Collaborative learning of deliberated preference

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1 Introduction

In a situation where a user has to choose an item among a set of possible items, recommender systems aim at recommending some item as most appropriate for the user. In the collaborative learning community, this is usually taken to mean: recommend an item among the most preferred for the user. In this article, we introduce another goal for recommender systems. We keep the global aim of recommending some item as most appropriate for a given user, but we propose a different understanding of appropriateness. The goal we propose is to recommend an item among those that are in the deliberated preference of the user, and to prove that it does. The deliberated preference of the user captures what she prefers when considering all arguments in favor or against all possible items. This differs from her intuitive preference in that the latter does not consider arguments. Thus, the goal we propose is appropriate when one is interested to help the user form a deliberated judgement about which item is best, rather than predict which item the user intuitively prefers.

Together with this new goal, it is necessary to propose a corresponding measure of the performance of the recommender system. In the classical approach, measuring the performance of the system amounts to compare its claims (the items it proposes as preferred) to reality (the preference of the user). Because the deliberated preference of the user is not directly observable, this direct check of correspondance becomes non applicable. However, as we will show, the problem of evaluating the quality of the recommendation in our sense can be framed as a prediction problem (although involving different objects), thereby recovering the possibility of empirical validation by confrontation between claims of the system and reality.

This article is structured as follows. We first present the task of recommendation that we propose in general: we define precisely the notion of deliberated preference, then, the goal of a recommender system, and finally, the way its recommendations can be (in)validated. Then, we study one possible way of undertaking this task, applicable in specific contexts: we suggest to recommend on the basis of some explicit principles coming from decision theory. This requires to focus on situations where it is a priori plausible that those principles can be used to find an item in the deliberated preference of the user, and, importantly, can be used to prove that the item does belong to her deliberated preference. We will present some such situations and describe those principles. Third, we want to provide first steps towards showing technically how a rec-

ommender system can be built in our situation of choice, and study our building procedure and resulting system empirically by (among others) comparing it to classical recommender systems on a real dataset. Finally, we discuss some of our choices and relate our approach to existing literature.

2 Deliberated preference and collaborative learning

2.1 Definition of deliberated preference

This section describes precisely the new understanding of preference we propose. We call it the deliberated preference of the user.

We consider given a set of items $\mathcal{J}$ among which a user i wants to choose. We also consider given a set S^* containing arguments that may possibly help u form a deliberated opinon about which item is best for her (elements of S^* are not detailed and can be conceived as strings in a natural language).

We will define the deliberated preference of i in that situation as the subset of items that are considered by i as having no item strictly preferred to them, considering all possible arguments. To define this properly, we assume we can observe the reaction of i to arguments, as follows.

Define $\triangleright$ as a binary relation over $\mathcal{J} \times S^*$, representing the results of the following experiments. We present (j, s) and (j', s') to i and let her decide, considering the arguments s and s', which item among j and j' she prefers, if any. We define $(j, s) \triangleright (j', s')$ iff i strictly prefers j to j', given s and s'. Note that if no strict preference holds, $\triangleright$ does not relate these two pairs (thus $(j, s) \not\triangleright (j', s')$ and $(j', s') \not\triangleright (j, s)$).

The deliberated preference $J_i \subseteq \mathcal{J}$ of i in that situation contains an item j iff $\forall (j', s') \in \mathcal{J} \times S^*, \exists s \in S^* \mid (j', s') \not\triangleright (j, s)$.

2.2 Prediction of deliberated preference

We can now define the goal of a recommender system. A recommender system (as considered here) should be able, given a user, to single out some items as being in his deliberated preference, and some items as being not in his deliberated preference, and to prove its claims by arguing correspondingly.

Formally, a recommender system η has a *scope* I, a set of users it claims to be able to predict the deliberated preference of, and gives, for a user $i \in I$, a tuple $(J_\eta, f_\eta^{\text{def}}, R_\eta, f_\eta^{\text{att}})$,

where $J_\eta \subseteq \mathcal{J}$ is a set of items that η claims are among the deliberated preference of i, $f_\eta^{\text{def}} : J_\eta \times \mathcal{J} \to S^*$ is an argumentation strategy used to defend items in J_η, $R_\eta \subseteq \mathcal{J} \times \mathcal{J}$ is a binary relation that contains pairs of items (j, j') such that η claims that i deliberately prefers j to j', and $f_\eta^{\text{att}} : R_\eta \to S^*$ is an argumentation strategy used to support the claims represented by R_η.

