Non-arithmetic uniformization of metric spaces attached to unitary Shimura varieties

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Abstract

Abstract: We develop a new method of constructing non-arithmetic lattices in the projective orthogonal group $\mathrm{PO}(n,1)$ for every integer n larger than one. The technique is to consider anti-holomorphic involutions on a complex arithmetic ball quotient, glue their fixed loci along geodesic subspaces, and show that the resulting metric space carries canonically the structure of a complete real hyperbolic orbifold. The volume of various of these non-arithmetic orbifolds can be explicitly calculated.

Résumé: *Nous développons une nouvelle méthode de construction des réseaux non arithmétiques dans le groupe orthogonal projectif* $\mathrm{PO}(n,1)$ *pour tout entier positif* n. *La technique consiste à considérer des involutions anti-holomorphes sur un quotient arithmétique de la boule complexe, à recoller leurs lieux fixes le long de sous-espaces géodésiques, et à montrer que l'espace métrique résultant porte, de façon canonique, la structure d'un orbifold hyperbolique réel complet. Le volume de divers de ces orbifolds hyperboliques non arithmétiques peut être calculé explicitement.*

1 Introduction

To study locally symmetric spaces $M = \Gamma\backslash G/K$ of finite volume, one would like to classify lattices in semi-simple Lie groups. This difficult problem can be simplified somewhat by considering the notion of *arithmeticity*. On the one hand, arithmetic lattices can be classified in a way which is close to the classification of semi-simple algebraic groups over number fields, something which is well-understood. On the other hand, if a semi-simple Lie group G is non-compact and not isogenous to $\mathrm{PO}(n,1)$ or $\mathrm{PU}(n,1)$, then every irreducible lattice $\Gamma \subset G$ is arithmetic [Mar75; Cor92; GS92]. Thus, to describe all non-arithmetic lattices, one is left with the problem of classifying non-arithmetic lattices in $\mathrm{PO}(n,1)$ and in $\mathrm{PU}(n,1)$, a complicated task which remains completely open in general.

In various cases, when studying variations of Hodge structure attached to the cohomology of algebraic varieties, lattices in $\mathrm{PU}(n,1)$ arise in a natural way. Indeed, there are moduli spaces of complex varieties whose period map identifies it with a quotient of complex hyperbolic space. The ball quotients that arise in this way are often arithmetic [Pic83; Shi64; Kon00; ACT02b], but may happen to be non-arithmetic as well [DM86]. Similar constructions can be carried out to uniformize moduli of real varieties as quotients of real hyperbolic space by lattices $\Gamma \subset \mathrm{PO}(n,1)$, see e.g. [AY98; ACT06; ACT07; ACT10; Chu11; HR18].

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A striking result in the latter direction was given by Allcock, Carlson and Toledo, who identified the moduli space of stable real cubic surfaces with a four-dimensional non-arithmetic real ball quotient [ACT10]. This ball quotient is assembled from various pieces, each of which is a real arithmetic ball quotient that, over an open subset, parametrizes moduli of smooth real cubic surfaces of one topological type.

The starting point of this paper is the question whether one can glue the fixed loci of anti-holomorphic involutions on an arbitrary unitary Shimura variety, in a way that does not depend on real moduli theory, and such that the resulting space is naturally a real ball quotient. Our first goal is to prove that such glueing can be done, leading to a construction which generalizes [ACT06; ACT07; ACT10] and seems an orbifold analogue of the construction of Gromov–Piatetski-Shapiro [GPS87]. For indeed, our second goal is to show that many of the lattices that arise in this way, are in fact non-arithmetic.

To explain this, let $n \in \mathbb{Z}_{\geq 1}$. Consider a CM field K and a hermitian $\mathcal{O}_K$-lattice Λ_n of signature $(n, 1)$ for one infinite place of K and definite for others. To the lattice Λ_n, one can associate a complex ball quotient $\Gamma \setminus \mathbb{B}^n(\mathbb{C})$. Moreover, each anti-unitary involution $\alpha\colon \Lambda_n \to \Lambda_n$ defines a real ball quotient $\Gamma_\alpha \setminus \mathbb{B}^n_\alpha(\mathbb{R}) := \mathrm{Stab}_\Gamma(\mathbb{B}^n(\mathbb{C})^\alpha) \backslash \mathbb{B}^n(\mathbb{C})^\alpha$. We glue these real ball quotients together along a certain orthogonal hyperplane arrangement $\mathscr{H} \subset \mathbb{B}^n(\mathbb{C})$. The resulting space $X(\Lambda_n)$ is naturally a path metric space, see Proposition 4.7. Our first main result says that, in fact, $X(\Lambda_n)$ carries a complete hyperbolic orbifold structure:

Theorem 1.1 (c.f. Theorem 4.1)**.** *For each component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ there exists a lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ and a canonical isometry $X(\Lambda_n)^+ \cong \Gamma_n^+ \setminus \mathbb{B}^n(\mathbb{R})$.*

Applying Theorem 1.1 to the case where $K = \mathbb{Q}(\zeta_3)$ and

$$\Lambda_n = \mathbb{Z}[\zeta_3]^{n,1} = \left(\mathbb{Z}[\zeta_3]^{n+1}, \mathrm{diag}(-1, 1, \dots, 1)\right), \tag{1}$$

one obtains a sequence of real hyperbolic orbifolds $X(\Lambda_n)$ with interesting properties. Indeed, $X(\Lambda_2)$ is connected, its orbifold fundamental group Γ_2^+ is non-arithmetic, and $X(\Lambda_2)$ immerses totally geodesically into a connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$ for each $n \in \mathbb{Z}_{\geq 2}$ (c.f. Theorem 8.5). By [BC05, Proposition 15.2.2], this implies that $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ is non-arithmetic. Thus, we obtain:

Theorem 1.2 (c.f. Theorem 8.7)**.** *Let $n \in \mathbb{Z}_{\geq 2}$ and let $\Lambda_n = \mathbb{Z}[\zeta_3]^{n,1}$. There exists a connected component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$ is non-arithmetic.*

Theorem 1.1 also has applications to the theory of moduli spaces of real algebraic varieties. For some moduli stacks of GIT stable hypersurfaces $\mathcal{M}_s$, one can consider the hermitian lattice Λ_n that arises as the cohomology of the finite cover of projective space ramified along a member of the moduli space, and define an isomorphism $\mathcal{M}_s(\mathbb{R}) \cong X(\Lambda_n)$ for $n = \dim(\mathcal{M}_s)$. In such cases, an application of Theorem 1.1 gives a uniformization of $\mathcal{M}_s(\mathbb{R})$ by real hyperbolic space. For instance, for cubic surfaces and binary sextics, one retrieves the main results of [ACT06; ACT07; ACT10] in this way.

Applying Theorem 1.1 to $K = \mathbb{Q}(\zeta_5)$ and $\Lambda_n = (\mathbb{Z}[\zeta_5]^{n+1}, \mathrm{diag}(-\lambda, 1, \dots, 1))$, where $\lambda = \frac{\sqrt{5}-1}{2} \in \mathbb{R}$, yields further examples of non-arithmetic lattices. In a follow-up paper, see [GF23], we will pursue this direction for $n = 2$, by using the theory developed here to investigate the structure of the moduli space of stable real binary quintics. An analysis of the period map and an application of Theorem 1.1 will show that, also in that case, $X(\Lambda_2)$ is connected and underlies a non-arithmetic ball quotient. As before, this has the following consequence.

Theorem 1.3. *Let $K = \mathbb{Q}(\zeta_5)$. Let $\Phi = \{\sigma_1, \sigma_2 \colon K \to \mathbb{C}\}$ be the CM type with $\sigma_i(\zeta_5) = \zeta_5^i$ for $i = 1, 2$. Define a sequence of hermitian $\mathcal{O}_K$-lattices Λ_n as follows:*

$$\Lambda_n = \left(\mathbb{Z}[\zeta_5]^{n+1}, \mathrm{diag}(-\lambda, 1, \ldots, 1)\right) \quad \textit{with} \quad \lambda = \zeta_5 + \zeta_5^{-1} = (\sqrt{5} - 1)/2.$$

For each $n \in \mathbb{Z}_{\geq 2}$, there is a connected component $X(\Lambda_n)^+ \subset X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ attached to $X(\Lambda_n)^+$ via Theorem 1.1 is non-arithmetic.

The results of Sections 7 and 8 will reduce the proof of Theorem 1.3 to the case $n = 2$. The remaining part of the proof (i.e. the non-arithmeticity of Γ_2^+) requires a new technique, namely the use of equivariant Hodge theory to identify $X(\Lambda_2)$ with the moduli space of stable real binary quintics. This will be done in [GF23].

1.1 Outline of the glueing construction

Let (K, Ψ) be a CM type, where K is a CM field of degree $2g$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$, and let Λ be a hermitian $\mathcal{O}_K$-lattice. Thus, Λ is a finite free $\mathcal{O}_K$-module equipped with a non-degenerate hermitian form $h \colon \Lambda \times \Lambda \to \mathcal{O}_K$. Let $n \in \mathbb{Z}_{\geq 1}$ and assume that h has signature $(n, 1)$ with respect to one embedding $(\tau : K \to \mathbb{C}) \in \Psi$ and is positive definite for the other elements in Ψ. Let $\mathbb{C}H^n$ be the space of negative lines in $\Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C}$, define $\Gamma = \mathrm{Aut}(\Lambda)$ and let $P\Gamma = \mathrm{Aut}(\Lambda)/\mu_K$, where $\mu_K \subset \mathcal{O}_K^*$ is the group of finite units. Finally, let $P\mathscr{A}$ be the quotient of the set of anti-unitary involutions $\alpha : \Lambda \to \Lambda$ by μ_K.

Attached to the hermitian lattice Λ there is a Shimura variety $\mathrm{Sh}_K(G, X)$, see [Ach20, Section 5.3], with complex uniformization $\mathrm{Sh}_K(G, X)(\mathbb{C}) = \Gamma \setminus \mathbb{C}H^n$, see Corollary 5.9 or [Shi63, Theorem 2]. Consider the hyperplane arrangement

$$\mathscr{H} = \bigcup_{\substack{r \in \Lambda: \\ h(r,r)=1}} \langle r_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n,$$

and assume that $\mathscr{H}$ is an *orthogonal arrangement*, i.e. that the following holds:

Condition 1.4 (c.f. [ACT02a])**.** *Any two different hyperplanes $\langle r_{\mathbb{C}} \rangle^{\perp}, \langle t_{\mathbb{C}} \rangle^{\perp}$ in $\mathscr{H}$ intersect orthogonally or not at all, i.e. either $h(r, t) = 0$ or $\langle r_{\mathbb{C}} \rangle^{\perp} \cap \langle t_{\mathbb{C}} \rangle^{\perp} = \emptyset$.*

In many cases, Condition 1.4 is automatically satisfied:

Theorem 1.5 (c.f. Theorem 6.2)**.** *If the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a purely imaginary element $\eta \in \mathfrak{D}_K$ such that $\Im(\varphi(\eta)) > 0$ for every $\varphi \in \Psi$, then Condition 1.4 holds.*

The condition that $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a purely imaginary element $\eta \in \mathcal{O}_K$ holds whenever K is cyclotomic or imaginary quadratic, see Proposition 6.4.

The glueing construction is performed by assembling the different copies

$$\mathbb{R}H^n_{\alpha} \coloneqq (\mathbb{C}H^n)^{\alpha} \subset \mathbb{C}H^n, \quad \alpha \in P\mathscr{A}$$

of real hyperbolic space $\mathbb{R}H^n$ along the hyperplane arrangement $\mathscr{H}$. See Definition 3.9 and Remark 3.10 for the precise formulation of the equivalence relation. This gives a topological space $Y(\Lambda)$, acted upon by $P\Gamma$. Define $P\Gamma_\alpha \subset P\Gamma$ to be the stabilizer of $\mathbb{R}H^n_\alpha$. Theorem 1.1 can then be reformulated as follows.

Theorem 1.6 (c.f. Theorem 4.1)**.** *The space $Y(\Lambda)$ admits a path metric such that the natural map $Y(\Lambda) \to \mathbb{C}H^n$ is a local isometric embedding. This metric on $Y(\Lambda)$ induces a complete path metric on $X(\Lambda) = P\Gamma \setminus Y(\Lambda)$, which extends canonically to a real hyperbolic orbifold structure of finite volume. The subspace*

$$\coprod_{\alpha \in P\Gamma \setminus P\mathscr{A}} P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H}) \subset X(\Lambda)$$

is an open suborbifold, and for each connected component $X(\Lambda)^+ \subset X(\Lambda)$ there exists a lattice $\Gamma^+ \subset \mathrm{PO}(n,1)$ and an isomorphism of hyperbolic orbifolds

$$X(\Lambda)^+ \cong \Gamma^+ \setminus \mathbb{R}H^n.$$

Remarks 1.7. 1. The volume of $X(\Lambda)$ equals the volume of $\coprod_{\alpha \in P\Gamma \setminus P\mathscr{A}} P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha$, and hence the sum of the volumes of $P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha$ for $\alpha \in P\Gamma \setminus P\mathscr{A}$.

2. Crucial in the proof of the finiteness of the volume of $X(\Lambda)$ are: (i) the fact that $P\Gamma_\alpha \subset \mathrm{Isom}(\mathbb{R}H^n_\alpha)$ is an arithmetic lattice for $\alpha \in P\mathscr{A}$; (ii) the fact that the set $P\Gamma \setminus P\mathscr{A}$ is finite. We prove these facts in Theorem 2.7 and Proposition 2.9.

3. Our construction relies on Condition 1.4, saying that the hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$ is an orthogonal arrangement in the sense of [ACT02a]. In fact, there is always an orthogonal arrangement $\mathcal{H} \subset \mathbb{C}H^n$ attached to h in such a way that $\mathcal{H} = \mathscr{H}$ when $\mathscr{H}$ is orthogonal (c.f. Remark 6.5), and one can glue the different copies $\mathbb{R}H^n_\alpha$ of real hyperbolic space along the arrangement $\mathcal{H}$ to obtain a complete hyperbolic orbifold as in Theorem 1.6. We will not prove this.

4. The fact that $\mathscr{H} \subset \mathbb{C}H^n$ is an orthogonal arrangement implies that for $n > 1$, the orbifold fundamental $\pi_1^{\mathrm{orb}}(P\Gamma \setminus (\mathbb{C}H^n - \mathscr{H}))$ is not a lattice in any Lie group with finitely many connected components, see [ACT02a, Theorem 1.2].

1.2 Further remarks and questions

Remark 1.8. As mentioned above, our construction can be seen as an orbifold analogue of the construction of non-arithmetic lattices provided by Gromov and Piatetski–Shapiro [GPS87]. Apart from the analogy, their construction is technically quite different from ours. Gromov and Piatetski–Shapiro glue hyperbolic manifolds with boundary whose fundamental groups are Zariski dense in $\mathrm{PO}(n,1)$ and lie in non-commensurable arithmetic lattices. In this paper, we glue real hyperbolic orbifolds with boundary and corners whose orbifold fundamental groups are generally not Zariski dense in $\mathrm{PO}(n,1)$ (but do lie in arithmetic lattices that are generally not all commensurable). In light of these differences, it makes sense to ask whether the lattices we construct are essentially different, in the sense that they (or some of them) are not commensurable with any of the non-arithmetic lattices constructed via the method of [GPS87]. More generally, one could ask if our lattices are commensurable with any other non-arithmetic lattice currently known to exist.

Remark 1.9. Let $X(\Lambda)$ be a hyperbolic orbifold as constructed in Theorem 1.6. As indicated in Remark 1.7.1, we have $\mathrm{Vol}(X(\Lambda)) = \sum_{\alpha \in P\Gamma \backslash P\mathscr{A}} \mathrm{Vol}(P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha)$. Moreover, if $\mathscr{A} \neq \emptyset$ then $\mathrm{Gal}(K/F)$ acts on the group $P\Gamma$ and there is a bijection between $P\Gamma\backslash P\mathscr{A}$ and the first non-abelian Galois cohomology group $\mathrm{H}^1(G, P\Gamma)$, see Proposition 2.9. In various cases, this gives a way to explicitly calculate the volume of the hyperbolic orbifold $X(\Lambda)$, or of one of its connected components $X(\Lambda)^+$.

Remark 1.10. Let us elaborate on Remark 1.9 by analyzing the case $K = \mathbb{Q}(\zeta_3)$ and $\Lambda_n = \mathbb{Z}[\zeta_3]^{n,1}$, see (1). For $n \geq 2$, the volume of the connected non-arithmetic hyperbolic orbifold $X(\Lambda_n)^+ = X(\mathbb{Z}[\zeta_3]^{n,1})^+$ (see Theorem 1.2) can be bounded from below, and in some cases precisely determined, in the following way. For each $i \in \{0, \ldots, n\}$, define an anti-unitary involution $\alpha_i \colon \mathbb{Z}[\zeta_3]^{n,1} \to \mathbb{Z}[\zeta_3]^{n,1}$ as follows:

$$\alpha_i \colon \mathbb{Z}[\zeta_3]^{n,1} \to \mathbb{Z}[\zeta_3]^{n,1}, \quad \alpha_i(x_0, \ldots, x_n) = (\bar{x}_0, -\bar{x}_1, \ldots, -\bar{x}_i, \bar{x}_{i+1}, \ldots, \bar{x}_n).$$

Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ and $P\Gamma = \mathrm{PU}(\mathbb{Z}[\zeta_3]^{n,1})$. One can show that the $n+1$ classes $[\alpha_0], \ldots, [\alpha_n] \in P\Gamma \setminus P\mathscr{A} = \mathrm{H}^1(G, P\Gamma)$ are pairwise distinct, see Lemma 7.2. For $i = 0, \ldots, n$, consider then the integral quadratic form

$$\Psi_i(x_0, \ldots, x_n) = -x_0^2 + \sqrt{3}x_1^2 + \cdots + \sqrt{3}x_i^2 + x_{i+1}^2 + \cdots + x_n^2.$$

Then $P\Gamma_{\alpha_i} \setminus \mathbb{R}H^n_{\alpha_i} \cong \mathrm{PO}(\Psi_i, \mathbb{Z}) \setminus \mathbb{R}H^n$ and hence (see Theorem 1.6), we have

$$\coprod_{i=0}^{n} \mathrm{PO}(\Psi_i, \mathbb{Z}) \setminus (\mathbb{R}H^n - \mathscr{H}) \subset X(\Lambda_n)^+ = X(\mathbb{Z}[\zeta_3]^{n,1})^+.$$

In particular,

$$\sum_{i=0}^{n} \mathrm{Vol}\left(\mathrm{PO}(\Psi_i, \mathbb{Z}) \setminus \mathbb{R}H^n\right) \leq \mathrm{Vol}\left(X(\Lambda_n)^+\right). \tag{2}$$

Moreover, for each $n \geq 2$ such that the cardinality of $\mathrm{H}^1(G, P\Gamma)$ equals $n+1$, one has $X(\Lambda_n) = X(\Lambda_n)^+$ and the inequality in (2) is actually an equality. This happens for instance when $n = 2$ or $n = 4$. There are other values of $n \geq 3$ (such as $n = 3$, in which case $\#\mathrm{H}^1(G, P\Gamma) = n + 2$), for which one has equality in (2) even though $X(\Lambda_n)^+ \subsetneq X(\Lambda_n)$, i.e. $X(\Lambda_n)$ is not connected.

