Two principles for two-person social choice *

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We consider two-person ordinal collective choice from an axiomatic perspective. We identify two principles: minimal Rawlsianism (the chosen alternatives belong to the upper-half of both individuals' preferences) and the equal loss principle (the chosen alternatives ensure that both individuals concede "as equally as possible" from their highest ranked alternative). The equal loss principle has variants of different strength, depending on the precise definition of "as equally as possible". We consider all prominent ordinal two-person social choice rules of the literature and explore which of these principles they satisfy. Moreover, we show that minimal Rawlsianism is logically incompatible with one version of the equal loss principle that we call the minimal dispersion principle. On the other hand, there are social choice rules that satisfy the Rawlsian minimal dispersion principle where the minimal dispersion principle is restricted to alternatives within the upper-half of both individuals' preferences.

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1. Introduction

Two-person discrete social choice models allow a specific interpretation of collective decision making: bargaining over a finite set of alternatives. Since the seminal model of Nash [1950], for a long time, bargaining problems were formulated by assuming a convex set of alternatives. However, there are many instances where bargaining takes place over a finite set of alternatives. Thus, this simplifying assumption of Nash excludes several real-life situations.

Mariotti [1998] is among the first to relax this assumption by characterizing the Nash solution for a finite set of alternatives. His approach is followed by Nagahisa and Tanaka [2002] who, again in a finite setting, characterize the solution of Kalai and Smorodinsky [1975]. Both characterizations are built in a cardinal framework.

An ordinal framework of two-person finite bargaining problems is presented by Brams and Kilgour [2001] who introduce and analyze an ordinal solution, namely *fallback bargaining*, that is based on compromising where each of the two bargainers begins by claiming the best outcome with respect to his ranking of alternatives. When the claims of the two bargainers differ, they continue by falling back, in lockstep, to lower ranked alternatives until a mutually (hence unanimously) agreed outcome is found. A characterization of fallback bargaining is provided by de Clippel and Eliaz [2012]. As this solution is presented in a model that does not admit a disagreement point, fallback bargaining is rather an arbitration rule in the sense of Sprumont [1993] than being a bargaining solution. An analysis of fallback bargaining in a model with a disagreement point is made by Kıbrıs and Sertel [2007] who rebaptize the solution as *unanimity compromise* and define several variants of it.[1] In a recent paper, Barberà and Coelho [2022] use the term unanimity compromise in a framework with no disagreement point to refer to fallback bargaining in the sense of Brams and Kilgour [2001]. We adopt their terminology, to avoid proposing a bargaining solution without a disagreement outcome.

As a matter of fact, the compromising approach that underlies unanimity compromise was originally used to design voting rules in settings with more than two individuals with the required support varying from unanimity to simple majority, such as the *Kant-Rawls Social Compromise* by Hurwicz and Sertel [1999] and the *majoritarian compromise* by Sertel and Yılmaz [1999]. It also paved the way to new axioms for social choice, such as *majoritarian approval* and *majoritarian optimality* as well as *efficiency in the degree of compromise*

[1] They also observe that one of these variants, namely the imputational compromise, is the finite version of the equal length rule by Thomson [2019]. The imputational compromise is further studied by Conley and Wilkie [2012].

by Özkal-Sanver and Sanver [2004]. Merlin et al. [2019] present a recent comprehensive analysis of voting rules and axioms based on this compromising idea.

A closer look at this compromising idea and in particular at unanimity compromise reveals a principle for two-person social choice. Sprumont [1993] qualifies arbitration rules that maximize the welfare of the least happy individual as being *Rawlsian*. Congar and Merlin [2012] characterize the Rawlsian principle within the framework of social welfare functions. For social choice rules, Brams and Kilgour [2001] establish the equivalence between the Rawlsian principle and unanimity compromise. Moreover, they show that every individual ranks a unanimity compromise outcome in the upper-half of his ranking. Thus, at every preference profile, there is an alternative that both individuals rank in the upper-half of their preference. In other words, the least happy individual of the society can always be granted a welfare within the first half of his preference. We qualify a two-person social choice rule that complies with this possibility as *minimally Rawlsian.*[2]

Cailloux et al. [2022] propose a different conception of compromising based on an *equal loss principle* that favors outcomes where every individual concedes as equally as possible from his highest ranked alternative. They show that several two-person social choice rules fail this principle.

Although the minimal Rawlsian and equal loss principles cover many of the two-person social choice rules, the literature is missing an axiomatic analysis of these rules from this perspective, an observation which forms the subject matter of our paper. We consider the following rules, where m is the number of alternatives.

- Unanimity compromise, as defined by Brams and Kilgour [2001] (under the name of fallback bargaining).

- The *veto-rank rule* where, for m odd, each individual vetoes $(m-1)/2$ alternatives and ranks the remaining $(m+1)/2$. The outcome is the alternatives with the minimal sum of ranks among those that have not been vetoed.

- The *shortlisting rule* where, for m odd, one individual selects her best $(m+1)/2$ alternatives and the rule picks the best alternative of the other individual out of that shortlist.

[2] In a two-person collective choice framework with an interpretation that is more specific than ours, de Clippel et al. [2014] call this condition the *minimal satisfaction test.*

- The class of *Pareto-and-veto rules* where each individual i vetoes a fixed number v_i of alternatives with $v_1 + v_2$ being lower than m. The outcome is the set of Pareto optimal alternatives that are not vetoed.

The veto-rank rule and the shortlisting rule are used for the selection of arbitrators and their strategic aspects are comprehensively analyzed by de Clippel et al. [2014][3]. Our class of Pareto-and-veto rules generalizes the Pareto-and-veto rules analyzed by Laslier et al. [2021] which impose $v_1 + v_2 = m - 1$. These rules we consider cover most of the ordinal two-person social choice rules in the literature. The literature also admits various interesting real-life procedures expressed as extensive form games, such as those in Anbarci [1993, 2006] and Barberà and Coelho [2022]. However, as shown in these papers, the subgame perfect equilibrium outcomes of these games are always among the unanimity compromise alternatives.

A rule of specific interest is the Pareto-and-veto rule that gives the highest equal veto power to both individuals. This rule yields all Pareto efficient alternatives that are in the upper half of both individuals. In fact, it can be expressed the union all Pareto efficient and minimally Rawlsian rules. Unanimity compromise, the veto-rank rule and the shortlisting rule are all sub correspondences of this rule, hence they are minimally Rawlsian. When m is odd, Within the class of Pareto-and-veto rules, the one with highest equal veto power is unique in being minimally Rawlsian while when m is even, there are two other that give the highest almost equal veto power to both individuals.

The equal loss principle we consider favors outcomes that have the same rank for both individuals. Without imposing Pareto optimality separately, this principle may lead to Pareto dominated outcomes. Thus, we consider a Pareto efficient version that favors, among the Pareto optimal outcomes, the one that has the same rank for both individuals. Note that such an alternative, if it exists, will be unique. We define two versions of the Paretian equal loss principle, one being stronger than the other. The stronger version requires that the Pareto optimal alternative that has the same rank for both individuals must be uniquely chosen. Under the weaker version, it suffices that this alternative be among the outcomes. The veto-rank rule and the shortlisting rule both fail the weak (hence strong) version of the Paretian equal loss principle. While Pareto-and-veto rules that endow individuals with a veto power that does not exceed $\lfloor \frac{m}{2} \rfloor$ satisfy the weak Paretian equal loss principle, all of them fail the strong version. On the other hand, unanimity compromise satisfies the strong Paretian equal loss principle, thus showing that this principle is compatible

[3] De Clippel et al. [2014] present the shortlisting mechanism, whose equilibrium outcome corresponds to what we call the shortlisting rule.

with being minimally Rawlsian.

Within the spirit of equal loss, we propose the *minimal dispersion principle* as another strengthening of the (weak) Paretian equal loss principle. The *dispersion* of an alternative is the difference between the two ranks at which is it placed. The minimal dispersion principle requires that an alternative whose dispersion is minimal is among the outcomes. Not only unanimity compromise fails this principle, but the minimal dispersion principle turns out to be logically incompatible with the minimal Rawlsian principle. As a result, among the social choice rules we consider, the only candidates to satisfy the minimal dispersion principle are the Pareto-and-veto rules that fail the minimal Rawlsian principle. As a matter of fact, those that endow individuals with a veto power that does not exceed a third of the available alternatives turn out to satisfy the minimal dispersion principle.[4] Nevertheless, these Pareto-and-veto rules fail two stronger versions of the minimal dispersion principle, one that requires to pick all alternatives with minimal dispersion and the other that requires to pick only alternatives with minimal dispersion. As a result, we define the minimal dispersion rule which, at every preference profile, picks precisely the alternatives with minimal dispersion.[5]

Given the incompatibility between the two principles, we introduce the Rawlsian minimal dispersion principle that requires the outcome to contain the Rawlsian alternatives whose loss vectors have minimal dispersion. It turns out that it is failed by all social choice rules we consider, except the Pareto-and-veto rule that gives the highest equal veto power to both individuals. The satisfaction is at the expense of resoluteness, since this rule picks every Pareto efficient alternative within the first half. It is thus natural to seek for a stronger principle that imposes more resoluteness. To this end, we define the strong Rawlsian minimal dispersion principle that requires the outcome to lie within the Rawlsian alternatives whose loss vectors have minimal dispersion. No rule that we have examined satisfies this principle. Therefore, we define the strong Rawlsian minimal dispersion rule as the least resolute rule that satisfies this principle. This new rule appears as a reasonable compromise to satisfy the two main principles we analyze, namely that of being minimally Rawlsian and that of favoring small dispersion alternatives. Moreover, it is also reasonably

[4] Recall that this upper bound is a half for the satisfaction of the (weaker) Paretian equal loss principle.

[5] In a framework with a disagreement outcome, Kıbrıs and Sertel [2007] argue that the finite version of the *equal area rule* [Thomson, 1994] minimizes the difference between losses with respect to individually rational alternatives. They also show that this rule differs from their unanimity compromise. In our framework without a disagreement outcome, the equal area rule is equivalent to our minimal dispersion rule.

resolute as it never picks more than two alternatives.

