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Review papers

Cellular Automata and Finite Volume solvers converge for 2D shallow flow modelling for hydrological modelling

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ABSTRACT

Surface flows of hydrological interest, including overland flow, runoff, river and channel flow and flooding have received significant attention from modellers in the past 30 years. A growing effort to address these complex environmental problems is in place in the scientific community. Researchers have studied and favoured a plethora of techniques to approach this issue, ranging from very simple empirically-based mathematical models, to physically-based, deductive and very formal numerical integration of systems of partial-differential equations. In this work, we review two families of methods: cell-based simulators – later called Cellular Automata – and Finite Volume solvers for the Zero-Inertia equation, which we show to converge into a single methodology given appropriate choices. Furthermore, this convergence, mathematically shown in this work, can also be identified by critically reviewing the existing literature, which leads to the conclusion that two methods originating from different reasoning and fundamental philosophy, fundamentally converge into the same method. Moreover, acknowledging such convergence allows for some generalisation of properties of numerical schemes such as error behaviour and stability, which, importantly, is the same for the converging methodology, a fact with practical implications. Both the review of existing literature and reasoning in this work attempts to aid in the effort of synchronising and cross-fertilizing efforts to improve the understanding and the outlook of Zero-Inertia solvers for surface flows, as well as to help in clarifying the possible confusion and parallel developments that may arise from the use of different terminology originating from historical reasons. Moreover, synchronising and unifying this knowledge-base can help clarify model capabilities, applicability and modelling issues for hydrological modellers, specially for those not deeply familiar with the mathematical and numerical details.

1. Introduction

The simulation of spatially-distributed and time-dependant surface flow processes – with interests on flood modelling, runoff modelling and geomorphology – has been approached with different levels of complexity. One of the simplest and general approaches are cell-based methods, often (but not always) termed the Cellular Automata (CA) approach, originally proposed by Von Neumann (1966) in the context of computationally mimicking biological behaviours. In this approach, an individual automaton – a discrete entity with properties – communicates with its neighboring automata through some prescribed rules of interaction (Fonstad, 2006) – which may be argued to be fluxes. Clearly, this requires to define what is meant by "neighborhood" and what are such rules, which in turn, obviously depends on the intended application. Within the plethora of applications for CA models, only those on runoff and surface flow phenomena are of interest to this work. One of the earliest works in this context is that of Murray and Paola (1994) who attempted to model braided rivers using CA. In their approach, CA was used to discretise space and interaction rules were implemented for both water and sediment flows. Such rules depended basically on bed slope, and were rather convenient conceptual formulations. Other authors have chosen different intercell fluxes, depending on their interest. For example, Cai et al. (2014) chose a broad-crested weir rating curve to describe the intercell flux. Ghimire et al. (2013) proposed fluxes based on a cascade volume transfer strategy among pre-determined cell sets, including a relaxation parameter to aggressively damp numerical oscillations. On the other side, models spawning from simplified shallow-water dynamics have also been present. Solving the full shallow water equations (SWE) can be challenging and computationally costly, in particular for large domains typical of hydrological problems (García-Navarro, 2016). Although this issue is currently being addressed through the development of advanced numerical strategies and the use parallel and GPU (Graphical Processing Unit) computing (Kesserwani et al., 2016), alternative

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Table 1
CA and FV models.

