Matchings and Allocations under Preferences: Algorithms and Polytopes

My main work with Michel Balinski: 1997-2002

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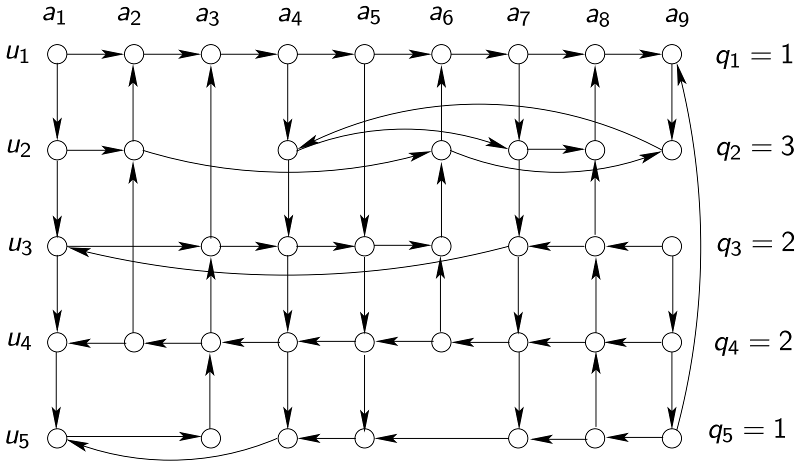
The Stable Admission Polytope



a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9
u_1
u_2
u_3
u_4
u_5
$q_1 = 1$
$q_2 = 3$
$q_3 = 2$
$q_4 = 2$
$q_5 = 1$

A *matching* μ in an admissions problem (Γ, q) is a set of nodes of Γ that has at most one per column and at most q_u in row u for each u in U.

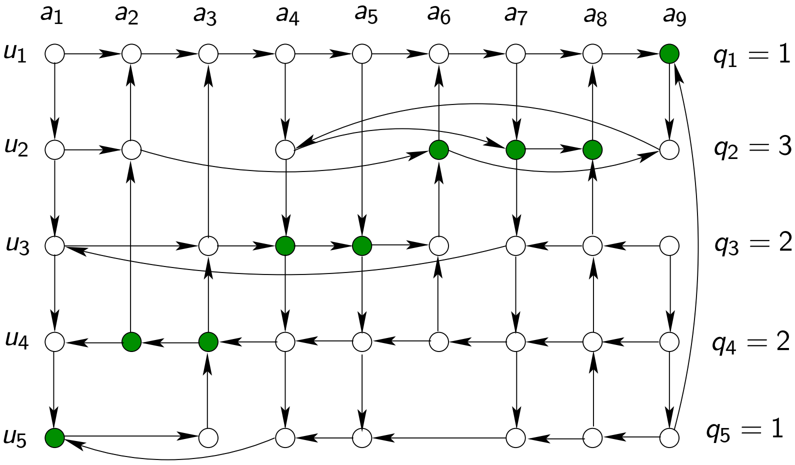
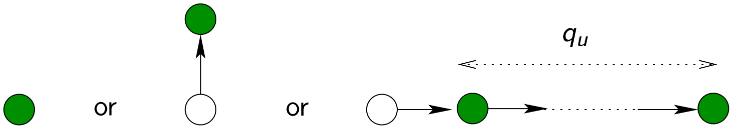


Figure – The set of green nodes is a matching.



(i) : $(u, a) \in \mu$ (ii) : $(u, a) \notin \mu$ (iii) : $(u, a) \notin \mu$

Figure – Stability via graphs : a matching μ (represented by green nodes in the figure) is stable if for any node (u, a) one of the assumptions (i), (ii) or (iii) hold.

