Eliciting a Suitable Voting Rule via Rank-Vectors

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https://github.com/oliviercailloux/eliciting-voting-rules

Introduction

Context

- A *committee* (a group of decision makers)
  - a panel attributing a research price
  - a management committee
- Recurring decisions
- A decision is taken using a voting rule
- Voting rule: a systematic way of aggregating different opinions and decide

Our goal

We want to help the committee choose a suitable voting rule.

Voting rule

Input

- A set of possible alternatives (options) $\mathcal{A}$
- Each voter $i \in N$ has a linear order of preference over $\mathcal{A}$
- A profile $\mathbf{R}$ associates each i to such an order.

Voting rule

Associates to each profile $\mathbf{R}$ winning alternatives $A \subseteq \mathcal{A}$.

$$\underbrace{\begin{array}{cc} R_1 & R_2 \\ \begin{array}{|cc|} \hline a & b \\ b & a \\ c & c \\ \hline \end{array} \end{array}}_{\mathbf{R}} \xrightarrow{f} \{\, a, b \,\}$$

Our goal

Making decisions involves two steps.

1. Establish a constitution: choose a voting rule.
2. Solve a decision problem: apply the voting rule.

Our goal

We focus on step 1: help the committee choose a voting rule.

- Class of functions $\mathcal{F}$ (the set of all voting rules)
- Preference elicitation in order to choose a function $f \in \mathcal{F}$.
- We want to ask *simple* questions: example-based.

A naïve attempt

A first attempt

Simply give a profile $\mathbf{R}$ and ask for $f(\mathbf{R})$. Then iterate.

$$\underbrace{\begin{array}{cc} R_1 & R_2 \\ \boxed{\begin{array}{cc} a & c \\ b & b \\ c & a \end{array}} \end{array}}_{\mathbf{R}} \xrightarrow{f} \{\,?\,\}$$

- Completely general: all functions in $\mathcal{F}$ can be reached.

But. . .

- One question brings very little information.
- Questions may be difficult to answer.

General idea

- Ask *good* (informative, example-based) questions.
- Restrict the class of a priori acceptable functions to $\mathcal{F}' \subset \mathcal{F}$.

Outline

Outline

A different view of a profile

- We want to ask more informative questions about f
- We look at profiles under a different angle
- A *rank-vector* maps voters to ranks, $x : N \to [1, m]$
- All rank-vectors: $[1, m]^N$

$$\underbrace{\begin{array}{cc} R_1 & R_2 \\ a & a \\ b & c \\ c & b \end{array}}_{\mathbf{R}}$$

$$\begin{array}{c} \\ \\ \\ \mathcal{A} \\ \\ \end{array} \underbrace{\begin{array}{c|cc|} & \multicolumn{2}{c}{N} \\ & 1 & 2 \\ \hline a & 1 & 1 \\ \hline b & 2 & 3 \\ \hline c & 3 & 2 \\ \hline \end{array}}_{\mathbf{x_R}} \begin{array}{l} \\ \\ \leftarrow \mathbf{x_R}(a) \in [1, m]^N \\ \leftarrow \mathbf{x_R}(b) \in [1, m]^N \\ \leftarrow \mathbf{x_R}(c) \in [1, m]^N \end{array}$$

Rank-profile function

- Rank-profile $\mathbf{x} \in \left([1, m]^N\right)^{\mathcal{A}}$ maps alternatives to rank-vectors
- To each profile $\mathbf{R}$ corresponds a *rank-profile* $\mathbf{x_R}$
- Voting rule f maps $\mathbf{R}$ to $A \subseteq \mathcal{A}$
- Rank-profile voting rule $f_{\text{r-p}}$ maps $\mathbf{x}$ to $A \subseteq \mathcal{A}$
- Rank-profile voting rule $f_{\text{r-p}}$ corresponds to voting rule f iff $f_{\text{r-p}}(\mathbf{x_R}) = f(\mathbf{R})$

Rank-profiles correspond to *some* combinations of rank-vectors

- Some sets of rank-vectors do *not* form a rank-profile
- We assume preferences are strict
- Thus, for a given voter: ranks must be all different

Not a rank-profile:

$\boxed{1 \quad 1}$

$\boxed{2 \quad 3}$

$\boxed{2 \quad 2}$

Symmetries of rank-profile functions

- A rule is *neutral* iff it treats the alternatives equally:
- after renaming alternatives, f selects the renamed alternatives
- In that case, $f_{\textsf{r-p}}$ only requires a *set* of rank-vectors:
  $f_{\textsf{r-p}}(x^1 = \boxed{1 \quad 2}, x^2 = \boxed{2 \quad 1}) = \dots$
- A rule satisfies *anonymity* iff it treats the voters equally:
- renaming the voters does not change the winners
- No similar simplification of the input of $f_{\textsf{r-p}}$

