Arguing about voting rules

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Outline

Introduction

Context

- Voting rule: a systematic way of aggregating different opinions and decide
- Multiple reasonable ways of doing this
- Different voting rules have different interesting properties
- None satisfy all desirable properties
- Some are more reasonable than others (example)

Our goal

We want to easily communicate about strength and weaknesses of voting rules.

Voting rule

Alternatives $\mathscr{A} = \{ a, b, c, d, \dots \}$

Possible voters $\mathscr{N} = \{ 1, 2, \dots \}$

Voters $\emptyset \subset N \subseteq \mathscr{N}$

Profile function $\mathbf{R}$ from N to linear orders on $\mathscr{A}$.

Voting rule function f mapping each $\mathbf{R}$ to winners $\emptyset \subset A \subseteq \mathscr{A}$.

$$\begin{array}{c} \mathbf{R}(1) \quad \mathbf{R}(2) \\ \underbrace{\boxed{\begin{array}{cc} a & b \\ b & a \\ c & c \end{array}}}_{\mathbf{R}} \end{array} \xrightarrow{f} A = \{ a, b \}$$

Borda

Given a profile $\mathbf{R}$:

- count the score of each alternative;
- the highest scores win.
- Score of $a \in \mathscr{A}$ is the number of alternatives it beats.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- score a is...?

Borda

Given a profile $\mathbf{R}$:

- count the score of each alternative;
- the highest scores win.
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$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- score a is…? $3 + 1 + 2 = 6$
- score b is $0 + 3 + 3 = 6$
- score c is $1 + 2 + 1 = 4$
- score d is $2 + 0 + 0 = 2$

Winners are $\{ a, b \}$.

Condorcet's principle

Condorcet's principle

We ought to take the Condorcet winner as sole winner if it exists.

- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}$$. Who wins?

Condorcet's principle

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- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\mathbf{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix}. \text{ Who wins?} b$$

How are voting rules analyzed?

- Examples featuring counter-intuitive results for some voting rules.
- Properties of voting rules, e.g. Borda does not satisfy Condorcet's principle.
- Axiomatization of a voting rule: accepting such principles lead to a unique voting rule.

Our objective

- Different voting rules
- Arguments in favor or against rules
- Dispersed in the literature
- Using mathematical formalism

We propose

- Common language
- Instantiate arguments on concrete examples

Goal: help understand strengths and weaknesses of given rules.

Outline

Example

Who should win?

Voter 1: $a \succ b \succ c$
Voter 2: $a \succ b \succ c$
Voter 3: $c \succ b \succ a$

- Veto rule chooses b
- Borda rule chooses a

Voter 1:	$a \succ b \succ c$
Voter 2:	$a \succ b \succ c$
Voter 3:	$c \succ b \succ a$

System: Take the *red subprofile*. Here, *a should win*, right? [unanimity]

User: Obviously!

System: Now consider the *green subprofile*. For symmetry reasons, there should be a *three-way tie*, right? [cancellation]

User: Sounds reasonable.

System: So, as there was a three-way tie for the green part, the red part should decide the overall winner, right? [reinforcement]

User: Yes.

System: To summarise, you agree that a should win.

Language

We use propositional logic (with connectives $\neg, \vee, \wedge, \rightarrow$).

Atoms

- One atom for each $(\mathbf{R}, A)$, $\emptyset \subset A \subseteq \mathscr{A}$.
- An atom talks about assigning winners A to $\mathbf{R}$.
- Written $[\mathbf{R} \longmapsto A]$.

Semantics

Semantics v_f, given a voting rule f:

$$v_f([\mathbf{R} \longmapsto A]) = \mathrm{T} \text{ iff } f(\mathbf{R}) = A.$$

Dominance l-axiom

Dom

L-axiom Dom: for each $\mathbf{R}$,

$$[\mathbf{R} \xmapsto{\in} \mathscr{P}_{\emptyset}(U_{\mathbf{R}})],$$

with $U_{\mathbf{R}}$ the set of alternatives in $\mathbf{R}$ that are not dominated.

