Algebraic Geometry II : Part 2

Lecture notes

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Contents

Chapter 1

Quasi-coherent sheaves on the projective spectrum of a graded ring

1.1 Lecture 14: Quasi-coherent sheaves and the proj construction

Definition 1.1.1. A *graded ring* is a ring S with a decomposition $S = \oplus_{d \geq 0} S_d$ of the underlying abelian group into abelian subgroups $S_d \subset S$, such that $S_d \cdot S_e \subset S_{d+e}$. A $\mathbb{Z}$*-graded ring* is a ring S with a decomposition $S = \oplus_{d \in \mathbb{Z}} S_d$ of the underlying abelian group into abelian subgroups $S_d \subset S$, such that $S_d \cdot S_e \subset S_{d+e}$.

Goal of this lecture: For a graded ring S, consider the scheme $X = \mathrm{Proj}(S)$, and define a functor

$$M \mapsto \widetilde{M}$$

from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules, as in the affine case.

Recall. *A* graded abelian group *is an abelian group M together with a decomposition $M = \oplus_{d \in \mathbb{Z}} M_d$ into abelian subgroups $M_d \subset M$.*

Recall. *Let $S = \oplus S_d$ be a graded ring, which is either graded or $\mathbb{Z}$-graded.*

(1) A graded S-module *is an S-module M with the structure of a graded abelian group $M = \oplus M_d$, such that the gradings of S and M are compatible in the sense that $S_d \cdot M_e \subset M_{d+e}$ for all $d, e \in \mathbb{Z}$.*

(2) An element $x \in M$ is called homogeneous *if $x \in M_d$ for some $d \in \mathbb{Z}$.*

(3) A graded submodule *of a graded S-module M is a submodule $N \subset M$ which is generated by homogeneous elements.*

(4) A morphism *of graded S-modules $\varphi \colon M \to N$ is a morphism of S-modules such that $\varphi(M_d) \subset N_d$ for $d \in \mathbb{Z}$.*

Question 1.1.2. *(1) In which ways can you turn $R = \mathbb{Z}$ into a graded ring?*

(2) Consider the graded ring structure such that $R = R_0$. Is a graded R-module the same thing as a graded abelian group?

Example 1.1.3. Let $M = \oplus M_d$ be a graded S-module. For $n \in \mathbb{Z}$, define a new graded S-module $M(n)$ as follows:

$$M(n)_d := M_{d+n}, \qquad M(n) := \oplus M(n)_d.$$

In particular, we have the graded S-module $S(n)$ for $n \in \mathbb{Z}$.

Lemma 1.1.4. *Let S be a graded ring and M a graded S-module.*

(1) An S-submodule $N \subset M$ is a graded submodule if and only if $N = \oplus N_d$ for $N_d := N \cap M_d$.

(2) If $N \subset M$ is a graded submodule, then M/N is naturally a graded S-module.

(3) Let $\varphi \colon M \to N$ be a morphism of graded S-modules. Then the kernel, image and cokernel of φ are graded S-modules in a natural way.

Proof. (1) Consider a submodule $N \subset M$, and define $N_d = N \cap M_d$ for $d \in \mathbb{Z}$. By definition, N is graded if and only if N is generated by the submodules $N_d \subset N$ for $d \in \mathbb{Z}$. As $N_d \cap N_{d'} = 0$ for $d \neq d'$, this happens if and only if $N = \oplus N_d$.

(2) Define $(M/N)_d = \mathrm{Im}(M_d \to M/N)$. Then the natural map

$$\oplus (M/N)_d \longrightarrow M/N$$

is surjective. We need to show it is injective. In other words, we need to show, for $d \neq e \in \mathbb{Z}$, that $(M/N)_d \cap (M/N)_e = 0$. Let

$$x \in (M/N)_d \cap (M/N)_e.$$

There exists $m_d \in M_d$ and $m_e \in M_e$ which both have image $x \in M/N$. Hence,

$$m_d \equiv m_e \mod N.$$

In other words, $m_d - m_e \in N$. Since N is graded, we can write $m_d - m_e = \sum_{k \in \mathbb{Z}} n_k$ as a sum of homogeneous elements $n_k \in N_k$. We have $N_k \subset M_k$, and it follows that $n_k = 0$ for $k \neq d, e$, and that $m_d = n_d$ and $m_e = -n_e$. In particular, $m_d, m_e \in N$, so that $x = 0 \in M/N$.

(3) In view of item (2), it suffices to prove the statement for the kernel $\mathrm{Ker}(\varphi)$ of $\varphi \colon M \to N$. Indeed, we have $\mathrm{Im}(\varphi) = M/\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi) = N/\mathrm{Im}(\varphi)$. Thus, let us show that $K := \mathrm{Ker}(\varphi)$ is a graded S-module. Let $x \in K$. Write $x = \sum m_d$ for $m_d \in M_d$. Then

$$0 = \varphi(x) = \sum \varphi(m_d).$$

As $\varphi(m_d) \in N_d$, this implies $\varphi(m_d) = 0$ for each $d \in \mathbb{Z}$. Hence $m_d \in K$.

This proves the lemma. $\square$

Remark 1.1.5. Let S be a graded ring, and M a graded S-module. Let $\mathfrak{p} \in \operatorname{Proj}(S)$. As in Section 5, consider the multiplicatively closed subset $T \subset S$ containing all homogeneous elements in $S \setminus \mathfrak{p}$. Then $T^{-1}M$ is naturally a graded $T^{-1}S$-module: we have

$$\begin{aligned} T^{-1}M &= \oplus \left(T^{-1}M\right)_k, \qquad \text{with} \\ \left(T^{-1}M\right)_k &= \left\{\frac{m}{t} \in T^{-1}M \colon m \text{ homogeneous of degree } k + \deg(t)\right\}. \end{aligned}$$

Definition 1.1.6. Consider the notation in Remark 1.1.5. We define

$$M_{(\mathfrak{p})} \coloneqq \left(T^{-1}M\right)_0.$$

Notice that $M_{(\mathfrak{p})}$ is an $R_{(\mathfrak{p})}$-module in a natural way.

Definition 1.1.7. Let M be a graded S-module. Let $U \subset \operatorname{Proj}(S)$ be open, and define

$$\widetilde{M}(U) = \left\{(s(\mathfrak{p})) \in \prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})} \colon \text{ condition } (\star) \text{ holds}\right\},$$

where $(\star)$ is the condition that for each $\mathfrak{p} \in U$, there exists an open neighbourhood $\mathfrak{p} \in V_{\mathfrak{p}} \subset U$ of $\mathfrak{p}$ in U, together with homogeneous elements $m \in M, f \in S$ of the same degree, such that for all $\mathfrak{q} \in V_{\mathfrak{p}}$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_{(\mathfrak{q})}$.

Proposition 1.1.8. *Let $X = \operatorname{Proj}(S)$ for a graded ring S, and let M be a graded S-module. Then the following holds:*

(1) For all $\mathfrak{p} \in \operatorname{Proj}(S)$, we have a canonical isomorphism

$$\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{(\mathfrak{p})}.$$

(2) Let $f \in S_+$ homogeneous, and consider the canonical isomorphism

$$\varphi \colon D_+(f) \xrightarrow{\sim} \operatorname{Spec} S_{(f)}.$$

Then there is a canonical isomorphism

$$\widetilde{M}|_{D_+(f)} \cong \varphi^*\left(\widetilde{M_{(f)}}\right).$$

Here, $M_{(f)}$ denotes the degree zero part of M_f (note that $M_{(f)}$ is an $S_{(f)}$-module in a natural way) and $\widetilde{M_{(f)}}$ is the affine tilde construction.

(3) $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M finitely generated, then $\widetilde{M}$ is coherent.

Proof. (1). We have

$$\left(\widetilde{M}\right)_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U\subset X} \widetilde{M}(U).$$

For $U \subset X$ open with $\mathfrak{p} \in U$, define a map

$$f_U \colon \widetilde{M}(U) \to M_{(\mathfrak{p})}, \quad (s(\mathfrak{q})) \mapsto s(\mathfrak{p}).$$

These maps are compatible with restrictions $\widetilde{M}(U) \to \widetilde{M}(V)$ for $\mathfrak{p} \in V \subset U$ open, and hence we get a well-defined map

$$f \colon \left(\widetilde{M}\right)_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U\subset X} \widetilde{M}(U) \to M_{(\mathfrak{p})}. \tag{1.1}$$

We claim that (1.1) is an isomorphism. As for the surjectivity, let $m/f \in M_{(\mathfrak{p})}$ with m, f homogeneous, $f \notin \mathfrak{p}$ and $\deg(m) = \deg(f)$. Then for each $\mathfrak{q} \in D_+(f)$, put $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$. Then we get a section

$$s := (s(\mathfrak{q})) \in \widetilde{M}(D_+(f)),$$

and we have $f_{D_+(f)}(s) = m/f \in M_{(\mathfrak{p})}$. Thus, the map (1.1) is surjective.

To prove the injectivity, let $s, t \in (\widetilde{M})_{\mathfrak{p}}$ such that $f(s) = f(t)$. We can find an open neighbourhood $\mathfrak{p} \in U \subset X$ and $\bar{s}, \bar{t} \in \widetilde{M}(U)$ that map to $s, t \in (\widetilde{M})_{\mathfrak{p}}$. We have $\bar{s}(\mathfrak{p}) = \bar{t}(\mathfrak{p})$, and hence there exists an open neighbourhood $\mathfrak{p} \in V_{\mathfrak{p}} \subset U$ such that $\bar{s}|_{V_{\mathfrak{p}}} = \bar{t}|_{V_{\mathfrak{p}}}$. In particular, $s = t$, and we are done.

(2). Exercise.

(3). By (2), quasi-coherence is clear. If S is noetherian and M finitely generated, then $S_{(f)}$ is noetherian and $M_{(f)}$ is finitely generated, hence M is coherent by (2). □

Recall that for a scheme X and a sheaf $\mathcal{F}$ on X, one defines the support of $\mathcal{F}$ as

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Lemma 1.1.9. *For a graded S-module M,* $\mathrm{Supp}(\widetilde{M}) = \left\{\mathfrak{p} \in \mathrm{Proj}(S) \mid M_{(\mathfrak{p})} \neq 0\right\}$.

Proof. Clear from item (1) in Proposition 1.1.8. □

Lemma 1.1.10. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded S-modules. Then for each $d \in \mathbb{Z}$, the induced sequence*

$$0 \to A_d \to B_d \to C_d \to 0$$

is exact.

Proof. Everything apart from possibly the surjectivity of $B_d \to C_d$ is trivial. To prove the latter, let $x \in C_d$ and lift x to an element $y \in B$. Write $y = \sum_n y_n$. Then since y maps to x, y_n maps to zero for each $n \neq d$. Therefore, y_d maps to x, and $y_d \in B_d$. □

Lemma 1.1.11. *For a graded ring S, and $X = \mathrm{Proj}(S)$, the tilde construction $M \mapsto \widetilde{M}$ defines an exact functor from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of graded S-modules. Let $\mathfrak{p} \in \mathrm{Proj}(S)$. Then the sequence

$$0 \to (M_1)_\mathfrak{p} \to (M_2)_\mathfrak{p} \to (M_3)_\mathfrak{p} \to 0$$

is exact. In particular, in view of Lemma 1.1.10, the sequence

$$0 \to (M_1)_{(\mathfrak{p})} \to (M_2)_{(\mathfrak{p})} \to (M_3)_{(\mathfrak{p})} \to 0$$

is exact. By Proposition 1.1.8, we are done. □

Recall that, for a ring R and an R-module M, we have $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}\ R \mid M_\mathfrak{p} \neq 0\}$.

Lemma 1.1.12. *Let S be a graded ring and M, N graded S-modules.*

(1) Suppose that $\mathrm{Supp}(M) \subset V(S_+) \subset \mathrm{Spec}\ S$. Then $\widetilde{M} = 0$.

(2) Assume that $N_{>d} \cong M_{>d}$ for some $d \in \mathbb{Z}_{\geq 0}$. Then $\widetilde{M} \cong \widetilde{N}$.

Proof. (1). The assumption implies that $\mathrm{Supp}(M) \cap \mathrm{Proj}(S) = \emptyset$. Hence $M_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \mathrm{Proj}(S)$. In particular, $M_{(\mathfrak{p})} = 0$ for each $\mathfrak{p} \in \mathrm{Proj}(S)$. It follows that $(\widetilde{M})_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \mathrm{Proj}(S)$, see Proposition 1.1.8. Thus $\widetilde{M} = 0$.

(2). Since $M_{>d} \subset M$ is a graded submodule, the quotient $L := M/M_{>d}$ is graded (see Lemma 1.1.4). Note that $\mathrm{Supp}(L) \subset V(S_+)$. Therefore, $\widetilde{L} = 0$ by item (1). From Lemma 1.1.11, it follows that the sequence

$$0 \to \widetilde{M_{>d}} \to \widetilde{M} \to \widetilde{L} \to 0$$

is exact. Hence $\widetilde{M_{>d}} \cong \widetilde{M}$. Consequently,

$$\widetilde{M} \cong \widetilde{M_{>d}} \cong \widetilde{N_{>d}} \cong \widetilde{N}.$$

We are done. □

Example 1.1.13. Let $X = \mathrm{Proj}(S)$ with $S = k[x_0, x_1]$, where k is a field. Let M be the graded S-module $M = k[x_0, x_1]/(x_0^2, x_1^2)$. Then $\widetilde{M} = 0$. Indeed, we have $S_+ = (x_0, x_1)$. If $M_\mathfrak{p} \neq 0$ for some $\mathfrak{p} \in \mathrm{Spec}\ S$, then $r \cdot 1 \neq 0$ for each $r \notin \mathfrak{p}$. Thus, $r \notin (x_0^2, x_1^2)$ for each $r \notin \mathfrak{p}$. Thus, $(x_0^2, x_1^2) \subset \mathfrak{p}$. Hence $(x_0, x_1) \subset \mathfrak{p}$, so that $\mathfrak{p} \in V(S_+)$.

1.1.1 *Serre's twisting sheaf*

Definition 1.1.14. Let S be a graded ring and $X = \mathrm{Proj}(S)$. For $n \in \mathbb{Z}$, define

$$\mathcal{O}_X(n) := \widetilde{S(n)}.$$

We call $\mathcal{O}_X(n)$ the n-th *twisting sheaf* (of Serre). If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, we put

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n),$$

and call $\mathcal{F}(n)$ the n-th *twist* of $\mathcal{F}$.

Proposition 1.1.15. *Let S be a graded ring such that S is generated by S_1 as an S_0-algebra. Let $X = \mathrm{Proj}(S)$. Then:*

(1) The sheaf $\mathcal{O}_X(n)$ is invertible for all $n \in \mathbb{Z}$.

(2) Let M, N be graded S-modules. There is a canonical isomorphism

$$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}. \tag{1.2}$$

(3) For all graded S-modules M and $n \in \mathbb{Z}$, we have a canonical isomorphism

$$\widetilde{M}(n) \xrightarrow{\sim} \widetilde{M(n)}.$$

(4) We have canonical isomorphisms $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ for $n, m \in \mathbb{Z}$.

Proof. (1). With respect to the identification $D_+(f) = \mathrm{Spec}\ S_{(f)}$, we have a canonical isomorphism

$$\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}$$

of sheaves on $\mathrm{Spec}\ S_{(f)}$. For $n \in \mathbb{Z}$ and $f \in S_1$, we have an isomorphism

$$S_{(f)} \longrightarrow S(n)_{(f)}, \qquad s \mapsto f^n \cdot s.$$

Thus, $\mathcal{O}_X(n)|_{D_+(f)}$ is a free $\mathcal{O}_X|_{D_+(f)}$-module of rank one. Since S is generated by S_1 over S_0, we have $S = \langle f \mid f \in S_1 \rangle$, hence $\mathrm{Proj}(S) = \cup_{f \in S_1} D_+(f)$.

