The geometry and arithmetic of cubic hypersurfaces

Lecture notes

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[1]This is an incomplete, preliminary version of my lecture notes on cubic hypersurfaces. These notes will be updated weakly, see https://olivierfortman.github.io. For comments on the text, please write me an e-mail (*degaayfortman@math.uni-hannover.de*).

These are lectures notes for a course given at the Institute of Algebraic Geometry in Hannover, between October 2023 and February 2024. The goal of these lectures is to give an introduction to the theory of cubic hypersurfaces. In these notes, I will treat geometric as well as arithmetic aspects of the theory. Some of the topics covered:

(1) Topology of hypersurfaces.

(2) Hodge theory of cubic hypersurfaces.

(3) Lines on cubic hypersurfaces.

(4) Two-dimensional birational geometry, intersection theory, deformation theory.

(5) Cubic surfaces and cubic threefolds.

(6) Moduli spaces, algebraic stacks; period domains and period mappings.

(7) Étale cohomology and cubic hypersurfaces over finite fields.

Should you have any questions, or comments on these notes, do not hesitate to send me an e-mail[1].

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Chapter 1

Introduction

One of the goals of algebraic geometry is to study zero sets of systems of homogenous polynomials in multiple variables with coefficients in a field k. To do so, one is led to investigate the geometry of *algebraic varieties* over k. Among the simplest ways to obtain examples of an algebraic variety, is to consider a degree d hypersurface

$$X = Z(F) = \{F = 0\} \subset \mathbb{P}^m_k, \quad F \in k[x_0, \dots, x_m]_d, \quad d \in \mathbb{Z}_{\geq 1}.$$

It turns out that, although their definition is simple, hypersurfaces $X \subset \mathbb{P}^m_k$ are in general difficult objects to study.

To facilitate the study of hypersurfaces in $\mathbb{P}^m$, one can restrict to the *smooth hypersurfaces*, i.e. those for which the equation $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_m = 0$ has no solution in $\mathbb{P}^m(\bar{k})$. If $d = 1$ then $X \cong \mathbb{P}^m$ is a hyperplane. If $d = 2$ then X is a smooth quadric, which implies that F is projectively equivalent to $x_0^2 + \cdots + x_m^2 = 0$. When $d \geq 3$, degree d hypersurfaces in $\mathbb{P}^m$ for $m \geq 2$ come in positive dimensional families, and their investigation starts to become more complicated.

When $d = 3$, one enters the realm of *smooth cubic hypersurfaces*. For each value of $n = \dim(X)$, the class of cubic hypersurfaces of dimension n is very rich; however, only for small n, the theory is fairly well understood. When $n = \dim(X) = 1$ and X is equipped with a rational point $e \in X(k)$, then $E = (X, e)$ is called an *elliptic curve*. The fundamental theorem in the theory of elliptic curves says that there exists an algebraic group law $E \times E \to E$ in this case, turning E into a one-dimensional smooth projective group variety. If $n = \dim(X) = 2$, then $X = S$ is a *cubic surface*, and $S_{\bar{k}}$ turns out to contain exactly 27 lines over $\bar{k}$. In higher dimensions, cubic hypersurfaces provide a rich class of objects to test important conjectures in algebraic geometry on; think of the Hodge and Tate conjectures. Another example is provided by the Weil conjectures, that were proven for cubic threefolds before they were proven in general.

In the theory of cubic hypersurfaces, many beautiful areas in mathematics interact with one another, such as arithmetic geometry, algebraic topology, étale cohomology, Hodge theory and moduli theory. Open questions concern cycle class conjectures and rationality questions. The goal of these lectures is to dive into these theories, and use the developed techniques to study the geometry and arithmetic of cubic hypersurfaces.

Chapter 2

Preliminaries

To follow this course, it is useful, but not strictly necessary, to be familiar with the basic theory of schemes (as in [Mum88] or [Har77, Ch. I, §1-2]) and sheaf cohomology (see e.g. [Har77, Ch. II, §1-4]). In any case, the reader should have followed a first course in algebraic geometry.

Throughout the course, we will make use of some classical, fundamental results in algebraic geometry, without providing a proof. We collect these results this section [or in the appendix, to be added later]. Apart from this, we aim to make the body of the text will be as self-contained as possible; in particular, we try to avoid presenting a theorem without providing at least a sketch of its proof.

Chapter 3

Topology and differential forms

3.1 Lecture 1: Kähler differentials on hypersurfaces

Let k be a field.

3.1.1 Bott vanishing

Let $n \in \mathbb{Z}_{\geq 0}$ and $m = n+1$. Before we start to study hypersurfaces $X \subset \mathbb{P}^m_k$, we study the projective space $\mathbb{P}^m$ itself.

Lemma 3.1.1. *Let X be a scheme.*

(1) If $0 \to E \to F \to L \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^p E \to \bigwedge^p F \to \bigwedge^{p-1} E \otimes L \to 0.$$

(2) Similarly, if $0 \to L \to E \to F \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for each $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^{p-1} F \otimes L \to \bigwedge^p E \to \bigwedge^p F \to 0.$$

(3) Let E be a vector bundle and L a line bundle on X. Let $a > 0$ be an integer. There is a canonical isomorphism

$$\bigwedge^a (E \otimes L) = \left(\bigwedge^a E\right) \otimes L^{\otimes a}.$$

Proof. 1. Let Q be the cokernel of $\wedge^p E \to \wedge^p F$. Wedge the original sequence with $\wedge^{p-1} E$, and consider the canonical morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^{p-1} E \otimes E \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes F \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigwedge^p E \arrow[r] & \bigwedge^p F \arrow[r] & Q \arrow[r] & 0.
\end{tikzcd}$$

It suffices to show that the so-constructed natural map $\wedge^{p-1}E \otimes L \to Q$ is an isomorphism. For this, we may assume that $F = E \oplus L$. In this case, $\wedge^p F = \wedge^p(E \oplus L) = \oplus_{i+j=p} \wedge^i E \otimes \wedge^j L = (\wedge^{p-1}E \otimes L) \oplus \wedge^p E$, and hence $\wedge^p F / \wedge^p E = \wedge^{p-1}E \otimes L$.

2. Dualize the exact sequence $0 \to L \to E \to F \to 0$, use item 1, and then dualize the result.

3. The map

$$(E \otimes F)^{\otimes a} \to \left(\bigwedge^a E\right) \otimes L^{\otimes a}, \quad e_1 \otimes f_1 \otimes \cdots \otimes e_a \otimes f_a \mapsto (e_1 \wedge \cdots \wedge e_a) \otimes (f_1 \otimes \cdots \otimes f_a),$$

factors through a map

$$\bigwedge^a (E \otimes L) \to \left(\bigwedge^a E\right) \otimes L^{\otimes a},$$

which is an isomorphism (this can be verified on stalks, where this is clear). $\square$

Lemma 3.1.2. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\mathbb{P} = \mathbb{P}^m$. For each $p \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$, there is a canonical exact sequence*

$$0 \to \Omega^p_{\mathbb{P}}(k) \to \mathcal{O}_{\mathbb{P}}^{\oplus\binom{m+1}{p}}(k-p) \to \Omega^{p-1}(k) \to 0. \tag{3.1}$$

Proof. Consider the *Euler sequence*, which is the exact sequence

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}} \to 0. \tag{3.2}$$

It yields

$$0 \to \Omega_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

By item 1 in Lemma 3.1.1, this yields an exact sequence $0 \to \wedge^p(\Omega_{\mathbb{P}}(1)) \to \wedge^p(\mathcal{O}_{\mathbb{P}}^{\oplus(m+1)}) \to \wedge^{p-1}(\Omega_{\mathbb{P}}(1)) \to 0$. By item 3 in Lemma 3.1.1, we obtain:

$$\bigwedge^p (\Omega_{\mathbb{P}}(1)) = \left(\bigwedge^p \Omega_{\mathbb{P}}\right) \otimes \mathcal{O}(p) = \Omega^p(p).$$

The lemma follows. $\square$

Lemma 3.1.3. *Let X be a projective variety of dimension n over k, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let E a vector bundle of rank r on X. For $p \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, there is a canonical isomorphism $((\wedge^p E)(k))^* = (\wedge^r E)^* \otimes (\wedge^{r-p} E)(-k)$.*

Proof. We have

$$\left(\left(\bigwedge^p E\right) \otimes \mathcal{O}_X(k)\right)^* = \left(\bigwedge^p E\right)^* \otimes \mathcal{O}_X(-k).$$

Hence, it suffices to prove the lemma in the case $k = 0$. Consider the natural map

$$\bigwedge^p E \to \operatorname{Hom}\left(\bigwedge^{p-r} E, \bigwedge^r E\right) = \operatorname{Hom}\left(\bigwedge^{p-r} E, \mathcal{O}_X\right) \otimes \bigwedge^r E = \left(\bigwedge^{p-r} E\right)^* \otimes \bigwedge^r E.$$

We claim that this map is an isomorphism. This may be checked locally, in which case it is clear. As $(\wedge^p E)^* = \wedge^p E^*$, the lemma follows by duality. $\square$

Corollary 3.1.4. *Let X be a smooth projective variety of dimension n over k, with ample line bundle $\mathcal{O}_X(1)$. For $k \in \mathbb{Z}$, there are canonical isomorphisms*

$$(\Omega_X^p(k))^* \cong \omega_X^* \otimes \Omega_X^{r-p}(-k) \quad \textit{and} \quad H^q(X, \Omega_X^p(k)) \cong H^{n-q}(X, \Omega_X^{n-p}(-k))^\vee. \tag{3.3}$$

In particular, $h^q(X, \Omega^p(k)) = h^{n-q}(X, \Omega_X^{n-p}(-k))$ for each $k \in \mathbb{Z}$.

