Arguing about voting rules

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Outline

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1. Introduction to Social Choice
2. Voting rules
3. Analyzing voting rules
4. Analyzing the Borda rule
5. Our contribution

Definition

Definition (Social Choice)

Designing and analyzing methods for collective decision making

- Focus on: mathematical analysis usually non empirical
- (Usually) no focus on: collective social behavior

The Condorcet paradox

- Why should we analyze decision schemes mathematically?
- Here is a preliminary argument
- 3 alternatives : a, b, c
- 3 voters : $1, 2, 3$
- Compare alternatives pairwise
- a VS b: 2 votes for a
- b VS c: 2 votes for b
- Conclusion?

The Condorcet paradox

- Why should we analyze decision schemes mathematically?
- Here is a preliminary argument
- 3 alternatives : a, b, c
- 3 voters : $1, 2, 3$
- Compare alternatives pairwise
- a VS b: 2 votes for a
- b VS c: 2 votes for b
- Conclusion? $a \succ c$? Not so simple!

$$\begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

Two key moments: 1. Borda, Condorcet

- Nicolas de Condorcet [1743 – 1794]
- Jean-Charles de Borda [1733 – 1799]
- Start of *mathematical, systematic* analysis of voting schemes
- Rich period of constitution writing
- Condorcet initially involved in the writing of the French constitution
- Borda's rank-order method was first proposed orally at the French Academy of Science in 1770 [Suzumura, 2002]

Two key moments: 2. Arrow

- up to Arrow: mostly analysis of individual voting schemes
- Kenneth Arrow [1921 –]
- Analysis of *classes* of voting schemes
- According to properties

⇒ (for example) Nobel price in 1998 to Amartya Sen (welfare economics)

This talk VS reality

- Elections happen IRL (?

This talk VS reality

- Elections happen IRL (? in real life)

This talk VS reality

- Elections happen IRL (? in real life)
- Quite far from the setting examined here
- Preferences expressed in various ways
- Procedures may not fit this framework
- Social Choice in general relevant to this part of reality
- In this talk: *limited* connexion

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4. Analyzing the Borda rule
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Definition

- Voting rule: a systematic way of aggregating different opinions and decide
- Multiple reasonable ways of doing this
- Different voting rules have different interesting properties
- None satisfy all desirable properties

Voting rule

Alternatives	$\mathscr{A} = \{ a, b, c, d, \dots \}$
Voters	$\mathscr{N} = \{ 1, 2, \dots \}$
Profile	function $\boldsymbol{R}$ from $\mathscr{N}$ to linear orders on $\mathscr{A}$.
Voting rule	function f mapping each $\boldsymbol{R}$ to winners $\varnothing \subset A \subseteq \mathscr{A}$.

$$\underbrace{\begin{array}{cc} \boldsymbol{R}(1) & \boldsymbol{R}(2) \\ \hline a & b \\ b & a \\ c & c \end{array}}_{\boldsymbol{R}} \xrightarrow{f} A = \{ a, b \}$$

Example of a profile

	nb voters					
	33	16	3	8	18	22
1	a	b	c	c	d	e
2	b	d	d	e	e	c
3	c	c	b	b	c	b
4	d	e	a	d	b	d
5	e	a	e	a	a	a

Who wins?

Example of a profile

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Who wins?

- Most top-1: a
- c is in the top 3 for everybody
- delete worst first, lowest nb of pref: c, b, e, $a \Rightarrow d$
- delete worst first, from bottom: a, e, d, $b \Rightarrow c$
- Borda: b

Borda

Given a profile $\boldsymbol{R}$:

- score of $a \in \mathscr{A}$: number of alternatives it beats
- the highest scores win

$$\boldsymbol{R} = \begin{array}{ccccc} a & a & a & b & b \\ b & b & b & c & c \\ c & c & c & a & a \end{array}$$

- score a is…?