Section 2 introduces the basic notions and notation. Sections 4 and 5 are devoted to the minimal Rawlsian and equal loss principles, respectively. Section 6 introduces the Rawlsian minimal dispersion principle. Section 7 makes some concluding remarks.

2. Basic notions and notation

Let $N = \{1, 2\}$ be a set of two individuals and $\mathcal{A}$ be a set of alternatives, with $\#\mathcal{A} = m \geq 2$. For each $i \in N$, we denote by $\bar{i} \in N \setminus \{i\}$ the other individual. Let $\mathscr{P}^*(\mathcal{A})$ denote the set of non-empty subsets of $\mathcal{A}$ and $\mathscr{L}(\mathcal{A})$ be the set of linear orders over $\mathcal{A}$. We let $\succ_i \in \mathscr{L}(\mathcal{A})$ be the preference of individual $i \in N$ and $\boldsymbol{P} = (\succ_1, \succ_2) \in \mathscr{L}(\mathcal{A})^N$ a preference profile. A social choice rule (SCR) is a function $f : \mathscr{L}(\mathcal{A})^N \to \mathscr{P}^*(\mathcal{A})$. Viewing such functions as relations on $\mathscr{L}(\mathcal{A})^N \times \mathcal{A}$, we can write $f \subseteq f'$ to denote a rule that "refines" f', in the sense that $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : f(\boldsymbol{P}) \subseteq f'(\boldsymbol{P})$. When $f \subseteq f'$ and $f \neq f'$, we write $f \subsetneq f'$ and say that f is a proper sub-correspondence of f'.

A SCR f is anonymous iff, for any $(\succ_1, \succ_2) \in \mathscr{L}(\mathcal{A})^N$, $f(\succ_1, \succ_2) = f(\succ_2, \succ_1)$ while f is neutral iff for any permutation σ over $\mathcal{A}$ and any profile $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, $\sigma \circ f(\boldsymbol{P}) = f(\sigma \circ \boldsymbol{P})$.

Given $p, q \in \mathbb{R}$, let $[\![p, q]\!] = [p, q] \cap \mathbb{N}$ denote the interval of integers between p and q. The loss of individual i at $\boldsymbol{P}$ for alternative x equals the number of alternatives that i prefers to x: $\lambda_{\boldsymbol{P}}(x)_i = \#\{y \in \mathcal{A} \mid y \succ_i x\}$. The loss vector of x at $\boldsymbol{P}$ is denoted $\lambda_{\boldsymbol{P}}(x) = (\lambda_{\boldsymbol{P}}(x)_1, \lambda_{\boldsymbol{P}}(x)_2) \in [\![0, m-1]\!]^N$. Given two loss vectors $l, l' \in [\![0, m-1]\!]^N$, l is smaller than l', denoted $l \leq l'$, iff $l_i \leq l'_i$ $\forall i$. We also write $l < l'$ when l is strictly smaller than l', meaning, smaller and different. Let $\min \lambda_{\boldsymbol{P}}(x) = \min_{i \in N} \lambda_{\boldsymbol{P}}(x)_i \in \mathbb{N}$, $\max \lambda_{\boldsymbol{P}}(x) = \max_{i \in N} \lambda_{\boldsymbol{P}}(x)_i \in \mathbb{N}$ and $\sum \lambda_{\boldsymbol{P}}(x) = \sum_{i \in N} \lambda_{\boldsymbol{P}}(x)_i$ respectively denote the minimal loss, the maximal one and the sum of the losses in $\lambda_{\boldsymbol{P}}(x)$.

Let $PE(\boldsymbol{P}) = \{x \in \mathcal{A} \mid \nexists y \text{ s.t. } \lambda_{\boldsymbol{P}}(y) < \lambda_{\boldsymbol{P}}(x)\}$ be the set of Pareto efficient alternatives at $\boldsymbol{P}$. The SCR f satisfies the Pareto property iff it picks only Pareto efficient alternatives. Let $\mathcal{PE} = \{f : \mathscr{L}(\mathcal{A})^N \to \mathscr{P}^*(\mathcal{A}) \mid \forall \boldsymbol{P} : f(\boldsymbol{P}) \subseteq PE(\boldsymbol{P})\}$ denote the class of SCRs satisfying the Pareto property.[6]

In concordance with the ceiling established by Theorem 1 of Brams and Kilgour [2001], we use "best half" to refer to the loss values up to $\lfloor \frac{m}{2} \rfloor$. For any profile $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, we let $H(\boldsymbol{P}) = \{x \in \mathcal{A} \mid \lambda_{\boldsymbol{P}}(x) \leq (\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)\}$ denote the set of alternatives reaching the best half of every individual's preference, and

[6] We systematically use calligraphic letters to denote the class of rules satisfying a given property.

$H^i(\boldsymbol{P}) = \{x \in \mathcal{A} \mid \lambda_{\boldsymbol{P}}(x)_i \leq \lfloor \frac{m}{2} \rfloor\}$ those that reach the best half of i's preference. Given $\boldsymbol{P}$ and a loss level k, define $U(\boldsymbol{P}, k) = \{x \in \mathcal{A} \mid \max \lambda_{\boldsymbol{P}}(x) \leq k\}$ as the alternatives with a loss of at most k for both individuals. An alternative receives unanimous support at level k iff it belongs to $U(\boldsymbol{P}, k)$. Note that $\min_{x \in \mathcal{A}}\{\max \lambda_{\boldsymbol{P}}(x)\}$, the "minimal worst loss", represents the least loss level at which some alternative receives unanimous support; formally: $\min_{x \in \mathcal{A}}\{\max \lambda_{\boldsymbol{P}}(x)\} = \min\{k \in [\![0, m-1]\!] \mid U(\boldsymbol{P}, k) \neq \emptyset\}$.

3. SCRs in the literature

We start by defining the SCRs of the literature that we analyze in the paper and giving some preliminary observations about these rules.

Unanimity compromise or *UC* is the SCR that picks all alternatives with unanimous support at the minimal worst loss $\rho_{\boldsymbol{P}} = \min_{x \in \mathcal{A}}\{\max \lambda_{\boldsymbol{P}}(x)\}$; formally: $UC(\boldsymbol{P}) = U(\boldsymbol{P}, \rho_{\boldsymbol{P}}) = \arg\min_{x \in \mathcal{A}}\{\max \lambda_{\boldsymbol{P}}(x)\}$.

Remark 1. *UC* is anonymous and neutral. It satisfies the Pareto property. Moreover, $\forall \boldsymbol{P}$, $UC(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$. The final observation comes from the fact that $\forall \boldsymbol{P}$, $\min_{x \in \mathcal{A}}\{\max \lambda_{\boldsymbol{P}}(x)\} \leq \lfloor \frac{m}{2} \rfloor$, a consequence of the following proposition that is a restatement of Brams and Kilgour [2001, Theorem 1]. We include a proof for completeness.

Proposition 1 (Brams and Kilgour)**.** $[\forall \boldsymbol{P}, \exists x \in \mathcal{A} \mid \max \lambda_{\boldsymbol{P}}(x) \leq k] \Leftrightarrow k \geq \lfloor \frac{m}{2} \rfloor$.

Proof. To prove the "only if" part, let $k < \lfloor \frac{m}{2} \rfloor$ and consider the profile $\boldsymbol{P}$ composed of an arbitrary ordering $\succ_i$ over $\mathcal{A}$ and its inverse ordering $\succ_i^{-1}$ and observe that $\min\max \lambda_{\boldsymbol{P}} = \lfloor \frac{m}{2} \rfloor > k$.

To prove the "if" part, let $k \geq \lfloor \frac{m}{2} \rfloor$. For any $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, we show that for some $x \in \mathcal{A}$, $\max \lambda_{\boldsymbol{P}}(x) \leq k$. Define $A = \{a \in \mathcal{A} \mid 0 \leq \lambda_{\boldsymbol{P}}(a)_1 \leq \lfloor \frac{m}{2} \rfloor\}$. Define $B = \{a \in \mathcal{A} \mid \lfloor \frac{m}{2} \rfloor < \lambda_{\boldsymbol{P}}(a)_2 \leq m-1\}$. Observe that $\#A = \lfloor \frac{m}{2} \rfloor + 1$ and $\#B = (m-1) - \lfloor \frac{m}{2} \rfloor = (m-1) - \lceil \frac{m-1}{2} \rceil = \lfloor \frac{m-1}{2} \rfloor \leq \lfloor \frac{m}{2} \rfloor$, thus, $\#B < \#A$. Thus, $\exists x \in A \setminus B$. And $\lambda_{\boldsymbol{P}}(x)_1 \leq \lfloor \frac{m}{2} \rfloor$ (as $x \in A$) and $\lambda_{\boldsymbol{P}}(x)_2 \leq \lfloor \frac{m}{2} \rfloor$ (as $x \notin B$). It follows that $\max \lambda_{\boldsymbol{P}}(x) \leq \lfloor \frac{m}{2} \rfloor$. $\square$

In the **veto-rank rule** or *VR*, each individual vetoes her worst $\lfloor \frac{m-1}{2} \rfloor$ alternatives, and the Borda winners among the non vetoed alternatives are picked: $VR(\boldsymbol{P}) = \arg\min_{H(\boldsymbol{P})} \sum \lambda_{\boldsymbol{P}} = \{x \in H(\boldsymbol{P}) \mid \forall y \in H(\boldsymbol{P}) : \sum \lambda_{\boldsymbol{P}}(x) \leq \sum \lambda_{\boldsymbol{P}}(y)\}$.

In the **shortlisting rule** or *SL*, the best alternative of individual 1 that is not among the worst $\lfloor \frac{m-1}{2} \rfloor$ alternatives of individual 2 and the best alternative of 2 that is not among the worst $\lfloor\frac{m-1}{2}\rfloor$ alternatives of 1 are selected: $SL(\boldsymbol{P}) = \cup_{i\in N}(\arg\min_{x\in H^i(\boldsymbol{P})} \lambda_{\boldsymbol{P}}(x)_{\bar{i}})$.

