

# A note on descent for algebraic stacks

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## 1. Introduction

Let  $S' \rightarrow S$  be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor  $X \mapsto X \times_S S'$  defines an equivalence of categories between the category of  $S$ -schemes  $X$  and the category of pairs  $(X', \phi)$  where  $X'$  is an  $S'$ -scheme and  $\phi$  a descent datum for  $X'$  over  $S'$  such that  $X'$  admits an open covering by affine schemes which are stable under  $\phi$ . In case  $S = \operatorname{Spec}(k)$ ,  $S' = \operatorname{Spec}(k')$  and the morphism  $S' \rightarrow S$  corresponds to a finite Galois extension of fields  $k \subset k'$ , this is known as Galois descent, and due to Weil.

The goal of this note is to prove a similar statement for algebraic stacks. In the case of stacks, the analogue of the aforementioned descent-theory is a notion called *2-descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes  $S' \rightarrow S$  which is smooth and surjective, *every 2-descent datum for an algebraic stack is effective*. More precisely, we have the following result. For a scheme  $S$ , let  $(\operatorname{Sch}/S)_{\text{fppf}}$  be the big fppf site of  $S$  as in [Stacks, Tag 021S]; a *stack over  $S$*  is a stack in groupoids  $\mathcal{X} \rightarrow (\operatorname{Sch}/S)_{\text{fppf}}$  over  $(\operatorname{Sch}/S)_{\text{fppf}}$ , see [Stacks, Tag 0304].

**THEOREM 1.1.** *Let  $S' \rightarrow S$  be a faithfully flat morphism of schemes locally of finite presentation, and let  $\mathcal{X}'$  be a stack over  $S'$ . Let  $(\phi, \psi)$  be a 2-descent datum for the stack  $\mathcal{X}'$  over  $S'$ , see Definition 3.1. Then  $(\phi, \psi)$  is effective. That is, there exists a stack  $\mathcal{X}$  over  $S$ , an isomorphism of stacks over  $S'$*

$$\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

*and a 2-isomorphism  $\chi: p_2^* f \circ \text{can} \Rightarrow \phi \circ p_1^* f$  as in the following diagram:*

$$(1.1) \quad \begin{array}{ccc} p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\text{can}} & p_2^*(\mathcal{X} \times_S S') \\ \downarrow p_1^* \rho & \swarrow \phi & \downarrow p_2^* \rho \\ p_1^* \mathcal{X}' & \xrightarrow{\quad} & p_2^* \mathcal{X}', \end{array}$$

*such that the natural compatibility between  $\chi$  and  $\psi$  is satisfied. Moreover, if  $S' \rightarrow S$  is smooth, then  $\mathcal{X}'$  is an algebraic stack over  $S'$  if and only if  $\mathcal{X}$  is an algebraic stack over  $S$ . Finally, if  $S' \rightarrow S$  is étale, then  $\mathcal{X}'$  is a Deligne–Mumford stack over  $S'$  if and only if  $\mathcal{X}$  is a Deligne–Mumford stack over  $S$ .*

Note that even the case where  $\mathcal{X}'$  is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack  $\mathcal{X}$  is obtained by defining  $\mathcal{X}(T)$  as the groupoid of objects of  $\mathcal{X}'(T \times_S S')$  equipped with a descent datum relative to the 2-descent datum of  $\mathcal{X}'$ , for any scheme  $T$  over  $S$ . More precisely, the first assertion in the above theorem follows from the fact that the 2-fibred category  $\underline{\text{Stack}}_S$  over  $(\text{Sch}/S)_{fppf}$ , whose fibre over  $U \in (\text{Sch}/S)_{fppf}$  is the category  $\underline{\text{Stack}}(U)$  of stacks over  $U$ , is a 2-stack over  $S$  (see e.g. [Bre94, Example 1.11.(1)]). The other two assertions follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the smooth (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case  $S' \rightarrow S$  is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ , then for a stack  $\mathcal{X}'$  over  $S'$ , one can reformulate the notion of 2-descent datum for  $\mathcal{X}'$  over  $S'$  in terms of an action of  $\Gamma$  on  $\mathcal{X}'$  over the action of  $\Gamma$  on  $S'$  over  $S$ , as in the classical case. To explain this, for an element  $\sigma \in \Gamma$ , define  ${}^\sigma \mathcal{X}'$  as the pull-back of  $\mathcal{X}'$  along  $\sigma: S' \rightarrow S'$ .

