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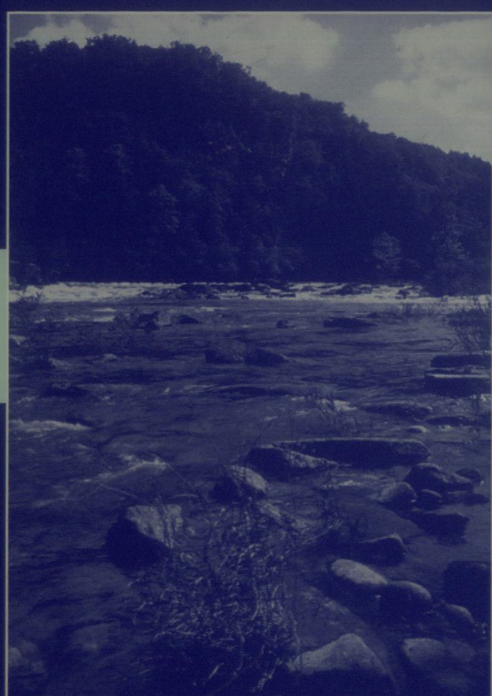

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Analytical Solutions for Two-Dimensional Transport Equation with Time-Dependent Dispersion Coefficients

By Mustafa M. Aral,[1] Member, ASCE, and Boshu Liao[2]

Abstract: Analytical solutions to advection-dispersion equations are of continuous interest because they present benchmark solutions to problems in hydrogeology, chemical engineering, and fluid mechanics. In this paper, we examine solutions to two-dimensional advection-dispersion equation with time-dependent dispersion coefficients. The time- and space-dependent nature of the dispersion coefficient in subsurface contaminant transport problems has been demonstrated in the literature in both field and laboratory scale studies. Analytical solutions given in this paper could be used to model the transport of solute in hydrogeologic systems characterized by dispersion coefficients that may vary as a function of travel time from the input source. In particular, in this paper we develop instantaneous and continuous point-source solutions for constant, linear, asymptotic, and exponentially varying dispersion coefficients. The relationship between the proposed general solution and the particular solutions given in the relevant literature are discussed. Examples are included to demonstrate the effect of time-dependent dispersion coefficients on solute transport.

INTRODUCTION

The migration of dissolved contaminant plumes in a subsurface environment is modeled by an advection-dispersion equation. In the literature, this equation is usually solved based on the assumption that the dispersion coefficient is time- and space-independent. However, field and laboratory scale experiments indicate that in subsurface transport problems the dispersivity parameter may be time-dependent. The apparent temporal and spatial variability of the dispersion coefficient is identified in the literature as the scale effect [Fried (1975)]. Stochastic analyses have shown that the dispersion coefficient may depend on travel time and that it may increase until it reaches an asymptotic value [Gelhar et al. (1979)]. Analytical solutions of the solute transport equation which time-dependent dispersion coefficients have been published, for restricted cases, by Pickens and Grisak (1981b), Yates (1992), and Barry and Sposito (1989). More recently, Basha and El-Habel (1993) published their research on the analytical solution of the one-dimensional transport equation with time-dependent dispersion coefficients for an infinite domain. In their paper, one-dimensional analytical solutions and numerical results were presented for four different functions, which may represent the relation between the dispersion coefficient and time. We show that these solutions are particular solutions of the general two-dimensional analytical solution developed in this study.

In the study presented here, a general analytical solution of the two-dimensional solute transport equation with time-dependent dispersion coefficients is given for an infinite domain aquifer. Particular solutions are presented for several dispersion coefficient functions where the source is represented by instantaneous injection and continuous injection at a constant rate, assuming that the initial concentration distribution in the aquifer domain is zero. In this approach the dispersion coefficient is assumed to depend on the travel time of the solute rather than on the mean travel distance as assumed by other researchers. These two approaches are the same for a constant velocity field since the mean travel distance is directly proportional to the mean travel time for such cases. In this study, the flow field is assumed to be steady and uniform. Analytical solutions developed are also extended to situations that involve special initial concentration distribution cases. For example, for several time-dependent dispersion coefficient functions, particular analytical solutions are discussed as an extension of the general solutions in which the injection rate is zero but the initial concentration distribution in the aquifer domain is nonzero. For the case of nonzero injection, as well as nonzero initial concentration distribution, superposition principles can be used to arrive at analytical solutions to these more complex cases. These solutions are discussed as possible applications of the primary solution developed. The analytical solutions presented may be used as practical tools in evaluating scale-dependent solute transport problems in subsurface systems.

GENERAL SOLUTION

The advection-dispersion equation analyzed in this study is defined for a steady-state two-dimensional velocity field with a first-order decay function and time-dependent dispersion coefficients. For a two-dimensional infinite aquifer domain, this equation can be given as

$$R\frac{\partial \bar{C}}{\partial \bar{t}} = \bar{D}_L(\bar{t})\frac{\partial^2 \bar{C}}{\partial \bar{x}^2} + \bar{D}_T(\bar{t})\frac{\partial^2 \bar{C}}{\partial \bar{y}^2} - \bar{u}\frac{\partial \bar{C}}{\partial \bar{x}} - \bar{v}\frac{\partial \bar{C}}{\partial \bar{y}} - \bar{\mu}R\bar{C}$$

$$+ \bar{q}(\bar{x}, \bar{y}, \bar{t});\ \bar{C}(\bar{x}, \bar{y}, 0) = \bar{f}(\bar{x}, \bar{y})\quad -\infty < \bar{x}, \bar{y} < \infty \qquad (1a,b)$$

In (1), R = retardation coefficient; $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ = components of the steady-state velocity vector in the $\bar{x}$- and $\bar{y}$-directions, respectively; $\bar{D}_L$ and $\bar{D}_T$ = longitudinal and transverse dispersion coefficients, which are functions of time $\bar{t}$; $\bar{C}$ = solute concentration; $\bar{q}(\bar{x}, \bar{y}, \bar{t})$ = mass injection rate; $\bar{\mu}$ = first-order decay coefficient; and $\bar{f}(\bar{x}, \bar{y})$ = a function representing the initial concentration distribution in the infinite domain aquifer.

The nondimensional form of (1) can be written as

$$R\frac{\partial C}{\partial t} = D_L(t)\frac{\partial^2 C}{\partial x^2} + D_T(t)\frac{\partial^2 C}{\partial y^2} - \mathbf{u}\frac{\partial C}{\partial x} - \mathbf{v}\frac{\partial C}{\partial y} - \mu RC$$

$$+ q(x, y, t);\ C(x, y, 0) = f(x, y)\quad -\infty < x, y < \infty \qquad (2a,b)$$

where

$$x = \frac{\bar{x}}{L},\ y = \frac{\bar{y}}{L},\ t = \bar{t}\frac{D_r}{L^2},\ C = \frac{\bar{C}}{C_r},\ D_L(t) = \frac{\bar{D}_L(\bar{t})}{D_r},$$

$$D_T(t) = \frac{\bar{D}_T(\bar{t})}{D_r},\ u = \bar{u}\frac{L}{D_r},\ v = \bar{v}\frac{L}{D_r},\ \mu = \bar{\mu}\frac{L^2}{D_r},$$

$$q(x, y, t) = \bar{q}(\bar{x}, \bar{y}, \bar{t})\frac{L^2}{D_rC_r},\ f(x, y) = \frac{\bar{f}(\bar{x}, \bar{y})}{C_r} \qquad (3a\text{–}k)$$

where L = a reference distance; C_r = a reference concentration; and D_r = a reference dispersion coefficient.

