The geometry and arithmetic of cubic hypersurfaces

Lecture notes

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[1]This is an incomplete, preliminary version of my lecture notes on cubic hypersurfaces. These notes will be updated weakly, see https://olivierfortman.github.io. For comments on the text, please write me an e-mail (*degaayfortman@math.uni-hannover.de*).

These are lectures notes for a course given at the Institute of Algebraic Geometry in Hannover, between October 2023 and February 2024. The goal of these lectures is to give an introduction to the theory of cubic hypersurfaces. In these notes, I will treat geometric as well as arithmetic aspects of the theory. Some of the topics covered:

1. Topology of hypersurfaces.
2. Hodge theory of cubic hypersurfaces.
3. Lines on cubic hypersurfaces.
4. Two-dimensional birational geometry, intersection theory, deformation theory.
5. Cubic surfaces and cubic threefolds.
6. Moduli spaces, algebraic stacks; period domains and period mappings.
7. Étale cohomology and cubic hypersurfaces over finite fields.

Should you have any questions, or comments on these notes, do not hesitate to send me an e-mail[1].

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Chapter 1

Introduction

One of the goals of algebraic geometry is to study zero sets of systems of homogenous polynomials in multiple variables with coefficients in a field k. To do so, one is led to investigate the geometry of *algebraic varieties* over k. Among the simplest ways to obtain examples of an algebraic variety, is to consider a degree d hypersurface

$$X = Z(F) = \{F = 0\} \subset \mathbb{P}^m_k, \quad F \in k[x_0, \ldots, x_m]_d, \quad d \in \mathbb{Z}_{\geq 1}.$$

It turns out that, although their definition is simple, hypersurfaces $X \subset \mathbb{P}^m_k$ are in general difficult objects to study.

To facilitate the study of hypersurfaces in $\mathbb{P}^m$, one can restrict to the *smooth hypersurfaces*, i.e. those for which the equation $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_m = 0$ has no solution in $\mathbb{P}^m(\bar{k})$. If $d = 1$ then $X \cong \mathbb{P}^m$ is a hyperplane. If $d = 2$ then X is a smooth quadric, which implies that F is projectively equivalent to $x_0^2 + \cdots + x_m^2 = 0$. When $d \geq 3$, degree d hypersurfaces in $\mathbb{P}^m$ for $m \geq 2$ come in positive dimensional families, and their investigation starts to become more complicated.

When $d = 3$, one enters the realm of *smooth cubic hypersurfaces*. For each value of $n = \dim(X)$, the class of cubic hypersurfaces of dimension n is very rich; however, only for small n, the theory is fairly well understood. When $n = \dim(X) = 1$ and X is equipped with a rational point $e \in X(k)$, then $E = (X, e)$ is called an *elliptic curve*. The fundamental theorem in the theory of elliptic curves says that there exists an algebraic group law $E \times E \to E$ in this case, turning E into a one-dimensional smooth projective group variety. If $n = \dim(X) = 2$, then $X = S$ is a *cubic surface*, and $S_{\bar{k}}$ turns out to contain exactly 27 lines over $\bar{k}$. In higher dimensions, cubic hypersurfaces provide a rich class of objects to test important conjectures in algebraic geometry on; think of the Hodge and Tate conjectures. Another example is provided by the Weil conjectures, that were proven for cubic threefolds before they were proven in general.

In the theory of cubic hypersurfaces, many beautiful areas in mathematics interact with one another, such as arithmetic geometry, algebraic topology, étale cohomology, Hodge theory and moduli theory. Open questions concern cycle class conjectures and rationality questions. The goal of these lectures is to dive into these theories, and use the developed techniques to study the geometry and arithmetic of cubic hypersurfaces.

Chapter 2

Preliminaries

To follow this course, it is useful, but not strictly necessary, to be familiar with the basic theory of schemes (as in [Mum88] or [Har77, Ch. I, §1-2]) and sheaf cohomology (see e.g. [Har77, Ch. II, §1-4]). In any case, the reader should have followed a first course in algebraic geometry.

Throughout the course, we will make use of some classical, fundamental results in algebraic geometry, without providing a proof. We collect these results this section [or in the appendix, to be added later]. Apart from this, we aim to make the body of the text will be as self-contained as possible; in particular, we try to avoid presenting a theorem without providing at least a sketch of its proof.

Chapter 3

Topology and Hodge theory

3.1 Lecture 1: Kähler differentials on hypersurfaces

Let k be a field.

3.1.1 Bott vanishing

Let $n \in \mathbb{Z}_{\geq 0}$ and $m = n+1$. Before we start to study hypersurfaces $X \subset \mathbb{P}_k^m$, we study the projective space $\mathbb{P}^m$ itself.