Reference	Termed	Discretisation		Flux	N	Application	Comment
		Space	Time				
Murray and Paola (1994)	CA	CA	E	Ad-hoc routing	DS	Braided river morphology	Routs to three downstream cells
Julien et al. (1995)	RB	FV	E	ZI	VN	Hydrological modelling	Known as CASC2D
Lal (1998)	FD	FD	E/I	ZI	–	Overland flow	Unstructured meshes
Bates and Roo (2000)	RBSC	FV/CA	E	ZI	VN	Flood modeling	Includes 1D kinematic wave solver
D'Ambrosio et al. (2001)	CA	CA	E	Gradient minimization	VN	Erosion	–
Thomas and Nicholas (2002)	Cellular routing	CA	E	Ad-hoc routing	DS	Braided river morphology	Routs to five downstream cells
D'Ambrosio et al. (2003)	CA	CA	E	Gradient minimization + kinetic energy	–	Debris flow	Hexagonal cells
Panday et al. (2004)	FD-ZI	FD	I	ZI	VN	Coupled surface-subsurface flow	Manning, Darcy and Chezy friction laws
Hunter et al. (2005)	RBSC	FD	E	ZI	VN	Flood modelling	Stability issues on complex topography
Kollet and Maxwell (2006)	FV	FV	I	Kinematic wave	VN	Coupled surface-subsurface flow	Rainfall-runoff assessment
Parsons and Fonstad (2007)	CA	CA	E	Manning	VN	Rainfall-runoff	–
Rinaldi et al. (2007)	CA	CA	E	Downslope routing	M	Flood modelling	–
Wiel et al. (2007)	CA	CA	SE	Water surface slope + Manning	VN	Alluvial Geomorphology	Flow-sweep algorithm
Weill et al. (2009)	FE	FE	I	ZI	–	Coupled surface-subsurface flow	ZI-Richards similarity (Generalized Richards)
Cea et al. (2010)	FV-DW	FV	E	ZI	VN	Urban rainfall-runoff	Square and triangular cells
Bates et al. (2010)	Inertial	FD	E	ZI + inertia	VN	Flood modelling	Keeps an inertial term
Aricò et al. (2011)	FV-DW	FV	E	ZI	–	Flood modelling	Fractional time-stepping
Dottori and Todini (2011)	CA	FV	E	ZI	VN	Flood modelling	Local time stepping
Wang et al. (2011)	ZI	FV	E	ZI	VN	Flood modelling	Low CFL may be enough for stability
Lopez-Barrera et al. (2012)	FV	FV	E	ZI	VN	Hydrological modelling	Hyperbolic-like approach
Baartman et al. (2012)	CA	CA	E	Gradient based	VN	Land evolution	–
Ghimire et al. (2013)	CA	CA	E	Ranked-cells outflow scheme	VN	Urban pluvial flood modelling	Velocity limited by Manning
Cai et al. (2014)	CA	CA	E	Weir rating curve	M	Flood modelling	–
Leandro et al. (2014)	FV	FV	E	ZI	VN	Flood modelling	Stability study and parallelization
Mendicino et al. (2015)	Macroscopic CA	CA	E	ZI	VN	Ecohydrological simulation	Stability discussion
Shao et al. (2015)	CA	CA	E	Travel time + Manning	M	Rainfall-runoff	–
Liu et al. (2015)	CA	CA	E	Gradient minimization + Manning	VN	Urban floods	–
Fernández-Pato and García-Navarro (2016)	FV-ZI	FV	E/I	ZI	–	Rainfall-runoff	Triangular cells
Costabile et al. (2017)	FV-DW	FV	E	ZI	–	Urban flood modelling	Triangular cells
Jahanbazi et al. (2017)	OFS-CA	FV	E	ZI	VN	Flood modelling and runoff	Novel corrections for stability and efficiency

CA: Cellular Automata, FV: Finite Volumes, FD: Finite Differences, FE: Finite Elements, RB: Raster-based, RBSC: Raster-based storage cell, ZI: Zero-Inertia, DW: Diffusive-wave, N: Neighborhood; VN: von Neumann, M: Moore, DS: Downslope, E: Explicit, I: Implicit, SE: Semi explicit.

approaches, which are mathematically, numerically and computationally simpler have also been historically adopted to make simulation of these types of problems feasible and accessible. Various studies have explored the Zero-Inertia (ZI) – also often inaccurately termed Diffusive Wave (Yen and Tsai, 2001) – model, with different numerical strategies (Costabile et al., 2017; Dottori and Todini, 2011; Fernández-Pato et al., 2016; Julien et al., 1995; Panday et al., 2004, e.g.).

The growing and very recent literature on both cellular-automata (CA) and finite-volumes (FV) based solvers clearly indicates that this remains an active field, and that an effort is required so that several communities and methods may effectively converge. Most importantly, the growing use of these models for sophisticated, spatially distributed, large scale hydrological simulation prompts the need to robustly identify key advantages and disadvantages of the underlying numerical approaches, and requires for modellers to be deeply aware of the applicability and assumptions of the models available to them and a working understanding of the underlying numerics of their computational tools. This needs motivate this review. In particular, we set out to draw attention and clearly show how some so-called Cellular-Automata solvers are exactly the same as the explicit Finite Volume solvers of the ZI equation. This is the main contribution of this work, which has the significant implication that a vast knowledge base can be brought together thus aiding hydrological modellers to better understand the available computational tools. In particular, it is noteworthy that stability and error properties of the two solvers are the same and well known, and that they differ from the less-studied stability properties of other – perhaps less formal – cell-based routing models. In order to advocate that effort, it is our goal to review and summarize the contributions from both communities to what is in fact, the same numerical approach to the same mathematical approximation of the shallow water equations. To do so, we derive both a CA simulator from fundamental discrete principles in Section 2, we derive the mathematical and numerical expressions for a FV-ZI solver in Section 3 and we finally show and discuss the equivalence between methods, review and classify a plethora of existing models reported as CA and FV in the literature, and compare and contrast them in Section 4. 5 briefly summarises key insights and outlook.