Borda

Given a profile $\boldsymbol{R}$:

- score of $a \in \mathscr{A}$: number of alternatives it beats
- the highest scores win

$$\boldsymbol{R} = \begin{matrix} a & a & a & b & b \\ b & b & b & c & c \\ c & c & c & a & a \end{matrix}$$

- score a is...? $2 + 2 + 2 = 6$
- score b is $1 + 1 + 1 + 2 + 2 = 7$
- score c is $1 + 1 = 2$

Winner: b.

Condorcet's principle

Condorcet's principle

We ought to take the Condorcet winner as sole winner if it exists.

- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\boldsymbol{R} = \begin{array}{ccccc} a & a & a & b & b \\ b & b & b & c & c \\ c & c & c & a & a \end{array}$$

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Who wins? a

Condorcet's principle and a voting rule

- Condorcet's principle does not define a voting rule. Why?

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- Condorcet's principle does not define a voting rule. Why?
- No winner is defined when no Condorcet winner

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Condorcet's principle and a voting rule

- Condorcet's principle does not define a voting rule. Why?
- No winner is defined when no Condorcet winner

$$\boldsymbol{R} = \begin{array}{ccc} b & c & d \\ c & d & b \\ a & b & a \\ d & a & c \end{array}$$

a loses against b; b against d; c against b; d against c

- Dodgson's method (1876): candidates "closest" to being Condorcet winners (in nb of swaps)

Outline

Problems do arise

- Example of a problem: Borda does not satisfy Condorcet's principle
- Even when knowing the precise rule, we may overlook some problems of it
- We would like some way of finding out systematically whether a rule is "well-behaved"

Axiomatics

> *Rather than dream up a multitude of arbitration schemes and determine whether or not each withstands the best of plausibility in a host of special cases, let us invert the procedure. Let us examine our subjective intuition of fairness and formulate this as a set of precise desiderata that any acceptable arbitration scheme must fulfil. Once these desiderata are formalized as axioms, then the problem is reduced to a mathematical investigation of the existence of and characterization of arbitration schemes which satisfy the axioms.*

Luce and Raiffa [1957, p. 121]
Some example of axioms?

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Luce and Raiffa [1957, p. 121]
Some example of axioms? Condorcet's principle!

What's an axiom?

- An axiom (for us) is a principle
- Expressed formally
- That dictates some behavior of a voting rule
- In some conditions
- Usually seen as something to be satisfied
- Ideally, some union of some such axioms define exactly one rule
- Some axioms can be shown to be incompatible

Example (Condorcet's principle)

We ought to take the Condorcet winner as sole winner if it exists.

Unanimity

Definition (Unanimity)

We ought to select as winner someone who has no unanimously preferred alternative

$$\boldsymbol{R} = \begin{array}{ccc} a & a & b \\ b & b & c \\ c & c & a \end{array}$$

Constraint?

Unanimity

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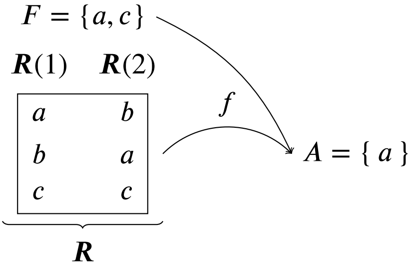
$$\boldsymbol{R} = \begin{array}{ccc} a & a & b \\ b & c & c \\ c & b & a \end{array}$$

Constraint? No constraint.

Social choice correspondence

- Some alternatives might be infeasible
- Preferences over $\mathscr{A}$, choose among F
- f a function that, given a profile and a set of feasible alternatives, chooses a subset of winners
- $f(\boldsymbol{R}, F) \subseteq F$



$F = \{a, c\}$

$\boldsymbol{R}(1)$	$\boldsymbol{R}(2)$
a	b
b	a
c	c

$\boldsymbol{R}$

f

$A = \{ a \}$

Independence of Infeasible Alternatives

Definition (Independence of Infeasible Alternatives)

When preferences change only about unfeasible alternatives, we ought not to change our decision