Proof. Lemma 3.1.3 shows that

$$((\wedge^p \Omega_X)(k))^* = (\wedge^n \Omega_X)^* \otimes (\wedge^{n-p} \Omega_X)(-k) = \omega_X^* \otimes \Omega_X^{n-p}(-k).$$

By Serre duality [reference], we obtain:

$$\begin{aligned} H^q(X, \Omega_X^p(k)) &= H^{n-q}(X, \omega_X \otimes (\Omega_X^p(k))^*)^\vee \\ &= H^q(X, \omega_X \otimes \omega_X^* \otimes \Omega_X^{n-p}(-k))^\vee = H^q(X, \Omega^{n-p}(-k))^\vee. \end{aligned}$$

The last statement follows readily from (3.3). $\square$

Theorem 3.1.5 (Bott vanishing)**.** *Consider the projective space $\mathbb{P} = \mathbb{P}_k^m$ of dimension $m > 0$ over k. Then $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in each of the following cases:*

(a) $p \neq q$ and $0 < q < m$;

(b) $p = q > 0$ and $k \neq 0$, and $k > 0$ if $p = q = m$;

(c) $q = 0$ and $k \leq p$, and $k < 0$ if $p = 0$;

(d) $q = m$ and $k \geq p - m$, and $k > 0$ if $p = m$.

Proof. We assume that we are in one of the cases (a) – (d); our goal is to prove that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$. By Serre duality, see Corollary 3.1.4, we may assume that $q \geq p$. We proceed by induction on p.

First, assume that $p = 0$. In this case, either $q = 0$ in which case $k < 0$ hence $H^q(\mathbb{P}, \mathcal{O}(k)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) = 0$, or $m > q > 0$ in which case $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$, or $m = q$ in which case $k \geq p - m = -m$ hence again $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$. We conclude that the assertion holds if $p = 0$.

Next, assume that $p > 0$. Then $q \geq p > 0$. Sequence (3.1) gives us a long exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(\mathcal{O}(k-p)^{\oplus\binom{m+1}{p}}) \to H^{q-1}(\Omega^{p-1}(k)) \to H^q(\Omega^p(k)) \to H^q(\mathcal{O}(k-p)^{\oplus\binom{m+1}{p}}) \\ \to H^q(\Omega^{p-1}(k)) \to H^{q+1}(\Omega^p(k)) \to \cdots \end{aligned} \tag{3.4}$$

We claim that $H^q\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-p)^{\oplus\binom{m+1}{p}}\right) = 0$. Indeed, this follows from the fact that $q > 0$, and $k - p \geq -m$ if $q = m$. Therefore, using the exact sequence (3.4), we conclude that the canonical map

$$H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) \to H^q(\mathbb{P}, \Omega^p(k)) \tag{3.5}$$

is surjective.

We claim that we may assume that $q > p$. To see this, suppose that $q = p$. If $q = p \geq 2$, then the induction hypothesis implies that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$ (since $k \neq 0$), hence by the surjection (3.5), we have $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in this case. Thus, suppose that $p = q = 1$. In this case, we have $k \neq 0$, and we want to show that $H^1(\mathbb{P}, \Omega^1(k)) = H^{m-1}(\mathbb{P}, \Omega^{m-1}(-k))^\vee = 0$.

To prove this, we proceed by induction on m. Suppose first that $m = 1 = p = q$. Then $k > 0$, and hence $H^1(\Omega^1(k)) = H^{m-1}(\Omega^{m-1}(-k))^\vee = H^0(\Omega^0(-k))^\vee = 0$. Next, assume $m \geq 2$. Then there are two cases to distinguish: $k > 0$ and $k < 0$. If $k < 0$, then the surjection (3.5) implies that $H^1(\Omega^1(k)) = 0$. Thus, assume that $k > 0$. We need to show that $H^{m-1}(\Omega^{m-1}(-k)) = 0$ for $k > 0$. We obtain a long exact sequence

$$\cdots \to H^{m-2}(\Omega^{m-2}(-k)) \to H^{m-1}(\Omega^{m-1}(-k)) \to H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}}) \to \cdots$$

The group $H^{m-2}(\Omega^{m-2}(-k))$ is zero by induction, and $H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}})$ vanishes as well, as $m \geq 2$. Therefore, $H^{m-1}(\Omega^{m-1}(-k)) = 0$ as desired.

By the above claim, we may assume $q > p \geq 1$. We can then apply the induction hypothesis (recall that we are still arguing by induction on p) to see that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$. Indeed, we have $0 < q - 1 < m$. Therefore, the surjection (3.5) implies that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$, and we are done. $\square$

Exercise 3.1.6. Show that the non-zero twisted Hodge numbers $h^q(\Omega^p(k))$ are:

(a) $h^p(\Omega^p) = 1$,

(b) $h^0(\Omega^p(k)) = \binom{m+k-p}{k} \cdot \binom{k-1}{p}$ if $k > p$,

(c) $h^m(\Omega^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{m-p}$ if $k < p - m$.

Exercise 3.1.7. Consider the projective space $\mathbb{P} = \mathbb{P}_{\mathbb{C}}^m$ of dimension m over $\mathbb{C}$. By Theorem 3.1.5, we have $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^p) = 0$ for $p > 0$. Show directly that there are no non-zero holomorphic one-forms on $\mathbb{P}^1(\mathbb{C})$.

3.1.2 Kähler differentials on hypersurfaces

Lemma 3.1.8. *Let $X \subset \mathbb{P}$ be a smooth hypersurface of degree $d > 0$. For each $k \in \mathbb{Z}$, there are canonical exact sequences*

$$0 \to \Omega_{\mathbb{P}}^p(k-d) \to \Omega_{\mathbb{P}}^p(k) \to \Omega_{\mathbb{P}}^p|_X(k) \to 0, \tag{3.6}$$

$$0 \to \mathcal{O}_X(k-d) \to \Omega_{\mathbb{P}}|_X(k) \to \Omega_X(k) \to 0, \tag{3.7}$$

$$0 \to \Omega_X^{p-1}(k-d) \to \Omega_{\mathbb{P}}^p|_X(k) \to \Omega_X^p(k) \to 0. \tag{3.8}$$

Proof. It suffices to take $k = 0$.

To prove (3.6), we may take $p = 0$. In this case, the result follows from the following exact sequence, where i denotes the inclusion $X \hookrightarrow \mathbb{P}$:

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to i_* \mathcal{O}_X \to 0. \tag{3.9}$$

One obtains (3.9) via the identification $\mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{I}_X$, where the latter denotes the ideal sheaf of $X \subset \mathbb{P}$, resulting from the isomorphisms $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}}(-X)$ (see [Har77, II, Proposition 6.18]) and $\mathcal{O}_{\mathbb{P}}(X) \cong \mathcal{O}_{\mathbb{P}}(d)$ (which holds because $\deg(X) = d$). Note by the way that (3.9) corresponds to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \xrightarrow{1 \mapsto F} \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_X(d) \to 0,$$

where $F \in \mathcal{O}_{\mathbb{P}}(d) = k[x_0, \ldots, x_{n_1}]_d$ is a polynomial that defines X.

To obtain the exact sequence (3.7), one combines the conormal exact sequence

$$0 \to \mathcal{N}^\vee_{Z/Y} \to \Omega_Y|_Z \to \Omega_Z \to 0$$

for any smooth hypersurface $i \colon Z \hookrightarrow Y$ in a smooth variety Y, where $\mathcal{N}^\vee_{Z/Y}$ is a sheaf on Z such that $i_* \mathcal{N}^\vee_{Z/Y} \cong I/I^2$ (see [Har77, II, Theorem 8.17]), and the canonical isomorphism

$$\mathcal{N}^\vee_{X/\mathbb{P}} = i^* \mathcal{O}_{\mathbb{P}}(-d) = \mathcal{O}_X(-d). \tag{3.10}$$

The second isomorphism in (3.10) being clear, it suffices to prove $\mathcal{N}^\vee_{X/\mathbb{P}} = i^* \mathcal{O}_{\mathbb{P}}(-d)$. This is again a general statement: if $i \colon Z \to Y$ is a closed immersion of schemes, then $i^* \mathcal{I}_Z$ has the property that $i_* i^* \mathcal{I}_Z = \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Y / \mathcal{I}_Z = \mathcal{I}_Z / \mathcal{I}_Z^2$ (to see this, reduce to the case where Y affine, where this is clear).

Finally, note that (3.8) follows from (3.7) together with Lemma 3.1.1. $\square$

Proposition 3.1.9. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$ with canonical bundle ω_X. Then $\omega_X \cong \mathcal{O}_X(d - n - 2)$. In particular,*

(1) ω_X is ample if $d > n + 2$;

(2) $\omega_X \cong \mathcal{O}_X$ if $d = n + 2$;

(3) ω_X^* is ample if $d < n + 2$.

Proof. Consider sequence (3.8) with $p = n + 1 = m$ and $k = d$. This gives

$$\omega_X \cong \omega_{\mathbb{P}}|_X(d) \cong \mathcal{O}_{\mathbb{P}}(-m-1)|_X \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(d - m - 1).$$

The remaining statement follow directly. $\square$

We proceed to prove:

Theorem 3.1.10. *Let $X \subset \mathbb{P} = \mathbb{P}^m = \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$. Then the following holds.*

(1) *Let $k \in \mathbb{Z}$ with $k < d$. The natural map*

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p(k))$$

is bijective for $p + q < n$ and injective for $p + q \leq n$.

(2) *We have*

$$H^q(X, \Omega^p(k - d)) = 0 \quad \textit{for} \quad p + q < n \quad \textit{and} \quad k < d. \tag{3.11}$$

(3) *We have $H^q(X, \Omega^p(k)) = 0$ for $(p + q < n, k < 0)$ and for $(p + q > n, k > 0)$.*

Proof. Throughout the proof, we will use Theorem 3.1.5 without mention. We first prove 1 and 2 by induction on p. Therefore, assume that $k < d$.