2. The Cellular Automata approach for surface flows

A general form for the CA state evolution rule (Cai et al., 2014) for a state variable $\mathscr{S}$ is

$$\mathscr{S}_i^{n+1} = f(\mathscr{S}_i^n, \mathscr{S}_j^n) \tag{1}$$

where j denotes all the neighboring cells surrounding cell i. Superindex n denotes the current time level, and $n+1$ the future time. Eq. (1) simply states that the future state of a cell depends on the current state of that cell and the states of its neighbors. How each neighbor influences the cell of interest is determined by the function f. Such function can be very simple and may only require to evaluate the states of the cells, without the need for a rate of information exchange. On the other hand, in the context of fluid motion, the state variables are indeed exchanged between cells at particular rates, called fluxes $\mathscr{F}$. By assuming that the interaction function f depends on such fluxes, and that information is exchanged during a time interval Δt, it is possible to write Eq. (1) as

$$\mathscr{S}_i^{n+1} = \mathscr{S}_i^n + \Delta t\left(\mathscr{I}_i^n + \sum_{j=1}^{N_j} \mathscr{F}_{ij}^n\right) \tag{2}$$

where i indicates a cell, which is connected to N_j other cells in a prescribed neighborhood (or stencil). Any neighboring cell j will exchange information with cell i through the local flux $\mathscr{F}_{ij}$ which may depend on the states of both cell i and j. In Eq. (2) information can also change due to sources and sinks through $\mathscr{I}$, which is a rate of change of the quantity $\mathscr{S}$. In general, the source term $\mathscr{I}$ and the flux $\mathscr{F}$ may or may not depend on the state variables.

Eq. (2) is the evolution equation at the core of CA-based models, in particular those concerned with flood problems. From a discretisation point of view, it is relevant to point out that Eq. (2) can be posed with no inherent knowledge of what information is being exchanged nor how, i.e., what exactly $\mathscr{S}$ or $\mathscr{F}$ are. This is a generic concept in the CA method. Moreover, the reader should note that Eq. (2) is an explicit time discretisation, since the state at time $n+1$ is computed based exclusively on states at time n. Moreover, if dealing with a spatial problem – as surface flow modelling is, cells represent regions in space. Typically, in two-dimensional CA models, square cells are chosen for convenience to discretise space, although there is no particular requirement for this (Ortigoza, 2015).

Particular CA methods, such as those listed in Table 1 can be obtained by choosing a suitable state variable $\mathscr{S}$ and formulating $\mathscr{F}$ accordingly and conveniently, thus defining the rules for information exchange. Herein the interest is on surface flow modelling, in particular models dealing with runoff and floods. To proceed in such direction, let the state variable be water volume V $[L^3]$ in a cell

$$V_i^{n+1} = V_i^n + \Delta t\left(\mathscr{I}_i^n + \sum_{j=1}^{N_j} \mathscr{F}_{ij}^n\right) \tag{3}$$

At this point, $\mathscr{F}$ and $\mathscr{I}$ *must* be volumetric fluxes (rates of change of volume) $[L^3/T]$ for the equation to be consistent. Let the problem be two-dimensional, therefore cells can be characterized by their area A $[L^2]$. Dividing Eq. (4) by the area

$$h_i^{n+1} = h_i^n + \frac{\Delta t}{A}\left(\mathscr{I}_i^n + \sum_{j=1}^{N_j} \mathscr{F}_{ij}^n\right) \tag{4}$$

where $h = V/A$ $[L]$ is the piece-wise constant water depth in a cell. Observe that the meaning of $\mathscr{I}$ and $\mathscr{F}$ remains unchanged, they are still volumetric fluxes. At this point, the choice of $\mathscr{F}$ creates a divergence between methods. A simple possibility is to select $\mathscr{F} = f(z(x,y))$, i.e., the volumetric flux depends only on surface elevation $z(x,y)$, or more precisely on the slopes ∇z (Baartman et al., 2012). In such case, the fluxes depend only on the topography itself and the definition of the neighborhood, which sometimes contains only four cells (direct neighbors in both Cartesian directions), referred to as the von Neumann neighborhood, and in other cases it contains eight cells – the Moore neighborhood, which expands on the von Neumann neighborhood to also include the direct neighbor cells at a 45 degree angle from Cartesian directions. Herein, only the von Neumann neighborhood is considered since it is the most common in overland flow models.