(2). Indeed, let $f \in S_1$, and consider the canonical isomorphism $D_+(f) = \mathrm{Spec}\ S_{(f)}$. Using Proposition 1.1.8, we can define isomorphisms

$$\widetilde{M \otimes_S N}|_{D_+(f)} \cong (M \otimes_S N)_{(f)} \to M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong \widetilde{M} \otimes \widetilde{N}|_{D_+(f)},$$
$$\frac{m \otimes n}{f^{\deg(m)+\deg(n)}} \mapsto \frac{m}{f^{\deg(m)}} \otimes \frac{n}{f^{\deg(n)}}.$$

These isomorphisms agree on overlaps $D_+(f) \cap D_+(f)$, hence glue to give (1.2).

(3). This follows from (2), by taking $N = \mathcal{O}_X(n)$.

(4). This follows from (2), by observing that there are canonical isomorphisms

$$S(n) \otimes_S S(m) \xrightarrow{\sim} S(n+m), \quad s \otimes t \mapsto s \cdot t.$$

$\square$

1.2 Lecture 15: Projective schemes

1.2.1 The associated graded module

In the affine case, we can recover M from $\mathcal{F} = \widetilde{M}$ by taking global sections. In the projective setting, this will not work, as for instance $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$. Instead, we will have to look at the various Serre twists $\mathcal{F}(d)$, $d \in \mathbb{Z}$.

Definition 1.2.1. Let S be a graded ring. Let $X = \operatorname{Proj}(S)$, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *graded S-module associated to $\mathcal{F}$* as

$$\Gamma_*(\mathcal{F}) := \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

In particular, from X we get an associated $\mathbb{Z}$-graded ring

$$\Gamma_*(\mathcal{O}_X) := \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)).$$

Question 1.2.2. *Note $R = \Gamma_*(\mathcal{O}_X)$ has a grading $R = \oplus_{d\in\mathbb{Z}} R_d$ indexed by the full set of integers $\mathbb{Z}$. Hence R is a $\mathbb{Z}$-graded ring in the sense of Definition 1.1.1. Is it always true that $R_d = 0$ for $d < 0$? In other words, is R actually a graded ring, or not?*

The S-module structures are defined as follows. Let M be a graded S-module. There is a canonical morphism

$$\alpha \colon M \longrightarrow \Gamma_*(\widetilde{M}). \tag{1.3}$$

To define α, let $m \in M_d$ for $d \in \mathbb{Z}$. We need to provide a global section $\alpha(m) \in \Gamma(X, \widetilde{M}(d))$. It suffices to provide sections $\alpha(m) \in \Gamma(D_+(f), \widetilde{M}(d))$ that agree on overlaps. We have

$$\Gamma(D_+(f), \widetilde{M}(d)) = (M(d))_{(f)},$$

and put

$$\alpha(m) := \frac{m}{1} \in (M(d))_{(f)} = \left(M_{(f)}\right)_d.$$

This defines the map (1.3).

In particular, we get a canonical morphism

$$\beta \colon S \longrightarrow \Gamma_*(\widetilde{S}) = \Gamma_*(\mathcal{O}_X) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)). \tag{1.4}$$

This turns $\Gamma_*(\mathcal{O}_X)$ into a $\mathbb{Z}$-graded S-algebra (with compatible gradings). Moreover, for each $\mathcal{O}_X$-module $\mathcal{F}$, we have that $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$-module in a canonical way. Indeed, by item (4) of Proposition 1.1.15, we have canonical isomorphisms

$$\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(e) = \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) \cong \mathcal{F}(d+e).$$

In particular, for $s \in \mathcal{O}_X(d)$ and $t \in \mathcal{F}(e)$, we get a canonical section $s \cdot t \in \mathcal{F}(d+e)$, which defines the graded $\Gamma_*(\mathcal{O}_X)$-module structure on $\Gamma_*(\mathcal{F})$. Via (1.4), we obtain the graded S-module structure on $\Gamma_*(\mathcal{F})$.

Proposition 1.2.3. *Let A be a ring, and $S = A[x_0, \dots, x_r]$ for some $r \geq 1$. Let $X = \mathrm{Proj} S$ (projective r-space over A). Then (1.4) defines an isomorphism $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. Cover X by the open subsets $D_+(x_i) \subset X$. By the sheaf axiom for $\mathcal{O}_X(n)$, we get an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(n)) \to \oplus_{i=0}^r (S_{x_i})_n \to \bigoplus_{i,j} (S_{x_i x_j})_n.$$

Taking the direct sum over all $n \in \mathbb{Z}$, we get an exact sequence

$$0 \to \Gamma_*(\mathcal{O}_X) \to \bigoplus_{i=0}^r S_{x_i} \to \bigoplus_{i,j} S_{x_i x_j}.$$

As the $x_i \in S$ are non-zero divisors, the maps

$$S \to S_{x_i} \to S_{x_i x_j} \to S' := S_{x_0 \cdots x_r}$$

are all injective. We get

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^r S_{x_i} = S,$$

as subrings of S'. $\square$

Exercise 1.2.4. More generally, let S be a graded ring finitely generated over S_0 by non-zero divisors $x_0, \dots, x_r \in S_1$. Let $X = \mathrm{Proj}(S)$. Suppose that each x_i is a prime element. Show that $S = \Gamma_*(\mathcal{O}_X)$.

Corollary 1.2.5. *(1) Let $X = \mathbb{P}^r_k = \mathrm{Proj}(k[x_0, \dots, x_n])$. Then*

$$\Gamma(X, \mathcal{O}_X(n)) = (k[x_0, \dots, x_r])_n.$$

In particular,

$$\Gamma(X, \mathcal{O}_X(1)) = (k[x_0, \dots, x_r])_1 = \bigoplus_{i=0}^r k \cdot x_i.$$

(2) Let $X = \mathrm{Proj}(S)$ where S satisfies the assumptions in Exercise 1.2.4. Then $S_1 = \Gamma(X, \mathcal{O}_X(1))$.

Definition 1.2.6. Let A be a ring and $r \geq 0$. We let $x_0, \dots, x_r \in \mathcal{O}_{\mathbb{P}^r_A}(1)$ be the above global sections.

Lemma 1.2.7. *Let S be a graded ring, generated by S_1 as an S_0-module. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X = \mathrm{Proj}(S)$. Let $f \in S_1$. There are canonical isomorphisms*

$$\mathcal{F}(d)|_{D_+(f)} \cong f^d \cdot \mathcal{F}|_{D_+(f)}. \tag{1.5}$$

Proof. As $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(d)$, it suffices to prove the result for $\mathcal{F} = \mathcal{O}_X$. Notice that

$$S(d)_{(f)} = (S(d)_f)_0 = (S_f(d))_0\,,$$

that

$$S_f(d) = \bigoplus_{e \in \mathbb{Z}} (S_f)_{d+e}, \quad (S_f)_{d+e} = \left\{ \frac{x}{f^m} \mid x \in S_{m+d+e} \right\},$$

and that the map

$$\begin{aligned} S_{(f)} &\longrightarrow (S_f(d))_0 = (S_f)_d = \left\{ \frac{x}{f^m} \mid x \in S_{m+d} \right\}, \\ \frac{y}{f^m} &\mapsto \frac{f^d \cdot y}{f^m} \in (S_f)_d \end{aligned}$$

is an isomorphism. More precisely, we have

$$f^d \cdot S_{(f)} = (S_f)_d \subset S_f.$$

Therefore, we have

$$\mathcal{O}_X(d)|_{D_+(f)} = \widetilde{S(d)_{(f)}} = \widetilde{(S(d)_f)_0} = \widetilde{(S_f)(d)_0} = \widetilde{f^d \cdot S_{(f)}} = f^d \cdot \widetilde{S_{(f)}} = f^d \cdot \mathcal{O}_X|_{D_+(f)}.$$

This proves the lemma. $\square$

Proposition 1.2.8. *Let S be a graded ring such that S is generated by S_1 as an S_0-algebra. Let $X = \mathrm{Proj}(S)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then there is a natural isomorphism*

$$\psi\colon \widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}. \tag{1.6}$$

Proof. Let $f \in S_1$ and consider the scheme $D_+(f) = \mathrm{Spec}\ S_{(f)}$. We have

$$\Gamma(D_+(f), \widetilde{\Gamma_*(\mathcal{F})}) = (\Gamma_*(\mathcal{F}))_{(f)} = \left(\left(\bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F}(d)) \right)_f \right)_0$$

This is an $S_{(f)}$-module; an element of this module is given by an expression

$$x = \frac{s}{f^d}, \qquad s \in \Gamma(X, \mathcal{F}(d)).$$

The canonical isomorphism (1.5) shows that the section

$$s|_{D_+(f)} \in \Gamma(D_+(f), \mathcal{F}(d))$$

is of the form

$$s|_{D_+(f)} = f^d \cdot t \qquad \text{for some} \qquad t \in \Gamma(D_+(f), \mathcal{F}).$$

We define $\varphi_f(x) := t$, which gives a map

$$\varphi_f\colon \Gamma(D_+(f), \widetilde{\Gamma_*(\mathcal{F})}) \longrightarrow \Gamma(D_+(f), \mathcal{F}).$$

Since $D_+(f)$ is affine, and $\widetilde{\Gamma_*(\mathcal{F})}$ and $\mathcal{F}$ quasi-coherent, this yields a map

$$\psi_f\colon \widetilde{\Gamma_*(\mathcal{F})}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}.$$

It is straightforward to show that the maps ψ_f and ψ_g agree on overlaps $D_+(f \cdot g) = D_+(f) \cap D_+(g)$, hence glue to give the morphism (1.6). It is also readily checked that ψ_f is an isomorphism for each $f \in S_1$. The result follows. $\square$

Exercise 1.2.9. We have two functors

$$\begin{aligned} F = (-)^\sim \colon \mathrm{GrMod}_S &\longrightarrow \mathrm{QCoh}(X), \\ G = \Gamma_*\colon \mathrm{QCoh}(X) &\longrightarrow \mathrm{GrMod}_S, \end{aligned}$$

with $F \circ G \cong \mathrm{id}$ as functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$.

(1) Show that this implies that the functor G is fully faithful, and that the functor F is essentially surjective.

(2) Show that we do not in general have an isomorphism of functors $G \circ F \cong \mathrm{id}$.

Proof. (1). Essential surjectivity of F is clear: any object $\mathcal{M} \in \mathrm{QCoh}(X)$ is isomorphic to $(F \circ G)(\mathcal{M}) = F(G(\mathcal{M}))$. As for the faithfulness of G: this holds, as we have maps

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}(G(\mathcal{M}), G(\mathcal{N})) \longrightarrow \mathrm{Hom}(FG(\mathcal{M}), FG(\mathcal{N})) \cong \mathrm{Hom}(\mathcal{M}, \mathcal{N})$$

whose composition is the identity. Hence the first map in the composition is injective.

(2). We give an example of a graded module M with $\Gamma_*(\widetilde{M}) \ncong M$. Let M be any non-zero graded S-module such that $\mathrm{Supp}(M) \subset V(S_+)$. Then $\widetilde{M} = 0$ hence $\Gamma_*(\widetilde{M}) = 0$. This finishes the proof. $\square$

1.2.2 Projective schemes

Definition 1.2.10. Let A be a ring. A scheme X over A is *projective* if there exists an integer $r \geq 0$ such that the structure morphism $X \to \operatorname{Spec} A$ factors through a closed immersion $X \hookrightarrow \mathbb{P}^r_A$ of schemes over A.

Lemma 1.2.11. *Let S be a graded ring. Let S' be another graded ring, and $\varphi\colon S \to S'$ is a surjective morphism of graded rings.*

(1) We have $S_+ \not\subset \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Proj}(S')$. In particular, $\mathrm{Bs}(\varphi) = \emptyset$, and we get a morphism of schemes $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$.

(2) The above morphism of schemes $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ is a closed immersion.

Proof. As for part (1), note that for $\mathfrak{p} \in \operatorname{Spec} S'$ homogeneous, we have

$$S'_+ \subset \mathfrak{p} \iff \varphi^{-1}(S'_+) \subset \varphi^{-1}(\mathfrak{p}) \iff S_+ \subset \varphi^{-1}(\mathfrak{p}),$$

where we use the fact that φ is surjective.

As for part (2), note that the morphism is locally given by the maps

$$\operatorname{Spec}\,(S'_{(\varphi(f))}) \to \operatorname{Spec}\,(S_{(f)}), \qquad f \in S.$$

These are induced by the ring maps

$$S_{(f)} \longrightarrow S'_{(\varphi(f))}. \tag{1.7}$$

In turn, the latter is induced via restriction by

$$S_f \longrightarrow S'_{\varphi(f)}.$$

This map is surjective: let $x/\varphi(f)^n \in S'_{\varphi(f)}$; then we can find $y \in S$ with $\varphi(y) = x$, so that

$$\varphi(y/f^n) = \varphi(y)/\varphi(f)^n \in S'_{\varphi(f)}.$$

Hence (1.7) is surjective (see Lemma (1.1.10)), proving (2). $\square$

Proposition 1.2.12. *Let A be a ring.*

(1) Let X be a closed subscheme of $\mathbb{P}^r_A$. Then there exists a homogeneous ideal $I \subset A[x_0, \ldots, x_r]$ such that X is the closed subscheme determined by the surjective morphism of graded rings $A[x_0, \ldots, x_r] \to A[x_0, \ldots, x_r]/I$.

(2) A scheme X over $\operatorname{Spec} A$ is projective if and only if $X \cong \operatorname{Proj}(S)$ for some graded ring S such that $A = S_0$ and S is finitely generated by S_1 as an S_0-algebra.

Proof. (1). Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r_A}$ be the corresponding quasi-coherent ideal sheaf. By Proposition 1.2.8, there is a canonical isomorphism of graded S-modules

$$\widetilde{\Gamma_*(\mathcal{I})} \cong \mathcal{I}.$$

Moreover, the map

$$\Gamma_*(\mathcal{I}) \to \Gamma_*(\mathcal{O}_{\mathbb{P}^r_A})$$

is injective and identifies $\Gamma_*(\mathcal{I})$ with an ideal

$$I \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^r_A}) = A[x_0, \ldots, x_r],$$

where the canonical isomorphism $\Gamma_*(\mathcal{O}_{\mathbb{P}^r_A}) = A[x_0, \ldots, x_r]$ was provided in Proposition 1.2.3. Hence we have

$$\mathcal{I} = \widetilde{I} \subset \widetilde{R} = \mathcal{O}_{\mathbb{P}^r_A}, \qquad R := A[x_0, \ldots, x_r].$$

Item (1) follows from this.

(2). Suppose that X is projective. Then there is a closed immersion $X \hookrightarrow \mathbb{P}^r_A$ of schemes over A, for some $r \geq 0$. By item (1), we get that $X \cong \mathrm{Proj}(A[x_0, \ldots, x_r]/I)$ for some homogeneous ideal $I \subset A[x_0, \ldots, x_r]$. Conversely, if $X = \mathrm{Proj}(S)$ for some graded ring S with $A = S_0$ and S finitely generated by S_1 as S_0-algebra, then we can find elements $y_0, \ldots, y_r \in S_1$ that generate S as an A-algebra. This gives a surjective morphism of graded A-algebras

$$A[x_0, \ldots, x_r] \longrightarrow S, \qquad x_i \mapsto y_i,$$

yielding a closed immersion $\mathrm{Proj}(S) \hookrightarrow \mathbb{P}^r_A$ of schemes over A. □

Definition 1.2.13. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module for a scheme X. We say $\mathcal{F}$ is *generated by global sections* if there is an index set I and a surjective map of $\mathcal{O}_X$-modules

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}.$$

Note that to give such a morphism is to give global sections $s_i \in \mathcal{F}$ for $i \in I$. We say that $\mathcal{F}$ is *globally generated* by the sections s_i.

Exercise 1.2.14. Let $S = k[u^4, u^3v, uv^3, v^4] \subset k[u, v]$, where the generators of S are considered as to have degree one (i.e. $\deg(u^4) = 1, \deg(u^3v) = 1$, etc.). Note that $\dim S_1 = 4$. Show that $\dim \Gamma(X, \mathcal{O}_X(1)) = 5$. Conclude that the canonical map $S_1 \to \Gamma(X, \mathcal{O}_X(1))$ is not surjective.