Lemma 3.1.1. *Let X be a scheme.*

1. *If $0 \to E \to F \to L \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence*

$$0 \to \bigwedge^p E \to \bigwedge^p F \to \bigwedge^{p-1} E \otimes L \to 0.$$

2. *Similarly, if $0 \to L \to E \to F \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for each $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence*

$$0 \to \bigwedge^{p-1} F \otimes L \to \bigwedge^p E \to \bigwedge^p F \to 0.$$

3. *Let E be a vector bundle and L a line bundle on X. Let $a > 0$ be an integer. There is a canonical isomorphism*

$$\bigwedge^a (E \otimes L) = \left(\bigwedge^a E\right) \otimes L^{\otimes a}.$$

Proof. 1. Let Q be the cokernel of $\wedge^p E \to \wedge^p F$. Wedge the original sequence with $\wedge^{p-1} E$, and consider the canonical morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^{p-1} E \otimes E \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes F \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigwedge^p E \arrow[r] & \bigwedge^p F \arrow[r] & Q \arrow[r] & 0.
\end{tikzcd}$$

It suffices to show that the so-constructed natural map $\wedge^{p-1}E \otimes L \to Q$ is an isomorphism. For this, we may assume that $F = E \oplus L$. In this case, $\wedge^p F = \wedge^p(E \oplus L) = \oplus_{i+j=p} \wedge^i E \otimes \wedge^j L = (\wedge^{p-1}E \otimes L) \oplus \wedge^p E$, and hence $\wedge^p F / \wedge^p E = \wedge^{p-1} E \otimes L$.

2. Dualize the exact sequence $0 \to L \to E \to F \to 0$, use item 1, and then dualize the result.

3. The map

$$(E \otimes F)^{\otimes a} \to \left(\bigwedge^a E\right) \otimes L^{\otimes a}, \quad e_1 \otimes f_1 \otimes \cdots \otimes e_a \otimes f_a \mapsto (e_1 \wedge \cdots \wedge e_a) \otimes (f_1 \otimes \cdots \otimes f_a),$$

factors through a map

$$\bigwedge^a (E \otimes L) \to \left(\bigwedge^a E\right) \otimes L^{\otimes a},$$

which is an isomorphism (this can be verified on stalks, where this is clear). $\square$

Lemma 3.1.2. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\mathbb{P} = \mathbb{P}^m$. For each $p \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$, there is a canonical exact sequence*

$$0 \to \Omega^p_{\mathbb{P}}(k) \to \mathcal{O}_{\mathbb{P}}^{\oplus \binom{m+1}{p}}(k-p) \to \Omega^{p-1}(k) \to 0. \tag{3.1}$$

Proof. Consider the *Euler sequence*, which is the exact sequence

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}} \to 0. \tag{3.2}$$

It yields

$$0 \to \Omega_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

By item 1 in Lemma 3.1.1, this yields an exact sequence $0 \to \wedge^p(\Omega_{\mathbb{P}}(1)) \to \wedge^p(\mathcal{O}_{\mathbb{P}}^{\oplus(m+1)}) \to \wedge^{p-1}(\Omega_{\mathbb{P}}(1)) \to 0$. By item 3 in Lemma 3.1.1, we obtain:

$$\bigwedge^p (\Omega_{\mathbb{P}}(1)) = \left(\bigwedge^p \Omega_{\mathbb{P}}\right) \otimes \mathcal{O}(p) = \Omega^p(p).$$

The lemma follows. $\square$

Lemma 3.1.3. *Let X be a projective variety of dimension n over k, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let E a vector bundle of rank r on X. For $p \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, there is a canonical isomorphism $((\wedge^p E)(k))^* = (\wedge^r E)^* \otimes (\wedge^{r-p} E)(-k)$.*

Proof. We have

$$\left(\left(\bigwedge^p E\right) \otimes \mathcal{O}_X(k)\right)^* = \left(\bigwedge^p E\right)^* \otimes \mathcal{O}_X(-k).$$

Hence, it suffices to prove the lemma in the case $k = 0$. Consider the natural map

$$\bigwedge^p E \to \operatorname{Hom}\left(\bigwedge^{p-r} E, \bigwedge^r E\right) = \operatorname{Hom}\left(\bigwedge^{p-r} E, \mathcal{O}_X\right) \otimes \bigwedge^r E = \left(\bigwedge^{p-r} E\right)^* \otimes \bigwedge^r E.$$

We claim that this map is an isomorphism. This may be checked locally, in which case it is clear. As $(\wedge^p E)^* = \wedge^p E^*$, the lemma follows by duality. $\square$

Corollary 3.1.4. *Let X be a smooth projective variety of dimension n over k, with ample line bundle $\mathcal{O}_X(1)$. For $k \in \mathbb{Z}$, there are canonical isomorphisms*

$$(\Omega_X^p(k))^* \cong \omega_X^* \otimes \Omega_X^{r-p}(-k) \quad \textit{and} \quad H^q(X, \Omega_X^p(k)) \cong H^{n-q}(X, \Omega_X^{n-p}(-k))^\vee. \tag{3.3}$$

In particular, $h^q(X, \Omega^p(k)) = h^{n-q}(X, \Omega_X^{n-p}(-k))$ for each $k \in \mathbb{Z}$.

Proof. Lemma 3.1.3 shows that

$$((\wedge^p \Omega_X)(k))^* = (\wedge^n \Omega_X)^* \otimes (\wedge^{n-p}\Omega_X)(-k) = \omega_X^* \otimes \Omega_X^{n-p}(-k).$$

By Serre duality [reference], we obtain:

$$\begin{aligned} H^q(X, \Omega_X^p(k)) &= H^{n-q}(X, \omega_X \otimes (\Omega_X^p(k))^*)^\vee \\ &= H^q(X, \omega_X \otimes \omega_X^* \otimes \Omega_X^{n-p}(-k))^\vee = H^q(X, \Omega^{n-p}(-k))^\vee. \end{aligned}$$

The last statement follows readily from (3.3). $\square$

Theorem 3.1.5 (Bott vanishing)**.** *Consider the projective space $\mathbb{P} = \mathbb{P}_k^m$ of dimension $m > 0$ over k. Then $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in each of the following cases:*

(a) $p \neq q$ and $0 < q < m$;

(b) $p = q > 0$ and $k \neq 0$, and $k > 0$ if $p = q = m$;

(c) $q = 0$ and $k \leq p$, and $k < 0$ if $p = 0$;

(d) $q = m$ and $k \geq p - m$, and $k > 0$ if $p = m$.