Example

$$\boldsymbol{R} = \begin{array}{ccc} b & c & d \\ c & d & b \\ a & b & a \\ d & a & c \end{array}, \quad f(\boldsymbol{R}, \{b, c, d\}) = \{\, b \,\},$$

$$\boldsymbol{R}' = \begin{array}{ccc} b & a & d \\ c & c & a \\ d & d & b \\ a & b & c \end{array}, \quad f(\boldsymbol{R}', \{b, c, d\})\ ?$$

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Arrow's choice axiom

Definition (Arrow's choice axiom)

When feasible alternatives are restricted, with no change of preference, we ought not to change our decision if possible

Example

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Dictatorial social choice correspondences

Definition (Dictatorial social choice correspondence)

f is *dictatorial* for a given voter iff f always selects her preferred alternative among the feasible ones

Example

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$$\boldsymbol{R}' = \begin{array}{ccc} b & a & d \\ c & b & a \\ d & d & b \\ a & c & c \end{array}, \quad f(\boldsymbol{R}', \{b, c, d\}) \mathrel{?} \{\, b \,\}$$

Arrow’s impossibility theorem

Theorem (Arrow’s impossibility theorem)

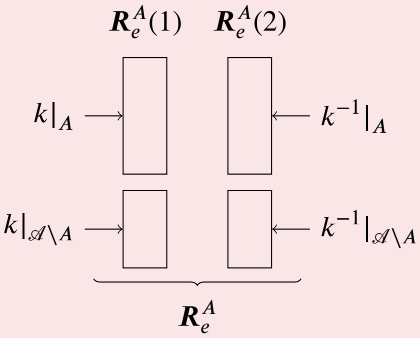
When there are at least three alternatives, there is no social choice correspondence that satisfies Unanimity, Independence of Infeasible alternatives, Arrow’s Choice Axiom, and that is not dictatorial.

Outline

Elementary profile

Fix an arbitrary linear order k on $\mathscr{A}$. Given $A \subseteq \mathscr{A}$, define $\boldsymbol{R}_e^A$.

Definition (Elementary profile)



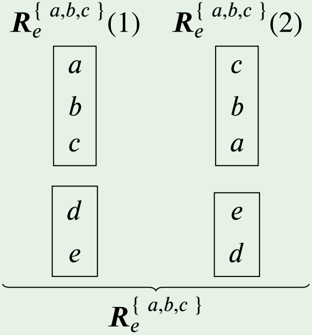
$\boldsymbol{R}_e^A(1)$ $\boldsymbol{R}_e^A(2)$

$k|_A \longrightarrow$ $\longleftarrow k^{-1}|_A$

$k|_{\mathscr{A}\setminus A} \longrightarrow$ $\longleftarrow k^{-1}|_{\mathscr{A}\setminus A}$

$\boldsymbol{R}_e^A$

Example



$\boldsymbol{R}_e^{\{a,b,c\}}(1)$ $\boldsymbol{R}_e^{\{a,b,c\}}(2)$

a	c
b	b
c	a
d	e
e	d

$\boldsymbol{R}_e^{\{a,b,c\}}$

Elementary profile: axiom

Definition (Axiom about elementary profiles)

$f(\boldsymbol{R}_e^A) = A$

Cyclic profile

Definition (Axiom about cyclic profiles)

When given a cyclic profile, we ought to select all winners as ex-æquo

Example

$$f\begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix} = \mathscr{A}$$

Cancellation

Definition (Cancellation)

When all pairs of alternatives (a, b) in a profile are such that a is preferred to b as many times as b to a, we ought to select all winners as ex-æquo

Example

$$f\begin{pmatrix} a & b & c & c \\ b & a & a & b \\ c & c & b & a \end{pmatrix} = \mathscr{A}$$

Reinforcement

Definition (Reinforcement)

When joining two sets of voters, exactly those winners that each set accepts should be selected, if possible

Example

$$\boldsymbol{R}_1 = \begin{matrix} a & b \\ b & a \\ c & c \end{matrix}, A_1 = \{ a, b \}, \boldsymbol{R}_2 = \begin{matrix} a & b & a \\ b & a & c \\ c & c & b \end{matrix}, A_2 = \{ a \},$$

$$\boldsymbol{R} = \begin{matrix} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{matrix}. \text{ Winners?}$$

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$$\boldsymbol{R} = \begin{matrix} a & b & a & b & a \\ b & a & b & a & c \\ c & c & c & c & b \end{matrix}. \text{ Winners? } \{ a \}$$

Characterisation of Borda

Theorem

The Borda rule is the only voting rule that satisfies Elementary and Cyclic profile axioms, Cancellation and Reinforcement.