Example 1.2.15. (1) Let A be a ring, $X = \mathrm{Spec}\, A$, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F} \cong \widetilde{M}$ for some A-module M, and any set of generators for $M \cong \Gamma(X, \mathcal{F})$ will generate $\mathcal{F}$.

(2) Let S be a graded ring generated over S_0 by a subset $I \subset S_1$. Then the map

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)$$

induced by the map $\beta \colon S_1 \to \Gamma(X, \mathcal{O}_X(1))$, is surjective.

Proof. Exercise. As for (2), suppose for instance that $S = A[x_0, \ldots, x_r]$, with $S_0 = A$. Then for each x_i, we have that

$$S(1)_{(f)} = A[x_0, \ldots, x_r](1)_{(x_i)} = (A[x_0, \ldots, x_r]_{x_i})_1$$

is generated by the x_i as an $A[x_0, \ldots, x_r]_{(x_i)}$-module. In fact, the map

$$S_{(x_i)} \longrightarrow S(1)_{(x_i)} = (S_{x_i})_1, \qquad s \mapsto x_i \cdot s$$

is an isomorphism of $S_{(x_i)}$-modules, with inverse $t \mapsto x_i^{-1} \cdot t$. Therefore, for each $i \in \{0, \ldots, r\}$, the images of the elements $x_0, \ldots, x_r \in S_1$ in $S(1)_{x_i} = (S_{x_i})_1$ generate $S(1)_{x_i}$ as an $S_{(x_i)}$-module. Thus, the map

$$\bigoplus_{i=0}^{r} S \longrightarrow S(1), \qquad (0, \ldots, 1, \ldots, 0) \mapsto x_i,$$

yields a surjection $\bigoplus_{i=0}^r \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)$. □

Lemma 1.2.16. *Let A be a ring, let $r \in \mathbb{Z}_{\geq 0}$ and consider a morphism of A-schemes $\varphi\colon X \to \mathbb{P}^r_A$. Then the global sections $x_0, \ldots, x_r \in \mathcal{O}_{\mathbb{P}^r_A}(1)$, see Definition 1.2.6, give rise to global sections*

$$s_i = \varphi^*(x_i) \in L := \varphi^*(\mathcal{O}_{\mathbb{P}^r_A}(1)), \qquad i = 0, \ldots, r,$$

that satisfy the property that L is globally generated by the sections s_i.

The following result shows that the converse is also true. An *isomorphism* between pairs $(L, (s_i))$ and $(M, (t_i))$, where L and M are line bundles on a scheme X and $s_0, \ldots, s_r, t_0, \ldots, t_r$ global sections, is an isomorphism $f\colon L \to M$ such that $s_i = f^*(t_i)$.

Theorem 1.2.17. *Let A be a ring. Let X be a scheme over A, and let L be a line bundle globally generated by sections $s_0, \ldots, s_r \in L$. Then there is a unique morphism*

$$\varphi\colon X \longrightarrow \mathbb{P}^r_A$$

such that

$$(\varphi^*(\mathcal{O}(1)), \varphi^*(x_0), \ldots, \varphi^*(x_r)) \cong (L, s_0, \ldots, s_r).$$

Corollary 1.2.18. *Let A be a ring. Consider the functor*

$$\begin{aligned} F\colon \mathsf{Sch}/A &\longrightarrow \mathsf{Set}, \\ X &\mapsto \{(L, s_0, \ldots, s_r) \mid L \textit{ line bundle globally generated by the } s_i\} / \cong . \end{aligned}$$

This functor is representable by $\mathbb{P}^r_A$. More precisely, the association

$$\varphi \mapsto \left(\varphi^*(\mathcal{O}_{\mathbb{P}^r_A}(1)), \varphi^*(x_0), \ldots, \varphi^*(x_r)\right)$$

defines a bijection

$$\mathrm{Hom}(X, \mathbb{P}^r_A) \xrightarrow{\sim} F(X)$$

for each A-scheme X, compatible with morphisms of A-schemes $X \to Y$.

For schemes X and T over $\mathbb{C}$, we define $X(T) := \mathrm{Hom}_{\mathsf{Sch}/\mathbb{C}}(T, X)$ as the set of morphisms $T \to X$ of schemes over $\mathbb{C}$.

Example 1.2.19. We make the following observations and definitions:

(1) For a finite dimensional complex vector space V, we get a graded ring $S = \mathrm{Sym}^*(V) = \oplus_{d \geq 0} \mathrm{Sym}^d(V)$ with $S_0 = \mathbb{C}$. If we choose a basis $\{e_0, \ldots, e_r\}$ for V, we get a set $\{x_0, \ldots, x_r\} \subset S_1 = \mathrm{Sym}^1(V) = V$ of generators for S as an $S_0 = \mathbb{C}$-algebra, in a way that $S = \mathbb{C}[x_0, \ldots, x_r]$.

(2) We define
$$\mathbb{P}(V) := \mathrm{Proj}(\mathrm{Sym}^*(V)).$$
This gives back $\mathbb{P}^r_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^{r+1})$.

(3) We define
$$\check{\mathbb{P}}^r_{\mathbb{C}} := \mathbb{P}((\mathbb{C}^{r+1})^\vee).$$

(4) Using Corollary 1.2.18, we can show that there is a canonical bijection
$$\check{\mathbb{P}}^r_{\mathbb{C}}(\mathbb{C}) \cong \left\{\text{lines } \ell \subset \mathbb{C}^{r+1}\right\}.$$

Proof. Exercise. □

Chapter 2

Cohomology

2.1 Lecture 16: Cech cohomology of sheaves on a scheme

Goal of this lecture: For an abelian sheaf $\mathcal{F}$ on a scheme X, define cohomology groups $\mathrm{H}^i(X, \mathcal{F})$, such that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of abelian sheaves, then one gets a long exact sequence:

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to \mathrm{H}^1(X, \mathcal{F}_1) \to \mathrm{H}^1(X, \mathcal{F}_2) \to \cdots .$$

Thus, the cohomology measures the failure of the right exactness of the global sections functor $\Gamma(X, -)$. Moreover, if $(X_i, \mathcal{F}_i)$ $(i = 1, 2)$ are schemes with sheaves on them, and if $\phi\colon X_1 \to X_2$ is an isomorphism with $\phi^*\mathcal{F}_2 \cong \mathcal{F}_1$, then one has an isomorphism $\mathrm{H}^p(X_1, \mathcal{F}_1) \cong \mathrm{H}^p(X_2, \mathcal{F}_2)$ for each $p \geq 0$. Thus, sheaf cohomology forms an invariant of the pair $(X, \mathcal{F})$, and this invariant turns out to be very important.

2.1.1 Some homological algebra

Definition 2.1.1. A *complex of abelian groups* $A^\bullet$ is a sequence of groups A^i indexed by $\mathbb{Z}$ together with maps d_A^i between them as follows:

$$\cdots \xrightarrow{d_A^{i-2}} A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \xrightarrow{d_A^{i+1}} \cdots ,$$

such that $d_A^i \circ d_A^{i-1} = 0$. A *morphism* of complexes

$$f^\bullet\colon A^\bullet \to B^\bullet$$

is a collection of maps $f_p\colon A^p \to B^p$ such that $f_i \circ d_A^{i-1} = d_B^{i-1} \circ f_{i-1}$ for each $i \in \mathbb{Z}$. In this way, we can talk about *kernels, images, cokernels* and *exact sequences* of complexes of abelian groups. We define

$$\mathrm{H}^p(A^\bullet) := \mathrm{Ker}(d_A^p)/\mathrm{Im}(d_A^{p-1}).$$

Lemma 2.1.2. *Let $0 \to F^\bullet \to G^\bullet \to H^\bullet \to 0$ be an exact sequence of complexes of abelian groups. Then there is an associated long exact sequence of cohomology groups*

$$\cdots \to H^p(F^\bullet) \to H^p(G^\bullet) \to H^p(H^\bullet) \to H^{p+1}(F^\bullet) \to \cdots .$$

Proof. We have a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & F^p \arrow[r] \arrow[d] & G^p \arrow[r] \arrow[d] & H^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1} \arrow[r] & G^{p+1} \arrow[r] & H^{p+1} \arrow[r] & 0
\end{tikzcd}$$

By the Snake lemma, we get an exact sequence

$$0 \to \operatorname{Ker}(d_F^p) \to \operatorname{Ker}(d_G^p) \to \operatorname{Ker}(d_H^p) \to F^{p+1}/\operatorname{Im}(d_F^p) \to \cdots .$$

Consider now the diagram

$$\begin{tikzcd}
 & F^p/\operatorname{Im}(d^{p-1}) \arrow[r] \arrow[d, "d"] & G^p/\operatorname{Im}(d^{p-1}) \arrow[r] \arrow[d] & H^p/\operatorname{Im}(d_H^p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d^{p+1}) \arrow[r] & \operatorname{Ker}(d^{p+1}) \arrow[r] & \operatorname{Ker}(d_H^{p+1}).
\end{tikzcd}$$

It has exact rows by the previous argument. Applying the Snake lemma again, gives an exact sequence

$$H^p(F^\bullet) \to H^p(G^\bullet) \to H^p(H^\bullet) \to H^{p+1}(F^\bullet) \to H^{p+1}(G^\bullet) \to H^{p+1}(H^\bullet).$$

Since this sequence is exact for every $p \in \mathbb{Z}$, the result follows. □

Let $f\colon C^\bullet \to D^\bullet$ be a morphism of complexes $C^\bullet$ and $D^\bullet$. Then, since $f \circ d_C = d_D \circ f$, the map f induces a well-defined map on cohomology groups

$$f\colon H^i(C^\bullet) \to H^i(D^\bullet).$$

Definition 2.1.3. A *chain homotopy* between two morphisms $f, g\colon C^\bullet \to D^\bullet$ is a collection of maps $h\colon C^n \to D^{n-1}$ such that

$$f - g = d_D \circ h + h \circ d_C.$$

Lemma 2.1.4. *If there exists a chain homotopy between f and g, then f and g induce the same map $H^i(C^\bullet) \to H^i(D^\bullet)$.*

Proof. Let $c \in \operatorname{Ker}(C^i \to C^{i+1})$. Then $[f(c) - g(c)] = [d_D(h(c))] = 0 \in H^i(D^\bullet)$. □

Exercise 2.1.5. Let $C^\bullet$ be a complex.

(1) Show that $C^\bullet$ is exact if and only if $H^i(C^\bullet) = 0$ for all i.

(2) Assume that there exists a chain homotopy $h\colon C^n \to C^{n-1}$ between the identity $\mathrm{id}\colon C^\bullet \to C^\bullet$ and the zero map $0\colon C^\bullet \to C^\bullet$. Show that $c = d^p \circ h(c) + h \circ d(c)$ for every $c \in C^{p+1}$. Show that $H^i(C^\bullet) = 0$ for each i, hence that $C^\bullet$ is exact.

2.1.2 Cech cohomology

Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, indexed by some set I. By the well-ordering theorem, there exists a well-ordering I, which we choose once and for all. For any finite set of indices $i_0, \ldots, i_p \in I$, we denote

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

For a sheaf $\mathcal{F}$ on X, we have the sheaf sequence

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j).$$

Definition 2.1.6. Let X and $\mathcal{U}$ be as above. Let $\mathcal{F}$ be a sheaf on X. We define the *Cech complex* of $\mathcal{F}$ (with respect to $\mathcal{U}$) as the complex $C^\bullet(\mathcal{U}, \mathcal{F})$ with

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,u_p}).$$

Thus, to given an element $\alpha \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is to give a $(p+1)$-tuple of elements

$$\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p})$$

for *each* strictly increasing $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I. We define the coboundary map $d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ as the map that sends $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ to the element $d\alpha \in C^{p+1}(\mathcal{U}, \mathcal{F})$ with

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \in \mathcal{F}(U_{i_0,\ldots,i_{p+1}}).$$

Here, the notation $\widehat{i_k}$ means that we omit i_k.

Let $\alpha \in C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in I} \mathcal{F}(U_i)$. Then

$$(d\alpha)_{i_0,i_1} = \alpha_{i_1}|_{U_{i_0,i_1}} - \alpha_{i_0}|_{U_{i_0,i_1}} \in \mathcal{F}(U_{i_0,i_1}).$$

Hence, for each $i_0, i_1, i_2 \in I$ with $i_0 < i_1 < i_2$, we have:

$$\begin{aligned}
(d^2\alpha)_{i_0,i_1,i_2} &= (d\alpha)_{i_1,i_2}|_{U_{i_0,i_1,i_2}} - (d\alpha)_{i_0,i_2}|_{U_{i_0,i_1,i_2}} + (d\alpha)_{i_0,i_1}|_{U_{i_0,i_1,i_2}} \\
&= \left(\left(\alpha_{i_2}|_{U_{i_1,i_2}} - \alpha_{i_1}|_{U_{i_1,i_2}} \right) - \left(\alpha_{i_2}|_{U_{i_0,i_2}} - \alpha_{i_0}|_{U_{i_0,i_2}} \right) + \left(\alpha_{i_1}|_{U_{i_0,i_1}} - \alpha_{i_0}|_{U_{i_0,i_1}} \right) \right) |_{U_{i_0,i_1,i_2}} \\
&= \left(\alpha_{i_2}|_{U_{i_0,i_1,i_2}} - \alpha_{i_1}|_{U_{i_0,i_1,i_2}} \right) - \left(\alpha_{i_2}|_{U_{i_0,i_1,i_2}} - \alpha_{i_0}|_{U_{i_0,i_1,i_2}} \right) + \left(\alpha_{i_1}|_{U_{i_0,i_1,i_2}} - \alpha_{i_0}|_{U_{i_0,i_1,i_2}} \right) \\
&= 0.
\end{aligned}$$

In particular, we get $d \circ d = 0$ as maps $C^0(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$. This generalizes as follows.

Lemma 2.1.7. *We have $d^{p+1} \circ d^p = 0$ as maps $C^p(\mathcal{U}, \mathcal{F}) \to C^{p+2}(\mathcal{U}, \mathcal{F})$.*

Proof. Exercise. □

Definition 2.1.8. The *p-th Cech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ is the group

$$\mathrm{H}^p(\mathcal{U}, \mathcal{F}) \coloneqq \mathrm{H}^p(C^\bullet(\mathcal{U}, \mathcal{F})) = \mathrm{Ker}(d^p)/\mathrm{Im}(d^{p-1}).$$

Notice that a sheaf homomorphism $\mathcal{F} \to \mathcal{G}$ induces morphisms $C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{G})$, and it is not hard to show that these induce morphisms

$$\mathrm{H}^p(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^p(\mathcal{U}, \mathcal{G}).$$

This gives functors $\mathrm{H}^p(\mathcal{U}, -)$ from abelian sheaves on X to abelian groups.

Example 2.1.9. Notice that

$$\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}\left(\prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j)\right) = \mathcal{F}(X).$$

Example 2.1.10. The group $\mathrm{H}^1(\mathcal{U}, \mathcal{F})$ is the group of sections $\sigma_{ij} \in \prod_{i<j} \mathcal{F}(U_{ij})$ such that $\sigma_{ik}|_{U_{ijk}} = \sigma_{ij}|_{U_{ijk}} + \sigma_{jk}|_{U_{ijk}}$, modulo the sections σ_{ij} of the form $\sigma_{ij} = \tau_j|_{U_{ij}} - \tau_i|_{U_{ij}}$.

Example 2.1.11. Consider a short exact sequence of abelian sheaves on X:

$$0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{f} \mathcal{C} \to 0.$$

Let $c \in \mathcal{C}(X)$. Let $\mathcal{U} = \{U_i\}_i$ be an open covering of X such that $c|_{U_i} = f(b_i)$ for some $b_i \in \mathcal{B}(U_i)$. Define

$$\sigma_{ij} \coloneqq b_j|_{U_{ij}} - b_i|_{U_{ij}} \in \mathcal{A}(U_{ij}).$$

(1) We have $\sigma_{ik}|_{U_{ijk}} = \sigma_{ij}|_{U_{ijk}} + \sigma_{jk}|_{U_{ijk}}$.

