Preference models that relax completeness or transitivity: why, how?

Olivier Cailloux[1] and Sébastien Destercke[2]

[1] LAMSADE
olivier.cailloux@dauphine.fr
[2] HDS

Abstract. Literature involving preferences of artificial agents or human beings often assume their preferences can be represented using a complete transitive binary relation. Much has been written however on more complex, or more interesting, models of preferences. In this article we review some of the reasons that have been put forward to justify more complex modeling, and review some of the techniques that have been proposed to obtain models of such preferences. (Optional: we connect to various related literature about argumentation, …)

1 Introduction

Preferences of agents are usually assumed to be linear, that is complete, transitive and antisymmetric, or representable with a weak order (a complete and transitive binary relation). In practice, such assumptions limit the scope of models (of uncertainty, of preferences) one can consider as legitimate, and are also falsified by observed empirical evidences. While models relaxing antisymmetry to allow incompleteness between options have been around for some time *** ELECTRE ?***, models relaxing the assumptions of completeness or transitivity are more recent.

It is clear that our paper will not be complete, yet we think it gives a fair overview of why one would like to drop some of the above assumption. In particular, we will review both empirical evidence that preferences do not always follow the above assumption, thus relating them to more descriptive approaches, as well as formal models that allows one to induce intransitive or incomplete preferences.

There are two canonical fields where the modelling of preferences is a central topic: choosing an alternative when it is evaluated according to different aspects (a.k.a. multi-criteria decision making, or MCDM), and picking an alternative whose quality depends on states of the world that are uncertainly known. Although the two frameworks are formally similar to some extent, they also present some key differences, if not formally then at least conceptually.

In MCDM, the common assumption is that the alternatives, i.e., the state of the world is known without ambiguity, and the difficulty is to determine what are the user preferences over these different, but well-defined states. In decision making under uncertainty (DMU), the preferences of the user are usually assumed to be well-known, or at least to have been previously assessed in the form of utility functions, and the problem is to recommend an alternative given our uncertainty about the world.

2 Completeness and transitivity in classical and less classical settings

In this section, we are going to recall the main models that consider completeness and transitivity of preferences as a consequence of natural requirements, if not as pre-requisite of any preference modelling. We will also recall normative views and descriptive views of these concepts.

2.1 MCDM

In MCDM, the classical approach is to assume that the alternatives are evaluated using a set of criteria $\mathcal{G}$, each having an evaluation scale X_g. The set of all possible alternatives is $\mathcal{X} = \prod_{g \in G} X_g$, that is, every combination of evaluations are considered possible. We are interested in a preference relation $\succeq$ defined as a binary relation over $\mathcal{X}$.

Example 1. Say the Decision Maker (DM) must choose what to plant in her garden. The set of alternatives $\mathcal{X}$ are all possible vegetables, the criteria $\mathcal{G} = \{g_1, g_2, g_3\}$ measure the taste, quantity, and price of each vegetable, and $A \subseteq \mathcal{X}$ are the vegetables that are available this year for planting. $X_{g_1} = \{A, B, C, D\}$, a set of labels, with x_1 representing the taste of the vegetable $x \in \mathcal{X}$ as considered by the DM (A is the worst taste, D the best), $X_{g_2} = [0, 100]$, with x_2 representing the number of meals that the DM would enjoy if deciding to plant x, and $X_{g_3} = \mathbb{R}$, where x_3 indicates the price to pay for planting x.

Typical approaches in MCDM assume that there is some real-valued function $v : \mathcal{X} \to \mathbb{R}$ mapping alternatives to their values, and that $x \succeq y$ iff $v(x) \geq v(y)$.

2.2 DMU

In the simplest form of DMU considered here (SDMR, for Simple Decision Making under Risk), we consider a set S of possible states of the world, a finite set of consequences C, and each act $x : S \to C$ is modeled as a function where $x(s)$ is the consequence of performing x when s is the actual state of the world. Define $\mathcal{X}$ in this case as the set of possible acts (thus $\mathcal{X} = C^S$). In SDMR, uncertainty is modeled by a probability measure p over the power set of S, $\mathscr{P}(S)$, thus with $p(s) \in [0, 1]$ indicating the probability of occurence of s (with $s \subseteq S$), and $p(S) = 1$. We consider a preference relation $\succeq$ defined as a binary relation over $\mathcal{X}$. Given an act x and a probability measure p, it is usually convenient to view x as p_x, a probability mass over the consequences: define $p_x : C \to [0, 1]$ as $p_x(c) = p(x^{-1}(c))$, where $x^{-1}(c)$ designate the set of states in which x leads to the consequence c. Such a p_x is usually called a lottery.

