_1 - (1 + D_0 u_0)t]^2}{4D_0 t}\right).$$

Taking logarithms of both sides then

$$\ln(A) = \ln\left(\frac{x_1}{x_0}\right) - \frac{[X_2 - X_0 - (1 - D_0 u_0)t]^2 + [Y_2 - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 u_0^2 t}$$
$$+ \ln\left(\frac{y_1}{y_0}\right) + \frac{[X_2 - X_1 - (1 - D_0 u_0)t]^2 + [Y_2 - Y_1 - (1 + D_0 u_0)t]^2}{4D_0 t}.$$

If the conjugate point (x_1, y_1) lies on a line passing through the origin and the point (x_0, y_0), as shown in Fig. 6 then $x_1/x_0 = y_1/y_0$ and

$$\ln(A) = 2\ln\left(\frac{x_1}{x_0}\right) - \frac{[X_2 - X_0 - (1 - D_0 u_0)t]^2 + [Y_2 - Y_0 - (1 + D_0 u_0)t]^2}{4D_0 t}$$
$$+ \frac{[X_2 - X_1 - (1 - D_0 u_0)t]^2 + [Y_2 - Y_1 - (1 + D_0 u_0)t]^2}{4D_0 t}.$$

This can be simplified so that

$$\ln(A) = 2\ln\left(\frac{x_1}{x_0}\right) + \left\{\frac{(2X_2 - X_0 - X_1 - 2(1 - D_0 u_0)t)(X_0 - X_1)}{4D_0 t}\right.$$
$$\left. + \frac{(2Y_2 - Y_0 - Y_1 - 2(1 + D_0 u_0)t)(Y_0 - Y_1)}{4D_0 t}\right\}. \tag{22}$$

Since, $x_1 y_0 = x_0 y_1$ and using Eq. (3) then $X_0 - X_1 = -(Y_0 - Y_1)$. Eq. (22) becomes

$$\ln(A) = 2\ln\left(\frac{x_1}{x_0}\right) + \frac{(2X_2 - 2Y_2 - X_1 - X_0 + Y_0 + Y_1 + 4D_0 u_0 t)(X_0 - X_1)}{4D_0 t}. \tag{23}$$

In order to satisfy Eq. (21) $2X_2 - 2Y_2 - X_1 - X_0 + Y_0 + Y_1 = 0$ which occurs when $x_0 y_1 = x_1 y_0 = x_2 y_2 = a^2$. Therefore Eq. (23) yields

$$A = \frac{x_1}{x_0}.$$

Using $x_0 y_1 = x_1 y_0 = x_2 y_2 = a^2$ then the required form of the strength of the conjugate sink that is required in Eq. (21) is given by

$$A = \sqrt{\frac{x_1 y_1}{x_0 y_0}}.$$

The location of the conjugate sink, (x_1, y_1) can be determined using

$$x_1 y_0 = a^2 \quad \text{and} \quad x_0 y_1 = a^2. \tag{24}$$

Writing Eq. (21) explicitly, the solution for an instantaneous unit line release at (x_0, y_0) in two-dimensional corner flow with an impermeable flow boundary along $xy = a^2$ on which C is zero is

$$\begin{aligned} C(x,y,t) = {} & \frac{1}{4\pi D_0 u_0^2 t\sqrt{xyx_0y_0}}\left(\frac{xy_0}{x_0 y}\right)^{1/(2u_0D_0)} \exp\left(\frac{-\rho_1^2 - 2(1 + D_0^2 u_0^2)t^2}{4D_0 t}\right) \\ & - \frac{1}{4\pi D_0 u_0^2 t\sqrt{xyx_0y_0}}\left(\frac{xy_1}{x_1 y}\right)^{1/(2u_0D_0)} \exp\left(\frac{-\rho_2^2 - 2(1 + D_0^2 u_0^2)t^2}{4D_0 t}\right), \end{aligned} \tag{25}$$