(2) Let

$$\sigma(c) \in \mathrm{H}^1(\mathcal{U}, \mathcal{A})$$

be the Cech cohomology class induced by the c_{ij}. Then $\sigma(c) = 0$ if and only if there exists an element $b \in \mathcal{B}(X)$ with $f(b) = c$.

Definition 2.1.12. Let P be a property that a morphism of schemes can have. For instance, P can be being a closed immersion, an open immersion, surjective, an isomorphism, etc. We say that the property P is *stable under base change* if for any morphism of schemes $X \to Y$ that has property P, any scheme T and any morphism of schemes $T \to Y$, the resulting morphism of schemes $X \times_Y T \to T$ has property P.

Lemma 2.1.13. *The property of being a closed immersion is stable under base change.*

Proof. Let $f\colon X \to Y$ be a closed immersion. We consider a morphism of schemes $T \to Y$; the goal is to show that $\pi\colon X \times_Y T \to T$ is a closed immersion. It suffices to provide an affine open covering $\{T_i\}$ of T such that $\pi^{-1}(T_i)$ is affine and $\pi^{-1}(T_i) \to T_i$ is a closed immersion. We start with an affine open covering $\{Y_i\}$ of Y, which gives an open covering of T (by taking inverse images under $T \to Y$) which we refine to an affine open covering $\{T_j\}$ of T. Thus, for each $j \in J$ there is an $i \in I$ such that T_j maps into Y_i under $T \to Y$. Then $\pi^{-1}(T_j) = f^{-1}(Y_i) \times_{Y_i} T_j$ is affine, and the map $\mathcal{O}(T_j) \to \mathcal{O}(f^{-1}(Y_i)) \otimes_{\mathcal{O}(Y_i)} \mathcal{O}(T_j)$ is surjective as $\mathcal{O}(Y_i) \to \mathcal{O}(f^{-1}(Y_i))$ is surjective. □

Lemma 2.1.14. *Let X be a separated scheme. Let $U \subset X$ and $V \subset X$ be affine opens. Then $U \cap V$ is affine.*

Proof. Notice that $U \cap V = U \times_X V$. This is naturally a closed subscheme of $U \times_{\mathbb{Z}} V$, since it sits inside the cartesian diagram

$$\begin{tikzcd} U \times_X V \arrow[r, hook] \arrow[d] & U \times_{\mathbb{Z}} V \arrow[d] \\ X \arrow[r, hook] & X \times_{\mathbb{Z}} X, \end{tikzcd}$$

and closed immersions are stable under base change by Lemma 2.1.13. Moreover, $U \times_{\mathbb{Z}} V = U \times_{\mathrm{Spec}\,(\mathbb{Z})} V$ is affine, because U, V and Spec $(\mathbb{Z})$ are all affine. As closed subschemes of affine schemes are affine, we are done. □

Theorem 2.1.15. *Let X be a noetherian separated scheme. Let $\mathcal{U} = \{U_0, U_1, \ldots, U_r\}$ be a finite covering of X by affine opens $U_i \subset X$. Then all the intersections $U_{i_0,\ldots,i_p}$ are affine, and moreover:*

(1) The Cech cohomology groups define functors $\mathrm{H}^i(\mathcal{U}, -)\colon \mathrm{AbSh}_X \to \mathrm{Ab}$.

(2) We have $\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

(3) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then there is an associated long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_1) \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_2) \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_3) \to \mathrm{H}^{i+1}(\mathcal{U}, \mathcal{F}_1) \to \mathrm{H}^{i+1}(\mathcal{U}, \mathcal{F}_2) \to \cdots.$$

(4) If $\mathcal{V} = \{V_j\}$ is another finite covering of X by affine opens, then there is a canonical isomorphism

$$\mathrm{H}^p(\mathcal{U}, \mathcal{F}) = \mathrm{H}^p(\mathcal{V}, \mathcal{F})$$

for every $p \geq 0$ and every quasi-coherent sheaf $\mathcal{F}$ on X.

(5) If X has dimension n, then $\mathrm{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every integer $p > n$.

Proof. Finite intersections of affines on separated scheme are affine. Indeed, this follows from Lemma 2.1.14 above.

(1) & (2). We have already observed this above.

(3). Note that if $U \subset X$ is an affine open subset, then the sequence

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$$

is exact, because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ from quasi-coherent $\mathcal{O}_U$-modules to $\mathcal{O}_X(U)$-modules is exact as U is affine. It follows that for each $p \geq 0$ and each $i_0 < \cdots < i_p \in I$, the sequence

$$0 \to \mathcal{F}_1(U_{i_0,\ldots,i_p}) \to \mathcal{F}_2(U_{i_0,\ldots,i_p}) \to \mathcal{F}_3(U_{i_0,\ldots,i_p}) \to 0$$

is exact (again since $U_{i_0,\dots,i_p}$ is affine). Therefore, the sequence

$$0 \to C^p(\mathcal{U}, \mathcal{F}_1) \to C^p(\mathcal{U}, \mathcal{F}_2) \to C^p(\mathcal{U}, \mathcal{F}_3) \to 0$$

is exact for each $p \geq 0$, so that we get an exact sequence of complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}_1) \to C^\bullet(\mathcal{U}, \mathcal{F}_2) \to C^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0.$$

Hence the desired long exact sequence comes from Lemma 2.1.2.

(4). We do not prove this here.

(5). We only prove this in case X is quasi-projective of finite type over a noetherian ring A. In this case, X admits an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ consisting of $m \leq n+1$ affine open subsets $U_i \subset X$, see Exercise 2.1.16 below. In particular, $C^p(\mathcal{U}, \mathcal{F}) = 0$ for $p \geq m$, since there are no $(p+1)$-tuples $i_0 < \cdots < i_p \in I$, for $p \geq m$. □

Exercise 2.1.16. Let X be a quasi-projective scheme of finite type over a noetherian ring A. Let $n = \dim(X)$. Then X admits an affine open cover $\mathcal{U}$ consisting of at most $n+1$ affine open subsets $U_i \subset X$.

Proof. Hint: Suppose that $X \subset Z \subset \mathbb{P}^r_A$, where Z is a closed subscheme of $\mathbb{P}^r_A$ and X is an open subscheme of Z. Write $W = Z - X$. Write $Z = \cup_i Z_i$ as a union of its irreducible components. If $Z_i \subset W$, then $X = Z - W \subset Z - Z_i$, so that $X \cap Z_i = \emptyset$, hence $X \subset \cup_{j\neq i} Z_j$. Therefore, one may assume that the irreducible components of Z are not contained in W. Using induction on the dimension, one can prove that X is covered by $n+1$ open affines induced from open affines in $\mathbb{P}^r_A$. □

2.2 Lecture 17: Examples & Cohomology via resolutions

2.2.1 *Some examples*

Recall. *Let k be a field. We consider $\mathbb{P}^1 := \mathbb{P}^1_k = \mathrm{Proj}k[x_0, x_1]$. Then there is a natural isomorphism between $\mathbb{P}^1$ and the scheme obtained by glueing together $U_0 = \mathrm{Spec}\ k[t]$ and $U_1 = \mathrm{Spec}\ k[t^{-1}]$ along $\mathrm{Spec}\ k[t, t^{-1}]$.*

Proof. We have isomorphism

$$U_i := D_+(x_i) \cong \mathrm{Spec}\ k[x_0, x_1]_{(x_i)}$$

for $i = 0, 1$. Moreover, there is a map of k-algebras

$$\varphi_0 \colon k[t] \to k[x_0, x_1]_{(x_0)}, \quad t \mapsto \frac{x_1}{x_0}.$$

Then φ_0 is an isomorphism, with inverse $s \mapsto s(1, t)$. Similarly, we have

$$\varphi_1 \colon k[t^{-1}] \cong k[x_0, x_1]_{(x_1)}, \quad t^{-1} \mapsto \frac{x_0}{x_1}.$$

Finally, $D_+(x_0x_1) = \mathrm{Spec}\ k[x_0, x_1]_{(x_0x_1)}$, and there is an isomorphism $k[t, t^{-1}] \cong k[x_0, x_1]_{(x_0x_1)}$ defined as $t \mapsto x_0^2/(x_0x_1)$ and $t^{-1} \mapsto x_1^2/(x_0x_1)$. □

Example 2.2.1. Consider the projective line $\mathbb{P}^1 = \mathbb{P}^1_k$ as above; it is covered by the open affines $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$ with intersection $U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$. Let $\mathcal{U} = \{U_0, U_1\}$. For the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, the Cech complex

$$0 \to C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \to C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \to 0$$

takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1) \arrow[r] & 0 \\
0 \arrow[r] & k[t] \times k[t^{-1}] \arrow[u] \arrow[r, "d"] & k[t, t^{-1}] \arrow[u] \arrow[r] & 0,
\end{tikzcd}$$

with

$$d\left(f(t), g(t^{-1})\right) = g(t^{-1}) - f(t).$$

If $f(t) = g(t^{-1}) \in k[t, t^{-1}]$, then $f = g \in k$. In other words,

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathrm{H}^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Ker}(d) = k.$$

Furthermore, each element $s \in k[t, t^{-1}]$ is a sum of a polynomial in t and a polynomial in t^{-1}. Therefore, d is surjective, so that

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathrm{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = 0.$$

Example 2.2.2. Let $m \in \mathbb{Z}$, consider $\mathbb{P}^1 := \mathbb{P}^1_k$, the projective line over a field k, and the sheaf $\mathcal{O}(m) := \mathcal{O}_{\mathbb{P}^1}(m)$. Let $S = k[x_0, x_1]$. We have

$$\mathcal{O}(m)(D_+(x_i)) = S(m)_{(x_i)} = x_i^m \cdot S_{(x_i)}$$

for $i = 1, 2$. Under the isomorphisms

$$\begin{aligned}
S_{(x_0)} &\to k[t], \quad f \mapsto f(1, t) \\
S_{(x_1)} &\to k[t^{-1}], \quad f \mapsto f(t^{-1}, 1), \\
S_{(x_0 x_1)} &\to k[t, t^{-1}], \quad f \mapsto f(1, t) = f(t^{-1}, 1),
\end{aligned}$$

see Example 2.2.1, the Cech complex takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(m)(U_0) \times \mathcal{O}_{\mathbb{P}^1}(m)(U_1) \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(m)(U_0 \cap U_1) \arrow[r] & 0 \\
0 \arrow[r] & x_0^m \cdot S_{(x_0)} \times x_1^m \cdot S_{(x_1)} \arrow[u] \arrow[r, "d"] & x_1^m \cdot S_{(x_0 x_1)} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & S_{(x_0)} \times (\frac{x_1}{x_0})^m \cdot S_{(x_1)} \arrow[u] \arrow[r] & (\frac{x_1}{x_0})^m \cdot S_{(x_0 x_1)} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & k[t] \times t^m \cdot k[t^{-1}] \arrow[u] \arrow[r] & t^m \cdot k[t, t^{-1}] \arrow[u] \arrow[r] & 0.
\end{tikzcd}$$

Here, we have

$$d\colon x_0^m \cdot S_{(x_0)} \times x_1^m \cdot S_{(x_1)} \longrightarrow x_1^m \cdot S_{(x_0x_1)}, \quad d\left(x_0^m \cdot f, x_1^m \cdot g\right) = x_1^m \cdot g - \frac{x_0^m}{x_1^m} \cdot x_1^m \cdot f,$$

corresponding to the map

$$d\colon k[t] \times t^m \cdot k[t^{-1}] \longrightarrow t^m \cdot k[t, t^{-1}] = k[t, t^{-1}], \quad d(f(t), t^m \cdot g(t^{-1})) \mapsto t^m \cdot g - f.$$

Suppose that $m \geq 0$. Then the elements

$$(t^m, t^m \cdot 1), (t^{m-1}, t^m \cdot t^{-1}), \ldots, (t^0, t^m \cdot t^{-m})$$

are linearly independent elements that generate the kernel of d. Therefore,

$$\dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1)) = \dim \mathrm{H}^0(\mathcal{U}, \mathcal{O}(1)) = \dim \mathrm{Ker}(d) = m + 1.$$

If $m < 0$, then $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1)) = 0$.

Example 2.2.3. Next, we compute the dimension of $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m))$. If $m \geq 0$, then any polynomial in $k[t, t^{-1}]$ can be written in the form $t^m g(t^{-1}) - f(t)$ for $f(t) \in k[t]$ and $g(t^{-1}) \in k[t^{-1}]$. We claim the same holds if $m = -1$. Indeed, let $t^{-k} \in k[t, t^{-1}]$ for some $k \geq 1$ (for the non-negative powers of t, the claim is clear). Then $t^{-k} = t^{-1} \cdot t^{-k+1}$, with $t^{-(k-1)} \in k[t^{-1}]$ as $k - 1 \geq 0$. Therefore, the map

$$k[t] \times t^m \cdot k[t^{-1}] \to t^m \cdot k[t, t^{-1}], \qquad (f, t^m \cdot g) \mapsto t^m \cdot g - f$$

is surjective if $m \geq -1$. Hence $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m)) = 0$ for $m \geq -1$.

If $m \leq -2$, then no linear combinations of the monomials

$$t^{-1}, t^{-2}, \ldots, t^{m+1} = t^{-(-m-1)}$$

lies in the image of d, but combinations of all the others do. It follows that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m))$ is a k-vector space of dimension $-m - 1$ in this case.

Example 2.2.4. We now consider an example from topology. Let $X = S^1$ be the unit circle, with the standard euclidean topology. Let $\mathcal{U} = \{U, V\}$, where U and V are connected open intervals that intersect in two connected open intervals W_1 and W_2. Let $\mathcal{F} = \mathbb{Z}_X$ be the constant sheaf associated to $\mathbb{Z}$. Then, we have

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U) \times \mathcal{F}(V) = \mathbb{Z} \times \mathbb{Z}, \qquad C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U \cap V) = \mathcal{F}(W_1 \sqcup W_2) = \mathbb{Z} \times \mathbb{Z}.$$

Under these identifications, the map $d\colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is given by

$$d\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}, \qquad (a, b) \mapsto (b, b) - (a, a) = (b - a, b - a).$$

Hence:

$$\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(d) = \mathrm{Im}(\mathbb{Z} \xrightarrow{x \mapsto (x,x)} \mathbb{Z} \times \mathbb{Z}) \cong \mathbb{Z}.$$

and

$$\mathrm{H}^1(\mathcal{U}, \mathcal{F}) = (\mathbb{Z} \times \mathbb{Z}) \,/ \mathrm{Im}(d) \cong \mathbb{Z}.$$

This gives the same answer as singular cohomology.

Remark 2.2.5. This is no coincidence: the groups $H^p(\mathcal{U}, \mathbb{Z})$ agree with the usual singular cohomology groups $H^p_{sing}(X, \mathbb{Z})$ for any topological space X homotopy equivalent to a CW complex, provided that the open sets in the covering $\mathcal{U}$ are contractible.

Exercise 2.2.6. Let X be a topological space and let $\mathcal{U}$ be an open cover of X. Assume that $U_i = X$ for some $i \in I$. Show that $H^p(\mathcal{U}, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ on X and every integer $p \geq 1$.

Example 2.2.7. Let X be an irreducible topological space. Then X is connected and any non-empty open subset $U \subset X$ is irreducible, hence connected. Let A_X be the constant sheaf associated to an abelian group A. Then $A_X(U) = A$ for any non-empty open $U \subset X$ (so that A_X agrees with the constant presheaf associated to A).

Let $\mathcal{U}$ be an open covering of X whose index set I is well-ordered. The Cech complex takes the form

$$0 \to \prod_{i_0 \in I} A \to \prod_{i_0 < i_1} A \to \prod_{i_0 < i_1 < i_2} A \to \cdots,$$

where for $\alpha \in \prod_{i_0 < \cdots < i_p} A$, we have its coordinate $\alpha_{i_0, \ldots, i_p} \in A$, and:

$$d(\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0,\ldots,p+1} (-1)^k \alpha_{i_0, \ldots, \widehat{i_k}, \ldots, i_p} \in A.$$

Note also that $H^p(\mathcal{U}, \mathcal{F}) = 0$ in view of Exercise 2.2.6. Indeed, by the above, the Cech complex does not depend on the U_i, only on the index set I. Hence we may assume $U_i = X$ for some i.