Both *VR* and *SL* are defined in de Clippel et al. [2014] for m odd only.

Remark 2. Both *VR* and *SL* are anonymous and neutral. They satisfy the Pareto property. Moreover, $\forall \boldsymbol{P}$, $VR(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$ and $SL(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$.

Remark 3. There is some $\boldsymbol{P}$ such that $UC(\boldsymbol{P}) \cap VR(\boldsymbol{P}) = \emptyset$, $UC(\boldsymbol{P}) \cap SL(\boldsymbol{P}) = \emptyset$ and $VR(\boldsymbol{P}) \cap SL(\boldsymbol{P}) = \emptyset$. Consider the following profile $\boldsymbol{P}$:

$$\begin{array}{ccccccc|cccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \\ g & h & i & d & b & j & a & c & e & f & k & l & m \end{array}, \tag{1}$$

where alternatives are ranked from left to right and the bar shows the "half" position. The proposition is proven by noting that $UC(\boldsymbol{P}) = \{d\}$, $VR(\boldsymbol{P}) = \{b\}$ and $SL(\boldsymbol{P}) = \{a, g\}$.

The class of **Pareto-and-veto rules**, $\mathcal{PV}$, contains rules parametrized by $v_1, v_2 \in [\![0, m-1]\!]$ with $v_1 + v_2 \leq m - 1$ where v_i represents the number of alternatives vetoed by individual $i \in N$ (individuals veto the alternatives at the bottom of their preference). Given $v_i \in [\![0, m-1]\!]$, define $a_i = m - v_i - 1 \in [\![0, m-1]\!]$ as the highest acceptable loss level for individual i. For $v = (v_1, v_2)$, the rule $PV^v(\boldsymbol{P}) = \cap_{i\in N}\{x \in \mathcal{A} \mid \lambda_{\boldsymbol{P}}(x)_i \leq a_i\} \cap PE(\boldsymbol{P})$ picks all Pareto-efficient alternatives that no individual vetoes. The class $\mathcal{PV} = \{PV^v \mid v_1, v_2 \in [\![0, m-1]\!]$ with $v_1 + v_2 \leq m - 1\}$ is the set of those rules, and the class $\mathcal{PV}^b = \{PV^v \mid v_1, v_2 \in [\![0, m-1]\!]$ with $v_1 + v_2 = m - 1\} \subseteq \mathcal{PV}$ is the set of rules where the inequality is binding.

Remark 4. All PV^v rules differ. To see why, consider a profile $\boldsymbol{P}$ with a preference ordering $(a_1, \dots, a_m)$ and the inverse preference ordering $(a_m, \dots, a_1)$: $\{a_{v_2+1}, \dots, a_{m-v_1}\} = PV^v(\boldsymbol{P}) \neq PV^{v'}(\boldsymbol{P}) = \{a_{v'_2+1}, \dots, a_{m-v'_1}\}$.

Remark 5. All SCRs in $\mathcal{PV}$ are neutral, while $PV^v \in \mathcal{PV}$ is anonymous iff $v_1 = v_2$.

We define the rule $PV^= = PV^{(\lfloor\frac{m-1}{2}\rfloor, \lfloor\frac{m-1}{2}\rfloor)}$ as the rule in $\mathcal{PV}$ that gives the highest equal veto power to both individuals. Thus, under $PV^=$, we have $v_1 = v_2 = \lfloor\frac{m-1}{2}\rfloor$, implying $v_1 = v_2 = \frac{m-1}{2}$ when m is odd and $v_1 = v_2 = \frac{m}{2} - 1$ when m is even. Note that $PV^= \in \mathcal{PV}^b$ iff m is odd. The rule $PV^=$ is of special interest regarding its relation to *UC*, *VR* and *SL* as well as to the minimal Rawlsian principle that will be introduced in the next section.

Theorem 1. *For each* $\boldsymbol{P}$, $PV^=(\boldsymbol{P}) = H(\boldsymbol{P}) \cap PE(\boldsymbol{P})$.

Proof. At every $\boldsymbol{P}$, $PV^{=}$ selects all Pareto efficient alternatives that are not among the worst $\lfloor \frac{m-1}{2} \rfloor$ alternatives for every individual, in other words, those that are among the best $\lceil \frac{m-1}{2} \rceil + 1 = \lfloor \frac{m}{2} \rfloor + 1$ ones. This is equivalent to say that $PV^{=}$ selects every Pareto efficient alternative x with $\lambda_{\boldsymbol{P}}(x) \leq (\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$. Yet, this is precisely $H(\boldsymbol{P}) \cap PE(\boldsymbol{P})$, as desired. $\square$

The next proposition shows that each of UC, VR and SL, as well as their union is a strict sub-correspondence of $PV^{=}$.

Proposition 2. $UC \cup VR \cup SL \subsetneq PV^{=}$.

Proof. Theorem 1, together with Remarks 1 and 2, establishes that $UC, VR, SL \subseteq PV^{=}$. To see the strictness of the inclusion, consider the profile

$$\begin{array}{ccccc|cccc} a & b & c & d & e & f & g & h & i \\ e & c & d & b & a & f & g & h & i \end{array}$$

where $UC(\boldsymbol{P}) = \{c\}$, $VR(\boldsymbol{P}) = \{c\}$, $SL(\boldsymbol{P}) = \{a, e\}$ but $PV^{=}(\boldsymbol{P}) = \{a, b, c, e\}$. $\square$

4. The minimal Rawlsian principle

This section focuses on the notion of Rawlsianism, which aims at maximizing the welfare of the worst-off individual.

Definition 1 (k-Rawlsianism)**.** Given $k \in [\![0, m-1]\!]$, a SCR is k-Rawlsian iff it selects among alternatives whose losses are within $[\![0, k]\!]$ for both individuals, whenever such alternatives exist. Formally, $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N, x, y \in \mathcal{A}$: $\max \lambda_{\boldsymbol{P}}(x) \leq k < \max \lambda_{\boldsymbol{P}}(y) \Rightarrow y \notin f(\boldsymbol{P})$.

k-Rawlsianism is binding at exactly those profiles where there is an alternative x whose loss vector is in $[\![0, k]\!]^N$ (equivalently, satisfies $\max \lambda_{\boldsymbol{P}}(x) \leq k$), and is void at the other profiles. It is thus natural to wonder which values of k make the constraint binding at every profile, so as to guarantee a minimal possible loss to every individual whatever the profile.[7]

[7] A variant of k-Rawlsianism that may come to mind is to mandate that the rule be k-strict-Rawlsian iff it selects its winners among those alternatives whose losses are within $[\![0, k]\!]$ for both individuals. Formally: $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N, f(\boldsymbol{P}) \subseteq \lambda_{\boldsymbol{P}}^{-1}([\![0, k]\!]^N)$. (We actually thought about that version first. We thank Miguel Ballester for the improved version.) By the reasoning above, $\lfloor \frac{m}{2} \rfloor$-Rawlsianism is equivalent to $\lfloor \frac{m}{2} \rfloor$-strict-Rawlsianism, whereas for $k < \lfloor \frac{m}{2} \rfloor$, no rule satisfies k-strict-Rawlsianism.

It follows from Proposition 1 that k-Rawlsianism is binding at every profile if and only if k is at least "half". Thus, the strongest version of k-Rawlsianism that is binding at every profile is when k equals $\lfloor \frac{m}{2} \rfloor$. On the other hand, this choice of k reflects a general bound that makes k-Rawlsianism systematically constraining while there are several profiles where the minimal loss is lower than k. Thus, we qualify $\lfloor \frac{m}{2} \rfloor$-Rawlsianism as "minimal Rawlsianism", which we formally define as follows.

Definition 2 (Minimal Rawlsianism (MR))**.** A SCR f satisfies MR iff $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N, f(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$.

Considering two SCRs f and f', let $f \cup f'$ denote the rule $(f \cup f')(\boldsymbol{P}) = f(\boldsymbol{P}) \cup f'(\boldsymbol{P})$. Given any non empty class of SCRs F, let $\bigcup F$ denote the maximal (least resolute) SCR that can be formed by unions of rules of F.

Theorem 2. $\bigcup(\mathcal{PE} \cap \mathcal{MR}) = PV^{=}$.

Proof. Observe that at each profile, $\bigcup(\mathcal{PE} \cap \mathcal{MR})$ picks all Pareto alternatives, and only those, that are in the first half of both individuals' preferences. This observation combined with Theorem 1 establishes the result. $\square$

Theorem 2 admits the corollary below.

Corollary 1. *A SCR $f \in \mathcal{PE} \cap \mathcal{MR}$ if and only if $f \subseteq PV^{=}$.*

Corollary 1 and Theorem 1 lead to the following corollary:

Corollary 2. *UC, VR, $SL \in \mathcal{PE} \cap \mathcal{MR}$.*

We now discuss the relationship of the class $\mathcal{PV}$ to the MR property. The following result determines which rules in $\mathcal{PV}$ satisfy MR.

Theorem 3. *Among the class $\mathcal{PV}$, when m is odd, only the rule $PV^{=}$ satisfies MR, and when m is even, only the three rules $\{PV^{=}, PV^{(\frac{m}{2},\frac{m}{2}-1)}, PV^{(\frac{m}{2}-1,\frac{m}{2})}\}$ satisfy it.*

Proof. Note that $[\forall i : v_i \geq \lfloor \frac{m-1}{2} \rfloor \wedge \sum_i v_i \leq m-1]$ is equivalent to $[\exists i \mid v_i = \lfloor \frac{m-1}{2} \rfloor \wedge \lfloor \frac{m-1}{2} \rfloor \leq v_{\bar{i}} \leq \lceil \frac{m-1}{2} \rceil = \lfloor \frac{m}{2} \rfloor]$. Thus, the claim is equivalent to the claim $\mathcal{PV} \cap \mathcal{MR} = \{PV^v \in \mathcal{PV} \mid \forall i : v_i \geq \lfloor \frac{m-1}{2} \rfloor\}$, which we now prove.

