A note on descent for algebraic stacks

Olivier de Gaay Fortman

1. Introduction

Let $S' \to S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor $X \mapsto X \times_S S'$ defines an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X' over S' such that X' admits an open covering by affine schemes which are stable under ϕ. In case $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(k')$ and the morphism $S' \to S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil.

The goal of this note is to prove a similar statement for algebraic stacks. In the case of stacks, the analogue of the aforementioned descent-theory is a notion called 2-*descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes $S' \to S$ which is smooth and surjective, *every* 2-*descent datum for an algebraic stack is effective.* More precisely, we have the following result. For a scheme S, let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of S as in [Stacks, Tag 021S]; a *stack over* S is a stack in groupoids $\mathcal{X} \to (\mathrm{Sch}/S)_{fppf}$ over $(\mathrm{Sch}/S)_{fppf}$, see [Stacks, Tag 0304].

Theorem 1.1. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and let $\mathcal{X}'$ be a stack over S'. Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S', see Definition 3.1. Then (ϕ, ψ) is effective. That is, there exists a stack $\mathcal{X}$ over S, an isomorphism of stacks over S'*

$$\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

and a 2-isomorphism $\chi \colon p_2^* f \circ \mathrm{can} \Rightarrow \phi \circ p_1^* f$ as in the following diagram:

$$\begin{tikzcd} p_1^*(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\rho"'] \arrow[dr, phantom, "\swarrow"] & p_2^*(\mathcal{X} \times_S S') \arrow[d, "p_2^*\rho"] \\ p_1^*\mathcal{X}' \arrow[r, "\phi"] & p_2^*\mathcal{X}', \end{tikzcd} \tag{1.1}$$

such that the natural compatibility between χ and ψ is satisfied. Moreover, if $S' \to S$ is smooth, then $\mathcal{X}'$ is an algebraic stack over S' if and only if $\mathcal{X}$ is an algebraic stack over S. Finally, if $S' \to S$ is étale, then $\mathcal{X}'$ is a Deligne–Mumford stack over S' if and only if $\mathcal{X}$ is a Deligne–Mumford stack over S.

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Note that even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack $\mathcal{X}$ is obtained by defining $\mathcal{X}(T)$ as the groupoid of objects of $\mathcal{X}'(T \times_S S')$ equipped with a descent datum relative to the 2-descent datum of $\mathcal{X}'$, for any scheme T over S. More precisely, the first assertion in the above theorem follows from the fact that the 2-fibred category $\underline{Stack}_S$ over $(\mathrm{Sch}/S)_{fppf}$, whose fibre over $U \in (\mathrm{Sch}/S)_{fppf}$ is the category $\underline{Stack}(U)$ of stacks over U, is a 2-stack over S (see e.g. [Bre94, Example 1.11.(1)]). The other two assertions follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the smooth (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, then for a stack $\mathcal{X}'$ over S', one can reformulate the notion of 2-descent datum for $\mathcal{X}'$ over S' in terms of an action of Γ on $\mathcal{X}'$ over the action of Γ on S' over S, as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define ${}^{\sigma}\mathcal{X}'$ as the pull-back of $\mathcal{X}'$ along $\sigma \colon S' \to S'$.

Definition 1.2. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be a stack over S'. A* Galois 2-descent datum *consists of:*

(1) a family of 1-isomorphisms $f_\sigma \colon {}^{\sigma}\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ $(\sigma \in \Gamma)$;
(2) a family of 2-isomorphisms $\psi_{\sigma,\tau} \colon f_\sigma \circ {}^{\sigma}(f_\tau) \implies f_{\sigma\tau}$ $(\sigma, \tau \in \Gamma)$;

such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms

$$\begin{tikzcd}
f_\sigma \circ {}^{\sigma}f_\tau \circ {}^{\sigma\tau}f_\gamma \arrow[r, Rightarrow, "({}^{\sigma\tau}f_\gamma)^*(\psi_{\sigma,\tau})"] \arrow[d, Rightarrow, "f_{\sigma*}({}^{\sigma}\psi_{\tau,\gamma})"] & f_{\sigma\tau} \circ {}^{\sigma\tau}f_\gamma \arrow[d, Rightarrow, "\psi_{\sigma\tau,\gamma}"] \\
f_\sigma \circ {}^{\sigma}f_{\tau\gamma} \arrow[r, Rightarrow, "\psi_{\sigma,\tau\gamma}"] & f_{\sigma\tau\gamma}
\end{tikzcd}$$

is commutative.

