Covering and partitioning with bicliques

Denis Cornaz

University Paris-Dauphine

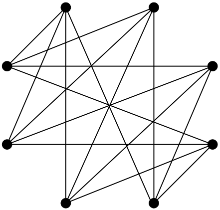
Plan :

1. Covering with bicliques
2. Partitioning with bicliques
3. Optimization

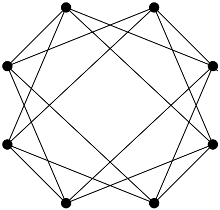
Ô mathématique sévère, vous m'apparûtes, avec votre longue robe, flottante comme une vapeur, et vous m'attirâtes vers vos fières mamelles, qui ont existé avant l'univers et qui se maintiendront après lui. Alors, j'accourus avec empressement, mes mains crispées sur votre gorge blanche.

Lautréamont, Maldoror II.10

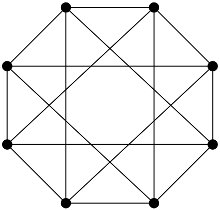
00001111



00110011



01010101



Def. Given a simple undirected graph G..

$bc(G)$
is the minimum number of *bicliques covering* the edge-set of G
(K_n is the complete graph on n vertices)

Thm. $bc(K_n) = \lceil \log_2 n \rceil$ (Harary, Hsu, Miller, JGT1 (77))

Cor. Any G can be covered by $\lceil \log_2 \chi(G) \rceil$ bipartite subgraphs

n	16	32	64	128	256	512	1024	2048	4096	8192	16384
log	4	5	6	7	8	9	10	11	12	13	14

Pf. $bc(K_n) \leq \lceil \log_2 n \rceil$

$bc(K_n) \leq bc(K_{2^{\lceil \log_2 n \rceil}})$
$bc(K_{2^p}) \leq p$ since $bc(K_{2^0}) = 0$ and $bc(K_{2^p}) \leq 1 + bc(K_{2^{p-1}})$

Pf. $bc(K_n) \geq \lceil \log_2 n \rceil$

Construct $M \in \{0,1\}^{k \times n}$ from a $k-$biclique-covering of K_n

Two distinct columns of M must be different

Hence $n \leq 2^k$

Def. Given a simple undirected graph G..

$bp(G)$
is the minimum number of *bicliques partitioning* the edges of G

Thm. $bp(K_n) = n - 1$ (Tverberg, JGT6 (82))

(K_n is the complete graph on n vertices)

Pf. $bp(K_n) \leq n - 1$

It follows from $bp(G) \leq \tau(G) \leq |V(G)| - 1$
(In other words, we can always partition with stars)

Pf. $bp(K_n) \geq n-1$

Construct k polynomials from a $k-$biclique-partition of K_n

$$p_i(x)q_i(x) := \Big(\sum_{v\in 0} x_v\Big)\Big(\sum_{v\in 1} x_v\Big) \qquad \forall i \in [k]$$

For any solution $\bar{x} \in \mathbb{R}^n$ of

$$p_1(x) = p_2(x) = \ldots = p_k(x) = \sum_v x_v = 0$$

then

$$||\bar{x}||^2 = \Big(\sum_v \bar{x}_v\Big)^2 - 2\sum_{uv} \bar{x}_u\bar{x}_v = 0 - 2\sum_i p_i(\bar{x})q_i(\bar{x}) = 0$$

Hence $k+1 \geq n$

Approximating Max-Cut

(Max-Cut) Given a weight-vector $w \in \mathbb{R}_+^{E(K_n)}$..

Find a biclique of K_n maximizing

$$\sum_{\substack{u \in 0 \\ v \in 1}} w_{uv}$$

Obs. $\frac{1}{2}^\top w \leq$ Max-Cut $\leq \mathbf{1}^\top w$

$(x_u, x_v) = (0,0)$ or $(0,1)$ or $(1,0)$ or $(1,1)$
for any biclique $x \in \{0,1\}^n$ of K_n

Approximating Max-Cut

Max-Cut Formulation

$$\max_{x \in \{-1,+1\}^n} \sum_{uv} \frac{1}{2} w_{uv}(1 - x_u x_v)$$

Relaxation

$$\max_{x \in (\mathbb{R}^d)^n :\, ||x_v||=1} \sum_{uv} \frac{1}{2} w_{uv}(1 - x_u^\top x_v)$$

Thm. 0.87856 approximation (Goemans, Williamson, STOC (94))

$$\left(\min_{\theta \in (0,\pi]} \frac{\theta}{1 - \cos\theta} \frac{2}{\pi} \right) \times \text{Relaxation} \leq \text{Max-Cut} \leq \text{Relaxation}$$

Approximating Max-Cut

Pr. Let $\bar{x} = \arg\max_{x\in(\mathbb{R}^d)^n:\,||x_v||=1} \sum_{uv} \frac{1}{2} w_{uv}(1 - x_u^\top x_v)$

There is a biclique with weight at least :

$$\sum_{uv} \frac{\theta(\bar{x}_u, \bar{x}_v)}{\pi} w_{uv} \geq \rho \times \sum_{uv} \frac{1}{2} w_{uv}(1 - \cos\theta(\bar{x}_u, \bar{x}_v))$$

where $\rho := \min_{\theta\in(0,\pi]} \frac{\theta}{1-\cos\theta}\frac{2}{\pi}$

Chromatic characterization of biclique covers

Obs. $bc(G) = \chi(H(G))$

$V(H(G)) = E(G)$
$E(H(G)) = \{C \subseteq E(G) : \text{no biclique of } G \text{ contains } C\}$

Rk. We can do $H(G) := H^{\min}(G)$

$V(H^{\min}) = V(H)$
$E(H^{\min}) = \{C \in E(H) : \text{there is no } D \in E(H) \text{ with } D \subset C\}$

Chromatic characterization of biclique covers

Thm. No triangle $\Rightarrow H(G)$ is a graph (Fishburn, Hammer DM160 (96))

In other words, every edge of $H(G)$ has size 2, if the maximum clique of G has size $\omega(G) = 2$

Thm. Max edge-size of $H(G)$ is $\omega(G)$ (Conjecture Fishburn, Hammer)

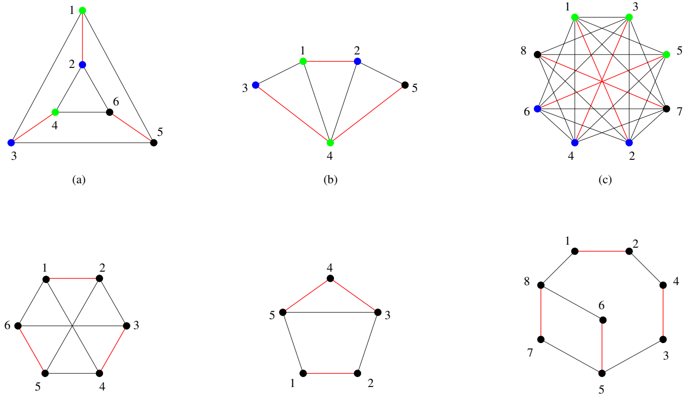
Formally, $C \subseteq E(G)$ is an edge of $H(G)$ if

(i) $C \not\subseteq B$ ($\forall B$ biclique of G)

(ii) $C' \subseteq B$ ($\forall C' \subset C$, $\exists B$ biclique of G)

Pr. Step 1

1. Recognize subset of edges which are contained in the edge-set of some biclique



(a) (b) (c)

(d) (e) (f)

Pr. Step 2 and 3

2. Characterize the inclusion-wise minimal edge-subsets which are not contained in the edge-set of some biclique
3. Find the big clique