To show the "if" part, note that the condition $\forall i : v_i \geq \lfloor \frac{m-1}{2} \rfloor$ suffices to guarantee that $PV^v(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$ for any profile .

To see the "only if" part, consider an arbitrary ordering $\succ_i$ over $\mathcal{A}$, let $\succ_i^{-1}$ denote its inverse, and consider the profile $\boldsymbol{P} = (\succ_i, \succ_i^{-1})$. Observe that $PV^v(\boldsymbol{P})$ will exclusively pick winners in the first half of individual i only if $v_i \geq \lfloor \frac{m-1}{2} \rfloor$. $\square$

5. The equal loss principle

This section introduces the second fairness criteria of this work, based on the idea of equal sacrifice by both players or equal loss.

Given $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, define $S(\boldsymbol{P}) = \{x \in \mathcal{A} \mid \lambda_{\boldsymbol{P}}(x)_1 = \lambda_{\boldsymbol{P}}(x)_2\}$ as the alternatives ranked at the same position by both individuals.

Definition 3 (Equal loss compatibility (ELC))**.** $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : [S(\boldsymbol{P}) \neq \emptyset] \Rightarrow f(\boldsymbol{P}) \cap S(\boldsymbol{P}) \neq \emptyset$.

Equal loss compatibility is logically incompatible with Pareto efficiency, as formally stated by the next proposition.

Proposition 3. $\forall m \geq 3 : \mathcal{PE} \cap \mathcal{ELC} = \emptyset$.

Proof. Consider the following profile:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_1 & a_3 & \dots & a_m \end{array}.$$

Observe that a_1 and a_2 are the only Pareto efficient alternatives. Thus, all the alternatives that are ranked at the same position by both individuals, namely $\mathcal{A} \setminus \{a_1, a_2\}$, are Pareto dominated. □

Thus, Pareto efficiency is a fortiori incompatible with the following stronger version of equal loss compatibility.

Definition 4 (Equal loss)**.** $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : [S(\boldsymbol{P}) \neq \emptyset] \Rightarrow f(\boldsymbol{P}) \subseteq S(\boldsymbol{P})$.

We now embed Pareto efficiency into the equal loss requirement by mandating f to pick the (unique) Pareto efficient alternative that is ranked equally by both individuals, if any.

Definition 5 (Paretian equal loss compatibility (PEL-compatibility))**.** $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : [S(\boldsymbol{P}) \cap PE(\boldsymbol{P}) \neq \emptyset] \Rightarrow f(\boldsymbol{P}) \cap S(\boldsymbol{P}) \cap PE(\boldsymbol{P}) \neq \emptyset$.

Proposition 4. *VR and SL are not PEL-compatible.*

Proof. Consider the profile $\boldsymbol{P}$ stated in (1), where $VR(\boldsymbol{P}) = \{b\}$ and $SL(\boldsymbol{P}) = \{a, g\}$:

$$\begin{array}{ccccccc|cccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \\ g & h & i & d & b & j & a & c & e & f & k & l & m \end{array}.$$

PEL-compatibility requires to choose d. □

The following result will be useful throughout.

Lemma 1. *For any $\boldsymbol{P}$, if $x \in PE(\boldsymbol{P})$ then $\lambda_{\boldsymbol{P}}(x)_1 + \lambda_{\boldsymbol{P}}(x)_2 \leq m - 1$.*

Proof. Since $x \in PE(\boldsymbol{P})$, $\{y \in \mathcal{A} \mid y \succ_1 x\} \cap \{y \in \mathcal{A} \mid y \succ_2 x\} = \emptyset$, which implies $\#\{y \in \mathcal{A} \mid y \succ_1 x\} + \#\{y \in \mathcal{A} \mid y \succ_2 x\} \leq m - 1$. □

Some rules from the class $\mathcal{PV}$ are PEL-compatible, $PV^{=}$ being among those.

Proposition 5. *For any $m \geq 3$, a rule $PV^{(v_1,v_2)} \in \mathcal{PV}$ is PEL-compatible iff its veto levels are both at most $\lfloor \frac{m}{2} \rfloor$, thus, iff $\max_{i \in \{1,2\}} v_i \leq \frac{m}{2}$.*

Proof. For all $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, if $x \in S(\boldsymbol{P}) \cap PE(\boldsymbol{P})$, then x is not among the last $\lfloor \frac{m}{2} \rfloor$ ranks, as follows from Lemma 1. A PV rule with veto parameters at most $\lfloor \frac{m}{2} \rfloor$ will thus pick all such alternatives $S(\boldsymbol{P}) \cap PE(\boldsymbol{P})$, as required by PEL-compatibility.

For the other direction, observe that there exists $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$ such that for some $x \in \mathcal{A}$, $x \in S(\boldsymbol{P}) \cap PE(\boldsymbol{P})$ and x is positioned just better than the last $\lfloor \frac{m}{2} \rfloor$ ranks (thus $\exists \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N, x \in \mathcal{A} \mid \forall i \in \{1,2\} : \lambda_{\boldsymbol{P}}(x)_i = \lfloor \frac{m-1}{2} \rfloor$, leaving $\lfloor \frac{m}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$ positions behind x). A PV rule such that $\max_{i \in \{1,2\}} v_i > \lfloor \frac{m}{2} \rfloor$ will thus not include x in the set of winners, hence, the rule will fail PEL-compatibility. □

Proposition 5 leads to the following corollary for the binding Pareto-and-veto rules.

Corollary 3. *For any $m \geq 3$, a rule $PV^{(v_1,v_2)} \in \mathcal{PV}^b$ is PEL-compatible iff $v_i = \lfloor \frac{m-1}{2} \rfloor$ and $v_{\bar{i}} = \lfloor \frac{m}{2} \rfloor$ for any $\{i, \bar{i}\} = \{1, 2\}$.*

The rules that fail PEL-compatibility will a fortiori fail the following stronger version of the Paretian equal loss property which requires that the Pareto efficient alternative ranked at the same position by both individuals, if it exists, be the unique outcome.

Definition 6 (Paretian equal loss (PEL))**.** $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : [S(\boldsymbol{P}) \cap PE(\boldsymbol{P}) \neq \emptyset] \Rightarrow f(\boldsymbol{P}) = S(\boldsymbol{P}) \cap PE(\boldsymbol{P})$.

Thus, *VR*, *SL* and those rules in $\mathcal{PV}$ that are not PEL-compatible all fail PEL. Furthermore, as we state and show below, even the rules in $\mathcal{PV}$ that are PEL-compatible fail PEL.

Proposition 6. *When $m \geq 4$, all rules in $\mathcal{PV}$ fail PEL.*

Proof. Let $\mathcal{A} = \{a, b, c, a_4, a_5, \ldots\}$. Consider the following profile $\boldsymbol{P}$:

$$\begin{array}{cccccc} a & b & c & a_4 & a_5 & \ldots \\ c & b & a & a_4 & a_5 & \ldots \end{array}.$$

PEL requires to pick solely b. To have $f \in \mathcal{PV}$ and $f(\boldsymbol{P}) = \{b\}$ requires that $v_1 \geq m-2$ (for $c \notin f(\boldsymbol{P})$) and $v_2 \geq m-2$ (for $a \notin f(\boldsymbol{P})$), which implies $2m-4 \leq \sum v_i$. Also, the definition of $\mathcal{PV}$ requires that $\sum v_i \leq m-1$. However, $2m-4 \leq m-1$ is satisfied only when $m \leq 3$. □

Remark 6. When $m = 3$, $PV^{(1,1)}$ satisfies PEL.

Proposition 7. *UC satisfies PEL.*

Proof. Pick any profile $\boldsymbol{P}$ and any $x \in PE(\boldsymbol{P})$ with $\lambda_{\boldsymbol{P}}(x) = (k, k)$ for some $k \in [\![0, m-1]\!]$. Let us show that $UC(\boldsymbol{P}) = \{x\}$. Consider any $y \in UC(\boldsymbol{P})$. As UC minimizes the maximal loss, $\max \lambda_{\boldsymbol{P}}(y) \leq \max \lambda_{\boldsymbol{P}}(x) = k$. Since $x \in PE(\boldsymbol{P})$, we have $\max \lambda_{\boldsymbol{P}}(y) \geq k$ (otherwise $\min \lambda_{\boldsymbol{P}}(y) \leq \max \lambda_{\boldsymbol{P}}(y) < k$ and y Pareto-dominates x). Hence, $\max \lambda_{\boldsymbol{P}}(y) = k$, thus $x = y$. □

PEL, failed by all rules we consider but one, allows to distinguish UC from the rest. Moreover, by satisfying both conditions, UC establishes the compatibility between PEL and MR. However, as discussed below, this compatibility vanishes when another stronger version of PEL-compatibility is adopted.

Call the dispersion of a loss vector l at $\boldsymbol{P}$ the value $d(l) = |l_1 - l_2|$. Thus, $(d \circ \lambda_{\boldsymbol{P}})(x) = \max \lambda_{\boldsymbol{P}}(x) - \min \lambda_{\boldsymbol{P}}(x)$. We show in Appendix A that d coincides with several commonly used spread measures.

Given a profile $\boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$, define $\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ as the minimal dispersion obtained by loss vectors of Pareto efficient alternatives in that profile, and $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ as the Pareto efficient alternatives whose loss vectors have minimal dispersion among Pareto efficient alternatives.

Define the minimal dispersion (MD) condition as follows.

Definition 7 (Minimal dispersion)**.** $f(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$, $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N$.

MD requires the outcome to contain some Pareto efficient alternatives whose loss vectors have minimal dispersion. As such, MD is another strengthening of PEL-compatibility while it is logically independent of PEL. Nevertheless, although there are rules that satisfy both PEL and MR, MD turns out to be logically incompatible with MR (except for sufficiently low values of m).