To ensure that a recommender system is valid, we require that its functions f_η^{def} and f_η^{att} indeed justify its claims, in the following sense. A recommender system η is said to be *correct* about a user i iff $\forall j \in J_\eta, j' \in \mathcal{J}, s' \in S^* : (j', s') \not\rhd (j, f_\eta^{\text{def}}(j, j'))$ and $\forall (j, j') \in R_\eta, s' \in S^* : (j, f_\eta^{\text{att}}(j, j')) \rhd (j', s')$. A recommender system is *valid* iff it is correct about all users in its scope.

Importantly, validity implies the correctness of the claims concerning which items belong or do not belong to the deliberated preference of i. This fact is stated formally below. (The proof is omitted.)

Fact 1. *If a recommender system η is valid, then $\forall i \in I : J_\eta \subseteq J_i$ and $J_\eta^\downarrow \subseteq \mathcal{J} \setminus J_i$, where $J_\eta^\downarrow = R_\eta(\mathcal{J}) = \{ j' \in \mathcal{J} \mid (j, j') \in \mathbb{R}_\eta \text{ for some } j \in \mathcal{J} \}$.*

Validity can't be operationally checked if S^* is too large to allow for an exhaustive verification. We propose to use this notion of validity to *compare* recommender systems, in the following way. Given two recommender systems η and η', consider a user i in both their scopes and obtain two tuples, $(J_\eta, f_\eta^{\text{def}}, R_\eta, f_\eta^{\text{att}})$ and $(J_{\eta'}, f_{\eta'}^{\text{def}}, R_{\eta'}, f_{\eta'}^{\text{att}})$. Assume they strongly disagree on the prediction of the deliberated preference of i, meaning that for some item $j \in \mathcal{J}$, the first system claims that j is in the deliberated preference of i, and the second system claims it is not. Thus, assume that $j \in J_\eta$ and $\exists j' \mid (j', j) \in R_{\eta'}$. Suffices now to let the two systems play against each other and use i as a judge. That is, we obtain an argument from the first system in defense of its claim, $s = f_\eta^{\text{def}}(j, j')$, and an argument from the second system, $s' = f_{\eta'}^{\text{att}}(j', j)$. We present (j, s) and (j', s') to i and accordingly obtain $(j', s') \rhd (j, s)$ or $(j', s') \not\rhd (j, s)$. In the first case, the first system is falsified, in the second one, the second system is falsified.

3 Recommend on the basis of explicit principles

One plausible approach for building valid recommenders in the context of collaborative learning rests on using explicit and sound principles. We propose here to use Multiple Attribute Value Theory (MAVT), a well-known set of principles for choice coming from decision theory.

Those principles are applicable in specific contexts, to which we restrict our attention. In our case of interest, the items among which the user aims to choose can be exhaustively described by a set of objective descriptors of the items, known a priori, and which each evaluate the desirability of the items on some aspect. Such descriptors are called criteria. Exhaustively described means that nothing else than the performance of the items on the criteria is considered relevant for the recommendation by the user. In this case, we say that the recommendation task is justifiable, given that set of criteria. Thus, a recommendation task is justifiable, given a set of criteria,

when two items are equally recommendable whenever they are equally valued on this set of criteria.

Consider as an example the choice of an apartment to rent for holidays. The set of criteria might be: surface of the living room, number of beds, distance to city center, modernity of equipment, presence of a washing machine, price. The recommendation task is justifiable given these criteria if, when two apartments are equally valued on all these criteria (have the same surface of the living room, the same number of beds, and so on), they are equally recommendable. Thus, when the set of criteria completely describes the desirability of an apartment.

We acknowledge that justifiability is, in many case, an approximation. We merely assume in this article that it is an appropriate approximation to consider some recommendation tasks as justifiable given some known set of criteria. Other examples include: buying a smartphone, buying a computer, choosing a flight to reach a given place, or even, possibly, choosing who to give a "best student" award among a given classroom, locating a new factory, or choosing a university to study at.

By contrast, a recommendation task is not justifiable (or, more precisely, it is unknown how to make it justifiable) if no set of criteria is known that fully describe the desirability of an item. For example, a choice of movie to watch this evening. Note that a movie is certainly describable on some set of descriptors (for example, the singleton set including its serial number, or a set including its title, release date, and other attributes that together uniquely identify a movie), but those are not criteria as defined above, as those descriptors do not each evaluate the desirability of a movie on some aspect.

