Arguing about voting rules

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https://github.com/oliviercailloux/Arguing-about-voting-rules

Outline

Introduction

Context

- Voting rule: a systematic way of aggregating different opinions and decide
- Multiple reasonable ways of doing this
- Different voting rules have different interesting properties
- None satisfy all desirable properties
- Some are more reasonable than others

Our goal

We want to easily communicate about strength and weaknesses of voting rules.

Voting rule

Alternatives $\mathscr{A} = \{ a, b, c, d, \dots \}$

Possible voters $\mathscr{N} = \{ 1, 2, \dots \}$

Voters $\emptyset \subset N \subseteq \mathscr{N}$

Profile function $\mathbf{R}$ from N to linear orders on $\mathscr{A}$.

Voting rule function f mapping each $\mathbf{R}$ to winners $\emptyset \subset A \subseteq \mathscr{A}$.

$$\underbrace{\begin{array}{cc} \mathbf{R}(1) & \mathbf{R}(2) \\ \hline \multicolumn{1}{|c}{a} & \multicolumn{1}{c|}{b} \\ \multicolumn{1}{|c}{b} & \multicolumn{1}{c|}{a} \\ \multicolumn{1}{|c}{c} & \multicolumn{1}{c|}{c} \\ \hline \end{array}}_{\mathbf{R}} \xrightarrow{f} A = \{ a, b \}$$

Borda

Given a profile $\mathbf{R}$:

- count the score of each alternative;
- the highest scores win.
- Score of $a \in \mathscr{A}$ is the number of alternatives it beats.

$$\mathbf{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix}.$$

- score a is…?

Borda

Given a profile $\mathbf{R}$:

- count the score of each alternative;
- the highest scores win.
- Score of $a \in \mathscr{A}$ is the number of alternatives it beats.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- score a is...? $3 + 1 + 2 = 6$
- score b is $0 + 3 + 3 = 6$
- score c is $1 + 2 + 1 = 4$
- score d is $2 + 0 + 0 = 2$

Winners are $\{ a, b \}$.

Condorcet's principle

Condorcet's principle

We ought to take the Condorcet winner as sole winner if it exists.

- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}. \text{ Who wins?}$$

Condorcet's principle

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- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\mathbf{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix}. \text{ Who wins?} b$$

How are voting rules analyzed?

- Examples featuring counter-intuitive results for some voting rules.
- Properties of voting rules, e.g. Borda does not satisfy Condorcet's principle.
- Axiomatization of a voting rule: accepting such principles lead to a unique voting rule.

Our objective

- Different voting rules
- Arguments in favor or against rules
- Dispersed in the literature
- Using mathematical formalism

We propose

- Common language
- Instantiate arguments on concrete examples

Goal: help understand strengths and weaknesses of given rules.

Outline

Example

Who should win?

Voter 1: $a \succ b \succ c$
Voter 2: $a \succ b \succ c$
Voter 3: $c \succ b \succ a$

- Veto rule chooses b
- Borda rule chooses a

Voter 1:	$a \succ b \succ c$
Voter 2:	$a \succ b \succ c$
Voter 3:	$c \succ b \succ a$

System: Take the *red subprofile*. Here, *a should win*, right? [unanimity]

User: Obviously!

System: Now consider the *green subprofile*. For symmetry reasons, there should be a *three-way tie*, right? [cancellation]

User: Sounds reasonable.

System: So, as there was a three-way tie for the green part, the red part should decide the overall winner, right? [reinforcement]

User: Yes.

System: To summarise, you agree that a should win.

Language

We use propositional logic (with connectives $\neg, \vee, \wedge, \rightarrow$).

Atoms

- One atom for each $(\mathbf{R}, A)$, $\emptyset \subset A \subseteq \mathscr{A}$.
- An atom talks about assigning winners A to $\mathbf{R}$.
- Written $[\mathbf{R} \longmapsto A]$.

Semantics

Semantics v_f, given a voting rule f:

$$v_f([\mathbf{R} \longmapsto A]) = \mathrm{T} \text{ iff } f(\mathbf{R}) = A.$$

Dominance l-axiom

Dom

L-axiom Dom: for each $\mathbf{R}$,

$$[\mathbf{R} \xhookrightarrow{\in} \mathcal{P}_{\emptyset}(U_{\mathbf{R}})],$$

with $U_{\mathbf{R}}$ the set of alternatives in $\mathbf{R}$ that are not dominated.