Theorem 4. *$[\exists f \in \mathcal{MR} \cap \mathcal{MD}] \Leftrightarrow [m \leq 6 \vee m = 8]$.*

Proof. We prove the theorem through Lemmas 2 to 5.

Lemma 2 states that a Pareto efficient alternative that minimizes dispersion among the top $\lceil \frac{2m-4}{3} \rceil$ of both individuals always exists.

Lemma 2. *Given* $m \geq 3, \forall \boldsymbol{P} : \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, \lceil \frac{2m-4}{3} \rceil \rrbracket \times \llbracket 0, \lceil \frac{2m-4}{3} \rceil \rrbracket) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$.

Proof. Consider any $x \in \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ and let i denote any individual such that $\max \lambda_{\boldsymbol{P}}(x) = \lambda_{\boldsymbol{P}}(x)_i$. In the sake of brevity, define $t = \lceil \frac{2m-4}{3} \rceil$.

If $\lambda_{\boldsymbol{P}}(x)_i \leq t$ then $\lambda_{\boldsymbol{P}}(x)_{\bar{i}} \leq t$ so that $x \in \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, t \rrbracket \times \llbracket 0, t \rrbracket) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$, proving the claim.

If $\lambda_{\boldsymbol{P}}(x)_i > t$, let $A = \{a \mid \lambda_{\boldsymbol{P}}(a)_i \leq t\}$ be the $t+1$ top alternatives for i. Define $y = \arg\min_A \lambda_{\boldsymbol{P}}(.)_{\bar{i}}$ as the best alternative for $\bar{i}$ in A. Observe that $y \in PE(\boldsymbol{P})$: for i, only the alternatives in A can be preferred to y, and those are less preferred than y for $\bar{i}$.

Since $\lambda_{\boldsymbol{P}}(x)_i > t$ and $y \in A$, we see that $\lambda_{\boldsymbol{P}}(y)_i < \lambda_{\boldsymbol{P}}(x)_i$. It follows that $\lambda_{\boldsymbol{P}}(x)_{\bar{i}} < \lambda_{\boldsymbol{P}}(y)_{\bar{i}}$ to avoid that y Pareto-dominates x. Moreover, if $\lambda_{\boldsymbol{P}}(y)_{\bar{i}} \leq \lambda_{\boldsymbol{P}}(y)_i$, then

$$\lambda_{\boldsymbol{P}}(x)_{\bar{i}} < \lambda_{\boldsymbol{P}}(y)_{\bar{i}} \leq \lambda_{\boldsymbol{P}}(y)_i < \lambda_{\boldsymbol{P}}(x)_i$$

which implies that $d(\lambda_{\boldsymbol{P}}(y)) < d(\lambda_{\boldsymbol{P}}(x))$, contradicting $x \in \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$. Thus, we have $\lambda_{\boldsymbol{P}}(y)_i < \lambda_{\boldsymbol{P}}(y)_{\bar{i}}$.

Observe that $2(m-t-1) \leq t+2$. Indeed, $\frac{2m-4}{3} \leq t = \lceil \frac{2m-4}{3} \rceil$, thus $2m-4 \leq 3t$.

Also, when $m \geq 3$, $m-t-1 \leq t$. Indeed, define $\epsilon = t - \frac{2m-4}{3}$, then $m-1 \leq 2t$ iff $3m-3 \leq 6(\epsilon + \frac{2m-4}{3})$ iff $5 \leq m + 6\epsilon$ which holds as when $m=3$, $\epsilon = 1/3$ and when $m=4$, $\epsilon = 2/3$.

By construction, $\forall a \neq y \in A : \lambda_{\boldsymbol{P}}(y)_{\bar{i}} < \lambda_{\boldsymbol{P}}(a)_{\bar{i}}$, hence, $\lambda_{\boldsymbol{P}}(y)_{\bar{i}} \leq m - 1 - \#(A \setminus \{y\}) = m-t-1$. We obtain that $\lambda_{\boldsymbol{P}}(y)_i < \lambda_{\boldsymbol{P}}(y)_{\bar{i}} \leq m-t-1 \leq t$, which yields that $y \in \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, t \rrbracket \times \llbracket 0, t \rrbracket)$ and that $d(\lambda_{\boldsymbol{P}}(y)) \leq m-t-1$.

We also have that $m-t-1 \leq d(\lambda_{\boldsymbol{P}}(x))$: from $2(m-t-1) \leq t+2$, $m-t-1 \leq t+2-(m-t-1) = 2(t+1)-m+1 \leq 2\lambda_{\boldsymbol{P}}(x)_i - m + 1 = 2\max \lambda_{\boldsymbol{P}}(x) - m + 1$; and by Lemma 1, $\max \lambda_{\boldsymbol{P}}(x) \leq m-1-\min \lambda_{\boldsymbol{P}}(x)$, whence $2\max \lambda_{\boldsymbol{P}}(x) - m + 1 \leq \max \lambda_{\boldsymbol{P}}(x) - m + 1 + m - 1 - \min \lambda_{\boldsymbol{P}}(x) = d(\lambda_{\boldsymbol{P}}(x))$.

To conclude, note that $d(\lambda_{\boldsymbol{P}}(y)) \leq m-t-1 \leq d(\lambda_{\boldsymbol{P}}(x))$ implies that $y \in \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$. □

Lemma 3 shows the existence of some profile where there is a Pareto efficient and dispersion minimizing alternative among the top t alternatives of one of the individuals, as long as $t < \frac{2m-4}{3}$.

Lemma 3. *For any $t \in \mathbb{N}$ with $t < \frac{2m-4}{3}$, we have the following two implications:*

- $\exists \boldsymbol{P} \mid \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, t \rrbracket \times \llbracket 0, m-1 \rrbracket) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \emptyset$;
- $\exists \boldsymbol{P} \mid \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, m-1 \rrbracket \times \llbracket 0, t \rrbracket) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \emptyset$.

Proof. We prove only the first implication: the other one admits a similar proof by swapping the roles of individuals 1 and 2.

Note that $\forall k \in \mathbb{N}$, $k < \frac{2m-4}{3} \Leftrightarrow k \leq \lceil \frac{2m-7}{3} \rceil$: if $\frac{2m-4}{3} \in \mathbb{N}$, $k < \frac{2m-4}{3} = \lceil \frac{2m-4}{3} \rceil \Leftrightarrow k \leq \lceil \frac{2m-4}{3} \rceil - 1$; and otherwise, for some $0 < \epsilon < 1$, $k < \frac{2m-4}{3} = \lceil \frac{2m-4}{3} \rceil - \epsilon \Leftrightarrow k \leq \lceil \frac{2m-4}{3} \rceil - 1$.

When $2 \leq m \leq 3$, the claim holds because $\llbracket 0, \lceil \frac{2m-7}{3} \rceil \rrbracket \times \llbracket 0, m-1 \rrbracket = \emptyset$.

Defining $t' = \max\{\lfloor \frac{m}{2} \rfloor, \lceil \frac{2m-7}{3} \rceil\}$ and considering $4 \leq m$, we will define a profile $\boldsymbol{P}$ such that $\lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, t' \rrbracket \times \llbracket 0, m-1 \rrbracket) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \emptyset$. This proves the claim as, with $t < \frac{2m-4}{3}$ (thus $t \leq \lceil \frac{2m-7}{3} \rceil$), $\llbracket 0, t \rrbracket \times \llbracket 0, m-1 \rrbracket \subseteq \llbracket 0, t' \rrbracket \times \llbracket 0, m-1 \rrbracket$.

Observe that $\lceil \frac{2m-7}{3} \rceil < \frac{2m-4}{3} \leq m-2$ (the latter inequality being equivalent to $2 \leq m$) and $\lfloor \frac{m}{2} \rfloor \leq \frac{m}{2} \leq m-2$ (the latter inequality being equivalent to $4 \leq m$). Thus, $0 \leq m - t' - 2$.

Observe that $m = t' + 1 + (m - t' - 2) + 1$. We label the alternatives as $\{a_1, \ldots, a_{t'+1}, c_1, \ldots, c_{m-t'-2}, x\}$ and consider the sequences $A = (a_1, \ldots, a_{t'+1})$ and $C = (c_1, \ldots, c_{m-t'-2})$. Define $\boldsymbol{P}$ as associating to individual 1 (resp. individual 2) the preference order (A, x, C) (resp. (C, x, A)).

Observe that as $\lambda_{\boldsymbol{P}}(x)_1 = t'+1$, $x \notin \lambda_{\boldsymbol{P}}^{-1}(\llbracket 0, t' \rrbracket \times \llbracket 0, m-1 \rrbracket)$, and $x \in PE(\boldsymbol{P})$, so the claim is proved by showing that $\forall y \in A \cup C : d(\lambda_{\boldsymbol{P}}(y)) > d(\lambda_{\boldsymbol{P}}(x))$, so that $\forall y \in A \cup C : y \notin \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$.

Note that $\#A - \#C \geq 0$ as $\#A - \#C = t' + 1 - (m - t' - 2) = 2t' + 3 - m \geq 2\lfloor \frac{m}{2} \rfloor + 3 - m \geq 2\frac{m-1}{2} + 3 - m = 2$ where we rely on the equality $\lfloor \frac{m}{2} \rfloor = \lceil \frac{m-1}{2} \rceil$ which holds for any integer m.

Thus, $d(\lambda_{\boldsymbol{P}}(x)) = \#A - \#C = 2t' + 3 - m$.

For any $a_i \in A$, observe that $\lambda_{\boldsymbol{P}}(a_i) = (i-1, \#C + 1 + i - 1)$ which implies that $d(\lambda_{\boldsymbol{P}}(a_i)) = m - t' - 1$. It follows that $d(\lambda_{\boldsymbol{P}}(a_i)) > d(\lambda_{\boldsymbol{P}}(x))$, equivalently, $m - t' - 1 > 2t' + 3 - m$, as $3t' < 2m - 4$ by hypothesis.

