

# Arguing about voting rules

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<https://github.com/oliviercailloux/Arguing-about-voting-rules>



# Outline

- 1 Context
- 2 Two examples
- 3 Arguing for Borda
- 4 Goal: Build argumentative and adaptative recommender systems

# Introduction

## Context

- Voting rule: a systematic way of aggregating different opinions and decide
- Multiple reasonable ways of doing this
- Different voting rules have different interesting properties
- None satisfy all desirable properties
- Some are more reasonable than others

## Our goal

We want to easily communicate about strength and weaknesses of voting rules.

# Voting rule

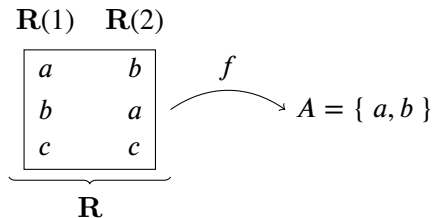
Alternatives  $\mathcal{A} = \{ a, b, c, d, \dots \}$

Possible voters  $\mathcal{N} = \{ 1, 2, \dots \}$

Voters  $\emptyset \subset N \subseteq \mathcal{N}$

Profile function  $\mathbf{R}$  from  $N$  to linear orders on  $\mathcal{A}$ .

Voting rule function  $f$  mapping each  $\mathbf{R}$  to winners  $\emptyset \subset A \subseteq \mathcal{A}$ .



# Borda

Given a profile  $\mathbf{R}$ :

- count the score of each alternative;
- the highest scores win.
- Score of  $a \in \mathcal{A}$  is the number of alternatives it beats.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array} .$$

- score  $a$  is...?

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$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- score  $a$  is...?  $3 + 1 + 2 = 6$
- score  $b$  is  $0 + 3 + 3 = 6$
- score  $c$  is  $1 + 2 + 1 = 4$
- score  $d$  is  $2 + 0 + 0 = 2$

Winners are  $\{ a, b \}$ .

# Condorcet's principle

## Condorcet's principle

We ought to take the Condorcet winner as sole winner if it exists.

- $a$  *beats*  $b$  iff more than half the voters prefer  $a$  to  $b$ .
- $a$  is a *Condorcet winner* iff  $a$  beats every other alternatives.

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array} . \text{ Who wins?}$$

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$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array} . \text{ Who wins? } b$$



# How are voting rules analyzed?

- Examples featuring counter-intuitive results for some voting rules.
- Properties of voting rules, e.g. Borda does not satisfy Condorcet's principle.
- Axiomatization of a voting rule: accepting such principles lead to a unique voting rule.

# Our objective

- Different voting rules
- Arguments in favor or against rules
- Dispersed in the literature
- Using mathematical formalism

## We propose

- Common language
- Instantiate arguments on concrete examples

Goal: help understand strengths and weaknesses of given rules.

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## Example

Who should win?

Voter 1:  $a \succ b \succ c$

Voter 2:  $a \succ b \succ c$

Voter 3:  $c \succ b \succ a$

- Veto rule chooses  $b$
- Borda rule chooses  $a$

Voter 1:  $a \succ b \succ c$

Voter 2:  $a \succ b \succ c$

Voter 3:  $c \succ b \succ a$

**System:** Take the *red subprofile*. Here, *a should win*, right? [unanimity]

**User:** Obviously!

**System:** Now consider the *green subprofile*. For symmetry reasons, there should be a *three-way tie*, right? [cancellation]

**User:** Sounds reasonable.

**System:** So, as there was a three-way tie for the green part, the red part should decide the overall winner, right? [reinforcement]

**User:** Yes.

**System:** To summarise, you agree that *a* should win.

# Language

We use propositional logic (with connectives  $\neg, \vee, \wedge, \rightarrow$ ).

## Atoms

- One atom for each  $(\mathbf{R}, A)$ ,  $\emptyset \subset A \subseteq \mathcal{A}$ .
- An atom talks about assigning winners  $A$  to  $\mathbf{R}$ .
- Written  $[\mathbf{R} \mapsto A]$ .

## Semantics

Semantics  $v_f$ , given a voting rule  $f$ :

$$v_f([\mathbf{R} \mapsto A]) = \text{T iff } f(\mathbf{R}) = A.$$

# Dominance I-axiom

## Dom

L-axiom DOM: for each  $\mathbf{R}$ ,

$$[\mathbf{R} \vdash \mathcal{P}_{\emptyset}(U_{\mathbf{R}})],$$

with  $U_{\mathbf{R}}$  the set of alternatives in  $\mathbf{R}$  that are not dominated.