2.2.2 Cohomology as right derived functor

Definition 2.2.8. (1) Let A be an abelian group. Then A is *injective* if the contravariant functor $\mathrm{Hom}(-, A)$ from Ab to Ab, is exact. This is equivalent to saying that it is right exact. In other words, for any injective morphism $B_1 \hookrightarrow B_2$ of abelian groups, and any morphism $B_1 \to A$, there should exist a morphism $B_2 \to A$ that makes the obvious triangle commute.

(2) Let $\mathcal{F}$ be an abelian sheaf on a topological space X. Then $\mathcal{F}$ is *injective* if the contravariant functor $\mathrm{Hom}(-, \mathcal{F})$ from $\mathrm{AbSh}(X)$ to Ab, is exact. This is equivalent to saying that it is right exact. In other words, for any injective morphism $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ of abelian sheaves, and any morphism $\mathcal{B}_1 \to \mathcal{F}$, there should exist a morphism $\mathcal{B}_2 \to \mathcal{F}$ that makes the obvious triangle commute.

Exercise 2.2.9. (1) Show that an abelian group A is injective if and only if it is *divisible*: for each $n \in \mathbb{Z}_{\geq 1}$ and each $x \in A$ there exists $y \in A$ such that $n \cdot y = x$.

(2) Give an example of a divisible abelian group A such that for each $a \in A$ there exists $n \in \mathbb{Z}_{\geq 1}$ such that $n \cdot a = 0$.

(3) Show that a finite abelian group which is divisible, is zero.

(4) Show that the quotient of a divisible abelian group is divisible.

Proposition 2.2.10. *Let X be a topological space. Then any abelian sheaf $\mathcal{F}$ admits an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective abelian sheaf $\mathcal{I}$.*

Proof. We first prove the proposition in the case where $X = \{x\}$ is a point. Then $\mathcal{F}$ corresponds to an abelian group A, and we need to find an injective morphism $A \hookrightarrow I$ into a divisible abelian group I (see the above exercise). Consider the morphism

$$F := \bigoplus_{a \in A} \mathbb{Z} \longrightarrow A, \qquad \sum_a n_a \mapsto \sum_a n_a \cdot a.$$

This is clearly a surjective group homomorphism. Let K be the kernel. There is an embedding

$$F \hookrightarrow F \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{a \in A} \mathbb{Q},$$

and hence an embedding

$$A = F/K \hookrightarrow (F \otimes_{\mathbb{Z}} \mathbb{Q})/K.$$

As $(F \otimes_{\mathbb{Z}} \mathbb{Q})/K$ is divisible, being the quotient of a divisible abelian group (see the above exercise), we are done in the case $X = \{x\}$.

In the general case, for each $x \in X$, choose an injective abelian group I_x and an embedding $\mathcal{F}_x \hookrightarrow I_x$. For each $x \in X$, let $\varphi_x \colon \{x\} \hookrightarrow X$ denote the natural inclusion. We define

$$\mathcal{I} := \prod_{x \in X} (\varphi_x)_*(I_x).$$

We have

$$\operatorname{Hom}(\mathcal{F}, \mathcal{I}) = \prod_{x \in X} (\mathcal{F}, (\varphi_x)_* I_x) = \prod_{x \in X} \operatorname{Hom}(\mathcal{F}_x, I_x).$$

This yields a natural morphism of sheaves $\mathcal{F} \to \mathcal{I}$, which is injective since it is so on each stalk. It is also easily checked that $\mathcal{I}$ is injective. We are done. ☐

Definition 2.2.11. Let $\mathcal{F}$ be an abelian sheaf on a topological space X. An *injective resolution* of $\mathcal{F}$ is a complex $\mathcal{I}^\bullet$, defined in degrees $i \geq 0$, together with a morphism $\epsilon \colon \mathcal{F} \to \mathcal{I}^0$ such that $\mathcal{I}^i$ is injective for each $i \geq 0$ and such that the sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

is exact.

Corollary 2.2.12. *Let X be a topological space. Then any abelian sheaf $\mathcal{F}$ on X admits an injective resolution.* ☐

Lemma 2.2.13. *Let X be a topological space and let $\mathcal{F} \to \mathcal{I}^\bullet$ and $\mathcal{F} \to \mathcal{J}^\bullet$ be two injective resolutions. Then there are morphisms of complexes $f \colon \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and $g \colon \mathcal{J}^\bullet \to \mathcal{I}^\bullet$ whose compositions are homotopic to the identity (see Definition 2.1.3).*

Proof. Exercise. □

Note that if $\mathcal{I}^\bullet$ is an injective resolution of an abelian sheaf $\mathcal{F}$ on X, we get a complex $\Gamma(X, \mathcal{I}^\bullet)$ whose terms are $\Gamma(X, \mathcal{I}^i) = \mathcal{I}^i(X)$ for $i \geq 0$.

Definition 2.2.14. Let X be a topological space. For each abelian sheaf $\mathcal{F}$ on X, choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, and define $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(\Gamma(X, \mathcal{I}^\bullet))$.

Theorem 2.2.15. *Let X be a topological space.*

(1) For each $i \geq 0$, $\mathcal{F} \mapsto \mathrm{H}^i(X, \mathcal{F})$ defines a functor from $\mathrm{AbSh}(X)$ to Ab. Moreover, this functor is, up to natural isomorphism of functors, independent of the choices of injective resolutions made.

(2) We have $\mathrm{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$.

(3) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a short exact sequence of abelian sheaves. Then there is an associated long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^i(X, \mathcal{F}_1) \to \mathrm{H}^i(X, \mathcal{F}_2) \to \mathrm{H}^i(X, \mathcal{F}_3) \to \mathrm{H}^{i+1}(X, \mathcal{F}_1) \to \mathrm{H}^{i+1}(X, \mathcal{F}_2) \to \cdots .$$

Proof. Exercise. *Hint:* Use Lemmas 2.2.13 and 2.1.2. □

Theorem 2.2.16. *Let X be a noetherian separated scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a canonical isomorphism between the group $\mathrm{H}^p(X, \mathcal{F})$ introduced in Definition 2.2.14 and the Cech cohomology group $\mathrm{H}^p(\mathcal{U}, \mathcal{F})$ introduced in Definition 2.1.8, where $\mathcal{U} = \{U_0, \ldots, U_r\}$ is a finite cover of affine opens $U_i \subset X$.*

Proof. Exercise. □

2.3 Lecture 18: Coherent sheaves on projective schemes

2.3.1 Cohomology of twisting sheaves on projective space

Recall. *See Examples 2.2.1, 2.2.2 and 2.2.3. We have $\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}(m)) = k[x_0, x_1]_m$, $\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}(m)) = 0$ for $m \geq -1$, and $\dim \mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}(m)) = -m - 1$ for $m \geq -2$.*

We would like to generalize this to projective spaces of arbitrary dimension $n \geq 1$.

Theorem 2.3.1. *Let $\mathbb{P}^n_A = \mathrm{Proj}\, A[x_0, \ldots, x_n]$ where A is a noetherian ring. Then:*

(1) For each $m \in \mathbb{Z}$, $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m)) = A[x_0, \ldots, x_n]_m$.

(2) For all $0 < p < n$ and all $m \in \mathbb{Z}$, $\mathrm{H}^p(\mathbb{P}^n_A, \mathcal{O}(m)) = 0$.

(3) For each $m \in \mathbb{Z}$,

$$\mathrm{H}^n(\mathbb{P}^n_A, \mathcal{O}(m)) = \left(x_0^{-1} \cdots x_n^{-1} \cdot A[x_0^{-1}, \ldots, x_n^{-1}]\right)_m .$$

In particular, $\mathrm{H}^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) = A$.

Proof. We consider the open cover $\mathcal{U} = \{U_i\}$ with $U_i = D_+(x_i)$. This gives

$$I = \{0, \dots, n\}.$$

We get

$$\mathcal{C}^p(\mathcal{U}, \mathcal{O}(m)) = \prod_{i_0 < \cdots < i_p} \Big(A[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}} \Big)_m.$$

The Cech complex takes the form

$$\prod_i (A[x_0, \dots, x_n]_{x_i})_m \xrightarrow{d_0} \prod_{i<j} \big(A[x_0, \dots x_n]_{x_i x_j}\big)_m \xrightarrow{d_1} \prod_{i<j<k} \big(A[x_0, \dots, x_n]_{x_i x_j x_k}\big)_m \xrightarrow{d_2} \cdots.$$

For each $i_0 < \cdots < i_p \in I$, we have a decomposition

$$\Big(A[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}} \Big)_m = \bigoplus_{\substack{e \in \mathbb{Z}^{n+1} :\ \deg(e) = m \\ e_j \geq 0\ \forall j \notin \{i_0, \dots, i_p\}}} A x_0^{e_0} \cdots x_n^{e_n}.$$

This gives a decomposition

$$C^p(\mathcal{U}, \mathcal{O}(m)) = \prod_{i_0 < \cdots < i_p} \Big(A[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}} \Big)_m = \prod_{i_0 < \cdots < i_p} \bigoplus_{\substack{e \in \mathbb{Z}^{n+1} :\ \deg(e) = m \\ e_j \geq 0\ \forall j \notin \{i_0, \dots, i_p\}}} A x_0^{e_0} \cdots x_n^{e_n}.$$

Note that (1) follows from Proposition 1.2.3. Let us prove (2) and (3). We have:

$$(A[x_0, \dots, x_n]_{x_0 \cdots x_n})_m = \bigoplus_{\sum e_i = m} A x_0^{e_0} \cdots x_n^{e_n}.$$

More generally:

$$C^p(\mathcal{U}, \mathcal{O}(m)) = \bigoplus_{e \in \mathbb{Z}^{n+1}} C^p(\mathcal{U}, \mathcal{O}(m))_e,$$

with

$$\mathcal{C}^p(\mathcal{U}, \mathcal{O}(m))_e = \prod_{i_0 < \cdots < i_p :\ e_j \geq 0\ \forall j \notin \{i_0, \dots, i_p\}} (x_0^{e_0} \cdots x_n^{e_n} A)_m.$$

Therefore, to prove (ii), it suffices to prove that the complex $C^\bullet(\mathcal{U}, \mathcal{O}(m))_e$ is exact in the range $0 < p < n$, for each $e \in \mathbb{Z}^{n+1}$. For $\deg(e) \neq m$, the complex is zero. For $\deg(e) = m$ and $0 \leq p \leq n$, we have a canonical split embedding

$$\prod_{\substack{i_0 < \cdots < i_p \leq n \\ e_j \geq 0\ \ \forall j \notin \{i_0, \dots, i_p\}}} x_0^{e_0} \cdots x_n^{e_n} A \hookrightarrow \prod_{i_0 < \cdots < i_p \leq n} x_0^{e_0} \cdots x_n^{e_n} A,$$

and the complex

$$\to \prod_{i_0 < \cdots < i_{p-1} \leq n} x_0^{e_0} \cdots x_n^{e_n} A \to \prod_{i_0 < \cdots < i_p \leq n} x_0^{e_0} \cdots x_n^{e_n} A \to \prod_{i_0 < \cdots < i_{p+1} \leq n} x_0^{e_0} \cdots x_n^{e_n} \cdot A \to \cdots$$

identifies with the complex $C^\bullet$ with $C^p = \prod_{i_0<\cdots<i_p} A$, that is, with

$$\to \prod_{i_0<\cdots<i_{p-1}\le n} A \to \prod_{i_0<\cdots<i_p\le n} A \to \cdots \to \prod_{i_0<i_1<\cdots<i_n} A = A.$$

The latter is exact in degrees $0 < p < n$ (see Example 2.2.7), hence the former is exact in those degrees as well. This proves (2).

To prove (3), observe that

$$\mathcal{C}^n(\mathcal{U}, \mathcal{O}(m)) = (A[x_0, \ldots, x_n]_{x_0\cdots x_n})_m$$

is a free graded A-module spanned by the monomials of the form $x_0^{e_0} \cdots x_n^{e_n}$ with $\sum e_i = m$. The image of d^{n-1} is spanned by the monomials $x_0^{e_0} \cdots x_n^{e_n}$ with $\sum e_i = m$ and at least one $e_j \ge 0$. Hence

$$\begin{aligned}\mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(m)) = \mathrm{Coker}(d^{n-1}) &= A\left\{x_0^{e_0} \cdots x_n^{e_n} \mid e_i < 0\ \forall i \text{ and } \sum e_i = m\right\}\\ &= \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}]\right)_m.\end{aligned}$$

This gives

$$\mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}]\right)_{-n-1} = A \cdot x_0^{-1} \cdots x_n^{-1}.$$

The proof is finished. □

Corollary 2.3.2. *Let k be a field. For $m \ge 0$, we have*

$$\dim \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{m+n}{m}, \qquad \dim \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(-m)) = \binom{m-1}{n}.$$

We have $\mathrm{H}^p(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for all other (p, m).

Proof. Exercise. □

2.3.2 Cohomology of coherent sheaves on projective schemes

Theorem 2.3.3. *Let A be a noetherian ring. Let $X \subset \mathbb{P}^r_A$ be a projective scheme over A. For $n \in \mathbb{Z}$, consider the sheaf $\mathcal{O}_X(n)$ on X. Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(1) The cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are finitely generated A-modules for each $i \ge 0$.

(2) There exists an $n_0 > 0$ such that

$$\mathrm{H}^i(X, \mathcal{F}(n)) = 0 \qquad (\textit{where } \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$$

for all $n \ge n_0$ and $i > 0$.

To prove this, we need a couple of results.

Lemma 2.3.4. *Let X be a topological space and let $i\colon Z \subset X$ be a closed subset. Let $\mathcal{U}$ be an open cover of X, and let $\mathcal{U}_Z$ be the induced open cover of Z. Then for any sheaf $\mathcal{F}$ on Z and any $p \geq 0$, we have $\mathrm{H}^p(Z, \mathcal{F}) = \mathrm{H}^p(X, i_*\mathcal{F})$.*

Proof. This follows from the fact that for each open $U \subset X$, $\Gamma(U \cap Z, \mathcal{F}) = \Gamma(U, i_*\mathcal{F})$, so the two cohomolgy groups arise from the same Cech complexes. □

Lemma 2.3.5. *Let $f\colon X \to Y$ be a morphism of schemes. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{L}$ be a line bundle on Y. Then there exists an isomorphism*

$$\varphi\colon f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{\sim} f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right). \tag{2.1}$$

Proof. Let $\{U_i\}$ be an open cover of Y such that for each $i \in I$ there exists an isomorphism $\rho_i\colon \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. For $i \in I$, define an isomorphism

$$\varphi_i\colon\ \left(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L}\right)|_{U_i} \xrightarrow{\sim} \left(f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right)\right)|_{U_i}$$

as the composition

$$\left(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L}\right)|_{U_i} \cong f_*(\mathcal{F})|_{U_i} \cong \left(f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right)\right)|_{U_i}.$$

Note that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. Thus, the φ_i glue to an isomorphism (2.1). □

Lemma 2.3.6. *Let S be a graded ring and let M be a finitely generated graded S-module. Then M is generated by finitely many homogeneous elements, and there is a set of integers $a_1, \ldots, a_n \in \mathbb{Z}$ and a surjection of graded S-modules $\oplus_i S(-a_i) \to M$.*

Proof. First observe that there exists a set of generators $\{m_1, \ldots, m_n\} \subset M$ for M over S such that each m_i is homogeneous. Let $a_i = \deg(m_i)$. The map $S(-a_i) \to M$ that sends $1 \in S(a_i)_{a_i} = S_0$ to the element m_i is a morphism of graded S-modules. Moreover, the resulting map of graded S-modules $\oplus_i S(-a_i) \to M$ is surjective. □

Proof of Theorem 2.3.3. Let $i\colon X \hookrightarrow \mathbb{P}^r_A$ be the given closed embedding into $\mathbb{P}^r_A$. Then $i_*\mathcal{F}$ is coherent and

$$\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(\mathbb{P}^r_A, i_*\mathcal{F}),$$

see Lemma 2.3.4. Moreover, by Lemma 2.3.5, we have $\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n) = i_*\left(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)\right)$, so that

$$\begin{aligned}
\mathrm{H}^i(X, \mathcal{F}(n)) &= \mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{O}_X(n)) \\
&= \mathrm{H}^i(X, \mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)) \\
&= \mathrm{H}^i(\mathbb{P}^r, i_*\left(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)\right) \\
&= \mathrm{H}^i(\mathbb{P}^r_A, i_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r_A}(n)).
\end{aligned}$$

This reduces the theorem to the case $X = \mathbb{P}^r_A$.