Likewise, for any $c_i \in C$, $\lambda_{\boldsymbol{P}}(c_i) = (\#A + 1 + (i-1), i-1)$ thus $d(\lambda_{\boldsymbol{P}}(c_i)) = t' + 2$. Using $\#A \geq \#C$, it follows that $d(\lambda_{\boldsymbol{P}}(c_i)) = \#A + 1 \geq \#C + 1 = d(\lambda_{\boldsymbol{P}}(a_i)) > d(\lambda_{\boldsymbol{P}}(x))$, thus $d(\lambda_{\boldsymbol{P}}(c_i)) > d(\lambda_{\boldsymbol{P}}(x))$. □

The next lemma states that to ensure the existence of an alternative within the first t alternatives for both players that minimizes dispersion among the Pareto efficient ones, it is necessary and sufficient to set t at least $\frac{2m-4}{3}$. This follows from Lemmas 2 and 3.

Lemma 4. *Given $m \geq 3$, $\forall t \in [\![0, m-1]\!]$, the following two statements are logically equivalent:*

1. $t \geq \frac{2m-4}{3}$,
2. $\forall \boldsymbol{P} : \lambda_{\boldsymbol{P}}^{-1}([\![0,t]\!] \times [\![0,t]\!]) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$.

Lemma 4 in turn permits to characterize the number of alternatives in which a Pareto efficient alternative minimizing dispersion is guaranteed to be found in the first half, as stated by Lemma 5.

Lemma 5. $\forall \boldsymbol{P} : H(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset \Leftrightarrow m \leq 6 \vee m = 8$

Proof. When $m = 2$, $H(\boldsymbol{P}) = \lambda_{\boldsymbol{P}}^{-1}([\![0,1]\!] \times [\![0,1]\!]) = \boldsymbol{P}$ thus $H(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$.

When $m \geq 3$, fix $t = \lfloor \frac{m}{2} \rfloor$ and use Lemma 4 to obtain:

$$m \leq \frac{3\lfloor \frac{m}{2} \rfloor + 4}{2} \Leftrightarrow \forall \boldsymbol{P} : H(\boldsymbol{P}) \cap \operatorname*{arg\,min}_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset.$$

The left hand side is equivalent, when m is odd, to $m \leq \frac{3m+5}{4}$ thus $m \leq 5$, in other words, $m \in \{3, 5\}$, and when m is even, to $m \leq \frac{3m+8}{4}$ thus $m \leq 8$, that is, $m \in \{4, 6, 8\}$. $\square$

It follows from Lemma 5 that when $m \leq 6$ or $m = 8$, the SCR $H(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ is well-defined and satisfies MR and MD by construction; and when $m = 7$ or $m \geq 9$, there is no $f \in \mathcal{MR}$ that satisfies MD. $\square$

The minimal Rawlsian and minimal dispersion principles being logically incompatible, the SCRs that satisfy MR (namely *UC*, *VR*, *SL* and the rules in $\mathcal{PV}$ identified by Theorem 3) all fail MD.

Remark 7. As *UC* has a strong egalitarian flavor, it may be surprising that it fails *MD*. The following profile $\boldsymbol{P}$ (built using the proof of Lemma 4) illustrates this failure for $m = 7$.

$$\begin{array}{ccccccc} y & a_1 & a_2 & a_3 & x & c_1 & c_2 \\ c_1 & c_2 & x & y & a_1 & a_2 & a_3 \end{array}.$$

Observe that $UC(\boldsymbol{P}) = \{y\}$ while the *MD* principle mandates x to be picked.

Since Theorem 3 shows that most rules in $\mathcal{PV}$ fail MR, the following result determines the subclass of $\mathcal{PV}$ satisfying MD.

Proposition 8. *For $m \geq 3$, $PV^{(v_1,v_2)}$ satisfies MD iff* $\max_{i \in \{1,2\}} v_i \leq \frac{m+1}{3}$.

Proof. Define $t = \min_{i\in\{1,2\}}(m-1-v_i) \in \mathbb{N}$. Observe that $\max_{i\in\{1,2\}} v_i \leq \frac{m+1}{3} \Leftrightarrow \forall i \in \{1,2\} : v_i \leq \frac{m+1}{3} \Leftrightarrow \forall i \in \{1,2\} : m-1-v_i \geq \frac{2m-4}{3} \Leftrightarrow t \geq \frac{2m-4}{3}$.

If $t = m-1-v_1$, define $S = [\![0,t]\!] \times [\![0,m-1]\!]$, otherwise (implying that $t = m-1-v_2$), define $S = [\![0,m-1]\!] \times [\![0,t]\!]$. By definition, $\forall \boldsymbol{P} : PV^{(v_1,v_2)}(\boldsymbol{P}) = \lambda_{\boldsymbol{P}}^{-1}([\![0,m-1-v_1]\!] \times [\![0,m-1-v_2]\!]) \cap PE(\boldsymbol{P})$. It follows that $\forall \boldsymbol{P}$:

$$\lambda_{\boldsymbol{P}}^{-1}([\![0,t]\!] \times [\![0,t]\!]) \cap PE(\boldsymbol{P}) \subseteq PV^{(v_1,v_2)}(\boldsymbol{P})$$

and

$$PV^{(v_1,v_2)}(\boldsymbol{P}) \subseteq \lambda_{\boldsymbol{P}}^{-1}(S) \cap PE(\boldsymbol{P}),$$

therefore (intersecting all sets with $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ and using $PE(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$):

$$\lambda_{\boldsymbol{P}}^{-1}([\![0,t]\!] \times [\![0,t]\!]) \cap \operatorname*{arg\,min}_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \subseteq PV^{(v_1,v_2)}(\boldsymbol{P}) \cap \operatorname*{arg\,min}_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$$

and

$$PV^{(v_1,v_2)}(\boldsymbol{P}) \cap \operatorname*{arg\,min}_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \subseteq \lambda_{\boldsymbol{P}}^{-1}(S) \cap \operatorname*{arg\,min}_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}).$$

It follows from Lemmas 3 and 4 that $t \geq \frac{2m-4}{3} \Leftrightarrow \forall \boldsymbol{P} : \lambda_{\boldsymbol{P}}^{-1}([\![0,t]\!] \times [\![0,t]\!]) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset \Leftrightarrow \forall \boldsymbol{P} : \lambda_{\boldsymbol{P}}^{-1}(S) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$, and thus $\max_{i\in\{1,2\}} v_i \leq \frac{m+1}{3} \Leftrightarrow t \geq \frac{2m-4}{3} \Leftrightarrow \forall \boldsymbol{P} : PV^{(v_1,v_2)}(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$. □

Remark 8. As MD implies PEL-compatibility, the class of PV rules that satisfy MD is a subclass of those that are PEL-compatible. This relationship can be more precisely observed by comparing Proposition 5 and Proposition 8.

Remark 9. Proposition 8 shows that PV rules with $\max_i v_i \leq \frac{m+1}{3}$ satisfy MD. On the other hand, these rules fail two stronger versions of MD, one which mandates to pick *all* minimal dispersion alternatives (thus $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \subseteq f(\boldsymbol{P})$) and the other which mandates to pick *only* minimal dispersion alternatives (thus $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \supseteq f(\boldsymbol{P})$). To illustrate this for the former, for $m = 5$, consider the rule $PV^{(2,2)}$ and the profile $\boldsymbol{P}$.

$$\begin{matrix} a & b & c & d & e \\ c & d & e & b & a \end{matrix}.$$

Here, $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \{b, c\}$ while $PV^{(2,2)}(\boldsymbol{P}) = \{c\}$.

For the latter, consider the profile $\boldsymbol{P}$ with $m = 7$:

$$\begin{matrix} a & b & c & d & e & f & g \\ g & f & e & d & c & b & a \end{matrix}.$$

Observe that $\arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) = \{d\}$ whereas $PV^{(2,2)}(\boldsymbol{P}) = \{c, d, e\}$.

Remark 10. To ensure $\arg\min_{PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}}) \subseteq f(\boldsymbol{P})$ and $\arg\min_{PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}}) \supseteq f(\boldsymbol{P})$, one can define the rule $MD(\boldsymbol{P}) = \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ which selects all Pareto efficient alternatives that minimize the dispersion and only those. As mentioned in Footnote 5, this rule is the finite version of Thomson's [1994] equal area rule in our framework with no disagreement outcome. As the rules UC, VR and SL fail the MD principle, they are distinct from the rule MD. Also, the rule MD is not in $\mathcal{PV}$: MD satisfies PEL while no rule in $\mathcal{PV}$ does (for $m \geq 4$), as shown by Proposition 6. It follows that satisfying MD does not imply being a Pareto-and-veto rule.

We end this section by showing that every profile admits some Pareto efficient alternative with a dispersion of at most $\lfloor \frac{m}{2} \rfloor$. Moreover, there is no guarantee of finding an efficient alternative with a dispersion lower than $\lfloor \frac{m}{2} \rfloor$. In other words, $\lfloor \frac{m}{2} \rfloor$ is a general upper bound on dispersion.

Proposition 9. $[\forall \boldsymbol{P}, \exists x \in PE(\boldsymbol{P}) \mid d(\lambda_{\boldsymbol{P}}(x)) \leq k] \Leftrightarrow k \geq \lfloor \frac{m}{2} \rfloor$.