where

$$\rho_1 = (1/u_0)\sqrt{\ln^2(x/x_1) + \ln^2(y/y_1)},$$

$$\rho_2 = (1/u_0)\sqrt{\ln^2(x/x_2) + \ln^2(y/y_2)}.$$

To illustrate the transport of an instantaneous release of a unit line source in two-dimensional corner flow with an impermeable flow boundary, consider the following problem. The impermeable flow boundary is located at $\psi = 20$ in two-dimensional flow, where $u_0 = 1$ and $D_0 = 0.2$. An instantaneous release occurs at $x_0 = 7.5$ and $y_0 = 9$ and the evolution with time of the concentration plume is illustrated in Fig. 7, where $t = 0.1$ in (a) and $t = 1$ in (b). From Eq. (24) the conjugate source is located at $x_1 = 20/9$ and $y_1 = 20/7.5$.

The shaded region represents an impermeable region. Along its edge $u \neq 0$ and $v \neq 0$ but $C(x,y,t) = 0$. As the plume evolves, the maximum concentration attenuates and the plume spreads as it is swept downstream.

3.3.2. Steady source

Using the same approach used to derive Eq. (25), the following analytical solution is obtained for a steady unit line source:

$$\begin{aligned} C(x,y,x_1,y_1,a) = {} & \frac{1}{2\pi D_0 u_0^2\sqrt{xyx_0y_0}}\left(\frac{xy_0}{x_0 y}\right)^{1/(2u_0D_0)} K_0\left(\frac{\rho\sqrt{1 + D_0^2 u_0^2}}{\sqrt{2}D_0}\right) \\ & - \frac{1}{2\pi D_0 u_0^2\sqrt{xyx_0y_0}}\left(\frac{xy_1}{x_1 y}\right)^{1/(2u_0D_0)} K_0\left(\frac{\rho\sqrt{1 + D_0^2 u_0^2}}{\sqrt{2}D_0}\right). \end{aligned} \tag{26}$$

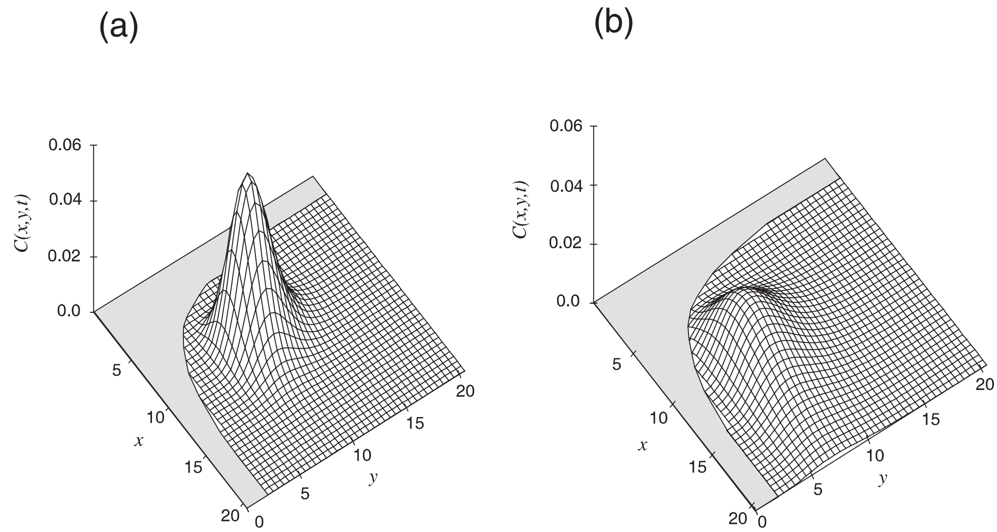

Fig. 7. Concentration profile, $C(x, y, t)$ for an instantaneous unit line release located at $x_0 = 7.5, y_0 = 9$ in two-dimensional corner flow with $u_0 = 1, D_0 = 0.2$, an impermeable flow boundary located at $\psi = xy = 20$ and $t = 0.1$ in (a) and $t = 1$ in (b).

Using the same parameters that were used to produce the results shown in Fig. 7, the results for a steady line source are shown in Fig. 8. In Fig. 8(a) $D_0 = 2$ and in Fig. 8(b) $D_0 = 0.2$.