Proof. We assume that we are in one of the cases (a) – (d); our goal is to prove that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$. By Serre duality, see Corollary 3.1.4, we may assume that $q \geq p$. We proceed by induction on p.

First, assume that $p = 0$. In this case, either $q = 0$ in which case $k < 0$ hence $H^q(\mathbb{P}, \mathcal{O}(k)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) = 0$, or $m > q > 0$ in which case $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$, or $m = q$ in which case $k \geq p - m = -m$ hence again $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$. We conclude that the assertion holds if $p = 0$.

Next, assume that $p > 0$. Then $q \geq p > 0$. Sequence (3.1) gives us a long exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(\mathcal{O}(k-p)^{\oplus\binom{m+1}{p}}) \to H^{q-1}(\Omega^{p-1}(k)) \to H^q(\Omega^p(k)) \to H^q(\mathcal{O}(k-p)^{\oplus\binom{m+1}{p}}) \\ \to H^q(\Omega^{p-1}(k)) \to H^{q+1}(\Omega^p(k)) \to \cdots \end{aligned} \tag{3.4}$$

We claim that $H^q\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-p)^{\oplus\binom{m+1}{p}}\right) = 0$. Indeed, this follows from the fact that $q > 0$, and $k - p \geq -m$ if $q = m$. Therefore, using the exact sequence (3.4), we conclude that the canonical map

$$H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) \to H^q(\mathbb{P}, \Omega^p(k)) \tag{3.5}$$

is surjective.

We claim that we may assume that $q > p$. To see this, suppose that $q = p$. If $q = p \geq 2$, then the induction hypothesis implies that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$ (since $k \neq 0$), hence by the surjection (3.5), we have $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in this case. Thus, suppose that $p = q = 1$. In this case, we have $k \neq 0$, and we want to show that $H^1(\mathbb{P}, \Omega^1(k)) = H^{m-1}(\mathbb{P}, \Omega^{m-1}(-k))^\vee = 0$.

To prove this, we proceed by induction on m. Suppose first that $m = 1 = p = q$. Then $k > 0$, and hence $H^1(\Omega^1(k)) = H^{m-1}(\Omega^{m-1}(-k))^\vee = H^0(\Omega^0(-k))^\vee = 0$. Next, assume $m \geq 2$. Then there are two cases to distinguish: $k > 0$ and $k < 0$. If $k < 0$, then the surjection (3.5) implies that $H^1(\Omega^1(k)) = 0$. Thus, assume that $k > 0$. We need to show that $H^{m-1}(\Omega^{m-1}(-k)) = 0$ for $k > 0$. We obtain a long exact sequence

$$\cdots \to H^{m-2}(\Omega^{m-2}(-k)) \to H^{m-1}(\Omega^{m-1}(-k)) \to H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}}) \to \cdots$$

The group $H^{m-2}(\Omega^{m-2}(-k))$ is zero by induction, and $H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}})$ vanishes as well, as $m \geq 2$. Therefore, $H^{m-1}(\Omega^{m-1}(-k)) = 0$ as desired.

By the above claim, we may assume $q > p \geq 1$. We can then apply the induction hypothesis (recall that we are still arguing by induction on p) to see that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$. Indeed, we have $0 < q - 1 < m$. Therefore, the surjection (3.5) implies that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$, and we are done. □

Exercise 3.1.6. Show that the non-zero twisted Hodge numbers $h^q(\Omega^p(k))$ are:

(a) $h^p(\Omega^p) = 1$,

(b) $h^0(\Omega^p(k)) = \binom{m+k-p}{k} \cdot \binom{k-1}{p}$ if $k > p$,

(c) $h^m(\Omega^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{m-p}$ if $k < p - m$.

Exercise 3.1.7. Consider the projective space $\mathbb{P} = \mathbb{P}_{\mathbb{C}}^m$ of dimension m over $\mathbb{C}$. By Theorem 3.1.5, we have $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^p) = 0$ for $p > 0$. Show directly that there are no non-zero holomorphic one-forms on $\mathbb{P}^1(\mathbb{C})$.

3.1.2 Kähler differentials on hypersurfaces

Lemma 3.1.8. *Let $X \subset \mathbb{P}$ be a smooth hypersurface of degree $d > 0$. For each $k \in \mathbb{Z}$, there are canonical exact sequences*

$$0 \to \Omega_{\mathbb{P}}^p(k-d) \to \Omega_{\mathbb{P}}^p(k) \to \Omega_{\mathbb{P}}^p|_X(k) \to 0, \tag{3.6}$$

$$0 \to \mathcal{O}_X(k-d) \to \Omega_{\mathbb{P}}|_X(k) \to \Omega_X(k) \to 0, \tag{3.7}$$

$$0 \to \Omega_X^{p-1}(k-d) \to \Omega_{\mathbb{P}}^p|_X(k) \to \Omega_X^p(k) \to 0. \tag{3.8}$$

Proof. It suffices to take $k = 0$.