Symmetric cancellation I-axiom

Sym

For each $\mathbf{R}$ consisting of a linear order and its inverse,

$$[\mathbf{R} \longmapsto \mathscr{A}].$$

$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}$, constraints? $f(\mathbf{R}_1) =$

Symmetric cancellation I-axiom

Sym

For each $\mathbf{R}$ consisting of a linear order and its inverse,

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$$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \text{ constraints? } f(\mathbf{R}_1) = \mathscr{A} = \{\, a, b, c \,\}.$$

Symmetric cancellation I-axiom

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$$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \text{ constraints? } f(\mathbf{R}_1) = \mathscr{A} = \{ a, b, c \}.$$

$$\mathbf{R}_2 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, \text{ constraints?}$$

Symmetric cancellation I-axiom

SYM

For each $\mathbf{R}$ consisting of a linear order and its inverse,

$$[\mathbf{R} \longmapsto \mathscr{A}].$$

$$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}$$, constraints? $f(\mathbf{R}_1) = \mathscr{A} = \{ a, b, c \}$.

$$\mathbf{R}_2 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}$$, constraints? None.

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\mathbf{R}_1 + \mathbf{R}_2$ must be $A_1 \cap A_2$.

Example

$$\mathbf{R}_1 = \begin{matrix} a & b \\ b & a \\ c & c \end{matrix}, A_1 = \{ a, b \},$$

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
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Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{\, a, b \,\}, \mathbf{R}_2 = \begin{array}{ccc} a & b & a \\ b & a & c \\ c & c & b \end{array}, A_2 = \{\, a \,\},$$

$$\mathbf{R} = \begin{array}{ccccc} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{array}. \text{ Winners?}$$

Reinforcement axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
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Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{ a, b \}, \mathbf{R}_2 = \begin{array}{ccc} a & b & a \\ b & a & c \\ c & c & b \end{array}, A_2 = \{ a \},$$

$$\mathbf{R} = \begin{array}{ccccc} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{array}. \text{Winners? } \{ a \}$$

Reinforcement I-axiom

Classical reinforcement axiom: consider $\mathbf{R}_1$, $\mathbf{R}_2$,

- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\mathbf{R}_1 + \mathbf{R}_2$ must be $A_1 \cap A_2$.

Reinf

For each $\mathbf{R}_1, \mathbf{R}_2, A_1, A_2 \subseteq \mathscr{A}, A_1 \cap A_2 \neq \emptyset$:

$$([\mathbf{R}_1 \longmapsto A_1] \wedge [\mathbf{R}_2 \longmapsto A_2]) \rightarrow [\mathbf{R}_1 + \mathbf{R}_2 \longmapsto A_1 \cap A_2].$$

A simple argument

Claim

Consider:

- $\mathbf{R} = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}$,
- $J = \{ \textsc{Dom}, \textsc{Sym}, \textsc{Reinf} \}$.

We can prove that for f compliant with J:

$$[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}].$$

See how?

A simple argument

Claim

Consider:

- $\mathbf{R} = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}$,
- $J = \{\, \textsc{Dom}, \textsc{Sym}, \textsc{Reinf} \,\}$.

We can prove that for f compliant with J:

$$[\mathbf{R} \xmapsto{\in} \{\, \{\, a \,\}, \{\, b \,\}, \{\, a, b \,\} \,\}].$$

See how? Consider $\mathbf{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}$, $\mathbf{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}$, $\mathbf{R} = \mathbf{R}_D + \mathbf{R}_S$.

Example proof

$$\mathbf{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}, \mathbf{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}.$$

1. $[\mathbf{R}_D \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (DOM)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (SYM)
3. $((1) \land (2)) \rightarrow [\mathbf{R} \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (REINF-SETS)
4. $[\mathbf{R} \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

Outline

Argument building for Borda

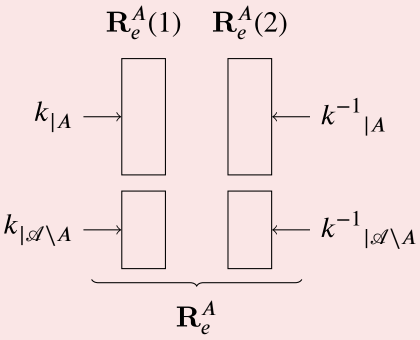
Write f_B for the Borda rule.