Economists classically model such an SDMR situation as follows. Define a function that represents the utility of consequences, $u_1 : C \to \mathbb{R}$. Now define a function $u : \mathcal{X} \to \mathbb{R}$ that represents the utility of an act, such that $u(x) = \sum_{s \in S} p(s) u_1(x(s))$.

[OC: En lisant mieux, je me rends compte que tu as raison, ta formulation englobe effectivement le cas dont je voulais parler (désolé pour la confusion). Je me permets simplement de noter u_1 ton a et $u_1(x(s))$ au lieu de $a(s)$ qui me semble un peu perturbant (vu que la fonction est définie sur C d'une part et que je trouve bizarre qu'elle ne fasse pas apparaitre x dans sa notation alors qu'elle en dépend). Et je reformule légèrement pour pouvoir dire ensuite que ces fonctions sont parfois déduites de $\succeq$.] In most DMU frameworks, consequences can be mapped to a real-valued reward or utility through a function $u_1 : C \to \mathbb{R}$, in which case $u_1(x(s))$ denotes the utility of performing x in state s, and acts can be evaluated using a utility function $u : \mathcal{X} \to \mathbb{R}$ defined as $u(x) = \sum_{s \in S} p(s) u_1(x(s))$, such that $u(x) \geq u(y)$ iff $x \succeq y$. It follows from this definition that u and u_1 are coherent, in the following sense: given an act x that brings a consequence c with probability one, $u(x) = u_1(c)$.

Example 2. Assume you want to go out and wonder about the weather state ($S = \{A, B, C, D\}$), going from $A =$"shiny" to $D =$"stormy". Given this state, two actions are x_1: "leave the umbrella home" and x_2: "take the umbrella", and the consequences are c_1: "unhappy", c_2: "neither happy nor unhappy" and c_3: "happy". If the weather is nice ($x_1(A) = c_3, x_2(A) = c_1$), not having an umbrella is enjoyable, and having to bear one for nothing will make you unhappy. On the contrary, if it is stormy ($x_1(D) = c_1, x_2(D) = c_3$), you will be happy to have taken your umbrella, but quite unhappy if you end up being very wet. In the other situation, you are quite indifferent to both actions. Assuming that $p(A) = 0.2, p(D) = 0.6$ (you think it has a bad taste), $p(B) = p(C) = 0.1$, and that $u_1(c_i) = i - 2$, we then have

$$u(x_1) = 0.2 \cdot 1 + 0.6 \cdot -1 < u(x_2) = 0.2 \cdot -1 + 0.6 \cdot 1$$

[OC: Je me dis maintenant qu'on devrait laisser le polissage de cette partie pour plus tard : on a intérêt à voir d'abord la place qui nous reste pour savoir à quel niveau de détail on va parler de ces approches.] It has been justified axiomatically by different authors, the main ones being Savage (ref), De Finetti and Von Neumann Morgenstern. It is worth noticing that these settings are quite different.

- In De Finetti setting, utilities or rewards of actions are supposed to be given in the form of random variable, and it is assumed that a precise price can be associated to these uncertain rewards. That these prices should follow the rule of probabilistic calculus, and should in particular be expected utilities, follow from two rationality axioms: linearity and boundedness of those prices. In short, in de Finetti completeness of preferences is a consequence of the axioms.
- von Neumann and Morgenstern postulate conditions on $\succeq$ that ensures that utility functions u and u_1 satisfying the above conditions exist. The axioms also assume completeness of the preferences. The probabilities are assumed to be known.

– In Savage setting, the basic assumption is the one recalled above of SDMR, and both probabilities and expected utility criterion follow from axioms about preferences between acts. In particular, the first axiom (P1) is that any pair of act should be comparable. In contrast with the De Finetti setting, completeness of preferences is here postulated in the axioms, and expected utility together with probabilistic modeling follow from it.

While these theoretical constructs have set very strong foundations for the use of probabilities, in practice experiments such as the Ellsberg urn (that contradicts Savage second axiom of the sure-thing principle) suggest that people do not always act according to expected utility computed from a unique probability. And even before, researchers such as Keynes or Boole suggested that getting precise probabilities in practice was not a very realistic assumption.