A comparison of these figures indicates that the behaviour of the concentration plume is similar to the steady line source shown in Fig. 5 for the corner flow problem. In Fig. 8(a) diffusion also has a significant influence in the transport process. The concentration plume is uniformly distributed in the x- and y-directions. This is in contrast to the plume shown in Fig. 5(b). In this case the plume is skewed in the direction of the flow and the concentration contours accumulating

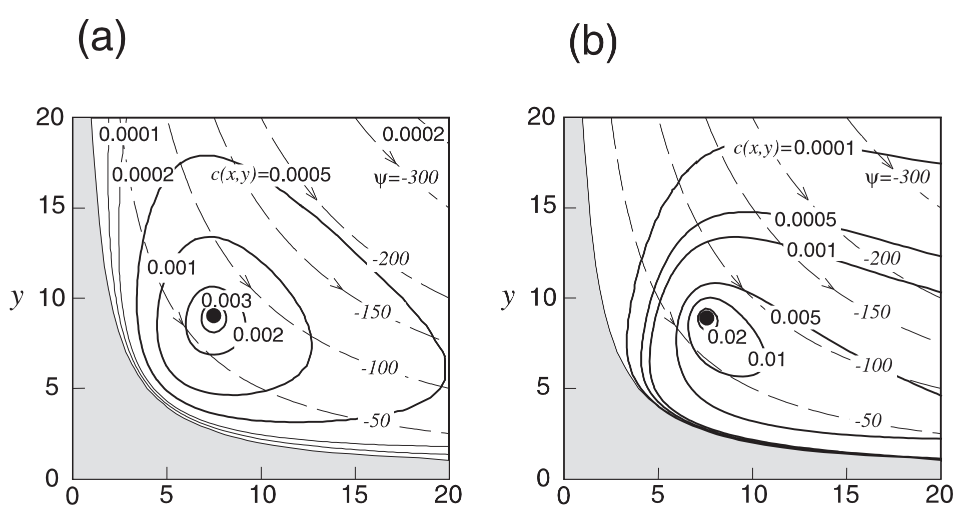

Fig. 8. Contours of the concentration profile, $C(x, y)$ for a steady unit line source located at $x_0 = 7.5, y_0 = 9$ in two-dimensional corner flow with $u_0 = 1$, an impermeable boundary located at $a^2 = 20$ and $D_0 = 2$ in (a) and $D_0 = 0.2$ in (b).

along the impermeable boundary. This suggests that the flow dominates the transport process here.

The solution to these problems is useful for testing numerical schemes designed to solve problems with curved boundaries, such as finite element and finite volume techniques.

4. Three-dimensional solutions

Using the same approach that was used to obtain two-dimensional analytical solutions, analytical solutions have been found for three-dimensional corner flows. The details have been omitted here, with only the results being presented. We provide analytical solutions to Eq. (6) for an instantaneous unit point release and unit point steady source of contaminant in the positive octant of three-dimensional corner flow.

4.1. Instantaneous release

The solution with time for an instantaneous point source released when $t = 0$ at the position (x_0, y_0, z_0) is given by

$$C(x,y,z,t) = \frac{1}{8|u_0 v_0 w_0|\sqrt{(\pi D_0 t)^3 (xyzx_0 y_0 z_0)}} \times \left(\frac{x}{x_0}\right)^{1/(2u_0 D_0)} \left(\frac{y}{y_0}\right)^{1/(2v_0 D_0)} \left(\frac{z}{z_0}\right)^{1/(2w_0 D_0)} \exp\left\{\frac{-R^2 - \left(3 + D_0^2 U^2\right)t^2}{4D_0 t}\right\},$$

in which

$$R = \sqrt{(1/u_0^2)\ln^2(x/x_0) + (1/v_0^2)\ln^2(y/y_0) + (1/w_0^2)\ln^2(w/w_0)},$$
$$U = \sqrt{u_0^2 + v_0^2 + w_0^2}.$$

The concentration surface, $C(x,y,z,t) = 0.0001$ predicted by the above equation for an instantaneous point unit source located at $x_0 = 5, y_0 = 5,\ z_0 = 5$ in three-dimensional corner flow with $u_0 = 0.5,\ v_0 = 0.5,\ w_0 = -1,\ D_0 = 0.1$ when $t = 0.1$ and $t = 1$ is illustrated in Fig. 9(a) and (b) respectively.