- We want to produce an argument justifying Borda's output.
- Given $\mathbf{R}$, we want an argument with claim $[\mathbf{R} \longmapsto f_B(\mathbf{R})]$.
- Basis: Young (1974)'s axiomatization of the Borda rule.
- Our l-axiomatization uses three simple profile types plus REINF.

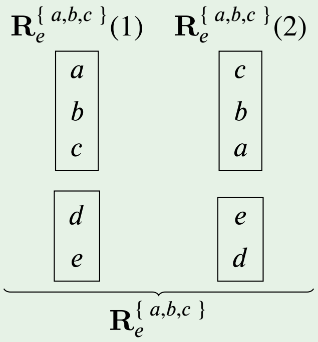
Elementary profile

Fix an arbitrary linear order k on $\mathscr{A}$. Given $A \subseteq \mathscr{A}$, define $\mathbf{R}_e^A$.

Elementary profile



Example



Cyclic profiles

Given S a complete cycle in $\mathscr{A}$, define $\mathbf{R}_c^S$.

Cyclic profile

$\mathbf{R}_c^S$ is the profile composed by all $|\mathscr{A}|$ possible linearizations of S as preference orderings.

$$\mathbf{R}_c^{\langle a,b,c,d \rangle} = \begin{array}{cccc} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{array}.$$

Borda l-axiomatization

ELEM for all A: $[\mathbf{R}_e^A \longmapsto A]$.

CYCL for all S: $[\mathbf{R}_c^S \longmapsto \mathscr{A}]$.

REINF as previously but generalized to any number of summed profiles.

CANC cancellation: when all pairs of alternatives (a, b) in a profile are such that a is preferred to b as many times as b to a, the set of winners must be $\mathscr{A}$.

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\mathbf{R}$ defined as:

$$\mathbf{R} = \begin{matrix} a & c \\ b & b \\ d & a \\ c & d \end{matrix} .$$

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\mathbf{R}$ defined as:

$$\mathbf{R} = \begin{array}{cc} a & c \\ b & b \\ d & a \\ c & d \end{array}.$$

We want to justify that $f_B(\mathbf{R}) = \{ a, b \}$.

Sketch

- Consider any $\mathbf{R}' = q_1\mathbf{R}_e^{a,b} + q_2\mathbf{R}_e^{a,b,c} + \sum_{S\in\mathscr{S}} q_S\mathbf{R}_c^S$, $q_1, q_2, q_S \in \mathbb{N}, \mathscr{S}$ some set of cycles.
- In $\mathbf{R}'$, $W = \{ a, b \}$ must win.
- Find $k \in \mathbb{N}$ such that $\overline{k\mathbf{R}} + \mathbf{R}'$ cancel.
- Then $k\mathbf{R}$ has winners W. (Skipping details.)
- Then $\mathbf{R}$ has winners W.

Our task: find $\mathbf{R}'$ a combination of elementary and cyclic profiles such that $\overline{k\mathbf{R}} + \mathbf{R}'$ cancel.
Good news: this is always possible.

Application on the example

Define $\mathbf{R}' = \mathbf{R}_e^{a,b} + 2\mathbf{R}_e^{a,b,c} + \mathbf{R}_c^{\langle c,b,a,d\rangle} + \mathbf{R}_c^{\langle b,d,c,a\rangle}$.