When a recommendation task is justifiable, it is a plausible theory that the user's deliberated preference is representable by an MAVT model and that this model be useable to prove the claims of a recommender system based on such representation.

We think a good starting point, in order to direct the learning process of decision models, is to suppose that the usual behavior of the user is a noisy version of its deliberated behavior. Thus, we will attempt to learn the decision model that fits as closely as possible its usual behavior.

We leave the important issue of argument generation aside in this article. Existing literature proposes to generate arguments in natural language that explain, on the basis of an MAVT model, why an alternative is preferred to another one given this model and could provide a starting point for this step.

3.1 Sufficient conditions for an MAVT representation

A simple approach to build a recommender system of the sort of interest in this article is to build a weak order $\succeq \subseteq \mathcal{J} \times \mathcal{J}$, meaning a complete (thus reflexive) and transitive binary relation, whose maximal elements correspond to J_i. (Note however that our definition tolerates other possibilities.)

We are interested in defining sufficient conditions for being able to represent such a relation $\succeq$ as an MAVT model.

Condition 1: a finite set of descriptors exist, $b_c : \mathcal{J} \to X_c, c \in \mathcal{C}$, that each describe the items on a specific aspect c (called criterion), and which are known a priori. They are such that whether $j \succeq j'$ depend only on the descriptions on

the items according to $\{b_c, c \in \mathcal{C}\}$. We also mandate for simplicity that there be at least three descriptors and that the scales X_c be finite.

In the sequel we treat $\succeq$ as a relation over $X = \prod_{c \in \mathcal{C}} X_c$, which we can do thanks to condition 1. Define $\sim$ as the symmetric part of $\succeq$ and $\succ$ its asymmetric part. Thus, $\sim$ is an equivalence relation. With $C = \mathcal{C} \setminus \{c\}$ for some $c \in \mathcal{C}$, thus C contains all criteria but one, let $(x_C, x_c) \in X$ represent the performance of an item $j \in \mathcal{J}$ where $x_C \in \prod_{d \in C} X_d$ describes the item j on the criteria C and $x_c \in X_c$ is the remaining description. Similar notation is used when C contain all criteria but two.

Condition 2: Trade-offs between two criteria must be independent of levels on other criteria: $\forall C = \mathcal{C} \setminus \{c_1, c_2\}, c_1 \neq c_2, x_1, y_1 \in X_{c_1}, x_2, y_2 \in X_{c_2}, x_C, y_C \in \prod_{c \in C} X_c$: $(x_1, x_2, x_C) \succ (y_1, y_2, x_C) \Rightarrow (x_1, x_2, y_C) \succ (y_1, y_2, y_C)$. (TODO check that this is enough.)

Theorem (to be checked): if the preference of the user $\succeq$ satisfies all the above conditions, then there exists a value function $v : \mathcal{J} \to \mathbb{R}$ and partial value functions $v_c : X_c \to \mathbb{R}$ for each criterion c such that for any two items, $j \succeq j' \Leftrightarrow v(j) \geq v(j')$ and $v(j) = \sum_{c \in \mathcal{C}} v_c(b_c(j))$.

4 Estimating an MAVT model

Assume we have a set of users I of size $m \in \mathbb{N}$, a set of items J of size $n \in \mathbb{N}$, a set of observed pairs $O \subseteq I \times J$ and a relation r mapping those observed pairs $(i, j) \in O$ to some rating r_{ij}, an integer in $[\![1, 5]\!]$ (throughout the article, this notation designates intervals in the integers). We also have descriptions of the users in a vector space A of dimension $d_A \in \mathbb{N}$: user i is described by $a_i \in A$. The items are described by a set of descriptors whose indices are in $\mathcal{C} = [\![1, \ldots, c, \ldots, d]\!]$. The descriptor $B_c : J \to X_c, B_c \in \mathcal{B}$ evaluates each item according to a scale X_c. Let $X = \prod_{c \in \mathcal{B}}$ denote the cardinal product space where descriptions of items lie, and let $b_j \in X$ denote the description of an item j.

4.1 Classical collaborative learning

In classical collaborative learning, we wish to obtain compressed descriptions of the users and the items, $\{u_i, i \in I\}$ and $\{v_j, j \in J\}$, with all u_i and v_j being vectors of some size $k \in \mathbb{N}$. We write U the matrix of size (m, k) that has all u_i^T as rows, and V the matrix of size (k, n) that has all v_j as columns. The compressed descriptions should be such that $r_{ij} \approx u_i^T v_j$.