It is also possible to formulate the flux as $\mathscr{F} = f(h(x,y), z(x,y))$, meaning that both topography and water depth play a role. This, in turn, allows for different levels of complexity in the representation of the physical processes, leading to different reliability and applicability of models. It is at this point in the mathematical modelling process where many of the surface-flow-oriented CA models diverge. Broadly, two sets of models can be distinguished. Firstly, those attempting to capture physically the competing forces which drive the flow – gravity, friction, inertia, and those which neglect explicitly recalling the driving forces but prefer simpler proxies and in the process assuming particular flow conditions for which such proxies are valid. In the first group, notably, many authors developing CA models have chosen to keep gravity and friction forces, but have neglected inertia, thus assuming pseudo-steady flow conditions in which friction and gravity balance each other. Thus, the flow rates can be represented with well-known friction laws such as Chezy, Darcy-Weisbach, and – most frequently – Manning's equation. In general terms, such friction laws yield a flux in the form of

$$\mathscr{F} = \alpha(h) L \sqrt{\frac{\delta(h+z)}{d}} \tag{5}$$

where α depends on the selected friction formulation. Most CA models are based on Manning's equation, which imposes

$$\alpha(h) = \frac{h^{5/3}}{n} \tag{6}$$

allowing to write the flux as

$$\mathscr{F} = \frac{L h^{5/3}}{n} \sqrt{\frac{\delta(h+z)}{d}} \tag{7}$$

which, substituted in (4) results in

$$h_i^{n+1} = h_i^n + \frac{\Delta t}{A_i}\left(\mathscr{I}_i^n + \sum_{j=1}^{N_j} \frac{L h^{5/3}}{n} \sqrt{\frac{\delta(h+z)}{d}}\right) \tag{8}$$

By defining the water surface gradient between two cells as

$$Z_\omega^n = \frac{h_j - h_i + z_j - z_i}{d_\omega} \tag{9}$$

and manipulating together with Eq. (8) slightly, yields

$$h_i^{n+1} = h_i^n + \Delta t I_i^n + \frac{\Delta t}{A_i} \sum_{j=1}^{N_j} \frac{h^{5/3}}{n\sqrt{||Z_\omega^n||}} Z_\omega^n L_\omega \tag{10}$$

Defining the surface gradient vector $\mathbf{Z}_\omega$ across an edge ω with normal vector $\boldsymbol{n}_\omega$, so that

$$\mathbf{Z}_\omega^n \cdot \boldsymbol{n}_\omega = Z_\omega \tag{11}$$

allows to rewrite Eq. (10) into vector form

$$h_i^{n+1} = h_i^n + \Delta t I_i^n + \frac{\Delta t}{A_i} \sum_{j=1}^{N_j} \frac{h^{5/3}}{n\sqrt{||\mathbf{Z}_\omega^n||}} \mathbf{Z}_\omega^n \cdot \boldsymbol{n}_\omega L_\omega \tag{12}$$

where $I = \mathscr{I}/A$ are source term contributions per unit area $[L/T]$, which can account for rain rate r $[L/T]$ and infiltration/exfiltration rate i $[L/T]$, so that $I = r - i$, for example. Eq. (12) has been termed often in the literature as Cellular Automata solver of the diffusive-wave equation (CA-DW). It should be noted that the accurate terminology for Eq. (12) should be a CA solver for the Zero-Inertia equation (CA-ZI), as was pointed out by Yen and Tsai (2001), issue on which more details are discussed later on.

3. A Finite Volume approach for surface flows

3.1. Zero-Inertia approximation

The Zero-Inertia (ZI) assumption to simplify the shallow-water equations requires to neglect acceleration terms in the momentum equation (Fernández-Pato and García-Navarro, 2016), thus resulting in a steady momentum equation, in which only hydrostatic, bed and friction terms appear

$$\mathbf{Z} = \sigma \tag{13}$$

where $\mathbf{Z} = -\nabla(h + z)$ is the surface gradient, h $[L]$ is depth, z $[L]$ is bed elevation and σ $[-]$ is the friction slope. Eq. (13) accompanies the mass conservation equation

$$\frac{\partial h}{\partial t} + \nabla(h\boldsymbol{q}) = I \tag{14}$$

where $\boldsymbol{q} = (qx, qy)$ $[L^2/T]$ is the unit-discharge (momentum), $I\,[L/T]$ is a source term which can account, for example, for rain $r\,[L/T]$ and infiltration $i\,[L/T]$ rates.