The relationship between the time-dependent dispersion coefficients in longitudinal and transverse directions may be assumed to be

$$D_L(t) = a^2D_T(t) \qquad (4)$$

where a = a constant of proportionality. Substituting (4) in (2) and using the transformation

$$x' = x,\quad y' = ay,\quad v' = av \qquad (5a\text{–}c)$$

Eq. (2) can be given as

$$R\frac{\partial C}{\partial t} = D_L(t)\left(\frac{\partial^2 C}{\partial x'^2} + \frac{\partial^2 C}{\partial y'^2}\right) - u\frac{\partial C}{\partial x'} - v'\frac{\partial C}{\partial y'}$$

$$- \mu RC + q(x', y'/a, t) \qquad (6)$$

Eq. (6) can be written as (7) by dividing both sides by R, which is assumed to be constant in this study

$$\frac{\partial C}{\partial t} = D(t)\left(\frac{\partial^2 C}{\partial x'^2} + \frac{\partial^2 C}{\partial y'^2}\right) - U\left(\frac{\partial C}{\partial x'}\right) - V\left(\frac{\partial C}{\partial y'}\right)$$

$$- \mu C + q(x', y'/a, t)/R \qquad (7)$$

where

$$D(t) = D_L(t)/R,\quad U = u/R,\quad V = v'/R = av/R \qquad (8a\text{–}c)$$

Eq. (7) may now be simplified using a series of transformations. If we let

$$C = C_1\exp(-\mu t) \qquad (9)$$

then (7) can be written as

$$\frac{\partial C_1}{\partial t} = D(t)\left(\frac{\partial^2 C_1}{\partial x'^2} + \frac{\partial^2 C_1}{\partial y'^2}\right) - U\frac{\partial C_1}{\partial x'} - V\frac{\partial C_1}{\partial y'}$$

$$+ \frac{q(x', y'/a, t)\exp(\mu t)}{R} \qquad (10)$$

If we let,

$$X = x' - Ut,\quad Y = y' - Vt \qquad (11a,b)$$

then we may obtain

$$\frac{\partial C_1}{\partial t} = D(t)\left(\frac{\partial^2 C_1}{\partial X^2} + \frac{\partial^2 C_1}{\partial Y^2}\right)$$

$$+ \frac{q[X + Ut, (Y + Vt)/a, t]\exp(\mu t)}{R} \qquad (12)$$

Utilizing the following transformation

$$T = \alpha(t) = \int_0^t D(t')\,dt' \qquad (13)$$

Eq. (12) can be reduced to the following nonhomogeneous equation:

$$\frac{\partial C_1}{\partial T} = \frac{\partial^2 C_1}{\partial X^2} + \frac{\partial^2 C_1}{\partial Y^2} + Q(X, Y, T) \qquad (14)$$

where

$$C_1(X, Y, 0) = f(X, Y/a),\quad -\infty < X, Y < \infty \qquad (15a)$$

$$Q(X, Y, T) = \frac{q\{X + U\alpha^{-1}(T), [Y + V\alpha^{-1}(T)]/a, \alpha^{-1}(T)\}\exp[\mu\alpha^{-1}(T)]}{RD[\alpha^{-1}(T)]} \qquad (15b)$$

The analytical solution of (14) can be obtained using the superposition principle. This solution can be given as [Haberman (1987), p. 412]

$$C_1 = \int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q(\xi, \eta, \tau)}{4\pi(T-\tau)}\exp\left[-\frac{(X-\xi)^2 + (Y-\eta)^2}{4(T-\tau)}\right]$$

$$\cdot d\xi\,d\eta\,d\tau + \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta/a)$$

$$\cdot\exp\left[-\frac{(X-\xi)^2 + (Y-\eta)^2}{4T}\right]d\xi\,d\eta \qquad (16)$$

On substituting (15) into (16), one may obtain

$$C_1 = \int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\cdot\frac{q\{\xi + U\alpha^{-1}(\tau), [\eta + V\alpha^{-1}(\tau)]/a, \alpha^{-1}(\tau)\}\exp[\mu\alpha^{-1}(\tau)]}{4\pi R(T-\tau)D[\alpha^{-1}(\tau)]}$$

$$\cdot\exp\left[-\frac{(X-\xi)^2 + (Y-\eta)^2}{4(T-\tau)}\right]d\xi\,d\eta\,d\tau + \frac{1}{4\pi T}$$

$$\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta/a)\exp\left[-\frac{(X-\xi)^2 + (Y-\eta)^2}{4T}\right]d\xi\,d\eta \qquad (17)$$

If we let $\tau = \alpha(t_0)$, where $\tau = 0$ for $t_0 = 0$; and $\tau = T$ for $t_0 = t$, then (17) can be written as

$$C_1 = \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q[\xi + Ut_0, (\eta + Vt_0)/a, t_0]\exp(\mu t_0)}{4\pi R[T - \alpha(t_0)]}$$

$$\cdot\exp\left\{-\frac{(X-\xi)^2 + (Y-\eta)^2}{4[T - \alpha(t_0)]}\right\}d\xi\,d\eta\,dt_0$$

$$+ \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta/a)$$

$$\cdot\exp\left[-\frac{(X-\xi)^2 + (Y-\eta)^2}{4T}\right]d\xi\,d\eta \qquad (18)$$

Using (5), (9), (11), and (13), one may reduce (18) to the following form:

$$C(x, y, t) = \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\cdot\frac{q[\xi + Ut_0, (\eta + Vt_0)/a, t_0]\exp[-\mu(t - t_0)]}{4\pi R[\alpha(t) - \alpha(t_0)]}$$

$$\cdot\exp\left\{-\frac{(x - Ut - \xi)^2 + (ay - Vt - \eta)^2}{4[\alpha(t) - \alpha(t_0)]}\right\}d\xi\,d\eta\,dt_0$$

$$+ \frac{\exp(-\mu t)}{4\pi\alpha(t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta/a)$$

$$\cdot\exp\left[-\frac{(x - Ut - \xi)^2 + (ay - Vt - \eta)^2}{4\alpha(t)}\right]d\xi\,d\eta \qquad (19)$$

Eq. (19) is the general solution of the partial differential (2) in the form of a multiintegral expression. This solution may now be used to define particular solutions for various forms of dispersion coefficient $D_L(t)$, various injection function def-

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initions $q(x, y, t)$, and various initial concentration distributions $f(x, y)$. These particular solutions are given in the following sections.

SOLUTIONS FOR INSTANTANEOUS POINT INJECTION WITH ZERO INITIAL CONCENTRATION DISTRIBUTION

In this case, $C(x, y, 0) = f(x, y) = 0$, and the instantaneous nondimensional injection of a tracer is given by

$$q(x, y, t) = \frac{M}{n}\delta(x)\delta(y)\delta(t) \quad (20)$$

where M = nondimensional mass injected; n = porosity; and $\delta(\ )$ = Dirac delta function. Utilizing the initial condition and (19), the analytical solution can be given as

$$C(x, y, t) = \frac{aM}{4\pi n R\alpha(t)} \cdot \exp\left[-\mu t - \frac{(x - Ut)^2 + (ay - Vt)^2}{4\alpha(t)}\right] \quad (21)$$

where $\alpha(t)$ = a function of $D(t)$, which is defined by (13). Eq. (21) is an analytical expression, which can be used to describe solutions to several dispersion coefficient functions for the particular problem considered. These special solutions are given as follows:

Constant Dispersion Coefficient

A constant nondimensional dispersion coefficient can be defined as

$$D_L(t) = D_0 + D_m \quad (22)$$

Given (22), $\alpha(t)$ can be obtained from (13) as

$$\alpha(t) = \frac{D_0 + D_m}{R} t \quad (23)$$

For this case, $C(x, y, t)$ can be given as follows:

$$C(x, y, t) = \frac{aM}{4\pi n(D_0 + D_m)t} \cdot \exp\left[-\mu t - \frac{[x - (u/R)t]^2 + a^2[y - (v/R)t]^2}{4(D_0 + D_m)t/R}\right] \quad (24)$$