Recall (see Proposition 1.2.8) that in this case, the coherent sheaf $\mathcal{F}$ on $X = \mathbb{P}^r_A$ is of the form $\mathcal{F} = \widetilde{M}$ for some finitely generated graded S-module M, where $S =$

$A[x_0, \ldots, x_n]$. Both parts of the theorem are trivially satisfied when $i > \dim \mathbb{P}^r_A = r + \dim(A)$. We take this as the base case, and proceed by downwards induction on i.

(1). As M is finitely generated, we may pick a surjection of graded A-modules

$$\bigoplus_k A(-a_k) \longrightarrow M.$$

The kernel K of this surjection is graded and finitely generated (see Lemma 1.1.4), so that we get an exact sequence of finitely generated graded A-modules

$$0 \to K \to \bigoplus_k A(-a_k) \to M \to 0.$$

Applying the tilde functor, which is exact by Lemma 1.1.11, we get an exact sequence of coherent sheaves

$$0 \to \mathcal{K} = \widetilde{K} \to \bigoplus_k \mathcal{O}_{\mathbb{P}^r_A}(-a_k) \to \mathcal{F} \to 0. \tag{2.2}$$

Taking the long exact sequence in cohomology yields:

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_k \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^r_A}(-a_k)) \to \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F}) \to \mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}) \to \cdots .$$

By the induction hypothesis, we have that $\mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K})$ is a finitely generated A-module. The A-module $\bigoplus_k \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^r_A}(-a_k))$ is also finitely generated, see Theorem 2.3.1. Hence, we get that $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated.

(2). It suffices to prove that for each $i > 0$, there exists $n_0 > 0$ such that $\mathrm{H}^i(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$ for all $n \geq n_0$. Indeed, one then takes the max of all such n_0 defined for the various $0 < i \leq r + \dim(A)$.

Twist the exact sequence (2.2) by $\mathcal{O}_{\mathbb{P}^r_A}(n)$ and take cohomology, to get an exact sequence

$$\cdots \to \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{K}(n)) \to \bigoplus_k \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n-a_k)) \to \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{F}(n)) \to \mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}(n)) \to \cdots .$$

Again, by downward induction on $i > 0$, we get some n_0 such that $\mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}(n)) = 0$ for $n \geq n_0$, and enlarging n_0 if necessary, we may assume $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}(n - a_k)) = 0$ for $n \geq n_0$ and all k (see Theorem 2.3.1. This gives $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F}(n)) = 0$ for $n \geq n_0$. $\square$

2.4 Lecture 19: Hypersurfaces

2.4.1 Field-valued points of schemes

Let k be a field and let X be a scheme over k.

Definition 2.4.1. For a scheme T over k, we write $X(T) = \mathrm{Hom}_{\mathsf{Sch}/k}(T, X)$. This is the set of morphisms of k-schemes $T \to X$. If $T = \operatorname{Spec} A$ is affine, we write $X(A) = X(T)$.

Note that for affine k-schemes $X = \operatorname{Spec} R$ and $T = \operatorname{Spec} A$, we have that $X(T) = X(A)$ is naturally in bijection with the set of morphisms of k-algebras $R \to A$.

Lemma 2.4.2. *Suppose that $X = \operatorname{Spec} R$ with*

$$R = k[t_1, \dots, t_n]/(f_1, \dots, f_m), \qquad f_i \in k[t_1, \dots, t_n].$$

Let $T = \operatorname{Spec} A$ be an affine scheme over k. Then there are natural bijections

$$\begin{aligned} X(A) = X(T) &= \operatorname{Hom}_{\mathsf{Sch}/k}(T, X) \\ &= \operatorname{Hom}_{k\text{-}\mathsf{Alg}}(R, A) = \{\alpha \in A^n \mid f_i(\alpha) = 0 \ \forall i \in \{1, \dots, m\}\}. \end{aligned}$$

Proof. Exercise. $\square$

Examples 2.4.3. (1) Let $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2)$. Then $X(\mathbb{R}) = \emptyset$.

(2) Let $X = \operatorname{Spec} \mathbb{R}[x, y]/(x + y, x - y)$. Then $X(\mathbb{R}) = 0 \in \mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2$.

Example 2.4.4. Let k be a field. Let $V = k^{n+1}$. Then there is a natural isomorphism of k-vector spaces $V \xrightarrow{\sim} V^\vee$ given by $e_i \mapsto e_i^\vee$. This gives an isomorphism

$$\mathbb{P}_k^n = \check{\mathbb{P}}_k^r,$$

where we recall that

$$\check{\mathbb{P}}_k^r = \mathbb{P}(V^\vee) \quad \text{and that} \quad \mathbb{P}(W) = \operatorname{Proj}(\operatorname{Sym}^*(W))$$

for a finite dimensional k-vector space W. For each field extension $k' \supset k$, one gets a canonical bijection (see also Example 1.2.19):

$$\mathbb{P}_k^n(k') = \left\{\text{lines } \ell \subset (k')^{n+1}\right\}.$$

2.4.2 Hypersurfaces in projective space

Definition 2.4.5. (1) A hypersurface is a closed subscheme $X \subset \mathbb{P}_k^n$ defined as

$$X = V(F) = \operatorname{Proj}(k[x_0, \dots, x_n]/(F)),$$

for some homogeneous polynomial $F \in k[x_0, \dots, x_n]$ of positive degree. The *degree* of this hypersurface is the degree of F.

(2) A *complete intersection of two hypersurfaces* $X \subset \mathbb{P}_k^n$ is a closed subscheme

$$X = V(F) \cap V(G) = V(F, G) \subset \mathbb{P}_k^n$$

defined by two homogeneous polynomials $F, G \in k[x_0, \dots, x_n]$ of positive degrees $d > 0, e > 0$ such that $V(F)$ and $V(G)$ have no irreducible component in common.

Example 2.4.6. Continue with the notation from Example 2.4.4. Let $X = V(F) \subset \mathbb{P}^n_k$ be a hypersurface. Then for each field extension $k' \supset k$, we have:

$$X(k') = \{\alpha = [x_0 \colon \cdots : x_n] \in \mathbb{P}^n(k') \mid F(\alpha) = 0\} \subset \mathbb{P}^n(k').$$

Exercise 2.4.7. For a hypersurface $X = V(F) \subset \mathbb{P}^n_k$ of degree $d > 0$, show that:

(1) $\dim(X) = n - 1$;

(2) the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is canonically isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(-d)$.

Exercise 2.4.8. For a complete intersection $X = V(F) \cap V(G) = V(F, G) \subset \mathbb{P}^n_k$, where $\deg(F) = d > 0$ and $\deg(G) = e > 0$, show that:

(1) $\dim(X) = n - 2$:

(2) for $R = k[x_0, \ldots, x_n]$, the sequence of graded R-modules

$$0 \to R(-d-e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} (F, G) \to 0$$

is exact, where $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1 F + h_2 G$. Applying the tilde functor, we get an exact sequence of $\mathcal{O}_{\mathbb{P}^n_k}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_X \to 0,$$

where $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is the ideal sheaf of $X \subset \mathbb{P}^n_k$.

2.4.3 Genus of a plane curve

Definition 2.4.9. Let $X \to \mathrm{Spec}\ (k)$ be a scheme of finite type over a field k. We say that X is *geometrically integral* (resp. *irreducible, reduced*) if $X_{\bar{k}} = X \times_k \bar{k}$ is integral (resp. irreducible, reduced).

Definition 2.4.10. Let k be a field. A *curve* over k is a geometrically integral and projective scheme C over k with $\dim(X) = 1$. The *genus* $g(C)$ of a curve C is the dimension of the k-vector space $\mathrm{H}^1(C, \mathcal{O}_C)$. This dimension is finite by Theorem 2.3.3. A *plane curve* is a hypersurface $C \subset \mathbb{P}^2_k$ which is geometrically irreducible.

Example 2.4.11. We have that $\mathbb{P}^1_k$ is a curve with $g(\mathbb{P}^1_k) = 0$.

Definition 2.4.12. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a plane curve defined by a homogeneous polynomial $F \in k[x_0, x_1, x_2]$ of positive degree. We say that C is *smooth* if there is no point $p \in C(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ such that $\partial F / \partial x_i(p) = 0$ for each $i = 0, 1, 2$. In other words, C is smooth if there is no $p \in \mathbb{P}^2(\mathbb{C})$ such that

$$F(p) = \partial F / \partial x_0(p) = \partial F / \partial x_1(p) = \partial F / \partial x_2(p) = 0.$$

Proposition 2.4.13. *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve. Then, with respect to the natural complex manifold structure of $\mathbb{P}^2(\mathbb{C})$, we have that $C(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ is a complex submanifold of dimension one.*

In particular, $C(\mathbb{C})$ is a connected and compact Riemann surface in a natural way.

Proof. Exercise. $\square$

Fact 2.4.14. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve. Then $g(C)$ equals the (topological) genus of the Riemann surface $X(\mathbb{C})$. In particular, $\mathrm{rank}_{\mathbb{Z}} H^1(C(\mathbb{C}), \mathbb{Z}) = 2 \cdot g(C)$.

Lemma 2.4.15. *Let $n \in \mathbb{Z}_{\geq 3}$ and let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact complex of finite dimensional vector spaces V^i over a field k. Then $\sum_{i=1}^n (-1)^i \dim(V_i) = 0$.*

Proof. First assume $n = 3$. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of finite dimensional vector spaces, then there exists a injective linear map $V_3 \to V_2$ whose composition with the given map $V_2 \to V_3$ is the identity: the sequence *splits*. Thus $V_2 \cong V_1 \oplus V_3$ in this case, whence the result.

We assume $n \geq 4$ and apply induction on n, assuming the lemma to be true for $n-1$. Let $W_{n-1} = \mathrm{Coker}(V_{n-3} \to V_{n-2})$. Then we have exact sequences $0 \to V_1 \to \cdots \to V_{n-3} \to V_{n-2} \to W_{n-1} \to 0$ and $0 \to W_{n-1} \to V_{n-1} \to V_n \to 0$. By the induction hypothesis, we have

$$\sum_{i=1}^{n-2} (-1)^i \dim(V_i) + (-1)^{n-1} \dim(W_{n-1}) = 0.$$

Moreover, the $n = 3$ case gives $(-1)^{n-1} \dim(W_{n-1}) = (-1)^{n-1} (\dim(V_{n-1}) - \dim(V_n))$. Hence,

$$\begin{aligned} 0 &= \sum_{i=1}^{n-2} (-1)^i \dim(V_i) + (-1)^{n-1} \dim(W_{n-1}) \\ &= \sum_{i=1}^{n-2} (-1)^i \dim(V_i) + (-1)^{n-1} (\dim(V_{n-1}) - \dim(V_n)) \\ &= \sum_{i=1}^{n-1} (-1)^i \dim(V_i) + (-1)^n \dim(V_n) \\ &= \sum_{i=1}^{n} (-1)^i \dim(V_i). \end{aligned}$$

We are done. $\square$

Theorem 2.4.16. *Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d > 0$. Then*

$$g(C) = (d-1)(d-2)/2.$$

Proof. Let $i \colon C \hookrightarrow \mathbb{P}^2_k$ be the natural closed immersion. Consider the ideal sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^2} \to i_* \mathcal{O}_C \to 0.$$

Using Lemma 2.3.4, we get a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(-d)) &\to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{H}^0(C, \mathcal{O}_C) \to \\ \to \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(-d)) &\to \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{H}^1(C, \mathcal{O}_C) \to \\ \to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}(-d)) &\to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to 0. \end{aligned}$$

In view of Lemma 2.4.15 and Corollary 2.3.2, this gives:

$$0 - 1 + 1 - 0 + 0 - g(X) + \binom{d-1}{2} - 0 = 0.$$

Therefore,

$$g(X) = \binom{d-1}{2} = \frac{(d-1)!}{2!(d-3)!} = \frac{(d-1)(d-2)}{2}.$$

This proves the proposition. □

Example 2.4.17. Let $C = V(ZY^2 - X^3 - Z^3) \subset \mathbb{P}^2_{\mathbb{C}}$. Then C is smooth (see Definition 2.4.12), and the Riemann surface $C(\mathbb{C})$ is topologically a torus. Hence $g(C) = 1$ (see Fact 2.4.14). This is compatible with Theorem 2.4.16, since $1 = (3-1)(3-2)/2$.

Chapter 3

Divisors

3.1 Lecture 20 : Bézout's theorem and Weil divisors

3.1.1 Bézout's theorem

Let k be an algebraically closed field. Let $C \subset \mathbb{P}^2_k$ and $D \subset \mathbb{P}^2_k$ be two plane curves of degrees $d > 0$ and $e > 0$, that have no irreducible component in common. This implies that the scheme-theoretic intersection

$$Z = C \times_{\mathbb{P}^2_k} D \subset \mathbb{P}^2_k$$

is a zero-dimensional subscheme of $\mathbb{P}^2_k$. In particular, the underlying topological space $|Z|$ of Z consists of finitely many closed points $p_1, \ldots, p_r \in |\mathbb{P}^2_k|$. Note that there exists an automorphism $\phi \in \mathrm{Aut}(\mathbb{P}^2_k)$ such that $\phi(|Z|)$ is contained in the affine open

$$U_0 \coloneqq D_+(x_0) = \mathrm{Spec}\ (k[x_0, x_1, x_2]_{(x_0)}) \cong \mathrm{Spec}\ (k[x, y]).$$

Replacing C by $\phi(C)$ and D by $\phi(D)$, we get that $Z \subset U_0 \subset \mathbb{P}^2_k$. Let

$$\mathfrak{m}_i \subset k[x, y]$$

be the maximal ideal associated to the closed point $p_i \in U_0 = \mathrm{Spec}\ k[x, y] = \mathbb{A}^2_k$.

Theorem 3.1.1 (Bézout's theorem)**.** *Under the above notation and assumptions,*

$$\dim \mathrm{H}^0(Z, \mathcal{O}_Z) = \sum_{i=1}^{r} \dim_k \left(\frac{k[x, y]}{(f, g)}\right)_{\mathfrak{m}_{z_i}} = d \cdot e.$$

Example 3.1.2. Let $C = V(x_1 - x_2)$ and $D = V(x_1 + x_2)$. Then $Z = C \times_{\mathbb{P}^2_k} D = V(x_1 - x_2, x_1 + x_2) = V(x_1, x_2) \subset U_0$. We get $Z = \mathrm{Spec}\ k$ with closed embedding $\mathrm{Spec}\ k \hookrightarrow U_0 = \mathbb{A}^2_k$ given by $0 \in \mathbb{A}^2_k(k) = \mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}\ k, \mathbb{A}^2_k)$, see Lemma 2.4.2.