Symmetric cancellation I-axiom

SYM

For each $\mathbf{R}$ consisting of a linear order and its inverse,

$$[\mathbf{R} \longmapsto \mathscr{A}].$$

$\mathbf{R}_1 = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}$, constraints? $f(\mathbf{R}_1) =$

Symmetric cancellation I-axiom

Sym

For each $\mathbf{R}$ consisting of a linear order and its inverse,

$$[\mathbf{R} \longmapsto \mathscr{A}].$$

$$\mathbf{R}_1 = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}\text{, constraints? } f(\mathbf{R}_1) = \mathscr{A} = \{\, a, b, c \,\}.$$

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For each $\mathbf{R}$ consisting of a linear order and its inverse,

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$$\mathbf{R}_1 = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}\text{, constraints? } f(\mathbf{R}_1) = \mathscr{A} = \{\, a, b, c \,\}.$$

$$\mathbf{R}_2 = \begin{matrix} a & b \\ b & a \\ c & c \end{matrix}\text{, constraints?}$$

Symmetric cancellation I-axiom

Sym

For each $\mathbf{R}$ consisting of a linear order and its inverse,

$$[\mathbf{R} \longmapsto \mathscr{A}].$$

$\mathbf{R}_1 = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}$, constraints? $f(\mathbf{R}_1) = \mathscr{A} = \{ a, b, c \}$.

$\mathbf{R}_2 = \begin{matrix} a & b \\ b & a \\ c & c \end{matrix}$, constraints? None.

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\mathbf{R}_1 + \mathbf{R}_2$ must be $A_1 \cap A_2$.

Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{\, a, b \,\},$$

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
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then winners in $\mathbf{R}_1 + \mathbf{R}_2$ must be $A_1 \cap A_2$.

Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{\, a, b \,\}, \mathbf{R}_2 = \begin{array}{ccc} a & b & a \\ b & a & c \\ c & c & b \end{array}, A_2 = \{\, a \,\},$$

$$\mathbf{R} = \begin{array}{ccccc} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{array}. \text{ Winners?}$$

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

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Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{\, a, b \,\}, \mathbf{R}_2 = \begin{array}{ccc} a & b & a \\ b & a & c \\ c & c & b \end{array}, A_2 = \{\, a \,\},$$

$$\mathbf{R} = \begin{array}{ccccc} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{array}. \text{ Winners? } \{\, a \,\}$$

Reinforcement I-axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\mathbf{R}_1 + \mathbf{R}_2$ must be $A_1 \cap A_2$.

Reinf

For each $\mathbf{R}_1, \mathbf{R}_2, A_1, A_2 \subseteq \mathscr{A}, A_1 \cap A_2 \neq \emptyset$:

$$([\mathbf{R}_1 \longmapsto A_1] \wedge [\mathbf{R}_2 \longmapsto A_2]) \rightarrow [\mathbf{R}_1 + \mathbf{R}_2 \longmapsto A_1 \cap A_2].$$

A simple argument

Claim

Consider:

- $\mathbf{R} = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}$,
- $J = \{ \textsc{Dom}, \textsc{Sym}, \textsc{Reinf} \}$.

We can prove that for f compliant with J:

$$[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}].$$

See how?

A simple argument

Claim

Consider:

- $\mathbf{R} = \begin{array}{cccc} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{array}$,
- $J = \{\, \text{Dom}, \text{Sym}, \text{Reinf} \,\}$.

We can prove that for f compliant with J:

$$[\mathbf{R} \xmapsto{\in} \{\, \{\, a \,\}, \{\, b \,\}, \{\, a, b \,\} \,\}].$$

See how? Consider $\mathbf{R}_D = \begin{array}{cc} a & b \\ b & c \\ c & a \end{array}, \mathbf{R}_S = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S$.