Proof. Theorem 1 establishes, $\forall \boldsymbol{P}$, the existence of an alternative $x \in PE(\boldsymbol{P}) \cap H(\boldsymbol{P})$. Thus $d(\lambda_{\boldsymbol{P}}(x)) \leq \lfloor \frac{m}{2} \rfloor$, proving the "if" part. To show the "only if" part, consider the profile

$$\begin{array}{llllllll} a_1 & a_2 & \dots & a_{\lfloor \frac{m}{2} \rfloor} & a_{\lfloor \frac{m}{2} \rfloor+1} & a_{\lfloor \frac{m}{2} \rfloor+2} & \dots & a_m \\ a_{\lfloor \frac{m}{2} \rfloor+1} & a_{\lfloor \frac{m}{2} \rfloor+2} & \dots & a_m & a_1 & a_2 & \dots & a_{\lfloor \frac{m}{2} \rfloor} \end{array}$$

where only a_1 and $a_{\lfloor \frac{m}{2} \rfloor+1}$ are Pareto efficient. Their dispersion is $\lfloor \frac{m}{2} \rfloor$. Thus, one cannot ensure the existence of an efficient alternative with $k < \lfloor \frac{m}{2} \rfloor$. ☐

Remark 11. Given that there is an upper bound on dispersion, one can define the following condition. An SCR f satisfies k-bounded dispersion iff $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N, \forall x \in f(\boldsymbol{P}), d(\lambda_{\boldsymbol{P}}(x)) \leq k$. The condition gets weaker with increasing k. Moreover, by Proposition 9, it is satisfiable if and only if $k \geq \lfloor \frac{m}{2} \rfloor$. Therefore, its strongest satisfiable version is $\lfloor \frac{m}{2} \rfloor$-bounded dispersion, which is implied by MR.

6. Reconciling the two principles

Given the incompatibility between MR and MD, one can attempt to reconcile the two principles by imposing minimal dispersion over the alternatives that are minimally Rawlsian.

Given a profile, let us call "Rawlsian minimal dispersion (RMD) alternatives" those that minimize dispersion among the Pareto efficient alternatives within the first half. As every profile admits RMD alternatives, we can define the RMD principle as the requirement that the social choice always contains an RMD alternative while remaining within the first half.

Definition 8 (Rawlsian minimal dispersion)**.** $\forall \boldsymbol{P} \in \mathscr{L}(\mathcal{A})^N : f(\boldsymbol{P}) \cap \arg\min_{H(\boldsymbol{P}) \cap PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$ and $f(\boldsymbol{P}) \subseteq H(\boldsymbol{P})$.

RMD strengthens MR. On the other hand, the relationship between RMD and MD depends on m: when m is small, RMD is stronger than MD while when m is large, no rule can satisfy both RMD and MD (because there are profiles where all Pareto efficient dispersion minimizers are out of the first half, by Lemma 5). Summarizing, when m is small enough ($m \leq 6$ or $m = 8$), and only then, first, RMD implies MD, and second, some rules are both RMD and MD.

Proposition 10. *$[m \leq 6 \vee m = 8] \Leftrightarrow \mathcal{RMD} \subseteq \mathcal{MD} \Leftrightarrow \mathcal{RMD} \cap \mathcal{MD} \neq \emptyset$.*

Proof. First consider $[m \leq 6 \vee m = 8] \Rightarrow \mathcal{RMD} \subseteq \mathcal{MD}$. Lemma 5 indicates that $\forall \boldsymbol{P} : H(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset \Leftrightarrow m \leq 6 \vee m = 8$. And $[H(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset] \Rightarrow [\arg\min_{H(\boldsymbol{P}) \cap PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \subseteq \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})]$. Given $f \in \mathcal{RMD}$, $f(\boldsymbol{P}) \cap \arg\min_{H(\boldsymbol{P}) \cap PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$, thus, when $m \leq 6 \vee m = 8$, $f(\boldsymbol{P}) \cap \arg\min_{PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}}) \neq \emptyset$.

Second, observe that $\mathcal{RMD} \subseteq \mathcal{MD} \Rightarrow \mathcal{RMD} \cap \mathcal{MD} \neq \emptyset$.

To conclude, we prove that $\mathcal{RMD} \cap \mathcal{MD} \neq \emptyset \Rightarrow [m \leq 6 \vee m = 8]$. As $\mathcal{RMD} \subseteq \mathcal{MR}$, we have $[\mathcal{RMD} \cap \mathcal{MD} \neq \emptyset] \Rightarrow [\mathcal{MR} \cap \mathcal{MD} \neq \emptyset]$. And from Theorem 4, the latter implies $[m \leq 6 \vee m = 8]$. $\square$

Theorem 5. *UC, VR, SL fail RMD, and $\mathcal{PV} \cap \mathcal{RMD} = \{PV^=\}$.*

Proof. Consider the following profile $\boldsymbol{P}$ with 11 alternatives, $\{a, b, c, d, e, f, g, h, w, x, y\}$ (the bar indicates the "half" position).

$$\begin{array}{cccccc|cc} x & a & b & c & d & y & w & \dots \\ e & f & g & y & x & h & w & \dots \end{array}.$$

$H(\boldsymbol{P}) = \{x, y\}$ which are both Pareto efficient. For a SCR f to satisfy RMD, we must have $y \in f(\boldsymbol{P})$. As $UC(\boldsymbol{P}) = VR(\boldsymbol{P}) = \{x\}$, both rules fail RMD.

To see that SL fails RMD, consider the following profile $\boldsymbol{P}$ with $SL(\boldsymbol{P}) = \{a, c\}$ while RMD requires $b \in SL(\boldsymbol{P})$.

$$\begin{array}{ccc|cc} a & b & c & d & e \\ c & b & a & d & e \end{array}.$$

We now turn to the rules in $\mathcal{PV}$. As RMD implies MR, $\mathcal{PV} \cap \mathcal{RMD} \subseteq \mathcal{PV} \cap \mathcal{MR}$, and by Theorem 3, $\mathcal{PV} \cap \mathcal{MR} = \{PV^=\}$ when m is odd and $\mathcal{PV} \cap \mathcal{MR} = \{PV^=, PV^{(\frac{m}{2},\frac{m}{2}-1)}, PV^{(\frac{m}{2}-1,\frac{m}{2})}\}$ when m is even. Thus, we need only prove that for m even, $PV^{(\frac{m}{2},\frac{m}{2}-1)}$ and $PV^{(\frac{m}{2}-1,\frac{m}{2})}$ fail RMD. To see this, consider the profile $\boldsymbol{P}$

$$\begin{array}{cccc} a & b & c & d \\ d & c & a & b \end{array}$$

where $PV^{(\frac{m}{2},\frac{m}{2}-1)}(\boldsymbol{P}) = \{a\}$ while RMD requires $c \in PV^{(\frac{m}{2},\frac{m}{2}-1)}(\boldsymbol{P})$. A similar argument shows that $PV^{(\frac{m}{2}-1,\frac{m}{2})}$ fails RMD.

By Theorem 2, $PV^=(\boldsymbol{P}) = H(\boldsymbol{P}) \cap PE(\boldsymbol{P})$, thus, it satisfies RMD. $\square$

$PV^=$ satisfies the RMD principle at the expense of resoluteness, since it picks every Pareto efficient alternative within the first half. It is thus natural to seek for a stronger principle that imposes more resoluteness. To this end, we define the Strong RMD principle, which requires to pick only RMD alternatives.

Definition 9 (Strong Rawlsian minimal dispersion)**.** $\forall \boldsymbol{P} \in \mathcal{L}(\mathcal{A})^N : f(\boldsymbol{P}) \subseteq \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$.

The following result is the counterpart to Proposition 10 for SRMD.

Proposition 11. $[m \leq 6 \vee m = 8] \Leftrightarrow \mathcal{SRMD} \subseteq \mathcal{MD} \Leftrightarrow \mathcal{SRMD} \cap \mathcal{MD} \neq \emptyset$.

Proof. First, it follows from $[m \leq 6 \vee m = 8] \Rightarrow \mathcal{RMD} \subseteq \mathcal{MD}$ (by Proposition 10) and $\mathcal{SRMD} \subseteq \mathcal{RMD}$ that $[m \leq 6 \vee m = 8] \Rightarrow \mathcal{SRMD} \subseteq \mathcal{MD}$.

Second, observe that $\mathcal{SRMD} \subseteq \mathcal{MD} \Rightarrow \mathcal{SRMD} \cap \mathcal{MD} \neq \emptyset$.

To conclude, we prove that $\mathcal{SRMD} \cap \mathcal{MD} \neq \emptyset \Rightarrow [m \leq 6 \vee m = 8]$. As $\mathcal{SRMD} \subseteq \mathcal{MR}$, we have $[\mathcal{SRMD} \cap \mathcal{MD} \neq \emptyset] \Rightarrow [\mathcal{MR} \cap \mathcal{MD} \neq \emptyset]$. And from Theorem 4, the latter implies $[m \leq 6 \vee m = 8]$. $\square$

It also follows from Theorem 2 that $PV^=$ fails Strong RMD. Thus, no rule that we have examined so far satisfies Strong RMD. Now, we define the Strong RMD rule $f(\boldsymbol{P}) = \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d \circ \lambda_{\boldsymbol{P}})$ as the least resolute rule satisfying Strong RMD. The Strong RMD rule (or SRMD rule) might appear as a reasonable compromise to satisfy the two main concepts exposed here (besides Pareto), that of being minimally Rawlsian and that of favoring small dispersion alternatives. It is also reasonably resolute.

Proposition 12. *The SRMD rule f selects at most two alternatives:* $\forall \boldsymbol{P} \in \mathcal{L}(\mathcal{A})^N : 1 \leq \# f(\boldsymbol{P}) \leq 2$.