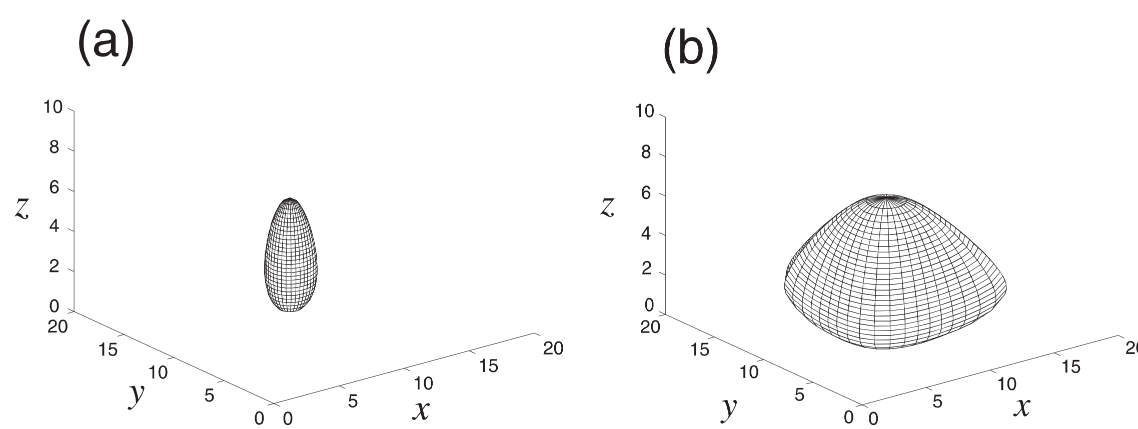

Fig. 9. Concentration surface, $C(x,y,z,t) = 0.0001$ for an instantaneous point unit release located at $x_0 = 5, y_0 = 5, z_0 = 5$ in three-dimensional corner flow with $u_0 = 0.5, v_0 = 0.5, w_0 = -1, D_0 = 0.1$ and $t = 0.1$ in (a) and $t = 1$ in (b).

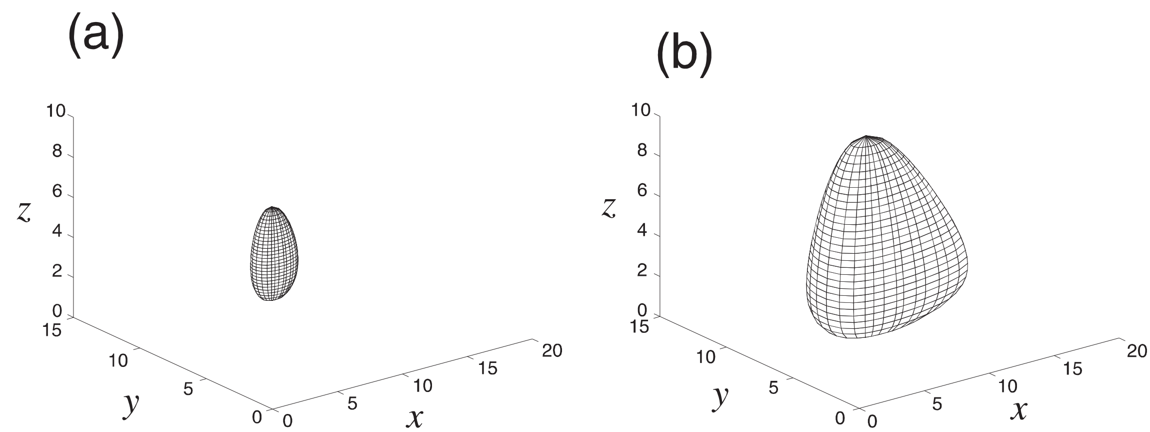

Fig. 10. Concentration surface, $C(x,y,z,t) = 0.0001$ for an instantaneous point unit release located at $x_0 = 5, y_0 = 5$, $z_0 = 5$ in three-dimensional corner flow with $u_0 = 0.5, v_0 = 0.25, w_0 = -0.75, D_0 = 0.2$ and $t = 0.1$ in (a) and $t = 0.5$ in (b).