Since then, many different extensions have been proposed, some modifying the utilities (ref to prospect theory) or the probabilities (ref to rank dependent utility). Others simply propose to relax the probabilistic assumption, for instance by considering a possibilistic setting where rewards can be non-numerical (see *** ref Dubois/Fargier ***, where it is shown that one can derive a possibilistic decision theory from Savage-like axioms), by considering sets of probabilities such as in the literature concerning decision under ambiguity (ref to Jean-marc Tallon and co), or by simply considering completely missing information, such as Wald (citation) and his celebrated maximin criterion (widely used in decision under risk and in robust optimisation). Some other works also consider dropping the precision of utility functions (Gul/pesendorfer). However, not knowing the utility function is somehow similar to problems treated in MCDM, as the utility function does not express our lack of information about the state of the world, but our preferences about different options.

However, while these extensions do address some of the observed behaviours that do not comply with expected utility, their axiomatic or models keep assuming that every acts can or should be comparable between each others, hence that preferences should be complete. Yet, if one considers that probabilities or utilities may be incompletely known, the next natural step is to consider that preferences derived from them could also be incomplete.

All models presented or advocated thus far assume that it is possible to order alternatives according to their value, as given by a real function. Thus, they require $\succeq$ to be complete and transitive (by which we mean that if $\succeq$ is incomplete or not transitive, then no suitable function exists).

The two contexts we have introduced, multiple criteria and risk, use as a basis the notion of a preference relation. We need to say a word about what those preferences really represent and what the goal of modeling those may be. This is required to give meaning to discussion about which conditions are reasonable to postulate about $\succeq$, and how to check whether they hold.

2.3 Descriptive approach

In the descriptive approach to preferences, the goal of the model is to reflect the normal behavior of a DM. Typically, we would have collected a database of sample choices of the DM, say, of choices of food products in his favorite store, and we would try to obtain the model that best reflects his choice attitude. Or, we would query an individual's preference about pairs of objects, and we would try to build a predictive relation on the whole set of possible pairs of alternatives. (Nowadays this would be called an "active learning" approach, but this term would be anachronic as the literature about this in decision theory pre-dates the field of machine learning.) Such a model may be used to predict his behavior, e.g. for marketing or regulation purposes. It may also be applied with the aim of replacing the DM by automating his decision procedure, although it is debatable whether the normative approach would better suit this use case.

Here is what von Neumann and Morgenstern say about the preference relation.

"It is clear that every measurement — or rather every claim of measurability — must ultimately be based on some immediate sensation, which possibly cannot and certainly need not be analyzed any futher. [Such as the sensations of light, heat, muscular effort, etc., in the corresponding branches of physics.] In the case of utility the immediate sensation of preference — of one object or aggregate of objects as against another — provides this basis." (3.1.2) (The square brackets indicate a footnote.)

"Let us for the moment accept the picture of an individual whose system of preferences is all-embracing and complete, i.e. who, for any two objects or rather for any two imagined events, possesses a clear intuition of preference. More precisely we expect him, for any two alternative events which are put before him as possibilities, to be able to tell which of the two he prefers." (3.3.2) (The "events" here correspond to what we call alternatives.)

As the preference relation is considered basic under a purely descriptive approach, the preference itself should be easily observable, and uncontroversial. This permits to test conditions postulated on $\succeq$. Either by asking the individual what she prefers (in which case we "observe" her answer), or by presenting a choice set and observing what she picks from that set (in real life or in a laboratory experiment). In the first case, we have however to be clear about what is meant by "prefers", which is a reason why many experimenters prefer to go for the latter strategy.

2.4 Normative approach

Under the normative approach, the goal is to reflect on how the DM ought to choose. Either according to external norms that are somehow (collectively) considered as representing rational behavior, or using some norms that the DM himself accepts after careful reflexion. Hence, the decision outcome using such approach may differ from that the DM would have chosen in his daily life. Consider as an example a responsible person in an enterprise who wants to model the procedure used to recruit employees. After having collected the figures, it may appear that, for some (possibly unconscious) reason, the recrutement is biased against some particular socio-economic or gender category, even considering equal competences. It may be, further, that the DM himself is not happy about this situation. Thus, he may try to find a strategy of selecting employees that would avoid such biases, therefore actively trying *not* to build a perfectly descriptive model of the normal selection attitude.

