

# Preference models that relax completeness or transitivity: why, how?

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**Abstract.** Literature involving preferences of artificial agents or human beings often assume their preferences can be represented using a complete transitive binary relation. Much has been written however on more complex, or more interesting, models of preferences. In this article we review some of the reasons that have been put forward to justify more complex modeling, and review some of the techniques that have been proposed to obtain models of such preferences. (Optional: we connect to various related literature about argumentation, ...)

## 1 Introduction

Preferences of agents are usually assumed to be linear, that is complete, transitive and antisymmetric, or representable with a weak order (a complete and transitive binary relation). In practice, such assumptions limit the scope of models (of uncertainty, of preferences) one can consider as legitimate, and are also falsified by observed empirical evidences. While models relaxing antisymmetry to allow incompleteness between options have been around for some time \*\*\* ELECTRE ?\*\*\*, models relaxing the assumptions of completeness or transitivity are more recent.

It is clear that our paper will not be complete, yet we think it gives a fair overview of why one would like to drop some of the above assumption. In particular, we will review both empirical evidence that preferences do not always follow the above assumption, thus relating them to more descriptive approaches, as well as formal models that allows one to induce intransitive or incomplete preferences.

There are two canonical fields where the modelling of preferences is a central topic: choosing an alternative when it is evaluated according to different aspects (a.k.a. multi-criteria decision making, or MCDM), and picking an alternative whose quality depends on states of the world that are uncertainly known. Although the two frameworks are formally similar to some extent, they also present some key differences, if not formally then at least conceptually.

In MCDM, the common assumption is that the alternatives, i.e., the state of the world is known without ambiguity, and the difficulty is to determine what are the user preferences over these different, but well-defined states. In decision making under uncertainty (DMU), the preferences of the user are usually assumed to be well-known, or at least to

have been previously assessed in the form of utility functions, and the problem is to recommend an alternative given our uncertainty about the world.

We will adopt the following basic scheme for the two settings (MCDM and DMU), that we will question later on, but that are necessary to set some notations. [OC: Je me permets de les considérer hors d'un environnement itemize pour qu'on soit plus à l'aise pour exposer un peu plus longuement les bases, ça me semble valoir la peine car je me dis que certains lecteurs connaîtront peut-être très bien seulement un des deux contextes et pas du tout l'autre. Il ne faudrait pas les perdre dès le début. Je me permets aussi de ré-intégrer mon exemple de jardinage, mais sens-toi libre de l'effacer si tu supposes que les lecteurs sauront bien assez de quoi il s'agit. Je mesure mal ce que notre audience dans cette conférence sait déjà.]

In MCDM, the classical approach is to assume that the alternatives are evaluated using a set of criteria  $\mathcal{G}$ , each having an evaluation scale  $X_g$ . The set of all possible alternatives is  $\mathcal{X} = \times_{g \in \mathcal{G}}$ , that is, every combination of evaluations are considered possible. [OC: As-tu une forte préférence pour la notation  $\times_{g \in \mathcal{G}}$ ? Sinon, je proposerais d'utiliser  $\prod_{g \in \mathcal{G}}$ , qui me semble plus courant. Mais je peux me tromper, et je ne suis pas rigide sur cette question.] Typical approaches assume that there is some real-valued function  $v : \mathcal{X} \rightarrow \mathbb{R}$  mapping alternatives to their values, and that  $x \geq y$  iff  $v(x) \geq v(y)$ .

*Example 1.* Say the Decision Maker (DM) must choose what to plant in her garden. The set of alternatives  $\mathcal{X}$  are all possible vegetables, the criteria  $\mathcal{G} = \{g_1, g_2, g_3\}$  measure the taste, quantity, and price of each vegetable, and  $A \subseteq \mathcal{X}$  are the vegetables that are available this year for planting.  $X_{g_1} = \{A, B, C, D\}$ , a set of labels, with  $x_1$  representing the taste of the vegetable  $x \in \mathcal{X}$  as considered by the DM ( $A$  is the best taste,  $D$  the worst),  $X_{g_2} = [0, 100]$ , with  $x_2$  representing the number of meals that the DM would enjoy if deciding to plant  $x$ , and  $X_{g_3} = \mathbb{R}$ , where  $x_3$  indicates the price to pay for planting  $x$ .

In the simplest form of DMU considered here (SDMR, for Simple Decision Making under Risk), we consider a finite space  $S$  of possible states of the world, a set of consequences  $C$ , and each act  $x : S \rightarrow C$  is modelled as a function where  $x(s)$  is the consequence of performing  $x$  when  $s$  is the actual state of the world. Define  $\mathcal{X}$  in this case as the set of possible acts (thus  $\mathcal{X} = C^S$ ). Given  $c \in C$ , let  $x_c$  denote the act that constantly yields  $c$ , thus,  $x_c(s) = c, \forall s \in S$ . In SDMR, uncertainty is modelled by a probability measure  $p$  over the power set of  $S$ ,  $\mathcal{P}(S)$ , thus with  $p(s) \in [0, 1]$  indicating the probability of occurrence of  $s$ , and  $\sum_{s \in S} p(s) = 1$ . Finally, SDMR assumes the existence of a utility function  $u : \mathcal{X} \rightarrow \mathbb{R}$  such that  $x \geq y$  iff  $u(x) \geq u(y)$ , and such that  $u(x) = \sum_{s \in S} p(s)x_{a(s)}$ . It is sometimes considered that  $C$  is the set of reals, and that an element  $c \in C$  (thus  $c \in \mathbb{R}$ ) directly represents the utility of the consequence. In that case, the utility function becomes simply  $u(x) = \sum_{s \in S} p(s)a(s)$ .