The friction slope $\sigma = [\sigma_x, \sigma_y]$, and its equality to the water surface slope (Eq. (13)) allow to express unit-discharge $\boldsymbol{q}$ – generically – in terms of the water surface gradient $\mathbf{Z}$

$$\boldsymbol{q} = \alpha(h)\frac{\mathbf{Z}}{||\mathbf{Z}||} \tag{15}$$

where $\alpha(h)$ is a non-linear coefficient which depends on the friction formulation of choice. For Manning's friction, with a Manning's roughness coefficient n $[TL^{-1/3}]$ the coefficient α is

$$\alpha(h) = \frac{h^{5/3}}{n} \tag{16}$$

For the Darcy-Weisbach friction law, with friction coefficient f $[-]$

$$\alpha(h) = h^{3/2}\left(\frac{8g}{f}\right)^{1/2} \tag{17}$$

And for the Chezy equation, with Chezy roughness coefficient C $[L^{1/2}/T]$

$$\alpha(h) = Ch^{3/2} \tag{18}$$

The standard approach in the literature for the ZI equation relies on Manning's formulation. Thus, substituting Eqs. (16) and (15) in the mass conservation Eq. (14) yields the classical 2D ZI equation based on Manning's friction.

$$\frac{\partial h}{\partial t} + \nabla\left(\frac{h^{5/3}}{n\sqrt{||\mathbf{Z}||}}\mathbf{Z}\right) = I \tag{19}$$

3.2. Finite Volume discretisation

The ZI can be solved by means of the Finite Volume (FV) method. To do so, Eq. (19) is first integrated in a finite control volume A

$$\int_A \frac{\partial h}{\partial t}dA + \int_A \nabla\left(\frac{h^{5/3}}{n\sqrt{||\mathbf{Z}||}}\mathbf{Z}\right)dA = \int_A I dA \tag{20}$$

Invoking Gauss theorem yields

$$\int_A \frac{\partial h}{\partial t}dA + \int_S \frac{h^{5/3}}{n\sqrt{||\mathbf{Z}||}}\mathbf{Z}\cdot\boldsymbol{n}\,dl = \int_A I dA \tag{21}$$

where S is the surface enclosing volume A and $\boldsymbol{n}$ is the outer-pointing vector normal to S. Finally, choosing polygonal cells comprised of N_ω sides as control volumes, and choosing a piece-wise constant representation of functions in the cells, the semi-discretisation for any cell i is achieved

$$\frac{\partial h}{\partial t} = I_i - \frac{1}{A_i}\sum_{\omega=1}^{N_\omega}\frac{h_\omega^{5/3}}{n_\omega\sqrt{||\mathbf{Z}_\omega||}}\mathbf{Z}_\omega\cdot\boldsymbol{n}_\omega L_\omega \tag{22}$$

where ω denotes the edge between cells i and j, h_ω is the estimated depth at the cell edge ω, A is the cell area, L_ω is the length of the edge ω, $\mathbf{Z}_\omega$ is the water surface gradient across the edge with normal vector $\boldsymbol{n}_\omega$. The gradient across an edge $\mathbf{Z}_\omega$ is defined as

$$\mathbf{Z}_\omega = \nabla\mathbf{Z}|_\omega = \left(\frac{\partial(h+z)}{\partial x}, \frac{\partial(h+z)}{\partial y}\right)|_\omega \tag{23}$$

so that, effectively

$$\mathbf{Z}_\omega\cdot\boldsymbol{n}_\omega = \frac{h_j + z_j - h_i - z_i}{d_\omega} \tag{24}$$

with d_ω the distance between cell centers perpendicular to edge ω. Note that $\mathbf{Z}_\omega$ is well defined at the edge, but h_ω is not, and must be estimated to cope with its bi-valuated nature at such edge. Approximating the value at the edge has implications in both the accuracy and efficiency of the method (Dottori and Todini, 2011; Hunter et al., 2005; Mendicino et al., 2015).