When the focus is on providing decision help to a DM by letting him think about and adopt the norms he prefers, rather than considering them as external norms that he ought to follow under threat of being considered irrational, the approach is often termed prescriptive, or constructive. There are important differences between these terms, and different authors use them somewhat differently. (Give refs.) The focus here being not on those differences, we refer to … and will use the general normative term as an umbrella.

When building recommender systems in the literature in artificial intelligence, the focus is often on descriptive approaches. This is usually left implicit, with no discussion about possible alternatives. We think however that an interesting path is offered by normative approaches. Descriptive approaches will, by design, reflect cognitive limitations exhibited by us, normal human beings. Those limitations are numerous and sometimes obviously not in agreement with what a more thoughtful and knowledgeable person would do, as is well known and will be illustrated in this article (although there is debate about how such a sentence is to be interpreted exactly, more about this later). Providing (more) normative-based automatic recommendations might help provide sound advices, help increase serendipity, and possibly reduce incitations to merchants to exploit imperfections on the human reasoning abilities by using marketing techniques that may lead to choices that the DM himself would possibly reject when thinking more carefully.

3 Dropping completeness

[OC: Je me permets d'inverser les sections à propos de completeness et de transitivity, car je suis nettement plus inspiré sur la question de la complétude. Mais cette inversion est négociable et potentiellement temporaire, bien sûr.]

3.1 Defining and testing incompleteness

A precise definition of what the preference relation $\succeq$ represents is required for discussing conditions postulated about it. For "preference", in its everyday usage, is very ambiguous. (**?** cites seven reasonable interpretations of the phrase "to want to"; this exercice transposes, *mutatis mutandis*, to the concept of preference.)

Expanding on the proposition of nVM, we define that the DM *prefers* a to b when he expresses an intuitive attraction towards a when presented with a and b, or an equal attraction towards a and b; and this attraction is stable over time (at least within a short time span) and does not change with irrelevant changes in the context. The second part of our definition makes it applicable also in case the intuitive attraction felt by the DM changes for no apparent reason, or depends on the context at the moment of presenting a and b to the DM, for example whether it is currently raining. Here we are interested in the multicriteria and SDMR settings, thus we assume that a, b are alternatives in $\mathscr{X}$ described by their evaluations on the criteria (in the first case) or by the relevant probability distributions and consequences (in the second case), and consider irrelevant changes in the context as anything that does not change those descriptions. This is by no means the only reasonable definition of a preference relation. We discuss alternative definitions of the preference relation later.

Under this definition, postulating completeness of $\succeq$ amounts to state that the intuitive attractions of the DM towards alternatives do not vary within short time spans and do not depend on irrelevant changes in the context. We think this is one reasonable way of capturing the essence of completeness, and it makes the condition empirically testable.

It appears that this completeness assumption is too strong, for lack of stability over time for some pairs of alternatives. This is well known in the field of experimental psychology, and is not even presented as a remarkable fact. As Tversky (1969) writes, individuals "are not perfectly consistent in their choices. When faced with repeated choices between x and y, people often choose x in some instances and y in others. Furthermore, such inconsistencies are observed even in the absence of systematic changes in the decision maker's taste which might be due to learning or sequential effects. It seems, therefore, that the observed inconsistencies reflect inherent variability or momentary fluctuation in the evaluative process."

?, p. 630 themselves considered completeness as a strong condition, as the following often cited quote indicates: "it is very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient". The question thus may be considered to be not whether in reality preferences are complete, but rather whether a model of complete preferences may provide a useful approximation of it. Let us examine, to discuss this hypothesis, a second reason for failure of completeness.

In order to discuss this hypothesis, we turn to the second (and much more interesting) reason for failure of completeness, which is also given by the literature in empirical psychology. It appears that preferences change in systematic ways according to changes in the presentation of the alternatives or the context that should have no impact from a normative point of view.

3.2 Empirical evidence of incompleteness

[OC: Ici je n'ai pas encore décrit les expériences en détail, juste l'idée pour que tu voies où je veux en venir. Faudra voir de combien de place on dispose pour cet aspect. J'ai l'impression d'en avoir déjà fait des tonnes sur cette section.]