Finally, there is a closed formula for the volume $\mathrm{Vol}\left(\mathrm{PO}(\Psi_i, \mathbb{Z}) \setminus \mathbb{R}H^n\right)$ of the arithmetic orbifold $\mathrm{PO}(\Psi_i, \mathbb{Z}) \setminus \mathbb{R}H^n$, c.f. [Pra89], [Bel04, Section 3] and [BE12].

1.3 Overview of this paper

This paper is structured as follows. In Section 2 we introduce some notation and prove several preliminary results, which will be needed in Sections 3 and 4. In Section 3, we construct the glued space $X(\Lambda) = P\Gamma\backslash Y(\Lambda)$, starting from a hermitian $\mathcal{O}_K$-lattice Λ of hyperbolic signature. In Section 4, we provide the glued space $X(\Lambda)$ with a real hyperbolic orbifold structure. The goal of Section 5 will be to provide the ball quotient $P\Gamma \setminus \mathbb{C}H^n$ with a modular interpretation, which will be used in Section 6 to prove Theorem 1.5. In Sections 7 and 8, we prove Theorem 1.2.

1.4 Acknowledgements

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2 Preliminaries

The goal of Sections 2.1 - 2.3 is to introduce the definitions needed for the glueing construction, which will be carried out in Section 3. We also establish some preliminary results. In Section 2.4, we will provide examples of hermitian $\mathcal{O}_K$-lattices Λ satisfying the signature condition, together with anti-unitary involutions $\alpha \colon \Lambda \to \Lambda$.

2.1 Arithmetic ball quotients and anti-unitary involutions

Let n be a positive integer. Let K be a CM field over $\mathbb{Q}$, with totally real subfield $F \subset K$ and non-trivial element $\sigma \in \mathrm{Gal}(K/F)$. Let $\mathcal{O}_K$ (resp. $\mathcal{O}_F$) be the ring of integers of K (resp. F), and assume the following:

Condition 2.1. *The different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by an element $\eta \in \mathfrak{D}_K$ that satisfies $\sigma(\eta) = -\eta$.*

For an element $x \in K$, we will sometimes use the notation $\overline{x} = \sigma(x)$. Fix an element $\eta \in \mathcal{O}_K$ as in Condition 2.1, and let Ψ be a set of embeddings

$$\Psi \subset \mathrm{Hom}(K, \mathbb{C}) \qquad \text{such that} \qquad \Psi \cup \Psi\sigma = \mathrm{Hom}(K, \mathbb{C})$$

and such that

$$\Im\left(\psi(\eta)\right) > 0 \qquad \text{for each} \qquad \psi \in \Psi.$$

Fix an embedding

$$\tau \colon K \to \mathbb{C}$$

with $\tau \in \Psi$. Let Λ be a free $\mathcal{O}_K$-module of rank $n+1$ equipped with a non-degenerate hermitian form

$$h \colon \Lambda \times \Lambda \to \mathcal{O}_K$$

of the following signature (r_ψ, s_ψ) with respect to Ψ: we have

$$(r_\psi, s_\psi) \quad = \quad \begin{cases} (n,1) & \text{for } \psi = \tau \in \Psi, \\ (n+1, 0) & \text{for } \psi \in \Psi \text{ with } \psi \neq \tau. \end{cases} \tag{3}$$

In other words: h is an $\mathcal{O}_F$-bilinear form, $\mathcal{O}_K$-linear in its first argument, satisfies $h(y,x) = \sigma(h(x,y))$ for $x, y \in \Lambda$, and the complex vector space $\Lambda \otimes_{\mathcal{O}_K, \psi} \mathbb{C}$ admits a basis $\{e_i\}$ such that $(h^\psi(e_i,e_j))_{ij}$ is a diagonal matrix with r_ψ diagonal entries equal to 1 and s_ψ to -1. Here, $h^\psi\colon \Lambda\otimes_{\mathcal{O}_K,\psi}\mathbb{C}\times\Lambda\otimes_{\mathcal{O}_K,\psi}\mathbb{C}\to\mathbb{C}$ is the hermitian form attached to h via the embedding ψ. Define

$$V=\Lambda\otimes_{\mathcal{O}_K,\tau}\mathbb{C}.$$

Let m be the largest positive integer for which the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$ can be embedded in K, where $\zeta_m=e^{2\pi i/m}\in\mathbb{C}$. Let $\zeta\in K$ be a primitive m-th root of unity in K, and define

$$\mu_K=\langle\zeta\rangle\subset\mathcal{O}_K^*\subset\mathcal{O}_K.$$

Moreover, define Γ to be the unitary group of Λ, and $P\Gamma$ as its quotient by μ_K:

$$\Gamma=U(\Lambda)(\mathcal{O}_K)=\mathrm{Aut}_{\mathcal{O}_K}(\Lambda,h)\quad\text{and}\quad P\Gamma=\Gamma/\mu_K.$$

A norm one vector $r\in\Lambda$ is called a *short root*. Let $\mathscr{R}\subset\Lambda$ be the set of short roots. For $r\in\mathscr{R}$, define isometries $\phi_r^i\colon V\to V$ as follows:

$$\phi_r(x)=x-(1-\zeta)h(x,r)\cdot r,\quad \phi_r^i(x)=x-(1-\zeta^i)h(x,r)\cdot r,\quad i\in(\mathbb{Z}/m)^*.$$

Note that $\phi_r^i\in\Gamma$ for $r\in\mathscr{R}$, and that $\phi_r^i=\phi_r\circ\cdots\circ\phi_r$ (i times). In particular, $\phi_r^m=\mathrm{id}$. Let $\mathbb{P}(V)$ be the projective space of lines in V, and let

$$\mathbb{C}H^n=\{\ell=[v]\in\mathbb{P}(V)\mid h(v,v)<0\}\subset\mathbb{P}(V)$$

be the space of negative lines in V. Define

$$H_r=\{x\in\mathbb{C}H^n: h(x,r)=0\}\ \text{ for }\ r\in\mathscr{R},\qquad\text{and}\qquad \mathscr{H}=\bigcup_{r\in\mathscr{R}}H_r\quad\subset\quad\mathbb{C}H^n.$$

2.2 Orthogonality of the hyperplane arrangement

It turns out that Condition 2.1 implies that the following condition is satisfied.

Condition 2.2. *For $r,t\in\mathscr{R}$ with $H_r\neq H_t$ and $H_r\cap H_t\neq\emptyset$, one has $h(r,t)=0$.*

The fact that Condition 2.2 follows from Condition 2.1 will be proved in Sections 5 and 6, see Theorem 6.2.

Remarks 2.3. 1. Condition 2.1 is satisfied by quadratic and cyclotomic CM fields K, see Proposition 6.4 in Section 6.

2. Our glueing construction depends heavily on Condition 2.2; the only reason that we assume Condition 2.1 is that it implies Condition 2.2 (c.f. Theorem 6.2). We do not know whether the converse implication holds true.

3. There is a way to ensure that Condition 2.2 is automatic, see Remark 6.5, but the definitions become more complicated. We do not pursue this direction here.

2.3 Anti-unitary involutions

Define an $\mathcal{O}_F$-linear map $\alpha\colon \Lambda \to \Lambda$ to be *anti-unitary* if for all $x, y \in \Lambda$ and $b \in \mathcal{O}_K$, one has $\alpha(b \cdot x) = \sigma(b) \cdot \alpha(x)$ and $h(\alpha(x), \alpha(y)) = \sigma(h(x, y)) \in \mathcal{O}_K$. Define Γ' to be the group of unitary and anti-unitary $\mathcal{O}_F$-linear bijections $\Lambda \xrightarrow{\sim} \Lambda$. Let $\mathscr{A} \subset \Gamma'$ be the set of anti-unitary involutions $\alpha\colon \Lambda \to \Lambda$. Then

$$\mu_K \subset \Gamma \subset \Gamma' \quad - \quad \text{define} \quad P\Gamma' = \Gamma'/\mu_K. \tag{4}$$

Let $x \in K^*$. Observe that

$$(x \in \mathcal{O}_K^* \text{ and } x \cdot \sigma(x) = 1) \quad \iff \quad (x \in \mu_K)\,. \tag{5}$$

Indeed, for an embedding $\varphi\colon K \to \mathbb{C}$, one has $|\varphi(x)|^2 = \varphi(x \cdot \sigma(x))$; if $x \in \mathcal{O}_K$, then $|\varphi(x)| = 1$ for each φ if and only if x is a root of 1 [Mil08, Corollary 5.6].

Lemma 2.4. *Let* $\mathrm{Isom}(\mathbb{C}H^n)$ *be the group of isometries* $f\colon \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}H^n$. *The natural homomorphism* $P\Gamma' \to \mathrm{Isom}(\mathbb{C}H^n)$ *is injective.*

Proof. This follows readily from (5). □

The group μ_K acts on $\mathscr{A}$ by multiplication; define

$$P\mathscr{A} = \mu_K \setminus \mathscr{A}, \qquad \text{and} \qquad C\mathscr{A} = P\Gamma \setminus P\mathscr{A},$$

where $P\Gamma$ acts on $P\mathscr{A}$ by conjugation. Any $\alpha \in P\mathscr{A}$ defines an anti-holomorphic involution

$$\alpha : \mathbb{C}H^n \to \mathbb{C}H^n; \qquad \text{define} \qquad \mathbb{R}H^n_\alpha = (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n.$$

For any element $\alpha \in \mathscr{A}$, the quadratic form $h|_{V^\alpha}$ on the real vector space

$$V^\alpha = \Lambda^\alpha \otimes_{\mathcal{O}_F, \tau|_F} \mathbb{R} \tag{6}$$

has hyperbolic signature. The following lemma is readily proved:

Lemma 2.5. *For* $\alpha \in \mathscr{A}$, *let* $\mathbb{P}(V^\alpha)$ *be the real projective space of lines in* V^α, *and let* $\mathbb{R}H(V^\alpha) \subset \mathbb{P}(V^\alpha)$ *be the space of negative lines in* V^α. *The canonical isomorphism* $\mathbb{P}(V^\alpha) \cong \mathbb{P}(V)^\alpha$ *restricts to an isomorphism* $\mathbb{R}H(V^\alpha) \cong \mathbb{R}H^n_\alpha$. □

We conclude that $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$ is isometric to the real hyperbolic space of dimension n. We define

$$P\Gamma_\alpha = \mathrm{Stab}_{P\Gamma}(\mathbb{R}H^n_\alpha) \subset P\Gamma \qquad (\text{the stabilizer of } \mathbb{R}H^n_\alpha \text{ in } P\Gamma).$$

Lemma 2.6. *We have* $P\Gamma_\alpha = N_{P\Gamma}(\alpha) \coloneqq \{\gamma \in P\Gamma \mid \gamma \circ \alpha = \alpha \circ \gamma\} \subset P\Gamma$.

Proof. See the proof of [HR18, Proposition 4.9]. □

Let $\alpha \in P\mathscr{A}$, consider the lattice $\Lambda^\alpha \subset V^\alpha$ (see (6)) and the space of negative lines $\mathbb{R}H(V^\alpha) \subset \mathbb{P}(V^\alpha)$. We have $\mathrm{Isom}(\mathbb{R}H(V^\alpha)) \cong \mathrm{Isom}(\mathbb{R}H^n) = \mathrm{PO}(n,1)$ by Lemma 2.4. Let $\mathrm{O}(\Lambda^\alpha)(\mathcal{O}_F)$ be the group of $\mathcal{O}_F$-linear isometries $\gamma\colon \Lambda^\alpha \xrightarrow{\sim} \Lambda^\alpha$, and define $\mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F) = \mathrm{O}(\Lambda^\alpha)(\mathcal{O}_F)/\{\pm 1\}$. There are canonical embeddings

$$P\Gamma_\alpha \hookrightarrow \mathrm{Isom}(\mathbb{R}H(V^\alpha)) = \mathrm{PO}(V^\alpha) \quad \text{and} \quad \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F) \hookrightarrow \mathrm{PO}(V^\alpha).$$

Theorem 2.7. *The groups $P\Gamma_\alpha$ and $\mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)$ are commensurable.*

Proof. We follow the ideas of the proof of [HR18, Theorem 4.10]. Let $g \in \Gamma$ such that $[g] \in P\Gamma_\alpha$. Recall that $\zeta \in K$ is a primitive m-th root of unity in K that generates $\mu_K \subset \mathcal{O}_K^*$, see Section 2.1, and write $m = 2^a \cdot k$ with $k \in \mathbb{Z}_{\geq 1}$ odd. Let $\alpha \in \mathscr{A}$ be a representative of $\alpha \in P\mathscr{A}$, recall that $P\Gamma_\alpha = N_{P\Gamma}(\alpha)$ by Lemma 2.6, and define a subgroup $P\Gamma_\alpha^I \subset P\Gamma_\alpha$ as follows:

$$P\Gamma_\alpha^I = \left\{[g] \in P\Gamma_\alpha \mid g\alpha g^{-1} = \zeta^i \cdot \alpha \ \text{ with } \ i \in 2\mathbb{Z}_{\geq 0} \ \text{ for every } \ g \in [g]\right\}.$$

Note that for each $[g] \in P\Gamma_\alpha^I$ there is an element $h \in [g]$, unique up to sign, such that $h\alpha h^{-1} = \alpha$. Consequently, there is a natural embedding

$$P\Gamma_\alpha^I \hookrightarrow \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F). \tag{7}$$

In fact, one has $P\Gamma_\alpha \cap \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F) = P\Gamma_\alpha^I \subset \mathrm{PO}(V^\alpha)$, and this intersection has index at most two in $P\Gamma_\alpha$. Indeed, either $P\Gamma_\alpha^I = P\Gamma_\alpha$ or there exists an element $[g] \in P\Gamma_\alpha$ such that for every $g \in [g]$ one has $g\alpha g^{-1} = \zeta^i\alpha$ for some odd $i \in \mathbb{Z}$, and for any two such $g_i \in [g_i] \in P\Gamma_\alpha$, $i = 1, 2$, one has $[g_1 g_2^{-1}] \in P\Gamma_\alpha^I \subset P\Gamma_\alpha$.

It remains to show that $P\Gamma_\alpha^I$ has finite index in $\mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)$. To prove this, let $i\colon \Lambda^\alpha \hookrightarrow \Lambda$ denote the inclusion. The induced map $i\colon \Lambda^\alpha \otimes_{\mathcal{O}_F} \mathcal{O}_K \hookrightarrow \Lambda$ has finite cokernel; let d be its order. There is a map $\varphi\colon \Lambda \to \Lambda^\alpha \otimes_{\mathcal{O}_F} \mathcal{O}_K$ such that $i \circ \varphi = d$ and $\varphi \circ i = d$, and such that the induced map $\varphi_K\colon \Lambda \otimes_{\mathcal{O}_K} K \to \Lambda^\alpha \otimes_{\mathcal{O}_F} K$ is d times the inverse i^{-1} of $i\colon \Lambda^\alpha \otimes_{\mathcal{O}_F} K \xrightarrow{\sim} \Lambda \otimes_{\mathcal{O}_K} K$. Define

$$\mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I = \left\{[M] \in \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F) \mid i \circ M_{\mathcal{O}_K} \circ \varphi = d \cdot \overline{M} \text{ for some } \overline{M} \in \Gamma\right\}.$$

Let $[g] \in P\Gamma_\alpha^I$ and let $g \in [g]$ such that $g\alpha = \alpha g$. Let $M_g = g|_{\Lambda^\alpha}$ be the induced map $\Lambda^\alpha \to \Lambda^\alpha$; we have $[M_g] \in \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)$. Furthermore, as $i \circ (M_g)_{\mathcal{O}_K} \circ \varphi = d \cdot g\colon \Lambda \to \Lambda$, the embedding $P\Gamma_\alpha^I \hookrightarrow \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)$ defined in (7) factors as

$$P\Gamma_\alpha^I \hookrightarrow \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I \hookrightarrow \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F).$$

We claim that the first inclusion is an equality. Namely, let $[M] \in \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I$. There exists an element $\overline{M} \in \Gamma$ such that $\overline{M}(\Lambda^\alpha) = \Lambda^\alpha$ and $\overline{M}|_{\Lambda^\alpha} = M$. Since $[\overline{M}] \in P\Gamma$ stabilizes $\mathbb{R}H^n_\alpha$, we have $[\overline{M}] \in P\Gamma_\alpha$ by Lemma's 2.5 and 2.6. Thus $\overline{M}\alpha(\overline{M})^{-1} = \zeta^i \cdot \alpha$ for some $i \in \mathbb{Z}_{\geq 0}$. The element $\overline{M} \in [\overline{M}]$ satisfies $\overline{M}(\Lambda^\alpha) = \Lambda^\alpha$, which implies that $i \equiv 0 \bmod m$. Therefore, $[\overline{M}] \in P\Gamma_\alpha^I$, proving the claim.

We conclude that

$$P\Gamma_\alpha^I = \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I \subset \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F). \tag{8}$$

Pick a basis for Λ^α over $\mathcal{O}_F$. This yields an isomorphism $(\Lambda^\alpha, h|_{\Lambda^\alpha}) \cong (\mathcal{O}_F^{n+1}, Q_\alpha)$ for some quadratic form Q_α on $\mathcal{O}_F^{n+1}$. If $[M] \in \mathrm{PO}(Q_\alpha)(\mathcal{O}_F)$ satisfies $M \equiv \mathrm{Id} \bmod d$, we can write $M = \mathrm{Id} + d \cdot N$ for some element N in the matrix algebra $M_{n+1}(\mathcal{O}_F)$. This gives $i \circ M_{\mathcal{O}_K} \circ \varphi = d \cdot (\mathrm{Id} + i \circ N_{\mathcal{O}_K} \circ \varphi) \in d \cdot M_{n+1}(\mathcal{O}_K)$, which implies that $[M] \in \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I$. Therefore,

$$\{[M] \in \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F) \mid M \equiv \mathrm{Id} \bmod d\} \subset \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)^I = P\Gamma_\alpha^I \subset \mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F).$$

In particular, $P\Gamma_\alpha^I$ has finite index in $\mathrm{PO}(\Lambda^\alpha)(\mathcal{O}_F)$, and we are done. $\square$

Corollary 2.8. *The subgroup* $P\Gamma_\alpha \subset \mathrm{PO}(V^\alpha) \cong \mathrm{PO}(n,1)$ *is an arithmetic lattice in* $\mathrm{PO}(V^\alpha)$. $\square$

We will also need the following finiteness result.