The semi-discretisation in Eq. (22) still requires to integrate in time. Adopting an explicit time discretisation approach – a standard forward Euler approximation, the complete numerical scheme reads

$$h_i^{n+1} = h_i^n + \Delta t I_i^n - \frac{\Delta t}{A_i}\sum_{\omega=1}^{N_\omega}\frac{(h_\omega^n)^{5/3}}{n_\omega\sqrt{||\mathbf{Z}_\omega^n||}}\mathbf{Z}_\omega^n\cdot\boldsymbol{n}_\omega L_\omega \tag{25}$$

Eq. (25) is the explicit, finite volume solution to the ZI approximation. Thus, this is termed the FV-ZI solver, as reported in the literature (Cea et al., 2010; Fernández-Pato et al., 2016; Leandro et al., 2014; Wang et al., 2011). Note again that this solver is often also inaccurately termed FV-DW solver (Yen and Tsai, 2001). Furthermore, Eq. (25) does not require any particular shape or arrangement for the computational cells. It allows for fully unstructured meshes with convex polygonal cells. A particular case of this is of course a structured mesh of squared cells.

4. Review and discussion

A key interest of this paper is to strongly point out that the FV-ZI solver in Eq. (25) (Cea et al., 2010; Fernández-Pato and García-Navarro, 2016; Leandro et al., 2014; Wang et al., 2011) is *identical* to the cellular-automata approach described by Eq. (12) (Dottori and Todini, 2011; Mendicino et al., 2015; Parsons and Fonstad, 2007), which is henceforth denoted CA-ZI, and also identical to the so-called *raster-based* solvers (Bates and Roo, 2000; Hunter et al., 2005) or cell-based solvers (Julien et al., 1995). In other words, CA-ZI solvers are, at the core, no different from FV-ZI solvers, although obtained by reasoning which originates from a discrete system. Somewhat different technical approximations to different components may be the sole differences, but this is also the case of different FV-ZI approaches by various authors. This identity has been seldom pointed out, which, together with the wide variety of CA approaches (see Table 1) have lead to parallel literature from researchers in Hydrology and Geology and researchers in Fluid Dynamics and Computational Hydraulics. This issue of terminology is also entangled with the misuse of the Diffusive-Wave term for what the Zero-Inertia equation. Although the ZI equation is a DW equation, not all DW equations are ZI (Sivapalan et al., 1997; Yen and Tsai, 2001).

It is relevant to point out that the definition of flux $\mathscr{F}$ is critical to the meaning and capabilities of CA solvers. CA-ZI solvers (identical to FV-ZI) solvers are obtained when the flux $\mathscr{F}$ corresponds to the unit-discharge (momentum) $\boldsymbol{q}$, i.e., Eq. (7) is equivalent to (15) evaluated with (16). Another relevant choice in the numerical formulation common to models termed CA-ZI or FV-ZI is that depth must be estimated at the edge between two cells, where it is bi-valuated. Several studies from the CA literature (Dottori and Todini, 2011; Mendicino et al., 2015; Hunter et al., 2005) have shown showed that the *naïve* arithmetic mean between neighboring cells is computationally inefficient due to numerical stability issues, in particular in the presence of complex topography (Dottori and Todini, 2011), and have proposed a (more) stable estimation (Hunter et al., 2005)

$$h_\omega = \max(h_i + z_i, h_j + z_j) - \max(z_i, z_j) \quad (26)$$

which has been also adopted in FV models identified as FV-ZI, hinting the relevance of finding a common ground between the research communities.

Similar to the FV-ZI solver in Eq. (19) is also that of Lopez-Barrera et al. (2012) which attempts to incorporate elements of the hyperbolicity present in the SWE, requiring a redefinition of the fluxes. On the other hand, the FV-ZI solver in Eq. (25) differs obviously from implicit approaches, relying either on FV (Fernández-Pato et al., 2016) or FE (Panday et al., 2004; Weill et al., 2009) since it requires no matrix inversion and allows for direct parallelization. The FV-ZI solver also differs clearly from other CA based approaches which input convenient (yet not necessarily formal or physical) expressions in the flux terms – i.e., choosing a flux $\mathscr{F}$ different than the one spawning from Manning's equation (Eq. 7) or alternative friction laws – which do not really account for the physics as described in the SWE (D'Ambrosio et al., 2001; Ghimire et al., 2013; Parsons and Fonstad, 2007; Rinaldi et al., 2007; Shao et al., 2015, e.g) or are not really DW/ZI simplifications (Bates et al., 2010).