? have collected preferences of individuals over alternatives in a risk setting, representing monetary bets. Their subjects are business executives. One of the most striking result of the study is that individuals order some pairs of alternatives differently depending on which other alternatives they are presented with. The study is run as follows. Each subject is presented instructions and three sets of five alternatives. (The same three sets of five alternatives are presented to every individual.) The subjects are asked to rank each alternative by order of preference. Each alternative has exactly two possible consequences (monetary outcomes), and can thus be fully described using three numbers: best outcome, probability of winning the best outcome, and worst outcome. We focus on two of these sets of five alternatives, labelled set B and C in the paper. Set B contains the alternatives $(\$5, 1, \$5)$, $(\$20, 0.692, \$3{,}90)$, $(\$20, 0.2752, \$-0{,}70)$, $(\$20, 0.6185, \$-19{,}30)$, $(\$20, 0.9046, \$-137{,}00)$. Set C contains $(\$5, 1, \$5)$, $(\$10, 0.6185, \$-3{,}10)$, $(\$15, 0.6185, \$-11{,}20)$, $(\$20, 0.6185, \$-19{,}30)$, $(\$25, 0.6185, \$-27{,}40)$. The first and fourth of these alternatives are identical in sets B and C. However, 9 out of 40 subjects give those pairs of alternatives different relative positions depending on which set they are ranking (4 of them prefer B1 to B4 but C4 to C1, and vice-versa for the other 5). This effect might be thought to be due to the individuals not taking the task seriously, but as the authors note, this appears implausible because of the very regular choice patterns observed in the rest of the analysis.

When working under utility theory, it is common to ask questions about probability equivalent loteries, or about certainty equivalents. Systematic differences appear in the preferences exhibited between each mode of questioning. (Better description needed here.)

In situations of risk, a famous study [**?**] showed an important effect of framing. The subjects, split in two groups, have to choose a preferred program to prepare against an epidemic outspring which will, if no action is taken, result in the death of 600 persons. The first group must choose between program A, which saves 200 persons, and program B, which saves 600 persons with 1 chance on 3, and otherwise saves nobody. Most persons in that group choose program A. The second group must choose between program A', which lets 400 persons die, and B', which results in nobody dying with 1 chance on 3, and otherwise the death of 600 persons. Most persons in the second group choose program B'. Observe that both choices are identical up to phrasing. This illustrates an effect well-known in psychology, according to which choices phrased as losses are evaluated differently than choices phrased as gains.

In multicriteria contexts, when the decision involves trade-offs, psychologists have shown systematic differences between choice and matching elicitation procedures [**?**]. Assume you want to know which of two alternatives a, b the DM prefers, in a problem involving two criteria. You can present both and directly ask for a choice. Alternatively, the matching procedure consists in presenting alternative a with its two evaluations $g_1(a), g_2(a)$, and present alternative b with only one evaluation $g_1(b)$, and ask the DM to state the value $g_2'(b)$ which would make b indifferent to a. Assuming $\succeq$ satisfies dominance and transitivity, you then know that $a \succeq b$ iff $g_2'(b) \geq g_2(b)$.

Other discussions and presentations of descriptive studies in the multicriteria and risk case are presented in **?**, Ch. 2, **??**, …

Physchologists talk about preference reversal effects when referring to situations where an individual ends up stating, indirectly, that he prefers a to b or b to a depending on the way the question is asked (possibly while assuming transitivity and dominance, as illustrated previously). Many advocate to view such effects as illustrating that preferences are constructed (by the process of elicitation), rather than given.

This shows that the $\succeq$ relation as defined above is in general not complete. One way wish however to model a preference relation as complete anyway, in order, for example, to benefit from increased simplicity, tractability, or mathematical elegance of such models. One path for doing this is to consider the context as fixed, for example, consider the way of presenting the alternatives to the DM as fixed. Thus, for example, the model would represent the preference as stated by the DM when asked questions in terms of binary choices, and not when he is interrogated in terms of matching. Furthermore, the model would represent not a preference understood as the expression of clear intuitive attraction from the DM towards some alternative, but rather as the a noisy intuitive attraction, known only imperfectly by the DM.

In some cases, this is a perfectly reasonable path to follow. (This strategy is commonly followed in experimental psychology, for example, **?** indicates that his book is about studying preferences in terms of choice and not of judgment; **?** indicates that some kind of loteries should not be considered to belong to the scope of the model.) We of course do not claim that models of preference relations that assume completeness have no validity whatsoever, as the validity of a model depends on its purposes. In the case however not of psychologists but of researchers building recommender systems, it is unclear that this path should be systematically followed. Indeed, in some cases we may want the recommendations of the recommender system to be invariant to the particular details about how alternatives have been presented to the individual. Not only because we may not know how the alternatives have been presented (what the context is), but more fundamentally, because the individual may find the advices of a recommender system more sound if it is independent of minute details of the context. (Adopting this view of course makes the system closer to the normative approach.)