# Symmetric cancellation I-axiom

## SYM

For each  $\mathbf{R}$  consisting of a linear order and its inverse,

$$[\mathbf{R} \mapsto \mathcal{A}].$$

$$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \text{ constraints? } f(\mathbf{R}_1) =$$



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$$\mathbf{R}_2 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, \text{ constraints?}$$

# Symmetric cancellation I-axiom

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For each  $\mathbf{R}$  consisting of a linear order and its inverse,

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$$\mathbf{R}_1 = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \text{ constraints? } f(\mathbf{R}_1) = \mathcal{A} = \{ a, b, c \}.$$

$$\mathbf{R}_2 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, \text{ constraints? None.}$$

# Reinforcement axiom

Classical reinforcement axiom: consider  $\mathbf{R}_1, \mathbf{R}_2$ ,

- having winners  $A_1, A_2$ ,
- with  $A_1 \cap A_2 \neq \emptyset$ ;

then winners in  $\mathbf{R}_1 + \mathbf{R}_2$  must be  $A_1 \cap A_2$ .

## Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{a, b\},$$

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## Example

$$\mathbf{R}_1 = \begin{array}{cc} a & b \\ b & a \\ c & c \end{array}, A_1 = \{a, b\}, \mathbf{R}_2 = \begin{array}{ccc} a & b & a \\ b & a & c \\ c & c & b \end{array}, A_2 = \{a\},$$

$$\mathbf{R} = \begin{array}{ccccc} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{array}. \text{Winners?}$$

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## Reinforcement I-axiom

Classical reinforcement axiom: consider  $\mathbf{R}_1, \mathbf{R}_2$ ,

- having winners  $A_1, A_2$ ,
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then winners in  $\mathbf{R}_1 + \mathbf{R}_2$  must be  $A_1 \cap A_2$ .

### REINF

For each  $\mathbf{R}_1, \mathbf{R}_2, A_1, A_2 \subseteq \mathcal{A}, A_1 \cap A_2 \neq \emptyset$ :

$$([\mathbf{R}_1 \mapsto A_1] \wedge [\mathbf{R}_2 \mapsto A_2]) \rightarrow [\mathbf{R}_1 + \mathbf{R}_2 \mapsto A_1 \cap A_2].$$

# A simple argument

## Claim

$$\bullet \mathbf{R} = \begin{array}{cccc} & a & b & a & c \\ b & c & b & b & , \\ c & a & c & a & \end{array}$$

Consider:

$$\bullet J = \{ \text{DOM}, \text{SYM}, \text{REINF} \}.$$

We can prove that for  $f$  compliant with  $J$ :

$$[\mathbf{R} \vdash^{\mathbf{E}} \{ \{ a \}, \{ b \}, \{ a, b \} \}].$$

See how?



# A simple argument

## Claim

$$\bullet \mathbf{R} = \begin{array}{cccc} & a & b & a & c \\ b & & c & b & b \\ c & a & c & a & \end{array},$$

Consider:

$$\bullet J = \{ \text{DOM}, \text{SYM}, \text{REINF} \}.$$

We can prove that for  $f$  compliant with  $J$ :

$$[\mathbf{R} \mapsto \{ \{a\}, \{b\}, \{a,b\} \}].$$

See how? Consider  $\mathbf{R}_D = \begin{array}{cc} a & b \\ b & c \\ c & a \end{array}, \mathbf{R}_S = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S.$

## Example proof

$$\mathbf{R}_D = \begin{array}{cc} a & b \\ b & c \\ c & a \end{array}, \mathbf{R}_S = \begin{array}{cc} a & c \\ b & b \\ c & a \end{array}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{array}{cccc} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{array}.$$

- ①  $[\mathbf{R}_D \vdash^{\subseteq} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (DOM)
- ②  $[\mathbf{R}_S \vdash \{ a, b, c \}]$  (SYM)
- ③  $((1) \wedge (2)) \rightarrow [\mathbf{R} \vdash^{\subseteq} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (REINF-SETS)
- ④  $[\mathbf{R} \vdash^{\subseteq} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

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## Argument building for Borda

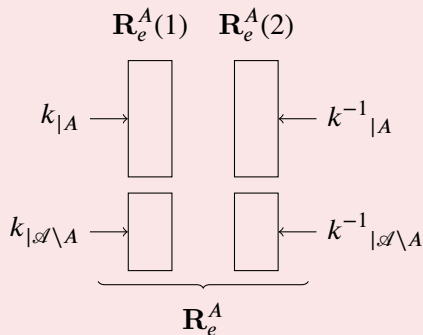
Write  $f_B$  for the Borda rule.