(We proved this variant of a characterisation given by Young [1974])

Fishburn-against-Condorcet argument

Fishburn [1974, p. 544] argument against the Condorcet principle (see also `http://rangevoting.org/FishburnAntiC.html`).

Condorcet winner

w VS $\mu, \mu \in \{ a, \dots, h \}$?

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

	ranks								
	1	≤ 2	≤ 3	≤ 4	≤ 5	≤ 6	≤ 7	≤ 8	≤ 9
w	0	30	30	51	51	82	82	82	101
a	50	50	80	80	101	101	101	101	101

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Fishburn [1974, p. 544] argument against the Condorcet principle (see also `http://rangevoting.org/FishburnAntiC.html`).

Condorcet winner

w VS $\mu, \mu \in \{a, \dots, h\}$? 51/101

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
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Goal

- Better understanding of rules
- Illustrate reasoning of $\neq$ rules
- On examples
- On the basis of axioms
- Long-term goal: compare rules by letting them argue

Approach

- Express some axioms in a simple formal language
- Compute automatically their applications on examples
- Apply to the Borda rule

Formal language

- We use propositional logic
- A restricted language for easier deductions
- Contains $\neg, \vee, \wedge, \rightarrow$ but not $\forall, \exists$
- We translate axioms into this language

Example profile

Who should win?

$$\boldsymbol{R} = \begin{array}{ccc} a & a & c \\ b & b & b \\ c & c & a \end{array}$$

- Veto rule chooses b
- Borda rule chooses a

What we aim at

$$\boldsymbol{R} = \begin{matrix} a & a & c \\ b & b & b \\ c & c & a \end{matrix}$$

System: Take the *red subprofile*. Here, *a should win*, right? [unanimity]

User: Obviously!

System: Now consider the *green subprofile*. For symmetry reasons, there should be a *three-way tie*, right? [cancellation]

User: Sounds reasonable.

System: So, as there was a three-way tie for the green part, the red part should decide the overall winner, right? [reinforcement]

User: Yes.

System: To summarise, you agree that a should win.

Provided

- Our system can automatically explain the outcome of Borda
- Starting from any profile
- Requires solving a system of equation to find right intermediate profiles
- Does not use natural language (for the moment)

Conclusion

- No perfect voting rule
- Strengths can be expressed using axioms
- We use a specific language to express some of these axioms
- We can then instanciate concrete arguments (example-based)
- May render some arguments in the specialized literature accessible to non experts
- Extensions may permit to *debate* about voting rules
- Provides a way to study appreciation of arguments

Thank you for your attention!

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Naïve attempt

⇒ Examine outcome of a rule on every possible profile?

1. We may not see what the problem is on a given example
2. Too time consuming

4 alternatives, 6 voters: at least $(4!)^6/4!/6!$ profiles ≈ 10000 deducing obvious symetries for anonymity and neutrality

Anonymity

Definition (Anonymity)

The identity of the voters should not matter

$$\boldsymbol{R}_1 = \begin{array}{ccc} 1 & 2 & 3 \\ a & b & b \\ b & a & c \\ c & c & a \end{array}, \boldsymbol{R}_2 = \begin{array}{ccc} 3 & 1 & 2 \\ a & b & b \\ b & a & c \\ c & c & a \end{array}$$

Constraint?

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Constraint? Same winners.

Cyclic profiles

Given S a complete cycle in $\mathscr{A}$, define $\boldsymbol{R}_c^S$.