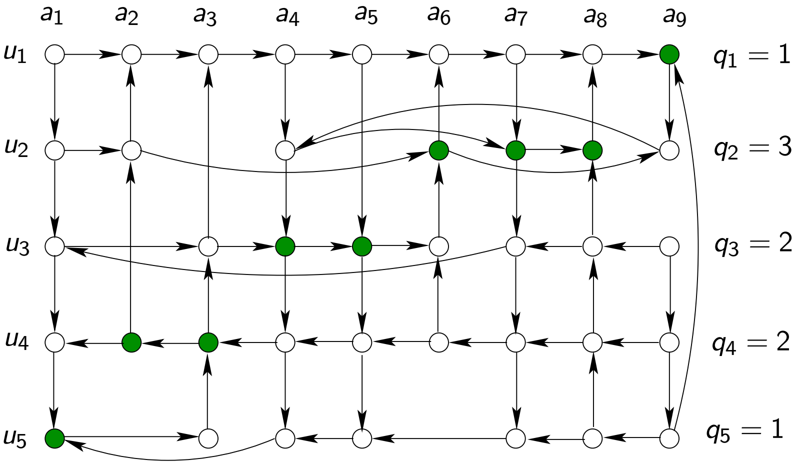


Figure – The set of green nodes is a *non stable matching*.

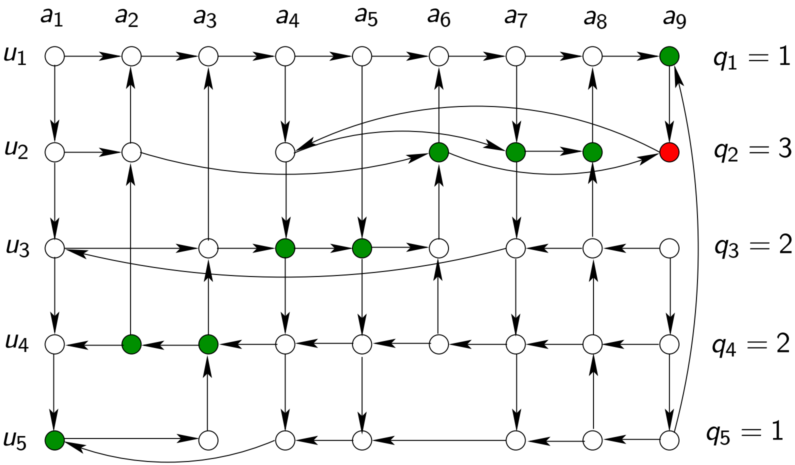


Figure – The pair (u_2, a_9) is a *blocking pair*.

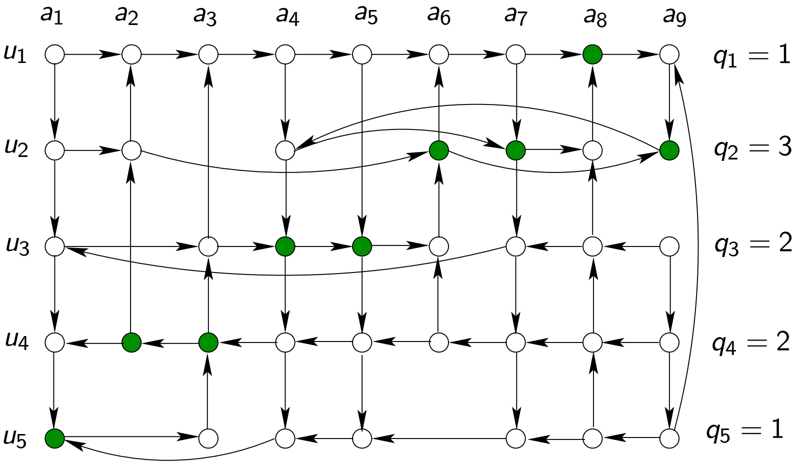


Figure – The set of green nodes is a stable matching.

Polyhedra

- The *incidence matrix* x^{μ} of a subset μ of the nodes of Γ is defined by : $x^{\mu}_{ua} = 1$ if $(u, a) \in \mu$ and $x^{\mu}_{ua} = 0$ otherwise.
- The *stable admissions polytope*, $SAP(\Gamma, q)$, of an admissions problem (Γ, q) is the convex hull of the incidence matrices of the stable matchings in (Γ, q) :

$$SAP(\Gamma, q) = \mathrm{conv}\{x^{\mu} : \ \mu \text{ is a stable matching in } (\Gamma, q)\}.$$

Let us first concentrate on the case where $q_u = 1$ for each $u \in U$. This is the stable marriage problem.