All models presented thus far assume that it is possible to order alternatives according to their value, as given by a real function. Thus,

they require  $\succeq$  to be complete and transitive (by which we mean that if  $\succeq$  is incomplete or not transitive, then no suitable function exists).

The two contexts we have introduced, multiple criteria and risk, use as a basis the notion of a preference relation. We need to say a word about what those preferences really represent and what the goal of modeling those may be. This is required to give meaning to discussion about which conditions are reasonable to postulate about  $\succeq$ , and how to check whether they hold.

## 2 Different approaches to preference modeling

### 2.1 Descriptive approach

In the descriptive approach to preferences, the goal of the model is to reflect the normal behavior of a DM. Typically, we would have collected a database of sample choices of the DM, say, of choices of food products in his favorite store, and we would try to obtain the model that best reflects his choice attitude. Or, we would ask an individual questions of preference about pairs of objects, and we would try to build a predictive relation on the whole set of possible pairs of alternatives. (Which nowadays would be called an “active learning” approach.) Such a model may be used to predict his behavior, e.g. for marketing or regulation purposes. It may also be applied with the aim of replacing the DM by automating his decision procedure, although it is debatable whether the normative approach would better suit this use case.

Here is what von Neumann and Morgenstern say about the preference relation.

“It is clear that every measurement — or rather every claim of measurability — must ultimately be based on some immediate sensation, which possibly cannot and certainly need not be analyzed any further. [Such as the sensations of light, heat, muscular effort, etc., in the corresponding branches of physics.] In the case of utility the immediate sensation of preference — of one object or aggregate of objects as against another — provides this basis.” (3.1.2) (The square brackets indicate a footnote.)

“Let us for the moment accept the picture of an individual whose system of preferences is all-embracing and complete, i.e. who, for any two objects or rather for any two imagined events, possesses a clear intuition of preference. More precisely we expect him, for any two alternative events which are put before him as possibilities, to be able to tell which of the two he prefers.” (3.3.2) (The “events” here correspond to what we call alternatives.)

As the preference relation is considered basic under a purely descriptive approach, the preference itself should be easily observable, and uncontroversial. This permits to test conditions postulated on  $\succeq$ . Either by asking the individual what she prefers (in which case we “observe” her answer), or by presenting a choice set and observing what she picks from that set (in real life or in a laboratory experiment). In the first case, we have however to be clear about what is meant by “prefers”, which is a reason why many experimenters prefer to go for the latter strategy.

## 2.2 Normative approach

Under the normative approach, the goal is to reflect on how the DM ought to choose. Either according to external norms that are somehow (collectively) considered as representing rational behavior, or using some norms that the DM himself accepts after careful reflexion. Hence, the decision outcome using such approach may differ from that the DM would have chosen in his daily life. Consider as an example a responsible person in an enterprise who wants to model the procedure used to recruit employees. After having collected the figures, it may appear that, for some (possibly unconscious) reason, the recruitment is biased against some particular socio-economic or gender category, even considering equal competences. It may be, further, that the DM himself is not happy about this situation. Thus, he may try to find a strategy of selecting employees that would avoid such biases, therefore actively trying *not* to build a perfectly descriptive model of the normal selection attitude.

When the focus is on providing decision help to a DM by letting him think about and adopt the norms he prefers, rather than considering them as external norms that he ought to follow under threat of being considered irrational, the approach is often termed prescriptive, or constructive. There are important differences between these terms, and different authors use them somewhat differently. (Give refs.) The focus here being not on those differences, we refer to ... and will use the general normative term as an umbrella.

When building recommender systems in the literature in artificial intelligence, the focus is often on descriptive approaches. This is usually left implicit, with no discussion about possible alternatives. We think however that an interesting path is offered by normative approaches. Descriptive approaches will, by design, reflect cognitive limitations exhibited by us, normal human beings. Those limitations are numerous and sometimes obviously not in agreement with what a more thoughtful and knowledgeable person would do, as is well known and will be illustrated in this article (although there is debate about how such a sentence is to be interpreted exactly, more about this later). Providing (more) normative-based automatic recommendations might help provide sound advices, help increase serendipity, and possibly reduce incitations to merchants to exploit imperfections on the human reasoning abilities by using marketing techniques that may lead to choices that the DM himself would possibly reject when thinking more carefully.

## 3 Dropping transitivity

### 3.1 Empirical evidences of intransitivity

\* Fisher, ...[\[OC: Je suis curieux et impatient de connaître l'opinion de Fisher là-dessus! Je n'étais pas tombé là-dessus dans mes lectures.\]](#)

When conditions about the  $\geq$  preference relation are assumed to bear on what DMs normally do, it makes it very easy to test them. Very numerous laboratory experiments have tackled this question.