Example proof

$$\mathbf{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}, \mathbf{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}.$$

1. $[\mathbf{R}_D \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (DOM)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (SYM)
3. $((1) \wedge (2)) \rightarrow [\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (REINF-SETS)
4. $[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

Outline

Argument building for Borda

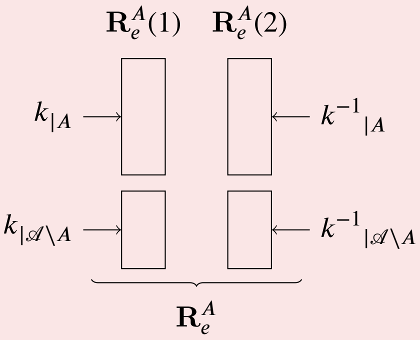
Write f_B for the Borda rule.

- We want to produce an argument justifying Borda's output.
- Given $\mathbf{R}$, we want an argument with claim $[\mathbf{R} \longmapsto f_B(\mathbf{R})]$.
- Basis: Young (1974)'s axiomatization of the Borda rule.
- Our l-axiomatization uses three simple profile types plus REINF.

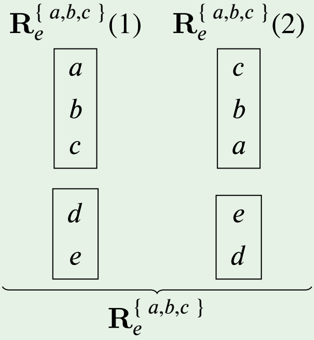
Elementary profile

Fix an arbitrary linear order k on $\mathscr{A}$. Given $A \subseteq \mathscr{A}$, define $\mathbf{R}_e^A$.

Elementary profile



Example



Cyclic profiles

Given S a complete cycle in $\mathscr{A}$, define $\mathbf{R}_c^S$.

Cyclic profile

$\mathbf{R}_c^S$ is the profile composed by all $|\mathscr{A}|$ possible linearizations of S as preference orderings.

$$\mathbf{R}_c^{\langle a,b,c,d \rangle} = \begin{matrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{matrix}.$$

Borda l-axiomatization

ELEM for all A: $[\mathbf{R}_e^A \longmapsto A]$.

CYCL for all S: $[\mathbf{R}_c^S \longmapsto \mathscr{A}]$.

REINF as previously but generalized to any number of summed profiles.

CANC cancellation: when all pairs of alternatives (a, b) in a profile are such that a is preferred to b as many times as b to a, the set of winners must be $\mathscr{A}$.

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\mathbf{R}$ defined as:

$$\mathbf{R} = \begin{matrix} a & c \\ b & b \\ d & a \\ c & d \end{matrix}.$$

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\mathbf{R}$ defined as:

$$\mathbf{R} = \begin{array}{cc} a & c \\ b & b \\ d & a \\ c & d \end{array}.$$

We want to justify that $f_B(\mathbf{R}) = \{ a, b \}$.

Sketch

- Consider any $\mathbf{R}' = q_1\mathbf{R}_e^{a,b} + q_2\mathbf{R}_e^{a,b,c} + \sum_{S \in \mathcal{S}} q_S \mathbf{R}_c^S$, $q_1, q_2, q_S \in \mathbb{N}, \mathcal{S}$ some set of cycles.
- In $\mathbf{R}'$, $W = \{ a, b \}$ must win.
- Find $k \in \mathbb{N}$ such that $\overline{k\mathbf{R}} + \mathbf{R}'$ cancel.
- Then $k\mathbf{R}$ has winners W. (Skipping details.)
- Then $\mathbf{R}$ has winners W.

Our task: find $\mathbf{R}'$ a combination of elementary and cyclic profiles such that $\overline{k\mathbf{R}} + \mathbf{R}'$ cancel.
Good news: this is always possible.

Application on the example

Define $\mathbf{R}' = \mathbf{R}_e^{a,b} + 2\mathbf{R}_e^{a,b,c} + \mathbf{R}_c^{\langle c,b,a,d\rangle} + \mathbf{R}_c^{\langle b,d,c,a\rangle}$.