The rather loose use of terminology in different communities seems to have encouraged parallel developments, with researchers not benefiting from the knowledge base from related fields. Many studies in geological and hydrological related fields are based on the FV/CA-ZI approach, mainly because of the convenient simplicity of the equation and its numerical solution, and some pointing out stability as a relevant issue (Mendicino et al., 2015). However, it has been pointed out that the price for simplicity is computational efficiency, i.e., the CA-ZI approach can often be computationally more expensive than SWE solvers (Cea et al., 2010; Fernández-Pato and García-Navarro, 2016; Lopez-Barrera et al., 2012; Wang et al., 2011), although studies have found the opposite (Dottori and Todini, 2011; Martins et al., 2017). The cost of the explicit ZI solvers is related to stability issues, thus justifying implicit approaches (Fernández-Pato and García-Navarro, 2016; Panday et al., 2004; Weill et al., 2009), which are not equivalent to CA-ZI solvers in terms of time discretisation, although they solve the same partial-differential equation (PDE). It has been suggested that the high cost of solving ZI explicitly is inherent to the equation (Wang et al., 2011), and various researchers have independently proposed and studied ways to reduce the computational cost of the ZI approximation by addressing the underlying stability issues. Some of the reasoning is obtained by following ideas from the stability of hyperbolic systems, i.e., the Courant-Friedrich-Lewy (CFL) condition (Lopez-Barrera et al., 2012; Wang et al., 2011; Dottori and Todini, 2011) and von Neumann stability analysis (Hunter et al., 2005), which is similar and consistent to stability analysis of parabolic-elliptic equations such as the Richards equation (Caviedes-Voullième et al., 2013; Mendicino et al., 2015) which shows a similar (non-linear) behaviour as the ZI approach. Interestingly, it has also been pointed out that adding an inertial term to the ZI equation (without reverting to the full SWE) also reduces stability constrains (Bates et al., 2010; Dottori and Todini, 2011), an idea which should be further pursued.

Table 1 presents a (non-exhaustive) chronological review of many CA models reported in the literature for different types of surface flow simulation. These CA models have smartly and conveniently selected fluxes, which might represent the physics to some extent under certain conditions. The table also includes a chronological review of many CFD-based models, most of which were developed for flood modelling. In the table it can be seen that some models referred to as CA models, are in fact, FV solvers of a PDE-based reasoning which leads to ZI equation. The reported models are classified in terms of how they were termed by their authors, how in fact they are spatially and temporally discretised, how the flux $\mathscr{F}$ is formulated, the definition of the neighborhood and the application intended by the authors. Some additional comments are included for clarifications and particular highlights. Authors might term their models CA, but the model may in fact be a FV discretisation. Similarly, authors might term their model to solve de DW equation, but the flux formulation is stated as ZI in the table, in an effort to generate a consistent terminology.

One of the interesting observations to be drawn from Table 1 is that there has been a continuous development of CA, CA-ZI and FV-ZI solvers in the past 25 years. Nonetheless, a number of recent publications have indeed tackled some of the complex issues such as stability (Mendicino et al., 2015) or computational efficiency through local time stepping (Dottori and Todini, 2011). Very recent publications have also provided new observations on the performance of the methods in contrast to SWE solvers (Costabile et al., 2017; Fernández-Pato et al., 2016; Jahanbazi et al., 2017). Interestingly, as a group, these very recent papers still carry with them the mix of terminology pointed out in this paper, even though some of them (Dottori and Todini, 2011; Jahanbazi et al., 2017) actually do realise and comment on the equivalent nature of the methodologies. Other very recent publications do still approach the problem from a strictly CA (non FV-ZI) approach, exploring ad hoc flux formulations and algorithms (Liu et al., 2015; Ghimire et al., 2013; Shao et al., 2015).

