A note on descent for algebraic stacks

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1. Introduction

Let $S' \to S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor $X \mapsto X \times_S S'$ defines an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X' over S' such that X' admits an open covering by affine schemes which are stable under ϕ. In case $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(k')$ and the morphism $S' \to S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil.

The goal of this note is to prove a similar statement for algebraic stacks. In the case of stacks, the analogue of the aforementioned descent-theory is a notion called 2-*descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes $S' \to S$ which is smooth and surjective, *every 2-descent datum for an algebraic stack is effective.* More precisely, we have the following result. For a scheme S, let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of S as in [Stacks, Tag 021S]; a *stack over S* is a stack in groupoids $\mathcal{X} \to (\mathrm{Sch}/S)_{fppf}$ over $(\mathrm{Sch}/S)_{fppf}$, see [Stacks, Tag 0304].

THEOREM 1.1. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and let $\mathcal{X}'$ be a stack over S'. Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S', see Definition 3.1. Then (ϕ, ψ) is effective. That is, there exists a stack $\mathcal{X}$ over S, an isomorphism of stacks over S'*

$$\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

and a 2-isomorphism $\chi \colon p_2^* f \circ \mathrm{can} \Rightarrow \phi \circ p_1^* f$ as in the following diagram:

(1.1)
$$
\begin{array}{ccc}
p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & p_2^*(\mathcal{X} \times_S S') \\
\Big\downarrow{\scriptstyle p_1^*\rho} & \swarrow & \Big\downarrow{\scriptstyle p_2^*\rho} \\
p_1^*\mathcal{X}' & \xrightarrow{\phi} & p_2^*\mathcal{X}',
\end{array}
$$

such that the natural compatibility between χ and ψ is satisfied. Moreover, if $S' \to S$ is smooth, then $\mathcal{X}'$ is an algebraic stack over S' if and only if $\mathcal{X}$ is an algebraic stack over S. Finally, if $S' \to S$ is étale, then $\mathcal{X}'$ is a Deligne–Mumford stack over S' if and only if $\mathcal{X}$ is a Deligne–Mumford stack over S.

Note that even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack $\mathcal{X}$ is obtained by defining $\mathcal{X}(T)$ as the groupoid of objects of $\mathcal{X}'(T \times_S S')$ equipped with a descent datum relative to the 2-descent datum of $\mathcal{X}'$, for any scheme T over S. More precisely, the first assertion in the above theorem follows from the fact that the 2-fibred category $\underline{Stack}_S$ over $(\mathrm{Sch}/S)_{fppf}$, whose fibre over $U \in (\mathrm{Sch}/S)_{fppf}$ is the category $\underline{Stack}(U)$ of stacks over U, is a 2-stack over S (see e.g. [Bre94, Example 1.11.(1)]). The other two assertions follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the smooth (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, then for a stack $\mathcal{X}'$ over S', one can reformulate the notion of 2-descent datum for $\mathcal{X}'$ over S' in terms of an action of Γ on $\mathcal{X}'$ over the action of Γ on S' over S, as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define $^\sigma\mathcal{X}'$ as the pull-back of $\mathcal{X}'$ along $\sigma \colon S' \to S'$.

DEFINITION 1.2. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be a stack over S'. A Galois 2-descent datum consists of:*

(1) a family of 1-isomorphisms $f_\sigma \colon {}^\sigma\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ ($\sigma \in \Gamma$);
(2) a family of 2-isomorphisms $\psi_{\sigma,\tau} \colon f_\sigma \circ {}^\sigma(f_\tau) \implies f_{\sigma\tau}$ ($\sigma, \tau \in \Gamma$);

such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms

$$
\begin{array}{ccc}
f_\sigma \circ {}^\sigma f_\tau \circ {}^{\sigma\tau} f_\gamma & \xrightarrow{\;({}^{\sigma\tau} f_\gamma)^*(\psi_{\sigma,\tau})\;} & f_{\sigma\tau} \circ {}^{\sigma\tau} f_\gamma \\
{\scriptstyle f_{\sigma*}({}^\sigma \psi_{\tau,\gamma})} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \psi_{\sigma\tau,\gamma}} \\
f_\sigma \circ {}^\sigma f_{\tau\gamma} & \xrightarrow{\;\;\psi_{\sigma,\tau\gamma}\;\;} & f_{\sigma\tau\gamma}
\end{array}
$$

is commutative.