Theorem (Vande Vate (89), Rothblum (92))

For the case of the marriage problem $SAP(\Gamma, q)$ is characterized by the following inequalities :

$$\sum_{(u,a)\in\Gamma} x_{ua} \leq 1 \qquad \text{for all } u \in U, \text{ row inequalities}$$

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\sum_{(u,a)\in\Gamma} x_{ua} \leq 1 &\qquad \textit{for all } u \in U, \textit{ row inequalities} \\
\sum_{(u,a)\in\Gamma} x_{ua} \leq 1 &\qquad \textit{for all } a \in A, \textit{ column inequalities} \\
x_{ua} + x_{ua\uparrow} + x_{ua\longrightarrow} \geq 1 &\qquad \textit{for all } (u,a) \in \Gamma, \textit{ stablity inequalities.}
\end{aligned}$$

Illustration.

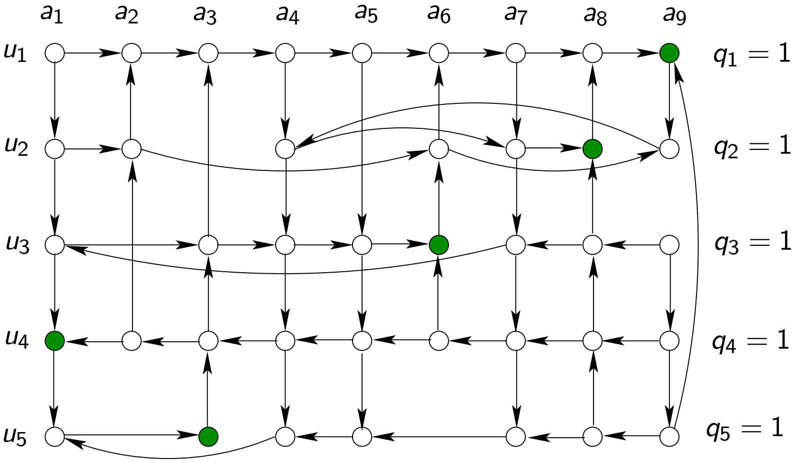
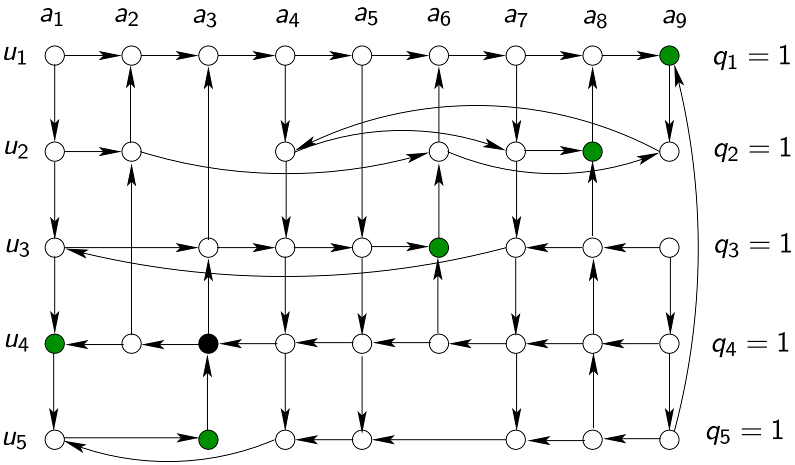


Figure – The set of green nodes is a stable matching of the marriage problem.

Illustration.