4.2 Using given knowledge

However, we also wish that our compressed descriptions u_i be in a simple relationship with the provided descriptions a_i, thus, we search for $u_i \approx f(a_i)$, and similarly, for $v_j \approx g(b_j)$, with f and g "simple". More precisely, we search for a matrix W_A of size (k, d_A) such that $W_A a_i \approx u_i$, and W_B of size (k, d_B) such that $W_B b_j \approx v_j$. We call $W_A a_i$ and $W_B b_j$ our explainable representation of the users and the items.

We could define $u_i = W_A a_i$ and $v_j = W_B b_j$ and obtain $r_{ij} \approx a_i^T(W_A^T W_B) b_j$, thus we could optimize by searching for the best $(W_A^T W_B)$. But as this could be too constrained, we will also try to relax these constraints and search for a

representation that takes into account that u_i should be close to $W_A a_i$, but not necessarily equal.

4.3 Predict ratings with logits

Assume we are given a loss function L that compares the ratings r_{ij} to our approximations of the ratings, for $(i, j) \in O$.

Define $\sigma(\alpha, \beta, x) = \frac{1}{1 + \alpha e^{-\beta x}}$.

We want to find $\{w_i^c, i \in I, c \in \mathcal{C}\}, \{\alpha^c, c \in \mathcal{C}\}, \{\beta^c, c \in \mathcal{C}\}$ that optimize the following objective:

$$\min_{w, \alpha, \beta} \sum_{(i,j) \in O} L(r_{ij}, \sum_{c \in \mathcal{C}} w_i^c \sigma(\alpha^c, \beta^c, b_j^c)) + \lambda \sum_{i,c} \|w_i^c\|_2^2.$$

We force w positive. And α is forced positive to respect the preference direction: we assume the data is encoded so that higher b_j^c means preferred performance. Where b_j^c is the performance of item b_j on criterion c.

4.4 Predict ratings

We could want to check whether we are able to predict the right ratings. In that case, we would assume we are given loss functions L, that compare the ratings r_{ij} to our approximations of the ratings, for $(i, j) \in O$; L_I, that compare the vectors u_i to our explainable representation of them; and L_J, that compare the vectors v_j to our explainable representation of them.

We would want to find matrices U, V, W_A, W_B that optimize the following objective:

$$\min_{U, V, W_A, W_B} \sum_{(i,j) \in O} L(r_{ij}, u_i^T v_j) + \sum_{i \in I} L_I(u_i, W_A a_i) + \sum_{j \in J} L_J(v_j, W_B b_j).$$

4.5 Predict orders

We are interested in comparing our predictions to orderings (because that's what we have underhand).

We assume we are given loss function L, that compares preorders (transitive and reflexive binary relations), and L_I and L_J, as above. We assume we have a preorder $\geq_i$ for each user indicating her true preference over some pairs of items among J.

We want to find matrices U, V, W_A, W_B that optimize the following objective:

$$\min_{U, V, W_A, W_B} \sum_{i \in i} L(\geq_i, \{(v_j, v_{j'}) \mid u_i^T v_j \geq u_i^T v_{j'}\}) + \sum_{i \in I} L_I(u_i, W_A a_i) + \sum_{j \in J}$$

4.6 Data set: Sushi

We experiment using the Sushi data set (ref...)

Here J is a set of 100 sushi types [1]. The original authors collected the sushi from the menu of 25 restaurants, computed the frequency of each sushi type occurring on a menu, and took the 100 most frequent. Write P_J the frequency (thus a multiple of $1/25$).

Write $J_{\text{top}} \subset J$ the 10 most frequent sushi types [2].

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[1] $X^* = X_B$ in the original notation.
[2] X_A

The set I has cardinal $m = 5000$. The authors drew randomly a set of 10 sushi types among J, for each user i, according to the distribution P_J. Let us denote this by $J_i \subset J$ [3].