Proof. Consider any $x \in \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$, any $i \mid \min\lambda_{\boldsymbol{P}}(x) = \lambda_{\boldsymbol{P}}(x)_i$ and any $y \in \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$. We first prove that $\min\lambda_{\boldsymbol{P}}(y) = \lambda_{\boldsymbol{P}}(y)_i$ iff $y = x$. Equivalently, we prove that $\forall a \in H(\boldsymbol{P}) : \lambda_{\boldsymbol{P}}(a)_i \neq \lambda_{\boldsymbol{P}}(x)_i \Rightarrow [a \notin \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}}) \vee \min\lambda_{\boldsymbol{P}}(a) \neq \lambda_{\boldsymbol{P}}(a)_i]$. Indeed, consider any $a \in H(\boldsymbol{P}) \mid \lambda_{\boldsymbol{P}}(a)_i < \lambda_{\boldsymbol{P}}(x)_i$, then $\lambda_{\boldsymbol{P}}(a)_{\bar{i}} > \lambda_{\boldsymbol{P}}(x)_{\bar{i}}$ (because $x \in PE(\boldsymbol{P})$), so $\lambda_{\boldsymbol{P}}(a)_i < \lambda_{\boldsymbol{P}}(x)_i \leq \lambda_{\boldsymbol{P}}(x)_{\bar{i}} < \lambda_{\boldsymbol{P}}(a)_{\bar{i}}$ and $(d\circ\lambda_{\boldsymbol{P}})(a) > (d\circ\lambda_{\boldsymbol{P}})(x)$; and considering now any $a \in H(\boldsymbol{P}) \mid \lambda_{\boldsymbol{P}}(a)_i > \lambda_{\boldsymbol{P}}(x)_i$, then $\lambda_{\boldsymbol{P}}(a)_{\bar{i}} < \lambda_{\boldsymbol{P}}(x)_{\bar{i}}$ is required for $a \in PE(\boldsymbol{P})$, and if $\min\lambda_{\boldsymbol{P}}(a) = \lambda_{\boldsymbol{P}}(a)_i$ then $\lambda_{\boldsymbol{P}}(x)_i < \lambda_{\boldsymbol{P}}(a)_i \leq \lambda_{\boldsymbol{P}}(a)_{\bar{i}} < \lambda_{\boldsymbol{P}}(x)_{\bar{i}}$ and $(d\circ\lambda_{\boldsymbol{P}})(a) < (d\circ\lambda_{\boldsymbol{P}})(x)$, contradicting $x \in \arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$, so that again $\min\lambda_{\boldsymbol{P}}(a) \neq \lambda_{\boldsymbol{P}}(a)_i$.

To conclude, observe that if $\arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$ contain at least two elements (say, x and y), one of them, say, x, has $\min\lambda_{\boldsymbol{P}}(x) = \lambda_{\boldsymbol{P}}(x)_i$, the other one has $\min\lambda_{\boldsymbol{P}}(y) = \lambda_{\boldsymbol{P}}(y)_{\bar{i}}$, and thus there cannot be a third element z in $\arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$ as neither $\min\lambda_{\boldsymbol{P}}(z) = \lambda_{\boldsymbol{P}}(z)_i$ nor $\min\lambda_{\boldsymbol{P}}(z) = \lambda_{\boldsymbol{P}}(z)_{\bar{i}}$ is possible, by the above result. □

One can define a refinement of the SRMD rule through the notion of *anonymous dominance* of loss vectors.

Given two loss vectors $l, l' \in [\![0, m-1]\!]^N$, say that l dominates l', denoted $l \lhd l'$, iff $\min l \leq \min l' \wedge \max l \leq \max l'$ with one of these inequalities being strict. For any profile $\boldsymbol{P}$, $E(\boldsymbol{P})$ denotes the alternatives whose loss vectors are not dominated: $E(\boldsymbol{P}) = \{x \in \mathcal{A} \mid \nexists y \in \mathcal{A} : \lambda_{\boldsymbol{P}}(y) \lhd \lambda_{\boldsymbol{P}}(x)\}$.

The rule SRMD* that selects $\arg\min_{H(\boldsymbol{P})\cap PE(\boldsymbol{P})\cap E(\boldsymbol{P})}(d\circ\lambda_{\boldsymbol{P}})$ is then a refinement of the SRMD rule, since the anonymous dominance is used as a tie-breaker, and thus picks at most two alternatives.

Note also that this rule is equivalent, within the set of alternatives with minimal dispersion, to minimize the sum of losses (equivalently, pick the Borda preferred alternative). This is also equivalent to minimize the best loss and in turn equivalent to minimize the worst loss: given two losses l, l' with $d(l) = d(l')$, l dominates l' iff $\sum l < \sum l'$ iff $\min l < \min l'$ iff $\max l < \max l'$.

7. Concluding remarks

Axiomatic analysis of social choice rules with or without strategic concerns presents two strands of the literature that complement each other. This complementarity appears less balanced in two-person collective choice problems where there is a clear focus on a strategic analysis that usually adopts sub-

	MR	PEL-compatible	PEL	MD	RMD
UC	✓	✓	✓		
VR	✓				
SL	✓				

Table 1: Summary of the results about UC, VR and SL. A tick means that the SCR satisfies the corresponding property for any value of $m \geq 4$ and n.

game perfect equilibrium as the solution concept.[8]

The richness of the non-strategic axiomatic analysis of collective choice with three or more individuals is accompanied by a wealthy list of conceived social choice rules. On the other hand, for two individuals, it is hard to name a prominent social choice rule beyond unanimity compromise, the shortlisting rule, the veto-rank rule, and the class of Pareto-and-veto rules. Moreover, these social choice rules are conceived under different interpretations of the two-person collective choice model, thus being analyzed from somewhat different perspectives.

We bring a consideration based on a common interpretation and perspective. The axiomatic analysis we propose is free of strategic concerns and relies on two basic principles that we identify: the minimally Rawlsian principle and the equal loss principle. These two principles that emerge from the existing literature exhibit an incompatibility. More precisely, no minimally Rawlsian social choice rule can ensure to pick an alternative that minimize the dispersion of the loss vector. Tables 1 and 2 summarize our findings.[9]

In front of this incompatibility, the literature seems to favor the minimally Rawlsian principle, as unanimity compromise, the shortlisting rule and the veto-rank rule are minimally Rawlsian. Moreover, within the class of Pareto-and-veto rules, perhaps the most prominent ones, namely those which give both individuals the highest equal or almost equal veto power are minimally Rawlsian. By the established incompatibility, these rules cannot minimize

[8]While we will not restate here the relevant papers already cited in the introduction, we wish to remark that the focus on subgame perfect equilibrium can be explained by the classical result of Hurwicz and Schmeidler [1978] and Maskin [1999] on the impossibility of Nash implementing Pareto efficient two-person social choice rules.

[9]It should be noted that the axioms listed on Table 1 do not characterize the SCRs that we discuss. This is somehow expected, as these axioms impose conditions at a given profile without alluding to the consequences when the given profile is modified, i.e., they are "punctual" in the sense of Thomson [2012]. We thank Sylvain Béal for this comment.

	MR	PEL-comp.	PEL	MD	RMD
$\mathcal{PV}$	$v_1, v_2 \geq \lfloor \frac{m-1}{2} \rfloor$	$\max v_i \leq \frac{m}{2}$		$\max v_i \leq \frac{m+1}{3}$	$v_1 = v_2 = \lfloor \frac{m-1}{2} \rfloor$

Table 2: Summary of the results about the class $\mathcal{PV}$. Each cell specifies those members that satisfy the corresponding property.

loss dispersion but, except the Pareto-and-veto rule that gives the highest equal veto power, they do not even minimize loss dispersion among the Pareto efficient alternatives that are ranked within the first half of both individuals.

Among the rules we consider, those that minimize loss dispersion are the Pareto-and-veto rules that give each individual a veto power that does not exceed a third of the total number of alternatives. These rules will typically make coarse choices with several tied alternatives. Moreover, they fail two stronger versions of the minimal dispersion condition, one which mandates to pick all minimal dispersion alternatives and the other which mandates to pick only minimal dispersion alternatives.

Therefore, it is of interest to introduce rules that are compatible with the minimal dispersion principle. One possibility is to pick at every preference profile the Pareto efficient alternatives that minimize loss dispersion and only those. This rule is the finite version of Thomson's [1994] equal area rule in our framework with no disagreement outcome. By the incompatibility we establish, this rule will be incompatible with the minimal Rawlsian principle. Another possibility that allows to comply with the minimal Rawlsian principle is to pick the Pareto efficient alternatives within the first half of both individuals that minimize loss dispersion. This rule is somehow a counterpart of the veto rank rule that picks the Pareto efficient alternatives within the first half of both individuals that maximize average rank. However, it is more resolute, as it picks at most two alternatives at any given preference profile. Seemingly none of these two rules are considered in the literature and their properties are not known, which suggests that there is room to conceive and analyze new solutions to the two-person social choice problem.

A. Spread measures and minimal dispersion

Recall that we call the dispersion of a loss vector l at $\boldsymbol{P}$ the value $d(l) = |l_1 - l_2|$.

We reuse the following definitions from Cailloux et al. [2022], letting n denote the number of individuals, $l \in \mathbb{R}^n$ denote a generalized loss vector and $\bar{l} = \sum_{i=1}^{n} l_i / n$ denote the arithmetic mean of the losses:

- the mean absolute difference $\sigma_{mad}(l) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |l_i - l_j|$;
- the average absolute deviation $\sigma_{ad}(l) = \frac{\sum_{i=1}^{n} |l_i - \bar{l}|}{n}$;
- the standard deviation $\sigma_{sd}(l) = \sqrt{\frac{\sum_{i=1}^{n} (l_i - \bar{l})^2}{n}}$;
- the Gini coefficient $\sigma_G(l) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |l_i - l_j|}{2n \sum_{i=1}^{n} l_i}$.

When $n = 2$ as in our case, it is plain that $\forall i \in \{1, 2\} : |l_i - \bar{l}| = \frac{|l_1 - l_2|}{2}$; $\sum_{i=1}^{n} \sum_{j=1}^{n} |l_i - l_j| = 2|l_1 - l_2|$; $\forall l \in \mathbb{R}^n : \sigma_{mad}(l) = \sigma_{ad}(l) = \sigma_{sd}(l) = \frac{d(l)}{2}$ and $\sigma_G(l) = \frac{|l_1 - l_2|}{2(l_1 + l_2)}$. Thus, σ_{mad}, σ_{ad} and σ_{sd} coincide with d, but σ_G does not. For example, the Gini coefficient considers $(49, 51)$ as less unequal than $(0, 1)$ whereas d orders these vectors reversely.

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