To prove (3.6), we may take $p = 0$. In this case, the result follows from the following exact sequence, where i denotes the inclusion $X \hookrightarrow \mathbb{P}$:

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0. \tag{3.9}$$

One obtains (3.9) via the identification $\mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{I}_X$, where the latter denotes the ideal sheaf of $X \subset \mathbb{P}$, resulting from the isomorphisms $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}}(-X)$ (see [Har77, II, Proposition 6.18]) and $\mathcal{O}_{\mathbb{P}}(X) \cong \mathcal{O}_{\mathbb{P}}(d)$ (which holds because $\deg(X) = d$). Note by the way that (3.9) corresponds to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \xrightarrow{1 \mapsto F} \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_X(d) \to 0,$$

where $F \in \mathcal{O}_{\mathbb{P}}(d) = k[x_0, \ldots, x_{n_1}]_d$ is a polynomial that defines X.

To obtain the exact sequence (3.7), one combines the conormal exact sequence

$$0 \to \mathcal{N}^\vee_{Z/Y} \to \Omega_Y|_Z \to \Omega_Z \to 0$$

for any smooth hypersurface $i\colon Z \hookrightarrow Y$ in a smooth variety Y, where $\mathcal{N}^\vee_{Z/Y}$ is a sheaf on Z such that $i_*\mathcal{N}^\vee_{Z/Y} \cong I/I^2$ (see [Har77, II, Theorem 8.17]), and the canonical isomorphism

$$\mathcal{N}^\vee_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d) = \mathcal{O}_X(-d). \tag{3.10}$$

The second isomorphism in (3.10) being clear, it suffices to prove $\mathcal{N}^\vee_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d)$. This is again a general statement: if $i\colon Z \to Y$ is a closed immersion of schemes, then $i^*\mathcal{I}_Z$ has the property that $i_*i^*\mathcal{I}_Z = \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{I}_Z = \mathcal{I}_Z/\mathcal{I}_Z^2$ (to see this, reduce to the case where Y affine, where this is clear).

Finally, note that (3.8) follows from (3.7) together with Lemma 3.1.1. ☐

Proposition 3.1.9. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$ with canonical bundle ω_X. Then $\omega_X \cong \mathcal{O}_X(d - n - 2)$. In particular,*

1. *ω_X is ample if $d > n + 2$;*
2. *$\omega_X \cong \mathcal{O}_X$ if $d = n + 2$;*
3. *ω_X^* is ample if $d < n + 2$.*

Proof. Consider sequence (3.8) with $p = n + 1 = m$ and $k = d$. This gives

$$\omega_X \cong \omega_{\mathbb{P}}|_X(d) \cong \mathcal{O}_{\mathbb{P}}(-m-1)|_X \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(d - m - 1).$$

The remaining statement follow directly. ☐

We proceed to prove:

Theorem 3.1.10. *Let $X \subset \mathbb{P} = \mathbb{P}^m = \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$. Then the following holds.*

1. *Let $k \in \mathbb{Z}$ with $k < d$. The natural map*

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p(k))$$

is bijective for $p+q < n$ and injective for $p+q \leq n$.

2. *We have*

$$H^q(X, \Omega^p(k-d)) = 0 \quad \textit{for} \quad p+q < n \quad \textit{and} \quad k < d. \tag{3.11}$$

3. *We have $H^q(X, \Omega^p(k)) = 0$ for $(p+q < n, k < 0)$ and for $(p+q > n, k > 0)$.*

Proof. Throughout the proof, we will use Theorem 3.1.5 without mention. We first prove 1 and 2 by induction on p. Therefore, assume that $k < d$.

Suppose first that $p = 0$. Then (3.6) yields the following exact sequence:

$$H^q(\mathcal{O}_{\mathbb{P}}(k-d)) \longrightarrow H^q(\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^q(\mathcal{O}_X(k)) \longrightarrow H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d))$$

For $q \leq n < m$, we have $H^q(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ because $k-d < 0$. Thus, $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is injective for $q \leq n$ and $k-d < 0$. Moreover, if $q < n$ then $q+1 \leq n < m$, hence $H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ for $q < n$ and $k-d < 0$. This implies that $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is bijective $q < n$ and $k-d < 0$. In particular,

$$H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-d)) = H^q(X, \mathcal{O}_X(k-d)) = 0 \qquad \text{for} \quad (q < n, k-d < 0).$$

This proves that 1 and 2 hold whenever $p = 0$.

Next, let $p > 0$. Notice that in this case, $p+q \leq n$ implies $q < n$. Similarly, $p+q < n$ implies $q < n-1$. Notice also that (3.7) and (3.8) yield the following diagram, in which the rows are exact:

$$\begin{tikzcd}
H^q(\Omega^p_{\mathbb{P}}(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}(k)) \arrow[r, "f(p,q)"] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r] \arrow[d, equal] & H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) & \\
& H^q(\Omega^{p-1}_X(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r, "g(p,q)"] & H^q(\Omega^p_X(k)) \arrow[r] & H^{q+1}(\Omega^{p-1}_X(k-d)).
\end{tikzcd}$$

If $p+q \leq n < m$, then $q < m$ hence $H^q(\Omega^p_{\mathbb{P}}(k-d)) = 0$ as $k-d < 0$. This implies that $f(p,q)$ is injective if $p+q \leq n$. Moreover, if $p+q \leq n < m$ then $(p-1)+q < n$, hence $H^q(\Omega^{p-1}_X(k-d)) = 0$ by the induction hypothesis, as $k-d < 0$. Therefore, the maps $f(p,q)$ and $g(p,q)$ in the above diagram are both injective if $p > 0$ and $p+q \leq n$. This implies that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is injective for all $p, q \geq 0$ such that $p+q \leq n$.