One can show that to give a Galois 2-descent datum on $\mathcal{X}'$ over S' is to give a group action (in the sense of [Rom05]) of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for $\mathcal{X}'$ over S', see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

THEOREM 1.3. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be an algebraic stack over S', equipped with a Galois 2-descent datum (f_σ ($\sigma \in \Gamma$), $\psi_{\sigma,\tau}$ ($\sigma, \tau \in \Gamma$)). There exists an algebraic stack $\mathcal{X}$ over S and an isomorphism $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over S'. The stack $\mathcal{X}$ is Deligne–Mumford if and only if $\mathcal{X}'$ is.*

Observe that the statement in Theorem 1.3 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an

isomorphism of stacks $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ over S' as well as a family of 2-isomorphisms $\chi_\sigma \colon \rho \circ \mathrm{can} \implies f_\sigma \circ {}^\sigma\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$
\begin{array}{ccc}
{}^\sigma(\mathcal{X} \times_S S') & \xrightarrow{\ \mathrm{can}\ } & \mathcal{X} \times_S S' \\
\downarrow & \mathrel{\rotatebox{45}{$\Rightarrow$}} & \downarrow \\
{}^\sigma\mathcal{X}' & \xrightarrow{\hspace{2cm}} & \mathcal{X}',
\end{array}
$$

such that the obvious compatibility conditions are satisfied.

EXAMPLE 1.4. *Let k be a field and let $k \subset k'$ be a degree two field extension; one may think of $\mathbb{R} \subset \mathbb{C}$ or $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ for a prime power q. Let $\mathcal{X}'$ be a stack over k' equipped with a 1-isomorphism $\sigma \colon \mathcal{X}' \to \mathcal{X}'$ and a 2-isomorphism $F \colon \sigma^2 \implies \mathrm{id}_{\mathcal{X}'}$ between σ^2 and the identity functor, such that σ commutes with the functor $(\mathsf{Sch}/k') \to (\mathsf{Sch}/k')$ defined as $T \mapsto T \times_{k',\sigma} k'$. One obtains a stack $\mathcal{X}$ over k by defining, for $T \in (\mathsf{Sch}/k)$, $\mathcal{X}(T)$ as the groupoid of pairs (x, φ) with $x \in \mathcal{X}'(T_{k'})$ and $\varphi \colon x \to \sigma(x)$ an isomorphism such that the composition*

$$
x \xrightarrow{\varphi} \sigma(x) \xrightarrow{{}^\sigma\varphi} \sigma^2(x) \xrightarrow{F} x
$$

is the identity. There is a natural isomorphism $\mathcal{X} \times_k k' \cong \mathcal{X}'$ of stacks over k'.

2. Descending schemes

Let

$$
p \colon S' \to S
$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$
S'' := S' \times S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \to S,
$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$
S''' \underset{\rightarrow}{\overset{\rightarrow}{\rightarrow}} S'' \rightrightarrows S' \to S
$$

where the three arrows $S''' \to S''$ are p_{12}, p_{13} and p_{23}.

Let X' be a scheme over S'. Define

$$
p_i^* X' = X' \times_{S',p_i} S'', \quad p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'',p_{jk}} S'''
$$

and note that

$$
p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'',p_{jk}} S''' = \left(p_i \circ p_{jk}\right)^* X'.
$$

Recall that a *descent datum* for X'/S' is an S''-isomorphism

$$
\phi \colon p_1^* X' \xrightarrow{\sim} p_2^* X'
$$

such that the following diagram commutes:

$$
\begin{array}{ccccccc}
p_{12}^* p_1^* X' & \xrightarrow{p_{12}^*\phi} & p_{12}^* p_2^* X' & =\!=\!= & p_{23}^* p_1^* X' & \xrightarrow{p_{23}^*\phi} & p_{23}^* p_2^* X' \\
\| & & & & & & \| \\
p_{13}^* p_1^* X' & & & \xrightarrow{\hspace{1cm} p_{13}^*\phi \hspace{1cm}} & & & p_{13}^* p_2^* X'.
\end{array}
$$