Condorcet property

Condorcet property

- A rank-profile voting rule satisfies Condorcet iff it picks the Condorcet winner if it exists
- x^1 beats x^2 iff more than half of the positions satisfy $x^1_i < x^2_i$
- x is a Condorcet winner in $\mathbf{x}$ iff it beats all other $x' \in \mathbf{x}$

Condorcet with 3 voters, 3 alternatives

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ a & \boxed{\begin{array}{ccc} 1 & 2 & 2 \end{array}} \\ b & \boxed{\begin{array}{ccc} 2 & 3 & 1 \end{array}} \\ c & \boxed{\begin{array}{ccc} 3 & 1 & 3 \end{array}} \end{array} \Rightarrow ?$$

Condorcet property

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Condorcet with 3 voters, 3 alternatives

	1	2	3
a	1	2	2
b	2	3	1
c	3	1	3

$\Rightarrow ?\ a$

Informational view about profiles

- Sen [1977]
- Voter i has evaluation function $W_i : \mathcal{A} \to \mathbb{R}$
- Social welfare functional: associates $\{W_i\}$ to ranking over $\mathcal{A}$
- Subject to invariance requirement
- Example: changing $\{W_i\}$ but respecting order does not change the output

Representing the preferences of the committee

- We can now ask for the preference status of, e.g., $\boxed{1 \quad 3}$ versus $\boxed{2 \quad 2}$
- Sets of such questions permit to identify a voting rule
- *Assuming* the committee reasons in a specific way

- We assume the committee can answer each such question
- With one of $>, \sim, <$
- Meaning: when $x^1 > x^2$, the voting rule must select x^1 rather than x^2 if both are present (and similarly for $x^1 < x^2$)
- The preference $\succeq \, = \, > \cup \sim$ of the committee over rank-vectors is transitive

Weak-order based rules

Weak-order based rules

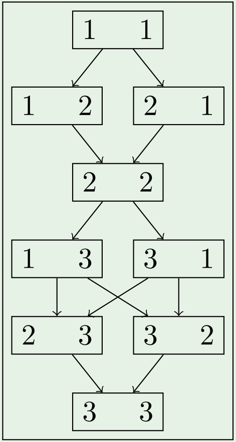
- $\succeq$ a weak-order (transitive, reflexive, connected) over $[1, m]^N$.
- The rule $f_{\succeq}$, at $\mathbf{x}$, selects those alternatives having maximal rank-vectors in $\mathbf{x}$ according to $\succeq$.

A rule f is *weak-order based* (WOB) if there exists $\succeq$ st $f = f_{\succeq}$.

Example of a WOB rule

Example (A WOB rule)

$\succeq$ =



	$\mathbf{x}_1$	$f_{\succeq}(\mathbf{x}_1)$
a	1 2	✓
b	2 1	✓
c	3 3	

	$\mathbf{x}_2$	$f_{\succeq}(\mathbf{x}_2)$
a	1 3	
b	2 2	✓
c	3 1	

Incomplete question sets

- We do not want to ask every possible questions!
- Can we get away with only *some* answers?

Robust rules

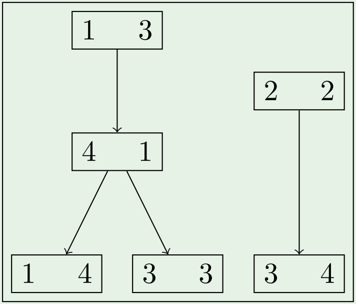
- $\succsim$ a *preorder* (transitive, reflexive) over $[1, m]^N$.
- Look at all weak-orders $\succeq$ extending $\succsim$.
- The *robust* rule $F_\succsim$, at $\mathbf{R}$, selects those alternatives winning in *some* $f_\succeq$ (for some $\succeq$ extension of $\succsim$).

A rule f is *robust* if there exists $\succsim$ st $f = F_\succsim$.