Let us come back to the topic of completeness. These effects of framing are of course not everywhere [**?**]. You can't make someone choose whatever you want just by presenting it in the right way. (Or, at least, and luckily, we do not yet know how to do that.) Thus, another way of viewing the experimental highlights discussed here above is that a more meaningful model may be built on some subset of pairs that exhibit a stable preference. By forcing completeness in the model, we may build models that are partly irrelevant to the perception of the DM we try to help: for some pairs of alternatives, there may be no sensible way, even conceptually, to determine whether some alternative is really preferred by the DM to some other one.

Talk here about lack of information leading to incomplete models even though the preference is intrisically complete?

3.3 Incompleteness in MCDM

Some approaches in MCDM in the family of outranking methods permit to represent incomparabilities. A much used idea is to take into account two points of view and accordingly build two weak-orders, $\succeq^1$, $\succeq^2$, then define $\succeq = \succeq^1 \cap \succeq^2$. Thus, when the points of view of the two weak-orders strongly disagree about some pair of objects, the resulting relation does not take position about the related preference. As an example, let us consider (a simplification of) the ELECTRE III method (our description is much simplified because we want to focus on the way incomparabilities may arise). It builds a concordance relation $\boldsymbol{C}$ that determines whether alternative x is sufficiently better than y, by accounting only for the criteria in favor of x. For example, a model compatible with ELECTRE III could declare that $x\boldsymbol{C}y$ iff x is better than or equal to y according to at least two criteria. It also defines a discordance relation $\boldsymbol{D}$. In ELECTRE III, the performance of x on a given criterion g may be considered as "unacceptably low" compared to the performance of y on g (depending on parameters of the model). For example, a model could consider that the performance of x on g is unacceptably low compared to y iff $y_g - x_g \geq 2$, where $\boldsymbol{X}_g$ would be numerical. Then define $x \succeq y$ iff $x\boldsymbol{C}y$ and $\neg(x\boldsymbol{D}y)$.

Consider an example with $\mathscr{X} = \mathbb{R}^3$, each criteria to be maximized, and the following two alternatives: $x = (0, 0, 2), y = (1, 1, 0)$. Using the example definitions given above for $\boldsymbol{C}$ and $\boldsymbol{D}$, we obtain $x\boldsymbol{C}y$ and $x\boldsymbol{D}y$, thus x and y are incomparable in the resulting model.

A recent trends aims at axiomatizing outranking approaches [**?**]…

Robust methods in MCDM exist that propose to distinguish conclusions that hold for sure, given some information about the preferences of the DM, and conclusions that possibly hold. Such robust methods typically start from a class of possible models, that represents an assumption about the way the DM reasons in problem considered. (For readers used to the machine learning terminology, this corresponds to the version space.) A robust method, given a class of models $\boldsymbol{M}$ that contains all preference relations considered possible a priori, and a set of constraints $\boldsymbol{C}$ (such as examples of comparisons), will consider that a is necessarily preferred to b, $a \succeq^N b$, whenever $a \succeq b$ for all relations $\succeq$ in $\boldsymbol{M}$ that satisfy $\boldsymbol{C}$ [**?**].

Example 3. Assume that the only thing you know about the DM is that she prefers $x = (0, 0, 2)$ to $y = (0, 4, 0)$, and you assume that $\succeq$ satisfies preferencial independence, meaning that the way two alternatives compare does not change when changing equal values on a given criterion. Thus, $\boldsymbol{M}$ contains all relations that satisfy preferencial independence, and $\boldsymbol{C}$ is the constraint $x \succ y$. You may then conclude that $a = (3, 0, 2)$ is preferred to $b = (3, 4, 0)$, thus, $a \succ^N b$, but you ignore whether $c = (1, 1, 1)$ is preferred to $d = (0, 2, 2)$, thus, $\neg(c \succeq^N d)$ and $\neg(d \succeq^N c)$.

Such approach permits to represent incomparabilities. They are used to represent incomparabilities that stem from incomplete knowledge of the analyst, rather than to represent intrinsic incomparabilities.

* Other MCDM model that induce incomplete orders (CP-net... others?)