Linear Dispersion Coefficient

The nondimensional dispersion coefficient varying linearly with respect to time can be defined as

$$D_L(t) = D_0\frac{t}{k} + D_m \quad (25)$$

where k = an arbitrary constant, different from zero. Given (25), $\alpha(t)$ can be obtained from Equation (13) as,

$$\alpha(t) = \frac{D_0}{2Rk}t^2 + \frac{D_m}{R}t \quad (26)$$

Using (26), (21) can be given as

$$C(x, y, t) = \frac{aM}{4\pi n[(D_0/2k)t^2 + D_m t]} \cdot \exp\left\{-\mu t - \frac{[x - (u/R)t]^2 + a^2[y - (v/R)t]^2}{4[(D_0/2k)t^2 + D_m t]/R}\right\} \quad (27)$$

Asymptotic Dispersion Coefficient

An asymptotic nondimensional dispersion coefficient can be defined as

$$D_L(t) = D_0\frac{t}{k + t} + D_m \quad (28)$$

Given (28), $\alpha(t)$ can be obtained from (13) as

$$\alpha(t) = \frac{D_0 + D_m}{R}t - \frac{D_0 k}{R}\ln\left(1 + \frac{t}{k}\right) \quad (29)$$

Utilizing (29), (21) can be given as

$$C(x, y, t) = \frac{aM}{4\pi n[(D_0 + D_m)t - D_0 k \ln(1 + t/k)]} \cdot \exp\left\{-\mu t - \frac{[x - (u/R)t]^2 + a^2[y - (v/R)t]^2}{4[(D_0 + D_m)t - D_0 k\ln(1 + t/k)]/R}\right\} \quad (30)$$

Exponential Dispersion Coefficient

The exponential nondimensional dispersion coefficient may be represented as

$$D_L(t) = D_0[1 - \exp(-t/k)] + D_m \quad (31)$$

Given (31), $\alpha(t)$ can be obtained as

$$\alpha(t) = \frac{D_0 + D_m}{R}t + \frac{D_0 k}{R}\left[\exp\left(-\frac{t}{k}\right) - 1\right] \quad (32)$$

The solution for $C(x, y, t)$ can then be given as

$$C(x, y, t) = \frac{aM}{4\pi n\{(D_0 + D_m)t + D_0 k[\exp(-t/k) - 1]\}} \cdot \exp\left(-\mu t - \frac{[x - (u/R)t]^2 + a^2[y - (v/R)t]^2}{4\{(D_0 + D_m)t - D_0 k[\exp(-t/k) - 1]\}/R}\right) \quad (33)$$

PARTICULAR SOLUTIONS FOR CONTINUOUS POINT SOURCE WITH ZERO INITIAL CONCENTRATION DISTRIBUTION

A continuous nondimensional point source at $x = 0$, $y = 0$ can be represented by

$$q(x, y, t) = C_0\delta(x)\delta(y) \quad (34)$$

Substituting $f(x, y) = 0$ and (34) in (19), the solution may be given as

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi R[\alpha(t) - \alpha(t_0)]}\exp[-\mu(t - t_0)] - \frac{[x - U(t - t_0)]^2 + [ay - V(t - t_0)]^2}{4[\alpha(t) - \alpha(t_0)]} dt_0 \quad (35)$$

Eq. (35) describes the general solution for the problem where $\alpha(t)$ is again defined by (13). Particular cases of this solution, for the four dispersion coefficient functions discussed earlier, are given in the following section.

Constant Dispersion Coefficient

For constant nondimensional dispersion coefficients, $\alpha(t)$ is given by (23). For this case, the solution for $C(x, y, t)$ can be given as

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi(D_0 + D_m)(t - t_0)}\exp\left\{-\mu(t - t_0) - \frac{[x - (u/R)(t - t_0)]^2 + a^2[y - (v/R)(t - t_0)]^2}{4(D_0 + D_m)(t - t_0)/R}\right\} dt_0 \quad (36)$$

or

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi(D_0 + D_m)t_0} \cdot \exp\left(-\mu t_0 - \frac{[x - (u/R)t_0]^2 + a^2[y - (v/R)t_0]^2}{4(D_0 + D_m)t_0/R}\right) dt_0 \quad (37)$$

Linear Dispersion Coefficient

For linear nondimensional dispersion (coefficients, $\alpha(t)$ is given by (26). Substituting (26) in (35), the solution for $C(x, y, t)$ becomes

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi\left[\frac{D_0}{k}(t^2 - t_0^2) + D_m(t - t_0)\right]} \cdot \exp\left\{-\mu(t - t_0) - \frac{\left[x - \frac{u}{R}(t - t_0)\right]^2 + a^2\left[y - \frac{v}{R}(t - t_0)\right]^2}{\frac{4}{R}\left[\frac{D_0}{2k}(t^2 - t_0^2) + D_m(t - t_0)\right]}\right\} dt_0 \quad (38)$$

or this equation may be written as

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi\left(\frac{D_0}{2k}(2t - t_0) + D_m\right)t_0} \cdot \exp\left[-\mu t_0 - \frac{\left(x - \frac{u}{R}t_0\right)^2 + a^2\left(y - \frac{v}{R}t_0\right)^2}{\frac{4}{R}\left(\frac{D_0}{2k}(2t - t_0) + D_m\right)t_0}\right] dt_0 \quad (39)$$

Asymptotic Dispersion Coefficient

For asymptotic nondimensional dispersion coefficient, $\alpha(t)$ is given by (29). Substituting (29) in (35), $C(x, y, t)$ can be defined as

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi\left[(D_0 + D_m)(t - t_0) - D_0 k\ln\left(\frac{k + t}{k + t_0}\right)\right]} \cdot \exp\left\{-\mu(t - t_0) - \frac{\left[x - \frac{u}{R}(t - t_0)\right]^2 + a^2\left[y - \frac{v}{R}(t - t_0)\right]^2}{\frac{4}{R}\left[(D_0 + D_m)(t - t_0) - D_0 k\ln\left(\frac{k + t}{k + t_0}\right)\right]}\right\} dt_0 \quad (40)$$

Eq. (40) can be given as follows by integral variable transformation

$$C(x, y, t) = \int_0^t \frac{aC_0}{4\pi\left[(D_0 + D_m)t_0 - D_0 k\ln\left(\frac{k + t}{k + t - t_0}\right)\right]} \cdot \exp\left\{-\mu t_0 - \frac{\left(x - \frac{u}{R}t_0\right)^2 + a^2\left(y - \frac{v}{R}t_0\right)^2}{\frac{4}{R}\left[(D_0 + D_m)t_0 - D_0 k\ln\left(\frac{k + t}{k + t - t_0}\right)\right]}\right\} dt_0 \quad (41)$$

Exponential Dispersion Coefficient

By substituting (32) in (35), one may obtain

$$C(x, y, t) = \int_0^t \left(aC_0/4\pi\left\{(D_0 + D_m)(t - t_0) + D_0 k \cdot \left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t_0}{k}\right)\right]\right\}\right) \cdot \exp\left[-\mu(t - t_0) - \left(\left\{\left[x - \frac{u}{R}(t - t_0)\right]^2 + a^2 \cdot \left[y - \frac{v}{R}(t - t_0)\right]^2\right\} \Big/ \frac{4}{R}\left\{(D_0 + D_m)(t - t_0) + D_0 k \cdot \left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t_0}{k}\right)\right]\right\}\right)\right] dt_0 \quad (42)$$

Eq. (42) may also be written as the following by integral variable transformation:

$$C(x, y, t) = \cdot\int_0^t \frac{aC_0}{4\pi\left\{(D_0 + D_m)t_0 + D_0 k\left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t - t_0}{k}\right)\right]\right\}} \cdot \exp\left[-\mu t_0 - \left(\left(x - \frac{u}{R}t_0\right)^2 + a^2\left(y - \frac{v}{R}t_0\right)^2 \Big/ \frac{4}{R} \cdot \left\{(D_0 + D_m)t_0 + D_0 k\left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t - t_0}{k}\right)\right]\right\}\right)\right] dt_0 \quad (43)$$

POINT INITIAL CONCENTRATION DISTRIBUTION WITH ZERO INJECTION

In this case, the injection rate is assumed to be zero $[q(x, y, t) = 0]$; it is further assumed that the initial nondimensional concentration distribution in the solution domain is zero except at point ($x = 0$, $y = 0$). This condition may be represented as

$$f(x, y) = \frac{M}{nR}\delta(x)\delta(y) \quad (44)$$

On substituting (44) and $q(x, y, t) = 0$ in (19), one may obtain

$$C(x, y, t) = \frac{aM}{4\pi n R\alpha(t)} \cdot \exp\left[-\mu t - \frac{(x - Ut)^2 + (ay - Vt)^2}{4\alpha(t)}\right] \quad (45)$$

Eq. (45) is the same as (21). Thus, solutions for different dispersion coefficient functions can be derived as described earlier.