Suppose first that $p = 0$. Then (3.6) yields the following exact sequence:

$$H^q(\mathcal{O}_{\mathbb{P}}(k-d)) \longrightarrow H^q(\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^q(\mathcal{O}_X(k)) \longrightarrow H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d))$$

For $q \leq n < m$, we have $H^q(\mathcal{O}_{\mathbb{P}}(k - d)) = 0$ because $k - d < 0$. Thus, $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is injective for $q \leq n$ and $k - d < 0$. Moreover, if $q < n$ then $q+1 \leq n < m$, hence $H^{q+1}(\mathcal{O}_{\mathbb{P}}(k - d)) = 0$ for $q < n$ and $k - d < 0$. This implies that $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is bijective $q < n$ and $k - d < 0$. In particular,

$$H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k - d)) = H^q(X, \mathcal{O}_X(k - d)) = 0 \qquad \text{for} \quad (q < n, k - d < 0).$$

This proves that 1 and 2 hold whenever $p = 0$.

Next, let $p > 0$. Notice that in this case, $p + q \leq n$ implies $q < n$. Similarly, $p + q < n$ implies $q < n - 1$. Notice also that (3.7) and (3.8) yield the following diagram, in which the rows are exact:

$$\begin{tikzcd}[column sep=small]
H^q(\Omega^p_{\mathbb{P}}(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}(k)) \arrow[r, "f(p,q)"] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r] \arrow[d, equal] & H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) & \\
& H^q(\Omega^{p-1}_X(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r, "g(p,q)"] & H^q(\Omega^p_X(k)) \arrow[r] & H^{q+1}(\Omega^{p-1}_X(k-d)).
\end{tikzcd}$$

If $p + q \leq n < m$, then $q < m$ hence $H^q(\Omega^p_{\mathbb{P}}(k - d)) = 0$ as $k - d < 0$. This implies that $f(p, q)$ is injective if $p + q \leq n$. Moreover, if $p + q \leq n < m$ then $(p - 1) + q < n$, hence $H^q(\Omega^{p-1}_X(k - d)) = 0$ by the induction hypothesis, as $k - d < 0$. Therefore, the maps $f(p, q)$ and $g(p, q)$ in the above diagram are both injective if $p > 0$ and $p + q \leq n$. This implies that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is injective for all $p, q \geq 0$ such that $p + q \leq n$.

Still assume $p > 0$. If $p + q < n$, then $q < n$, hence $q + 1 < n + 1 = m$. Therefore, $H^{q+1}(\Omega^p_{\mathbb{P}}(k - d)) = 0$ as $k - d < 0$. Moreover, if $p + q < n$ then $(p - 1) + (q + 1) < n$, hence $H^{q+1}(X, \Omega^{p-1}_X(k - d)) = 0$ by induction, as $k - d < 0$. Therefore, the maps

$f(p,q)$ and $g(p,q)$ in the above diagram are both bijective if $p>0$ and $p+q<n$. We conclude that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is bijective for all $p, q \geq 0$ such that $p+q<n$.

Continue to assume that $k<d$. Let $p, q \geq 0$ such that $p+q<n$. By what we have already proved, we have $H^q(X, \Omega^p_X(k-d)) = H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k-d))$, and this is zero because $q<m$ and $k-d<0$.

It remains to prove assertion 3. Notice that (3.11) implies $H^q(X, \Omega^p_X(k)) = 0$ for $(p+q<n, k<0)$. This also implies, via Corollary 3.1.4, that

$$H^q(X, \Omega^p(k)) \cong H^{n-q}(X, \Omega^{n-p}(-k))^\vee = 0 \quad \text{if} \quad (p+q>n, k>0).$$

This finishes the proof of the theorem. $\square$

Corollary 3.1.11. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d . If $n>2$, then* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$. *Similarly, for $n=2$ and $d \leq 3$, one has* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$.

Proof. Consider the *exponential exact sequence* of abelian sheaves on $X(\mathbb{C}) = X^{an}$:

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2i\pi} \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}^*_{X^{an}} \to 0.$$

Taking cohomology gives an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^*_X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X). \tag{3.12}$$

As $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X)$, see Exercise 3.1.13 below, it suffices to prove the following:

Claim 3.1.12. *If $n>2$ or $n=2$ and $d \leq 3$, then $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

On the one hand, by Theorem 3.1.5, we have $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = H^2(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = 0$ for $n>1$. On the other hand, by Theorem 3.1.10, we see that if $n>1$, then $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^1(X, \mathcal{O}_X)$ and if $n>2$ then $H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^2(X, \mathcal{O}_X)$. Therefore, for $n>1$, we have $H^1(X, \mathcal{O}_X) = 0$ and for $n>2$, we have $H^2(X, \mathcal{O}_X) = 0$.

By Corollary 3.1.4, we have $h^i(X, \mathcal{O}_X) = h^{n-i}(X, \omega_X))$, and by Proposition 3.1.9, we have $h^{n-i}(X, \omega_X) = h^{n-i}(X, \mathcal{O}_X(d-(n+2))$. Thus, for $n=2$, this gives

$$h^i(X, \mathcal{O}_X) = h^{2-i}(X, \mathcal{O}_X(d-4)) = 0 \quad \text{for} \quad i \in \{1,2\} \quad \text{and} \quad d \leq 3.$$

This proves the claim, and thereby the corollary. $\square$

Exercise 3.1.13. Sketch a proof of the fact that, for a locally ringed space $(X, \mathcal{O}_X)$, there is a natural isomorphism $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X)$. Use this to conclude that if X is a smooth projective variety over $\mathbb{C}$, then $H^1(X, \mathcal{O}^*_X) = H^1(X^{an}, \mathcal{O}^*_{X^{an}})$. Give an example of a sheaf $\mathcal{F}$ on a smooth projective variety X over $\mathbb{C}$ such that the natural map $H^1(X, \mathcal{F}) \to H^1(X^{an}, \mathcal{F}^{an})$ is not an isomorphism.

Exercise 3.1.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n and degree d. Provide all (n, d) for which the homomorphism $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Analyze the group which measures the possible failure of the injectivity of c_1.

Exercise 3.1.15. Consider a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$. Let $C \subset S$ be a curve contained in S. Prove that

$$[C] = c_1(\mathcal{O}_S(k)) \in H^2(S, \mathbb{Z})$$

if and only if there exists a hypersurface $Y \subset \mathbb{P}^3_{\mathbb{C}}$ of degree k such that $C = Y \cap S$.

3.2 Lecture 2: Lefschetz hyperplane theorem

To prove the Lefschetz hyperplane theorem, we will need some Morse theory. Let M be a smooth manifold of dimension n. Let $f \colon M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Then $0 \in M$ is called a *critical point* if $(df)_0 = 0$ as maps $T_0M \to T_{f(0)}\mathbb{R}$; in this case $f(0)$ is called a *critical value*. Consider the bilinear map

$$\mathrm{Hess}(f)_0 = (d^2 f)_0 \colon T_0M \times T_0M \to \mathbb{R}$$

defined as follows. Choose coordinates $x_1, \dots, x_n$ on M centred around 0, and put

$$\mathrm{Hess}(f)_0 \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Lemma 3.2.1. *Show that the function $Hess(f)_0$ does not depend on the choice of coordinates around 0. Show that $Hess(f)_0$ defines a symmetric bilinear form on T_0M.*

Proof. Exercise. □

We say that a critical point $0 \in M$ is *non-degenerate* if $\mathrm{Hess}(f)_0$ is non-degenerate. By Lemma 3.2.1, if $0 \in M$ is a non-degenerate critical point, then $\mathrm{Hess}(f)_0$ defines a non-degenerate quadratic form, which can be diagonalized; define $\lambda_0(f)$ as the number of negative eigenvalues in this case. The Morse lemma, see [Mil63, Lemma 2.2], states that in suitable local coordinates $x_1, \dots, x_n$ around a non-degenerate critical point $0 \in M$ of $f \colon M \to \mathbb{R}$, the function f can be written as the quadratic function

$$f(x) = f(0) - \sum_{i=1}^{\lambda_0(f)} x_i^2 + \sum_{i=\lambda_0(f)+1}^{n} x_i^2.$$

In particular, non-degenerate critical points (resp. values) are isolated in M (resp. $\mathbb{R}$).

We call f a *Morse function* if $f^{-1}(-\infty, a] \subset M$ is compact for each $a \in \mathbb{R}$, and f each critical point of f is non-degenerate. If f is a Morse function, then f is proper and its fibres $M_a = f^{-1}(a)$ are compact. Moreover, each critical value corresponds to a finite number of critical points, and the set of critical values is discrete in $\mathbb{R}$. In particular, for each $a \in \mathbb{R}$, there exist only finitely many critical values in $(-\infty, a] \subset \mathbb{R}$.

The basic theorem of Morse theory [Mil63, Theorem 3.5] says that if $f\colon M \to \mathbb{R}$ is a Morse function, then M has the homotopy type of a CW complex with one cell of dimension $\lambda_p(f)$ for each critical point $p \in M$.

Assume $M \subset \mathbb{R}^N$ is a closed submanifold of dimension n. By [Mil63, Theorem 6.6], for almost all (all but a set of measure 0) points $p \in \mathbb{R}$, the distance function

$$\varphi_p\colon M \to \mathbb{R}, \quad \varphi_p(x) = \|x - p\|^2$$

is a Morse function. We are now ready to prove:

Theorem 3.2.2 (Andreotti–Frankel [AF59])**.** *A closed n-dimensional complex submanifold $X \subset \mathbb{C}^r$ has the homotopy type of a CW complex of dimension $\leq n$.*

Proof. Let $c \in \mathbb{C}^r$ be a point such that the distance function $\varphi_c\colon X \to \mathbb{R}$ has only non-degenerate critical points.

Claim $(*)$. Let $p \in X$ be a critical point of $\varphi_c\colon X \to \mathbb{R}$. Then $\lambda_p(\varphi_c) \leq n$.