1. $[\mathbf{R}_e^{a,b} \longmapsto \{ a, b \}]$ (ELEM)
2. $[\mathbf{R}_e^{a,b,c} \longmapsto \{ a, b, c \}]$ (ELEM)
3. $[\mathbf{R}_c^{\langle c,b,a,d\rangle} \longmapsto \mathscr{A}]$ (CYCL)
4. $[\mathbf{R}_c^{\langle b,d,c,a\rangle} \longmapsto \mathscr{A}]$ (CYCL)
5. $[\mathbf{R}' \longmapsto \{ a, b \}]$ (REINF, 1, 2, 3, 4)
6. $[4\mathbf{R} + \overline{4\mathbf{R}} \longmapsto \mathscr{A}]$ (CANC)
7. $[4\mathbf{R} + \overline{4\mathbf{R}} + \mathbf{R}' \longmapsto \{ a, b \}]$ (REINF, 5, 6)
8. $[\overline{4\mathbf{R}} + \mathbf{R}' \longmapsto \mathscr{A}]$ (CANC)
9. $[4\mathbf{R} \longmapsto \{ a, b \}]$ (REINF, 7, 8)
10. $[\mathbf{R} \longmapsto \{ a, b \}]$ (REINF, 9)

Outline

Counter-argument against Borda

Counter-argument against Borda

Not Condorcet-consistent!

Example

$$\mathbf{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix}.$$

- Argument against Borda: use a Cond I-axiom
- Counter-argue with FvsC.

Building argumentative recommender systems

General goal

- Recommend complex objects
- Recommend *and* argue

Complex objects

- Voting rule
- Planning
- Strategy (game, negociation, …)
- Travel itinerary

Multi-level argumentation:

- NOT persuasion
- NOT predicting the natural user choice

Conclusion

- A language to express desirable properties of voting rules.
- We can then instanciate concrete arguments (example-based).
- May render some arguments in the specialized literature accessible to non experts.
- Extensions may permit to *debate* about voting rules.
- Provides a way to study appreciation of arguments.

Thank you for your attention!

Example of axiom

- Dominance: if a dominates b in $\mathbf{R}$, then b may not win.
- We want a language to express this kind of axioms.

L-axioms

- Now: “translate” axioms into language-axioms.
- An *l-axiom* is a set of formulæ.

Shortcut notations

$\mathcal{P}_\emptyset(\mathscr{A})$ the set of subsets of $\mathscr{A}$, excluding the empty set.

Let $\alpha \subseteq \mathcal{P}_\emptyset(\mathscr{A})$ be a set of possible winning alternatives.

Uni-profile clause

$[\mathbf{R} \xmapsto{\in} \alpha]$ shortcut for:

$$\bigvee_{A \in \alpha} [\mathbf{R} \longmapsto A].$$

- Intuitive content.
- Called a uni-profile clause.

Domain knowledge

- We need some formulæ encoding the voting rule concept.
- Define κ as the set of all those formulæ.

Domain knowledge κ

1. a voting rule can't select more than one set of winners: for all $\mathbf{R}$ and all $\emptyset \subset A \neq B \subseteq \mathscr{A}$,
$$[\mathbf{R} \longmapsto A] \land [\mathbf{R} \longmapsto B] \rightarrow \bot.$$
2. a voting rule must select at least one set of winners: for all $\mathbf{R}$,
$$[\mathbf{R} \overset{\in}{\longmapsto} \mathcal{P}_{\emptyset}(\mathscr{A})].$$

Fishburn-against-Condorcet argument

Fishburn (1974, p. 544) argument against the Condorcet principle (see also http://rangevoting.org/FishburnAntiC.html).

Condorcet winner

w VS $\mu, \mu \in \{a, \ldots, h\}$?

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

Fishburn-against-Condorcet argument

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Condorcet winner

w VS $\mu, \mu \in \{a, \ldots, h\}$? 51/101

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

Fishburn-against-Condorcet argument

Fishburn (1974, p. 544) argument against the Condorcet principle (see also http://rangevoting.org/FishburnAntiC.html).

Condorcet winner

w VS $\mu, \mu \in \{a, \dots, h\}$? 51/101

nb voters

	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

ranks

	1	≤ 2	≤ 3	≤ 4	≤ 5	≤ 6	≤ 7	≤ 8	≤ 9
w	0	30	30	51	51	82	82	82	101
a	50	50	80	80	101	101	101	101	101

Fishburn-versus-Condorcet l-axiom

Define $\mathbf{R}_F$ the profile shown in the previous slide.