The concentration profile moves rapidly in the negative-z direction and settles near the x–y plane, along which $w_0 = 0$. The concentration profile spreads laterally with time and for the chosen velocity field, the concentration surface remains symmetrical about the $x = y$ plane. This is not the case in the second example shown in Fig. 10. Here, the concentration surface, $C(x,y,z,t) = 0.0001$ for an instantaneous point unit source located at $x_0 = 5, y_0 = 5$, $z_0 = 5$ in three-dimensional corner flow with $u_0 = 0.5, v_0 = 0.25, w_0 = -0.75, D_0 = 0.2$ and $t = 0.05$ in (a) and $t = 0.5$ in (b), has been plotted. Again there is rapid spreading of the concentration in the negative-z direction. Near the x–y plane, material moves in the dominant flow direction. Therefore, the concentration profile is skewed in the x-direction. In both cases, the transport of the concentration profile is dominated by the velocity field.

4.2. Steady source

The corresponding analytical solution for a steady point unit source located at (x_0, y_0, z_0) in three-dimensional corner flow is given by

$$C(x,y,z) = \frac{1}{4\pi D_0 R|u_0 v_0 w_0|\sqrt{xyzx_0y_0z_0}} \times \left(\frac{x}{x_0}\right)^{1/(2u_0D_0)} \left(\frac{y}{y_0}\right)^{1/(2v_0D_0)} \left(\frac{z}{z_0}\right)^{1/(2w_0D_0)} \exp\left\{\frac{-R\sqrt{3 + D_0^2U^2}}{2D_0}\right\}.$$

The solution of this equation for a steady source located at $x_0 = 5, y_0 = 5$, $z_0 = 5$ in a three-dimensional corner flow with $u_0 = 1$, $v_0 = 1$, $w_0 = -2$ and $D_0 = 0.1$ is illustrated in Fig. 11. Plots of the concentration surface, $C(x,y,z) = 0.005$ and $C(x,y,z) = 0.001$ are shown in Fig. 11(a) in (b) respectively. In this case the concentration profile is symmetrical about the x–y plane. Material is being swept towards and along the x–y plane. This is in contrast to the results shown in Fig. 12. In this case $D_0 = 1$ and the concentration surface $C(x,y,z) = 0.001$ and $C(x,y,z) = 0.0005$ are shown in Fig. 12(a) and (b) respectively. Although material has spread in the negative-z direction, it is not as pronounced as in Fig. 11. Here the high concentration of material can only be found close to the source whereas high concentrations can be found over a larger area in Fig. 11. In Fig. 12 the velocity field plays a minor part in the transport process and is dominated by diffusion, whilst in Fig. 11 the transport process is dominated by the velocity field.