1. $[\mathbf{R}_e^{a,b} \longmapsto \{ a, b \}]$ (ELEM)
2. $[\mathbf{R}_e^{a,b,c} \longmapsto \{ a, b, c \}]$ (ELEM)
3. $[\mathbf{R}_c^{\langle c,b,a,d\rangle} \longmapsto \mathscr{A}]$ (CYCL)
4. $[\mathbf{R}_c^{\langle b,d,c,a\rangle} \longmapsto \mathscr{A}]$ (CYCL)
5. $[\mathbf{R}' \longmapsto \{ a, b \}]$ (REINF, 1, 2, 3, 4)
6. $[4\mathbf{R} + \overline{4\mathbf{R}} \longmapsto \mathscr{A}]$ (CANC)
7. $[4\mathbf{R} + \overline{4\mathbf{R}} + \mathbf{R}' \longmapsto \{ a, b \}]$ (REINF, 5, 6)
8. $[\overline{4\mathbf{R}} + \mathbf{R}' \longmapsto \mathscr{A}]$ (CANC)
9. $[4\mathbf{R} \longmapsto \{ a, b \}]$ (REINF, 7, 8)
10. $[\mathbf{R} \longmapsto \{ a, b \}]$ (REINF, 9)

Outline

Counter-argument against Borda

Counter-argument against Borda

Not Condorcet-consistent!

Example

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- Argument against Borda: use a Cond I-axiom
- Counter-argue with FvsC.

Building argumentative recommender systems

General goal

- Recommend complex objects
- Recommend *and* argue

Complex objects

- Voting rule
- Planning
- Strategy (game, negociation, ...)
- Travel itinerary

Multi-level argumentation:

- NOT persuasion
- NOT predicting the natural user choice

Conclusion

- A language to express desirable properties of voting rules.
- We can then instanciate concrete arguments (example-based).
- May render some arguments in the specialized literature accessible to non experts.
- Extensions may permit to *debate* about voting rules.
- Provides a way to study appreciation of arguments.

Thank you for your attention!

Example of axiom

- Dominance: if a dominates b in $\mathbf{R}$, then b may not win.
- We want a language to express this kind of axioms.

L-axioms

- Now: “translate” axioms into language-axioms.
- An *l-axiom* is a set of formulæ.

Shortcut notations

$\mathscr{P}_{\emptyset}(\mathscr{A})$ the set of subsets of $\mathscr{A}$, excluding the empty set.

Let $\alpha \subseteq \mathscr{P}_{\emptyset}(\mathscr{A})$ be a set of possible winning alternatives.

Uni-profile clause

$[\mathbf{R} \overset{\in}{\longmapsto} \alpha]$ shortcut for:

$$\bigvee_{A \in \alpha} [\mathbf{R} \longmapsto A].$$

- Intuitive content.
- Called a uni-profile clause.

Domain knowledge

- We need some formulæ encoding the voting rule concept.
- Define κ as the set of all those formulæ.

Domain knowledge κ

1. a voting rule can't select more than one set of winners: for all $\mathbf{R}$ and all $\emptyset \subset A \neq B \subseteq \mathscr{A}$,
$$[\mathbf{R} \longmapsto A] \land [\mathbf{R} \longmapsto B] \rightarrow \bot.$$
2. a voting rule must select at least one set of winners: for all $\mathbf{R}$,
$$[\mathbf{R} \xmapsto{\in} \mathscr{P}_{\emptyset}(\mathscr{A})].$$

Fishburn-against-Condorcet argument

Fishburn (1974, p. 544) argument against the Condorcet principle (see also http://rangevoting.org/FishburnAntiC.html).

Condorcet winner

w VS $\mu, \mu \in \{a, \dots, h\}$?

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

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Condorcet winner

w VS $\mu, \mu \in \{a, \dots, h\}$? 51/101

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

Fishburn-against-Condorcet argument

Fishburn (1974, p. 544) argument against the Condorcet principle (see also http://rangevoting.org/FishburnAntiC.html).

Condorcet winner

w VS $\mu, \mu \in \{a, \dots, h\}$? 51/101

nb voters

	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

ranks

	1	≤ 2	≤ 3	≤ 4	≤ 5	≤ 6	≤ 7	≤ 8	≤ 9
w	0	30	30	51	51	82	82	82	101
a	50	50	80	80	101	101	101	101	101

Fishburn-versus-Condorcet l-axiom

Define $\mathbf{R}_F$ the profile shown in the previous slide.