Still assume $p > 0$. If $p+q < n$, then $q < n$, hence $q+1 < n+1 = m$. Therefore, $H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) = 0$ as $k-d < 0$. Moreover, if $p+q < n$ then $(p-1)+(q+1) < n$, hence $H^{q+1}(X, \Omega^{p-1}_X(k-d)) = 0$ by induction, as $k-d < 0$. Therefore, the maps

$f(p,q)$ and $g(p,q)$ in the above diagram are both bijective if $p>0$ and $p+q<n$. We conclude that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is bijective for all $p,q \geq 0$ such that $p+q<n$.

Continue to assume that $k<d$. Let $p,q \geq 0$ such that $p+q<n$. By what we have already proved, we have $H^q(X, \Omega^p_X(k-d)) = H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k-d))$, and this is zero because $q<m$ and $k-d<0$.

It remains to prove assertion 3. Notice that (3.11) implies $H^q(X, \Omega^p_X(k)) = 0$ for $(p+q<n, k<0)$. This also implies, via Corollary 3.1.4, that

$$H^q(X, \Omega^p(k)) \cong H^{n-q}(X, \Omega^{n-p}(-k))^\vee = 0 \quad \text{if} \quad (p+q>n, k>0).$$

This finishes the proof of the theorem. □

Corollary 3.1.11. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d . If $n>2$, then* $\mathrm{Pic}(X) = H^2(X,\mathbb{Z})$. *Similarly, for $n=2$ and $d \leq 3$, one has* $\mathrm{Pic}(X) = H^2(X,\mathbb{Z})$.

Proof. Consider the *exponential exact sequence* of abelian sheaves on $X(\mathbb{C}) = X^{an}$:

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2i\pi} \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}^*_{X^{an}} \to 0.$$

Taking cohomology gives an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^*_X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X). \tag{3.12}$$

As $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X)$, see Exercise 3.1.13 below, it suffices to prove the following:

Claim 3.1.12. *If $n>2$ or $n=2$ and $d \leq 3$, then $H^1(X,\mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

On the one hand, by Theorem 3.1.5, we have $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = H^2(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = 0$ for $n>1$. On the other hand, by Theorem 3.1.10, we see that if $n>1$, then $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^1(X, \mathcal{O}_X)$ and if $n>2$ then $H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^2(X, \mathcal{O}_X)$. Therefore, for $n>1$, we have $H^1(X, \mathcal{O}_X) = 0$ and for $n>2$, we have $H^2(X, \mathcal{O}_X) = 0$.

By Corollary 3.1.4, we have $h^i(X, \mathcal{O}_X) = h^{n-i}(X, \omega_X))$, and by Proposition 3.1.9, we have $h^{n-i}(X, \omega_X) = h^{n-i}(X, \mathcal{O}_X(d-(n+2))$. Thus, for $n=2$, this gives

$$h^i(X, \mathcal{O}_X) = h^{2-i}(X, \mathcal{O}_X(d-4)) = 0 \quad \text{for} \quad i \in \{1,2\} \quad \text{and} \quad d \leq 3.$$

This proves the claim, and thereby the corollary. □

Exercise 3.1.13. Sketch a proof of the fact that, for a locally ringed space $(X, \mathcal{O}_X)$, there is a natural isomorphism $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X)$. Use this to conclude that if X is a smooth projective variety over $\mathbb{C}$, then $H^1(X, \mathcal{O}^*_X) = H^1(X^{an}, \mathcal{O}^*_{X^{an}})$. Give an example of a sheaf $\mathcal{F}$ on a smooth projective variety X over $\mathbb{C}$ such that the natural map $H^1(X, \mathcal{F}) \to H^1(X^{an}, \mathcal{F}^{an})$ is not an isomorphism.

Exercise 3.1.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n and degree d. Provide all (n, d) for which the homomorphism $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Analyze the group which measures the possible failure of the injectivity of c_1.

Exercise 3.1.15. Consider a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$. Let $C \subset S$ be a curve contained in S. Prove that

$$[C] = c_1(\mathcal{O}_S(k)) \in H^2(S, \mathbb{Z})$$

if and only if there exists a hypersurface $Y \subset \mathbb{P}^3_{\mathbb{C}}$ of degree k such that $C = Y \cap S$.

3.2 Lecture 2: Lefschetz hyperplane theorem

To prove the Lefschetz hyperplane theorem, we will need some Morse theory. Let M be a smooth manifold of dimension n. Let $f \colon M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Then $0 \in M$ is called a *critical point* if $(df)_0 = 0$ as maps $T_0M \to T_{f(0)}\mathbb{R}$; in this case $f(0)$ is called a *critical value*. Consider the bilinear map

$$\mathrm{Hess}(f)_0 = (d^2f)_0 \colon T_0M \times T_0M \to \mathbb{R}$$

defined as follows. Choose coordinates $x_1, \dots, x_n$ on M centred around 0, and put

$$\mathrm{Hess}(f)_0 \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Exercise 3.2.1. Show that the function $\mathrm{Hess}(f)_0$ does not depend on the choice of coordinates around 0. Show that $\mathrm{Hess}(f)_0$ defines a bilinear symmetric form on T_0M.