Proposition 2.9. *Suppose that* $\mathscr{A} \neq \emptyset$. *Let* $\alpha_0 \in \mathscr{A}$, *and define* $G = \mathrm{Gal}(K/F)$.

1. *The group* G *acts on* Γ, *and there is a bijection* $\Gamma\backslash\mathscr{A} \cong \mathrm{H}^1(G,\Gamma)$.
2. *The sets* $\Gamma\backslash\mathscr{A}$ *and* $C\mathscr{A} = P\Gamma\backslash P\mathscr{A}$ *are finite.*

Proof. 1. We can let G act on Γ via the involution $\sigma\colon \gamma \mapsto \alpha_0 \circ \gamma \circ \alpha_0$. Consider

$$\mathrm{H}^1(G,\Gamma) = \mathrm{Z}^1(G,\Gamma)/_\sim = \{1\text{-cocycles } G \to \Gamma, f\colon s \mapsto a_s\}/_\sim$$

where $f \sim f'$ if there exists $b \in \Gamma$ such that $f'(s) = b^{-1}f(s)^s b \in \Gamma$ for all $s \in G$. See [Ser73, Chapitre I, Section 5.1]. If g denotes the generator of G, the map $f \mapsto f(g)$ defines a bijection between the set of 1-cocycles $\mathrm{Z}^1(G,\Gamma)$ and the set of elements $\gamma \in \Gamma$ that satisfy $\gamma \cdot \sigma(\gamma) = \mathrm{id}$. This yields a bijection

$$\mathscr{A} \xrightarrow{\sim} \mathrm{Z}^1(G,\Gamma), \quad \alpha \mapsto \alpha \circ \alpha_0,$$

with inverse given by $\gamma \mapsto \gamma \circ \alpha_0$. Let $\alpha, \beta \in \mathscr{A}$ and $g \in \Gamma$. Then $g^{-1}\beta g = \alpha$ if and only if $g^{-1}(\beta \circ \alpha_0)\,\sigma(g) = \alpha \circ \alpha_0$, hence item 1 is proved.

2. Since the canonical map $\Gamma\backslash\mathscr{A} \to C\mathscr{A}$ is surjective, it suffices to prove that $\Gamma\backslash\mathscr{A}$ is finite. By item 1, we need to show that $\mathrm{H}^1(G,\Gamma)$ is finite. Note that $\Gamma = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda) = U(\Lambda)(\mathcal{O}_K)$ arises as the group of $\mathcal{O}_K$-rational points of a linear algebraic group $U(\Lambda)_K \subset \mathrm{GL}(\Lambda \otimes_{\mathcal{O}_K} K)$ over K. Moreover, the involution $\sigma\colon \Gamma \to \Gamma$ defined in item 1 extends to an involution $\sigma\colon U(\Lambda)_K \to U(\Lambda)_K$, which implies that Γ is an *arithmetic* G*-group* in the sense of [BS64, Section 3.4]. In particular, the set $\mathrm{H}^1(G,\Gamma)$ is finite by [BS64, Proposition 3.8].

The proof is finished. $\square$

2.4 Example

The goal of Section 2.4 is to work out an example. More precisely, we will show that for any CM field K over $\mathbb{Q}$, any CM type $\Psi \subset \mathrm{Hom}(K,\mathbb{C})$ and any $n \in \mathbb{Z}_{\geq 1}$, there exist a non-degenerate hermitian form h on $\mathcal{O}_K^{n+1}$ that satisfies the signature condition (3) with respect to Ψ. We also prove that in these examples, the set $\mathscr{A}$ of anti-unitary involutions α on $(\mathcal{O}_K^{n+1}, h)$ is non-empty.

Let K be a CM field over $\mathbb{Q}$, and let $F \subset K$ be the maximal totally real subfield of K.

Lemma 2.10. *There exists an element* $\lambda \in \mathcal{O}_F$ *with* $\tau(\lambda) > 0$ *for some embedding* $\tau\colon F \to \mathbb{R}$ *and* $\psi(\lambda) < 0$ *for all other* $\psi \neq \tau \in \mathrm{Hom}(F,\mathbb{R})$.

Proof. Let $\tilde\lambda \in \mathcal{O}_F - \mathbb{Z}$. Note that we can write $\mathrm{Hom}(F,\mathbb{R}) = \{\tau, \psi_1, \dots, \psi_k\}$ with $\tau(\tilde\lambda) < \psi_1(\tilde\lambda) < \cdots < \psi_k(\tilde\lambda)$. Let $r \in \mathbb{Q}$ be a rational number such that $\tau(\tilde\lambda) < r < \psi_1(\tilde\lambda)$. Possibly after replacing $\tilde\lambda$ by $n \cdot \tilde\lambda$ for some $n \in \mathbb{Z}_{\geq 1}$, we may assume that $r \in \mathbb{Z}$. The element $\lambda = r - \tilde\lambda \in \mathcal{O}_F$ satisfies the requirements. $\square$

Let $\lambda \in F$ be as in Lemma 2.10. Let $\Lambda = (\mathcal{O}_K^{n+1}, h)$, where h is the hermitian form on $\mathcal{O}_K^{n+1}$ given by $\mathrm{diag}(-\lambda, 1, \dots, 1)$. Let $\Psi \subset \mathrm{Hom}(K, \mathbb{C})$ be any CM type and let $\tau \in \Psi$ be a lift of $\tau \colon F \to \mathbb{R}$. By construction of λ, the signature of h^ψ is $(n, 1)$ for $\psi = \tau$ and $(n+1, 0)$ for $\psi \neq \tau \in \Psi$. Thus, h satisfies condition (3).

To obtain examples of anti-unitary involutions $\alpha \colon \Lambda \to \Lambda$, consider the set $\{\pm 1\}^{n+1}$. Define, for each $\epsilon = (\epsilon_0, \dots, \epsilon_n) \in \{\pm 1\}^{n+1}$, an involution α_ϵ as follows:

$$\alpha_\epsilon \colon \Lambda \to \Lambda, \quad \alpha_\epsilon\left(x_0, \dots, x_n\right) = \left(\epsilon_0 \bar{x}_0, \dots, \epsilon_n \bar{x}_n\right).$$

For $x, y \in \Lambda, b \in \mathcal{O}_K$, one has $\alpha_\epsilon(b \cdot x) = \bar{b} \cdot \alpha_\epsilon(x)$ and $h(\alpha_\epsilon(x), \alpha_\epsilon(y)) = \sigma(h(x, y))$ because $\sigma(\lambda) = \lambda$. Hence α_ϵ is an anti-unitary involution for each $\epsilon \in \{\pm 1\}^{n+1}$.

3 Glueing real hyperbolic orbifolds

In this section, we construct a topological space by glueing together arithmetic hyperbolic pieces. Afterwards, in Section 4, we provide this topological space first with a path metric and then with a complete hyperbolic orbifold structure.

3.1 *Orthogonal hyperplanes and complex reflections*

We continue with the notation and assumptions of Section 2.1. Note that Condition 2.2 implies that if $H_{r_1}, \dots, H_{r_k}$ for $r_i \in \mathscr{R}$ are mutually distinct, and if their common intersection is non-empty, then $\cap_{i=1}^k H_{r_i} \subset \mathbb{C}H^n$ is a totally geodesic subspace of codimension k. Note also that for any $r \in \mathscr{R}$, the element $\phi_r \in \Gamma$ generates a finite subgroup $\langle \phi_r \rangle \subset \Gamma$ of order m, and that the restriction of the quotient map $\Gamma \to P\Gamma$ to this subgroup $\langle \phi_r \rangle \subset \Gamma$ is injective. We will abuse notation, by letting $\phi_r \in P\Gamma$ denote the image of $\phi_r \in \Gamma$ in $P\Gamma$.

Definition 3.1. Let $\mathcal{H} = \{H_r \mid r \in \mathscr{R}\}$. For $x \in \mathbb{C}H^n$, define

$$\mathcal{H}(x) = \{H \in \mathcal{H} \mid x \in H\}, \quad G(x) = \langle \phi_r^i \in P\Gamma \text{ with } r \in \mathscr{R}, i \in \mathbb{Z}/m \mid x \in H_r \rangle.$$

The hyperplanes $H \in \mathcal{H}(x)$ are called the *nodes* of x. We say that x *has k nodes* if the cardinality of $\mathcal{H}(x)$ is k.

Lemma 3.2. *Let $x \in \mathbb{C}H^n$ be an element with k nodes. Then $G(x) \cong (\mathbb{Z}/m)^k$.*

Proof. Let $r, t \in \mathscr{R}$. Then, for $z \in \Lambda$, one has

$$\begin{aligned} \phi_r^i(\phi_t^j(z)) = z &- (1 - \zeta^j) h(z, t) \cdot t - (1 - \zeta^i) h(z, r) \cdot r \\ &+ (1 - \zeta^i)(1 - \zeta^j) h(z, t) h(t, r) \cdot r. \end{aligned} \tag{9}$$

Suppose that $H_r, H_t \in \mathcal{H}(x)$, with $H_r \neq H_t$. By Condition 2.2, we have $h(r, t) = 0$; by equation (9), this implies that $\phi_r^i \circ \phi_t^j = \phi_t^j \circ \phi_r^i$ for each $i, j \in \mathbb{Z}/m$. We conclude that the group $G(x)$ is abelian. Suppose that $H_r = H_t \in \mathcal{H}(x)$. By Lemma 3.3 below, we have that $\phi_t = b \cdot \phi_r^i$ for some $i \in \mathbb{Z}/m$ and $b \in \mu_K$. □

Lemma 3.3. *Let $r \in \mathscr{R}$. Let $\phi : \mathbb{C}H^n \to \mathbb{C}H^n$ be an isometry of order m that restricts to the identity on $H_r \subset \mathbb{C}H^n$. Then $\phi = \phi_r^i$ for some $i \in \mathbb{Z}/m$.*

Proof. Let $\mathbb{H}^n_{\mathbb{C}}$ be the hyperbolic space attached to the standard hermitian space $\mathbb{C}^{n,1}$ of dimension $n+1$. It is classical that

$$\mathrm{Stab}_{U(n,1)}(\mathbb{H}^{n-1}_{\mathbb{C}}) = U(n-1,1) \times U(1).$$

Thus, any $\phi \in U(n,1)$ that fixes $\mathbb{H}^{n-1}_{\mathbb{C}}$ pointwise lies in $\mathbb{C}^* \times U(1)$, where $U(1) = \{z \in \mathbb{C}^* : |z|^2 = 1\}$. If $\phi^m \in \mathbb{C}^* \times \{1\}$, then $\phi \in \mathbb{C}^* \times \langle \zeta \rangle \subset U(n-1,1) \times U(1)$. $\square$

Lemma 2.4 allows us to view $P\mathscr{A}$ as a subset of $\mathrm{Isom}(\mathbb{C}H^n)$, and also to view the groups $G(x) \subset P\Gamma \subset P\Gamma'$ (for any $x \in \mathbb{C}H^n$) as subgroups of $\mathrm{Isom}(\mathbb{C}H^n)$. Define

$$\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha.$$

We will glue the different hyperbolic spaces $\mathbb{R}H^n_\alpha$, by defining an equivalence relation $\sim$ on $\widetilde{Y}$. Before we define it, we state and prove a couple of easy results.

Lemma 3.4. *Let $\alpha \in \mathscr{A}$ and $r \in \mathscr{R}$. Then $\alpha \circ \phi_r^i = \phi_{\alpha(r)}^{-i} \circ \alpha$.*

Proof. Indeed, $\alpha(\phi_r^i(x)) = \alpha(x - (1-\zeta^i)h(x,r)\cdot r) = \phi^{-i}_{\alpha(r)}(\alpha(x))$ for $x \in \Lambda$. $\square$

Lemma 3.5. *Let $x \in \mathbb{R}H^n_\alpha$ and write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$, $r_i \in \mathscr{R}$. For each $i \in \{1, \dots, k\}$ there is a unique $j \in \{1, \dots, k\}$ with $\alpha(H_{r_i}) = H_{\alpha(r_i)} = H_{r_j}$.*

Proof. Let $\beta \in \mathscr{A}$ and $r \in \mathscr{R}$. Then $\beta(H_r) = H_{\beta(r)}$. Since $x \in H_{r_i}$, we have $x = \alpha(x) \in \alpha(H_{r_i}) = H_{\alpha(r_i)}$ for every i. In particular, we have $H_{\alpha(r_i)} \in \mathcal{H}(x)$ (see Definition 3.1), so that $H_{\alpha(r_i)} = H_{r_j}$ for some j. $\square$

Proposition 3.6. *Let $r, t \in \mathscr{R}$. The following are equivalent:*

1. *One has $\phi_r = \phi_t \in \Gamma$.*
2. *There exist $i, j \in \mathbb{Z}/m - \{0\}$ such that $\phi_r^i = \phi_t^j \in \Gamma$.*
3. *There exist $a, b \in \mathcal{O}_K - \{0\}$ with $|a|^2 = |b|^2$ such that $a \cdot r = b \cdot t \in \mathcal{O}_K$.*

Proof. This follows readily from the definitions. $\square$

Definition 3.7. Let $\alpha \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$, see Definition 3.1. By Lemma 3.5, the involution α induces an involution on the set $\mathcal{H}(x)$. Define $\alpha\colon I \to I$ as the resulting involution on the set $I = \{1, \dots, k\}$.

Proposition 3.8. *Let $\alpha \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$, and let $g = \phi_{r_1}^{i_1} \circ \dots \circ \phi_{r_k}^{i_k} \in G(x)$ for some $i_\nu \in \mathbb{Z}/m$. The following are equivalent:*

1. *We have $g \circ \alpha \in P\mathscr{A}$. Equivalently: $g \circ \alpha$ is an involution.*
2. *For each $\nu \in I$, we have $i_\nu \equiv i_{\alpha(\nu)} \bmod m$.*

Proof. This follows from Lemma's 3.3, 3.4 and 3.5, and Proposition 3.6. $\square$

3.2 The definition of the glued space

We are now in position to formulate for which $\alpha, \beta \in P\mathscr{A}$ we glue the space $\mathbb{R}H^n_\alpha$ to the space $\mathbb{R}H^n_\beta$, and how we glue these spaces together.

Definition 3.9. Define a relation $R \subset \widetilde{Y} \times \widetilde{Y}$ as follows. An element

$$(x_\alpha, y_\beta) \in \mathbb{R}H^n_\alpha \times \mathbb{R}H^n_\beta \subset \widetilde{Y} \times \widetilde{Y}$$

is an element of R if the following conditions are satisfied:

1. The images of x_α and y_β in $\mathbb{C}H^n$ under the natural map $\widetilde{Y} \to \mathbb{C}H^n$ agree.
2. If $\alpha \neq \beta$, then $x_\alpha = y_\beta$ lies in $\mathscr{H}$ and $\beta = g \circ \alpha \in P\mathscr{A}$ for some element $g \in G(x_\alpha) = G(y_\beta)$ (c.f. Lemma 2.4).

Remark 3.10. Conditions 1 and 2 in Definition 3.9 say that we are identifying points of $\mathbb{R}H^n_\alpha \cap \mathscr{H}$ and $\mathbb{R}H^n_\beta \cap \mathscr{H}$ that have the same image in $\mathbb{C}H^n$. But we do not glue all such points: the real structures α and β are required to differ by complex reflections in the hyperplanes that pass through x. In fact, we will see below (see Lemma 4.2) that the glueing rules can be rephrased as follows: we glue $\mathbb{R}H^n_\alpha$ and $\mathbb{R}H^n_\beta$ along their intersection, provided that for some (equivalently, any) $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$, the real structures α and β differ by reflections in hyperplanes $H_r \subset \mathscr{H}$ that pass through x.

Lemma 3.11. *The relation R is an equivalence relation.*

Proof. Consider three elements $x_\alpha, y_\beta, z_\gamma \in \widetilde{Y}$. The fact that $x_\alpha \sim x_\alpha$ is clear.

Suppose that $x_\alpha \sim y_\beta$. If $\alpha = \beta$ then $x_\alpha = y_\beta \in \widetilde{Y}$ hence $y_\beta \sim x_\alpha$. If $\alpha \neq \beta$ then $x_\alpha = y_\beta \in \mathscr{H} \subset \mathbb{C}H^n$, and $\beta = g \circ \alpha$ for $g \in G(x_\alpha) = G(y_\beta)$ as in Definition 3.9. Since $\alpha = g^{-1} \circ \beta$ with $g^{-1} \in G(x_\alpha)$, this shows that $y_\beta \sim x_\alpha$.