Proof of Theorem 3.1.1. Since Z is a zero-dimensional subscheme of $U_0 = \mathrm{Spec}\ k[x, y]$, it is clear that

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^{r} \mathcal{O}_{Z, p_i},$$

and that

$$\mathcal{O}_{Z,p_i} = \mathcal{O}_{U_0,p_i}/\mathcal{I}_{Z,p_i} = (\mathcal{O}(U_0)/\mathcal{I}_Z(U_0))_{\mathfrak{m}_i} = \left(\frac{k[x,y]}{(f,g)}\right)_{\mathfrak{m}_{z_i}} \qquad \forall i \in \{1,\dots,r\}\,.$$

Moreover, for the natural closed immersion $i\colon Z \hookrightarrow \mathbb{P}^2_k$, we have the ideal sheaf sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2_k} \to i_*\mathcal{O}_Z \to 0$, which gives exact sequences

$$0 \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \to \mathrm{H}^0(Z, \mathcal{O}_Z) \to \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) \to 0$$

and

$$0 = \mathrm{H}^1(Z, \mathcal{O}_Z) = \mathrm{H}^1(\mathbb{P}^2, i_*\mathcal{O}_Z) \to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0,$$

where $\mathrm{H}^1(Z, \mathcal{O}_Z) = 0$ because $\dim(Z) = 0$. This gives:

$$\begin{aligned}\dim \mathrm{H}^0(Z, \mathcal{O}_Z) &= \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) - \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z),\\ \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) &= 0.\end{aligned}$$

Recall the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^2_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^2_k}(-e) \to \mathcal{I}_Z \to 0, \tag{3.1}$$

see Exercise 2.4.8. As $\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m)) = 0$ for each $m \in \mathbb{Z}$, see Corollary 2.3.2, we get an exact sequence

$$0 = \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^2_k}(-e)) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e)) = 0,$$

which shows that $\mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) = 0$. Hence

$$\dim \mathrm{H}^0(Z, \mathcal{O}_Z) = \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) = 1 + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z).$$

Furthermore, (3.1) gives a long exact sequence

$$\begin{aligned}0 \to \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e)) &\to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \oplus \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-e))\\ &\to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) = 0,\end{aligned}$$

where the vanishing $\mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) = 0$ has been shown above. We conclude that

$$\begin{aligned}\dim_k \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) &= \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e))\\ &\quad - \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) - \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-e))\\ &= \binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2},\end{aligned}$$

see Corollary 2.3.2. Now

$$\begin{aligned}&\binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2}\\ &\qquad = \frac{(d+e-1)(d+e-2)}{2} - \frac{(d-1)(d-2)}{2} - \frac{(e-1)(e-2)}{2}\\ &\qquad = \frac{1}{2} \cdot \Big(\big(d^2 + 2de - 3d + e^2 + 2\big) - \big(d^2 - 3d + 2\big) - \big(e^2 - 3e + 2\big)\Big)\\ &\qquad = \frac{2de - 2}{2} = de - 1.\end{aligned}$$

Therefore,

$$\dim_k \mathrm{H}^0(Z, \mathcal{O}_Z) = 1 + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) = 1 + de - 1 = de.$$

The theorem follows. $\square$

3.1.2 Definition of an algebraic variety

In this course, we follow the Stacks Project with our notion of *algebraic variety*:

Definition 3.1.3. Let k be a field. Then an *algebraic variety* (or simply a *variety*) over k is a scheme X over k such that X is integral, and such that the structure morphism $X \to \operatorname{Spec} k$ is separated and of finite type.

Remark 3.1.4. Suppose that k'/k is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_k k'$ is not necessarily a variety over k'. For instance, let $k = \mathbb{Q}$, let $X = \operatorname{Spec} \mathbb{Q}(i)$ and let $k' = \operatorname{Spec} \mathbb{Q}(i)$. Then

$$X_{k'} = \operatorname{Spec} \left(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i)\right) \cong \operatorname{Spec} \mathbb{Q}(i) \sqcup \operatorname{Spec} \mathbb{Q}(i).$$

Remark 3.1.5. The same counterexample shows that the product of two varieties need not be a variety. If the ground field is algebraically closed however, then the product of varieties X and Y over $k = \bar{k}$ is a variety over k. This statement readily reduces to the affine case, and in fact to the statement that for an algebraically closed field k and two finitely generated k-algebras A and B which are integral domains, the tensor product $A \otimes_k B$ is an integral domain. We leave this as an exercise.

Corollary 3.1.6. *Let $X \to \operatorname{Spec} k$ be a projective morphism, where k is a field and X is a scheme. Then X is separated and of finite type over k. In particular, if X is integral, then X is a variety over k.*

Proof. Indeed, the composition of two separated (resp. finite type) morphisms is separated (resp. of finite type), and $\mathbb{P}^n_k$ is separated and of finite type over k. $\square$

Example 3.1.7. Let C be a curve over a field k. Then C is an algebraic variety.

Example 3.1.8. Let $X = \operatorname{Spec} \mathbb{C}$ and consider the morphism $X \to \operatorname{Spec} \mathbb{R}$. This turns X into an algebraic variety over $\mathbb{R}$.

Non-Example 3.1.9. Let k be a field and consider the scheme $X = \operatorname{Spec} k[x]/(x^2)$ with its natural morphism $X \to \operatorname{Spec} k$. Then X is irreducible, separated and of finite type over k. However, X is not an algebraic variety over k, since X is not reduced.

3.1.3 Smooth varieties

Let k be a field. Let $A = k[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$ be a finitely generated k-algebra, with $f_i \in k[t_1, \ldots, t_n]$ for $i = 1, \ldots, m$. Note that for each $i \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, n\}$, we get a polynomial

$$\frac{\partial f_i}{\partial t_j} \in k[t_1, \ldots, t_n],$$

and hence an element $\frac{\partial f_i}{\partial t_j}(\alpha) \in \bar{k}$ for each $\alpha \in (\bar{k})^n$.

Definition 3.1.10. With the above notation, we say that A is *smooth* over k if for each $\alpha \in (\bar{k})^n$ such that $f_i(\alpha) = 0$ for each $i \in \{1, \ldots, m\}$, the rank of the $m \times n$-matrix

$$\left(\frac{\partial f_i}{\partial t_j}(\alpha) \right)_{i=1,\ldots,m, j=1,\ldots,n} \in \mathrm{M}_{m\times n}(\bar{k})$$

is maximal (that is, equal to $\min(m, n)$).

Definition 3.1.11. Let X be a variety over a field k. Then X is *smooth* over k if there exists an affine open covering $X = \cup_i U_i$ and for each i an isomorphism of k-schemes $U_i \cong \operatorname{Spec} A$ for some finitely generated k-algebra A which is smooth over k.

Lemma 3.1.12. *Let X be a variety over k. If X is smooth over k then each open subscheme $U \subset X$ is smooth over k.*

Proof. Exercise. □

Example 3.1.13. Let k be a field and let $X = V(F) \subset \mathbb{P}^n_k$ be a hypersurface. Then X is smooth over k if and only if for each

$$\alpha = [x_0 \colon \cdots : x_n] \in X(\bar{k}) \subset \mathbb{P}^n(\bar{k}),$$

there exists $i \in \{0, 1, \ldots, n\}$ such that $(\partial F/\partial x_i)(\alpha) \neq 0$. In particular, Definitions 2.4.12 and 3.1.11 are compatible.

Example 3.1.14. Let k be a field and let p be a prime number. Consider the curve $C \subset \mathbb{P}^2_k$ defined by the equation $x_0^p + x_1^p + x_2^p = 0$. In other words, $C = \operatorname{Proj}(k[x_0, x_1, x_2]/(x_0^p + x_1^p + x_2^p))$.

(1) If the characteristic of k is different from p, then C is smooth. Namely, we have $\partial F/\partial x_i = p \cdot x_i^{p-1}$ for $i = 0, 1, 2$, and if, for each $i \in \{0, 1, 2\}$, this homogeneous degree $p - 1$ polynomial $p \cdot x_i^{p-1}$ vanishes at some $\alpha = [a_0 \colon a_1 \colon a_2] \in \mathbb{P}^2(\bar{k})$, then $a_0 = a_1 = a_2 = 0$, which is absurd.

(2) If the characteristic of k equals p, then C is not smooth. Namely, we then have $\partial F/\partial x_i = p \cdot x_i^{p-1} = 0$ for $i = 0, 1, 2$. Thus for *any* $\alpha \in C(\bar{k})$, we get $F(\alpha) = \partial F/\partial x_i(\alpha) = 0$ for $i = 0, 1, 2$.

3.1.4 Normal schemes

We consider the following important notion in scheme theory.

Definition 3.1.15. (1) Let A be a ring which is a domain. Then A is called *normal* if A is integrally closed in its field of fractions $Q(A)$. This means that for each $\alpha \in Q(A)$ which is integral over A, we have $\alpha \in A$. Equivalently: for each monic polynomial $f \in A[x]$ and each $\alpha \in Q(A)$ with $f(\alpha) = 0$, we have $\alpha \in A$.

(2) A ring R is *normal* if for each prime ideal $\mathfrak{p} \subset R$, the localization $R_{\mathfrak{p}}$ is a normal domain.

(3) A scheme X is called *normal* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

Suppose $X = \operatorname{Spec} A$ is an affine scheme such that A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is noetherian, then this is the case (exercise).

Lemma 3.1.16. *Let X be a scheme. The following are equivalent:*

(1) The scheme X is normal.

(2) For every affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is normal.

(3) There exists an affine open covering $X = \cup_i U_i$ such that each ring $\mathcal{O}_X(U_i)$ is normal.

(4) There exists an open covering $X = \cup_i X_i$ such that the scheme X_i is normal for each i.

Moreover, if X is normal, then every open subscheme $U \subset X$ is normal.

Proof. Exercise. □

Lemma 3.1.17. *Let X be a normal integral scheme. Then for each non-empty open $U \subset X$, the scheme U is normal and integral, and $\mathcal{O}_X(U)$ is a normal integral domain.*

Proof. The fact that U is normal and integral is clear. Thus, it suffices to show that $\mathcal{O}_X(X)$ is a normal integral domain. For this, see e.g. [Stacks Project, tag 0358]. □

Theorem 3.1.18. *Let A be a noetherian local domain of dimension one, with maximal ideal $\mathfrak{m}$. The following are equivalent:*

(1) A is a discrete valuation ring;

(2) A is normal;

(3) $\mathfrak{m}$ is a principal ideal.

Proof. See Atiyah–Maconald (Proposition 9.2 on page 94). □

Corollary 3.1.19. *Let k be an algebraically closed field and let C be a curve over k. Then C is smooth over k if and only if C is normal.*

Proof. This uses: (1) any discrete valuation ring is a *regular* local ring of dimension one, and conversely; (2) since k is algebraically closed, any variety X over k is smooth over k if and only if for each $x \in X$ there exists an affine open neighbourhood $U \subset X$ such that the localizations $R_{\mathfrak{p}}$ of $R = \mathcal{O}_X(U)$ are all regular. Details omitted. □

In arbitrary dimensions, one has:

Proposition 3.1.20. *Let X be a smooth variety over a field k. Then X is normal.*

Proof. We will prove this later (see [insert future reference here]). □

3.1.5 Codimension

Definition 3.1.21. Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset of X. The *codimension* of Y in X, denoted by $\operatorname{codim}(Y, X)$, is the supremum of all integers n such that there exists a chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of irreducible closed subsets Y_i of X.

Proposition 3.1.22. *Let X be a scheme, let $x \in X$ and define $Y = \overline{\{x\}} \subset X$. Then Y is irreducible, and* $\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,x}$.

Proof. Since Y has a generic point, it is irreducible. Let $Y = Y_0 \subsetneq \cdots \subsetneq Y_n \subset X$ be a chain of irreducible closed subsets. Let $U \subset X$ be an affine open neighbourhood of x in X. Since $U \cap Y_i \neq \emptyset$ for each i, we have $\eta_i \in U$ for each i. Moreover, for each i, $Y_i \cap U$ is a closed subset in U, defined by a prime ideal $\mathfrak{p}_i \subset R$, where $R = \mathcal{O}_X(U)$. Thus we get a chain of prime ideals

$$\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0 = \mathfrak{p},$$

where $\mathfrak{p}$ is the prime ideal that defines $Y \cap U$ in U. Hence we have

$$\operatorname{codim}(Y, X) = \sup_n (\exists \mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0 = \mathfrak{p} \subset R) = \operatorname{height}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}).$$

As $R_{\mathfrak{p}} = \mathcal{O}_{X,x}$, we get $\dim \mathcal{O}_{X,x} = \dim R_{\mathfrak{p}} = \operatorname{codim}(Y, X)$, whence the result. □

Theorem 3.1.23. *Let k be a field and let X be a variety over k, with generic point $\eta \in X$. Let $k(X) = \mathcal{O}_{X,\eta}$ be the function field of X. Then:*

(1) the dimension of X agrees with the transcendence degree of $k(X)$ over k;

(2) for each non-empty open subset $U \subset X$, we have $\dim(U) = \dim(X)$*;*

(3) *if $Y \subset X$ is a closed subvariety, then all maximal chains of irreducible subvarieties*

$$Y \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subset X$$

have the same length;

(4) *we have* $\operatorname{codim}(Y, X) = \dim(X) - \dim(Y)$.

Proof. We will not prove this here. □

3.1.6 Weil divisors

Definition 3.1.24. Let X be a normal integral noetherian scheme.

(1) A *prime divisor* is an integral subscheme $Z \subset X$ of codimension one.

(2) A *Weil divisor* of X is an element of the free abelian group generated by the prime divisors of X. We denote the group of Weil divisors by $\operatorname{Div}(X)$. Thus, an element $D \in \operatorname{Div}(X)$ can be written as a formal linear combination of prime divisors

$$D = \sum_{Z \subset X \text{prime}} n_Z \cdot Z$$

with $n_Z \in \mathbb{Z}$ for each prime divisor $Z \subset X$, and such that $n_Z = 0$ for all but finitely many prime divisors $Z \subset X$.

(3) We say that a Weil divisor $D = \sum n_Z \cdot Z$ is *effective* if $n_Z \geq 0$ for each prime divisor Z.

(4) Any Weil divisor $D = \sum n_Z Z$ can be written as $D = \sum_{i=1}^{k} n_i \cdot Z_i$ where Z_i is a prime divisor and $n_i \in \mathbb{Z} - \{0\}$ for each $i \in \{1, \ldots, k\}$. This gives a closed subset $\cup_i Z_i \subset X$ called the *support* of the Weil divisor D.

(5) Given two Weil divisors $D = \sum_Z n_Z Z$ and $D' = \sum m_Z Z$, we say that $D \geq D'$ if $D - D'$ is effective, or equivalently, if $n_Z \geq m_Z$ for all prime divisors Z. This turns $\operatorname{Div}(X)$ into a partially ordered group.

Example 3.1.25. Let k be a field and let $X = \mathbb{P}^1_k$ be the projective line over k. Since C is a curve, any irreducible closed subset of codimension one on X is a closed point. For example, for any

$$f \in \operatorname{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k, \mathbb{P}^1_k) = \mathbb{P}^1_k(k) = \left\{\text{lines in } k^2\right\},$$

the image $f(\operatorname{Spec} k)$ in $\mathbb{P}^1_k$ is a closed point, and the map

$$\mathbb{P}^1_k(k) \to \left\{\text{closed points } x \in \mathbb{P}^1_k\right\}$$

is injective. In this way, we get some examples of Weil divisors on $\mathbb{P}^1_k$:

$$\begin{aligned} D_1 &:= 3 \cdot (1\colon 0) - 5 \cdot (0\colon 1), \\ D_2 &:= (1\colon 1) + 5 \cdot (0\colon 1), \\ D_1 + D_2 &= 3 \cdot (1\colon 0) + (1\colon 1). \end{aligned}$$

3.2 Lecture 21 : The divisor class group of a scheme

3.2.1 Principal Weil divisors

Let X be a normal integral noetherian scheme with generic point $\eta \in X$ and fraction field $K = k(X) = \mathcal{O}_{X,\eta}$. Since X is normal, for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a domain which is integrally closed in its field of fractions $Q(\mathcal{O}_{X,x}) = K$.

Lemma 3.2.1. *Let X be a normal integral noetherian scheme. Let $\xi \in X$ be a point such that* $\operatorname{codim}(\overline{\{\xi\}}, X) = 1$.

(1) The reduced closed subscheme $\overline{\{\xi\}} \subset X$ is a prime divisor, and every prime divisor arises uniquely in this way.

(2) The local ring $A = \mathcal{O}_{X,\xi}$ is a discrete valuation ring.