We say that a critical point $0 \in M$ is *non-degenerate* if $\mathrm{Hess}(f)_0$ is non-degenerate. By Exercise 3.2.1, if $0 \in M$ is a non-degenerate critical point, then $\mathrm{Hess}(f)_0$ defines a non-degenerate quadratic form, which can be diagonalized; define $\lambda_0(f)$ as the number of negative eigenvalues in this case. The Morse lemma, see [Mil63, Lemma 2.2], states that in suitable local coordinates $x_1, \dots, x_n$ around a non-degenerate critical point $0 \in M$ of $f \colon M \to \mathbb{R}$, the function f can be written as the quadratic function

$$f(x) = f(0) - \sum_{i=1}^{\lambda_0(f)} x_i^2 + \sum_{i=\lambda_0(f)+1}^{n} x_i^2.$$

In particular, non-degenerate critical points (resp. values) are isolated in M (resp. $\mathbb{R}$).

We call f a *Morse function* if $f^{-1}(-\infty, a] \subset M$ is compact for each $a \in \mathbb{R}$, and f each critical point of f is non-degenerate. If f is a Morse function, then f is proper and its fibres $M_a = f^{-1}(a)$ are compact. Moreover, each critical value corresponds to a finite number of critical points, and the set of critical values is discrete in $\mathbb{R}$. In particular, for each $a \in \mathbb{R}$, there exist only finitely many critical values in $(-\infty, a] \subset \mathbb{R}$.

The basic theorem of Morse theory [Mil63, Theorem 3.5] says that if $f \colon M \to \mathbb{R}$ is a Morse function, then M has the homotopy type of a CW complex with one cell of dimension $\lambda_p(f)$ for each critical point $p \in M$.

Assume $M \subset \mathbb{R}^N$ is a closed submanifold of dimension n. By [Mil63, Theorem 6.6], for almost all (all but a set of measure 0) points $p \in \mathbb{R}$, the distance function

$$\varphi_p \colon M \to \mathbb{R}, \quad \varphi_p(x) = \|x - p\|^2$$

is a Morse function. We are now ready to prove:

Theorem 3.2.2 (Andreotti–Frankel [AF59])**.** *A closed n-dimensional complex submanifold $X \subset \mathbb{C}^r$ has the homotopy type of a CW complex of dimension $\leq n$.*

Proof. Let $c \in \mathbb{C}^r$ be a point such that the distance function $\varphi_c \colon X \to \mathbb{R}$ has only non-degenerate critical points.

Claim $(*)$. Let $p \in X$ be a critical point of $\varphi_c \colon X \to \mathbb{R}$. Then $\lambda_p(\varphi_c) \leq n$.

Before we prove Claim $(*)$, we will show that it implies the theorem. Indeed, by the basic theorem of Morse theory, X has the homotopy type of a CW complex with one cell of dimension $\lambda_p(\varphi_c)$ for each critical point $p \in M$ of φ_c. By Claim $(*)$, we have $\lambda_p(\varphi_c) \leq n$ for each critical point $p \in M$. Hence X has the homotopy type of a CW complex with one cell of dimension $\leq n$ for each critical point $p \in M$ of φ_c.

It remains to prove Claim $(*)$. We need:

Claim 3.2.3. *There exist local holomorphic coordinates on $\mathbb{C}^r$ such that $p = 0$, $c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(n+1)$-st position, and such that there exist open neighborhoods $0 \in V_1 \subset \mathbb{C}^n$ and $0 \in V_2 \subset \mathbb{C}^{r-n}$ and a holomorphic function*

$$\mathbb{C}^n \supset V_1 \xrightarrow{f} V_2 \subset \mathbb{C}^{r-n}$$

with $M \cap (V_1 \times V_2) = \mathrm{Graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^{r-n}$, and such that $df_0 = 0$.

Proof of Claim 3.2.3. Applying a suitable change of coordinates of $\mathbb{C}^r$, we may assume that $p = 0 \in M \subset \mathbb{C}^r$. As $M \subset \mathbb{C}^r$ is a closed submanifold, there exists an open subset $U \subset \mathbb{C}^r$ containing $p = 0$, and holomorphic functions $g_1, \ldots, g_m \colon U \to \mathbb{C}$ such that $X \cap U = \{g_1 = \cdots = g_m = 0\} \subset \mathbb{C}^r$. This gives a holomorphic function $g = (g_1, \ldots, g_m) \colon U \to \mathbb{C}^m$ such that $X \cap U = g^{-1}(0) = \{g = 0\} \subset U$. Thus, the fibre $g^{-1}(0)$ is smooth, which implies that g has maximal rank at each point of $X = g^{-1}(0)$. Applying the implicit function theorem, we obtain a holomorphic function $f \colon V_1 \to V_2 \subset \mathbb{C}^{r-n}$ defined on a open neighborhood $V_1 \subset \mathbb{C}^n$ of 0, such that $f(0) = 0$, $V_1 \times V_2 \subset U$ and such that

$$X \cap V_1 \times V_2 = \{(x, f(x)) \mid x \in V_1\} \subset V_1 \times V_2 \subset \mathbb{C}^n \times \mathbb{C}^{r-n} = \mathbb{C}^r.$$