- We want to produce an argument justifying Borda's output.
- Given  $\mathbf{R}$ , we want an argument with claim  $[\mathbf{R} \mapsto f_B(\mathbf{R})]$ .
- Basis: Young [1974]'s axiomatization of the Borda rule.
- Our l-axiomatization uses three simple profile types plus REINF.

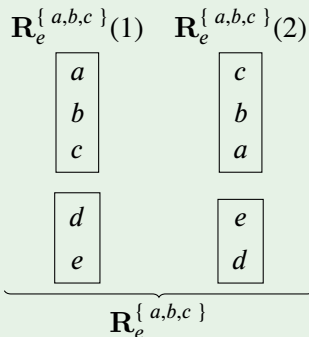
# Elementary profile

Fix an arbitrary linear order  $k$  on  $\mathcal{A}$ . Given  $A \subseteq \mathcal{A}$ , define  $\mathbf{R}_e^A$ .

## Elementary profile



## Example



# Cyclic profiles

Given  $S$  a complete cycle in  $\mathcal{A}$ , define  $\mathbf{R}_c^S$ .

## Cyclic profile

$\mathbf{R}_c^S$  is the profile composed by all  $|\mathcal{A}|$  possible linearizations of  $S$  as preference orderings.

$$\mathbf{R}_c^{\langle a,b,c,d \rangle} = \begin{array}{cccc} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{array}.$$

# Borda l-axiomatization

**ELEM** for all  $A$ :  $[\mathbf{R}_e^A \mapsto A]$ .

**CYCL** for all  $S$ :  $[\mathbf{R}_e^S \mapsto \mathcal{A}]$ .

**REINF** as previously but generalized to any number of summed profiles.

**CANC** cancellation: when all pairs of alternatives  $(a, b)$  in a profile are such that  $a$  is preferred to  $b$  as many times as  $b$  to  $a$ , the set of winners must be  $\mathcal{A}$ .

## An example

Consider  $\mathcal{A} = \{ a, b, c, d \}$  and a profile  $\mathbf{R}$  defined as:

$$\mathbf{R} = \begin{array}{cc} a & c \\ b & b \\ d & a \\ c & d \end{array} .$$



## An example

Consider  $\mathcal{A} = \{ a, b, c, d \}$  and a profile  $\mathbf{R}$  defined as:

$$\mathbf{R} = \begin{array}{cc} a & c \\ b & b \\ d & a \\ c & d \end{array} .$$

We want to justify that  $f_B(\mathbf{R}) = \{ a, b \}$ .

# Sketch

- Consider any  $\mathbf{R}' = q_1 \mathbf{R}_e^{a,b} + q_2 \mathbf{R}_e^{a,b,c} + \sum_{S \in \mathcal{S}} q_S \mathbf{R}_c^S$ ,  
 $q_1, q_2, q_S \in \mathbb{N}$ ,  $\mathcal{S}$  some set of cycles.
- In  $\mathbf{R}'$ ,  $W = \{a, b\}$  must win.
- Find  $k \in \mathbb{N}$  such that  $\overline{k\mathbf{R}} + \mathbf{R}'$  cancel.
- Then  $k\mathbf{R}$  has winners  $W$ . (Skipping details.)
- Then  $\mathbf{R}$  has winners  $W$ .

Our task: find  $\mathbf{R}'$  a combination of elementary and cyclic profiles such that  $\overline{k\mathbf{R}} + \mathbf{R}'$  cancel.

Good news: this is always possible.

## Application on the example

Define  $\mathbf{R}' = \mathbf{R}_e^{a,b} + 2\mathbf{R}_e^{a,b,c} + \mathbf{R}_c^{\langle c,b,a,d \rangle} + \mathbf{R}_c^{\langle b,d,c,a \rangle}$ .