Proof. Note that $\overline{\{\xi\}}$ is irreducible since it has a generic point, hence it is a prime divisor. For an arbitrary prime divisor $Z \subset X$, the generic point η_Z of Z gives a codimension one point $\eta_Z \in X$. As for part (2), this follows from Theorem 3.1.18. □

This has the following implication. By Theorem 3.1.18, for each codimension one point $\xi \in X$, the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring. Thus, this ring is equipped with an associated valuation

$$v \colon K \longrightarrow \mathbb{Z} \cup \{\infty\},$$

such that $A = v^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$.

In fact, one can define v explicitly as follows. Given $a \in A - \{0\}$, the ideal $(a) \subset A$ has the property that $(a) = \mathfrak{m}^n$ for some $n \in \mathbb{Z}_{\geq 0}$, and we define $v(a) = n$. This gives a function $v \colon A - \{0\} \to \mathbb{Z}$ which extends to $K^* = \left\{\frac{a}{b} \colon a, b \in A - \{0\}\right\}$ by putting $v(a/b) = v(a) - v(b)$, and then to a map $v \colon K \to \mathbb{Z} \cup \{\infty\}$ by putting $v(0) = \infty$.

Definition 3.2.2. Let $f \in K = k(X)$. For every prime divisor $Z \subset X$, we get by the above a valuation $v_Z \colon K \to \mathbb{Z} \cup \{\infty\}$, which allows us to define

$$\operatorname{ord}_{Z,X}(f) := v(f).$$

Lemma 3.2.3 (Algebraic Hartog's lemma)**.** *Let A be an integrally closed noetherian integral domain and let $x \in K$. Let $K = Q(A) = \operatorname{Frac}(A)$ be the fraction field of A. Then $x \in A$ if and only if $x \in A_{\mathfrak{p}} \subset K$ for all height one primes ideals $\mathfrak{p} \subset A$.*

Proof. We do not prove this here. □

Corollary 3.2.4. *Let A be a noetherian normal domain, and $f \in Q(A)$. Then $\operatorname{ord}_{V(\mathfrak{p}),\operatorname{Spec} A}(f) \geq 0$ for all primes $\mathfrak{p} \subset A$ of height one if and only if $f \in A$, and $\operatorname{ord}_{V(\mathfrak{p}),\operatorname{Spec} A}(f) = 0$ for all primes $\mathfrak{p} \subset A$ of height one if and only if $f \in A^*$.*

Proof. Let $f \in Q(A)^*$. Then apply Lemma 3.2.3 to f and to $f^{-1} \in Q(A)$. □

Lemma 3.2.5. *Suppose that X is a normal integral noetherian scheme with fraction field K and let $f \in K^*$. Then $\operatorname{ord}_{Z,X}(f) = 0$ for all but finitely many primes $Z \subset X$.*

Proof. We proceed in two steps:

Step 1: *Reduction to the case where $X = \operatorname{Spec} A$ is affine and $f \in A$:* Consider a non-empty affine open subset V of X. Let $R = \mathcal{O}_X(V)$. Then K is the fraction field of R, so that $f = a/b$ for some $a, b \in R$ which are both non-zero. We then look at the affine open $U \coloneqq D(b) \subset V \subset X$. This is an affine open where b is invertible, so that $f = a/b \in R_b = \Gamma(U, \mathcal{O}_X)$. The complement $W \coloneqq X - U$ is a closed subset of codimension at least one, since X is integral (which implies U is non-empty). Notice that

$$\sum_Z \operatorname{ord}_{Z,X}(f) \cdot Z = \sum_{Z \subset W} \operatorname{ord}_{Z,X}(f) Z + \sum_{Z \not\subset W} \operatorname{ord}_{Z,X}(f) Z,$$

and that there are only finitely many prime divisors $Z \subset X$ that satisfy $Z \subset W$. Thus, it suffices to show that $\operatorname{ord}_Z(f) = 0$ for almost all prime divisors $Z \subset X$ with $Z \cap U \neq \emptyset$. Notice that, for primes $Z \subset X$ with $Z \cap U \neq \emptyset$, we have

$$\operatorname{ord}_{Z,X}(f) = \operatorname{ord}_{Z \cap U, U}(f),$$

since $\mathcal{O}_{X,\xi} = \mathcal{O}_{U,\xi}$ for the generic point $\xi \in Z$. Now the sum $\sum_{Z \subset W} \operatorname{ord}_{Z,X}(f) Z$ is finite since W has finitely many irreducible components of codimension one. Hence it remains to show that $\operatorname{ord}_{Z \cap U, U}(f) = 0$ for $f \in \Gamma(U, \mathcal{O}_X)$ and almost all primes $Z \subset X$ with $Z \cap U \neq \emptyset$, so that indeed, we may assume that $X = \operatorname{Spec} A$ is affine and $f \in A$.

Step 2: *Case where $X = \operatorname{Spec} A$ is affine and $f \in A$:* We now have $\operatorname{ord}_Z(f) \geq 0$, and $\operatorname{ord}_Z(f) > 0$ if and only if $\mathfrak{p} | (f)_\mathfrak{p} \subset A_\mathfrak{p}$ for all $\mathfrak{p}$ of height one in Z if and only if $f \in \mathfrak{p}$ for all primes $\mathfrak{p}$ in Z if and only if Z is contained in $V(f) \subset \operatorname{Spec} A$. Since $V(f)$ has finitely many irreducible components of codimension one, we are done. $\square$

Definition 3.2.6. Let X be a normal integral noetherian scheme with fraction field K. For $f \in K^*$, define its corresponding Weil divisor $\operatorname{div}(f)$ as

$$\operatorname{div}(f) \coloneqq \sum_Z \operatorname{ord}_{Z,X}(f) Z,$$

where the sum runs over all prime divisors. Any Weil divisor D of the form $D = \operatorname{div}(f)$ for some $f \in K^*$ is called a *principal* Weil divisor.

Example 3.2.7. Let A be a normal noetherian integral domain and let $X = \operatorname{Spec} A$. Let K be the fraction field of A. Then for any $f \in K^*$, we have

$$\operatorname{div}(f) = \sum_{\mathfrak{p} \text{ height } 1} \operatorname{ord}_{V(\mathfrak{p}), \operatorname{Spec} A}(f) \cdot V(\mathfrak{p}).$$

Example 3.2.8. Let A be a discrete valuation ring with maximal ideal $\mathfrak{m} \subset A$. Let $t \in A$ such that $\mathfrak{m} = (t) \subset A$. The underlying topological space $|\operatorname{Spec} A|$ consists of two points: $|\operatorname{Spec} A| = \{\eta, \mathfrak{m}\}$. The point $\mathfrak{m}$ is closed and the point $\eta = (0)$ is open. We have $(0) \subsetneq \mathfrak{m}$ and $\mathfrak{m}$ is the only prime ideal of height one. For $f \in K = \operatorname{Frac}(A)$, we can write $f = u \cdot t^n$ for some $n \in \mathbb{Z}$ and $u \in A^*$. Then $\operatorname{div}(f) = \operatorname{ord}_{V(\mathfrak{m}), \operatorname{Spec} A}(f)$ and this equals $v(f) = v(u \cdot t^n) = v(t^n) = n$, where $v \colon K \to \mathbb{Z} \cup \{\infty\}$ is the valuation.

Lemma 3.2.9. *Let X be a normal integral noetherian scheme. The set of principal Weil divisors forms a subgroup of* $\mathrm{Div}(X)$.

Proof. For $f, g \in K^*$, we have $\mathrm{div}(f) - \mathrm{div}(g) = \mathrm{div}(f/g)$. $\square$

In fact, the map $K^* \to \mathrm{Div}(X)$ sending f to $\mathrm{div}(f)$, is a group homomorphism. If $X = \mathrm{Spec}\ A$ is affine, then $\mathrm{div}(f) = 0$ if and only if $f \in A^*$ (see Corollary 3.2.4); thus we get an exact sequence $0 \to A^* \to K^* \to \mathrm{Div}(X)$ in that case.

3.2.2 Examples

Example 3.2.10. Let $X = \mathrm{Spec}\ \mathbb{Z}$ with function field $Q(\mathbb{Z}) = \mathbb{Q}$. We claim that the map $\mathbb{Q}^* \to \mathrm{Div}(X)$ is surjective. Indeed, any element $D \in \mathrm{Div}(X)$ is a finite sum $D = \sum_i n_i \cdot V(p_i)$, where the p_i are prime numbers and $n_i \in \mathbb{Z}$; we have $\mathrm{div}(\prod_i p_i^{n_i}) = D$.

Example 3.2.11. Let $X = \mathbb{A}^1_k$. Consider $f = t^2(t-1)^{-1} \in k(t) = k(\mathbb{A}^1_k)$. Then $\mathrm{div}(f) = 2 \cdot [0] - [1]$, where $0, 1 \in \mathbb{A}^1(k)$ give closed points of $\mathbb{A}^1_k$.

Example 3.2.12. Let k be a field and consider $X = \mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$, whose function field is $k(X) = k(t)$, where $t = x_1/x_0$. Consider the rational function

$$f = t^2(t-1)^{-1} \in K.$$

Notice that $\mathbb{P}^1_k - U_0 = \{\infty\}$, where $U_0 = D_+(x_0) = \mathrm{Spec}\ k[x_0, x_1]_{(x_0)} = \mathrm{Spec}\ k[t]$, and where $\infty = [0:1] \in U_1(k)$. Therefore:

$$\begin{aligned}\mathrm{div}(f) &= \sum_{p \in U_0} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) \cdot \infty \\ &= 2 \cdot [1\colon 0] - [1\colon 1] + \mathrm{ord}_\infty(f) \cdot \infty,\end{aligned}$$

because

$$\sum_{p \in U_0} \mathrm{ord}_p(f) = \sum_{p \in \mathrm{Spec}\ k[t]} \mathrm{ord}_p(f) = 2 \cdot [0] - [1]$$

by Example 3.2.11. Moreover, using the identification

$$U_1 = D_+(x_1) = \mathrm{Spec}\ k[x_0, x_1]_{(x_1)} = \mathrm{Spec}\ k[u]$$

with $u = x_0/x_1 = t^{-1}$, we get

$$f = t^2(t-1)^{-1} = u^{-2}(u^{-1}-1)^{-1} = \frac{1}{u^2(u^{-1}-1)} = \frac{1}{u-u^2}.$$

Therefore, if we let $g = (u-u^2)^{-1} = u^{-1}(1-u)^{-1} \in k(u)$, then

$$\mathrm{ord}_\infty(f) = \mathrm{ord}_0(g) = -1.$$

All in all, this gives

$$\mathrm{div}(f) = \sum_{p \in U_0} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) \cdot [0\colon 1] = 2 \cdot [1\colon 0] - [1\colon 1] - [0\colon 1].$$

3.2.3 The divisor class group

Definition 3.2.13. Let X be a noetherian integral normal scheme with function field K. We define the *divisor class group* of X (or simply the *class group* of X) as the group of Weil divisors modulo principal Weil divisors, and we denote it by $\mathrm{Cl}(X)$. Thus, we have

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\left\langle \mathrm{div}(f) \mid f \in K^* \right\rangle.$$

Two Weil divisors D and D' are said to be *linearly equivalent* (written $D \sim D'$) if they have the same image in $\mathrm{Cl}(X)$; in other words, if $D - D' = \mathrm{div}(f)$ for some $f \in K^*$.

Example 3.2.14. Let A be a noetherian normal domain with fraction field K. Write $\mathrm{Div}(A) = \mathrm{Div}(\mathrm{Spec}\, A)$ and $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Spec}\, A)$. In view of Corollary 3.2.4, there is an exact sequence of abelian groups

$$0 \longrightarrow A^* \longrightarrow K^* \longrightarrow \mathrm{Div}(A) \longrightarrow \mathrm{Cl}(A) \longrightarrow 0. \tag{3.2}$$

Remark 3.2.15. Let K be a number field. Then K is the fraction field of its ring of integers $\mathcal{O}_K$, and in this case, $\mathrm{Div}(\mathcal{O}_K)$ can be identified with the group of *fractional ideals* (these are non-zero finitely generated $\mathcal{O}_K$-submodules of K, which form a group under ideal multiplication), and $\mathrm{Cl}(\mathcal{O}_K)$ with the group of fractional ideals modulo the *principal fractional ideals* (these are the fractional ideals generated by an element of K^*). A classical result in number theory says that the group $\mathrm{Cl}(\mathcal{O}_K)$ is finite. Note that $\mathrm{Cl}(\mathcal{O}_K) = 0$ if and only if $\mathcal{O}_K$ is a unique factorization domain. For example, $\mathbb{Z}[\sqrt{-5}]$ is not a UFD since $2\cdot 3 = (1-\sqrt{-5})(1+\sqrt{-5})$, and in fact $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2$.

Example 3.2.16. Consider the ring $\mathbb{Z}$. Then $\mathrm{Cl}(\mathbb{Z}) = 0$, see Example 3.2.10.

This generalizes as follows:

Proposition 3.2.17. *Let A be a normal noetherian integral domain and let $X = \mathrm{Spec}\, A$. Then $\mathrm{Cl}(X) = 0$ if and only if A is a unique factorization domain.*

Proof. Suppose that A is a unique factorization domain. Let $Z \subset X$ be a non-zero prime divisor in X. Then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$ of height one. Take $f \in \mathfrak{p}$ non-zero, and let $f = f_1 \cdots f_n$ be a factorization of f into irreducible elements of A. Since $\mathfrak{p}$ is prime, we see that $f_i \in \mathfrak{p}$ for some i. Since A is a UFD, the element f_i is prime. Thus $\mathfrak{p}$ contains the prime ideal (f_i). As $\mathfrak{p}$ has height one, we have $\mathfrak{p} = (f_i)$. Thus gives $Z = V(\mathfrak{p}) = V(f_i) \subset X$. But note that $\mathrm{div}(f) = V(f_i)$. Therefore, $Z = \mathrm{div}(f_i)$, and we get that $\mathrm{Cl}(X) = 0$.

Conversely, assume $\mathrm{Cl}(X) = 0$. Then every height one prime ideal $\mathfrak{p}$ is principal. Indeed, there is an $f \in K^*$ such that $\mathrm{div}(f) = V(\mathfrak{p})$, one has $f \in A$ (in view of the exact sequence (3.2)), and one can show that $\mathfrak{p} = (f)$ (exercise). Now since A is noetherian, every non-zero non-unit element $a \in A$ has a factorization into irreducibles, hence it suffices to show that an irreducible element $a \in A$ is prime. Let $(a) \subset \mathfrak{p}$ be a minimal prime over (a). Then $\mathfrak{p}$ has height one (exercise). By the above, $\mathfrak{p}$ is principal, so that $\mathfrak{p} = (b)$ for some $b \in A$. Hence $a \in (b)$ so that $a = bc$ for some $c \in A$, which must be a unit because a is irreducible. Thus, $(a) = (b) = \mathfrak{p}$ is prime, and we win. $\square$

Corollary 3.2.18. *Let k be a field and let $n \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.* $\square$

3.2.4 Class group of projective space

Let k be a field and consider $\mathbb{P}^n_k = \mathrm{Proj}(R)$ with $R = k[x_0, \ldots, x_n]$. Prime divisors on $\mathbb{P}^n_k$ are given by homogeneous height one prime ideals $\mathfrak{p} \subset R$. For such a prime ideal $\mathfrak{p}$ we have $\mathfrak{p} = (g)$ for some non-zero irreducible homogeneous polynomial $g \in R$ (see the proof of Proposition 3.2.17). The generator g is unique up to scalar, so the degree $\deg(\mathfrak{p}) \coloneqq \deg(g)$ of a height one prime ideal $\mathfrak{p}$ is well-defined. This gives a group homomorphism

$$\deg\colon \mathrm{Div}(\mathbb{P}^n_k) \longrightarrow \mathbb{Z}.$$

Exercise 3.2.19. (1) For a rational function $f \in K(\mathbb{P}^n_k)$, show that $\deg(\mathrm{div}(f)) = 0$.

(2) Show that deg factors through an isomorphism $\mathrm{Cl}(\mathbb{P}^n_k) \xrightarrow{\sim} \mathbb{Z}$.