In other words, one requires that

$$p_{23}^* \phi \circ p_{12}^* \phi = p_{13}^* \phi \quad \text{as morphisms} \quad p_{12}^* p_1^* X' \to p_{13}^* p_2^* X'.$$

THEOREM 2.1 (Grothendieck). *Let* $p \colon S' \to S$ *be a faithfully flat locally finitely presented morphism of affine schemes. The functor* $X \mapsto p^* X$ *defines an equivalence of categories between the category of* S*-schemes* X *and the category of pairs* (X', ϕ) *where* X' *is an* S'*-scheme and* ϕ *a descent datum for* X'/S' *such that* X' *admits an open covering by affine schemes stable under* ϕ.

Next, recall how to make this explicit in case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. For instance, S could be the spectrum of a field k, S' the spectrum of a finite field extension $k' \supset k$, and Γ the Galois group of k'/k. Let X' be a scheme over S' and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma \colon {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over S', for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau} X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of Γ on X' as a scheme over S is said to be *compatible with the action of* Γ *on* S' *over* S if for each $\sigma \in \Gamma$, the following diagram commutes:

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \sigma\ } & X' \\
\downarrow & & \downarrow \\
S' & \xrightarrow{\ \sigma\ } & S'.
\end{array}
$$

LEMMA 2.2. *Let* $S' \to S$ *be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group* Γ*, and let* X' *be a scheme over* S'*. To give a descent datum for* X' *over* S' *is to give a Galois descent datum for* X' *over* S'*. These notions are further equivalent to giving an action of* Γ *on* X' *compatible with the action of* Γ *on* S' *over* S*.*

PROOF. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4]. □

3. Descending algebraic stacks

Let $p \colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids on S', in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$(3.1) \qquad\qquad\qquad r_i \colon S'''' \to S'.$$

Similarly, S''' is equipped with three projections $q_i \colon S''' \to S'$. Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on S'. One has similar isomorphisms relating the other $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$ with $r_\mu^* \mathcal{X}'$, for $i, j, k \in \{1, 2, 3, 4\}$, $\alpha, \beta \in \{1, 2, 3\}$, $\nu \in \{1, 2\}$ and $\mu \in \{1, 2, 3, 4\}$.

Consider an isomorphism of S''-stacks (i.e. an equivalence of Sch/S''-categories):

$$\phi \colon p_1^* \mathcal{X}' \to p_2^* \mathcal{X}',$$

and let ψ be a 2-morphism

$$\psi \colon p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism $\Rightarrow$ in the following diagram:

(3.2)

$$
\begin{array}{ccccccc}
p_{12}^* p_1^* \mathcal{X}' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* \mathcal{X}' & =\!=\!= & p_{23}^* p_1^* \mathcal{X}' & \xrightarrow{p_{23}^* \phi} & p_{23}^* p_2^* \mathcal{X}' \\
\big\| & & & \swArrow \psi & & & \big\| \\
p_{13}^* p_1^* \mathcal{X}' & & & \xrightarrow{\quad p_{13}^* \phi \quad} & & & p_{13}^* p_2^* \mathcal{X}'.
\end{array}
$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234} \colon S'''' \to S''',$$

and note that

$$p_{123}^* \left(p_{23}^* \phi \circ p_{12}^* \phi \right) = p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and}$$
$$p_{123}^* p_{13}^* \phi = \pi_{13}^* \phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34} \colon S'''' \to S''$$

are the canonical morphisms. For $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$, define

$$\psi_{ijk} := p_{ijk}^* \psi.$$

For instance, pulling back ψ along p_{123} gives a 2-morphism

$$\psi_{123} = p_{123}^* \psi \colon \pi_{23}^* \circ \pi_{12}^* \phi \Rightarrow \pi_{13}^* \phi.$$