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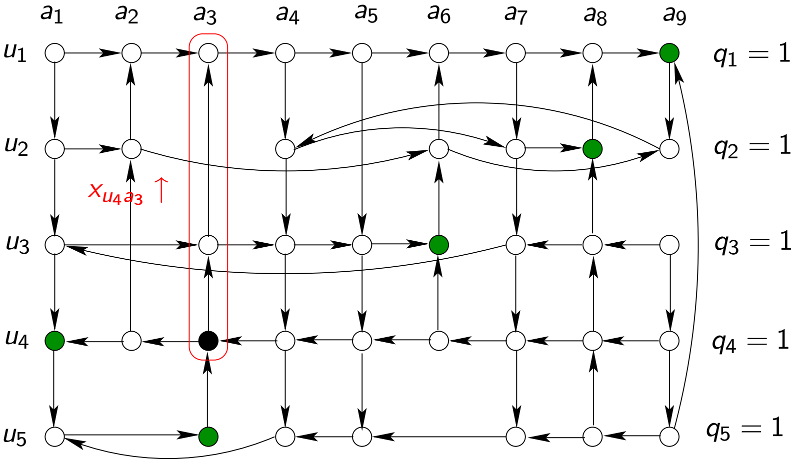

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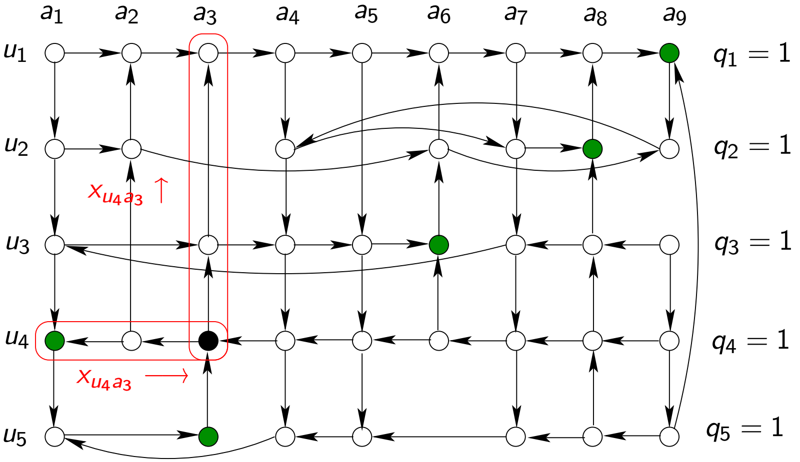

Figure – The set of green nodes is a stable matching of the marriage problem.

A natural extension of the theorem of [Vande Vate (89), Rothblum (92)] for the admissions polytope is the following

$$
\begin{aligned}
\sum_{(u,a)\in\Gamma} x_{ua} &\leq q_u, && \text{for all } u \in U,\\
\sum_{(u,a)\in\Gamma} x_{ua} &\leq 1, && \text{for all } a \in A,\\
q_u x_{ua} + q_u x_{ua}\uparrow + x_{ua}\longrightarrow &\geq q_u, && \text{for all } (u,a) \in \Gamma,\\
x_{au} &\geq 0, && \text{for all } (u,a) \in \Gamma.
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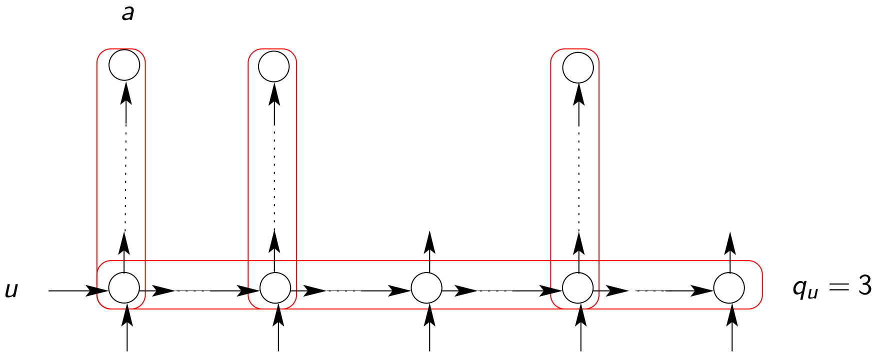
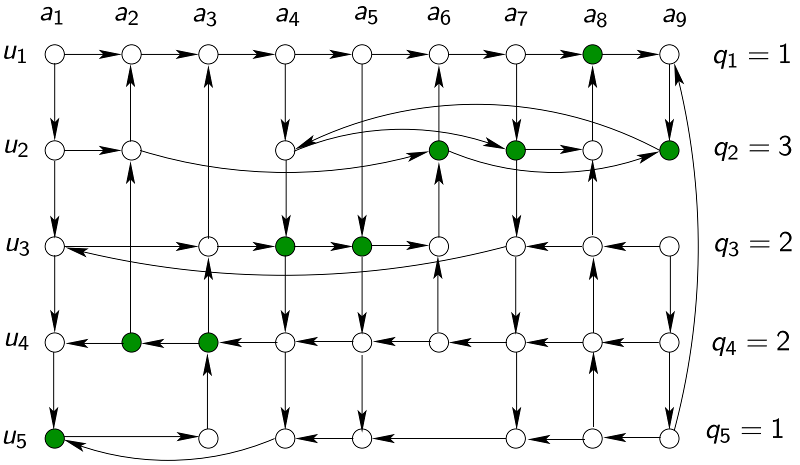
\end{aligned}
$$