Before we prove Claim $(*)$, we will show that it implies the theorem. Indeed, by the basic theorem of Morse theory, X has the homotopy type of a CW complex with one cell of dimension $\lambda_p(\varphi_c)$ for each critical point $p \in M$ of φ_c. By Claim $(*)$, we have $\lambda_p(\varphi_c) \leq n$ for each critical point $p \in M$. Hence X has the homotopy type of a CW complex with one cell of dimension $\leq n$ for each critical point $p \in M$ of φ_c.

It remains to prove Claim $(*)$. We need:

Claim 3.2.3. *There exist local holomorphic coordinates on $\mathbb{C}^r$ such that $p = 0$, $c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(n+1)$-st position, and such that there exist open neighborhoods $0 \in V_1 \subset \mathbb{C}^n$ and $0 \in V_2 \subset \mathbb{C}^{r-n}$ and a holomorphic function*

$$\mathbb{C}^n \supset V_1 \xrightarrow{f} V_2 \subset \mathbb{C}^{r-n}$$

with $M \cap (V_1 \times V_2) = \mathrm{Graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^{r-n}$, and such that $df_0 = 0$.

Proof of Claim 3.2.3. Applying a suitable change of coordinates of $\mathbb{C}^r$, we may assume that $p = 0 \in M \subset \mathbb{C}^r$. As $M \subset \mathbb{C}^r$ is a closed submanifold, there exists an open subset $U \subset \mathbb{C}^r$ containing $p = 0$, and holomorphic functions $g_1, \ldots, g_m\colon U \to \mathbb{C}$ such that $X \cap U = \{g_1 = \cdots = g_m = 0\} \subset \mathbb{C}^r$. This gives a holomorphic function $g = (g_1, \ldots, g_m)\colon U \to \mathbb{C}^m$ such that $X \cap U = g^{-1}(0) = \{g = 0\} \subset U$. Thus, the fibre $g^{-1}(0)$ is smooth, which implies that g has maximal rank at each point of $X = g^{-1}(0)$. Applying the implicit function theorem, we obtain a holomorphic function $f\colon V_1 \to V_2 \subset \mathbb{C}^{r-n}$ defined on a open neighborhood $V_1 \subset \mathbb{C}^n$ of 0, such that $f(0) = 0$, $V_1 \times V_2 \subset U$ and such that

$$X \cap V_1 \times V_2 = \{(x, f(x)) \mid x \in V_1\} \subset V_1 \times V_2 \subset \mathbb{C}^n \times \mathbb{C}^{r-n} = \mathbb{C}^r.$$

Now $0 \neq c \in \mathbb{C}^r$, hence c defines a basis element of $\mathbb{C}^r$, so that there exists a matrix $\alpha \in \mathrm{GL}_r(\mathbb{C})$ with $\alpha \cdot c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $(n+1)$-st position. As α is linear, we have $\alpha \cdot 0 = 0$. Finally, we claim that $df_0 = 0$. This follows from the fact that $\varphi_c\colon X \to \mathbb{R}$ is a distance function, with critical point $p = 0$. In other words, $(d\varphi_c)_0 = 0$, because $\varphi_c(x, f(x)) = \|(x, f(x)) - (0, 0, \ldots, 0, 1, 0, \ldots, 0)\|^2$. $\square$

As $|z-1|^2 = |x+iy-1|^2 = (x-1)^2+y^2 = (x^2+y^2)+(1-2x) = |z|^2+(1-2\cdot\Re(z))$, the distance function is now given by the formula

$$\varphi_c(z) = 1 - 2\cdot\Re(f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2 \,. \tag{3.13}$$

As $\mathrm{ord}_0(f_i) \geq 2$ for all i, the last sum in (3.13) does not contribute to $\mathrm{Hess}(\varphi_c)_0$. Write

$$f_1(z) = Q(z) + \text{terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \dots, z_n$. We obtain:

$$\mathrm{Hess}(\varphi_c)_0 = -2\cdot \mathrm{Hess}\left(\Re(Q(z))\right)_0 + 2\cdot \mathrm{Id}.$$

We claim that $\mathrm{Hess}(\Re(Q(z)))_0$ has at most n positive and at most n negative eigenvalues. Indeed, after a change of coordinates $z \mapsto w$, we can write

$$Q(w) = w_1^2 + \cdots + w_s^2, \qquad s \leq n;$$

writing $w_j = x_j + i\cdot y_j$, we obtain

$$\Re\left(Q(w)\right) = (x_1^2 - y_1^2) + \cdots + (x_s^2 - y_s^2).$$

This finishes the proof of Claim $(*)$, and thereby the proof of Theorem 3.2.2. $\square$

As a corollary, we obtain:

Theorem 3.2.4. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$. Let $Y = X \cap H$ be a hyperplane section such that $U := X \setminus Y$ is smooth of dimension n and let $j\colon Y \hookrightarrow X$ denote the canonical inclusion. The restriction map*

$$j^*\colon H^i(X,\mathbb{Z}) \to H^i(Y,\mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

Proof. For the proof, we need the following:

Claim 3.2.5. *We have a natural isomorphism $H^i(X,Y,\mathbb{Z}) \cong H_{2n-i}(U,\mathbb{Z})$.*

Assuming the claim, we obtain a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{i+1}(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{2n-i}(U,\mathbb{Z}) \arrow[r] & H^i(X,\mathbb{Z}) \arrow[r] & H^i(Y,\mathbb{Z}) \arrow[r] & H_{2n-i-1}(U,\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, to prove the theorem, we must show that $H_{2n-i}(U,\mathbb{Z}) = 0$ for $i \leq n-1$. As $i \leq n-1$ if and only if $2n-i \geq 2n-n+1 = n+1$, we need to prove that $H_i(U,\mathbb{Z}) = 0$ for $i \geq n+1$. Note that $U = X \setminus Y \subset \mathbb{P}^N \setminus H \cong \mathbb{A}^N_{\mathbb{C}}$ defines a closed submanifold

$U(\mathbb{C}) \subset \mathbb{C}^N$ of dimension n. By Theorem 3.2.2, $U(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\leq n$. In particular, $H_i(U, \mathbb{Z}) = 0$ for $i \geq n+1$, and Theorem 3.2.4 follows.

It remains to prove Claim 3.2.5; for this, we follow the exposition in [Voi02, page 306]. We admit the fact that Y admits a fundamental system of open neighborhoods $Y \subset Y_k \subset X$ that admit a deformation retract onto Y. It follows that the natural map

$$\varinjlim H^i(X, Y_k, \mathbb{Z}) \to H^i(X, Y, \mathbb{Z})$$

is an isomorphism. By excision, we have

$$H^i(X, Y_k, \mathbb{Z}) \cong H^i(U, U \cap Y_k, \mathbb{Z}).$$

If $K \subset U$ is a compact subset such that K is the deformation retract of an open subset $K \subset V \subset U$, then $H^i(U, U \setminus K, \mathbb{Z}) \cong H_{2n-i}(K, \mathbb{Z})$ (this is a refined version of Poincaré duality, see [Spa81, Section 6.2]). Applying this to

$$K_k := U \setminus (Y_k \cap U) = X \setminus Y_k,$$

which is a closed, hence compact, subset of X which admits a deformation retract of $X \setminus Y = U \supset K_k$, we obtain

$$H^i(U, Y_k \cap U, \mathbb{Z}) = H^i(U, U \setminus K_k, \mathbb{Z}) \cong H_{2n-i}(K_k, \mathbb{Z}).$$

As every singular chain on U is contained in one of the compact subsets $K_k \subset U$, the natural map $\varinjlim_k H_{2n-i}(K_k, \mathbb{Z}) \to H_{2n-i}(U, \mathbb{Z})$ is an isomorphism, and hence

$$H^i(X, Y, \mathbb{Z}) \cong \varinjlim H^i(U, U \cap Y_k, \mathbb{Z}) \cong \varinjlim H_{2n-i}(K_k, \mathbb{Z}) \cong H_{2n-i}(U, \mathbb{Z}),$$

proving Claim 3.2.5. $\square$

Remark 3.2.6. For a compact oriented n-dimensional manifold M, and a closed submanifold $N \subset M$, cup-product with the fundamental class defines an isomorphism $H^i(M, N, \mathbb{Z}) \cong H_{n-i}(M \setminus N, \mathbb{Z})$. This is relative Poincaré duality, cf. [Dol95, Section 7]. In particular, if $X \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective variety of dimension n and $Y = X \cap H$ a smooth hyperplane section, then it readily follows that $H^i(X, Y, \mathbb{Z}) \cong H_{2n-i}(X \setminus Y, \mathbb{Z})$.

Corollary 3.2.7. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a hypersurface.*

(1) The restriction map $H^i(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < n$. In particular, $H^i(X, \mathbb{Z}) = 0$ for i odd and $i < n$, and $H^{2i}(X, \mathbb{Z}) = \mathbb{Z} \cdot h^i$ for $2i < n$.

(2) Suppose X is smooth. Then $H^i(X, \mathbb{Z}) = 0$ for $i > n$ odd, and for each $j > n$ there is a unique $\alpha_{2j} \in H^{2j}(X, \mathbb{Z})$ such that $H^{2j}(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha_{2j}$ and $\alpha_{2j} \cup h^{n-j} = 1$.

Proof. Let d be the degree of X, and consider the d-th Veronese embedding $\mathbb{P}^{n+1}_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$. Then $X = \mathbb{P}^{n+1}_{\mathbb{C}} \cap H$ for a hyperplane $H \subset \mathbb{P}^N_{\mathbb{C}}$. Apply Theorem 3.2.4 to obtain the first assertion. The second assertion follows from the first via Poincaré duality. $\square$

Corollary 3.2.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then the restriction maps $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ and $\operatorname{Pic}(\mathbb{P}^{n+1}) \to \operatorname{Pic}(X)$ are isomorphisms. In particular, $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.*

Proof. The fact $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism is immediate from Theorem 3.2.4. From this, together with the commutative diagram

$$\begin{tikzcd} \operatorname{Pic}(\mathbb{P}^{n+1}) \arrow[r] \arrow[d] & \operatorname{Pic}(X) \arrow[d] \\ H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \arrow[r] & H^2(X, \mathbb{Z}), \end{tikzcd}$$

we deduce that $\operatorname{Pic}(\mathbb{P}^{n+1}) \to \operatorname{Pic}(X)$ is also an isomorphism, as the restriction map $\operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism by Corollary 3.1.11. □

Corollary 3.2.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p + q > n$ and $L \in \operatorname{Pic}(X)$ ample.* □

Remark 3.2.10. Later we will see that Corollary 3.2.9 remains valid for hypersurfaces over arbitrary fields k. Namely, if $X \subset \mathbb{P}^{n+1}_k$ is a smooth hypersurface of degree d, and if $n > 2$, then $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. See [refer to future section].