Table 1 also allows to visualize the range of modelling choices within the models. Most models reported in Table 1 are explicit in time, with the notable exceptions of Panday et al. (2004, 2006) and Weill et al. (2009), all of which were intended to solve the ZI equation coupled to Richards equation, which is commonly solved implicitly (Caviedes-Voullième et al., 2013; Mendicino et al., 2015). Notable as well are the studies by Lal (1998) who found that explicit solvers are less efficient than implicit solvers, and by Fernández-Pato et al. (2016) who compared explicit and implicit FV-ZI solvers, finding that implicit solvers may be more efficient than explicit solvers under certain conditions, but not always. It is striking that most of the ZI solvers take an explicit approach, since the parabolic nature of the ZI equation classically calls for implicit solvers (following von Neumann stability analysis) and to the significant problems that arise from the stability constrains of the explicit solution of ZI (Mendicino et al., 2015). This may be due to a combination the – at times – CA driven developments (Bates and Roo, 2000; Hunter et al., 2005, e.g.) in which the *ansatz* is always based on an explicit update scheme, contrary to what is classical and well-established in Numerical Analysis. Nonetheless, there is still an open debate on how to ensure the stability of the solver while keeping performance and efficiency, with most authors favouring formal stability constrains – a Peclet-like number depending on the square of the spatial resolution – (Cea et al., 2010; Leandro et al., 2014; Mendicino et al., 2015; Hunter et al., 2005), and others suggesting a less formal, alternative approach (Lopez-Barrera et al., 2012; Jahanbazi et al., 2017; Wang et al., 2011) of choosing a hyperbolic-like stability constrain: the Courant-Friedrich-Lewy (CFL) condition, proportional to the spatial resolution. There is merit to this idea, since the formal approach has been shown to both over-restrict the time step (Dottori and Todini, 2011) and not-strictly guarantee stability – giving rise to the issue known as checkerboarding (Hunter et al., 2005; Hunter et al., 2007; Lopez-Barrera et al., 2012), whereas keeping non-formal constrains (CFL number) has been shown to work reasonably, with significant less computational effort (Wang et al., 2011). Recently, Jahanbazi et al. (2017) attempted to reduce the stability constrains of the ZI approach, drawing from the CA philosophy and implementing variations on the flux formulation to control the issues, again making obvious for a need for efforts from both communities to converge. Arguably, the trend to favour explicit solutions is also supported by a simpler implementation, more intuitive interpretation of the scheme, and the easiness of parallelization (Dottori and Todini, 2011). This observation is also supported by the fact that most CA/FV-ZI solvers have been implemented for Cartesian meshes, again favouring simplicity of implementation, interpretation and parallelization.

Most of the reported models solving the ZI inertia equation – in particular those reported as CA – have been implemented for Cartesian (square) meshes, and make use of the von Neumann neighborhood. Two relevant points arise on this issue. The first is that few models allow for triangular (unstructured) meshes (Cea et al., 2010; Costabile et al., 2017; Fernández-Pato et al., 2016; Lal, 1998) whose use may be relevant to keep a low computational cost while ensuring accuracy (Caviedes-Voullième et al., 2012). The second is that none of the models reported as CA-ZI solvers use the Moore neighborhood – which can otherwise be found in CA models *not* based on ZI. This can be easily explained once CA-ZI solvers are seen as FV-ZI solvers, since the Moore neighborhood brings in an inconsistency: it calls to compute a flux across a vertex – a point, an entity with zero length – instead of across an edge. Other CA models which do not necessarily compute physical fluxes can and do use the Moore neighborhood.

5. Conclusions

This paper reviews the existing Cellular Automata and Finite Volume models reported in the past 25 years for the solution of surface flows. Such review leads to three key points: (i) FV solvers developed to solve the Zero-Inertia approximation to the shallow water equations are widely used and understood, (ii) there is a variety of CA solvers to address shallow surface flows, all of which rely on similar definitions of state variables but differ in the formulation of the fluxes, and (iii) there is a subset of CA solvers which are in fact identical to FV-ZI solver given a particular choice of flux – although originating from a different reasoning. These facts, which have seldom been pointed out in the literature should lead to some convergence from the contributions coming from different approaches, indeed converging to a single – widespread – modelling approach. Acknowledging this convergence allows for some generalization in terms of the properties of the numerical schemes, the meaning of the models and applicability of the approaches. In particular, it becomes clear that, of the approaches reported here, those which effectively solve the ZI equation can be interpreted with well-known numerical analysis, their stability properties can be formally established – although in practice such formal stability can be inefficient, and have a clear physical base and physical meaning. Notably, all ZI solvers (may they be termed CA or FV) make use of Neumann neighborhoods, since the Moore neighborhood lacks meaning when solving the ZI equation as it provides no length over which to integrate the flux across two cells. This allows to clearly distinguish cell-based models based on the ZI approach, and those *not* based on it (or the SWE equations for that matter). Complementary, those models which do not share the properties of CA-ZI/FV-ZI solvers must be analyzed differently (since they are not in the same category), with other tools – formal or not, and their stability, meaning and applicability must be critically assessed for each particular case since it cannot be extrapolated from the behaviour of alternative models. Furthermore, it becomes clear that even considering contributions from the different scientific communities involved, there are still open questions on how to efficiently and robustly solve the Zero-Inertia equation. To name a few, the range of applicability of the ZI equation is still under discussion, as very recent publications show. The choice of the temporal integration has relevant implications in the simplicity of the methodology, arguably its most desirable property. In the case of explicit solvers, the issue of stability seems not completely solved, although tested approaches have been mathematically rigorous and have provided insights. Finally, the advantages of choosing ZI solvers over shallow water solvers still remain debatable, as contradictory experiences and arguments have been recently reported.

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