Suppose that $x_\alpha \sim y_\beta$ and $y_\beta \sim z_\gamma$; we claim that $x_\alpha \sim z_\gamma$. We may and do assume that α, β and γ are different, which implies that $x_\alpha = y_\beta = z_\gamma \in \mathscr{H}$, that $\gamma = h \circ \beta$ for some $h \in G(y_\beta)$, and that $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$. We obtain $\gamma = h \circ \beta = h \circ g \circ \alpha$ for $h \circ g \in G(x_\alpha) = G(y_\beta) = G(z_\gamma)$. $\square$

Lemma 3.12. *The action of $P\Gamma$ on $\mathbb{C}H^n$ induces an action of $P\Gamma$ on $\widetilde{Y}$ which is compatible with the equivalence relation R. Therefore, $P\Gamma$ acts naturally on Y. Moreover, $P\Gamma \setminus \widetilde{Y} = \coprod_{\alpha \in C\mathscr{A}} P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha$.*

Proof. If $\phi \in P\Gamma$, then $\phi\left(\mathbb{R}H^n_\alpha\right) = \mathbb{R}H^n_{\phi\alpha\phi^{-1}}$ hence $P\Gamma$ acts on $\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha$, and

$$P\Gamma \setminus \widetilde{Y} = P\Gamma \setminus \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha = \coprod_{\alpha \in C\mathscr{A}} P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha.$$

Now suppose that $x_\alpha \sim y_\beta \in \widetilde{Y}$ and $f \in P\Gamma$. Then $f(x_\alpha) \in \mathbb{R}H^n_{f\alpha f^{-1}}$ and $f(y_\beta) \in \mathbb{R}H^n_{f\beta f^{-1}}$. We claim that $f(x_\alpha)_{f\alpha f^{-1}} \sim f(y_\beta)_{f\beta f^{-1}}$. For this, we may and do assume that $x_\alpha \neq y_\beta$, hence $x_\alpha = y_\beta \in \mathscr{H}$ and $\beta = g \circ \alpha$ for some $g \in G(x_\alpha)$ as in Definition 3.9. In particular, $f(x_\alpha) = f(y_\beta)$. Since $f \circ \phi_r^i \circ f^{-1} = \phi^i_{f(r)}$ for each $r \in \mathscr{R}$ and $i \in \mathbb{Z}/m$, and $h(x, r) = 0$ if and only if $h(f(x), f(r)) = 0$, we have $fG(x)f^{-1} = G(f(x))$ for each $x \in \mathbb{C}H^n$. This proves that $f\beta f^{-1} = f(g \circ \alpha)f^{-1} = fgf^{-1} \circ (f\alpha f^{-1})$ with $fgf^{-1} \in G(f(x_\alpha))$, and therefore $f(x_\alpha) \sim f(y_\beta) \in \widetilde{Y}$. $\square$

Definition 3.13. Define $Y(\Lambda)$ as the quotient of $\widetilde{Y}$ by the equivalence relation R introduced in Definition 3.9, and equip it with the quotient topology. By Lemma 3.12, the group $P\Gamma$ acts on $Y(\Lambda)$. We define

$$X(\Lambda) = P\Gamma \setminus Y(\Lambda),$$

and call $X(\Lambda)$ the *glued space* attached to the hermitian $\mathcal{O}_K$-lattice Λ.

4 The hyperbolic orbifold structure of the glued space

Section 4 is devoted to the proof of the following theorem.

Theorem 4.1. *1. The space $Y(\Lambda)$ admits a path metric such that the natural map $Y(\Lambda) \to \mathbb{C}H^n$ is a local isometric embedding. This metric on $Y(\Lambda)$ induces a complete path metric on the glued space $X(\Lambda) = P\Gamma \setminus Y(\Lambda)$.*

2. Each point $x \in X(\Lambda)$ admits an open neighborhood $U \subset X(\Lambda)$ which is isometric to the quotient of an open subset $V \subset \mathbb{R}H^n$ by a finite group of isometries.

3. The glued space $X(\Lambda)$ has a canonical realization of a real hyperbolic orbifold of finite volume.

4. One has $\coprod_{\alpha \in C\mathscr{A}} P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H}) \subset X(\Lambda)$ as an open suborbifold.

5. The connected components of the real hyperbolic orbifold $X(\Lambda)$ are uniformized by $\mathbb{R}H^n$: for each component $C \subset X(\Lambda)$ there exists a lattice $P\Gamma_C \subset \mathrm{PO}(n,1)$ and an isomorphism of real hyperbolic orbifolds $C \cong P\Gamma_C \setminus \mathbb{R}H^n$. Consequently,

$$X(\Lambda) \cong \coprod_{C \in \pi_0(P\Gamma \setminus K)} P\Gamma_C \setminus \mathbb{R}H^n.$$

In several cases, the glued space $X(\Lambda)$ is connected: this happens when $K = \mathbb{Q}(\zeta_3)$ and $\Lambda = \mathbb{Z}[\zeta_3]^{2,1}$ or $\Lambda = \mathbb{Z}[\zeta_3]^{4,1}$, see [ACT06; ACT10]. See [GF23] for an example with $K = \mathbb{Q}(\zeta_5)$. If $\Lambda = \mathbb{Z}[\zeta_3]^{3,1}$, then $X(\Lambda)$ has two components, see [ACT07, Remark 6].

4.1 The path metric of the glued space

We start with a lemma. We will need it in the proof of Lemma 4.4 below, which will be used to define a path metric on $X(\Lambda)$ making it locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries. It also serves as a sanity check: if there exists one element $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$, then one glues the entire space $\mathbb{R}H^n_\alpha$ to the space $\mathbb{R}H^n_\beta$ along their intersection in $\mathbb{C}H^n$.

Lemma 4.2. *1. Let $g = \prod_{\nu=1}^k \phi_{r_\nu}^{i_\nu} \in \Gamma$ for some set $\{r_\nu\}$ of mutually orthogonal short roots r_ν, where $i_\nu \not\equiv 0 \bmod m$ for each ν. Then $(\mathbb{C}H^n)^g = \cap_{\nu=1}^k H_{r_\nu}$.*

2. Let $\alpha, \beta \in P\mathscr{A}$ and $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$. Then $y_\alpha \sim y_\beta$ for every $y \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$.

3. The map $\widetilde{Y} \to \mathbb{C}H^n$ descends to a $P\Gamma$-equivariant map $\mathscr{P} \colon Y(\Lambda) \to \mathbb{C}H^n$.

Proof. 1. Let $y \in V$ be representing an element in $(\mathbb{C}H^n)^\phi$. Since the r_i are orthogonal, and $g(y) = \lambda$ for some $\lambda \in \mathbb{C}^*$, we have

$$g(y) = \prod_{\nu=1}^{k} \phi_{r_\nu}^{i_\nu}(y) = y - \sum_{\nu=1}^{k} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu = \lambda y, \tag{10}$$

hence $(1-\lambda)y = \sum_{i=1}^{\ell} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu \in V$. But y spans a negative definite subspace of V while the r_ν span a positive definite subspace, so that we must have $1 - \lambda = 0 = \sum_{\nu=1}^{k} \left(1 - \zeta^{i_\nu}\right) h(y, r_\nu) r_\nu$. Since the r_ν are mutually orthogonal, they are linearly independent; since $\zeta^{i_\nu} \neq 1$ we find $h(y, r_\nu) = 0$ for each ν. Conversely, if $x \in \cap H_{r_\nu}$, then $\phi_{r_\nu}^{i_\nu}(x) = x$ for each ν.

2. Since $x_\alpha \sim x_\beta$, there exists $g \in G(x)$ such that $\beta = g \circ \alpha$. Write $\mathcal{H}(x) = \{H_{r_1}, \dots, H_{r_k}\}$. Let $y \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$. Then $\alpha(y) = \beta(y) = y$ implies that $g(y) = y$. In particular, $y \in \cap_\nu H_{r_\nu}$ by Part 1, which implies that $\mathcal{H}(x) \subset \mathcal{H}(y)$, which in turn implies that $G(x) \subset G(y)$. We conclude that $g \in G(y)$. Hence $y_\alpha \sim y_\beta$.

3. If $x_\alpha \sim y_\beta$, then $x = y \in \mathbb{C}H^n$. $\square$

By Lemma 4.2, we obtain continuous maps

$$\mathscr{P} \colon Y(\Lambda) \to \mathbb{C}H^n \quad \text{and} \quad \overline{\mathscr{P}} : X(\Lambda) = P\Gamma \setminus Y(\Lambda) \to P\Gamma \setminus \mathbb{C}H^n.$$

Our next goal is to prove that each point $x \in Y(\Lambda)$ has a neighbourhood $V \subset Y(\Lambda)$ that maps homeomorphically onto a finite union $\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$. Hence x has an open neighbourhood $x \in U \subset V$ that identifies with an open set in a union of copies of $\mathbb{R}H^n$ in $\mathbb{C}H^n$ under the map $\mathscr{P}$. This allows us to define a metric on $Y(\Lambda)$ by pulling back the metric on $\mathbb{C}H^n$.

Lemma 4.3. *For each compact set $Z \subset \mathbb{C}H^n$ there are only finitely $\alpha \in P\mathscr{A}$ with $Z \cap \mathbb{R}H^n_\alpha \neq \emptyset$.*

Proof. The subgroup $P\Gamma' \subset \mathrm{Isom}(\mathbb{C}H^n)$ (see (4) and Lemma 2.4) acts properly discontinuously on $\mathbb{C}H^n$. So if S is the set of $\alpha \in P\mathscr{A}$ such that $\alpha Z \cap Z \neq \emptyset$, then S is finite. In particular, Z meets only finitely many sets $\mathbb{R}H^n_\alpha$. $\square$

Fix a point $f \in Y(\Lambda)$ and a point $x_\alpha \in \widetilde{Y}$ lying above f. Let $\alpha_1, \dots, \alpha_\ell$ be the elements in $P\mathscr{A}$ such that $x_{\alpha_i} \sim x_\alpha$ for each $i \in I := \{1, \dots, \ell\}$ (since the group $G(x)$ is finite by Lemma 3.2, these are finite in number).

Let $p : \widetilde{Y} \to Y(\Lambda)$ be the quotient map, and define

$$Y_f = p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right) \subset Y(\Lambda). \tag{11}$$

We prove that $Y(\Lambda)$ is locally isometric to opens in unions of real hyperbolic subspaces of $\mathbb{C}H^n$. Indeed, we have the following:

Lemma 4.4. *1. The set Y_f is closed in $Y(\Lambda)$.*

2. We have $\mathscr{P}(Y_f) = \cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$, and the map

$$\mathscr{P}_f : Y_f \to \bigcup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}$$

induced by $\mathscr{P}$ is a homeomorphism.

3. The set $Y_f \subset Y(\Lambda)$ contains an open neighborhood U_f of f in $Y(\Lambda)$.

Proof. 1. One has

$$p^{-1}(Y_f) = p^{-1}\left(p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)\right) = \bigcup_{i=1}^{\ell} p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right) \subset \widetilde{Y}.$$

Therefore, it suffices to show that $p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right)$ is closed in $\widetilde{Y}$. But notice that $x_\beta \in p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$ if and only if $x \in \mathbb{R}H^n_\alpha$ and $x_\alpha \sim x_\beta$, which implies (Lemma 4.2) that $\mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta \subset p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$. Hence for any $\alpha \in P\mathscr{A}$, one has

$$p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right) = \coprod_{\beta\sim\alpha} \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta,$$

where $\beta \sim \alpha$ if and only if there exists $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $x_\alpha \sim x_\beta$. It follows that $p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right) \cap \mathbb{R}H^n_\beta$ is closed in $\mathbb{R}H^n_\beta$ for every $\beta \in P\mathscr{A}$. But the $\mathbb{R}H^n_\beta$ are open in $\widetilde{Y}$ and cover $\widetilde{Y}$, so that $p^{-1}\left(p\left(\mathbb{R}H^n_\alpha\right)\right)$ is closed in $\widetilde{Y}$.

2. We have

$$\mathscr{P}_f(Y_f) = \mathscr{P}\left(p\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)\right) = \widetilde{\mathscr{P}}\left(\coprod_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right) = \bigcup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n.$$

To prove injectivity, let $x_{\alpha_i}, y_{\alpha_j} \in \widetilde{Y}$ and suppose that $x = y \in \mathbb{C}H^n$. Then indeed, $x_{\alpha_i} \sim y_{\alpha_j}$ because $\sim$ is an equivalence relation by Lemma 3.11.

Let $Z \subset \mathbb{C}H^n$ be a compact set. Let $\widetilde{\mathscr{P}} : \widetilde{Y} \to \mathbb{C}H^n$ be the canonical map. Remark that Z meets only finitely many of the $\mathbb{R}H^n_\alpha$ for $\alpha \in P\mathscr{A}$, see Lemma 4.3. Each $Z \cap \mathbb{R}H^n_\alpha$ is closed in Z since $\mathbb{R}H^n_\alpha$ is closed in $\mathbb{C}H^n$, so each $Z \cap \mathbb{R}H^n_\alpha$ is compact. We conclude that $\widetilde{\mathscr{P}}^{-1}(Z) = \coprod Z \cap \mathbb{R}H^n_\alpha$ is compact. In particular, $\widetilde{\mathscr{P}}$ is closed.

Finally, we prove that $\mathscr{P}_f$ is closed. Let $Z \subset Y_f$ be a closed set. Then Z is closed in Y by Part 1, hence $p^{-1}(Z)$ is closed in $\widetilde{Y}$, hence $\widetilde{\mathscr{P}}\left(p^{-1}(Z)\right)$ is closed in $\mathbb{C}H^n$, so that

$$\mathscr{P}_f(Z) = \mathscr{P}(Z) = \widetilde{\mathscr{P}}\left(p^{-1}(Z)\right) = \left(\widetilde{\mathscr{P}}\left(p^{-1}(Z)\right)\right) \cap \left(\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}\right)$$

is closed in $\cup_{i=1}^{\ell} \mathbb{R}H^n_{\alpha_i}$.

3. Let $x = \mathscr{P}(f) \in \mathbb{C}H^n$. Since $\mathbb{C}H^n$ is locally compact, there exists a compact set $Z \subset \mathbb{C}H^n$ and an open set $U \subset \mathbb{C}H^n$ with $x \in U \subset Z$. Since Z is compact, it meets only finitely many of the $\mathbb{R}H^n_\beta \subset \mathbb{C}H^n$ (Lemma 4.3). Consequently, the same holds for U; define $V = \mathscr{P}^{-1}(U) \subset Y(\Lambda)$. Define

$$\mathscr{B} = \{\beta \in P\mathscr{A} : U \cap \mathbb{R}H^n_\beta \neq \emptyset\}.$$

Also define, for $\beta \in P\mathscr{A}$, $Z_\beta = p\left(\mathbb{R}H^n_\beta\right) \subset Y(\Lambda)$. Then

$$f \in V \subset \bigcup_{\beta\in\mathscr{B}} Z_\beta = \bigcup_{\substack{\beta\in\mathscr{B}\\ \beta(x)=x}} Z_\beta \bigcup_{\substack{\beta\in\mathscr{B}\\ \beta(x)\neq x}} Z_\beta.$$

Since each Z_β is closed in $Y(\Lambda)$ by the proof of part 1, there is an open $V' \subset V$ with

$$f \in V' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x)=x}} Z_\beta = \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x)=x \\ x_\beta \not\sim x_\alpha}} Z_\beta$$

Hence again there exists an open subset $V'' \subset V'$ with

$$f \in V'' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta \subset \bigcup_{\substack{\beta \in P\mathscr{A} \\ \beta(x)=x \\ x_\beta \sim x_\alpha}} Z_\beta = Y_f.$$

Therefore, $U_f := V'' \subset Y$ satisfies the requirements. $\square$

Lemma 4.5. *The topological space $Y(\Lambda)$ is Hausdorff.*

Proof. Let $f, f' \in Y(\Lambda)$ be elements such that $f \neq f'$. First suppose that $f \notin Y_{f'}$. Since $Y_{f'}$ is closed in $Y(\Lambda)$ by Lemma 4.4, there is an open neighbourhood U of f such that $U \cap U_{f'} \subset U \cap Y_{f'} = \emptyset$.

Next, suppose that $f \in Y_{f'}$. Lift f and f' to elements $x_\alpha, y_\beta \in \widetilde{Y}$. Assume first that $x = y$. This means that $\mathscr{P}(f) = \mathscr{P}(f')$. Since $\mathscr{P} : Y_{f'} \to \mathbb{C}H^n$ is injective, this implies that $f = f'$, contradiction. So we have $x \neq y \in \mathbb{C}H^n$. But $\mathbb{C}H^n$ is Hausdorff, so there are open subsets $(U \subset \mathbb{C}H^n, V \subset \mathbb{C}H^n)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $\mathscr{P}^{-1}(U) \cap \mathscr{P}^{-1}(V) = \emptyset$. $\square$

Proposition 4.6. *$Y(\Lambda)$ is a path metric space, piecewise isometric to $\mathbb{R}H^n$.*

Proof. For each $f \in Y(\Lambda)$ there exists an open neighborhood $f \in U_f \subset Y(\Lambda)$ such that $\mathscr{P}$ induces a homeomorphism $Y \supset U_f \xrightarrow{\sim} \mathscr{P}(U_f) \subset \mathbb{C}H^n$. Indeed, this follows from Lemma 4.4. Pull back the metric on $\mathscr{P}(U_f)$ to obtain a metric on U_f. Then define a metric on $Y(\Lambda)$ as the largest metric which is compatible with the metric on each open set U_f and which preserves the lengths of paths. $\square$

Proposition 4.7. *The path metric on $Y(\Lambda)$ descends to a path metric on $X(\Lambda)$.*

Proof. The metric on $Y(\Lambda)$ descends in any case to a pseudo-metric on $X(\Lambda)$, and by [Gro07, Chapter 1], this is a metric if $P\Gamma$ acts by isometries on $Y(\Lambda)$ with closed orbits. This is true: the fact that $P\Gamma$ acts isometrically on $Y(\Lambda)$ comes from the $P\Gamma$-equivariance of $\mathscr{P} : Y \to \mathbb{C}H^n$ (Lemma 4.2) together with the construction of the metric on $Y(\Lambda)$ (Proposition 4.6). To check that the $P\Gamma$-orbits are closed in $Y(\Lambda)$, let $f \in Y(\Lambda)$ with representative $x_\alpha \in \widetilde{Y}$. By equivariance of $p : \widetilde{Y} \to Y(\Lambda)$, we have $p^{-1}(P\Gamma \cdot f) = P\Gamma \cdot (p^{-1} f)$, so since p is a quotient map, it suffices to show that

$$P\Gamma \cdot \left(p^{-1} f\right) = P\Gamma \cdot \bigcup_{x_\beta \sim x_\alpha} x_\beta = \bigcup_{x_\beta \sim x_\alpha} P\Gamma \cdot x_\beta$$

is closed in $\widetilde{Y}$, thus that each orbit $P\Gamma \cdot x_\beta$ is closed in $\widetilde{Y}$. Since $P\Gamma$ is discrete, it suffices to show that $P\Gamma$ acts properly on $\widetilde{Y}$. So let $Z \subset \widetilde{Y}$ be any compact set: we claim that $\{g \in P\Gamma : gZ \cap Z \neq \emptyset\}$ is a finite set. Indeed, for each $g \in P\Gamma$, one has $\widetilde{\mathscr{P}}(gZ \cap Z) \subset g\widetilde{\mathscr{P}}(Z) \cap \widetilde{\mathscr{P}}(Z)$, and the latter is non-empty for only finitely many $g \in P\Gamma$, by properness of the action of $P\Gamma$ on $\mathbb{C}H^n$.