3.4 Incompleteness in DMU

As recalled in Section 1, probability theory and expected utility are the most widely used tools when having to decide under uncertainty, and naturally induce completeness of preferences under uncertainty. It should however be noted that while these theoretical constructs have very strong foundations, early scholars were critical about the fact that completeness could hold in practice, including for instance Von Neumann and Morgenstern **?**, p. 19. Many attempts to relax the completeness axioms does so by considerings axioms leading to deal with sets of utilities and sets of probabilities (citer Anscombe/Aumann et autres), entangling together aspects about decision and about information modelling. We will mainly restrict ourselves to approaches focusing on the uncertainty part, rather than on the utility part (interested reader can consult provided references for other apprpoaches).

Keeping precise probabilities but not expected utility Even when having precise probabilities, there are alternatives to expected utility that induce incomplete preferences. One of them that is particularly interesting is the notion of stochastic dominance (citation H.Levy:stochastic dominance book). [OC: Vu les notations que je viens d'ajouter dans la présentation de SDMR, je suggère la formulation suivante. Assuming that the set of consequences is completely ordered by preference, which we denote by $\boldsymbol{C} = \{c_1, \ldots, c_n\}$ where c_i is preferred to c_{i-1}, then a lottery p_x is said to stochastically dominate p_y iff, etc. Si ça te va je changerai aussi la suite de l'exemple en fonction des nouvelles notations. Par ailleurs, faut-il ordonner depuis les pires ou depuis les meilleures conséquences ? Aucun des deux ne me semble clairement s'imposer sur l'autre.] Assuming that your probability is defined over a completely ordered space $\boldsymbol{Y} = \{y_1, \ldots, y_n\}$ (which can be your space of consequences), then an act modelled by probability p_1 is said to stochastically dominate another act modelled by probability p_2 iff for all y_i, we have

$$P_1(\{y_1, \ldots, y_i\}) = \sum_{j=1}^{i} p_1(y_j) \leq P_2(\{y_1, \ldots, y_i\}) = \sum_{j=1}^{i} p_2(y_j). \tag{1}$$

Since Inequality (1) can be satisfied for some y_i and not for others, possible incomparabilities immediately follow. It should be noticed that the probabilities p_1, p_2 may concern the same uncertain quantity, but may result from the mapping of the original probability p to the space of (ordered) consequences, such as in Example 2.

Example 4. Consider again the space $\boldsymbol{C} = \{c_1, \ldots, c_3\}$of consequences and the following probability masses (induced by different acts x_1, x_2, x_3) defined over them, given in vectorial forms: $p_1 = (0.2, 0.3, 0.5)$, $p_2 = (0.1, 0.3, 0.6)$ and $p_3 = (0.3, 0, 0.7)$. They induce the following cumulative probabilities:

$$P_1(\{c_1\}) = 0.2,\ P_1(\{c_1, c_2\}) = 0.5,\ P_1(\{c_1, c_2, c_3\}) = 1;$$

$$P_2(\{c_1\}) = 0.1,\ P_2(\{c_1, c_2\}) = 0.4,\ P_2(\{c_1, c_2, c_3\}) = 1;$$

$$P_3(\{c_1\}) = 0.3,\ P_3(\{c_1, c_2\}) = 0.3,\ P_3(\{c_1, c_2, c_3\}) = 1,$$

from which we can conclude that x_2 stochastically dominates x_1, while x_3 is incomparable to both x_1 and x_2, according to stochastic dominance.

The notion of stochastic dominance has some very attractive properties, as:

1. it does not necessitate to define utilities over consequences, and merely requires them to be linearly ordered;
2. it can be be perceived as a criterion allowing for utilities to be ill-defined, as p_1 stochastically dominates p_2, if and only if its expected value for any increasing utility function u defined over Y (that is, $u(y_i) \leq u(y_{i+1})$) is higher than the expected value of this utility function according to p_2 (trouver la référence à la preuve?).

Incompleteness from non-precise probabilities In the past few decades, different scholars have challenged the need for precise probabilities associated to classical axiomatics, advocating the use of imprecisely defined prices (expected values) or of imprecisely defined probabilities. To mention but a few:

- Levi advocates the uses of sets of probabilities within a logical interpretation of probabilities;
- Walley extends de Finetti axioms by assuming that given an act x and a (postulated) linear utility function, an agent would give different buying and selling prices for this act, therefore allowing indecision if the given price is between these two bounds;
- Shafer and Vovk explores a probabilistic setting centered on the notion of Martingale that naturally gives rise to expectation bounds rather than precisely defined one, inheriting ideas from Jean Ville and Kolmogorov complexity.