Exercise 3.2.11. Provide the equation of a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$ such that $\operatorname{Pic}(S) \ncong \mathbb{Z}$. See also Exercise 3.1.15. Define $V = H^0(\mathbb{P}^3, \mathcal{O}(d))$ and let $\mathbb{P}(V)$ be its projectivization. Let $\mathbb{P}(V)_0 \subset \mathbb{P}(V)$ be the locus of classes $[F] \in \mathbb{P}(V)$ such that $S_F := \{F = 0\}$ is smooth. Show that $\mathbb{P}(V)_0$ is Zariski open in the projective space $\mathbb{P}(V)$. Show that the locus of $[F] \in \mathbb{P}(V)_0$ such that $\operatorname{Pic}(S_F) \ncong \mathbb{Z}$ is a countable union $\mathscr{H} = \cup_n Z_n$ of closed algebraic subvarieties $Z_n \subset \mathbb{P}(V_0)$. Show that $\mathscr{H} \neq \mathbb{P}(V_0)$.

Exercise 3.2.12. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n. Suppose that $n \geq 2$. Show that X is simply connected.

Exercise 3.2.13. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of degree $d = 3$. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of arbitrary degree $d \geq 4$.

Exercise 3.2.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of dimension $n \geq 2$. Let $C \subset X \subset \mathbb{P}^{n+1}$ be a smooth curve contained in X, and consider the Gysin map

$$\varphi \colon \mathbb{Z} = H^0(C, \mathbb{Z}) \cong H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \cong H^{2n-2}(X, \mathbb{Z}).$$

Define $[C] = \varphi(1) \in H^{2n-2}(X, \mathbb{Z})$. Consider the class $\alpha_{2n-2} \in H^{2n-2}(X, \mathbb{Z})$, see Corollary 3.2.7. Show that $[C] = \alpha_{2n-2}$ if and only if C intersects a general hyperplane $H \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ in a unique point with multiplicity one. Given equations for a cubic surface $X = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and a curve $C = \{F = G = 0\} \subset X \subset \mathbb{P}^3_{\mathbb{C}}$ such that $[C] = \alpha_2$.

3.3 Lecture 3: Betti numbers of hypersurfaces

Convention 3.3.1. We assume all topological manifolds to be second-countable and Hausdorff. In particular, they are paracompact and admit partitions of unity subordinate to any open cover.

3.3.1 Chern classes in topology

Let X be a topological manifold. Let $E \to X$ be a complex vector bundle of rank r. We would like to define the *Chern classes*

$$c_i(E) \in E^{2i}(X, \mathbb{Z}), \quad 1 \leq i \leq r$$

of X. We put $c_0(E) = 1$ and $c_i(E) = 0$ for $i > r = \mathrm{rank}(E)$, and introduce the *Chern polynomial*

$$c(E) = \sum_{i=0}^{r} c_i(E) \cdot t^i \in H^\bullet(X, \mathbb{Z})[t]$$

whose coefficients we shall now define. Consider the exponential exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2i\pi} \mathscr{C}^0 \xrightarrow{\exp} (\mathscr{C}^0)^* \to 0, \tag{3.14}$$

where $\mathscr{C}^0$ is the sheaf of continuous complex-valued functions on X, and $(\mathscr{C}^0)^*$ the subsheaf of invertible functions. The sequence (3.14) defines a morphism

$$c_1 \colon \{\text{complex line bundles } L \text{ on } X\} /_{\cong} = H^1(X, (\mathscr{C}^0)^*) \to H^2(X, \mathbb{Z}). \tag{3.15}$$

In particular, if E is a vector bundle of rank $r = 1$ on X, we obtain an element $c_1(E) \in H^2(X, \mathbb{Z})$ such that $c_1(E) = c_1(E')$ if $E \cong E'$ as vector bundles on X.

Lemma 3.3.2. *Let X be a topological space and $E \to X$ a vector bundle of rank r on X. Let $\psi \colon \mathbb{P}(E) \to X$ be the associated projective bundle. Let $S \subset \psi^*E$ be the tautological line bundle, and define $h = c_1(S^*) \in H^2(\mathbb{P}(E), \mathbb{Z})$. Then $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(X, \mathbb{Z})$, with basis $1, h, \dots, h^{r-1}$.*

Proof. This follows from the Leray–Hirsch theorem (see [Hat02, Theorem 4D.1]). ☐

Lemma 3.3.3. *Let X be a topological manifold. Let $E \to X$ be a complex vector bundle on X. Then E admits a hermitian metric $E \times E \to \mathbb{C}$.*

Proof. Exercise. ☐

Theorem 3.3.4. *Let X be a topological manifold, and let $K(X)$ be the set of isomorphism classes of complex vector bundles of finite rank on X. There exists a unique function*

$$c_t \colon VB(X) \to H^\bullet(X, \mathbb{Z})[t], \quad E \mapsto c_t(E) = \sum_i c_i(E) \cdot t^i,$$

such that $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $E \in VB(X)$, $c_0(E) = 1$, $c_i(E) = 0$ for $i > r =$ $\mathrm{rank}(E)$, and such that the following conditions are satisfied:

(1) *(Compatibility with (3.15).) If $r = \mathrm{rank}(E) = 1$, then $c_t(E) = 1 + c_1(E) \cdot t$.*

(2) *(Functoriality.) If $\phi\colon Y \to X$ is continuous, then $c_t(\phi^*(E)) = \phi^*(c_t(E))$ for $E \in VB(X)$, where $\phi^*\colon H^\bullet(X, \mathbb{Z}) \to H^\bullet(Y, \mathbb{Z})$ is the pull-back of ϕ.*

(3) *(Turning exact sequences into products.) If $0 \to F \to E \to G \to 0$ is an exact sequence, then $c_t(E) = c_t(F) \cdot c_t(G)$.*

Proof of uniqueness. This follows readily from the following:

Claim 3.3.5. *Let $E \to X$ be a complex vector bundle. There exists a topological manifold Y and a continuous map $\phi\colon Y \to X$ such that $\phi^*\colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ is injective for each i, and such that $\phi^* E$ is a direct sum of line bundles.*

To prove the claim, consider the projective bundle $\psi\colon \mathbb{P}(E) \to X$. The morphism $\psi^*\colon H^*(X, \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z})$ turns $H^*(\mathbb{P}(E), \mathbb{Z})$ into a free module over $H^*(X, \mathbb{Z})$, see Lemma 3.3.2. In particular, ψ^* is injective. Consider the tautological line bundle $S \subset \psi^*(E)$; it has fibre $S_x = \Delta_x \subset E_x$ above the point $x = [\Delta_x] \in \mathbb{P}(E_x)$ corresponding to a line $\Delta_x \subset E_x$. Put a hermitian metric h on $\psi^*(E)$ (cf. Lemma 3.3.3) and define Q as the orthogonal complement of S with respect to h; then $\psi^*(E) \cong S \oplus Q$. By induction on the rank of E, the claim follows.

To see why uniqueness follows, let $\phi\colon Y \to X$ as in the claim. We obtain an isomorphism $\phi^*(E) \cong L_1 \oplus \cdots \oplus L_n$ for some line bundles L_i on Y. Suppose that

$$c_t(E) = 1 + c_1(E) \cdot t + c_2(E) \cdot t^2 + \cdots + c_r(E) \cdot t^r = \sum_{i=0}^{r} c_i(E) \cdot t^i.$$

Then

$$\sum_{i=0}^{r} \phi^*\left(c_i(E)\right) \cdot t^i = \phi^*\left(c_t(E)\right) = c_t\left(\phi^*(E)\right) = c_t\left(L_1 \oplus \cdots \oplus L_n\right) = \prod_{i=1}^{n} \left(1 + c_1(L_i) \cdot t\right).$$

Thus, if c_t' is another map $VB(X) \to H^\bullet(X, \mathbb{Z})[t]$ with the desired properties, then $\phi^*(c_i'(E)) = \phi^*(c_i(E))$ for each i; as ϕ^* is injective, we get $c_i(E) = c_i'(E)$ for each i. □

Proof of existence. Let $\psi\colon \mathbb{P}(E) \to X$ be the projective bundle associated to E, and let $S \subset \psi^*(E)$ be the tautological line bundle. Define $h = c_1(S^*) \in H^2(\mathbb{P}(E), \mathbb{Z})$. By Lemma 3.3.2, $H^*(\mathbb{P}(E), \mathbb{Z})$ is free as a module over $H^*(X, \mathbb{Z})$, and the elements $1, h, \ldots, h^{r-1}$ form a basis for $H^*(\mathbb{P}(E), \mathbb{Z})$ over $H^*(X, \mathbb{Z})$. Therefore, there are elements $a_i \in H^{2i}(X, \mathbb{Z})$ such that

$$h^r + \psi^*(a_1) \cdot h^{r-1} + \cdots + \psi^*(a_{r-1}) \cdot h + \phi^*(a_r) = 0 \qquad \text{in} \qquad H^{2r}(\mathbb{P}(E), \mathbb{Z}).$$

We put $c_0(E) = 1$, $c_i(E) = a_i$ for $1 \leq i \leq r$, and $c_i(E) = 0$ for $i > r$. We leave it to the reader to verify that conditions (1) – (3) are satisfied. □

Exercise 3.3.6. Let X be a topological manifold. Show that $H^1(X, \mathscr{C}^0) = H^2(X, \mathscr{C}^0) = 0$. Conclude that the morphism $c_1\colon H^1(X, (\mathscr{C}^0)^*) \to H^2(X, \mathbb{Z})$ is an isomorphism.