Since $Y(\Lambda)$ is a path metric space, the same holds for $X(\Lambda)$, see [Gro07]. $\square$

4.2 The orbifold structure of the glued space

The next step is to prove that the glued space $X(\Lambda) = P\Gamma \setminus Y(\Lambda)$ (see Definition 3.13) is locally isometric to quotients of open sets in $\mathbb{R}H^n$ by finite groups of isometries.

Definition 4.8. Let $f \in Y(\Lambda)$ with representative $x_\alpha \in \widetilde{Y}$. Thus, x is an element in $\mathbb{C}H^n$, and $\alpha \in P\mathscr{A}$ is the class of an anti-unitary involution such that $\alpha(x) = x$.

1. The *nodes* of f are by definition the nodes of x_α (see Definition 3.1). Thus, these are the hyperplanes $H \in \mathcal{H}(x)$, i.e. the hyperplanes $H_r \in \mathcal{H}$ defined by short roots $r \in \mathscr{R}$ such that $x \in H_r$ (equivalently, such that $h(x,r) = 0$).

2. The number of nodes of f is the cardinality of $\mathcal{H}(x)$.

3. The anti-unitary involution $\alpha \in P\mathscr{A}$ induces an involution on the set $\mathcal{H}(x)$ by Lemma 3.5. Let $H \in \mathcal{H}(x)$ be a node. We call H a *real node* of f if $\alpha(H) = H$. We call $(H, \alpha(H))$ a *pair of complex conjugate nodes* of f if $\alpha(H) \neq H$.

4. If k is the number of nodes of f, we generally write $k = 2a+b$, with a the number of pairs of complex conjugate nodes of f, and b the number of real nodes of f.

Fix again a point $f \in Y(\Lambda)$ and a point $x_\alpha \in \widetilde{Y}$ lying above f. Let $k = 2a + b$ be the number of nodes of f. Thus $x \in \mathbb{R}H^n_\alpha$, and there exist $r_1, \ldots, r_k \in \mathscr{R}$ such that

$$\mathcal{H}(x) = \{H_{r_1}, \ldots, H_{r_k}\}, \quad G(x) = \langle \phi_{r_1}, \ldots, \phi_{r_\ell} \rangle \cong (\mathbb{Z}/m)^k.$$

For $\beta \in P\mathscr{A}$, observe that $x_\beta \sim x_\alpha$ if and only if $\alpha \circ \beta \in G(x)$. We relabel the r_i so that they satisfy the following condition:

$$\alpha(H_{r_i}) = \begin{cases} H_{r_{i+1}} \text{ for } i \text{ odd and } i \leq 2a, \\ H_{r_{i-1}} \text{ for } i \text{ even and } i \leq 2a, \text{ and} \\ H_{r_i} \text{ for } i \in \{2a+1, \ldots, k\}. \end{cases} \tag{12}$$

In other words, H_{r_i} is a real root if and only if $i > 2a$, and $\left(H_{r_i}, H_{r_{i+1}}\right)$ is a pair of complex conjugate roots if and only if $i < 2a$ is odd.

Lemma 4.9. *Continue with the notation from above.*

1. Let $\beta \in P\mathscr{A}$ be such that $x_\beta \sim x_\alpha$. Then

$$\beta = \prod_{i=1}^{a} \left(\phi_{r_{2i-1}} \circ \phi_{r_{2i}}\right)^{j_i} \circ \prod_{i=2a+1}^{k} \phi_{r_i}^{j_i} \circ \alpha$$

for some $j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k \in \mathbb{Z}/m$. In particular, there are m^{a+b} such β.

2. There is an isometry $\mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$ identifying x with the origin, ϕ_{r_i} with the map

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \ldots, t_i, \ldots, t_n) \mapsto (t_1, \ldots, \zeta t_i, \ldots, t_n),$$

and α with the map defined by

$$t_i \mapsto \begin{cases} \bar{t}_{i+1} & \text{for } i \text{ odd and } i \leq 2a \\ \bar{t}_{i-1} & \text{for } i \text{ even and } i \leq 2a \\ \bar{t}_i & \text{for } i > 2a. \end{cases} \tag{13}$$

Proof. 1. This follows readily from Proposition 3.8.

2. Since the H_{r_i} are orthogonal by Condition 2.2, and their intersection contains x, we can find coordinates $t_1, \dots, t_{n+1}$ on V that induce an identification $(V, h) \cong \mathbb{C}^{n,1} := \left(\mathbb{C}^{n+1}, H\right)$ with $H(x,x) = |x_1|^2 + \dots + |x_n|^2 - |x_{n+1}|^2$, in such a way that $H_{r_i} \subset V$ is identified with the hyperplane $\{t_i = 0\} \subset \mathbb{C}^{n+1}$ and $x \in \cap_i H_{r_i}$ with the point $(0, 0, \dots, 0, 1)$. We will do this in the following way. Define

$$T = \langle x \rangle \oplus \langle r_1 \rangle \oplus \dots \oplus \langle r_k \rangle \subset V, \quad W = T^\perp = \{w \in V \mid h(w,t) = 0 \;\; \forall t \in T\}.$$

For each $i \in I = \{1, \dots, k\}$, we have $\alpha(r_i) = \lambda_i \cdot r_{\alpha(i)}$ for some $\lambda_i \in K$ (see Lemma 3.5, Definition 3.7, and Lemmas 3.3 and 3.2). Observe that $\alpha(W) = W$. Since $W \subset \langle x \rangle^\perp$, the hermitian space $(W, h|_W)$ is positive definite. Let $\{w_1, \dots, w_{n-k}\} \subset W$ be an orthonormal basis such that $\alpha(w_i) = w_i$. Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of $\mathbb{C}^{n+1}$, and define a $\mathbb{C}$-linear isomorphism

$$\Phi \colon V \xrightarrow{\sim} \mathbb{C}^{n+1}, \quad \left(\frac{x}{h(x,x)} \mapsto e_{n+1}, \quad r_i \mapsto e_i, \quad w_i \mapsto e_i\right). \tag{14}$$

By (12), we have that $\alpha(r_i) = \lambda_i \cdot r_{i+1}$ for i odd and $i \leq 2a$, that $\alpha(r_i) = \lambda_i \cdot r_{i-1}$ for i even and $i \leq 2a$, and that $\alpha(r_i) = \lambda_i \cdot r_i$ for $i > 2a$. We conclude that the anti-linear involution on $\mathbb{C}^{n+1}$ induced by α and (14) corresponds to the matrix

$$\alpha = \operatorname{diag}(A_{12}, A_{34}, \dots, A_{2a-1,2a}, \alpha_{2a+1}, \dots, \alpha_n), \quad A_{i,i+1} = \begin{pmatrix} 0 & \alpha_i \\ \alpha_{i+1} & 0 \end{pmatrix},$$

where each α_i is an anti-linear involution $\mathbb{C} \to \mathbb{C}$, and $\alpha_i = \alpha_{i+1}$ for $i < 2a$ odd. If $\alpha_i(1) = \mu_i \in \mathbb{C}^*$, then $\mu_i^{-1} \cdot \alpha_i$ is the complex conjugation map $\mathbb{C} \to \mathbb{C}$. Since $|\mu_i| = 1$, there exists $\rho_i \in \mathbb{C}$ such that $\mu_i = \overline{\rho_i}/\rho_i$ and $|\rho_i| = 1$. This gives $\mu_i^{-1} \cdot \alpha_i = \rho_i \cdot \alpha_i \cdot \rho_i^{-1} = \operatorname{conj} \colon \mathbb{C} \to \mathbb{C}$. The composition of $\Phi \colon V \to \mathbb{C}^{n+1}$ with the diagonal linear transformation $\operatorname{diag}(\rho_1, \dots, \rho_{n+1}) \colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces an isomorphism $\mathbb{C}H^n \cong \mathbb{B}^n(\mathbb{C})$ with the required properties. $\square$

Definition 4.10. 1. Define $A_f = \operatorname{Stab}_{P\Gamma}(f)$ to be the subgroup of $P\Gamma$ fixing $f \in Y(\Lambda)$. This contains the group $G(x) \cong (\mathbb{Z}/m)^k$.

2. Define B_f as the subgroup of $G(x)$ generated by the order m complex reflections associated to the real nodes of f. Hence $B_f = \langle \phi_{r_i} \rangle_{i > 2a} \cong (\mathbb{Z}/m)^b$.

Consider the quotient map $p \colon \widetilde{Y} \to Y(\Lambda)$, and the subset $Y_f \subset X(\Lambda)$, see (11).

Lemma 4.11. *The stabilizer A_f of $f \in Y(\Lambda)$ preserves the subset $Y_f \subset Y(\Lambda)$.*

Proof. Let $\psi \in A_f$, with $f = p(x_\alpha) \in Y(\Lambda)$, $x \in \cap_i H_{r_i}$. Then $\psi(x)_{\psi\alpha\psi^{-1}} \sim x_\alpha$. Now let $p(y_\beta) \in Y_f$. Then $\beta(x) = x$ and $x_\alpha \sim x_\beta$. Hence $x_\alpha \sim \psi \cdot x_\alpha \sim \psi \cdot x_\beta = \psi(x)_{\psi\beta\psi^{-1}}$. This implies that $\psi\beta\psi^{-1} \circ \alpha \in G(x)$, so that $p\left(\psi(y)_{\psi\beta\psi^{-1}}\right) \in Y_f$. $\square$

We also need the following lemma. Write $m = 2^a k$ with $k \neq 0 \bmod 2$.

Lemma 4.12. *Let $T = \{t \in \mathbb{C} : t^m \in \mathbb{R}\}$. Then $G = \langle \zeta_m \rangle$ acts on T by multiplication. Moreover, each element in T/G has a unique representative of the form $\zeta_{2^{a+1}}^\epsilon \cdot r \in T$ for $r \geq 0$ and $\epsilon \in \{0, 1\}$.*

Proof. Observe that $a \geq 1$. Moreover, for $t \in \mathbb{C}$, one has that $t = r\zeta_{2m}^{j}$ for some $j \in \mathbb{Z}$ and $r \in \mathbb{R}$ if and only if $t^m \in \mathbb{R}$. Since $\gcd(2,k) = 1$, we have $\zeta_{2^{a+1}} \cdot \zeta_{2^a k} = (\zeta_{2^{a+1}k})^{k+2}$. Raising both sides to the power $b = (k+2)^{-1} \in (\mathbb{Z}/m)^*$ gives $\zeta_{2m} = \zeta_{2^{a+1}}^{b} \cdot \zeta_{m}^{b}$. Consequently, $t^m \in \mathbb{R}$ if and only if $t = r \cdot \zeta_{2^{a+1}}^{bj} \cdot \zeta_{m}^{bj}$ for some $r \in \mathbb{R}$. Finally, $\zeta_{2^{a+1}}^{u} \cdot \zeta_{2^a}^{v} = \zeta_{2^{a+1}}^{u+2v}$ hence $\langle \zeta_{2^{a+1}} \rangle / \langle \zeta_{2^a} \rangle \cong \mathbb{Z}/2$. □

We obtain the key to Theorem 4.1.

Proposition 4.13. *With the above notation, the following assertions are true.*

1. If f has no nodes, then $G(x) = B_f$ is trivial, and $Y_f = \mathbb{R}H^n_\alpha \cong \mathbb{B}^n(\mathbb{R})$.

2. If f has only real nodes, then $B_f \setminus Y_f$ is isometric to $\mathbb{B}^n(\mathbb{R})$.

3. If f has a pairs of complex conjugate nodes ($k = 2a$), and no other nodes, then $B_f \setminus Y_f = Y_f$ is the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$, any two of which meet along a $\mathbb{B}^{2c}(\mathbb{R})$ for some integer c with $0 \leq c \leq a$.

4. If f has $2a$ complex conjugate nodes and b real nodes, then there is an isometry between $B_f \setminus Y_f$ and the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$ identified along common $\mathbb{B}^{2c}(\mathbb{R})'s$, that is, the set Y_f of case 3 above.

5. In each case, A_f acts transitively on the indicated copies of $\mathbb{B}^n(\mathbb{R})$. If $\mathbb{B}^n(\mathbb{R})$ is any one of them, and $\Gamma_f = (A_f/B_f)_{\mathbb{B}^n(\mathbb{R})}$ its stabilizer, then the natural map

$$\Gamma_f \setminus \mathbb{B}^n(\mathbb{R}) \to (A_f/B_f) \setminus (B_f \setminus Y_f) = A_f \setminus Y_f$$

is an isometry of path metrics.

Proof. 1. This is clear.

2. Suppose then that f has k real nodes. Then in the local coordinates t_i of Lemma 4.9.2, we have that $\alpha : \mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C})$ is defined by $\alpha(t_i) = \bar{t}_i$. Part 1 of the same lemma shows that any $\beta \in P\mathscr{A}$ fixing x such that $x_\alpha \sim x_\beta$ is of the form

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \ldots, t_i, \ldots, t_n) \mapsto (\bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \ldots, \bar{t}_n).$$

Since f has k real nodes and no complex conjugate nodes, we have (writing $j = (j_1, \ldots, j_k)$ and $\alpha_j = \prod_{i=1}^{k} \phi_{r_i}^{j_i} \circ \alpha$):

$$Y_f \cong \bigcup_{j_1,\ldots,j_k=1}^{m} \mathbb{R}H^n_{\alpha_j} \cong \{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_1^m, \ldots, t_k^m, t_{k+1}, \ldots, t_n \in \mathbb{R}\}.$$

Each of the 2^k subsets

$$K_{f,\epsilon_1,\ldots,\epsilon_k} := \left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : \zeta_{2^{a+1}}^{-\epsilon_1} t_1, \ldots, \zeta_{2^{a+1}}^{-\epsilon_k} t_k \in \mathbb{R}_{\geq 0} \text{ and } t_{k+1}, \ldots, t_n \in \mathbb{R}\right\},$$

indexed by $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, is isometric to the closed region in $\mathbb{B}^n(\mathbb{R})$ bounded by k mutually orthogonal hyperplanes. By Lemma 4.12, their union U is a fundamental domain for B_f, in the sense that it maps homeomorphically and piecewise-isometrically onto $B_f \setminus Y_f$. Under its path metric, $U = \cup K_{f,\epsilon_1,\ldots,\epsilon_k}$ is isometric to $\mathbb{B}^n(\mathbb{R})$ by the following map:

$$U \to \mathbb{B}^n(\mathbb{R}), \quad (t_1, \ldots, t_k) \mapsto \left((-\zeta_{2^{a+1}})^{-\epsilon_1} t_1, \ldots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \ldots, t_n\right).$$

This identifies $B_f \setminus Y_f$ with the standard $\mathbb{B}^n(\mathbb{R}) \subset \mathbb{B}^n(\mathbb{C})$.

3. Now suppose f has $k = 2a$ nodes $H_{r_1}, \dots, H_{r_{2a}}$. There are now m^a anti-isometric involutions α_{j_i} fixing x and such that $x_{\alpha_{j_i}} \sim x_\alpha$: they are given in the coordinates t_i as follows, taking $j = (j_1, \dots, j_a) \in (\mathbb{Z}/m)^a$:

$$\alpha_j : (t_1, \dots, t_n) \mapsto (\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \dots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}, \dots, \bar{t}_n).$$

So any fixed-point set $\mathbb{R}H^n_{\alpha_j}$ is identified with

$$\mathbb{B}^n(\mathbb{R})_{\alpha_j} = \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \colon t_i = \bar{t}_{i-1}\zeta^{j_i} \text{ for } i \leq 2a \text{ even, } t_i \in \mathbb{R} \text{ for } i > 2a\right\},$$

and we have

$$Y_f \cong \bigcup_j \mathbb{B}^n(\mathbb{R})_{\alpha_j} = \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \colon t_i^m = \bar{t}_{i-1}^m \text{if } i \leq 2a \text{ even, } t_i \in \mathbb{R} \text{ if } i > 2a\right\}.$$

These m^a copies of $\mathbb{B}^n(\mathbb{R})$ meet at the origin of $\mathbb{B}^n(\mathbb{C})$; in fact, for $j \neq j'$, the space $\mathbb{B}^n(\mathbb{R})_{\alpha_j}$ meets $\mathbb{B}^n(\mathbb{R})_{\alpha_{j'}}$ in a $\mathbb{B}^{2c}(\mathbb{R})$ if c is the number of pairs (j_i, j_i') with $j_i = j_i'$.

4. Now we treat the general case. In the local coordinates t_i, any anti-unitary involutions fixing x and equivalent to α is of the form

$$(t_1, \dots, t_n) \mapsto (\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \dots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}\zeta^{j_{2a+1}}, \dots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \dots, \bar{t}_n)$$

for some $j = (j_1, \dots, j_a, j_{2a+1}, \dots, j_k) \in (\mathbb{Z}/m)^{a+b}$. We now have $B_f \cong (\mathbb{Z}/m)^b$ acting by multiplying the t_i for $2a+1 \leq i \leq k$ by powers of ζ, and there are m^{a+b} anti-unitary involutions α_j. We have

$$Y_f \cong \bigcup_{j_1, \dots, j_k = 1}^{m} \mathbb{R}H^n_{\alpha_j} \cong$$
$$\left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^m = \bar{t}_1^m, \dots, t_{2a}^m = \bar{t}_{2a-1}^m, t_{2a+1}^m, \dots, t_k^m, t_{k+1}, \dots, t_n \in \mathbb{R}\right\}.$$

We look at subsets $K_{f,\epsilon_1,\dots,\epsilon_k} \subset Y_f$ again, this time defined as

$$\left\{t \in \mathbb{B}^n(\mathbb{C}) \mid t_i^m = \bar{t}_{i-1}^m \ i \leq 2a \text{ even, } \zeta_{2^{a+1}}^{-\epsilon_i} t_i \in \mathbb{R}_{\geq 0} \ 2a < i \leq k, \ t_i \in \mathbb{R}, i > k\right\}.$$

As before, the natural map $U \coloneqq \bigcup_\epsilon K_{f,\epsilon} \to B_f \setminus Y_f$ is an isometry. Define

$$\widetilde{Y}_f = \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \text{ for } i \leq 2a \text{ even, } t_i \in \mathbb{R}, \text{ for } i > 2a\right\}.$$

Under its path metric, $U = \cup_\epsilon K_{f,\epsilon_1,\dots,\epsilon_k}$ is isometric to $\widetilde{Y}_f$ by the following map:

$$U \to \widetilde{Y}_f, \quad (t_1, \dots, t_k) \mapsto$$
$$\left(t_1, \dots, t_{2a}, (-\zeta_{2^{a+1}})^{-\epsilon_1} t_{2a+1}, \dots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \dots, t_n\right).$$

Hence $B_f \setminus Y_f \cong \widetilde{Y}_f$; but since $\widetilde{Y}_f$ is what Y_f was in case 3, we are done.