Example of a robust rule

Example (A robust rule)

$\succsim =$



	$\mathbf{x}$	$F_{\succsim}(\mathbf{x})$
a	$\boxed{1 \quad 3}$	✓
b	$\boxed{2 \quad 2}$	✓
c	$\boxed{3 \quad 4}$	
d	$\boxed{4 \quad 1}$	

Outline

Our restriction over possible functions

- We assume the committee reasons in some specific way
- Restricts the class of rules
- Bad news: we are not fully general any more
- Good news: we have restricted our class of functions

The WOB class

- The committee has a weak-order $\succeq$ over rank-vectors "in mind"
- This represents a WOB rule $f_{\succeq}$
- $WOB = \left\{ f_{\succeq}, \succeq \text{ a weak-order over } [1, m]^N \right\}$ instead of $\mathcal{F}$

WOB rules are neutral

- If f is WOB, f is neutral
- Because $f_{\succeq}$ selects those alternatives with highest rank-vectors
- Thus, we care only about the set of rank-vectors as input of f
- And the rank-vector it selects

How to be a WOB rule?

f is a WOB rule iff f:

- Assigns a score $s(x) \in \mathbb{R}$ to each rank-vector $x \in [1, m]^N$
- Selects the rank-vectors having highest scores

Scoring rules are WOB rules

- Every scoring rule (e.g. Borda) is a WOB rule
- $s(x)$ is the sum of the partial-scores $s_r(i)$ of individual components of x
- Score of $\boxed{1 \quad 3 \quad 4} = s_r(1) + s_r(3) + s_r(4)$

Indifference to permutation is sufficient for anonymity

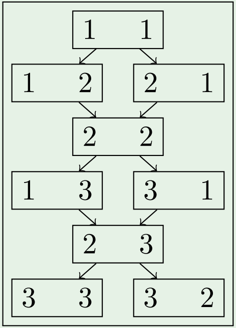
- Assume $\succeq$ is indifferent to permutations of x
- E.g. $\boxed{1 \quad 3} \sim \boxed{3 \quad 1}$
- It follows that $f_{\succeq}$ satisfies anonymity:
- Permuting the voters permutes all rank-vectors
- f must still select the same (reordered) rank-vectors

Weak-orders and WOB rules

- Relationship between $\succeq$ and $f_{\succeq}$ may be counter-intuitive!
- Is indifference to permutation in $\succeq$ required for anonymity of $f_{\succeq}$?

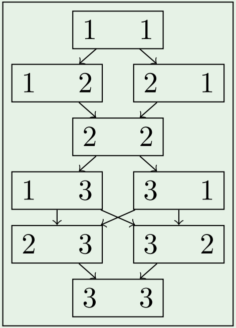
Example (A weak-order yielding a neutral WOB rule)

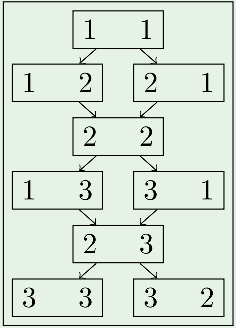
$\succeq =$



Two weak-orders, one WOB rule

Example (Two “equivalent” weak-orders)

$\succeq^1 =$ 

$\succeq^2 =$ 

$$f_{\succeq^1} = f_{\succeq^2}$$

Bucklin

Bucklin: a WOB rule that is not a scoring rule

Bucklin

- Look at rank r (starting with 1)
- Is there a majority for ranking an alternative at r or better?
- Iterate, stop when found a suitable rank r
- Select those alternatives that have most persons ranking them at r or better

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 & \\ a & a & b & b & c & \\ b & c & c & c & a & \Rightarrow ? \\ c & b & a & a & b & \end{array}$$

Bucklin

Bucklin: a WOB rule that is not a scoring rule

Bucklin

- Look at rank r (starting with 1)
- Is there a majority for ranking an alternative at r or better?
- Iterate, stop when found a suitable rank r
- Select those alternatives that have most persons ranking them at r or better

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ a & a & b & b & c \\ b & c & c & c & a \\ c & b & a & a & b \end{array} \Rightarrow ?\ c$$

Some WOB rules are not scoring rules

- Bucklin is a WOB rule
- Proof idea: let's build a score $s(x)$ to be *minimized*
- $s(x) =$ rank m_x required + frac. missing for unanimity at m_x
- Define m_x as the "median" of x lowest nb n st more than ½ the numbers are $\leq n$

$$s(x) = m_x + \frac{\#x_i > m_x}{\#x_i}$$

Example (Bucklin scores)

	1	2	3	4	5	m_x	$s(x)$
a	1	1	3	3	2	2	$2 + \frac{2}{5}$
b	2	3	1	1	3	2	$2 + \frac{2}{5}$
c	3	2	2	2	1	2	$2 + \frac{1}{5}$

WOB compared to other classes of rules

- Every scoring rule is a WOB rule
- Some WOB rules are not scoring rules
- Many Condorcet rules are not WOB rules

Some relationships between classes of rules

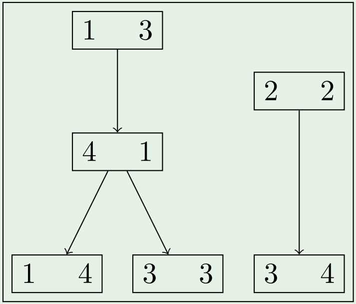
Scoring $\subset$ WOB; Condorcet $\cap$ WOB $= \emptyset$ (for $n = 3k, m \geq 4$).