LINE INITIAL CONCENTRATION DISTRIBUTION WITH ZERO INJECTION

In this case, the initial nondimensional distribution of the solute concentration is represented as

$$f(x, y) = \frac{C_0}{R}\delta(x) \quad (46)$$

This implies that the initial nondimensional concentration is not equal to zero along the y-axis. On substituting (46) and $q(x, y, t) = 0$ in (19), one may obtain

$$C(x, y, t) = \frac{C_0 \exp(-\mu t)}{4\pi R\alpha(t)} \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{(x - Ut)^2 + (ay - Vt - \eta)^2}{4\alpha(t)}\right] d\eta \tag{47}$$

This solution can be given as

$$C(x, y, t) = \frac{C_0 \exp\{-\mu t - [(x - Ut)^2/4\alpha(t)]\}}{4\pi R\alpha(t)} \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{(ay - Vt - \eta)^2}{4\alpha(t)}\right] d\eta \tag{48}$$

Let

$$\psi = \frac{ay - Vt - \eta}{\sqrt{4\alpha(t)}} \tag{49}$$

then, (48) can be written as

$$C(x, y, t) = \frac{C_0\sqrt{4\alpha(t)}}{4\pi R\alpha(t)} \exp\left[-\mu t - \frac{(x - Ut)^2}{4\alpha(t)}\right] \cdot \int_{-\infty}^{\infty} \exp[-\psi^2]\, d\psi \tag{50}$$

Since

$$\int_{-\infty}^{\infty} \exp[-\psi^2]\, d\psi = \sqrt{\pi} \tag{51}$$

Eq. (50) can be written as

$$C(x, y, t) = \frac{C_0}{R\sqrt{4\pi\alpha(t)}} \exp\left[-\mu t - \frac{(x - Ut)^2}{4\alpha(t)}\right] \tag{52}$$

From (52), we can see that $C(x, y, t)$ is not dependent on y. Eq. (52) is similar to the solution given by Bash and El-Habel (1993) for one-dimensional, time-dependent advection-dispersion equations with an initial point pulse at ($x = 0$) without injection. The following particular solutions can now be given for the dispersion-coefficient functions defined earlier.

Constant Dispersion Coefficient

Using (23), one may obtain

$$C(x, y, t) = \frac{C_0}{\sqrt{4\pi(D_0 + D_m)tR}} \cdot \exp\left[-\mu t - \frac{\left(x - \frac{u}{R}t\right)^2}{4(D_0 + D_m)t/R}\right] \tag{53}$$

Linear Dispersion Coefficient

Utilizing (26), one may obtain

$$C(x, y, t) = \frac{C_0}{\sqrt{4\pi\left(\frac{D_0}{2k}t + D_m\right)tR}} \cdot \exp\left[-\mu t - \frac{\left(x - \frac{u}{R}t\right)^2}{4\left(\frac{D_0}{2k}t + D_m\right)t/R}\right] \tag{54}$$

Asymptotic Dispersion Coefficient

Using (29), one may obtain

$$C(x, y, t) = \frac{C_0}{\sqrt{4\pi R\left[(D_0 + D_m)t + D_0 k \ln\left(1 + \frac{t}{k}\right)\right]}} \cdot \exp\left\{-\mu t - \frac{\left(x - \frac{u}{R}t\right)^2}{4\left[(D_0 + D_m)t + D_0 k \ln\left(1 + \frac{t}{k}\right)\right]\Big/R}\right\} \tag{55}$$

Exponential Dispersion Coefficient

Using (32), one may obtain

$$C(x, y, t) = \frac{C_0}{\sqrt{4\pi R\left[(D_0 + D_m)t + D_0 k\left(\exp\left(-\frac{t}{k}\right) - 1\right)\right]}} \cdot \exp\left[-\mu t - \frac{\left[x - \frac{u}{R}t\right]^2}{4\left\{(D_0 + D_m)t + D_0 k\left[\exp\left(-\frac{t}{k}\right) - 1\right]\right\}\Big/\mathrm{R}}\right] \tag{56}$$

EXACT SOLUTIONS FOR INSTANTANEOUS LINE INJECTION

Special Example for One-Dimensional Problem with Instantaneous Point Injection

In this case, $C(x, y, 0) = f(x, y) = 0$. We assume that the injection line is along the y-axis; then, the instantaneous nondimensional injection of a tracer can be defined by

$$q(x, y, t) = \frac{M}{n}\delta(x)\delta(t) \tag{57}$$

Where M = nondimensional mass injected; n = porosity; and $\delta(\ )$ = Dirac delta function. Substituting $f(x, y) = 0$ and (57) in (19), $C(x, y, t)$ can be given as

$$C(x, y, t) = \int_{-\infty}^{\infty} \frac{M \exp(-\mu t)}{4\pi nR\alpha(t)} \cdot \exp\left[-\frac{(x - Ut)^2 + (ay - Vt - \eta)^2}{4\alpha(t)}\right] d\eta \tag{58}$$

Or (58) can be written as

$$C(x, y, t) = \frac{M \exp\{-\mu t + (x - Ut)^2/4\alpha(t)\}}{4\pi nR\alpha(t)\}} \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{(ay - Vt - \eta)^2}{4\alpha(t)}\right] d\eta \tag{59}$$

Let

$$\psi = \frac{ay - Vt - \eta}{\sqrt{4\alpha(t)}} \tag{60}$$

then, (59) takes the following form:

$$C(x, y, t) = \frac{M\sqrt{4\alpha(t)}}{4\pi nR\alpha(t)} \cdot \exp\left[-\mu t - \frac{(x - Ut)^2}{4\alpha(t)}\right] \int_{-\infty}^{\infty} \exp[-\psi^2]\, d\psi \tag{61}$$

Utilizing (51), (61) can be written as

$$C(x, y, t) = \frac{M}{nR\sqrt{4\pi\alpha(t)}} \exp\left[-\mu t - \frac{(x - Ut)^2}{4\alpha(t)}\right] \tag{62}$$

Eq. (62) is similar to (52) if C_0 is replaced by (M/n). From (62), we can see that $C(x, y, t)$ is not dependent on y. Actually, (62) is also the solution for the one-dimensional, time-dependent advection-dispersion equation with a point pulse injection at $x = 0$, with zero initial concentration distribution (Bash and El-Habel 1993). For different dispersion-coefficient functions, the concentration distribution of the solute can be described as in the preceding section.