Definition (Cyclic profile)

$\boldsymbol{R}_c^S$ is the profile composed by all $|\mathscr{A}|$ possible linearizations of S as preference orderings.

Example

$$\boldsymbol{R}_c^{\langle a,b,c,d\rangle} = \begin{array}{cccc} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{array}.$$

Reinforcement I-axiom

Consider $\boldsymbol{R}_1$, $\boldsymbol{R}_2$,

- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\boldsymbol{R}_1 + \boldsymbol{R}_2$ must be $A_1 \cap A_2$.

Definition (Reinf)

For each $\boldsymbol{R}_1, \boldsymbol{R}_2, A_1, A_2 \subseteq \mathscr{A}, A_1 \cap A_2 \neq \emptyset$:

$$([\boldsymbol{R}_1 \longmapsto A_1] \land [\boldsymbol{R}_2 \longmapsto A_2]) \rightarrow [\boldsymbol{R}_1 + \boldsymbol{R}_2 \longmapsto A_1 \cap A_2].$$

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\boldsymbol{R}$ defined as:

$$\boldsymbol{R} = \begin{matrix} a & c \\ b & b \\ d & a \\ c & d \end{matrix}.$$

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\boldsymbol{R}$ defined as:

$$\boldsymbol{R} = \begin{array}{cc} a & c \\ b & b \\ d & a \\ c & d \end{array}.$$

We want to justify that $f_B(\boldsymbol{R}) = \{ a, b \}$.

Sketch

- Consider any $\boldsymbol{R}' = q_1 \boldsymbol{R}_e^{a,b} + q_2 \boldsymbol{R}_e^{a,b,c} + \sum_{S \in \mathcal{S}} q_S \boldsymbol{R}_c^S$, $q_1, q_2, q_S \in \mathbb{N}, \mathcal{S}$ some set of cycles.
- In $\boldsymbol{R}'$, $W = \{ a, b \}$ must win.
- Find $k \in \mathbb{N}$ such that $\overline{k\boldsymbol{R}} + \boldsymbol{R}'$ cancel.
- Then $k\boldsymbol{R}$ has winners W. (Skipping details.)
- Then $\boldsymbol{R}$ has winners W.

Our task: find $\boldsymbol{R}'$ a combination of elementary and cyclic profiles such that $\overline{k\boldsymbol{R}} + \boldsymbol{R}'$ cancel.
Good news: this is always possible.

Application on the example

Define $\boldsymbol{R}' = \boldsymbol{R}_e^{a,b} + 2\boldsymbol{R}_e^{a,b,c} + \boldsymbol{R}_c^{\langle c,b,a,d\rangle} + \boldsymbol{R}_c^{\langle b,d,c,a\rangle}$.

1. $[\boldsymbol{R}_e^{a,b} \longmapsto \{\, a, b \,\}]$ (ELEM)
2. $[\boldsymbol{R}_e^{a,b,c} \longmapsto \{\, a, b, c \,\}]$ (ELEM)
3. $[\boldsymbol{R}_c^{\langle c,b,a,d\rangle} \longmapsto \mathscr{A}]$ (CYCL)
4. $[\boldsymbol{R}_c^{\langle b,d,c,a\rangle} \longmapsto \mathscr{A}]$ (CYCL)
5. $[\boldsymbol{R}' \longmapsto \{\, a, b \,\}]$ (REINF, 1, 2, 3, 4)
6. $[4\boldsymbol{R} + \overline{4\boldsymbol{R}} \longmapsto \mathscr{A}]$ (CANC)
7. $[4\boldsymbol{R} + \overline{4\boldsymbol{R}} + \boldsymbol{R}' \longmapsto \{\, a, b \,\}]$ (REINF, 5, 6)
8. $[\overline{4\boldsymbol{R}} + \boldsymbol{R}' \longmapsto \mathscr{A}]$ (CANC)
9. $[4\boldsymbol{R} \longmapsto \{\, a, b \,\}]$ (REINF, 7, 8)
10. $[\boldsymbol{R} \longmapsto \{\, a, b \,\}]$ (REINF, 9)

License