But it does not work. The above polytope contains fractional extreme points.

The comb inequalities

- Shaft $S(u,a) = (u,a)$ and all of its successors in row u.
- Tooth $T(u,a) = (u,a)$ and all of its successors in column a.
- Comb $C(u,a) =$
  $S(u,a) \bigcup T(u,a) \bigcup_{(u,j)\in S(u,a)} (q_u - 1)$ teeth $T(u,j)$.





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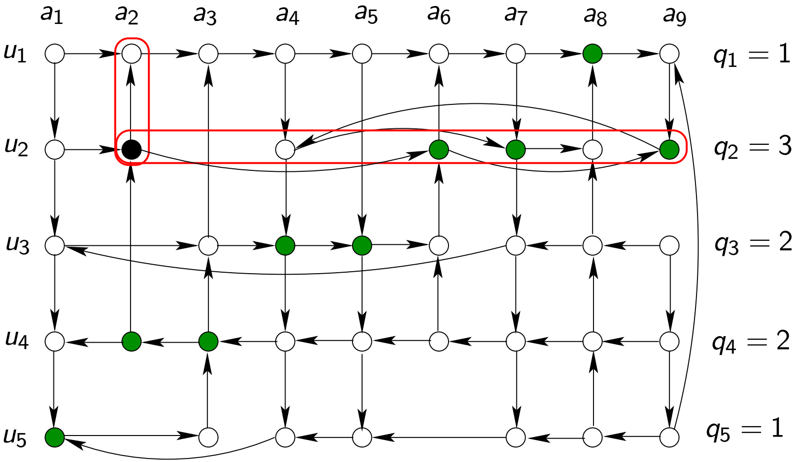


Figure – A Shaft $S(u, a)$ with a Teeth $T(u, a)$.