SOLUTIONS FOR CONTINUOUS LINE SOURCE

Special Example For One-Dimensional Problem with Continuous Point Injection

In this case we assume that the continuous nondimensional line source is located on the y-axis. This condition can be represented as

$$q(x, y, t) = C_0\delta(x) \tag{63}$$

Substituting $f(x, y) = 0$ and (63) into (19), we get

$$C(x, y, t) = \int_0^t \int_{-\infty}^{\infty} \frac{C_0 \exp[-\mu(t - t_0)]}{4\pi R[\alpha(t) - \alpha(t)_0)]} \cdot \exp\left\{-\frac{[x - U(t - t_0)]^2 + (ay - Vt - \eta)^2}{4[\alpha(t) - \alpha(t_0)]}\right\} d\eta\, dt_0 \tag{64}$$

This can be written as

$$C(x, y, t) = \int_0^t \frac{C_0 \exp\left\{-\mu(t - t_0) - \frac{[x - U(t - t_0)]^2}{4[\alpha(t) - \alpha(t_0)]}\right\}}{4\pi R[\alpha(t) - \alpha(t_0)]} \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{(ay - Vt - \eta)^2}{4[\alpha(t) - \alpha(t_0)]}\right\} d\eta\, dt_0 \tag{65}$$

Let

$$\psi = \frac{ay - Vt - \eta}{\sqrt{4[\alpha(t) - \alpha(t_0)]}} \tag{66}$$

then, (65) can be written as

$$C(x, y, t) = \int_0^t \frac{C_0 \exp\left\{-\mu(t - t_0) - \frac{[x - U(t - t_0)]^2}{4[\alpha(t) - \alpha(t_0)]}\right\}}{\pi R\sqrt{4[\alpha(t) - \alpha(t_0)]}} \cdot \int_{-\infty}^{\infty} \exp(-\psi^2)\, d\psi\, dt_0 \tag{67}$$

On substituting (51) in (67), $C(x, y, t)$ can be obtained as

$$C(x, y, t) = \int_0^t \frac{C_0}{R\sqrt{4\pi[\alpha(t) - \alpha(t_0)]}} \cdot \exp\left\{-\mu(t - t_0) - \frac{[x - U(t - t_0)]^2}{4[\alpha(t) - \alpha(t_0)]}\right\} dt_0 \tag{68}$$

$C(x, y, t)$ in this case is not dependent on y. Eq. (68) is also the solution for a one-dimensional problem with continuous injection at point $x = 0$, with concentration C_0. Special cases for this solution can be given as shown in the following section.

Constant Dispersion Coefficient

Using (23), one may obtain

$$C(x, y, t) = \int_0^t \frac{C_0}{R\sqrt{4\pi(D_0 + D_m)t_0/R}} \cdot \exp\left[-\mu t_0 - \frac{R(x - Ut_0)^2}{4(D_0 + D_m)t_0}\right] dt_0 \tag{69}$$

Linear Dispersion Coefficient

Utilizing (26), one may obtain

$$C(x, y, t) = \int_0^t \frac{C_0}{R\sqrt{4\pi\left[\frac{D_0}{2k}(t^2 - t_0^2) + D_m(t - t_0)\right]\Big/R}} \cdot \exp\left\{-\mu(t - t_0) - \frac{R[x - U(t - t_0)]^2}{4\left[\frac{D_0}{2k}(t^2 - t_0^2) + D_m(t - t_0)\right]}\right\} dt_0 \tag{70}$$

Asymptotic Dispersion Coefficient

Using (29), one may obtain

$$C(x, y, t) = \int_0^t \frac{C_0}{R\sqrt{4\pi\left[(D_0 + D_m)(t - t_0) - D_0 k \ln\left(\frac{k + t}{k + t_0}\right)\right]\Big/R}} \cdot \exp\left\{-\mu(t - t_0) - \frac{R[x - U(t - t_0)]^2}{4\left[(D_0 + D_m)(t - t_0) - D_0 k \ln\left(\frac{k + t}{k + t_0}\right)\right]}\right\} dt_0 \tag{71}$$

Exponential Dispersion Coefficient

Utilizing (32), one may obtain

$$C(x, y, t) = \int_0^t \left(C_0 \Big/ R\sqrt{4\pi\left\{(D_0 + D_m)(t - t_0) - D_0 k\left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t_0}{k}\right)\right]\right\}\Big/R}\right) \cdot \exp\left[-\mu(t - t_0) - \left(R[x - U(t - t_0)]^2/4\left\{(D_0 + D_m)(t - t_0) - D_0 k \cdot \left[\exp\left(-\frac{t}{k}\right) - \exp\left(-\frac{t_0}{k}\right)\right]\right\}\right)\right] dt_0 \tag{72}$$

NUMERICAL EXAMPLES

In the foregoing sections, a catalog of analytical solutions for a large class of solute transports problems are described. Due to space limitations numerical examples for all cases cannot be discussed in this paper. Thus, in order to evaluate contaminant migration patterns for the asymptotically varying dispersion-coefficient case, we have selected the analytical solution for the instantaneous point-source problem arbitrarily. The values of the parameters R, μ, D_m, D_0, M, n, u, and v used in this case study were 1.0, 0.0, 0.0, 1.0, 0.25, 0.25, 0.25, and 0.0, respectively.

The results shown in Figs. 1–6 correspond to the case of instantaneous nondimensional point source with zero initial nondimensional concentration distribution [(30)] for $a^2 = 1$. In Fig. 1, numerical results obtained for $y = 0$ are summarized. In this solution $k = 0$ corresponds to a constant nondimensional dispersion coefficient. Similarly, results obtained for $y = 2$ and $y = 5$ are given in Figs. 2 and 3, respectively. For the preceding parameters, analytical results obtained for

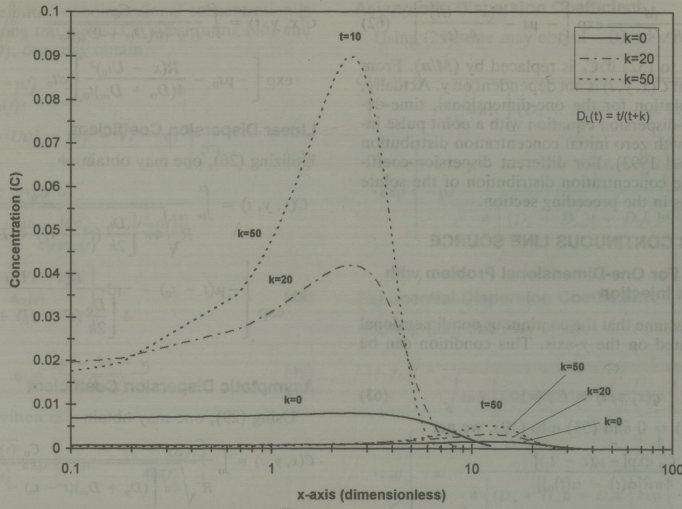

FIG. 1. Dimensionless Concentration Profiles as Function of Time and x at $y = 0$ for Instantaneous Point Source ($a^2 = 1$)

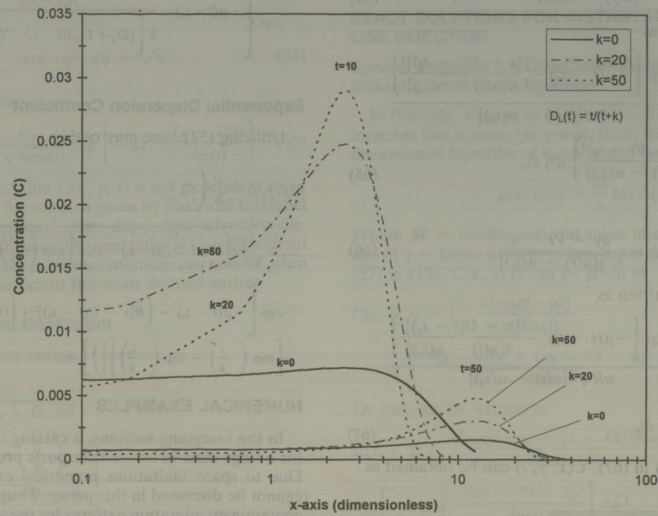

FIG. 2. Dimensionless Concentration Profiles as Function of Time and x at $y = 2$ for Instantaneous Point Source ($a^2 = 1$)