Fishburn-versus-Condorcet

The Fishburn-versus-Condorcet l-axiom FvsC is defined as:

$$[\mathbf{R}_F \xmapsto{\in} \mathscr{P}_{\emptyset}(\mathscr{A} \setminus \{\, w \,\})].$$

L-axiomatization

An l-axiomatization is a set of l-axioms.

Conforming to J

The rule f conforms to the l-axiomatization J iff v_f assigns the value $\mathtt{T}$ to all formulæ in j, for all $j \in J$.

An l-axiomatization is consistent iff there exists a voting rule conformant to it.

Arguments

Argument

An argument grounded on J is a pair (*claim*, *proof*),

- J an l-axiomatization,
- *claim* a uni-profile clause (thus of the form $[\mathbf{R} \xmapsto{\in} \alpha]$),
- *proof* a natural deduction proof of the claim grounded on J.

- The argument shows that for all voting rules f conformant to J, $f(\mathbf{R})$ selects a set of winners among α.
- The argument claims that it is only reasonable to choose the winners among α for $\mathbf{R}$ (provided J is accepted).
- *Consistent* arguments require a consistent l-axiomatization.

Example proof

$$\mathbf{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}, \mathbf{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}.$$

1. $[\mathbf{R}_D \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Dom)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (Sym)
3. $([\mathbf{R}_D \longmapsto \{ a \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ a \}]$ (Reinf)
4. $([\mathbf{R}_D \longmapsto \{ b \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ b \}]$ (Reinf)
5. $([\mathbf{R}_D \longmapsto \{ a, b \}] \wedge [\mathbf{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\mathbf{R} \longmapsto \{ a, b \}]$ (Reinf)
6. $[\mathbf{R}_D \longmapsto \{ a \}] \rightarrow [\mathbf{R} \longmapsto \{ a \}]$ (PR from 2 & 3)
7. $[\mathbf{R}_D \longmapsto \{ b \}] \rightarrow [\mathbf{R} \longmapsto \{ b \}]$ (PR from 2 & 4)
8. $[\mathbf{R}_D \longmapsto \{ a, b \}] \rightarrow [\mathbf{R} \longmapsto \{ a, b \}]$ (PR from 2 & 5)
9. $[\mathbf{R}_D \longmapsto \{ a \}] \vee [\mathbf{R}_D \longmapsto \{ b \}] \vee [\mathbf{R}_D \longmapsto \{ a, b \}]$ (rewrite 1)
10. $[\mathbf{R} \longmapsto \{ a \}] \vee [\mathbf{R} \longmapsto \{ b \}] \vee [\mathbf{R} \longmapsto \{ a, b \}]$ (PR from 6–9)
11. $[\mathbf{R} \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (rewrite 10)

Example shortened

Tweak l-axioms to skip steps which will seem intuitive to humans.

Reinforcement-sets

For each $\mathbf{R}_1$, $\mathbf{R}_2$, $\alpha_1, \alpha_2 \subseteq \mathscr{P}_\emptyset(\mathscr{A})$, ${}_{\cap Q \neq \emptyset, Q \in \alpha_1 \times \alpha_2}$:
$([\mathbf{R}_1 \xmapsto{\in} \alpha_1] \land [\mathbf{R}_2 \xmapsto{\in} \alpha_2]) \rightarrow$
$[\mathbf{R}_1 + \mathbf{R}_2 \xmapsto{\in} \bigcup_{A_1 \in \alpha_1, A_2 \in \alpha_2} \{ A_1 \cap A_2 \}]$.

1. $[\mathbf{R}_D \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Dom)
2. $[\mathbf{R}_S \longmapsto \{ a, b, c \}]$ (Sym)
3. $((1) \land (2)) \rightarrow [\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Reinf-sets)
4. $[\mathbf{R} \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

Soundness and completeness

Consider an l-axiomatization J and a claim $c = [\mathbf{R} \xmapsto{\in} \alpha]$.

Theorem (Soundness)

If there exists an argument (c, proof) grounded on J, the claim holds given J.

Theorem (Completeness)

If the claim holds given J, then there exists an argument (c, proof) grounded on J.

This is easily obtained from the soundness and completeness of natural deduction in propositional logic.

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