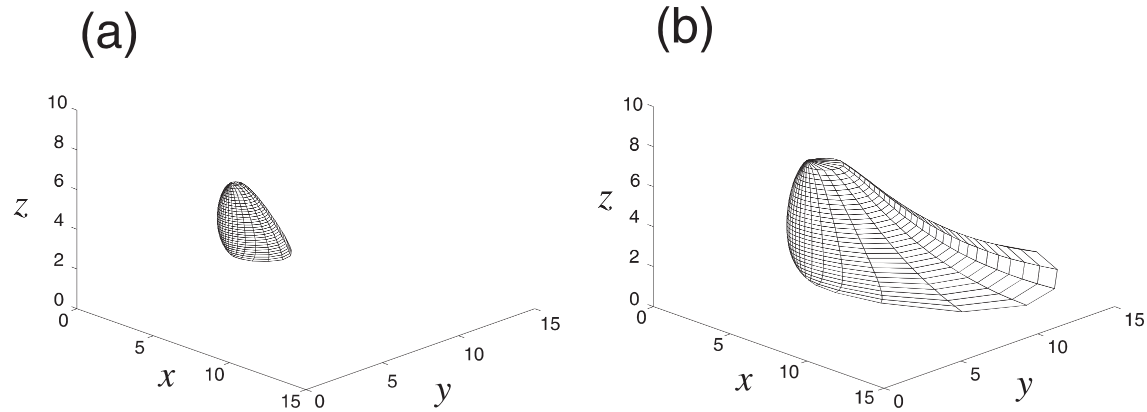

Fig. 11. Concentration surface for a steady point source located at $x_0 = 5, y_0 = 5$, $y_0 = 5$ in three-dimensional corner flow with $u_0 = 1, v_0 = 1, w_0 = -2$, $D_0 = 0.1$ and $C(x, y, z) = 0.005$ in (a) and $C(x, y, z) = 0.001$ in (b).

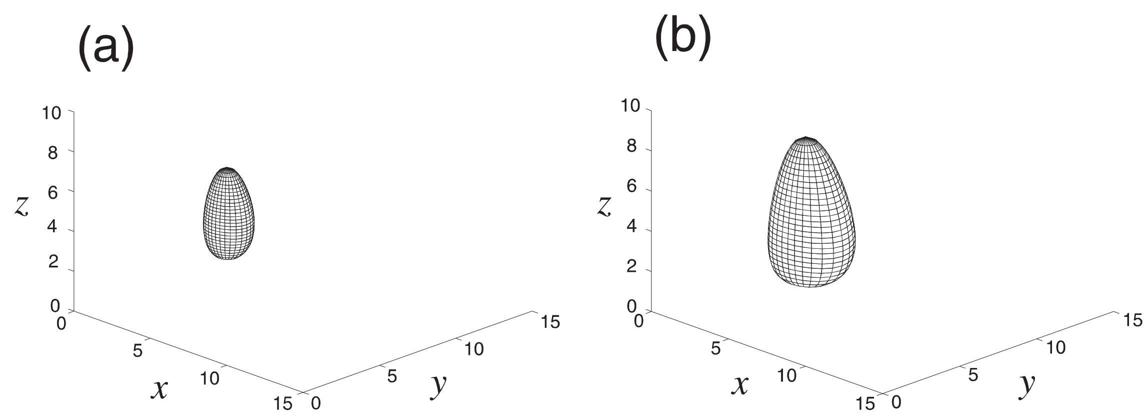

Fig. 12. Concentration surface for a steady point source located at $x_0 = 5, y_0 = 5$, $z_0 = 5$ in three-dimensional corner flow with $u_0 = 1, v_0 = 1, w_0 = -2$, $D_0 = 1$ and $C(x, y, z) = 0.001$ in (a) and $C(x, y, z) = 0.0005$ in (b).

Attempts to find analytical solutions to the three-dimensional corner flow with an impermeable flow boundary, similar to those found in two dimensions were unsuccessful.

5. Conclusions

Analytical solutions have been derived for the two- and three-dimensional advection–diffusion equation with a particular form of the spatially variable coefficients. Many other solutions can be built from these fundamental solutions. The analytical solutions are simple to evaluate and implement. They can be used to test numerical schemes developed to solve the advection–diffusion equation with problems containing non-uniform velocity fields and variable diffusion coefficients. Analytical solutions for two-dimensional corner flows containing an impervious flow boundary along a streamline are also provided. These analytical solutions are valuable for testing the performance of numerical schemes designed to cope with curved boundaries.

In the analytical solutions each velocity component is assumed to have a linear dependence on the distance and each diffusion coefficient is proportional to the square of the corresponding velocity component. With this particular form of the spatially variable coefficients, there exists a simple transformation which reduces the spatially variable coefficient equation into a linear equation with constant coefficients. There is available a large class of analytical solutions to the constant coefficient problem which can be used to obtain solutions to the spatially variable coefficient equation. In addition, the transformation can be used to obtain solutions for one-, two- and three-dimensional problems. The approach used to derive the analytical solutions for divergence free flows is also applicable for two- and three-dimensional problems where the divergence of the velocity is not equal to zero. For one-dimensional problems the divergence of the velocity is not zero.