One can show that to give a Galois 2-descent datum on $\mathcal{X}'$ over S' is to give a group action (in the sense of [Rom05]) of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for $\mathcal{X}'$ over S', see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

Theorem 1.3. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be an algebraic stack over S', equipped with a Galois 2-descent datum $(f_\sigma\ (\sigma \in \Gamma),\ \psi_{\sigma,\tau}\ (\sigma, \tau \in \Gamma))$. There exists an algebraic stack $\mathcal{X}$ over S and an isomorphism $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over S'. The stack $\mathcal{X}$ is Deligne–Mumford if and only if $\mathcal{X}'$ is.*

Observe that the statement in Theorem 1.3 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an isomorphism of stacks $\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ over S' as well as a family of 2-isomorphisms $\chi_\sigma\colon \rho \circ \mathrm{can} \implies f_\sigma \circ {}^\sigma\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$\begin{tikzcd} {}^\sigma(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathcal{X} \times_S S' \arrow[d] \\ {}^\sigma\mathcal{X}' \arrow[r] & \mathcal{X}', \end{tikzcd}$$

such that the obvious compatibility conditions are satisfied.

Example 1.4. *Let $\mathcal{X}'$ be a stack over $\mathbb{C}$ equipped with an isomorphism $\sigma\colon \mathcal{X}' \to \mathcal{X}'$ and a 2-isomorphism $F\colon \sigma^2 \implies \mathrm{id}$ between σ^2 and the identity functor, such that σ commutes with the functor $(\mathsf{Sch}/\mathbb{C}) \to (\mathsf{Sch}/\mathbb{C}), T \mapsto T^\sigma$ (complex conjugate scheme). One obtains a stack $\mathcal{X}$ over $\mathbb{R}$ by defining, for $T \in (\mathsf{Sch}/\mathbb{R})$, the groupoid $\mathcal{X}(T)$ as the groupoid of pairs (x, φ) where $x \in \mathcal{X}'(T_\mathbb{C})$ and $\varphi\colon x \to \sigma(x)$ is an isomorphism such that the composition*

$$x \xrightarrow{\varphi} \sigma(x) \xrightarrow{{}^\sigma\varphi} \sigma^2(x) \xrightarrow{F} x$$

is the identity $x \to x$. There is an isomorphism $\mathcal{X} \times_\mathbb{R} \mathbb{C} \cong \mathcal{X}'$ of stacks over $\mathbb{C}$.

2. Descending schemes

Let

$$p\colon S' \to S$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \to S,$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$S''' \substack{\rightrightarrows \\ \to} S'' \rightrightarrows S' \to S$$

where the three arrows $S''' \to S''$ are p_{12}, p_{13} and p_{23}.

Let X' be a scheme over S'. Define

$$p_i^*X' = X' \times_{S', p_i} S'', \quad p_{jk}^*p_i^*X' = (p_i^*X') \times_{S'', p_{jk}} S'''$$

and note that

$$p_{jk}^*p_i^*X' = (p_i^*X') \times_{S'', p_{jk}} S''' = (p_i \circ p_{jk})^* X'.$$

Recall that a *descent datum* for X'/S' is an S''-isomorphism

$$\phi\colon p_1^*X' \xrightarrow{\sim} p_2^*X'$$

such that the following diagram commutes:

$$\begin{tikzcd} p_{12}^*p_1^*X' \arrow[r, "p_{12}^*\phi"] \arrow[d, equal] & p_{12}^*p_2^*X' \arrow[r, equal] & p_{23}^*p_1^*X' \arrow[r, "p_{23}^*\phi"] & p_{23}^*p_2^*X' \arrow[d, equal] \\ p_{13}^*p_1^*X' \arrow[rrr, "p_{13}^*\phi"] & & & p_{13}^*p_2^*X'. \end{tikzcd}$$