Fishburn-versus-Condorcet

The Fishburn-versus-Condorcet l-axiom FvsC is defined as:

$$[\mathbf{R}_F \xmapsto{\in} \mathcal{P}_\emptyset(\mathscr{A} \setminus \{ w \})].$$

L-axiomatization

An l-axiomatization is a set of l-axioms.

Conforming to J

The rule f conforms to the l-axiomatization J iff v_f assigns the value $\mathtt{T}$ to all formulæ in j, for all $j \in J$.

An l-axiomatization is consistent iff there exists a voting rule conformant to it.

Arguments

Argument

An argument grounded on J is a pair (*claim*, *proof*),

- J an l-axiomatization,
- *claim* a uni-profile clause (thus of the form $[\mathbf{R} \xmapsto{\in} \alpha]$),
- *proof* a natural deduction proof of the claim grounded on J.

- The argument shows that for all voting rules f conformant to J, $f(\mathbf{R})$ selects a set of winners among α.
- The argument claims that it is only reasonable to choose the winners among α for $\mathbf{R}$ (provided J is accepted).
- *Consistent* arguments require a consistent l-axiomatization.

Example proof

$$\mathbf{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}, \mathbf{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}.$$

1. $[\mathbf{R}_D \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Dom)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (Sym)
3. $([\mathbf{R}_D \longmapsto \{ a \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ a \}]$ (Reinf)
4. $([\mathbf{R}_D \longmapsto \{ b \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ b \}]$ (Reinf)
5. $([\mathbf{R}_D \longmapsto \{ a, b \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ a, b \}]$ (Reinf)
6. $[\mathbf{R}_D \longmapsto \{ a \}] \rightarrow [\mathbf{R} \longmapsto \{ a \}]$ (PR from 2 & 3)
7. $[\mathbf{R}_D \longmapsto \{ b \}] \rightarrow [\mathbf{R} \longmapsto \{ b \}]$ (PR from 2 & 4)
8. $[\mathbf{R}_D \longmapsto \{ a, b \}] \rightarrow [\mathbf{R} \longmapsto \{ a, b \}]$ (PR from 2 & 5)
9. $[\mathbf{R}_D \longmapsto \{ a \}] \vee [\mathbf{R}_D \longmapsto \{ b \}] \vee [\mathbf{R}_D \longmapsto \{ a, b \}]$ (rewrite 1)
10. $[\mathbf{R} \longmapsto \{ a \}] \vee [\mathbf{R} \longmapsto \{ b \}] \vee [\mathbf{R} \longmapsto \{ a, b \}]$ (PR from 6–9)
11. $[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (rewrite 10)

Example shortened

Tweak l-axioms to skip steps which will seem intuitive to humans.

Reinforcement-sets

For each $\mathbf{R}_1$, $\mathbf{R}_2$, $\alpha_1, \alpha_2 \subseteq \mathcal{P}_{\emptyset}(\mathscr{A})$, ${}_{\cap Q \neq \emptyset, Q \in \alpha_1 \times \alpha_2}$:
$([\mathbf{R}_1 \xmapsto{\in} \alpha_1] \land [\mathbf{R}_2 \xmapsto{\in} \alpha_2]) \to [\mathbf{R}_1 + \mathbf{R}_2 \xmapsto{\in} \bigcup_{A_1 \in \alpha_1, A_2 \in \alpha_2} \{ A_1 \cap A_2 \}]$.

1. $[\mathbf{R}_D \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (DOM)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (SYM)
3. $((1) \land (2)) \to [\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (REINF-SETS)
4. $[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

Soundness and completeness

Consider an l-axiomatization $\boldsymbol{J}$ and a claim $c = [\mathbf{R} \xmapsto{\in} \alpha]$.

Theorem (Soundness)

If there exists an argument $(c, \textit{proof})$ grounded on $\boldsymbol{J}$, the claim holds given $\boldsymbol{J}$.

Theorem (Completeness)

If the claim holds given $\boldsymbol{J}$, then there exists an argument $(c, \textit{proof})$ grounded on $\boldsymbol{J}$.

This is easily obtained from the soundness and completeness of natural deduction in propositional logic.

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