5. The transitivity of A_f on the copies of $\mathbb{B}^n(\mathbb{R})$ follows from the fact that $G(x) \subset A_f$ contains transformations multiplying $t_1, \dots, t_{2a}$ by powers of ζ, hence the map $(t_i \mapsto \zeta^u t_i, t_{i-1} \mapsto t_{i-1})$ maps those (t_{i-1}, t_i) with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ to those (t_{i-1}, t_i) with $t_i = \bar{t}_{i-1}\zeta^{j_1+u}$. So if B is any one of the copies of $\mathbb{B}^n(\mathbb{R})$, and $G = (A_f/B_f)_H$ is its stabilizer, then it remains to prove that $G \setminus B \to A_f \setminus Y_f$ is an isometry. Surjectivity follows from the transitivity of A_f on the $\mathbb{B}^n(\mathbb{R})'s$. It is a piecewise isometry so we only need to prove injectivity. This will follow from the following elementary lemma.

Lemma 4.14. *Let a group G act on a set X, which is a union $X = \cup_{i\in I} Y_i$ of subsets $Y_i \subset X$. Suppose that for all $y \in X$, the stabilizer of y in G acts transitively on the sets Y_i containing y. Then for any $i_0 \in I$, if $H_{i_0} \subset G$ is the stabilizer of Y_{i_0}, then the natural map $H_{i_0} \setminus Y_{i_0} \to G \setminus X$ is injective.*

Proof. Let $x, y \in Y_{i_0}$ and $g \in G$ such that $g \cdot x = y$. Then $y = gx \in gY_{i_0}$. Since also $y \in Y_{i_0}$, there is an element $h \in \mathrm{Stab}_G(y)$ such that $hgY_{i_0} = Y_{i_0}$ and $hg(x) = h(y) = y$. Let $f = hg$; then $f \in H_{i_0}$ and $f \cdot x = y$. □

Let us use the lemma. Suppose that $y \in B_f \backslash Y_f$. We need to prove that $\mathrm{Stab}_{A_f/B_f}(y)$ acts transtivitely on the copies of $\mathbb{B}^n(\mathbb{R})$ containing y. There exists

$$j = (j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k) \in (\mathbb{Z}/m)^{a+b}$$

such that $y = (t_1, \ldots, t_n)$ with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $i \leq 2a$ even, $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $2a < i \leq k$, and $t_i \in \mathbb{R}$ for $i > k$. If all t_i are non-zero, then $y \in \cup_{j'} \mathbb{R}H^n_{\alpha_{j'}}$ is only contained in $\mathbb{R}H^n_{\alpha_j}$, so there is nothing to prove. Let us suppose that $t_1 = t_2 = 0$ and the other t_i are non-zero. Then y is contained in all the $\mathbb{R}H^n_{\alpha_{j'}}$ with $j'_i = j_i$ for $i \geq 2$; there are m of them. The stabilizer of y multiplies t_1 and t_2 by powers of ζ and leaves the other t_i invariant; it acts transitively on the $\mathbb{R}H^n_{\alpha_{j'}}$ containing y for if $t_2 = \bar{t}_1\zeta^{j'_1}$ then $\zeta^{(j''_1 - j'_1)} t_2 = \bar{t}_1 \zeta^{j''_1}$. The general case is similar. □

We need one more lemma before we can prove Theorem 4.1:

Lemma 4.15. *The maps $\mathscr{P} \colon Y(\Lambda) \to \mathbb{C}H^n$ and $\overline{\mathscr{P}} \colon X(\Lambda) \to P\Gamma \setminus \mathbb{C}H^n$ are proper with finite fibers.*

Proof. The map $\mathscr{P} \colon Y(\Lambda) \to \mathbb{C}H^n$ is proper with finite fibers because any compact set in $\mathbb{C}H^n$ meets only finitely many $(\mathbb{R}H^n_\alpha)'s$ for $\alpha \in P\mathscr{A}$ (see Lemma 4.3), and $\mathscr{P}$ carries each $p\,(\mathbb{R}H^n_\alpha) \subset Y(\Lambda)$ homeomorphically onto $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$. Since $\mathscr{P}$ is $P\Gamma$-equivariant, the induced map $\overline{\mathscr{P}} \colon X(\Lambda) \to P\Gamma \backslash \mathbb{C}H^n$ is proper with finite fibers as well. □

Proof of Theorem 4.1. 1. The path metric on $Y(\Lambda)$ was provided in Proposition 4.6, and the path metric on $X(\Lambda)$ was given in Proposition 4.7. The map $\mathscr{P} \colon Y \to \mathbb{C}H^n$ is a local embedding by Lemma 4.4, which was used to define the metric on $Y(\Lambda)$ (c.f. Proposition 4.6). Thus, almost by definition, $\mathscr{P}$ is a local isometric embedding. The space $P\Gamma \setminus \mathbb{C}H^n$ is complete, and $\mathscr{P}$ and $\overline{\mathscr{P}}$ are proper by Lemma 4.15. Therefore, the metric sapce $X(\Lambda)$ is complete as well.

2. $[f] \in X(\Lambda)$ be the image of $f \in Y(\Lambda)$. Then $[f]$ has an open neighborhood isometric to the quotient of an open set W in $\mathbb{R}H^n$ by a finite group of isometries Γ_f. Indeed, take $Y_f \subset Y(\Lambda)$ as in Equation (11), and $f \in U_f \subset Y_f$ as in Lemma 4.4.2. We let $A_f = P\Gamma_f$ be the stabilizer of f in $P\Gamma$ as before, and take an A_f-equivariant open neighborhood $V_f \subset U_f$ such that $A_f \setminus V_f \subset P\Gamma \setminus Y(\Lambda)$. By Proposition 4.13.5, we know that $A_f \setminus Y_f$ is isometric to $\Gamma_f \setminus \mathbb{R}H^n$ for some finite group of isometries of $\mathbb{R}H^n$. Thus, $A_f \setminus V_f$ is isometric to some open set W' in $\Gamma_f \setminus \mathbb{R}H^n$. Take $W \subset \mathbb{R}H^n$ to be the preimage of W'.

3. *Claim:* For any path metric space X locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries, there is a unique real-hyperbolic orbifold structure

on X whose path metric is the given one. *Proof of the Claim:* If U and U' are connected open subsets of $\mathbb{R}H^n$ and Γ and Γ' finite groups of isometries of $\mathbb{R}H^n$ preserving U and U' respectively, then any isometry $\bar{\phi} : \Gamma \setminus U \to \Gamma' \setminus U'$ extends to an isometry $\phi : \mathbb{R}H^n \to \mathbb{R}H^n$ such that $\phi(U) = U'$ and $\phi\Gamma\phi^{-1} = \Gamma' \subset \mathrm{Isom}(\mathbb{R}H^n)$. We conclude that $P\Gamma \setminus Y(\Lambda)$ is naturally a real hyperbolic orbifold.

We show that the complete real hyperbolic orbifold $X(\Lambda)$ has finite volume. For each $\alpha \in P\mathscr{A}$, the natural map $P\Gamma_\alpha \backslash \mathbb{R}H^n_\alpha \to X(\Lambda)$ is a closed immersion, and $X(\Lambda)$ is the union of the images of these maps, when α ranges over $C\mathscr{A} = P\Gamma \backslash P\mathscr{A}$. The hyperbolic orbifold $P\Gamma_\alpha \backslash \mathbb{R}H^n_\alpha$ has finite volume by Corollary 2.8, and the set $C\mathscr{A}$ is finite by Proposition 2.9. Thus, $X(\Lambda)$ has finite volume.

4. Let us show that

$$O \coloneqq \coprod_{\alpha \in C\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset X(\Lambda) = P\Gamma \setminus Y(\Lambda)$$

as hyperbolic orbifolds. It suffices to show the following:

Claim: For those $f = p(x_\alpha) \in Y(\Lambda)$ that have no nodes, the stabilizer $A_f = P\Gamma_f \subset P\Gamma$ of $f \in Y(\Lambda)$ and the stabilizer $P\Gamma_{\alpha,x} \subset P\Gamma_\alpha$ of $x \in \mathbb{R}H^n_\alpha$ agree as subgroups of $P\Gamma$. *Proof of the Claim:* To prove that $A_f = P\Gamma_{\alpha,x}$, we first observe that $p : \widetilde{Y} \to Y(\Lambda)$ induces an isomorphism between $P\Gamma_{x_\alpha}$, the stabilizer of $x_\alpha \in \widetilde{Y}$ and $P\Gamma_f$, the stabilizer of $f = [x, \alpha] \in Y(\Lambda)$. So it suffices to show that $P\Gamma_{x_\alpha} = P\Gamma_{\alpha,x}$. Indeed, the normalizer $N_{P\Gamma}(\alpha)$ and the stabilizer $P\Gamma_\alpha \subset P\Gamma$ of α in $P\Gamma$ are equal, which implies that $P\Gamma_{\alpha,x} = P\Gamma_{x_\alpha}$ because

$$\begin{aligned}\{g \in P\Gamma_\alpha : gx = x\} &= \{g \in N_{P\Gamma}(\alpha) : gx = x\} \\ &= \left\{g \in P\Gamma : g \cdot x_\alpha = (g(x), g\alpha g^{-1}) = x_\alpha\right\}.\end{aligned}$$

The claim is proved. Part 4 of the theorem can be deduced from it as follows. Let $f = p(x_\alpha) \in Y(\Lambda)$ have no nodes. We have $Y_f = \mathbb{R}H^n_\alpha$, hence

$$A_f \setminus \mathbb{R}H^n_\alpha = A_f \setminus Y_f = \Gamma_f \setminus \mathbb{R}H^n \quad \text{with} \quad \Gamma_f = A_f \setminus B_f = A_f.$$

By construction, an orbifold chart of the glued space $P\Gamma \setminus Y(\Lambda)$ is given by

$$W \to A_f \setminus W \subset P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha \subset Y(\Lambda)$$

for an invariant open subset W of $\mathbb{R}H^n_\alpha$ containing x. Because $A_f = P\Gamma_{\alpha,x}$ by the claim, this is also an orbifold chart for O at the point x_α.

5. The real-hyperbolic orbifold $X(\Lambda) = P\Gamma \setminus Y(\Lambda)$ is complete by Part 1, so the uniformization of the connected components of $X(\Lambda)$ follows from the Ehresmann–Thurston uniformization theorem for (G, X)-orbifolds, see [Thu80, Proposition 13.3.2] or [MMA91]. This concludes the proof of Theorem 4.1. □

5 Unitary Shimura varieties

The goal of this section is to prove Proposition 5.7, which describes the complex ball quotient $P\Gamma \backslash \mathbb{C}H^n$ defined in Section 2.1 in terms of moduli of abelian varieties with $\mathcal{O}_K$-action of hyperbolic signature, and Proposition 5.11, which interprets the divisor $P\Gamma \setminus \mathscr{H}$ as the locus of abelian varieties A that admit an $\mathcal{O}_K$-linear homomorphism $\mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$ of polarized $\mathcal{O}_K$-linear abelian varieties. This has two applications:

1. Consider a relative uniform cyclic cover (see e.g. [AV04]) $\mathscr{X} \to P \to S$, where $P = \mathbb{P}^1_S$ (resp. $\mathbb{P}^3_S$), the fibers of $\mathscr{X} \to S$ are curves (resp. threefolds with $\mathrm{H}^{0,3} = 0$) and the induced hermitian form on middle cohomology has hyperbolic signature. Since the image $\mathfrak{I} = P(S(\mathbb{C})) \subset P\Gamma \backslash \mathbb{C}H^n$ of the period map $P\colon S(\mathbb{C}) \to P\Gamma \backslash \mathbb{C}H^n$ is contained in the locus of principally polarized abelian varieties whose theta divisor is irreducible, one has $\mathfrak{I} \subset P\Gamma \backslash (\mathbb{C}H^n - \mathscr{H})$.

2. Using the theory of this section, we will be able to show that Condition 2.1 implies Condition 2.2, saying that the hyperplanes in the arrangement $\mathscr{H} \subset \mathbb{C}H^n$ are orthogonal along their intersection. See Theorem 6.2 in the next Section 6.

5.1 *Alternating and hermitian forms*

The goal of this subsection is to prove two lemmas. They will later be used to show that the ball quotient $P\Gamma \backslash \mathbb{C}H^n$ of Section 3 is a moduli space of abelian varieties, and then to interpret the divisor $P\Gamma \backslash \mathscr{H}$ in terms of moduli of abelian varieties. This will be the key to Theorem 6.2.

Let M be a CM field, with ring of integers $\mathcal{O}_M$, and let $\sigma\colon M \to M$ be the involution induced by complex conjugation on $\mathbb{C}$. Let Λ be a free $\mathcal{O}_M$-module of rank $n+1$ for some $n \in \mathbb{N}$. Let $\mathfrak{D}_M \subset \mathcal{O}_M$ be the different ideal.

Lemma 5.1. *The assignment $T \mapsto \mathrm{Tr}_{M/\mathbb{Q}} \circ T$ defines a bijection between:*

1. *The set of skew-hermitian forms $T : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to M$.*
2. *The set of alternating forms $E\colon \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ with $E(a \cdot x, y) = E(x, \sigma(a) \cdot y)$.*

Under this correspondence, $T(\Lambda, \Lambda) \subset \mathfrak{D}_M^{-1}$ if and only if $E(\Lambda, \Lambda) \subset \mathbb{Z}$.

Proof. Let $T\colon \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to M$ be as in 1. Define $E_T = \mathrm{Tr}_{M/\mathbb{Q}} \circ T$. Since T is skew-hermitian, we have, for each $x, y \in \Lambda_{\mathbb{Q}}$, that

$$\mathrm{Tr}_{M/\mathbb{Q}} T(x,y) = -\mathrm{Tr}_{M/\mathbb{Q}} \overline{T(y,x)}.$$

Since $M/\mathbb{Q}$ is separable, for $x \in M$, we have $\mathrm{Tr}_{M/\mathbb{Q}}(x) = \sum_{1 \leq i \leq g} (\tau_i(x) + \tau_i\sigma(x))$, see [Ste08, (7-1)]. Therefore, $\mathrm{Tr}_{M/\mathbb{Q}}(\sigma(x)) = \mathrm{Tr}_{M/\mathbb{Q}}(x)$, so that $E_T(x,y) = -E_T(y,x)$ for any $x, y \in \Lambda_{\mathbb{Q}}$. The property in 2 is easily checked.

Conversely, let $E : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ be as in 2. Choose a basis $\{b_1, \ldots, b_{n+1}\} \subset \Lambda$ for Λ over $\mathcal{O}_M$. Define Q to be the induced map $M^{n+1} \times M^{n+1} \to \mathbb{Q}$ and consider the map $M \to \mathbb{Q}$, $a \mapsto Q(a \cdot e_i, e_j)$. Since the trace pairing

$$M \times M \to \mathbb{Q}, \quad (x,y) \mapsto \mathrm{Tr}_{M/\mathbb{Q}}(xy)$$

is non-degenerate [Stacks, Tag 0BIE], there is a unique element $t_{ij} \in M$ such that $Q(a \cdot e_i, e_j) = \mathrm{Tr}_{M/\mathbb{Q}}(a \cdot t_{ij})$ for every $a \in M$. This gives a matrix $(t_{ij})_{ij} \in M_{n+1}(K)$ such that $\sigma(t_{ij}) = -t_{ji}$, and the basis $\{b_i\}$ induces a skew-hermitian form $T_E : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to M$. This provides the bijection; the other claim from the definition of $\mathfrak{D}_M^{-1} \subset M$: it is the trace dual of $\mathcal{O}_M$, see [Ser79, Chapter III, §3]. $\square$

Examples 5.2. 1. Suppose that $M = \mathbb{Q}(\sqrt{\Delta})$ is imaginary quadratic over $\mathbb{Q}$, with discriminant Δ and non-trivial Galois automorphism $a \mapsto \sigma(a)$. Let $E\colon \Lambda \times \Lambda \to \mathbb{Z}$

be an alternating form with $E(a\cdot x,y)=E(x,\sigma(a)\cdot y)$. The form $T\colon \Lambda\times\Lambda\to \mathfrak{D}_M^{-1}=(\sqrt{\Delta})^{-1}$ is defined as

$$T(x,y)=\frac{E(\sqrt{\Delta}\cdot x,y)+E(x,y)\sqrt{\Delta}}{2\sqrt{\Delta}}.$$

2. Suppose that $M=\mathbb{Q}(\zeta_p)$ where $\zeta_p=e^{2\pi i/p}\in\mathbb{C}$ for some prime number $p>2$. Let $E:\Lambda\times\Lambda\to\mathbb{Z}$ be an alternating form with $E(a\cdot x,y)=E(x,\sigma(a)\cdot y)$. Then $\mathfrak{D}_M=\left(p/(\zeta_p-\zeta_p^{-1})\right)$ and

$$T:\Lambda\times\Lambda\to\mathfrak{D}_M^{-1}\quad\text{is defined as}\quad T(x,y)=\frac{1}{p}\sum_{j=0}^{p-1}\zeta_p^j\cdot E\left(x,\zeta_p^j\cdot y\right).$$

Consider a pair $(E\colon \Lambda_{\mathbb{Q}}\times\Lambda_{\mathbb{Q}}\to\mathbb{Q}, T:\Lambda_{\mathbb{Q}}\times\Lambda_{\mathbb{Q}}\to K)$ as in Lemma 5.1, and suppose that E is non-degenerate. To any embedding $\phi\colon M\to\mathbb{C}$ be an embedding, one can associate a skew-hermitian form $T^\phi\colon \Lambda\otimes_{\mathcal{O}_M,\phi}\mathbb{C}\times\Lambda\otimes_{\mathcal{O}_M,\phi}\mathbb{C}\to\mathbb{C}$ by putting

$$T^\phi(x\otimes\lambda,y\otimes\mu)=\lambda\overline{\mu}\cdot\phi\left(T(x,y)\right)$$

and extending linearly. Remark that, on $\Lambda_{\mathbb{C}}=\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$, one has the skew-hermitian form $A(x,y)=E_{\mathbb{C}}(x,\bar{y})$. The composition $(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi\to\Lambda\otimes_{\mathbb{Z}}\mathbb{C}\to\Lambda\otimes_{\mathcal{O}_M,\phi}\mathbb{C}$ is an isomorphism; we define A^ϕ as the restriction of A to the subspace

$$(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi=\Lambda\otimes_{\mathcal{O}_M,\phi}\mathbb{C}\subset\Lambda_{\mathbb{C}}.$$

Note that $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}\cong\oplus_{\phi:M\to\mathbb{C}}(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi$. For $x\in\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$, let $x_\phi\in(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi$ denote the image of x under the projection $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}\to(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi$.