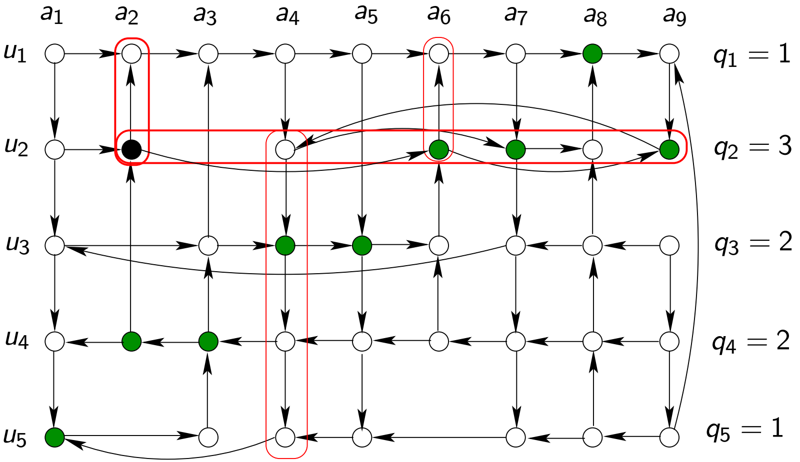
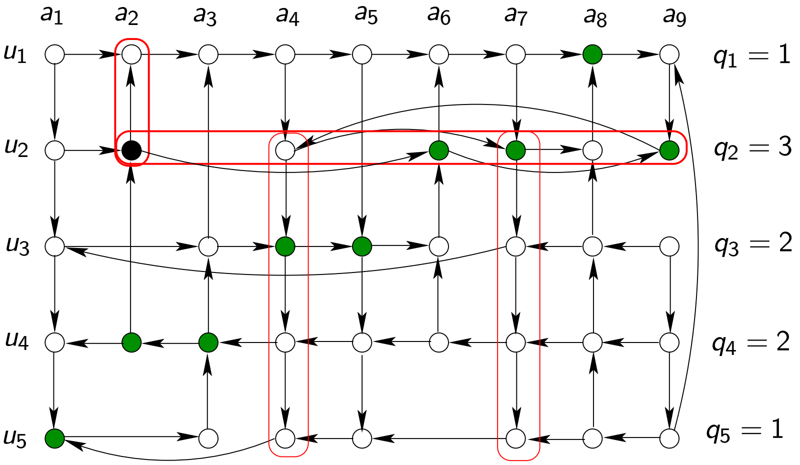


Figure – A comb $C(u_2, a_2)$.



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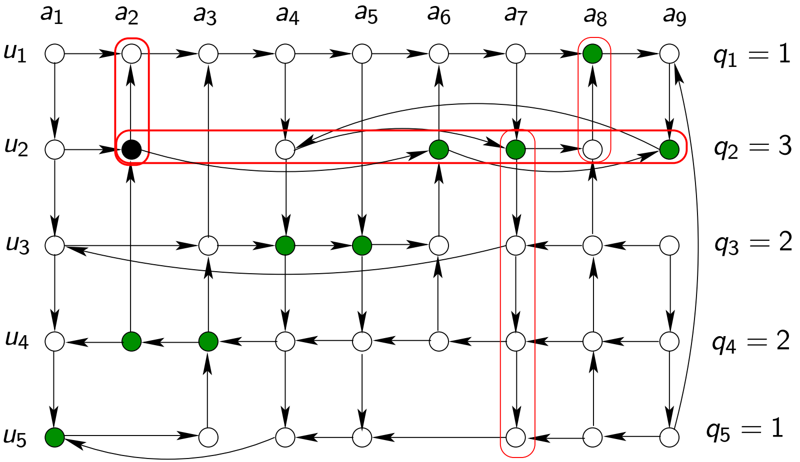


Figure – A comb $C(u_2, a_2)$.

Theorem (A new "linear" definition of stability)

A matching μ of an admissions problem (Γ, q) is stable if and only if every comb $C(u,a)$ contains at least q_u nodes of μ.

Theorem

For any stable admissions problem, $SAP(\Gamma, q)$ is described by the following set of linear inequalities :

$$
\begin{aligned}
\sum_{(u,a)\in\Gamma} x_{ua} &\leq q_u, && \text{for all } u \in U,\\
\sum_{(u,a)\in\Gamma} x_{ua} &\leq 1, && \text{for all } a \in A,\\
\sum_{(u,a)\in C(u,a)} x_{ua} &\geq q_u, && \text{for each comb } C(u,a) \text{of} \, \Gamma,\\
x_{au} &\geq 0, && \text{for all } (u,a) \in \Gamma.
\end{aligned}
$$

Stable allocations

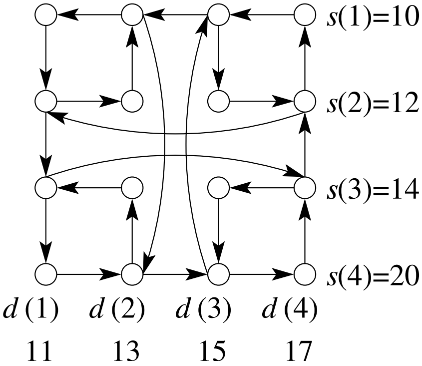


Figure – A comb $C(u_2, a_2)$.