- 1  $[\mathbf{R}_e^{a,b} \mapsto \{a, b\}]$  (ELEM)
- 2  $[\mathbf{R}_e^{a,b,c} \mapsto \{a, b, c\}]$  (ELEM)
- 3  $[\mathbf{R}_c^{\langle c,b,a,d \rangle} \mapsto \mathcal{A}]$  (CYCL)
- 4  $[\mathbf{R}_c^{\langle b,d,c,a \rangle} \mapsto \mathcal{A}]$  (CYCL)
- 5  $[\mathbf{R}' \mapsto \{a, b\}]$  (REINF, 1, 2, 3, 4)
- 6  $[4\mathbf{R} + \overline{4\mathbf{R}} \mapsto \mathcal{A}]$  (CANC)
- 7  $[4\mathbf{R} + \overline{4\mathbf{R}} + \mathbf{R}' \mapsto \{a, b\}]$  (REINF, 5, 6)
- 8  $[\overline{4\mathbf{R}} + \mathbf{R}' \mapsto \mathcal{A}]$  (CANC)
- 9  $[4\mathbf{R} \mapsto \{a, b\}]$  (REINF, 7, 8)
- 10  $[\mathbf{R} \mapsto \{a, b\}]$  (REINF, 9)

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# Counter-argument against Borda

## Counter-argument against Borda

Not Condorcet-consistent!

### Example

$$\mathbf{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array} .$$

- Argument against Borda: use a COND I-axiom
- Counter-argue with FvsC.

# Building argumentative recommender systems

## General goal

- Recommend complex objects
- Recommend *and* argue

## Complex objects

- Voting rule
- Planning
- Strategy (game, negotiation, ...)
- Travel itinerary

Multi-level argumentation:

- NOT persuasion
- NOT predicting the natural user choice

# Conclusion

- A language to express desirable properties of voting rules.
- We can then instantiate concrete arguments (example-based).
- May render some arguments in the specialized literature accessible to non experts.
- Extensions may permit to *debate* about voting rules.
- Provides a way to study appreciation of arguments.

*Thank you for your attention!*



## Example of axiom

- Dominance: if  $a$  dominates  $b$  in  $\mathbf{R}$ , then  $b$  may not win.
- We want a language to express this kind of axioms.

# L-axioms

- Now: “translate” axioms into language-axioms.
- An *l-axiom* is a set of formulæ.

## Shortcut notations

$\mathcal{P}_{\emptyset}(\mathcal{A})$  the set of subsets of  $\mathcal{A}$ , excluding the empty set.

Let  $\alpha \subseteq \mathcal{P}_{\emptyset}(\mathcal{A})$  be a set of possible winning alternatives.

### Uni-profile clause

$[\mathbf{R} \mapsto \alpha]$  shortcut for:

$$\bigvee_{A \in \alpha} [\mathbf{R} \mapsto A].$$

- Intuitive content.
- Called a uni-profile clause.

# Domain knowledge

- We need some formulæ encoding the voting rule concept.
- Define  $\kappa$  as the set of all those formulæ.

## Domain knowledge $\kappa$

- 1 a voting rule can't select more than one set of winners:  
for all  $\mathbf{R}$  and all  $\emptyset \subset A \neq B \subseteq \mathcal{A}$ ,

$$[\mathbf{R} \mapsto A] \wedge [\mathbf{R} \mapsto B] \rightarrow \perp.$$

- 2 a voting rule must select at least one set of winners:  
for all  $\mathbf{R}$ ,

$$[\mathbf{R} \mapsto \mathcal{P}_{\emptyset}(\mathcal{A})].$$

# Fishburn-against-Condorcet argument

Fishburn [1974, p. 544] argument against the Condorcet principle (see also <http://rangevoting.org/FishburnAntiC.html>).

Condorcet winner

$w$  VS  $\mu, \mu \in \{a, \dots, h\}$ ?

|   | nb voters |          |          |          |          |          |
|---|-----------|----------|----------|----------|----------|----------|
|   | 31        | 19       | 10       | 10       | 10       | 21       |
| 1 | <i>a</i>  | <i>a</i> | <i>f</i> | <i>g</i> | <i>h</i> | <i>h</i> |
| 2 | <i>b</i>  | <i>b</i> | <i>w</i> | <i>w</i> | <i>w</i> | <i>g</i> |
| 3 | <i>c</i>  | <i>c</i> | <i>a</i> | <i>a</i> | <i>a</i> | <i>f</i> |
| 4 | <i>d</i>  | <i>d</i> | <i>h</i> | <i>h</i> | <i>f</i> | <i>w</i> |
| 5 | <i>e</i>  | <i>e</i> | <i>g</i> | <i>f</i> | <i>g</i> | <i>a</i> |
| 6 | <i>w</i>  | <i>f</i> | <i>e</i> | <i>e</i> | <i>e</i> | <i>e</i> |
| 7 | <i>g</i>  | <i>g</i> | <i>d</i> | <i>d</i> | <i>d</i> | <i>d</i> |
| 8 | <i>h</i>  | <i>h</i> | <i>c</i> | <i>c</i> | <i>c</i> | <i>c</i> |
| 9 | <i>f</i>  | <i>w</i> | <i>b</i> | <i>b</i> | <i>b</i> | <i>b</i> |