Similarly, we obtain 2-morphisms

$$\psi_{124} \colon \pi_{24}^* \phi \circ \pi_{12}^* \phi \Rightarrow \pi_{14}^* \phi,$$
$$\psi_{134} \colon \pi_{34}^* \phi \circ \pi_{13}^* \phi \Rightarrow \pi_{14}^* \phi,$$
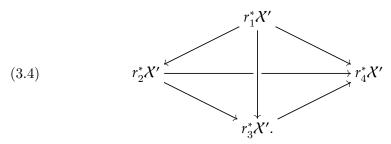$$\psi_{234} \colon \pi_{34}^* \phi \circ \pi_{23}^* \phi \Rightarrow \pi_{24}^* \phi.$$

Moreover, observe that under p_{123}, diagram (3.2) pulls back to the diagram

(3.3)

$$
\begin{array}{ccccccc}
r_1^* \mathcal{X}' & \xrightarrow{\pi_{12}^* \phi} & r_2^* \mathcal{X}' & =\!=\!= & r_2^* \mathcal{X}' & \xrightarrow{\pi_{23}^* \phi} & r_3^* \mathcal{X}' \\
\big\| & & & \swArrow & & & \big\| \\
r_1^* \mathcal{X}' & & & \xrightarrow{\quad \pi_{13}^* \phi \quad} & & & r_3^* \mathcal{X}',
\end{array}
$$

in which the 2-morphism $\Rightarrow$ is the 2-morphism ψ_{123} defined above (and with r_i is as in (3.1)). Using pull-backs by the other three $p_{ijk} \colon S'''' \to S'''$, we

thus obtain four triangles, that we may put together to form the following tetrahedron:

(3.4)

$$
\begin{array}{c}
r_1^* \mathcal{X}' \\
\\
r_2^* \mathcal{X}' \longrightarrow r_4^* \mathcal{X}' \\
\\
r_3^* \mathcal{X}'.
\end{array}
$$

DEFINITION 3.1. *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over S'. A 2-descent datum for $\mathcal{X}'$ over S' consists of:*

(1) an isomorphism of stacks (i.e. an equivalence of categories)

$$\phi\colon p_1^* \mathcal{X}' \to p_2^* \mathcal{X}'$$

over S'';

(2) a 2-isomorphism

$$\psi\colon p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi$$

as in diagram (3.2);

such that the following condition is satisfied: the 2-morphisms ψ_{ijk} between the several compositions in diagram (3.4) are compatible, in the sense that the following diagram of 2-morphisms commutes:

$$
\begin{array}{ccc}
\pi_{34}^* \phi \circ \pi_{23}^* \phi \circ \pi_{12}^* \phi & \xrightarrow{(\pi_{34}^* \phi)_* (\psi_{123})} & p_{34}^* \phi \circ p_{13}^* \phi \\
\Big\| {\scriptstyle (\pi_{12}^* \phi)^* (\psi_{234})} & & \Big\| {\scriptstyle \psi_{134}} \\
p_{24}^* \phi \circ p_{12}^* \phi & \xrightarrow{\quad \psi_{124} \quad} & p_{14}^* \phi.
\end{array}
$$

This gives the following result.

PROPOSITION 3.2 (Breen). *Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S'. Then there exists a stack $\mathcal{X}$ over S, an isomorphism*

$$\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over S', and a 2-isomorphism $\chi\colon p_2^* \rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^* \rho$ as in diagram

(3.5)

$$
\begin{array}{ccc}
p_1^* (\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & p_2^* (\mathcal{X} \times_S S') \\
\Big\downarrow {\scriptstyle p_1^* \rho} & \swarrow & \Big\downarrow {\scriptstyle p_2^* \rho} \\
p_1^* \mathcal{X}' & \xrightarrow{\quad \phi \quad} & p_2^* \mathcal{X}',
\end{array}
$$

such that the natural compatibility condition between χ and ψ is satisfied.
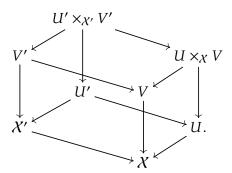
PROOF. This follows from [Bre94, Example 1.11.(i)].   □

To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-lemma, which should be well-known but which we include for convenience of the reader.