**DEFINITION 1.2.** *Let  $S' \rightarrow S$  be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ . Let  $\mathcal{X}'$  be a stack over  $S'$ . A Galois 2-descent datum consists of:*

- (1) *a family of 1-isomorphisms  $f_\sigma: {}^\sigma \mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$  ( $\sigma \in \Gamma$ );*
- (2) *a family of 2-isomorphisms  $\psi_{\sigma,\tau}: f_\sigma \circ {}^\sigma(f_\tau) \Rightarrow f_{\sigma\tau}$  ( $\sigma, \tau \in \Gamma$ );*

*such that for each  $\sigma, \tau, \gamma \in \Gamma$ , the diagram of 2-morphisms*

$$\begin{array}{ccc} f_\sigma \circ {}^\sigma f_\tau \circ {}^{\sigma\tau} f_\gamma & \xRightarrow{({}^{\sigma\tau} f_\gamma)^*(\psi_{\sigma,\tau})} & f_{\sigma\tau} \circ {}^{\sigma\tau} f_\gamma \\ \Downarrow f_{\sigma*}({}^\sigma \psi_{\tau,\gamma}) & & \Downarrow \psi_{\sigma\tau,\gamma} \\ f_\sigma \circ {}^\sigma f_{\tau\gamma} & \xRightarrow{\psi_{\sigma,\tau\gamma}} & f_{\sigma\tau\gamma} \end{array}$$

*is commutative.*

One can show that to give a Galois 2-descent datum on  $\mathcal{X}'$  over  $S'$  is to give a group action (in the sense of [Rom05]) of  $\Gamma$  on  $\mathcal{X}'$  as a stack over  $S$ , such that for each  $\sigma \in \Gamma$ , the composition  $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \rightarrow S'$  agrees with the composition  $\mathcal{X}' \rightarrow S' \xrightarrow{\sigma} S'$ ; this is also equivalent to giving 2-descent datum for  $\mathcal{X}'$  over  $S'$ , see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

**THEOREM 1.3.** *Let  $S' \rightarrow S$  be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ . Let  $\mathcal{X}'$  be an algebraic stack over  $S'$ , equipped with a Galois 2-descent datum  $(f_\sigma \ (\sigma \in \Gamma), \psi_{\sigma,\tau} \ (\sigma, \tau \in \Gamma))$ . There exists an algebraic stack  $\mathcal{X}$  over  $S$  and an isomorphism  $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$  of stacks over  $S'$ . The stack  $\mathcal{X}$  is Deligne–Mumford if and only if  $\mathcal{X}'$  is.*

Observe that the statement in Theorem 1.3 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an

isomorphism of stacks  $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$  over  $S'$  as well as a family of 2-isomorphisms  $\chi_\sigma: \rho \circ \text{can} \Rightarrow f_\sigma \circ {}^\sigma \rho$  for  $\sigma \in \Gamma$  as in the following diagram:

$$\begin{array}{ccc} {}^\sigma(\mathcal{X} \times_S S') & \xrightarrow{\text{can}} & \mathcal{X} \times_S S' \\ \downarrow & \Downarrow & \downarrow \\ {}^\sigma \mathcal{X}' & \longrightarrow & \mathcal{X}', \end{array}$$

such that the obvious compatibility conditions are satisfied.

EXAMPLE 1.4. Let  $k$  be a field and let  $k \subset k'$  be a degree two field extension; one may think of  $\mathbb{R} \subset \mathbb{C}$  or  $\mathbb{F}_q \subset \mathbb{F}_{q^2}$  for a prime power  $q$ . Let  $\sigma \in \text{Gal}(k'/k)$  be the generator of  $\text{Gal}(k'/k)$ . Let  $\mathcal{X}'$  be a stack over  $k'$  equipped with a 1-isomorphism  $\sigma: \mathcal{X}' \rightarrow \mathcal{X}'$  and a 2-isomorphism  $F: \sigma^2 \Rightarrow \text{id}_{\mathcal{X}'}$  between  $\sigma^2$  and the identity functor, such that  $\sigma$  commutes with the functor  $(\text{Sch}/k') \rightarrow (\text{Sch}/k')$  defined as  $T \mapsto {}^\sigma T = T \times_{k', \sigma} k'$ , and such that for each  $x \in \mathcal{X}'(T)$ ,  $T \in (\text{Sch}/k')$ , the isomorphism  $F(x): \sigma^2(x) \rightarrow x$  lies over the canonical isomorphism of schemes  ${}^\sigma({}^\sigma T) \rightarrow T$ . One obtains the descended stack  $\mathcal{X}$  over  $k$  by defining, for  $T \in (\text{Sch}/k)$ ,  $\mathcal{X}(T)$  as the groupoid of pairs  $(x, \varphi)$  with  $x \in \mathcal{X}'(T_{k'})$  and  $\varphi: x \rightarrow \sigma(x)$  an isomorphism such that the composition