An allocation $x = (x(i, j))$ for a problem (Γ, s, d, π) is a set od real-valued numbers satisfying

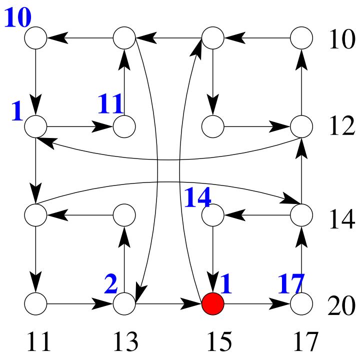
$$x(i, J) \leq s(i), \quad \text{for all } i \in I,$$

$$x(I, j) \leq d(j), \quad \text{for all } j \in J,$$

$$0 \leq x(i, j) \leq \pi(i, j), \quad \text{for all } (i, j) \in \Gamma,$$

An allocation x is stable is for every $(i,j) \in \Gamma$
$x(i,j) < \pi(i,j)$ implies

$$x(i,j) \longrightarrow= s(i) \text{ or } x(i,j) \uparrow= d(j).$$



non–stable allocation

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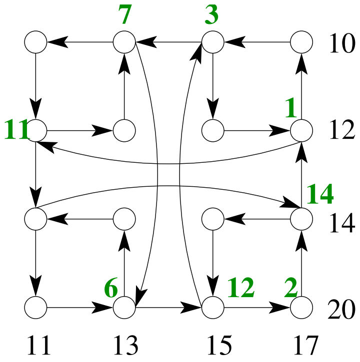


Stable allocation

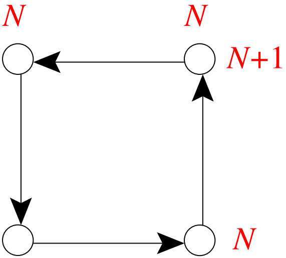
Optimal stable allocations

Row-greedy algorithm.

The row-greedy solution ρ is defined recursively, begining with i's prefered choice.

$$\rho(i,j) = \min\{s(i) - \rho(i, j^{>}),\ d(j),\ \pi(i,j)\}$$

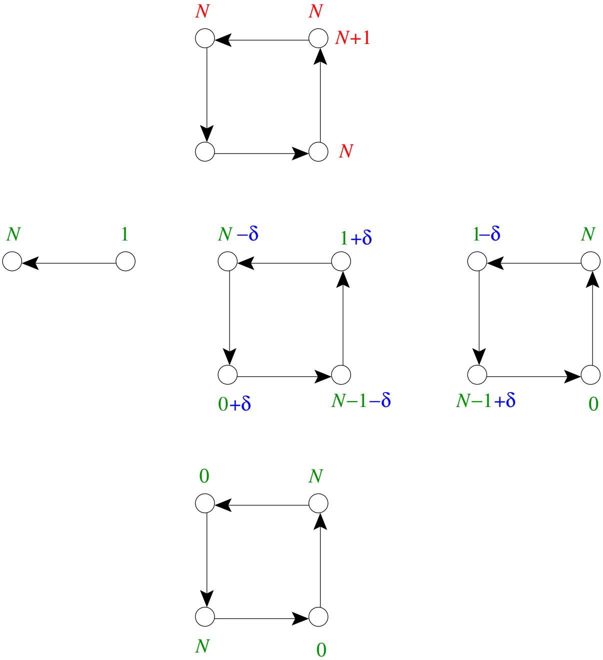
- Find the row-greedy solution ρ
- If ρ is not an allocation, then replace π by π^{ρ}, and repeat.



The unique stable allocation is
$\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$, obtained after $2N$ steps.