LEMMA 3.3. *Let S be a scheme. The following assertions are true.*

(1) *Let $\pi \colon \mathcal{X}' \to \mathcal{X}$ be a representable, smooth and surjective morphism of stacks in groupoids over S. If $\mathcal{X}'$ is algebraic, then $\mathcal{X}$ is algebraic. If in addition π is étale and $\mathcal{X}'$ is Deligne–Mumford, then $\mathcal{X}$ is Deligne–Mumford.*

(2) *Let $S' \to S$ be a smooth surjective morphism of schemes, let $\mathcal{X}$ be a stack in groupoids over S and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Suppose that $\mathcal{X}'$ is an algebraic stack over S'. Then $\mathcal{X}$ is an algebraic stack over S. If in addition $S' \to S$ is étale and $\mathcal{X}'$ is a Deligne–Mumford stack, then $\mathcal{X}$ is a Deligne–Mumford stack.*

PROOF. Let us first prove item (1). If U' is a scheme and $U' \to \mathcal{X}'$ a surjective and smooth morphism, then $U' \to \mathcal{X}' \to \mathcal{X}$ is surjective and smooth, and moreover étale if π and $U' \to \mathcal{X}'$ are étale. Therefore, it suffices to prove that the diagonal $\Delta \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. For this, it suffices to consider to schemes U and V, equipped with morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:

The morphism $\mathcal{X}' \to \mathcal{X}$ is representable, hence U' and V' are representable by algebraic spaces. Since $\mathcal{X}'$ is an algebraic stack, the morphism $V' \to \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \to U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \to U$ is étale and surjective, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that the morphism $U \times_{\mathcal{X}} V \to U$ is representable. As U is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, and we are done.

Next, we prove item (2). Via the composition $\mathcal{X}' \to S' \to S$, we may view $\mathcal{X}'$ as an algebraic stack over S, see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over S:

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

As $S' \to S$ is representable, surjective and étale, the same holds for $X' \to X$. The stack X' is algebraic, hence X is algebraic as well, see item (1). □

PROOF OF THEOREM 1.1. Proposition 3.2 yields the stack X over S together with 1-isomorphism $\rho \colon X \times_S S' \xrightarrow{\sim} X'$ and the 2-isomorphism $\chi \colon p_2^* \rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^* \rho$ that have the right compatibility properties with respect to ψ, so that we only need to prove that X is algebraic (resp. Deligne–Mumford if $S' \to S$ is surjective étale). This follows from Lemma 3.3. □

Even the case where X' is a scheme seems to yield a non-trivial result:

COROLLARY 3.4. *Let $S' \to S$ be a smooth surjective morphism of schemes, and let X' be a scheme over S' equipped with a descent datum ϕ as in Section 2. Then there exists an algebraic stack X over S and an S-morphism $\pi \colon X' \to X$ such that the diagram*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \pi\ } & X \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

is cartesian. The tuple $(X, \pi \colon X' \to X)$ is compatible with the descent datum ϕ in an appropriate sense, and this makes (X, π) unique up to isomorphism.

PROOF. This is a atraightforward consequence of Theorem 1.1. □

For a scheme S and a stack X, and a finite group Γ, a *group action of Γ on X over S* is an action of the functor in groups over S associated to Γ on the stack X over S, see [Rom05, Definition 1.3].

LEMMA 3.5. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, and let X' be a stack over S'. Then the following sets are in canonical bijection:*
- *(1) The set of 2-descent data (ϕ, ψ) for X' over S'.*
- *(2) The set of group actions of Γ on X' as a stack over S, such that for each $\sigma \in \Gamma$, the composition $X' \xrightarrow{\sigma} X' \to S'$ agrees with the composition $X' \to S' \xrightarrow{\sigma} S'$.*
- *(3) The set of Galois 2-descent data for X' over S'.*

PROOF. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] a the proof in the case of schemes. The stacky case is requires some straightforward generalizations; we leave the details to the reader. □

PROOF OF THEOREM 1.3. See Theorem 1.1 and Lemma 3.5. □

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