$$x \xrightarrow{\varphi} \sigma(x) \xrightarrow{{}^\sigma \varphi} \sigma^2(x) \xrightarrow{F} x$$

is the identity. There is a natural isomorphism  $\mathcal{X} \times_k k' \cong \mathcal{X}'$  of stacks over  $k'$ .

## 2. Descending schemes

Let

$$p: S' \rightarrow S$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times_{S'} S' \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} S' \rightarrow S,$$

and if  $S''' = S' \times_S S' \times_S S'$ , we can extend this to the diagram

$$S''' \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} S'' \rightrightarrows S' \rightarrow S$$

where the three arrows  $S''' \rightarrow S''$  are  $p_{12}$ ,  $p_{13}$  and  $p_{23}$ .

Let  $X'$  be a scheme over  $S'$ . Define

$$p_i^* X' = X' \times_{S', p_i} S'', \quad p_{jk}^* p_i^* X' = (p_i^* X') \times_{S'', p_{jk}} S'''$$

and note that

$$p_{jk}^* p_i^* X' = (p_i^* X') \times_{S'', p_{jk}} S''' = (p_i \circ p_{jk})^* X'.$$

Recall that a *descent datum* for  $X'/S'$  is an  $S''$ -isomorphism

$$\phi: p_1^* X' \xrightarrow{\sim} p_2^* X'$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
 p_{12}^* p_1^* X' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* X' & \xlongequal{\quad} & p_{23}^* p_1^* X' & \xrightarrow{p_{23}^* \phi} & p_{23}^* p_2^* X' \\
 \parallel & & & & & & \parallel \\
 p_{13}^* p_1^* X' & \xrightarrow{\quad p_{13}^* \phi \quad} & & & & & p_{13}^* p_2^* X'.
 \end{array}$$

In other words, one requires that

$$p_{23}^* \phi \circ p_{12}^* \phi = p_{13}^* \phi \quad \text{as morphisms} \quad p_{12}^* p_1^* X' \rightarrow p_{13}^* p_2^* X'.$$

**THEOREM 2.1** (Grothendieck). *Let  $p: S' \rightarrow S$  be a faithfully flat locally finitely presented morphism of affine schemes. The functor  $X \mapsto p^* X$  defines an equivalence of categories between the category of  $S$ -schemes  $X$  and the category of pairs  $(X', \phi)$  where  $X'$  is an  $S'$ -scheme and  $\phi$  a descent datum for  $X'/S'$  such that  $X'$  admits an open covering by affine schemes stable under  $\phi$ .*

Next, recall how to make this explicit in case  $S' \rightarrow S$  is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ . For instance,  $S$  could be the spectrum of a field  $k$ ,  $S'$  the spectrum of a finite field extension  $k' \supset k$ , and  $\Gamma$  the Galois group of  $k'/k$ . Let  $X'$  be a scheme over  $S'$  and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma: {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over  $S'$ , for  $\sigma \in \Gamma$ , satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau} X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of  $\Gamma$  on  $X'$  as a scheme over  $S$  is said to be *compatible with the action of  $\Gamma$  on  $S'$  over  $S$*  if for each  $\sigma \in \Gamma$ , the following diagram commutes:

$$\begin{array}{ccc}
 X' & \xrightarrow{\sigma} & X' \\
 \downarrow & & \downarrow \\
 S' & \xrightarrow{\sigma} & S'.
 \end{array}$$

**LEMMA 2.2.** *Let  $S' \rightarrow S$  be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ , and let  $X'$  be a scheme over  $S'$ . To give a descent datum for  $X'$  over  $S'$  is to give a Galois descent datum for  $X'$  over  $S'$ . These notions are further equivalent to giving an action of  $\Gamma$  on  $X'$  compatible with the action of  $\Gamma$  on  $S'$  over  $S$ .*