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$w$  VS  $\mu, \mu \in \{a, \dots, h\}$ ? 51/101

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|---|-----------|----------|----------|----------|----------|----------|
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| 1 | <i>a</i>  | <i>a</i> | <i>f</i> | <i>g</i> | <i>h</i> | <i>h</i> |
| 2 | <i>b</i>  | <i>b</i> | <i>w</i> | <i>w</i> | <i>w</i> | <i>g</i> |
| 3 | <i>c</i>  | <i>c</i> | <i>a</i> | <i>a</i> | <i>a</i> | <i>f</i> |
| 4 | <i>d</i>  | <i>d</i> | <i>h</i> | <i>h</i> | <i>f</i> | <i>w</i> |
| 5 | <i>e</i>  | <i>e</i> | <i>g</i> | <i>f</i> | <i>g</i> | <i>a</i> |
| 6 | <i>w</i>  | <i>f</i> | <i>e</i> | <i>e</i> | <i>e</i> | <i>e</i> |
| 7 | <i>g</i>  | <i>g</i> | <i>d</i> | <i>d</i> | <i>d</i> | <i>d</i> |
| 8 | <i>h</i>  | <i>h</i> | <i>c</i> | <i>c</i> | <i>c</i> | <i>c</i> |
| 9 | <i>f</i>  | <i>w</i> | <i>b</i> | <i>b</i> | <i>b</i> | <i>b</i> |

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Condorcet winner

$w$  VS  $\mu, \mu \in \{a, \dots, h\}$ ? 51/101

|   |     | nb voters |     |     |     |     |    |
|---|-----|-----------|-----|-----|-----|-----|----|
|   |     | 31        | 19  | 10  | 10  | 10  | 21 |
| 1 | $a$ | $a$       | $f$ | $g$ | $h$ | $h$ |    |
| 2 | $b$ | $b$       | $w$ | $w$ | $w$ | $g$ |    |
| 3 | $c$ | $c$       | $a$ | $a$ | $a$ | $f$ |    |
| 4 | $d$ | $d$       | $h$ | $h$ | $f$ | $w$ |    |
| 5 | $e$ | $e$       | $g$ | $f$ | $g$ | $a$ |    |
| 6 | $w$ | $f$       | $e$ | $e$ | $e$ | $e$ |    |
| 7 | $g$ | $g$       | $d$ | $d$ | $d$ | $d$ |    |
| 8 | $h$ | $h$       | $c$ | $c$ | $c$ | $c$ |    |
| 9 | $f$ | $w$       | $b$ | $b$ | $b$ | $b$ |    |

|     |    | ranks |          |          |          |          |          |          |          |          |
|-----|----|-------|----------|----------|----------|----------|----------|----------|----------|----------|
|     |    | 1     | $\leq 2$ | $\leq 3$ | $\leq 4$ | $\leq 5$ | $\leq 6$ | $\leq 7$ | $\leq 8$ | $\leq 9$ |
| $w$ | 0  | 30    | 30       | 51       | 51       | 82       | 82       | 82       | 101      |          |
| $a$ | 50 | 50    | 80       | 80       | 101      | 101      | 101      | 101      | 101      |          |

# Fishburn-versus-Condorcet I-axiom

Define  $\mathbf{R}_F$  the profile shown in the previous slide.

## Fishburn-versus-Condorcet

The Fishburn-versus-Condorcet I-axiom FvSC is defined as:

$$[\mathbf{R}_F \vdash^{\in} \mathcal{P}_{\emptyset}(\mathcal{A} \setminus \{w\})].$$



# L-axiomatization

An l-axiomatization is a set of l-axioms.

## Conforming to $J$

The rule  $f$  conforms to the l-axiomatization  $J$  iff  $v_f$  assigns the value T to all formulæ in  $j$ , for all  $j \in J$ .

An l-axiomatization is consistent iff there exists a voting rule conformant to it.