3.3.2 Hirzebruch–Riemann–Roch theorem

Let E be a vector bundle on a topological space X. Let $a_1, \ldots, a_r$ be the *formal Chern roots* of E. To be precise, we define them as formal symbols via the following formula:

$$c_t(E) = \sum_{i=0}^{r} c_i(E) \cdot t^i = \prod_{i=1}^{r} (1 + a_i \cdot t^i). \tag{3.16}$$

Thus, this means that the a_i are variables, subject to the following relations:

$$c_1(E) = \sum_{i=1}^{r} a_i, \quad c_2(E) = \sum_{1 \le i < j \le r} a_i \cdot a_j, \quad \ldots \quad , \quad c_r(E) = \prod_{i=1}^{r} a_i. \tag{3.17}$$

Define the *exponential Chern character* of E as the formal power series

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i}, \quad \text{where} \quad e^{a_i} = 1 + a_i + \frac{1}{2} a_i^2 + \cdots . \tag{3.18}$$

Similarly, define the *total Todd class* of E as the following formal power series, where the B_k are the Bernoulli numbers:

$$\begin{aligned} \mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}}, \quad \text{where} \quad \frac{a_i}{1 - e^{-a_i}} &= 1 + \frac{1}{2} a_i + \frac{1}{12} a_i^2 - \frac{1}{720} a_i^4 + \cdots \\ &= 1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_k}{(2k)!} t^{2k}. \end{aligned} \tag{3.19}$$

Lemma 3.3.7. *Let X be a topological manifold. Then (3.18) and (3.19) can be expressed as polynomials in the $c_i(E)$ with rational coefficients.*

Proof. Exercise. □

Let X be a topological manifold, and let E be a complex vector bundle on X. Define, for each i, the *i-th Chern character* and the *i-th Todd class* of E via the formulae

$$\begin{aligned} \mathrm{td}(E) &= \mathrm{td}_0(E) + \mathrm{td}_1(E) + \cdots , \qquad && \mathrm{td}_i(E) \in H^{2i}(X, \mathbb{Q}) \\ \mathrm{ch}(E) &= \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \ldots , \qquad && \mathrm{ch}_i(E) \in H^{2i}(X, \mathbb{Q}). \end{aligned}$$

For a complex manifold X of dimension n, with holomorphic tangent bundle $\mathcal{T}_X$, define the following invariants:

$$c_i(X) = c_i(\mathcal{T}_X), \qquad \mathrm{ch}_i(X) = \mathrm{ch}_i(\mathcal{T}_X) \qquad \text{and} \qquad \mathrm{td}_i(X) = \mathrm{td}_i(\mathcal{T}_X).$$

Moreover, if E is a holomorphic vector bundle on X, we put

$$\chi(X, E) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, E).$$

We then have the following fundamental result, whose proof we will omit.

Theorem 3.3.8 (Hirzebruch–Riemann–Roch). *Let E be a holomorphic vector bundle on a compact complex manifold X of dimension n. Consider the degree $2n$-component of $\mathrm{ch}(E) \cdot \mathrm{td}(X)$, defined as $(\mathrm{ch}(E) \cdot \mathrm{td}(X))_{2n} = \sum_{i=0}^{n} \mathrm{ch}_i(E)\mathrm{td}_{n-i}(X)$. Then*

$$\chi(X, E) = \int_X (\mathrm{ch}(E) \cdot \mathrm{td}(X))_{2n} \,.$$

Proof. See [BS58]. □

Exercise 3.3.9. Let E be a vector bundle Let E and F be vector bundles on a topological space X. Show that $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$, and that $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$. Show also that $c_i(E^\vee) = (-1)^i \cdot c_i(E)$ for each i.

Exercise 3.3.10. Let E be a holomorphic vector bundle on a complex compact manifold X. Deduce from Theorem 3.3.8 that $\chi(X, E)$ is independent of the holomorphic structure of E. In other words, prove that $\chi(X, E)$ depends only on the structure of E as a complex topological vector bundle.

3.3.3 Gauss–Bonnet formula

Let X be a compact complex manifold of dimension n. For integers $p, q \geq 0$, define $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$. The *Hirzebruch χ_y-genus* is the polynomial

$$\chi_y(X) = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) \cdot y^p. \tag{3.20}$$

The *Euler number* of X is defined as $e(X) = \sum_i (-1)^i b_i(X)$, where $b_i(X)$ is the i-th Betti number $b_i(X) = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ of X.

Corollary 3.3.11. *Let X be a compact Kähler manifold. Then $\chi_{y=-1}(X) = e(X)$.*

Proof. This will follow from Hodge theory, see Section ... Indeed, Hodge theory shows that $b_i(X) = \sum_{p=0}^{i} h^{p,i-p}(X)$. Therefore,

$$\chi_{y=-1}(X) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i \sum_{p+q=i} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i b_i(X),$$

proving the corollary. □

Corollary 3.3.12. *Let X be a compact complex manifold. Let $\gamma_1, \ldots, \gamma_n$ be the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$ of X, see (3.16). Then*

$$\chi_y(X) = \int_X \prod_{i=1}^{n} \left(1 + y \cdot e^{-\gamma_i}\right) \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

Proof. Exercise. □

Proposition 3.3.13. *Let X be a compact Kähler manifold of dimension n. Then*

$$e(X) = \int_X c_n(X).$$

Proof. By Corollary 3.3.11, we have $e(X) = \chi_{y=-1}(X)$, and by Corollary 3.3.12, we have $\chi_{y=-1}(X) = \int_X \prod_i \gamma_i$, where $\gamma_1, \ldots, \gamma_n$ are the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$. The proposition follows, as $\prod_i \gamma_i = c_n(X)$ by (3.17). □

3.3.4 Betti cohomology of smooth hypersurfaces

Let $X \subset \mathbb{P} = \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth complex hypersurface. Our goal now is to compute the middle Betti number $b_n(X) = \dim_{\mathbb{Q}} H^n(X, \mathbb{Q})$. Define the *Euler number of X* as follows:

$$e(X) = \sum_{i=0}^{2n} (-1)^i b_i(X) = \sum_{i=0, i \neq n}^{2n} (-1)^i b_i(X) + (-1)^n b_n(X).$$

Lemma 3.3.14. *Let $X \subset \mathbb{P} = \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth complex hypersurface. Then*

$$e(X) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n)\,. \tag{3.21}$$

Proof. By Corollary 3.2.7, we have $b_i(X) = 0$ for $i \neq n$ odd and $b_i(X) = 1$ for $i \neq n$ even. Hence

$$\begin{aligned} e(X) &= \sum_{i=0, 2i \neq n}^{n} (-1)^{2i} b_{2i}(X) + \sum_{i=1, 2i-1 \neq n}^{n} (-1)^{2i-1} b_i(X) + (-1)^n b_n(X) \\ &= \left(\sum_{i=0, 2i \neq n}^{n} 1 \right) + (-1)^n b_n(X) \\ &= \begin{cases} n + b_n(X) & \text{if } n \equiv 0(2), \\ n + 1 - b_n(X) & \text{if } n \equiv 1(2). \end{cases} \end{aligned}$$

This proves what we want. □

Proposition 3.3.15. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d and dimension $n \geq 0$. Let $b_n(X)$ be the n-th Betti number of X. Then*

$$b_n(X) = \frac{(-1)^n}{2d} \cdot \left(2 \cdot (1-d)^{n+2} + 3 \cdot d + (-1)^n \cdot d - 2\right). \tag{3.22}$$

Proof. See Section 3.3.1 above for an introduction to Chern classes. By Proposition 3.3.13, we have

$$e(X) = \int_X c_n(X), \quad \text{where} \quad c_n(X) = c_n(\mathcal{T}_X) \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Notice that sequence (3.2) yields an exact sequence

$$0 \to \Omega_{\mathbb{P}}|_X \to \mathcal{O}_X(-1)^{n+2} \to \mathcal{O}_X \to 0,$$

which, after dualizing, gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+2} \to \mathcal{T}_{\mathbb{P}}|_X \to 0.$$

We also consider the sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}|_X \to \mathcal{O}_X(d) \to 0,$$

that follows by dualizing (3.7). By item (3) in Theorem 3.3.4, we obtain

$$\begin{aligned} c(X) &= \sum_i c_i(X) = \sum_i c_i(\mathcal{T}_X) = c(\mathcal{T}_X) = c(\mathcal{T}_{\mathbb{P}}|_X) \cdot c(\mathcal{O}_X(d))^{-1} \\ &= c\left(\mathcal{O}_X(1)^{\oplus(n+2)}\right) \cdot c(\mathcal{O}_X(d))^{-1} = \frac{(1+h)^{n+2}}{(1+dh)}, \qquad h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z}). \end{aligned}$$

We now have the following:

Claim 3.3.16. *Let h be a variable, and consider the ring $R = \mathbb{Q}[h]/(h^{n+1})$. Then $(1+dh)$ is invertible in R hence $(1+dh)^{-1} \cdot (1+h)^{n+2}$ is a well-defined element in $\mathbb{Q}[h]/(h^{n+1})$. Moreover, its coefficient before h^n is $(1/d^2) \cdot ((1-d)^{n+2} + d \cdot (n+2) - 1)$.*

Proof. Exercise. □

By combining $\deg(h) = \int_X h^n = d$, equality (3.21) and Claim 3.3.16, we obtain:

$$e(X) = \frac{1}{d} \cdot \left((1-d)^{n+2} + d \cdot (n+2) - 1\right) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n).$$

In particular,

$$(-1)^n \cdot b_n(X) = \frac{2(1-d)^{n+2} + 2d \cdot (n+2) - 2 - 2nd - d + (-1)^n \cdot d}{d},$$

and equality (3.22) follows. □

Corollary 3.3.17. *The n-th Betti number $b_n(X_3)$ of a smooth cubic hypersurface $X_3 \subset \mathbb{P}^n_{\mathbb{C}}$ of dimension $n \geq 0$ is given by the following formula:*

$$b_n(X_3) = \frac{1}{6} \cdot \left(2^{n+3} + (-1)^n \cdot 7 + 3\right).$$

For instance, $b_0(X_3) = 3$, $b_1(X_3) = 2$, $b_2(X_3) = 7$ and $b_3(X_3) = 10$. □

Exercise 3.3.18. For a smooth projective variety X over $\mathbb{C}$, define $h^{p,q}(X) = h^q(X, \Omega^p_X)$. Calculate all the values $h^{p,q}(X)$ with $p+q = 3$ for a smooth cubic threefold $X \subset \mathbb{P}^4_{\mathbb{C}}$, and all the $h^{p,q}(X)$ with $p+q = 4$, for a smooth cubic fourfold $X \subset \mathbb{P}^5_{\mathbb{C}}$.