**PROOF.** This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4].  $\square$

### 3. Descending algebraic stacks

Let  $p: S' \rightarrow S$  be a faithfully flat locally finitely presented morphism of schemes. Let  $\mathcal{X}'$  be a stack in groupoids on  $S'$ , in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$(3.1) \quad r_i: S'''' \rightarrow S'.$$

Similarly,  $S'''$  is equipped with three projections  $q_i: S''' \rightarrow S'$ . Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on  $S'$ . One has similar isomorphisms relating the other  $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$  with  $r_\mu^* \mathcal{X}'$ , for  $i, j, k \in \{1, 2, 3, 4\}$ ,  $\alpha, \beta \in \{1, 2, 3\}$ ,  $\nu \in \{1, 2\}$  and  $\mu \in \{1, 2, 3, 4\}$ .

Consider an isomorphism of  $S''$ -stacks (i.e. an equivalence of  $\text{Sch}/S''$ -categories):

$$\phi: p_1^* \mathcal{X}' \rightarrow p_2^* \mathcal{X}',$$

and let  $\psi$  be a 2-morphism

$$\psi: p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism  $\Rightarrow$  in the following diagram:

$$(3.2) \quad \begin{array}{ccccc} p_{12}^* p_1^* \mathcal{X}' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* \mathcal{X}' & \xlongequal{\quad} & p_{23}^* p_1^* \mathcal{X}' & \xrightarrow{p_{23}^* \phi} & p_{23}^* p_2^* \mathcal{X}' \\ \parallel & & & \swarrow \psi & & & \parallel \\ p_{13}^* p_1^* \mathcal{X}' & \xrightarrow{\quad p_{13}^* \phi \quad} & & & & & p_{13}^* p_2^* \mathcal{X}'. \end{array}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234}: S'''' \rightarrow S''',$$

and note that

$$p_{123}^* (p_{23}^* \phi \circ p_{12}^* \phi) = p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and} \\ p_{123}^* p_{13}^* \phi = \pi_{13}^* \phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}: S'''' \rightarrow S''$$

are the canonical morphisms. For  $i, j, k \in \{1, 2, 3, 4\}$  with  $i < j < k$ , define

$$\psi_{ijk} := p_{ijk}^* \psi.$$

For instance, pulling back  $\psi$  along  $p_{123}$  gives a 2-morphism

$$\psi_{123} = p_{123}^* \psi: \pi_{23}^* \phi \circ \pi_{12}^* \phi \Rightarrow \pi_{13}^* \phi.$$

Similarly, we obtain 2-morphisms

$$\begin{aligned} \psi_{124}: \pi_{24}^* \phi \circ \pi_{12}^* \phi &\Rightarrow \pi_{14}^* \phi, \\ \psi_{134}: \pi_{34}^* \phi \circ \pi_{13}^* \phi &\Rightarrow \pi_{14}^* \phi, \\ \psi_{234}: \pi_{34}^* \phi \circ \pi_{23}^* \phi &\Rightarrow \pi_{24}^* \phi. \end{aligned}$$

Moreover, observe that under  $p_{123}$ , diagram (3.2) pulls back to the diagram

$$(3.3) \quad \begin{array}{ccccc} r_1^* \mathcal{X}' & \xrightarrow{\pi_{12}^* \phi} & r_2^* \mathcal{X}' & \xlongequal{\quad} & r_2^* \mathcal{X}' & \xrightarrow{\pi_{23}^* \phi} & r_3^* \mathcal{X}' \\ \parallel & & & \Downarrow & & & \parallel \\ r_1^* \mathcal{X}' & \xrightarrow{\quad \pi_{13}^* \phi \quad} & r_3^* \mathcal{X}' & & & & r_3^* \mathcal{X}' \end{array}$$

in which the 2-morphism  $\Rightarrow$  is the 2-morphism  $\psi_{123}$  defined above (and with  $r_i$  is as in (3.1)). Using pull-backs by the other three  $p_{ijk}: S'''' \rightarrow S'''$ , we thus obtain four triangles, that we may put together to form the following tetrahedron:

$$(3.4) \quad \begin{array}{ccccc} & & r_1^* \mathcal{X}' & & \\ & \swarrow & \downarrow & \searrow & \\ r_2^* \mathcal{X}' & \xleftarrow{\quad} & & \xrightarrow{\quad} & r_4^* \mathcal{X}' \\ & \searrow & \downarrow & \swarrow & \\ & & r_3^* \mathcal{X}' & & \end{array}$$