In other words, one requires that

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi \quad \text{as morphisms} \quad p_{12}^*p_1^*X' \to p_{13}^*p_2^*X'.$$

THEOREM 2.1 (Grothendieck). *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of affine schemes. The functor $X \mapsto p^*X$ defines an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X'/S' such that X' admits an open covering by affine schemes stable under ϕ.*

Next, recall how to make this explicit in case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. For instance, S could be the spectrum of a field k, S' the spectrum of a finite field extension $k' \supset k$, and Γ the Galois group of k'/k. Let X' be a scheme over S' and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma\colon {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over S', for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau}X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of Γ on X' as a scheme over S is said to be *compatible with the action of* Γ *on* S' *over* S if for each $\sigma \in \Gamma$, the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d] & X' \arrow[d] \\ S' \arrow[r, "\sigma"] & S'. \end{tikzcd}$$

LEMMA 2.2. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, and let X' be a scheme over S'. To give a descent datum for X' over S' is to give a Galois descent datum for X' over S'. These notions are further equivalent to giving an action of Γ on X' compatible with the action of Γ on S' over S.*

PROOF. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4]. □

3. Descending algebraic stacks

Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids on S', in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$r_i\colon S'''' \to S'. \tag{3.1}$$

Similarly, S''' is equipped with three projections $q_i\colon S''' \to S'$. Note that there are canonical isomorphisms

$$p_{12}^*p_1^*\mathcal{X}' = (p_1 \circ p_{12})^*\mathcal{X}' = q_1^*\mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^*p_{12}^*p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^*\mathcal{X}',$$

of algebraic stacks on S'. One has similar isomorphisms relating the other $p^*_{ijk}p^*_{\alpha\beta}p^*_\nu\mathcal{X}'$ with $r^*_\mu\mathcal{X}'$, for $i,j,k \in \{1,2,3,4\}$, $\alpha,\beta \in \{1,2,3\}$, $\nu \in \{1,2\}$ and $\mu \in \{1,2,3,4\}$.

Consider an isomorphism of S''-stacks (i.e. an equivalence of Sch/S''-categories):

$$\phi\colon p_1^*\mathcal{X}' \to p_2^*\mathcal{X}',$$

and let ψ be a 2-morphism

$$\psi\colon p_{23}^*\phi \circ p_{12}^*\phi \Rightarrow p_{13}^*\phi,$$

which we may picture as the 2-morphism $\Rightarrow$ in the following diagram:

$$\begin{tikzcd}
p_{12}^*p_1^*\mathcal{X}' \arrow[r, "p_{12}^*\phi"] \arrow[d, equal] & p_{12}^*p_2^*\mathcal{X}' \arrow[r, equal] \arrow[d, phantom, "\Swarrow\psi"] & p_{23}^*p_1^*\mathcal{X}' \arrow[r, "p_{23}^*\phi"] & p_{23}^*p_2^*\mathcal{X}' \arrow[d, equal] \\
p_{13}^*p_1^*\mathcal{X}' \arrow[rrr, "p_{13}^*\phi"] & & & p_{13}^*p_2^*\mathcal{X}'.
\end{tikzcd} \tag{3.2}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234}\colon S'''' \to S''',$$

and note that

$$p_{123}^*\left(p_{23}^*\phi \circ p_{12}^*\phi\right) = p_{123}^*p_{23}^*\phi \circ p_{123}^*p_{12}^*\phi = \pi_{23}^*\phi \circ \pi_{12}^*\phi, \quad \text{and}$$
$$p_{123}^*p_{13}^*\phi = \pi_{13}^*\phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}\colon S'''' \to S''$$

are the canonical morphisms. For $i,j,k \in \{1,2,3,4\}$ with $i<j<k$, define

$$\psi_{ijk} \coloneqq p_{ijk}^*\psi.$$

For instance, pulling back ψ along p_{123} gives a 2-morphism

$$\psi_{123} = p_{123}^*\psi\colon \pi_{23}^* \circ \pi_{12}^*\phi \Rightarrow \pi_{13}^*\phi.$$