3.4 Lecture 4: Cohomology of hypersurfaces and bilinear forms

Let X be a compact complex manifold of dimension n. Poincaré duality provides canonical isomorphisms $H^i(X,\mathbb{Z}) \cong H_{2n-i}(X,\mathbb{Z})$. Moreover, the universal coefficient theorem provides a canonical isomorphism $H^i(X,\mathbb{Z})/(\text{tors}) \cong \text{Hom}(H_i(X,\mathbb{Z}),\mathbb{Z})$. Combining the two assertions, one sees that the cap product pairing

$$H_i(X,\mathbb{Z})/(\text{tors}) \otimes H_{2n-i}(X,\mathbb{Z})/(\text{tors}) \to H_0(X,\mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing. Dually, the cup product pairing

$$H^i(X,\mathbb{Z})/(\text{tors}) \otimes H^{2n-i}(X,\mathbb{Z})/(\text{tors}) \to H^{2n}(X,\mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing.

Lemma 3.4.1. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 0$. Then $H^n(X,\mathbb{Z})$ is torsion-free.*

Proof. For $n = 0$, the claim is trivial, so we may assume $n \geq 1$. The universal coefficient theorem gives then an exact sequence

$$0 = \text{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^n(X,\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Here, $\text{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) = 0$ because $H_{n-1}(X,\mathbb{Z}),\mathbb{Z})$ is trivial or isomorphic to $\mathbb{Z}$, see Corollary 3.2.7. As $\text{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z})$ is torsion-free, the lemma follows. □

In particular, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, we obtain a perfect pairing

$$\cup\colon H^n(X,\mathbb{Z}) \otimes H^n(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z}) = \mathbb{Z}. \tag{3.23}$$

Recall that, for $\alpha,\beta \in H^n(X,\mathbb{Z})/(\text{tors})$, we have $\alpha\cup\beta = (-1)^n\cdot\beta\cup\alpha$. This implies that (3.23) is symmetric if n is even, and alternating if n is odd. The goal of this section is to study (3.23) in case $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth cubic hypersurface of dimension n.

3.4.1 Odd-dimensional cubic hypersurfaces

It turns out that if X is an odd-dimensional hypersurface, the intersection form on $H^n(X,\mathbb{Z})$ is quite easily calculated, as follows from the following lemma.

Lemma 3.4.2. *Let Λ be a free $\mathbb{Z}$-module of rank $n > 0$ and let*

$$E\colon \Lambda \otimes \Lambda \to \mathbb{Z} \tag{3.24}$$

be an alternating bilinear form on $\mathbb{Z}$, defining a perfect pairing. Then $n = 2g$ and there exists a basis $\{e_1,\ldots,e_g; f_1,\ldots f_g\}$ for Λ such that $E(e_i,e_j) = E(f_i,f_j) = 0$ for all i,j, and such that $E(e_i,f_i) = 1$ for all i and $E(e_i,f_j) = 0$ if $i \neq j$.

Proof. Notice that $n = \text{rank}(\Lambda) \geq 2$, for if $n = 1$ then $E(x, y) = 0$ for each $x, y \in \Lambda$. Suppose first that $n = 2$. Let $\{x, y\} \subset \Lambda$ be a basis for Λ. Let $M = (m_{ij})$ be the intersection matrix of E with respect to this basis. We have $m_{11} = E(x, x) = 0$, $m_{12} = E(x, y)$, $m_{21} = -E(x, y)$ and $m_{22} = E(y, y) = 0$. Thus, the determinant of M equals $E(x, y)^2$, which must be invertible in $\mathbb{Z}$. Hence $E(x, y) = \pm 1$, and the result in the case $n = 2$ follows.

Next, assume $n \geq 3$ is arbitrary. Let $y \in \Lambda$ and $W \subset \Lambda$ such that

$$\mathbb{Z} \cdot y \oplus W = \Lambda.$$

Define a linear map $f \colon \Lambda \to \mathbb{Z}$ by putting $f(y) = 1$ and $f(w) = 0$ for each $w \in W$, and extending linearly. As the pairing (3.24) is perfect, there exists an element $x \in \Lambda$ such that $E(x, -) = f$ as linear maps $\Lambda \to \mathbb{Z}$. This implies that $E(x, y) = 1$ and $E(x, w) = 0$ for each $w \in W$. Let $P = \langle x, y \rangle^{\perp}$ be the orthogonal complement of $\langle x, y \rangle$ in Λ with respect to E. We claim that

$$\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y \oplus P = \Lambda. \tag{3.25}$$

To prove this, let $\lambda \in \Lambda$. We must show that there exist unique $a, b \in \mathbb{Z}$ such that $\lambda - a \cdot x - b \cdot y \in P$. That is, we need to show there exist unique $a, b \in \mathbb{Z}$ such that

$$\begin{aligned} E\left(x, \lambda - a \cdot x - b \cdot y\right) &= E(x, \lambda) + b = 0, \\ E\left(y, \lambda - a \cdot x - b \cdot y\right) &= E(y, \lambda) - a = 0. \end{aligned}$$

We may simply put $b = -E(x, \lambda)$ and $a = E(y, \lambda)$. Decomposition (3.25) follows.

To finish the proof, we would like to show that the restriction of E to $P \otimes P$ defines a perfect pairing, i.e. a unimodular alternating bilinear form. To see this, observe that by choosing a basis $\{p_1, \dots, p - 2\}$ for P, the form E becomes associated to a $(n-2) \times (n-2)$-matrix $M_P := (E(p_i, p_j))$. Similarly, one attaches a matrix $M_{x,y}$ to the pairing that E defines on $\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y$. The basis $\{x, y, p_1, \dots, p_{n-2}\}$ for Λ then associates a matrix M_Λ to E, and we have

$$\det(M_P) \cdot \det(M_{x,y}) = \det(M) = \pm 1,$$

where the last equality holds because E is unimodular. Therefore, $\det(M_P) = \pm 1$, hence the restriction of E to $P \otimes P$ is unimodular. The lemma follows by induction on the rank n of Λ. $\square$

Corollary 3.4.3. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be an odd-dimensional smooth cubic hypersurface. Then $H^n(X, \mathbb{Z})$ is free of rank $b_n(X) = 2m$ over $\mathbb{Z}$, and admits a basis $\{\gamma_1, \dots, \gamma_{2m}\}$ with respect to which the intersection matrix of the pairing $H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$ has the following form, where $\mathrm{Id} \in \mathrm{GL}_m(\mathbb{Z})$ denotes the identity matrix:*

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Proof. Torsion-freeness follows from Lemma 3.4.1. As the dimension of X is odd, (3.23) is a unimodular, alternating bilinear form, and we can apply Lemma 3.4.2. $\square$

3.4.2 Even-dimensional cubic hypersurfaces

We are going to use the following result, without providing a proof:

Proposition 3.4.4. *If a smooth projective variety X over $\mathbb{C}$ (or, more generally, a compact Kähler manifold) has even dimension $2m$, and if $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$, then the intersection pairing on $H^n(X, \mathbb{R})$ has signature*

$$\operatorname{sgn}(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X).$$

Proof. See [Huy05, Corollary 3.3.18]. □

Corollary 3.4.5. *Let X be a smooth projective variety of dimension $n = 2m$ over $\mathbb{C}$. Consider the Hirzebruch χ_y-genus $\chi_y(X)$, see (3.20). Then $\chi_{y=1}(X) = \tau(X)$.* □

We shall also need the following beautiful result.

Theorem 3.4.6 (Hirzebruch)**.** *Let $X_n \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a sequence of smooth hypersurfaces of degree d. For each n, let $\chi_y(X_n)$ be the Hirzebruch χ_y-genus of X_n, cf. (3.20). Then*

$$\sum_{n=0}^{\infty} \chi_y(X_n) z^{n+1} = \frac{1}{(1+yz)(1-z)} \cdot \frac{(1+yz)^d - (1-z)^d}{(1+yz)^d + y(1-z)^d}. \tag{3.26}$$

Proof. See [Hir95, Theorem 22.1.1]. □

Notice that, by Proposition 3.4.4, for $y = 1$ and $d = 3$ we can rewrite (3.26) as

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(X_n) z^{n+1} &= \frac{1}{(1+z)(1-z)} \cdot \frac{(1+z)^3 - (1-z)^3}{(1+z)^3 + (1-z)^3} \\ &= (-1) \cdot \frac{z^3 + 3z}{3z^4 - 2z^2 - 1} \\ &= z \cdot \frac{3 + z^2}{(1+3z^2) \cdot (1-z^2)}.\end{aligned}$$

Lemma 3.4.7. *Consider the power series expansion*

$$z \cdot \frac{3 + z^2}{(1+3z^2) \cdot (1-z^2)} = z \cdot \sum_{i=0}^{\infty} a_i \cdot z^i.$$

Then $a_{2m} = (-1)^m \cdot 2 \cdot 3^m + 1$ for each $m \geq 0$.