Without entering into too much details, such theories can most of the time be associated to the use convex sets of probabilities, and recommend decision rules that extend expected utility but do allow incomparabilities to happen. Once we accept that a convex set $\mathcal{P}$ of probabilities (or a formally equivalent representation) can represent our knowledge, without having to be precise, incompleteness ensues easily.

A prototypical way to induce incompleteness between acts from incompleteness in probabilities is to adapt expected utility criterion, and among rules doing so, maximality is a popular one (it is championed by Walley, but is considered as early as the 60's by Ansombe and Aumann ***papier 63***). Given acts x_1, x_2, maximality says that

$$x_1 \succ x_2 \text{ iff } u(x_1) \geq u(x_2) \text{for all} p \in \mathcal{P},$$

still with $u(x) = \sum_{s \in S} p(s) u_1(x(s))$. It is clear that this reduces to expected utility when $\mathcal{P}$ is a singleton.

Example 5. Going back to Example 2, we have that

$$u(x_1) = p(A) - p(B) \text{ and } u(x_1) = p(B) - p(A)$$

which means that the two acts will be incomparable according to maximality as soon as $\mathcal{P}$ contains a probability where $p(A) = p(B)$ that is not on the border of $\mathcal{P}$ (i.e., it contains at least one mass where $p(A) < p(B)$, and another where $p(B) > p(A)$).

It should be noted that other authors have proposed different rules: for instance Levi as well as some of his followers (citer Seidenfeld) recommends to use a decision rule, often called E-admissibility, that does not give rise to an incomplete order between acts, but rather selects all the acts that are Bayes optimal according to at least one probability $p \in \mathcal{P}$. In terms of order, this comes down to consider a set of possible linear ordering, and to retain only those elements that are maximal for at least one of them.

Compared to maximality, E-admissibility implicitly uses the precise probabilities within the set $\mathcal{P}$, and therefore would not be consistent with Walley's view where the bounded (buying and selling) prices are not to be interpreted as incomplete information about an ideal precise price, but have to be considered as current state of knowledge that may not be reducible with additional information.

Working with sets of utilities **?** represents preferences over lotteries by means of a set of real-valued utility functions. Preference holds whenever the expected utility for the preferred alternative is higher, for all utility functions in the set. This idea has also been applied in other contexts [**??**].

? propose to view the preference relation as a completion of an intuitive partial preference relation: the DM knows intuitively the result of some comparisons, and compute the other ones by applying some reasoning process. They also obtain a preference relation that is representable using a set of utility functions. Observe that this approach directly tackles some of the shortcomings described in Section 3.1.

? use a utility function and a vagueness function, thus, represent the preference using intervals of utilities rather than real valued utilities. (Beyond DMU, also using this representation, see **?**, and **?** who assume that a specific alternative called the status quo alternative is prominently chosen whenever the DM faces a choice about which incomparability occur.)

Working with sets of probabilities and utilities [OC: Je proposerais de fusionner cette § avec la précédente, à moins que tu aies beaucoup de choses en tête à dire à ce sujet ? (Moi pas !)] **?** are interested in the Savage-like context where probabilities are unknown and represent an incomplete preference relation in uncertaintly using a set of pairs of probabilities and utilities.

Others **?**: axioms do not require completeness. $x \succeq y \Rightarrow u(x) \geq u(y)$ and $x \succ y \Rightarrow u(x) > u(y)$ but the reverse implications do not hold.

? propose to consider that an incomparability is observed whenever the DM is ready to pay a small price to postpone the decision, without receiving new information about the problem.

3.5 Incompleteness: absence of knowledge or knowledge of absence?

4 Dropping transitivity

4.1 Empirical evidences of intransitivity

4.2 Under a normative view

Fishburn is one of the prominent advocate about intransitivity holding even under normative approaches. Explain arguments...

Roy has very much insisted on incomparability being taken into account explicitly in preference modeling. Explain...

4.3 Approaches that admit intransitivity in MCDM

4.4 Approaches that admit intransitivity in MDU

* Statistical preferences, strongly related to the Condorcet paradox in social choice theory

4.5 Intransivity: a default to be repaired, or a fact to be modelled?