Similarly, we obtain 2-morphisms

$$\psi_{124}\colon \pi_{24}^*\phi \circ \pi_{12}^*\phi \Rightarrow \pi_{14}^*\phi,$$
$$\psi_{134}\colon \pi_{34}^*\phi \circ \pi_{13}^*\phi \Rightarrow \pi_{14}^*\phi,$$
$$\psi_{234}\colon \pi_{34}^*\phi \circ \pi_{23}^*\phi \Rightarrow \pi_{24}^*\phi.$$

Moreover, observe that under p_{123}, diagram (3.2) pulls back to the diagram

$$\begin{tikzcd}
r_1^*\mathcal{X}' \arrow[r, "\pi_{12}^*\phi"] \arrow[d, equal] & r_2^*\mathcal{X}' \arrow[r, equal] \arrow[d, phantom, "\Swarrow"] & r_2^*\mathcal{X}' \arrow[r, "\pi_{23}^*\phi"] & r_3^*\mathcal{X}' \arrow[d, equal] \\
r_1^*\mathcal{X}' \arrow[rrr, "\pi_{13}^*\phi"] & & & r_3^*\mathcal{X}',
\end{tikzcd} \tag{3.3}$$

in which the 2-morphism $\Rightarrow$ is the 2-morphism ψ_{123} defined above (and with r_i is as in (3.1)). Using pull-backs by the other three $p_{ijk}\colon S'''' \to S'''$, we thus obtain four triangles, that we may put together to form the following tetrahedron:

$$
\begin{tikzcd}
 & r_1^*\mathcal{X}' \arrow[dl] \arrow[dr] \arrow[dd] & \\
r_2^*\mathcal{X}' \arrow[rr] \arrow[dr] & & r_4^*\mathcal{X}' \\
 & r_3^*\mathcal{X}'. \arrow[ur] &
\end{tikzcd}
\tag{3.4}
$$

Definition 3.1. *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over S'. A* 2-descent datum *for $\mathcal{X}'$ over S' consists of:*

(1) an isomorphism of stacks (i.e. an equivalence of categories)

$$\phi\colon p_1^*\mathcal{X}' \to p_2^*\mathcal{X}'$$

over S'';

(2) a 2-isomorphism

$$\psi\colon p_{23}^*\phi \circ p_{12}^*\phi \Rightarrow p_{13}^*\phi$$

as in diagram (3.2);

such that the following condition is satisfied: the 2-morphisms ψ_{ijk} between the several compositions in diagram (3.4) are compatible, in the sense that the following diagram of 2-morphisms commutes:

$$
\begin{tikzcd}
\pi_{34}^*\phi \circ \pi_{23}^*\phi \circ \pi_{12}^*\phi \arrow[r, Rightarrow, "(\pi_{34}^*\phi)_*(\psi_{123})"] \arrow[d, Rightarrow, "(\pi_{12}^*\phi)^*(\psi_{234})"] & p_{34}^*\phi \circ p_{13}^*\phi \arrow[d, Rightarrow, "\psi_{134}"] \\
p_{24}^*\phi \circ p_{12}^*\phi \arrow[r, Rightarrow, "\psi_{124}"] & p_{14}^*\phi.
\end{tikzcd}
$$

This gives the following result.

Proposition 3.2 (Breen). *Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S'. Then there exists a stack $\mathcal{X}$ over S, an isomorphism*

$$\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over S', and a 2-isomorphism $\chi\colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ as in diagram

$$
\begin{tikzcd}
p_1^*(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\rho"'] \arrow[dr, phantom, "\Swarrow"] & p_2^*(\mathcal{X} \times_S S') \arrow[d, "p_2^*\rho"] \\
p_1^*\mathcal{X}' \arrow[r, "\phi"] & p_2^*\mathcal{X}',
\end{tikzcd}
\tag{3.5}
$$

such that the natural compatibility condition between χ and ψ is satisfied.

Proof. This follows from [Bre94, Example 1.11.(i)]. ◻

To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-lemma, which should be well-known but which we include for convenience of the reader.

LEMMA 3.3. *Let S be a scheme. The following assertions are true.*

(1) Let $\pi\colon \mathcal{X}' \to \mathcal{X}$ be a representable, smooth and surjective morphism of stacks in groupoids over S. If $\mathcal{X}'$ is algebraic, then $\mathcal{X}$ is algebraic. If in addition π is étale and $\mathcal{X}'$ is Deligne–Mumford, then $\mathcal{X}$ is Deligne–Mumford.