DEFINITION 3.1. *Let  $p: S' \rightarrow S$  be a faithfully flat locally finitely presented morphism of schemes. Let  $\mathcal{X}'$  be a stack in groupoids over  $S'$ . A 2-descent datum for  $\mathcal{X}'$  over  $S'$  consists of:*

- (1) *an isomorphism of stacks (i.e. an equivalence of categories)*

$$\phi: p_1^* \mathcal{X}' \rightarrow p_2^* \mathcal{X}'$$

*over  $S''$ ;*

- (2) *a 2-isomorphism*

$$\psi: p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi$$

*as in diagram (3.2);*

*such that the following condition is satisfied: the 2-morphisms  $\psi_{ijk}$  between the several compositions in diagram (3.4) are compatible, in the sense that the following diagram of 2-morphisms commutes:*

$$\begin{array}{ccc} \pi_{34}^* \phi \circ \pi_{23}^* \phi \circ \pi_{12}^* \phi & \xRightarrow{(\pi_{34}^* \phi)_* (\psi_{123})} & p_{34}^* \phi \circ p_{13}^* \phi \\ \parallel & & \parallel \\ (\pi_{12}^* \phi)^* (\psi_{234}) & & \psi_{134} \\ \downarrow & & \downarrow \\ p_{24}^* \phi \circ p_{12}^* \phi & \xRightarrow{\psi_{124}} & p_{14}^* \phi. \end{array}$$

This gives the following result.

PROPOSITION 3.2 (Breen). *Let  $(\phi, \psi)$  be a 2-descent datum for the stack  $\mathcal{X}'$  over  $S'$ . Then there exists a stack  $\mathcal{X}$  over  $S$ , an isomorphism*

$$\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over  $S'$ , and a 2-isomorphism  $\chi: p_2^* \rho \circ \text{can} \Rightarrow \phi \circ p_1^* \rho$  as in diagram

$$(3.5) \quad \begin{array}{ccc} p_1^*(X \times_S S') & \xrightarrow{\text{can}} & p_2^*(X \times_S S') \\ \downarrow p_1^* \rho & \swarrow & \downarrow p_2^* \rho \\ p_1^* X' & \xrightarrow{\phi} & p_2^* X', \end{array}$$

such that the natural compatibility condition between  $\chi$  and  $\psi$  is satisfied.

PROOF. This follows from [Bre94, Example 1.11.(i)].  $\square$

To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-known lemma, which should be well-known but which we include for convenience of the reader.

LEMMA 3.3. *Let  $S$  be a scheme. The following assertions are true.*

- (1) *Let  $\pi: X' \rightarrow X$  be a representable, smooth and surjective morphism of stacks in groupoids over  $S$ . If  $X'$  is algebraic, then  $X$  is algebraic. If in addition  $\pi$  is étale and  $X'$  is Deligne–Mumford, then  $X$  is Deligne–Mumford.*
- (2) *Let  $S' \rightarrow S$  be a smooth surjective morphism of schemes, let  $X$  be a stack in groupoids over  $S$  and define  $X' = X \times_S S'$ . Suppose that  $X'$  is an algebraic stack over  $S'$ . Then  $X$  is an algebraic stack over  $S$ . If in addition  $S' \rightarrow S$  is étale and  $X'$  is a Deligne–Mumford stack, then  $X$  is a Deligne–Mumford stack.*

PROOF. Let us first prove item (1). If  $U'$  is a scheme and  $U' \rightarrow X'$  a surjective and smooth morphism, then  $U' \rightarrow X' \rightarrow X$  is surjective and smooth, and moreover étale if  $\pi$  and  $U' \rightarrow X'$  are étale. Therefore, it suffices to prove that the diagonal  $\Delta: X \rightarrow X \times X$  is representable by algebraic spaces. For this, it suffices to consider to schemes  $U$  and  $V$ , equipped with morphisms  $U \rightarrow X$  and  $V \rightarrow X$ , and prove that the fibre product  $U \times_X V$  is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define  $U' = X' \times_X U$  and  $V' = X' \times_X V$ . We obtain the following cartesian diagram:

$$\begin{array}{ccccc} & & U' \times_{X'} V' & & \\ & \swarrow & \downarrow & \searrow & \\ V' & & & & U \times_X V \\ & \swarrow & \downarrow & \searrow & \\ & & U' & & V \\ & \swarrow & \downarrow & \searrow & \\ X' & & & & U \\ & \swarrow & \downarrow & \searrow & \\ & & X & & \end{array}$$