The authors obtained, for each user i in I, three preferential informations, defined as strict partial orders (transitive, asymmetric binary relations). First, they collected a preference ordering over the sushi types from J_{top} from most to least preferred, with no tie allowed. Let $>_i^{\text{top}}$ denote that preference ordering [4] ("top" designates the fact that the ordering is over the 10 most frequent sushi types). Second, they collected a rating, in $[\![1,5]\!]$, for each sushi type in J_i. Define O as the set of pairs (i,j) such that $j \in J_i$, and let r_{ij} denote that rating. Third, after two supplementary questions asking how oily the user think each sushi type is and how frequently they eat them, they collected a preference ordering over J_i, that we write $>_i^{\text{indiv}}$ [5] ("indiv" designates the fact that this relation orders a set of sushi that has been drawn specifically for that user).

Users are described in a space A comprising the attributes {gender, age category, origin-prefecture, origin-region, origin-east/west (binary), current-region, current-east/west (binary), origin-prefecture}. Items are described in a space B comprising the attributes {group, heaviness/oiliness in taste (in $[0,4]$), frequency of eating (in $[0,3]$), normalized price (in $[0,1]$), $P_J(j)$}.

We summarize the preference of i as follows. The ratings over J_i are transformed to preorders $\geq_i^{\text{rating}}$ that tolerate exæquos, in the obvious way (better ratings indicate preference). We then merge, for each user i, all preferential information, considering contradictions as indicating indifference, and obtain a preorder $\geq_i = [\geq_i^{\text{rating}} \cup >_i^{\text{top}} \cup >_i^{\text{indiv}}]$, where the brackets designate taking the transitive closure and adding the identity relation (to obtain reflexivity). For example, if $>_i^{\text{top}}$ represents the ordering $(j_1 > j_2 > j_3)$ and $>_i^{\text{indiv}}$ represents $(j_6 > j_3 > j_4 > j_2)$ and $\geq_i^{\text{rating}} = \emptyset$, then $\geq_i$ considers j_1 and j_6 as strictly preferred to j_2 to j_4, j_2 to j_4 ex-æquos, and j_1 incomparable to j_6.

5 Discussion

5.1 The need for another notion of preference

In this article we are interested in a notion of preference that possibly changes when confronted to arguments. That is, we assume that the user's knowledge of which item is best for him is not necessarily fixed a priori, and in particular, that the user may change his mind when presented with arguments in favor of, or counter-arguments against, various items.

By contrast, the notion of preference considered by classical recommender systems (usually left implicit) in the collaborative learning literature is that the user intuitively knows what he prefers, and that this is the right basis for him to decide. This notion can be related to the conception of preference put forth by von Neumann and Morgenstern [2004] in 1944 in their seminal work. They write that "It is clear that every measurement – or rather every claim of measurability – must

ultimately be based on some immediate sensation, which possibly cannot and certainly need not be analyzed any futher. In the case of utility the immediate sensation of preference – of one object or aggregate of objects as against another – provides this basis (. . .) Let us for the moment accept the picture of an individual whose system of preferences is all-embracing and complete, i.e. who, for any two objects or rather for any two imagined events, possesses a clear intuition of preference. More precisely we expect him, for any two alternative events which are put before him as possibilities, to be able to tell which of the two he prefers." When applied to collaborative filtering, this view suggests that the job of the system is to help the user single out which items are the best among all the possible ones (to relieve the user from having to search through the whole set). The user, once presented a pair of items (for example, a pair of movies), will know which one he prefers by simple introspection, possibly after having tried them both. Thus, although the user might need help for efficiently searching through a big set of items, he does not need help for the task of comparing a pair of item. Similarly, when the user has already provided some comparisons of preference, this is the final word about those pairs. (Admittedly, it is common to say that a user may make mistakes when reporting her preference, but what this means is usually not specified, and not much is done with this fact, as far as we know, in the literature of collaborative learning.)

We claim that this notion of preference, that we call intuitive preference, is not always the best basis to ground recommendation. Indeed, the user might be unwilling to judge what is best for her on the sole basis of her unaided intuition. First, there are cases where the user can't easily try out the items. An example is a non repeatable choice, such as choosing a university to study in. Second, in some situations, a notion of fairness is involved, or a similarly abstract notion not easily accessible by non-reflexive perception, such as determining who will receive a prize, or how to best distribute revenue in a society, or to which cause I should donate money. Third, even excluding moral considerations, the best choice might be the result of a complex thought process that should better ensure, as much as possible, that all relevant arguments have been considered, rather than rely on intuition. Consider the decision of which smartphone, or which computer, or which house, to buy. Fourth, the user might appreciate being pointed to some non-salient feature of the items under comparison: for example, a user might feel an intuitive attraction to the plane alternative when chosing a transportation means for holidays, seeing that the flight time is only one hour compared to three hours by train, but change her mind when being reminded that the total travel time should be taken into account. Finally, an important argument coming from psychology can be given in favor of considering arguments rather than intuitive preference to decide on what's best (in some contexts). It is that unaided intuition is known to be sensible, in some contexts, to framing effects that would perhaps not be considered relevant (by the user himself) when considering arguments and counter-arguments. An example that involves several kinds of the just enumerated concerns is obtaining a procedure to determine to which client a bank should lend money: first, the user probably wants to involve fairness considerations in order to avoid unjust discriminations (which might unconsciously appear if using unaided intuition); sec-