# Arguments

## Argument

An argument grounded on  $J$  is a pair (*claim*, *proof*),

- $J$  an l-axiomatization,
  - *claim* a uni-profile clause (thus of the form  $[\mathbf{R} \vdash \alpha]$ ),
  - *proof* a natural deduction proof of the claim grounded on  $J$ .
- 
- The argument shows that for all voting rules  $f$  conformant to  $J$ ,  $f(\mathbf{R})$  selects a set of winners among  $\alpha$ .
  - The argument claims that it is only reasonable to choose the winners among  $\alpha$  for  $\mathbf{R}$  (provided  $J$  is accepted).
  - *Consistent* arguments require a consistent l-axiomatization.

## Example proof

$$\mathbf{R}_D = \begin{array}{cc} a & b \\ b & c \end{array}, \mathbf{R}_S = \begin{array}{cc} a & c \\ b & b \end{array}, \mathbf{R} = \mathbf{R}_D + \mathbf{R}_S = \begin{array}{cccc} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{array}.$$

- ①  $[\mathbf{R}_D \vdash \subseteq \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (DOM)
- ②  $[\mathbf{R}_S \vdash \{ a, b, c \}]$  (SYM)
- ③  $([\mathbf{R}_D \vdash \{ a \}] \wedge [\mathbf{R}_S \vdash \{ a, b, c \}]) \rightarrow [\mathbf{R} \vdash \{ a \}]$  (REINF)
- ④  $([\mathbf{R}_D \vdash \{ b \}] \wedge [\mathbf{R}_S \vdash \{ a, b, c \}]) \rightarrow [\mathbf{R} \vdash \{ b \}]$  (REINF)
- ⑤  $([\mathbf{R}_D \vdash \{ a, b \}] \wedge [\mathbf{R}_S \vdash \{ a, b, c \}]) \rightarrow [\mathbf{R} \vdash \{ a, b \}]$   
(REINF)
- ⑥  $[\mathbf{R}_D \vdash \{ a \}] \rightarrow [\mathbf{R} \vdash \{ a \}]$  (PR from 2 & 3)
- ⑦  $[\mathbf{R}_D \vdash \{ b \}] \rightarrow [\mathbf{R} \vdash \{ b \}]$  (PR from 2 & 4)
- ⑧  $[\mathbf{R}_D \vdash \{ a, b \}] \rightarrow [\mathbf{R} \vdash \{ a, b \}]$  (PR from 2 & 5)
- ⑨  $[\mathbf{R}_D \vdash \{ a \}] \vee [\mathbf{R}_D \vdash \{ b \}] \vee [\mathbf{R}_D \vdash \{ a, b \}]$  (rewrite 1)
- ⑩  $[\mathbf{R} \vdash \{ a \}] \vee [\mathbf{R} \vdash \{ b \}] \vee [\mathbf{R} \vdash \{ a, b \}]$  (PR from 6–9)
- ⑪  $[\mathbf{R} \vdash \subseteq \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (rewrite 10)

## Example shortened

Tweak l-axioms to skip steps which will seem intuitive to humans.

### Reinforcement-sets

For each  $\mathbf{R}_1, \mathbf{R}_2, \alpha_1, \alpha_2 \subseteq \mathcal{P}_{\emptyset}(\mathcal{A}), \cap \emptyset \neq \emptyset, \emptyset \in \alpha_1 \times \alpha_2$ :

$$([\mathbf{R}_1 \vdash^{\in} \alpha_1] \wedge [\mathbf{R}_2 \vdash^{\in} \alpha_2]) \rightarrow [\mathbf{R}_1 + \mathbf{R}_2 \vdash^{\in} \bigcup_{A_1 \in \alpha_1, A_2 \in \alpha_2} \{ A_1 \cap A_2 \}].$$

- ①  $[\mathbf{R}_D \vdash^{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (DOM)
- ②  $[\mathbf{R}_S \vdash^{\in} \{ a, b, c \}]$  (SYM)
- ③  $((1) \wedge (2)) \rightarrow [\mathbf{R} \vdash^{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$  (REINF-SETS)
- ④  $[\mathbf{R} \vdash^{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

# Soundness and completeness

Consider an  $\mathcal{L}$ -axiomatization  $J$  and a claim  $c = [\mathbf{R} \vdash_{\mathcal{L}} \alpha]$ .

## Theorem (Soundness)

*If there exists an argument  $(c, \text{proof})$  grounded on  $J$ , the claim holds given  $J$ .*

## Theorem (Completeness)

*If the claim holds given  $J$ , then there exists an argument  $(c, \text{proof})$  grounded on  $J$ .*

This is easily obtained from the soundness and completeness of natural deduction in propositional logic.

# Bibliography I

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