Some properties of the analytical solutions have been investigated and reveal behaviour of the concentration plume that differs from that of a plume in a uniform velocity field with diffusion coefficient constant. In two-dimensional divergence free flow, the path of the maximum, the centroid of an instantaneous release of material and the fluid particle always lie on the same streamline when the diffusion coefficient, $D_0 = 0$. For this spatially dependent diffusion coefficient problem, the maximum concentration is always below and moves away from the streamline describing the path of the fluid particle. The centroid on the other hand, is always above the streamline. There are certain values of $D_0 > 0$ where the path of maximum and the centroid of the concentration plume move against the flow. These observations are in contrast to the behaviour of a plume in a uniform velocity field and when the diffusion coefficient is constant.

Nomenclature

A	strength of the conjugate sink in the two-dimensional corner flow with an impermeable flow boundary
A_1	strength of the one-dimensional source in the two-dimensional problem
A_2	strength of the one-dimensional source in the two-dimensional problem
$a^2 = xy$	a hyperbola which corresponds to a streamline
$C(x, y, z, t)$	contaminant concentration
C_0	constant concentration
Cr_x	Courant number in the x-direction ($Cr_x = u\Delta t/\Delta x$)
Cr_y	Courant number in the y-direction ($Cr_y = v\Delta t/\Delta y$)
$D_x(\mathbf{x}) = D_0 u_0^2 x^2$	diffusion coefficient in the x-direction
$D_y(\mathbf{x}) = D_0 v_0^2 y^2$	diffusion coefficient in the y-direction
$D_z(\mathbf{x}) = D_0 w_0^2 z^2$	diffusion coefficients in the z-direction
D_0	constant diffusion coefficient
$g_1(X, X_0, t)$	one-dimensional solution for the x-component of the two-dimensional problem
$g_2(Y, Y_0, t)$	one-dimensional solution for the y-component of the two-dimensional problem
$\mathrm{K}_0(r)$	modified Bessel function of the second kind of order zero and argument r
L	the length of the computational domain
R	$\sqrt{(1/u_0^2)\ln^2(x/x_0) + (1/v_0^2)\ln^2(y/y_0) + (1/w_0^2)\ln^2(w/w_0)}$
t	time
t_0	time of initial release of pollutant
U	$\sqrt{u_0^2 + v_0^2 + w_0^2}$

$\mathbf{u} = (u(\mathbf{x}), v(\mathbf{x}), w(\mathbf{x})$	vector of fluid velocity in the Cartesian coordinates
$u(\mathbf{x}) = u_0 x$	the velocity in the x-direction
u_0	constant velocity
$v(\mathbf{x}) = v_0 y$	the velocity in the y-direction
v_0	constant velocity
$w(\mathbf{x}) = w_0 z$	the velocity in the z-direction
w_0	constant velocity
$X = \ln(x)/u_0$	transformed x-direction
$\mathbf{x} = (x, y, z)$	vector of Cartesian coordinates, x, y and z
$\mathbf{x}_c(t)$	location of the centroid of the concentration plume
$\mathbf{x}_p(t)$	position of a particle when $D_0 = 0$
$\mathbf{x}_m(t)$	location of the maximum concentration
x	$\exp(u_0 X)$
x_0	location of the initial release of contaminant at time t in the x-direction
x_1	location of the conjugate source in the x-direction
x_2	location of the impermeable boundary in the x-direction
$Y = \ln(y)/v_0$	transformed y-direction
y	$\exp(v_0 Y)$
y_0	location of the initial release of contaminant at time t in the y-direction
y_1	location of the conjugate source in the y-direction
y_2	location of the impermeable boundary in the y-direction
$Z = \ln(z)/w_0$	transformed z-direction
z	$\exp(w_0 Z)$
z_0	location of the initial release of contaminant at time t in the z-direction
α	non-zero constant
Δx	grid spacing in the x-direction
Δt	time step
π	3.1416
ρ	$(1/u_0)\sqrt{\ln^2(x/x_0) + \ln^2(y/y_0)}$.

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