Now $0 \neq c \in \mathbb{C}^r$, hence c defines a basis element of $\mathbb{C}^r$, so that there exists a matrix $\alpha \in \mathrm{GL}_r(\mathbb{C})$ with $\alpha \cdot c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $(n+1)$-st position. As α is linear, we have $\alpha \cdot 0 = 0$. Finally, we claim that $df_0 = 0$. This follows from the fact that $\varphi_c \colon X \to \mathbb{R}$ is a distance function, with critical point $p = 0$. In other words, $(d\varphi_c)_0 = 0$, because $\varphi_c(x, f(x)) = \|(x, f(x)) - (0, 0, \ldots, 0, 1, 0, \ldots, 0)\|^2$. □

As $|z-1|^2 = |x+iy-1|^2 = (x-1)^2+y^2 = (x^2+y^2)+(1-2x) = |z|^2+(1-2\cdot\Re(z))$, the distance function is now given by the formula

$$\varphi_c(z) = 1 - 2\cdot\Re(f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2 . \tag{3.13}$$

As $\operatorname{ord}_0(f_i) \geq 2$ for all i, the last sum in (3.13) does not contribute to $\operatorname{Hess}(\varphi_c)_0$. Write

$$f_1(z) = Q(z) + \text{terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \ldots, z_n$. We obtain:

$$\operatorname{Hess}(\varphi_c)_0 = -2\cdot\operatorname{Hess}\left(\Re(Q(z))\right)_0 + 2\cdot\operatorname{Id}.$$

We claim that $\operatorname{Hess}(\Re(Q(z)))_0$ has at most n positive and at most n negative eigenvalues. Indeed, after a change of coordinates $z \mapsto w$, we can write

$$Q(w) = w_1^2 + \cdots + w_s^2, \qquad s \leq n;$$

writing $w_j = x_j + i\cdot y_j$, we obtain

$$\Re\left(Q(w)\right) = (x_1^2 - y_1^2) + \cdots + (x_s^2 - y_s^2).$$

This finishes the proof of Claim $(*)$, and thereby the proof of Theorem 3.2.2. $\square$

As a corollary, we obtain:

Theorem 3.2.4. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. Let $Y = X \cap H$ be a hyperplane section such that $U := X \setminus Y$ is smooth of dimension n and let $j\colon Y \hookrightarrow X$ denote the canonical inclusion. The restriction map*

$$j^*\colon H^i(X,\mathbb{Z}) \to H^i(Y,\mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

Proof. For the proof, we need the following:

Claim 3.2.5. *We have a natural isomorphism $H^i(X,Y,\mathbb{Z}) \cong H_{2n-i}(U,\mathbb{Z})$.*

Assuming the claim, we obtain a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{i+1}(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{2n-i}(U,\mathbb{Z}) \arrow[r] & H^i(X,\mathbb{Z}) \arrow[r] & H^i(Y,\mathbb{Z}) \arrow[r] & H_{2n-i-1}(U,\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, to prove the theorem, we must show that $H_{2n-i}(U,\mathbb{Z}) = 0$ for $i \leq n-1$. As $i \leq n-1$ if and only if $2n-i \geq 2n-n+1 = n+1$, we need to prove that $H_i(U,\mathbb{Z}) = 0$ for $i \geq n+1$. Note that $U = X \setminus Y \subset \mathbb{P}^N \setminus H \cong \mathbb{A}^N_{\mathbb{C}}$ defines a closed submanifold

$U(\mathbb{C}) \subset \mathbb{C}^N$ of dimension n. By Theorem 3.2.2, $U(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\leq n$. In particular, $H_i(U, \mathbb{Z}) = 0$ for $i \geq n+1$, and Theorem 3.2.4 follows.

It remains to prove Claim 3.2.5; for this, we follow the exposition in [Voi02, page 306]. We admit the fact that Y admits a fundamental system of open neighborhoods $Y \subset Y_k \subset X$ that admit a deformation retract onto Y. It follows that the natural map

$$\varinjlim H^i(X, Y_k, \mathbb{Z}) \to H^i(X, Y, \mathbb{Z})$$

is an isomorphism. By excision, we have

$$H^i(X, Y_k, \mathbb{Z}) \cong H^i(U, U \cap Y_k, \mathbb{Z}).$$

If $K \subset U$ is a compact subset such that K is the deformation retract of an open subset $K \subset V \subset U$, then $H^i(U, U \setminus K, \mathbb{Z}) \cong H_{2n-i}(K, \mathbb{Z})$ (this is a refined version of Poincaré duality, see [Spa81, Section 6.2]). Applying this to

$$K_k := U \setminus (Y_k \cap U) = X \setminus Y_k,$$

which is a closed, hence compact, subset of X which admits a deformation retract of $X \setminus Y = U \supset K_k$, we obtain

$$H^i(U, Y_k \cap U, \mathbb{Z}) = H^i(U, U \setminus K_k, \mathbb{Z}) \cong H_{2n-i}(K_k, \mathbb{Z}).$$