The class of robust rules

Some robust rules are not WOB rules

Example (A robust rule)

$\succsim =$



	$\mathbf{R}_1$	$F_{\succsim}(\mathbf{R}_1)$	$\mathbf{R}_2$	$F_{\succsim}(\mathbf{R}_2)$
a	1 3	✓	1 4	
b	2 2	✓	2 2	✓
c	3 4		3 3	
d	4 1		4 1	✓

Robust rules compared to other classes of rules

Some relationships between classes of rules

Scoring $\subset$ WOB $\subset$ Robust

Outline

1. Context
2. Asking good questions
3. Restrict the class of functions
4. Which questions to ask?
5. Conclusion

Which questions to ask?

- Assume the committee has a weak-order $\succeq$ in mind
- We want to discover much information using few questions
- Different questions bring different amount of information

Elicitation strategy

An elicitation strategy tells us which question should be asked considering our current knowledge

A strategy:

1. computes the fitness of asking about a pair of rank-vectors, for each pair
2. chooses the fittest pair

We ask q questions, then compare our approximation $F_{\succsim}$ to $f_{\succeq}$

Which strategy?

We tested three strategies

optimistic fitness of (x, y) proportional to the number of rank-vectors dominated by x or y, but not both

pessimistic a variant of the previous strategy, using the min operator rather than the sum

likelihood fitness proportional to the likelihood of a profile occurring where both rank-vectors are possible winners

(depends on the probability distribution over profiles, we assumed impartial culture)

We assume pareto-dominance and indifference to permutations

Comparison of strategies

- Optimistic not better than random!
- Likelihood much better than pessimistic

Number of questions

How many questions must be asked for a useful approximation?

- Our approximation has all the true winners: $f_{\succeq}(\mathbf{R}) \subseteq F_{\succsim}(\mathbf{R})$
- But it may have supplementary winners
- We are interested in the ratio of approximated VS true winners: $\frac{|F_{\succsim}(\mathbf{R})|}{|f_{\succeq}(\mathbf{R})|}$
- We average it over all profiles: $\frac{1}{|\boldsymbol{\mathcal{R}}|} \sum_{\mathbf{R} \in \boldsymbol{\mathcal{R}}} \frac{|F_{\succsim}(\mathbf{R})|}{|f_{\succeq}(\mathbf{R})|}$

For 6 voters, 6 alternatives, using the likelihood strategy:

	Target rule	
nb q	Borda	Random $\succeq$
0	1.9	2.2
25	1.3	1.7
99	1.0	1.3

Outline

Conclusion

We propose to help a committee choose a voting rule.

- We introduce a different look at a profile [see also Sen, 1977]
- We use it to ask simple questions to elicit preferences
- We analyse the class of rules reachable by our questioning process
- A robust voting rule may be defined to give all possible winners [inspired by Dias et al., 2002]
- We compare and analyse several elicitation strategies

Future work

- The committee could have a preorder in mind
- Or the stable part of the w-o might be a preorder
- Behavioural interpretation of the constraints given by the committee
- Further analysis of the classes WOB, Robust rules
- Explore approximation with robust rules more generally
- Better elicitation strategies with active learning techniques

Thank you for your attention!

References I

L. Dias, V. Mousseau, J. Figueira, and J. Clímaco. An aggregation/disaggregation approach to obtain robust conclusions with ELECTRE TRI. *European Journal of Operational Research*, 138(2):332–348, 2002. URL http://www.sciencedirect.com/science/article/pii/S0377221701002508.

A. Sen. On weights and measures: Informational constraints in social welfare analysis. *Econometrica*, 45(7):1539–1572, 1977. ISSN 0012-9682. doi:10.2307/1913949. URL http://www.jstor.org/stable/1913949.

More general aim

- Choose a rule: from axioms?
- Difficult to consider the implications of the axioms
- Incompatibilities, paradoxes. . .
- We want to help a commitee choose a voting rule
- Do not limit to ask which axioms are suitable
- We should use the power of the axiomatic analysis
- But leave the axioms implicit