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[3] X_i^B
[4] O_i^A
[5] O_i^B

ond, even if only pure profit considerations are to bear, the user might want the procedure to go beyond reflecting the bare intuition of an expert.

5.2 About the definition of the situation

The notion of deliberated preference is a simplification of the recently proposed concept of deliberated judgement [Cailloux and Meinard, 2017], applied to the notion of preference (instead of the less specific notion of judgement), and instanciated for collaborative learning.

We insist, to avoid turning recommendation into persuasion or manipulation, that the set of all arguments S^*, in our conception, must not be restricted to arguments judged as good arguments from some external point of view, but should contain a wide range of arguments that can possibly be considered by various (possibly disagreeing) sources as possibly influencing i. For example, S^* could be the set of all strings.

The relation $\triangleright$ is what plays in our approach the role of the basic observable von Neumann and Morgenstern talk about. There are multiple ways of instantiating the experiment represented by $\triangleright$, for example, we may consider stated attitude or revealed attitude, and we have to choose some way of presenting the arguments to i; those various ways would possibly define different deliberated preferences. Alternatively, the deliberated preference of a user could be considered as consisting in those items that resist counter-arguments considering various ways of presenting arguments. We do not discuss here the difficulties related to distinguishing strict preference from indifference in revealed preference approaches, and consider (as a possibly somewhat simplistic assumption) that strict preference can be equated to a willingness to pay a small amount to switch from one item to the preferred one, or that the experiment can be repeated and that strict preference holds if a systematic choice is observed.

We do not mandate that recommender systems be able to determine all the items in the deliberated preference of a user, but merely that the items it claims are in the deliberated preference of a user be indeed, and that its proofs of this fact be valid, and similarly for the items claimed as not belonging to the deliberated preference of a user. This is for two reasons: our requirement is enough to have a possibly useful system (provided $J_\eta \neq \emptyset$), and being able to completely predict the deliberated preference of a user might be extremely difficult.

Recommender systems may have very few claims, and accordingly, may have a low chance of being falsified. This reflects the usual difficulty that scientists face: general theories are more useful, but more difficult to get right.

5.3 Relation to existing literature

Other approaches exist which propose to estimate a decision-theoretic model that best approximates a user's observed behavior, thus, our proposal is not new in this respect. What is new is our proposal of a research agenda to develop such models not because they would best predict the user's normal behavior, but because they could best describe her deliberated judgment; and our instanciation to collaborative learning.

5.4 Going beyond

More elaborated developments of our proposal would incorporate biases known from experimental psychology to systematically attempt to correct possible mistakes of the user (the word "mistake" referring to the difference between her usual behavior and her own deliberated judgment).

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A Double-strategy

In the double-strategy, we use different dimensions: $k = d_{A'} + d_U = d_{B'} + d_V$. The matrix W_A has size $(d_{A'}, d_A)$ and U has size (m, d_U). W_B has size $(d_{B'}, d_B)$ and V has size (d_V, n). $r_{ij} = \mu_i^T \nu_j$ where μ_i is the k-dimensional vector composed of $W_A a_i$ then u_i and ν_j is composed of $W_B b_j$ then v_j.

We should compare this approach to the more classical one.

B Using raw item descriptions

The matrix W_B could be constrained to having mostly zeroes, to make sure that indeed the attributes of the items (b_j) are used in their μ-vectorial description. We could even take $d_{B'} = d_B$ and take W_B equal to identity. (Is there any benefit not to do so?)

This suggests the following simple approach: concatenate b_j to v_j to obtain ν_j. Don't use W_B. Obtain μ_i of size $k = d_B + d_V$, using $W_A a_i$ as its first components.

C Factorization machines

We should compare (theoretically or experimentally) our approach to factorization machines.