As every singular chain on U is contained in one of the compact subsets $K_k \subset U$, the natural map $\varinjlim_k H_{2n-i}(K_k, \mathbb{Z}) \to H_{2n-i}(U, \mathbb{Z})$ is an isomorphism, and hence

$$H^i(X, Y, \mathbb{Z}) \cong \varinjlim H^i(U, U \cap Y_k, \mathbb{Z}) \cong \varinjlim H_{2n-i}(K_k, \mathbb{Z}) \cong H_{2n-i}(U, \mathbb{Z}),$$

proving Claim 3.2.5. □

Remark 3.2.6. For a compact oriented n-dimensional manifold M, and a closed submanifold $N \subset M$, cup-product with the fundamental class defines an isomorphism $H^i(M, N, \mathbb{Z}) \cong H_{n-i}(M \setminus N, \mathbb{Z})$. This is relative Poincaré duality, cf. [Dol95, Section 7]. In particular, if $X \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective variety of dimension n and $Y = X \cap H$ a smooth hyperplane section, then it readily follows that $H^i(X, Y, \mathbb{Z}) \cong H_{2n-i}(X \setminus Y, \mathbb{Z})$.

Corollary 3.2.7. *Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface.*

1. *We have $H^i(X, \mathbb{Z}) = 0$ for i odd and $i < n$, and $H^{2i}(X, \mathbb{Z}) = \mathbb{Z} \cdot h^i$ for $2i < n$.*

2. *Suppose X is smooth. Then $H^i(X, \mathbb{Z}) = 0$ for $i > n$ odd, and for each $j > n$ there is a unique $\alpha_{2j} \in H^{2j}(X, \mathbb{Z})$ such that $H^{2j}(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha_{2j}$ and $\alpha_{2j} \cup h^{n-j} = 1$.*

Proof. Let d be the degree of X, and consider the d-th Veronese embedding $\mathbb{P}^{n+1}_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$. Then $X = \mathbb{P}^{n+1}_{\mathbb{C}} \cap H$ for a hyperplane $H \subset \mathbb{P}^N_{\mathbb{C}}$. Apply Theorem 3.2.4 to obtain the first assertion. The second assertion follows from the first via Poincaré duality. □

Corollary 3.2.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then the restriction maps $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ and $\mathrm{Pic}(\mathbb{P}^{n+1}) \to \mathrm{Pic}(X)$ are isomorphisms. In particular, $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.*

Proof. The fact $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism is immediate from Theorem 3.2.4. From this, together with the commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(\mathbb{P}^{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[d] \\ H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \arrow[r] & H^2(X, \mathbb{Z}), \end{tikzcd}$$

we deduce that $\mathrm{Pic}(\mathbb{P}^{n+1}) \to \mathrm{Pic}(X)$ is also an isomorphism, as the restriction map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism by Corollary 3.1.11. $\square$

Corollary 3.2.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p + q > n$ and $L \in \mathrm{Pic}(X)$ ample.* $\square$

Remark 3.2.10. Later we will see that Corollary 3.2.9 remains valid for hypersurfaces over arbitrary fields k. Namely, if $X \subset \mathbb{P}^{n+1}_k$ is a smooth hypersurface of degree d, and if $n > 2$, then $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. See [refer to future section].

Exercise 3.2.11. Provide the equation of a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$ such that $\mathrm{Pic}(S) \ncong \mathbb{Z}$. See also Exercise 3.1.15. Define $V = H^0(\mathbb{P}^3, \mathcal{O}(d))$ and let $\mathbb{P}(V)$ be its projectivization. Let $\mathbb{P}(V)_0 \subset \mathbb{P}(V)$ be the locus of classes $[F] \in \mathbb{P}(V)$ such that $S_F \coloneqq \{F = 0\}$ is smooth. Show that $\mathbb{P}(V)_0$ is Zariski open in the projective space $\mathbb{P}(V)$. Show that the locus of $[F] \in \mathbb{P}(V)_0$ such that $\mathrm{Pic}(S_F) \ncong \mathbb{Z}$ is a countable union $\mathscr{H} = \cup_n Z_n$ of closed algebraic subvarieties $Z_n \subset \mathbb{P}(V_0)$. Show that $\mathscr{H} \neq \mathbb{P}(V_0)$.

Exercise 3.2.12. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n. Suppose that $n \geq 2$. Show that X is simply connected.

Exercise 3.2.13. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of degree $d = 3$. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of arbitrary degree $d \geq 4$.

Exercise 3.2.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 1$. Show that $H^n(X, \mathbb{Z})$ is torsion-free. Deduce that $H^\bullet(X, \mathbb{Z})$ is torsion-free.

Exercise 3.2.15. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of dimension $n \geq 2$. Let $C \subset X \subset \mathbb{P}^{n+1}$ be a smooth curve contained in X, and consider the Gysin map

$$\varphi \colon \mathbb{Z} = H^0(C, \mathbb{Z}) \cong H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \cong H^{2n-2}(X, \mathbb{Z}).$$

Define $[C] = \varphi(1) \in H^{2n-2}(X, \mathbb{Z})$. Consider the class $\alpha_{2n-2} \in H^{2n-2}(X, \mathbb{Z})$, see Corollary 3.2.7. Show that $[C] = \alpha_{2n-2}$ if and only if C intersects a general hyperplane $H \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ in a unique point with multiplicity one. Given equations for a cubic surface $X = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and a curve $C = \{F = G = 0\} \subset X \subset \mathbb{P}^3_{\mathbb{C}}$ such that $[C] = \alpha_2$.

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