(2) Let $S' \to S$ be a smooth surjective morphism of schemes, let $\mathcal{X}$ be a stack in groupoids over S and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Suppose that $\mathcal{X}'$ is an algebraic stack over S'. Then $\mathcal{X}$ is an algebraic stack over S. If in addition $S' \to S$ is étale and $\mathcal{X}'$ is a Deligne–Mumford stack, then $\mathcal{X}$ is a Deligne–Mumford stack.

PROOF. Let us first prove item (1). If U' is a scheme and $U' \to \mathcal{X}'$ a surjective and smooth morphism, then $U' \to \mathcal{X}' \to \mathcal{X}$ is surjective and smooth, and moreover étale if π and $U' \to \mathcal{X}'$ are étale. Therefore, it suffices to prove that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. For this, it suffices to consider to schemes U and V, equipped with morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:

$$
\begin{tikzcd}
& U' \times_{\mathcal{X}'} V' \arrow[dl] \arrow[dd] \arrow[rr] & & U \times_{\mathcal{X}} V \arrow[dl] \arrow[dd] \\
V' \arrow[rr, crossing over] \arrow[dd] & & V & \\
& U' \arrow[dl] \arrow[rr] & & U. \arrow[dl] \\
\mathcal{X}' \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

The morphism $\mathcal{X}' \to \mathcal{X}$ is representable, hence U' and V' are representable by algebraic spaces. Since $\mathcal{X}'$ is an algebraic stack, the morphism $V' \to \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \to U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \to U$ is étale and surjective, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that the morphism $U \times_{\mathcal{X}} V \to U$ is representable. As U is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, and we are done.

Next, we prove item (2). Via the composition $\mathcal{X}' \to S' \to S$, we may view $\mathcal{X}'$ as an algebraic stack over S, see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over S:

$$
\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}
$$

As $S' \to S$ is representable, surjective and étale, the same holds for $\mathcal{X}' \to \mathcal{X}$. The stack $\mathcal{X}'$ is algebraic, hence $\mathcal{X}$ is algebraic as well, see item (1). □

Proof of Theorem 1.1. Proposition 3.2 yields the stack $\mathcal{X}$ over S together with 1-isomorphism $\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ and the 2-isomorphism $\chi\colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ that have the right compatibility properties with respect to ψ, so that we only need to prove that $\mathcal{X}$ is algebraic (resp. Deligne–Mumford if $S' \to S$ is surjective étale). This follows from Lemma 3.3. □

Even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result:

Corollary 3.4. *Let $S' \to S$ be a smooth surjective morphism of schemes, and let X' be a scheme over S' equipped with a descent datum ϕ as in Section 2. Then there exists an algebraic stack $\mathcal{X}$ over S and an S-morphism $\pi\colon X' \to \mathcal{X}$ such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & \mathcal{X} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is cartesian. The tuple $(\mathcal{X}, \pi\colon X' \to \mathcal{X})$ is compatible with the descent datum ϕ in an appropriate sense, and this makes $(\mathcal{X}, \pi)$ unique up to isomorphism.

Proof. This is a atraightforward consequence of Theorem 1.1. □

For a scheme S and a stack $\mathcal{X}$, and a finite group Γ, a *group action of* Γ *on* $\mathcal{X}$ *over* S is an action of the functor in groups over S associated to Γ on the stack $\mathcal{X}$ over S, see [Rom05, Definition 1.3].

Lemma 3.5. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, and let $\mathcal{X}'$ be a stack over S'. Then the following sets are in canonical bijection:*

(1) The set of 2-descent data (ϕ, ψ) for $\mathcal{X}'$ over S'.
(2) The set of group actions of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$.
(3) The set of Galois 2-descent data for $\mathcal{X}'$ over S'.

Proof. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] a the proof in the case of schemes. The stacky case is requires some straightforward generalizations; we leave the details to the reader. □

Proof of Theorem 1.3. See Theorem 1.1 and Lemma 3.5. □

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Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands

Email address: a.o.d.degaayfortman@uu.nl