Proof. Exercise. □

Combining the above, we obtain:

Proposition 3.4.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. Let $\tau(X)$ be the signature of the pairing $H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to \mathbb{R}$. Then $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.* □

Let Λ be a *lattice*, i.e. a free $\mathbb{Z}$-module equipped with a symmetric bilinear form $(\cdot\,,\cdot)$. We say that Λ is unimodular when the pairing is perfect, i.e. when the determinant of an intersection matrix is ± 1. We say that a unimodular lattice is *even* if $(\alpha, \alpha) \equiv 0 \bmod 2$ for all $\alpha \in \Lambda$; otherwise, we say that Λ is *odd*. For example, the rank one lattice $\mathbb{Z}(a)$ with $(1, 1) = a$ is odd if and only if a is odd.

If Λ is unimodular, odd, and indefinite, then for some positive integers r, s, we have

$$\Lambda \cong I_{r,s} := \mathbb{Z}(1)^{\oplus r} \oplus \mathbb{Z}(-1)^{\oplus s}.$$

For this, see for example [Ser73, Chapter V, Theorem 4].

Theorem 3.4.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. The intersection form on $H^n(X, \mathbb{Z})$ turns $H^n(X, \mathbb{Z})$ into a unimodular lattice, and there exists an isomorphism of lattices*

$$H^n(X, \mathbb{Z}) \cong \mathbb{Z}(1)^{\oplus b_n^+} \oplus \mathbb{Z}(-1)^{\oplus b_n^-} = I_{b_n^+, b_n^-}. \tag{3.27}$$

Here, $b_n^+ := b_n^+(X)$ is defined as the number of positive eigenvalues of an intersection matrix of the associated form on $H^n(X, \mathbb{R})$, and $b_n^- = b_n^-(X) = b_n(X) - b_n^+(X)$. The two integers b_n^+ and b_n^- can be calculated from the two equalities $b_n^+ + b_n^- = b_n(X) = (1/6) \cdot (2^{n+3} + (-1)^n \cdot 7 + 3)$ and $b_n^+ - b_n^- = \tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.

Proof. We prove that $H^n(X, \mathbb{Z})$ is odd. This is easy: the class $h^m = c_1(\mathcal{O}_X(1))$ in $H^n(X, \mathbb{Z})$ satisfies $(h^m, h^m) = \int_X h^n = d$. Moreover, it was shown in Corollary 3.3.17 that we have $b_n(X) = (1/6) \cdot (2^{n+3} + (-1)^n \cdot 7 + 3)$, and the fact that $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$ follows from Proposition 3.4.8. In particular, $b_n(X) \neq \pm\tau(X)$, hence $H^n(X, \mathbb{Z})$ is indefinite. The isomorphism (3.27) follows then by the above-mentioned classification of odd indefinite unimodular lattices. ☐

3.4.3 Cubic surfaces

Proposition 3.4.10. *Let X be a compact complex manifold of dimension two. Let L be a line bundle on X. Then*

$$\chi(X, \mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} \qquad \textit{and}$$
$$\chi(X, L) = \int_X \frac{c_1(L)^2 + c_1(L) \cdot c_1(X)}{2} + \chi(X, \mathcal{O}_X).$$

Proof. One calculates the value of $\operatorname{td}(X)$, which is $\operatorname{td}(X) = 1 + c_1(X)/2 + c_1(X)^2/12 + c_2(X)/12$. Moreover, $\operatorname{ch}(L) = e^{c_1(L)}$, and the result follows from Theorem 3.3.8. ☐

Lemma 3.4.11. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface, and let L be a line bundle on X. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$. Then*

$$\chi(X, L) = \frac{(L, L) + (L, h)}{2} + 1.$$

Proof. Let X be a smooth cubic hypersurface. Then

$$c(X) = \frac{(1+h)^{n+2}}{1+3h} = \left(1 - 3h + (3h)^2 \pm \cdots\right) \cdot \sum_{i=0}^{n} \binom{n+2}{i} h^i.$$

Hence, $c(X) = (1 - 3h + (3h)^2 \pm \cdots) \cdot \left(1 + (n+2) \cdot h + \binom{n+2}{2} \cdot h^2 + \cdots\right)$, which gives

$$\begin{aligned} c_1(X) &= (n+2) \cdot h - 3h = (n-1) \cdot h \\ c_2(X) &= \left(9 - 3 \cdot (n+2) + \binom{n+2}{2}\right) \cdot h^2. \end{aligned}$$

For $n = 2$, this becomes $c_1(X) = h \in H^2(X, \mathbb{Z})$ and $c_2(X) = 3 \cdot h^2$. Together with Proposition 3.4.10, this implies that $\chi(X, L) = (1/2) \cdot ((L, L) + (L, h)) + \chi(X, \mathcal{O}_X)$.

It remains to show that

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 \tag{3.28}$$

This follows immediately from the fact that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ by Claim 3.1.12. Alternatively, we can use the fact that $c_1(X)^2 + c_2(X) = h^2 + 3h^2 = 4h^2$; applying Proposition 3.4.10 yields

$$\chi(X, \mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} = \int_X \frac{4h^2}{12} = 1.$$

This proves (3.28), and hence the lemma. $\square$

Let Λ be an odd unimodular lattice. A primitive vector $\alpha \in \Lambda$ is called *characteristic* if $(\alpha, \beta) \equiv (\beta, \beta) \bmod 2$ for all $\beta \in \Lambda$.

Lemma 3.4.12. *Let Λ be an odd unimodular lattice. Let $\alpha \in \Lambda$ be primitive. Then α is characteristic if and only if its orthogonal complement $\alpha^\perp$ is an even lattice.*

Proof. Let $\alpha^\perp$ be the orthogonal complement of a characteristic vector $\alpha \in \Lambda$. Let $\beta \in \alpha^\perp$. Then $(\beta, \beta) \equiv (\alpha, \beta) \equiv 0 \bmod 2$. In particular, $\alpha^\perp$ is even.

Conversely, let $\alpha \in \Lambda$ be primitive such that $\alpha^\perp$ is even. Let $\beta_0 \in \Lambda$ with $(\alpha, \beta_0) = 1$. Then, for all $\beta \in \Lambda$,

$$(\alpha, \beta - (\alpha, \beta) \cdot \beta_0) = 0.$$

In other words, $\beta - (\alpha, \beta) \cdot \beta_0 \in \alpha^\perp$ for each $\beta \in \Lambda$; in particular, it is of even square because $\alpha^\perp$ is assumed to be even. Hence

$$(\beta - (\alpha, \beta) \cdot \beta_0, \beta - (\alpha, \beta) \cdot \beta_0) \equiv (\beta, \beta) + (\alpha, \beta)^2 \cdot (\beta_0, \beta_0) \equiv 0 \bmod 2,$$

which implies that, for each $\beta \in \Lambda$, we have $(\beta, \beta) \equiv (\alpha, \beta)^2 \cdot (\beta_0, \beta_0) \bmod 2$. As Λ is odd, there exists β with (β, β) odd, and hence β_0^2 must be odd. This proves that

$$(\beta, \beta) \equiv (\alpha, \beta)^2 \equiv (\alpha, \beta) \mod 2$$

for all β. In other words, α is characteristic. $\square$

We can now study the intersection form on $H^2(X,\mathbb{Z})$ for a smooth cubic surface $X \subset \mathbb{P}^3_{\mathbb{C}}$.

Lemma 3.4.13. *Let X be a smooth cubic surface. Then $H^2(X,\mathbb{Z}) \cong I_{1,6}$.*

Proof. We have $H^2(X,\mathbb{Z}) \cong I_{r,s}$ for some $r, s \in \mathbb{Z}_{\geq 1}$ by Theorem 3.4.9. We need to prove $r = 1$ and $s = 6$. This follows, as $\tau(X) = -5$, see Theorem 3.4.9. □

Consider the lattice $I_{1,6}$. Let $\alpha = (3,1,1,1,1,1,1)$ and define $e_1 = (0,1,-1,0,0,0,0)$. Similarly, define $e_2 = (0,0,1,-1,0,0,0)$, $e_3 = (0,0,0,1,-1,0,0)$, $e_4 = (1,0,0,0,1,1,1)$, $e_5 = (0,0,0,0,1,-1,0)$ and $(e_7 = (0,0,0,0,1,-1)$.

Lemma 3.4.14. *The elements e_i, $i \in \{1,2,3,4,5,7\}$, span $\alpha^\perp$, and their intersection matrix is an intersection matrix for the lattice $E_6(-1)$. In particular, $\alpha^\perp \cong E_6(-1)$.*

Proof. Exercise. □

Theorem 3.4.15. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X,\mathbb{Z})$, and consider the sublattice $H^2(X,\mathbb{Z})_{prim} := \langle h\rangle^\perp$ of $H^2(X,\mathbb{Z})$. The lattice $H^2(X,\mathbb{Z})_{prim}$ is isomorphic to $E_6(-1)$.*

Proof. We claim that $h \in H^2(X,\mathbb{Z})$ is characteristic. Indeed, as $\mathrm{Pic}(X) = H^2(X,\mathbb{Z})$ by Corollary 3.1.11, it suffices to show that $(L,L) \equiv (L,h) \bmod 2$ for every $L \in \mathrm{Pic}(X)$, which follows from Lemma 3.4.11. We then apply a general result for unimodular lattices: two primitive vectors $x, y \in \Lambda$ are in the same $O(\Lambda)$ orbit if and only if $(x,x) = (y,y)$ and either both are characteristic or both are not. It follows that the image of h in $I_{1,6}$ and $\alpha = (3,1,1,1,1,1,1) \in I_{1,6}$ are in the same $O(I_{1,6})$-orbit. In particular, $H^2(X,\mathbb{Z})_{prim} = \langle h\rangle^\perp \cong \langle\alpha\rangle^\perp \cong E_6(-1)$, see Lemma 3.4.14. □

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