$a^2 = 1$, indicate that for small values of time ($t = 10$), as k increases from zero to 20, peak concentration magnitudes increase approximately fivefold in the longitudinal direction ($y = 0$) and about sixfold in the transverse direction for ($y = 2$). Again, for small times ($t = 10$), for $k = 50$ this increase may reach to levels of tenfold in the longitudinal direction ($y = 0$) and fivefold for ($y = 2$). For $y = 2$, the peak concentration for the $k = 50$ case is less than the peak concentration for $k = 20$. The results summarized in Fig. 3 indicate that, relative to the $k = 0$ solution, the concentration magnitudes do not increase in the transverse direction for ($y = 5$). For the $k = 20$ and $k = 50$ cases, the peak concentrations are less than the results obtained for $k = 0$. For large distances or large time ($t = 50$), the increase in the concentration levels are not as large in the longitudinal direction ($y = 0$); however, in the transverse direction ($y = 2$) and ($y = 5$), fivefold and threefold increases are expected.

Thus, for early times and for the case of the asymptotically varying dispersion coefficient ($k = 0$ to $k = 50$), significant concentration increases are expected in the longitudinal direction, whereas concentration magnitudes do not increase as much, or even reduce, as one moves in the transverse direction. On the other hand, for large times, for the same variation in the dispersion coefficient, the increase in concentration magnitudes in the longitudinal direction is not significant, whereas the increase in the transverse direction becomes significant. The reversal of the increase in peak concentrations in the transverse direction, for large k-values and for small times, and the increase of transverse concentration for large times is a trend that is repeatedly observed for other solutions. In all cases, for $a^2 = 1$, the travel distance of the peak concentration was not altered.

This observation is illustrated in Figs. 4–6, in which the peak concentration is plotted as a function of dimensionless time for $y = 0$, $y = 2$, and $y = 5$, respectively. From Fig. 4, one can observe that peak concentrations are higher for early times in the longitudinal direction and as the time increases for all k-values, the peak concentration asymptotically reduces to the level of peak concentration for the case of $k = 0$. In the transverse direction the variation of peak concentration with time shows a different trend. The arrival of higher peak concentrations in the transverse direction, such as at $y = 2$ and $y = 5$, is not necessarily at early times, but may occur at much later stages of the transport period. An-

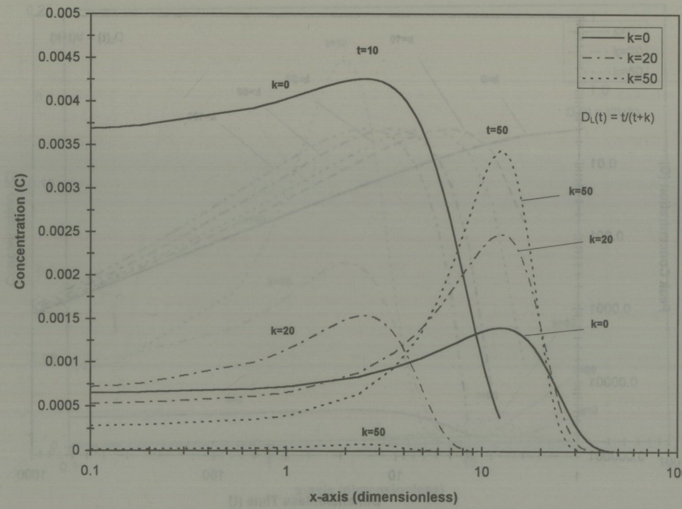

FIG. 3. Dimensionless Concentration Profiles as Function of Time and x at $y = 5$ for Instantaneous Point Source ($a^2 = 1$)

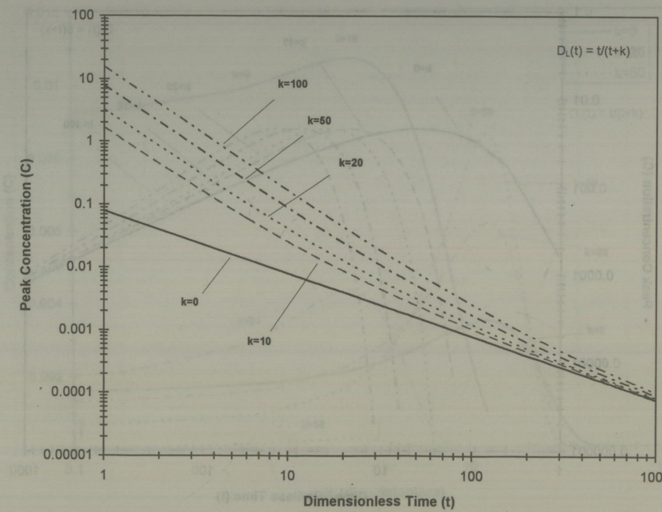

FIG. 4. Variation of Peak Dimensionless Concentration as Function of Time and k for Instantaneous Point Source ($a^2 = 1$)

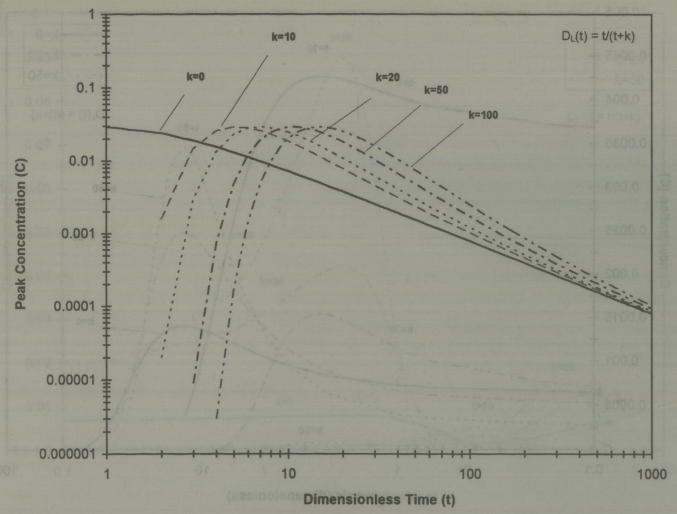

FIG. 5. Variation of Peak Dimensionless Concentration as Function of Time and k at $y = 2$ for Instantaneous Point Source ($a^2 = 1$)

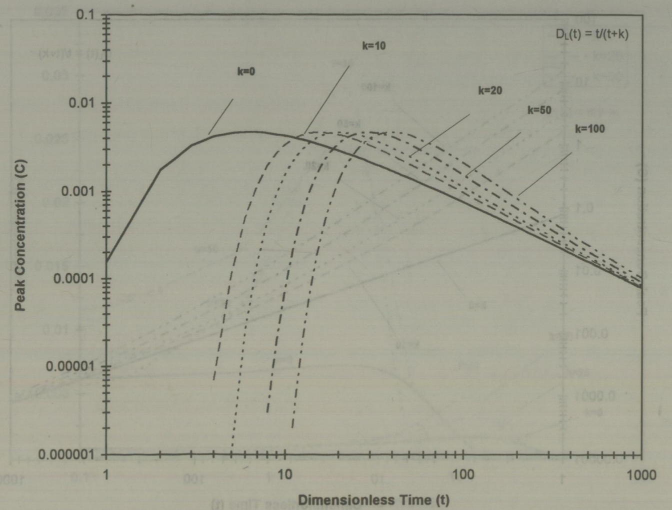

FIG. 6. Variation of Peak Dimensionless Concentration as Function of Time and k at $y = 5$ for Instantaneous Point Source ($a^2 = 1$)

other observation evident from Figs. 4–6 is that, for the case considered here, the time-dependent nature of the dispersion coefficient does not affect the concentration distribution solution at very large times.