The morphism  $X' \rightarrow X$  is representable, hence  $U'$  and  $V'$  are representable by algebraic spaces. Since  $X'$  is an algebraic stack, the morphism  $V' \rightarrow X'$  is representable by algebraic spaces, which implies that its base change  $U' \times_{X'} V' \rightarrow U'$  is representable by algebraic spaces. Finally, the morphism of algebraic spaces  $U' \rightarrow U$  is étale and surjective, hence an epimorphism. Using [LMB00,

Lemme 4.3.3], we conclude that the morphism  $U \times_X V \rightarrow U$  is representable. As  $U$  is scheme,  $U \times_X V$  is an algebraic space, and we are done.

Next, we prove item (2). Via the composition  $\mathcal{X}' \rightarrow S' \rightarrow S$ , we may view  $\mathcal{X}'$  as an algebraic stack over  $S$ , see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over  $S$ :

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

As  $S' \rightarrow S$  is representable, surjective and étale, the same holds for  $\mathcal{X}' \rightarrow \mathcal{X}$ . The stack  $\mathcal{X}'$  is algebraic, hence  $\mathcal{X}$  is algebraic as well, see item (1).  $\square$

PROOF OF THEOREM 1.1. Proposition 3.2 yields the stack  $\mathcal{X}$  over  $S$  together with 1-isomorphism  $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$  and the 2-isomorphism  $\chi: p_2^* \rho \circ \text{can} \Rightarrow \phi \circ p_1^* \rho$  that have the right compatibility properties with respect to  $\psi$ , so that we only need to prove that  $\mathcal{X}$  is algebraic (resp. Deligne–Mumford if  $S' \rightarrow S$  is surjective étale). This follows from Lemma 3.3.  $\square$

Even the case where  $\mathcal{X}'$  is a scheme seems to yield a non-trivial result:

COROLLARY 3.4. *Let  $S' \rightarrow S$  be a smooth surjective morphism of schemes, and let  $X'$  be a scheme over  $S'$  equipped with a descent datum  $\phi$  as in Section 2. Then there exists an algebraic stack  $\mathcal{X}$  over  $S$  and an  $S$ -morphism  $\pi: X' \rightarrow \mathcal{X}$  such that the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

*is cartesian. The tuple  $(\mathcal{X}, \pi: X' \rightarrow \mathcal{X})$  is compatible with the descent datum  $\phi$  in an appropriate sense, and this makes  $(\mathcal{X}, \pi)$  unique up to isomorphism.*

PROOF. This is a straightforward consequence of Theorem 1.1.  $\square$

For a scheme  $S$  and a stack  $\mathcal{X}$ , and a finite group  $\Gamma$ , a *group action of  $\Gamma$  on  $\mathcal{X}$  over  $S$*  is an action of the functor in groups over  $S$  associated to  $\Gamma$  on the stack  $\mathcal{X}$  over  $S$ , see [Rom05, Definition 1.3].

LEMMA 3.5. *Let  $S' \rightarrow S$  be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group  $\Gamma$ , and let  $\mathcal{X}'$  be a stack over  $S'$ . Then the following sets are in canonical bijection:*

- (1) *The set of 2-descent data  $(\phi, \psi)$  for  $\mathcal{X}'$  over  $S'$ .*
- (2) *The set of group actions of  $\Gamma$  on  $\mathcal{X}'$  as a stack over  $S$ , such that for each  $\sigma \in \Gamma$ , the composition  $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \rightarrow S'$  agrees with the composition  $\mathcal{X}' \rightarrow S' \xrightarrow{\sigma} S'$ .*
- (3) *The set of Galois 2-descent data for  $\mathcal{X}'$  over  $S'$ .*

PROOF. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] a the proof in the case of schemes. The stacky case is requires some straightforward generalizations; we leave the details to the reader.  $\square$

PROOF OF THEOREM 1.3. See Theorem 1.1 and Lemma 3.5.  $\square$



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