Lemma 5.3. *Let $\phi\colon M\to\mathbb{C}$ be an embedding. We have an equality $T^\phi=A^\phi$ of skew-hermitian forms $(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi\times(\Lambda\otimes_{\mathbb{Z}}\mathbb{C})_\phi\to\mathbb{C}$. More precisely, we have $A(x,y)=\sum_{\phi:M\to\mathbb{C}}T^\phi(x_\phi,y_\phi)$ for every $x,y\in\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$.*

Proof. This is straightforward and we leave the proof to the reader. □

5.2 Moduli of abelian varieties with an action by a CM field

In the rest of Section 5, we fix the following notation.

Notation 5.4. 1. Let M be a CM field over $\mathbb{Q}$, and define $\sigma\colon M\to M$ as the involution that corresponds to complex conjugation on $\mathbb{C}$.

2. Let $\eta\in\mathcal{O}_M$ be a non-zero element such that $\sigma(\eta)=-\eta$.

3. Let Λ be a free $\mathcal{O}_M$-module of rank $n+1$ for some $n\in\mathbb{Z}_{\geq 0}$ equipped with a non-degenerate hermitian form $\mathfrak{h}:\Lambda\times\Lambda\to\eta\cdot\mathfrak{D}_M^{-1}\subset K$.

4. Let $\Phi\subset\mathrm{Hom}(M,\mathbb{C})$ be the CM type such that $\Im\left(\phi(\eta)\right)>0$ for all $\phi\in\Phi$.

These data define a skew-hermitian form $T\colon\Lambda\times\Lambda\to\mathfrak{D}_M^{-1}$ by putting $T:=\eta^{-1}\cdot\mathfrak{h}$. The form T is in turn attached to an alternating form (see Lemma 5.1):

$$E\colon\Lambda\times\Lambda\to\mathbb{Z}\quad\text{such that}\quad E(ax,y)=E(x,\sigma(a)y)\quad\text{for all}\quad a\in\mathcal{O}_M,\ x,y\in\Lambda.$$

Write $V_\phi=\Lambda_{\mathbb{Q}}\otimes_{K,\phi}\mathbb{C}$ and define $\mathfrak{h}^\phi\colon V_\phi\times V_\phi\to\mathbb{C}$ to be the hermitian form restricting to $\phi\circ\mathfrak{h}$ on Λ. Let (r_ϕ,s_ϕ) be the signature of the hermitian form $\mathfrak{h}^\phi$.

Let A be a complex abelian variety, ι a homomorphism $\mathcal{O}_M \to \mathrm{End}(A)$, and λ a polarization $A \to A^\vee$, satisfying the following (compare [KR14, Part I, §2.1]):

Conditions 5.5. *1. We have $\iota(a)^\dagger = i(\sigma(a))$, where $\dagger\colon \mathrm{End}(A)_\mathbb{Q} \to \mathrm{End}(A)_\mathbb{Q}$ denotes the Rosati involution.*

2. We have $\mathrm{char}(t, \iota(a)|\mathrm{Lie}(A)) = \prod_{\phi\in\Phi}(t - a^\phi)^{r_\phi} \cdot (t - a^{\phi\sigma})^{s_\phi} \in \mathbb{C}[t]$.

Here, $\mathrm{char}(t, \iota(a)|\mathrm{Lie}(A)) \in \mathbb{C}[t]$ denotes the characteristic polynomial of $\iota(a)$. Any abelian variety A as above satisfies $\dim A = g(n+1)$, where $g = [M\colon \mathbb{Q}]$.

Define $E_A : \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A, \mathbb{Z}) \to \mathbb{Z}$ to be the alternating form corresponding to λ. The condition on the Rosati involution implies that $E_A(\iota(a)x, y) = E_A(x, \iota(\sigma(a))y)$ for $x, y \in \mathrm{H}_1(A, \mathbb{Q})$. Define a hermitian form $\mathfrak{h}_A\colon \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A, \mathbb{Z}) \to \eta \cdot \mathfrak{D}_M^{-1}$ as $\mathfrak{h}_A = \eta \cdot T_A$. Here, $T_A\colon \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A, \mathbb{Z}) \to \mathfrak{D}_M^{-1}$ is the skew-hermitian form attached to E_A via Lemma 5.1.

Definition 5.6. 1. Let $\widetilde{\mathrm{Sh}}_M(\Lambda)$ be the set of isomorphism classes of four-tuples (A, ι, λ, j), where (A, ι, λ) is as above and satisfies Conditions 5.5, and where $j\colon \mathrm{H}_1(A, \mathbb{Z}) \to \Lambda$ is an isomorphism of $\mathcal{O}_M$-modules compatible with the alternating forms E_A and E (equivalently, with the hermitian forms $\mathfrak{h}_A$ and $\mathfrak{h}$).

2. Let $\mathbb{D}(V_\phi)$ be the space of negative s_ϕ-planes in the hermitian space $(V_\phi, \mathfrak{h}^\phi)$.

We have the following proposition which is due to Shimura, see [Shi63, Theorems 1 & 2] or [Shi64, §1]. We give a different proof since it will imply Proposition 5.11 below, whereas we did not know how to deduce Proposition 5.11 from *loc. cit.* We remark that Shimura assumes Λ to be an R-module, for an order $R \subset \mathcal{O}_M$ which is not necessarily maximal; our proof carries over, but we do not need this.

Proposition 5.7. *There is a canonical bijection $\widetilde{\mathrm{Sh}}_M(\Lambda) \cong \prod_{\phi\in\Phi} \mathbb{D}(V_\phi)$.*

Proof. Let (A, ι, λ, j) be a representative of an isomorphism class that defines a point in $\widetilde{\mathrm{Sh}}_M(\Lambda)$. Consider the Hodge decomposition $\mathrm{H}_1(A, \mathbb{C}) = \mathrm{H}^{-1,0} \oplus \mathrm{H}^{0,-1}$. For each $\phi \in \Phi$, there is a decomposition

$$\mathrm{H}_1(A, \mathbb{C})_\phi = \mathrm{H}_\phi^{-1,0} \oplus \mathrm{H}_\phi^{0,-1}, \tag{15}$$

with $\dim \mathrm{H}_\phi^{-1,0} = r_\phi$ and $\dim \mathrm{H}_\phi^{0,-1} = s_\phi$. The latter holds because $\overline{\mathrm{H}_{\phi\sigma}^{-1,0}} = \mathrm{H}_\phi^{0,-1}$. By Lemma 5.3, the forms $\phi(\eta)E_{A,\mathbb{C}}(x, \bar{y})$ and $\mathfrak{h}_A^\phi(x, y)$ agree as hermitian forms on the complex vector space $\mathrm{H}_1(A, \mathbb{Z}) \otimes_{\mathcal{O}_M,\phi} \mathbb{C}$. Since $\Im(\phi(\eta)) > 0$ for every $\phi \in \Phi$, the decomposition of $\mathrm{H}_1(A, \mathbb{C})_\phi$ in (15) is a decomposition into a positive definite r_ϕ-dimensional subspace and a negative definite s_ϕ-dimensional subspace. The isomorphism $j\colon \mathrm{H}_1(A, \mathbb{Q}) \to \Lambda_\mathbb{Q}$ induces an isometry $j_\phi\colon \mathrm{H}_1(A, \mathbb{C})_\phi \to V_\phi$ for every ϕ, and so we obtain a negative s_ϕ-plane $j_\phi(\mathrm{H}_\phi^{0,-1})$ in the hermitian space V_ϕ for all ϕ. Reversing the argument shows that given a negative s_ϕ-plane $X_\phi \subset V_\phi$ for every ϕ, there is a canonical polarized abelian variety $A = \mathrm{H}^{-1,0}/\Lambda$, acted upon by $\mathcal{O}_M$ and inducing the planes $X_\phi \subset V_\phi$. □

Definition 5.8. 1. Let $\mathrm{Sh}_M(\Lambda)$ be the set of isomorphism classes of polarized abelian varieties with $\mathcal{O}_M$-action (A, ι, λ), satisfying Conditions 5.5, such that $\mathrm{H}_1(A, \mathbb{Z})$ is isomorphic to Λ as hermitian $\mathcal{O}_M$-modules.

2. Let $\Gamma(\Lambda) = \mathrm{Aut}_{\mathcal{O}_M}(\Lambda)$; this is the group of $\mathcal{O}_M$-linear automorphisms of Λ preserving the hermitian form $\mathfrak{h} : \Lambda \times \Lambda \to \eta \cdot \mathfrak{D}_M^{-1}$.

The bijection in Proposition 5.7 being $\Gamma(\Lambda)$-equivariant, we obtain the following:

Corollary 5.9. *There is a canonical bijection* $\mathrm{Sh}_M(\Lambda) \cong \Gamma(\Lambda) \setminus \prod_{i=1}^{g} \mathbb{D}(V_i)$. □

5.3 Moduli of abelian varieties in the hyperplane arrangement

The set of embeddings Φ introduced in Notation 5.4 defines a map $\Phi : \mathcal{O}_M \to \mathbb{C}^g$, which yields a complex torus $\mathbb{C}^g/\Phi(\mathcal{O}_M)$. The map

$$Q\colon M \times M \to \mathbb{Q}, \quad Q(x,y) = \mathrm{Tr}_{M/\mathbb{Q}}(\eta^{-1} x\bar{y})$$

is a non-degenerate $\mathbb{Q}$-bilinear form with $Q(ax,y) = Q(x,\sigma(a)y)$ for $a,x,y \in M$.

Lemma 5.10. *Suppose that the element $\eta \in \mathcal{O}_M$ of Notation 5.4.2 satisfies $\eta^{-1} \in \mathfrak{D}_M^{-1}$. Then Q defines a polarization on the complex torus $\mathbb{C}^g/\Phi(\mathcal{O}_M)$.*

Proof. Remark that $Q(\mathcal{O}_M, \mathcal{O}_M) \subset \mathbb{Z}$ because $\eta^{-1} \in \mathfrak{D}_M^{-1}$. By [Mil20, Example 2.9 & Footnote 16], the result follows. □

Suppose from now on that

$$(r_\phi, s_\phi) \quad = \quad \begin{cases} (n,1) & \text{if } \phi = \tau, \\ (n+1,0) & \text{if } \phi \neq \tau. \end{cases} \tag{16}$$

The space $W := \Lambda \otimes_{\mathcal{O}_M,\tau} \mathbb{C}$ is equipped with the hermitian form $\mathfrak{h}^\tau\colon W \times W \to \mathbb{C}$. Define

$$\mathbb{C}H(W) = \{\text{lines } \ell \subset W \mid \mathfrak{h}^\tau(x,x) < 0 \text{ for } 0 \neq x \in \ell\},$$

and let $\mathscr{H} = \cup_{\mathfrak{h}(r,r)=1} \langle r_{\mathbb{C}} \rangle^\perp \subset \mathbb{C}H(W)$.

Proposition 5.11. *Suppose that $\eta^{-1} \in \mathfrak{D}_M^{-1}$, so that $(\mathbb{C}^g/\Phi(\mathcal{O}_M), Q)$ is a polarized abelian variety by Lemma 5.10. Under the bijection $\widetilde{\mathrm{Sh}}_M(\Lambda) \cong \mathbb{C}H(W)$ of Proposition 5.7, the subset $\mathscr{H} \subset \mathbb{C}H(W)$ is identified with the set of isomorphism classes of marked polarized $\mathcal{O}_M$-linear abelian varieties (A,ι,λ,j) admitting an $\mathcal{O}_M$-linear homomorphism $f\colon \mathbb{C}^g/\Phi(\mathcal{O}_M) \to A$ of polarized abelian varieties.*

Proof. Consider an isomorphism class $[(A,\iota,\lambda,y)] \in \widetilde{\mathrm{Sh}}_M(\Lambda)$ corresponding to a point $[x] \in \mathbb{C}H^n$. We may assume that $A = \mathrm{H}^{-1,0}/\Lambda$ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{H}^{-1,0} \oplus \mathrm{H}^{0,-1}$, and that $T_A = T$. Let $f\colon \mathbb{C}^g/\Phi(\mathcal{O}_M) \to A$ be a homomorphism as in the proposition. We obtain a homomorphism $\mathcal{O}_M \to \Phi(\mathcal{O}_M) \to \mathrm{H}_1(A,\mathbb{Z}) = \Lambda$ which, for simplicity, we also denote by $f\colon \mathcal{O}_M \to \Lambda$. Let $r \in \Lambda$ be the image of $1 \in \mathcal{O}_M$. The fact that $Q = f^*E_A$ implies that $T_Q = f^*T_A = f^*T$. Therefore, we have $\eta^{-1} = T_Q(1,1) = T_A(f(1),f(1)) = T(f(1),f(1)) = T(r,r)$, so that $\mathfrak{h}(r,r) = \eta \cdot T(r,r) = 1$. We claim that $\mathfrak{h}(x,r_\tau) = 0$, where the element $r_\tau \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau$ is the image of $r \in \Lambda$. To see this, write $\Phi(\mathcal{O}_M) = L$, and $L \otimes \mathbb{C} = W^{-1,0} \oplus W^{0,-1}$. Notice that $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau = W_\tau^{-1,0}$. Consequently, since the composition

$$W_\tau^{-1,0} = (L \otimes_{\mathbb{Z}} \mathbb{C})_\tau \to (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = \mathrm{H}_\tau^{-1,0} \oplus \mathrm{H}_\tau^{0,-1}$$

factors through the inclusion of $\mathrm{H}_\tau^{-1,0}$ into $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau$, we see that $r_\tau = r_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0} = (\mathrm{H}_\tau^{0,-1})^\perp = \langle x \rangle^\perp$, and the claim follows.

Conversely, let $[x] \in \langle r_{\mathbb{C}} \rangle^\perp \subset \mathscr{H}$ with $r \in \Lambda$ such that $\mathfrak{h}(r,r) = 1$ and consider the marked abelian variety $A = \mathrm{H}^{-1,0}/\Lambda$ corresponding to $[x]$. Define a homomorphism $f\colon \mathcal{O}_M \to \Lambda$ by $f(1) = r$. Then f can be shown to be a morphism of Hodge structures using the fact that its $\mathbb{C}$-linear extension preserves the eigenspace decompositions. We obtain an $\mathcal{O}_M$-linear homorphism $f\colon \mathbb{C}^g/\Phi(\mathcal{O}_M) \to A$. The fact that $\mathfrak{h}(r,r) = 1$ implies that f preserves the polarizations on both sides. $\square$

6 Orthogonal hyperplane arrangements

Let M be a CM field of degree $2g$ over $\mathbb{Q}$, with involution $\sigma\colon M \to M$ that corresponds to complex conjugation on $\mathbb{C}$. Assume that M satisfies the following:

Condition 6.1. *There exists an element $\eta \in \mathcal{O}_M$ satisfying the condition that*

$$\mathfrak{D}_M = (\eta) \quad \text{and} \quad \sigma(\eta) = -\eta. \tag{17}$$

Fix an element $\eta \in \mathcal{O}_M$ that satisfies (17). Let $\Phi \subset \mathrm{Hom}(M, \mathbb{C})$ be the unique CM type such that $\Im(\varphi(\eta)) > 0$ for all $\varphi \in \Phi$. Let Λ be a free $\mathcal{O}_M$-module of rank $n+1$ equipped with a hermitian form $\mathfrak{h}\colon \Lambda \times \Lambda \to \mathcal{O}_M$ of signature $(n,1)$ with respect to one $\tau \in \Phi$ and of signature $(n+1,0)$ with respect to all other $\phi \neq \tau \in \Phi$. Let $W = \Lambda \otimes_{\mathcal{O}_M,\tau} \mathbb{C}$ and define $\mathbb{C}H(W)$ as the space of negative lines in the hermitian space W. For an element $r \in \Lambda$, let r_τ be its image in $(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = \Lambda \otimes_{\mathcal{O}_M,\tau} \mathbb{C}$. If $\mathfrak{h}(r,r) = 1$, we define $\langle r_\tau \rangle^\perp \subset \mathbb{C}H(W)$ as the set of negative lines $\ell \subset W$ with $\mathfrak{h}(r_\tau, \ell) = 0$.

Theorem 6.2. *Assume Condition 6.1. Let $r, t \in \Lambda$ such that $\mathfrak{h}(r,r) = \mathfrak{h}(t,t) = 1$. Suppose that $\langle r_\tau \rangle^\perp \cap \langle t_\tau \rangle^\perp \neq \emptyset$ and $\langle r_\tau \rangle^\perp \neq \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$. Then $\mathfrak{h}(r,t) = 0$.*

Proof. Consider the complex torus $B := \mathbb{C}^g/\Phi(\mathcal{O}_M)$. Since the different ideal $\mathfrak{D}_M$ is the principal ideal $(\eta) \subset \mathcal{O}_M$, we have

$$\{x \in M\colon \mathrm{Tr}_{M/\mathbb{Q}}\left(x\eta^{-1}\mathcal{O}_M\right) \subset \mathbb{Z}\} = \{x \in M\colon x \cdot \eta^{-1}\mathcal{O}_M \subset \eta^{-1}\mathcal{O}_M\} = \mathcal{O}_M.$$

Therefore, polarization on B attached to the alternating form

$$Q\colon \Phi(\mathcal{O}_M) \times \Phi(\mathcal{O}_M) \to \mathbb{Z}, \quad (\Phi(x), \Phi(y)) \mapsto \mathrm{Tr}_{M/\mathbb{Q}}\left(\eta^{-1} \cdot x \cdot y\right),$$

see Lemma 5.10, is a *principal* polarization on B.

Consider the moduli space $\mathrm{Sh}_M(\Lambda)$ attached to the hermitian $\mathcal{O}_M$-lattice Λ as in Definition (5.8). Let $[x] \in \langle r_\tau \rangle^\perp \cap \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$, and let $[(A, \iota, \lambda)]$ be the moduli point in $\mathrm{Sh}_M(\Lambda)$ corresponding to the image of $[x]$ in $P\Gamma \backslash \mathbb{C}H(W)$, see Corollary 5.9. By Proposition 5.11, the roots r and t induce $\mathcal{O}_M$-linear embeddings $f_1\colon B \hookrightarrow A$ and $f_2\colon B \hookrightarrow A$ of polarized abelian varieties. By [BL04, Corollary 5.3.13], there exist abelian subvarieties $C_i \subset A$, $i = 1, 2$ such that $A \cong B \times C_1$ and $A \cong B \times C_2$ as polarized abelian varieties, where B and C_i are endowed with the polarizations λ_B and λ_{C_i} induced by their embedding into A. Note that B is a simple abelian variety because $\mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{O}_M \otimes_{\mathbb{Z}} \mathbb{Q} = K$ is a field. In particular (B, λ_B) is non-decomposable as a principally polarized principally polarized abelian variety.