Similarly, numerical results obtained for the case $a^2 = 6$ are given in Figs. 7–12. Conclusions derived for this case follow the aforementioned pattern. These results show that the relation between time-dependent dispersivity values and contaminant distribution in the longitudinal and transverse directions is complex. The general trend observed is the reversal of the advective-dispersive expansion patterns as k or y increases as a function of time. For asymptotically varying dispersion coefficients, the general pattern is that concentration magnitudes increase in the longitudinal direction during early times. However, similar increases are expected in the transverse direction only in later solution periods and thus at large distances.

Numerical results for instantaneous point-source problem, based on the exponentially varying dispersivity coefficient, showed similar trends in our analysis. These results are not included here due to space limitations. Results for other problems may be obtained in a similar manner using the foregoing analytical solutions.

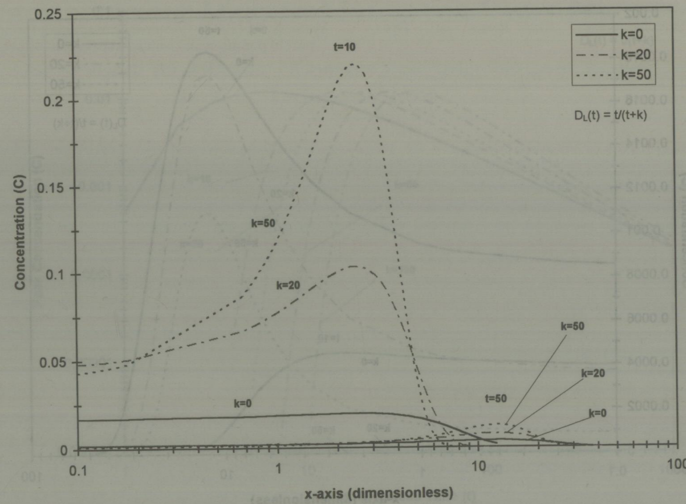

FIG. 7. Dimensionless Concentration Profiles as Function of Time and x at $y = 0$ for Instantaneous Point Source ($a^2 = 6$)

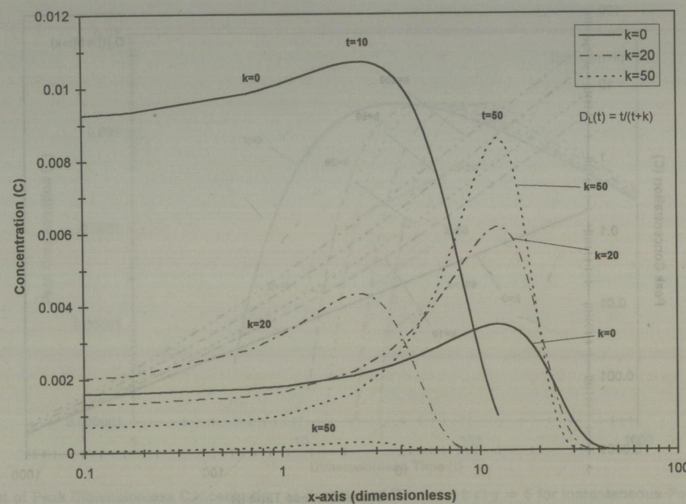

FIG. 8. Dimensionless Concentration Profiles as Function of Time and x at $y = 2$ for Instantaneous Point Source ($a^2 = 6$)

CONCLUSIONS

General analytical solutions for the two-dimensional advection-dispersion equation with time-dependent dispersion coefficients were presented. These solutions can be employed to obtain particular solutions for several time-dependent dispersion-coefficient functions and also for various injection and initial distribution functions. It was shown analytically that the point pulse initial distribution solution and the instantaneous point injection case tend toward similar solutions. As special cases, the analytical solutions for instantaneous line injection and continuous line injection cases were also given. It was shown that both these cases yield solutions similar to that of a one-dimensional problem with instantaneous point injection and continuous point injection. For these cases, it was shown that the analytical solutions presented for the four special dispersion-coefficient functions yield the same solutions as those given by Basha and El-Habel (1993).

In the experimental and analytical work conducted by several researchers, it was shown that there are two important scales in the analysis of the effects of dispersion: (1) The small

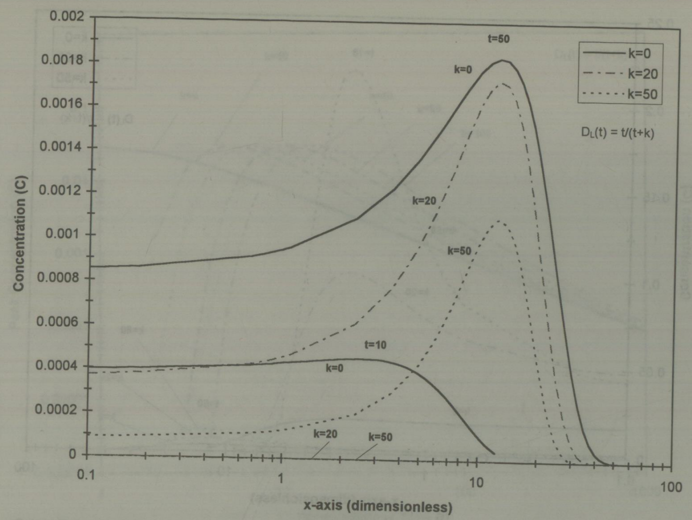

FIG. 9. Dimensionless Concentration Profiles as Function of Time and x at y = 5 for Instantaneous Point Source (a^2 = 6)

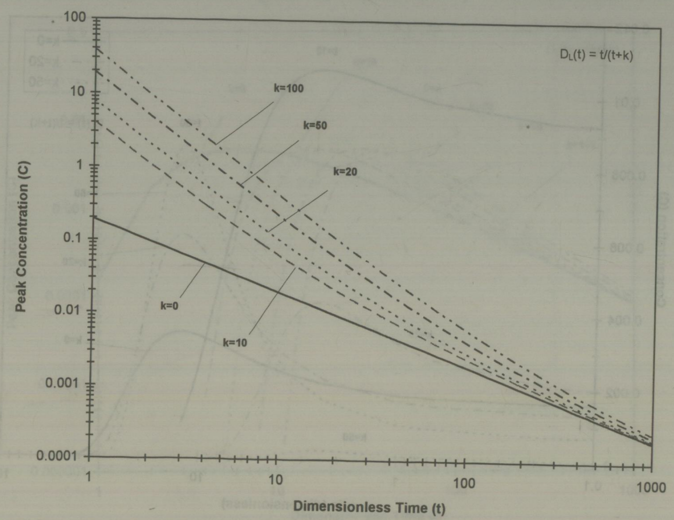

FIG. 10. Variation of Peak Dimensionless Concentration as Function of Time and k for Instantaneous Point Source (a^2 = 6)

time scale where the dispersivity grows with time and position; and (2) the larger time scale when the dispersion coefficient becomes constant. These studies also indicate that both longitudinal and transverse dispersions of contaminants will be influenced during the small time scale in which dispersivity is variable, and that the asymptotic behavior of the dispersion coefficient can be reached only for unrealistically large times or distances, and, thus the so-called preasymptotic period is important in many practical applications.