To make sure it is really the behavior of an individual that is captured with this relation, many experiments place subjects under real choices, presenting to them two objects among which the individual has to pick exactly one. This choice then represents his preference, by definition: here preference is meant to reflect the object the individual would choose when facing a choice. Such experiments face a first problem: individuals do not necessarily pick the same choice, when presented several times with the same choice pairs. This cannot simply be attributed to indifference between these pairs. (Cite studies.)

Interestingly, this effect is generally not considered in the literature in relation with incompleteness. Experimenters would choose to consider some stochastic version of preference instead, defining  $a \succeq b$  whenever the individual chooses  $a$  a majority of time when presented  $a$  and  $b$ , and then discuss whether this relation (complete by definition) satisfies transitivity. But it shows that, at least under the understanding quoted here above, completeness is not satisfied.

Such phenomenon is also exhibited in a more indirect fashion, in tests of utility theory. When working under utility theory, it is common to ask questions about probability equivalent loteries, or about certainty equivalents. Such answers show cycles in the exhibited preferences. (Better description needed here.)

Psychologists talk about preference reversal effects when referring to situations where an individual ends up stating, indirectly, that he prefers  $a$  to  $b$  or  $b$  to  $a$  depending on the way the question is asked (possibly while assuming transitivity and dominance, as illustrated previously). Many advocate to view such effects as illustrating that preferences are constructed (by the process of elicitation), rather than given. There has been debate about whether such effect is due to true intransitive effects, or if it can be attributed solely to differences of evaluations between probability equivalents and certainty equivalents (cite Luce, ...). But in any case, it can be also considered as an argument in favor of not using as a basis concept a complete preference relation, as if the DM was able to always compare two options reliably.

In fact, it must be noted, and it is often remarked by those authors working on relaxing completeness, that von Neumann and Morgenstern themselves considered completeness as a very strong condition. (Quote.) This is also visible from the extended description given by Fishburn of their work, although Fishburn does not focus on that aspect: the authors write that they assume the completeness of the relation, without defending the reasonableness of this hypothesis, whereas about other conditions they do not merely write that they assume them, but go at length about defending their reasonableness. About completeness, they merely state that it is also assumed by the concurrent theory which they propose to improve.

In situations of risk, a famous study [?] showed an important effect of framing. The subjects, split in two groups, have to choose a preferred program to prepare against an epidemic outspring which will, if no action is taken, result in the death of 600 persons. The first group must choose between program A, which saves 200 persons, and program B, which saves 600 persons with 1 chance on 3, and otherwise saves nobody. Most

persons in that group choose program A. The second group must choose between program A', which lets 400 persons die, and B', which results in nobody dying with 1 chance on 3, and otherwise the death of 600 persons. Most persons in the second group choose program B'. Observe that both choices are identical up to phrasing. This illustrates an effect well-known in psychology, according to which choices phrased as losses are evaluated differently than choices phrased as gains.

Closer to a multicriteria context, when the decision involves trade-offs, psychologists have shown systematic differences between choice and matching elicitation procedures [?]. Assume you want to know which of two alternatives  $a, b$  the DM prefers, in a problem involving two criteria. You can present both and directly ask for a choice. Alternatively, the matching procedure consists in presenting alternative  $a$  with its two evaluations  $g_1(a), g_2(a)$ , and present alternative  $b$  with only one evaluation  $g_1(b)$ , and ask the DM to state the value  $g'_2(b)$  which would make  $b$  indifferent to  $a$ . Assuming  $\geq$  satisfies dominance and transitivity, you then know that  $a \geq b$  iff  $g'_2(b) \geq g_2(b)$ .

Other discussions and presentations of descriptive studies in the multicriteria and risk case are presented in ?, Ch. 2, ?, ...

Savage assumes that every event can be compared infinitely precisely, talk about this?

Talk here about lack of information leading to incomplete models even though the preference is intrinsically complete?

### 3.2 Under a normative view

Fishburn is one of the prominent advocate about intransitivity holding even under normative approaches. Explain arguments...

Roy has very much insisted on incomparability being taken into account explicitly in preference modeling. Explain...

### 3.3 Intransitivity in MCDM

[OC: Tu veux qu'on parle ici des modèles MCDM qui admettent l'intransitivité, c'est bien ça?]

### 3.4 Intransitivity in MDU

\* Statistical preferences, strongly related to the Condorcet paradox in social choice theory

### 3.5 Intransitivity: a default to be repaired, or a fact to be modelled?

## 4 Dropping completeness

### 4.1 Empirical evidence of incompleteness

\* not clear what to put there... difficult to witness incompleteness?

## **4.2 Incompleteness in MCDM**

- \* Robust MCDM a la Greco etc...

- \* MCDM model that induce incomplete orders (CP-net... others?)

## **4.3 Incompleteness in DMU**

- \* Stochastic dominance inducing possibly incomparability

- \* Incompleteness from incompletely specified probabilities, including decision under risk

## **4.4 Incompleteness: absence of knowledge or knowledge of absence?**