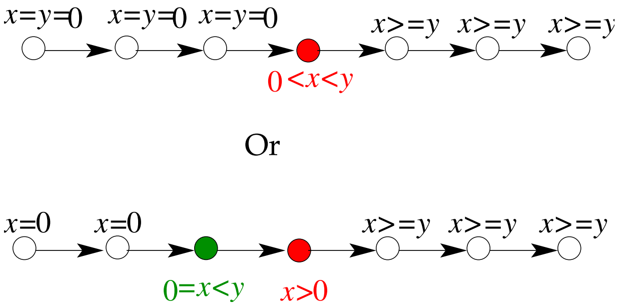
The inductive algorithm

- Suppose x^{-i_0} is a stable allocation for the subproblem $(\Gamma, s, d, \pi)^{-i_0}$, where the data relevant to row i_0 is suppressed.
- The inductive algorithm shows how to obtain a stable allocation for (Γ, s, d, π) given a stable allocation for $(\Gamma, s, d, \pi)^{-i_0}$.
- Give to row 1 the row-greedy stable allocation.


N
N
N+1
N
N
1
N −δ
1+δ
0+δ
N−1−δ
1–δ
N
N−1+δ
0
0
N
N
0

The structure of stable allocations

The situation between two stable allocations, x and y, is

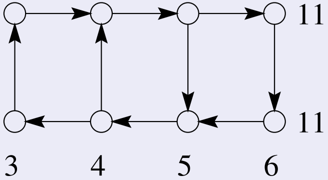


Let $i(x) = j^-$ if $x(i, j^-) > 0$ and $x(i, j) = 0$ for $j <_i j^-$.

Mechanisms

An allocation mechanism Φ is a function that selects exactly one stable allocation for any problem (Γ, s, d, π).

Efficiency



$$x_I = \begin{pmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \end{pmatrix}$$

Any allocation (stable or not) that gives 7 to rwo 1 and 11 to row 2, is better than x_I.

Generalized preferences to compare arbitrary allocation x and y

$$x \overset{\text{def}}{\succ_i} y \text{ if}$$

$$\left.\begin{array}{l} i(y) <_i i(x) \text{ or} \\ i(x) = i(y) = j^-, \\ x(i, j^-) < y(i, j^-) \end{array}\right\} \quad \text{when } x(i, J) = y(i, J) = s(i)$$

$$x(i, J) > y(i, J) \qquad \text{when } y(i, J) < s(i).$$

Theorem

There is no allocation (stable or not) y with $y \succ_i x_I$ for all $i \in I$.

Monotonicity

Let $P = (\Gamma, s, d, \pi)$ a stable allocation problem.
$P^h = (\Gamma^h, s, d, \pi^h)$ is a stable allocation problem defined as follows

$$\text{for } j \in J,$$

$$h >_j i \text{ in } P \text{ implies } h >_j i \text{ in } P^h \text{ and } \pi^h(h,j) \geq \pi(h,j).$$

A mechanism Φ is row-monotone if

$$\Phi(P^h) \succeq_h \Phi(P) \text{ for all } h \in I.$$

Theorem

Let χ_I the mechanism that produces x_I. χ_I is the unique row-monotone mechanism.

Strategy

If $P = (\Gamma, s, d, \pi)$ is the true problem, then $P' = (\Gamma', s', d, \pi')$ is an alternate problem for $I' \subset I$ if the two problems are identical except for row-agents I' who announce altered preferences and/or altered quotas s' and bounds π'.
A mechanism Φ is row-strategy-proof if

$$\Phi(P') \succ_i \Phi(P) \text{ for all } i \in I' \text{ is false,}$$

for any choice of $I' \subset I$.

Theorem

χ_I is the unique row-strategy-proof mechanism.

Degeneracy

- If $s(I') = d(J')$ for $I' \subseteq I$ and $J' \subseteq J$ with at least one of the subsets proper, then the problem (Γ, s, d, π) is said to be degenerate.
- A nondegenerate problem is strongly nondegenerate if $s(I) \neq d(J)$.

Theorem

A strongly nondegenerate problem has a unique stable allocation.