Based on the results of the aforementioned case studies, for the preasymptotic period, it can be concluded that the effect of time-dependent dispersion coefficients on the contaminant dispersion problem is not the same in both the longitudinal and transverse directions. While time-dependent dispersion coefficients increase the concentration magnitudes in the longitudinal direction in the preasymptotic period, similar increases are not observed in the transverse direction during the same periods. Instead, comparable increases in concentration levels occur at much larger times in the transverse direction. Thus, scale-dependence effects on contaminant dispersion in the longitudinal and transverse directions do not follow the same pattern. For the case studies discussed

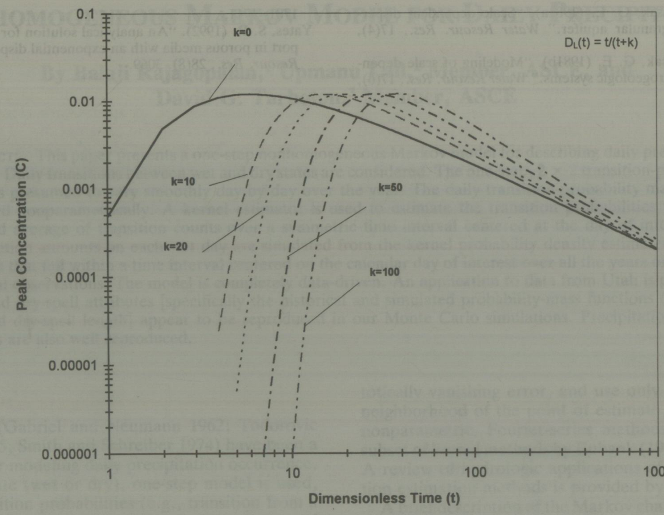

FIG. 11. Variation of Peak Dimensionless Concentration as Function of Time and k at y = 2 for Instantaneous Point Source (a^2 = 6)

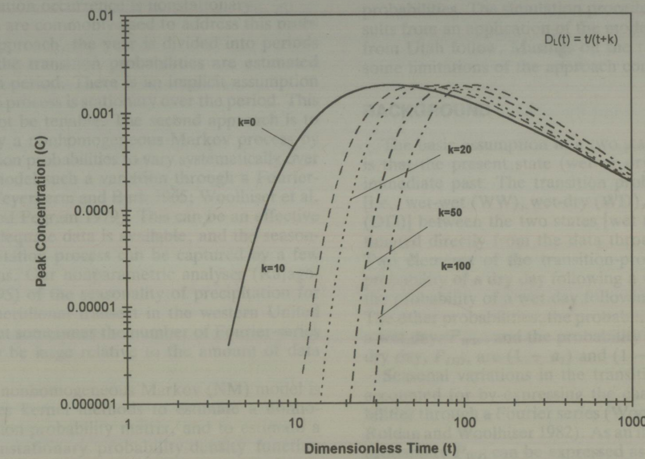

FIG. 12. Variation of Peak Dimensionless Concentration as Function of Time and k at y = 5 for Instantaneous Point Source (a^2 = 6)

here, for very large times, the analytical solutions indicate that the time-dependent nature of the dispersion equation do not significantly influence the contaminant migration pattern in both the longitudinal and transverse directions.

The analytical solutions presented in this paper are benchmark solutions for scale-dependent dispersivity problems for contaminant transport analysis in two-dimensional domains. These solutions may be used to analyze problems where such scale dependence is of concern. Thus, these solutions may also be used to provide tools to evaluate field data where the scale dependence of the dispersion coefficient is expected to influence contaminant migration patterns in an aquifer.

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Nonhomogeneous Markov Model for Daily Precipitation

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Abstract: This paper presents a one-step nonhomogeneous Markov model for describing daily precipitation at a site. Daily transitions between wet and dry states are considered. The one-step, 2 × 2 transition-probability matrix is presumed to vary smoothly day by day over the year. The daily transition-probability matrices are estimated nonparametrically. A kernel estimator is used to estimate the transition probabilities through a weighted average of transition counts over a symmetric time interval centered at the day of interest. The precipitation amounts on each wet day are simulated from the kernel probability density estimated from all wet days that fall within a time interval centered on the calendar day of interest over all the years of available historical observations. The model is completely data-driven. An application to data from Utah is presented. Wet- and dry-spell attributes [specifically the historical and simulated probability-mass functions (PMFs) of wet- and dry-spell length] appear to be reproduced in our Monte Carlo simulations. Precipitation amount statistics are also well reproduced.

INTRODUCTION

Markov chains (Gabriel and Neumann 1962; Todorovic and Woolhiser 1975; Smith and Schreiber 1974) have been a popular method for modeling daily precipitation occurrence. Typically a two-state (wet or dry), one-step model is used, and the state transition probabilities (e.g., transition from a wet day to a wet day, a wet day to a dry day) are estimated from the data. One problem with such a description is that the transition probabilities may vary over the year, i.e., the process of precipitation occurrence is nonstationary.

Two approaches are commonly used to address this problem. In the first approach, the year is divided into periods (or seasons) and the transition probabilities are estimated separately for each period. There is an implicit assumption that the occurrence process is stationary over the period. This assumption may not be tenable. The second approach is to consider essentially a nonhomogeneous Markov process by allowing the transition probabilities to vary systematically over the year, and to model such a variation through a Fourier-series expansion (Feyerherm and Bark 1965; Woolhiser et al. 1973; Woolhiser and Pegram 1979). This can be an effective approach where adequate data is available, and the seasonality in the precipitation process can be captured by a few Fourier-series terms. Our nonparametric analyses (Rajagopalan and Lall 1995) of the seasonality of precipitation for stations along a meridional transect in the western United States, suggests that sometimes the number of Fourier-series terms needed may be large relative to the amount of data available.

In this paper, a nonhomogeneous Markov (NM) model is presented that uses kernel methods to estimate a nonhomogeneous transition-probability matrix, and to estimate a corresponding nonstationary probability-density function (PDF) of daily precipitation amount. Kernel methods are local, weighted averages of the target function (relative frequency of occurrence in this case). Since they are capable of approximating a wide variety of target functions with asymptotically vanishing error, and use only data from a "small" neighborhood of the point of estimate, they are considered nonparametric. Fourier-series methods are shown to be a subset of kernel methods by Eubank (1988, secs. 3.4 and 4.1). A review of hydrologic applications of nonparametric function estimation methods is provided by Lall (1995).

A brief description of the Markov chain and its terminology is first presented as a background to motivate our formulation. The general structure of the NM model proposed is next outlined with the nonparametric estimators for the transition probabilities. The simulation procedure is then outlined. Results from an application of the model to a precipitation data from Utah follow. Musings on the results and discussion of some limitations of the approach conclude the paper.

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BACKGROUND

The basic assumption in a two state Markov-chain model is that the present state (wet or dry) depends only on the immediate past. The transition probabilities for transitions [i.e., wet-wet (WW), wet-dry (WD), dry-wet (DW), dry-dry (DD)] between the two states [wet (W) or dry (D)] are estimated directly from the data through a counting process. Two elements of the transition-probability matrix are the probability of a dry day following a wet day, $P_{WD} = a_1$, and the probability of a wet day following a dry day, $P_{DW} = a_2$. The other probabilities, the probability of a wet day following a wet day, P_{WW}, and the probability of a dry day following a dry day, P_{DD}, are $(1 - a_1)$ and $(1 - a_2)$, respectively.

Seasonal variations in the transition probabilities can be accounted for by expressing the changing transition probabilities through a Fourier series (Woolhiser and Pegram 1979; Roldan and Woolhiser 1982). As an illustration, the transition probability P_{WD} can be expressed as follows:

$$P_{WD}(t) = \bar{P}_{WD} + \sum_{k=1}^{m} c_k \sin(2\pi tk/365 + \theta_k), \quad t = 1, 2, \ldots, 365 \tag{1}$$

where m = maximum number of harmonics required to describe the seasonal variability of the transition probability; $\bar{P}_{WD}$ = annual mean value of the parameter; c_k = amplitude; and θ_k = phase angle in radians for the kth harmonic.

The means, amplitudes, and phase angles are estimated by numerical optimization of the log-likelihood function, as described by Woolhiser and Pegram (1979) and Roldan and Woolhiser (1982). Fourier-series representations of parameters of a first-order Markov chain for precipitation have been used (among others) by Feyerherm and Bark (1965), who