By [Deb96], the decomposition of (A, λ) into non-decomposable polarized abelian subvarieties is unique, in the strong sense that if (A_i, λ_i), $i \in \{1, \dots, r\}$ and (B_j, μ_j), $j \in \{1, \dots, m\}$ are polarized abelian subvarieties such that the natural homomorphisms $\prod_i (A_i, \lambda_i) \to (A, \lambda)$ and $\prod_j (B_j, \lambda_j) \to (A, \lambda)$ are isomorphisms, then $r = m$ and there exists a permutation σ on $\{1, \dots, r\}$ such that B_j and $A_{\sigma(j)}$ are *equal* as polarized abelian subvarieties of (A, λ), for every $j \in \{1, \dots, r\}$. Thus, for the abelian subvarieties $B_i = \phi_i(B) \subset A$, we have either that $B_1 = B_2 \subset A$, or that $B_1 \cap B_2 = \{0\} \subset A$.

Suppose that $B_1 = B_2$. Then $\mathcal{O}_M \cdot r = \phi_1(\mathcal{O}_M) = \phi_2(\mathcal{O}_M) = \mathcal{O}_M \cdot t \subset \Lambda$, hence $r = \lambda t$ for some $\lambda \in \mathcal{O}_M^*$; but then $\langle r_\tau \rangle^\perp = \langle t_\tau \rangle^\perp \subset \mathbb{C}H(W)$ which is absurd. Therefore, we must have $A = B_1 \times B_2 \times C$ as polarized abelian varieties, for some polarized abelian subvariety C of A. This implies that

$$\mathrm{H}^{-1,0} = \mathrm{Lie}(A) = \mathrm{Lie}(B_1) \times \mathrm{Lie}(B_2) \times \mathrm{Lie}(C), \tag{18}$$

which is orthogonal for the positive definite hermitian form $iE_{\mathbb{C}}(x, \bar{y})$ on $\mathrm{H}^{-1,0}$.

Observe that $r_\tau = r_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0}$ and $t_\tau = t_\tau^{-1,0} \in \mathrm{H}_\tau^{-1,0}$: see the proof of Proposition 5.11. By Lemma 5.3, we have

$$\mathfrak{h}(r,t) = \mathfrak{h}^\tau(r_\tau, t_\tau) = \tau(\eta) \cdot T_{\mathbb{C}}^\tau(r_\tau, t_\tau) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau, \bar{t_\tau}) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}).$$

Since $r_\tau^{-1,0} \in \mathrm{Lie}(B_1)$ and $t_\tau^{-1,0} \in \mathrm{Lie}(B_2)$, we have $iE_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}) = 0$ by the orthogonality of (18). We conclude that $\mathfrak{h}(r,t) = 0$ and the proof is finished. $\square$

Example 6.3. Suppose that $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ for some $d \in \mathbb{Z}$ with $d < 0$. Note that Condition 6.1 is satisfied. Moreover, there is an alternative, more elementary proof of Theorem 6.2 in this case, see [ACT02b, Lemma 7.29]. Namely, let $r, t \in \Lambda \subset \Lambda \otimes_{\mathcal{O}_K} \mathbb{C} = W$ and suppose that $\mathfrak{h}(r,r) = \mathfrak{h}(t,t) = 1$, that $\langle r \rangle^\perp \neq \langle t \rangle^\perp$ and that $\langle r \rangle^\perp \cap \langle t \rangle^\perp \neq \emptyset \subset \mathbb{C}H(W)$. Let $[x] \in \langle r \rangle^\perp \cap \langle t \rangle^\perp$ and consider the decomposition $W = \langle x \rangle \oplus \langle x \rangle^\perp$. Since $r, t \in \langle x \rangle^\perp$, and the signature of W is $(n, 1)$, they span a positive definite subspace $\langle r, t \rangle \subset W$. Therefore, the matrix

$$\begin{pmatrix} \mathfrak{h}(r,r) & \mathfrak{h}(r,t) \\ \mathfrak{h}(t,r) & \mathfrak{h}(t,t) \end{pmatrix} = \begin{pmatrix} 1 & \mathfrak{h}(r,t) \\ \mathfrak{h}(t,r) & 1 \end{pmatrix}$$

is positive definite, hence $|\mathfrak{h}(r,t)|^2 < 1$. Since $|\mathfrak{h}(r,t)|^2 \in \mathbb{Z}$, we have $\mathfrak{h}(r,t) = 0$.

We conclude Section 6 by showing that the condition on the different ideal $\mathfrak{D}_M \subset \mathcal{O}_M$ in Theorem 6.2 (see Condition 6.1) is satisfied in several natural cases.

Proposition 6.4. *Suppose that $M/\mathbb{Q}$ is an imaginary quadratic extension, or that $M = \mathbb{Q}(\zeta_n)$ is a cyclotomic field for some integer $n \geq 3$. Then $\mathfrak{D}_M = (\beta) \subset \mathcal{O}_M$ for some element $\beta \in \mathcal{O}_M$ such that $\sigma(\beta) = -\beta$.*

Proof. If $M/\mathbb{Q}$ is imaginary quadratic with discriminant Δ, then $\mathfrak{D}_M = (\sqrt{\Delta})$ and the assertion is immediate. Let $n \geq 3$ be an integer, and consider the fields $M = \mathbb{Q}(\zeta_n) \supset F = \mathbb{Q}(\alpha)$, where $\alpha = \zeta_n + \zeta_n^{-1}$. Since $\mathcal{O}_M = \mathbb{Z}[\zeta_n]$ we have $\mathcal{O}_M = \mathcal{O}_F[\zeta_n]$. Notice that $f(x) = x^2 - \alpha x + 1 \in \mathcal{O}_F[x]$ is the minimal polynomial of ζ_n over F. We have $f'(\zeta_n) = 2\zeta_n - \alpha = \zeta_n - \zeta_n^{-1}$ hence $\mathfrak{D}_{M/F} = (f'(\zeta_n)) = \left(\zeta_n - \zeta_n^{-1}\right)$, see [Neu99]. By [Lia76], we know that $\mathcal{O}_F = \mathbb{Z}[\alpha]$. Moreover, if $g(x) \in \mathbb{Z}[x]$ is the minimal polynomial of α over $\mathbb{Q}$, then $\mathfrak{D}_{F/\mathbb{Q}} = (g'(\alpha))$. Since $\mathfrak{D}_{M/\mathbb{Q}} = \mathfrak{D}_{M/F}\mathfrak{D}_{F/\mathbb{Q}}$ (c.f. [Neu99, Chapter III, Proposition 2.2]), we obtain $\mathfrak{D}_{M/\mathbb{Q}} = \mathfrak{D}_{M/F}\mathfrak{D}_{F/\mathbb{Q}} = \left(\left(\zeta_n - \zeta_n^{-1}\right) \cdot g'(\alpha)\right)$. $\square$

Remark 6.5. Consider triples (M, β, Λ) of the following form: M is a CM field with totally real subfield F, β is an element in $\mathcal{O}_M - \mathcal{O}_F$ such that $\beta^2 \in \mathcal{O}_F$, and Λ is a finite free $\mathcal{O}_M$-module equipped with a hermitian form $\mathfrak{h} \colon \Lambda \times \Lambda \to \beta \cdot \mathfrak{D}_M^{-1}$ of signature $(n, 1)$ with respect to one $\tau \in \Phi = \{\varphi \in \mathrm{Hom}(K, \mathbb{C}) \mid \Im(\varphi(\beta)) > 0\}$ and positive definite with respect to every $\varphi \neq \tau \in \Phi$. It turns out that can associate an orthogonal hyperplane arrangement

$$\mathcal{H}(M, \beta, \Lambda) = \bigcup_f H_f \subset \mathbb{C}H^n = \{\text{negative lines in } \Lambda \otimes_{\mathcal{O}_M, \tau} \mathbb{C}\}$$

to *every* such triple (M, β, Λ), in such a way that $\mathcal{H}(M, \beta, \Lambda) = \mathscr{H} = \cup_{\mathfrak{h}(r,r)=1} \langle r_{\mathbb{C}} \rangle^{\perp}$ if $\mathfrak{D}_M = (\beta)$. Indeed, for such a triple, let $\mathscr{S}$ be the set of fractional ideals $\mathfrak{a} \subset M$ for which there exist $b \in \mathcal{O}_F$ such that $\mathfrak{D}_M \mathfrak{a} \bar{\mathfrak{a}} = (b\beta)$. By [Wam99, Theorem 4], the set $\mathscr{S}$ is not empty. For each $\mathfrak{a} \in \mathscr{S}$, pick an element b as above and define $\eta(\mathfrak{a}) = b\beta \in \mathcal{O}_M$. The complex torus $B = \mathbb{C}^g / \Phi(\mathfrak{a})$ is polarized by the Riemann form $Q : \Phi(\mathfrak{a}) \times \Phi(\mathfrak{a}) \to \mathbb{Z}$, $(\Phi(x), \Phi(y)) \mapsto \mathrm{Tr}_{M/\mathbb{Q}}(\eta(\mathfrak{a})^{-1} \cdot x\bar{y})$, and this is a principal polarization [Wam99, Theorem 3]. Let $\mathscr{R}$ be the set of embeddings $f \colon \mathfrak{a} \to \Lambda$, $\mathfrak{a} \in \mathscr{S}$, such that $\mathfrak{h}(f(x), f(y)) = x\bar{y}$ for $x, y \in \mathfrak{a}$. For $f \in \mathscr{R}$, one obtains a hyperplane

$$H_f = \{x \in \mathbb{C}H^n \colon \mathfrak{h}^{\tau}(x, f(\mathfrak{a})) = 0\} \subset \mathbb{C}H^n.$$

Then the hyperplane arrangement $\mathcal{H}(M, \beta, \Lambda) = \cup_{f \in \mathscr{R}} H_f$ is orthogonal by arguments similar to those used to prove Proposition 5.11 and Theorem 6.2.

7 Hermitian lattices over the Eisenstein integers

The goal of this section is to study particular examples of the spaces $X(\Lambda)$ obtained via Theorem 4.1, namely those obtained by considering the field $\mathbb{Q}(\zeta_3)$ and the hermitian lattice $\mathbb{Z}[\zeta_3]^{n,1}$. The ultimate goal is to prove Theorem 1.2.

7.1 Eisenstein integers

In Section 7, we fix the following notation: we let $K = \mathbb{Q}(\zeta_3)$, and for each $n \in \mathbb{Z}_{\geq 2}$, we define a lattice $\Lambda_n = \left(\mathbb{Z}[\zeta_3]^{n+1}, h_n\right)$ with $h_n = \mathrm{diag}(-1, 1, \dots, 1)$. As in Sections 2.1 - 2.3, we make the following definitions, although this time we label each object by the integer $n \in \mathbb{Z}_{\geq 2}$ that is naturally attached to it:

$$\Gamma(n) = \mathrm{Aut}(\Lambda_n) = \mathrm{Aut}_{\mathbb{Z}[\zeta_3]}\left(\mathbb{Z}[\zeta_3]^{n+1}, h_n\right), \qquad P\Gamma(n) = \mu_K \backslash \Gamma(n)$$
$$\mathscr{A}_n = \{\text{anti-unitary involutions } \alpha \colon \Lambda_n \to \Lambda_n\}, P\mathscr{A}_n = \mu_K \backslash \mathscr{A}_n, C\mathscr{A}_n = P\Gamma(n) \backslash P\mathscr{A}_n.$$

By Theorem 6.2 and Proposition 6.4, the hyperplane arrangement $\mathscr{H}_n = \cup_{r \in \mathscr{R}_n} H_r$ is an orthogonal arrangement, where $\mathscr{R}_n = \{r \in \Lambda_n \mid h_n(r, r) = 1\}$. We can thus perform the glueing construction of Section 3 to obtain a sequence of metric spaces $X(\Lambda_n)$, with $n \in \mathbb{Z}_{\geq 2}$. For each $n \in \mathbb{Z}_{\geq 2}$, the metric on $X(\Lambda_n)$ extends to a complete real hyperbolic orbifold structure, hence its connected components are quotients of $\mathbb{R}H^n$ by discrete groups of isometries (see Theorem 4.1.5). The goal of Section 7 is to prove Theorem 1.2: for every $n \geq 2$, there exists a connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$ such that the lattice $\Gamma_n^+ \subset \mathrm{PO}(n, 1)$ underlying $X(\Lambda_n)^+$ is non-arithmetic.

For $i = 0, 1, 2$, define $\epsilon_i = (\epsilon_{i,1}, \epsilon_{1,2}, \epsilon_{i,3}) \in \{\pm 1\}^3$ by $\epsilon_0 = (1, 1, 1), \epsilon_1 = (1, -1, 1)$ and $\epsilon_2 = (1, -1, -1)$ (compare Section 2.4). For each $i \in \{0, 1, 2\}$, we define an anti-unitary involution α_i on Λ_2 as follows:

$$\alpha_i \colon \Lambda_2 \to \Lambda_2, \quad \alpha_i(x_0, x_1, x_2) = (\epsilon_{0,i} \cdot \bar{x}_0, \epsilon_{1,i} \cdot \bar{x}_1, \epsilon_{2,i} \cdot \bar{x}_2). \tag{19}$$

Lemma 7.1. *The involutions $\alpha_0, \alpha_1, \alpha_2$ are pairwise not $\Gamma(2)$-conjugate, and each anti-unitary involution of Λ_2 is $\Gamma(2)$-conjugate to exactly one of the $\pm\alpha_i$. Moreover, the composition $\coprod_{i=0}^{2} \mathbb{R}H^2_{\alpha_i} \to Y(\Lambda_2) \to P\Gamma(2) \setminus Y(\Lambda_2) = X(\Lambda_2)$ is surjective, and the orbifold $X(\Lambda_2)$ is connected.*

Proof. Let $\theta = \zeta_3 - \zeta_3^{-1} = \sqrt{-3} \in \mathbb{Z}[\zeta_3]$ and consider the vector space $W_2 = \Lambda_2/\theta\Lambda_2$. For $\alpha \in \mathscr{A}$, let $\alpha \colon W_2 \to W_2$ be the induced involution, let $q_2 \colon W_2 \to \mathbb{F}_3$ be the quadratic form $q_2(x) = h_2(x, x) \bmod \theta$, and define $D(\alpha) = \dim(V^\alpha)$ and $T(\alpha) = \det(q_2|_{V^\alpha}) \in \mathbb{F}_3^*/(\mathbb{F}_3^*)^2 = \{\pm 1\}$. The restrictions of q_2 to the fixed spaces $V^{\alpha_i} \subset V$ for α_0, α_1 and α_2 have pairwise distinct conjugacy invariant $(D(\alpha_i), T(\alpha_i))$ (see Lemma 7.2 below), which proves the first statement. Next, one observes that the subspace $\coprod_{\alpha \in C\mathscr{A}_2} P\Gamma(2)_\alpha \setminus \left(\mathbb{R}H^2_\alpha - \mathscr{H}_2\right) \subset X(\Lambda_2)$ (see Theorem 4.1.4) is naturally homeomorphic to the moduli space of stable real binary sextics with one double root at ∞ and no other double roots (c.f. [ACT06, Section 5]). As in the proof of Theorem 4 of [ACT10], one deduces that each anti-unitary involution of Λ_2 must be $\Gamma(2)$-conjugate to exactly one of the $\pm\alpha_i$. The statements follow. □

For each $i \in \{0, 1, \dots, n\}$, define an anti-unitary involution β_i as follows:

$$\beta_i \colon \Lambda_n \to \Lambda_n, \quad \beta_i(x_0, \dots, x_n) = (\bar{x}_0, -\bar{x}_1, \dots, -\bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n)\,. \tag{20}$$

Lemma 7.2. *The elements $\pm\beta_i \in \mathscr{A}$ for $i = 0, \dots, n$ are pairwise non $\Gamma(n)$-conjugate. In particular, the classes $\beta_i \in P\mathscr{A}_n$ are pairwise non $P\Gamma(n)$-conjugate.*

Proof. Consider the element $\theta = \zeta_3 - \zeta_3^{-1}$ and define $W_n = \Lambda_n/\theta\Lambda_n$. Consider the quadratic form $q_n \colon W_n \to \mathbb{F}_3$ defined as $q_n(x) = h_n(x, x) \bmod \theta$. Note that, for $i \geq 1$ and $x = (x_0, x_{i+1} \dots, x_n) \in W_n^{\beta_i}$, one has $q_n(x) = -x_0^2 + x_{i+1}^2 + \cdots + x_n^2$. Similarly, for $x = (x_1, \dots, x_i) \in V^{-\beta_i}$, one has $q_n(x) = x_1^2 + \cdots + x_i^2$. For an anti-unitary involution $\alpha \colon \Lambda_n \to \Lambda_n$, define $D(\alpha) = \dim(V^\alpha)$ and $T(\alpha) = \det(q_n|_{V^\alpha})$, which are $\Gamma(n)$-conjugacy invariants. The above shows that $(D(\beta_i), T(\beta_i)) = (n - i + 1, -1)$ and $(D(-\beta_i), T(-\beta_i)) = (i, 1)$ for $i = 0, 1, \dots, n$. The lemma follows. □

7.2 Maps between glued spaces

Let Ψ be the embedding of hermitian $\mathcal{O}_K$-lattices $\Lambda_2 \to \Lambda_n$ that sends $(x_0, x_1, x_2) \in \Lambda_2$ to $(x_0, x_1, x_2, 0, 0, \dots, 0) \in \Lambda_n$. View Λ_2 as a sublattice of Λ_n via Ψ, and write

$$\Lambda_n = \Lambda_2 \oplus (\Lambda_2)^\perp\,. \tag{21}$$

Using the canonical basis of Λ_n, we may view $\Gamma(n) = \mathrm{Aut}(\Lambda_n)$ as a subgroup of $\mathrm{GL}_{n+1}(\mathcal{O}_K)$. This gives an embedding

$$j \colon \Gamma(2) \to \Gamma(n), \quad M \mapsto (M, \mathrm{Id}). \tag{22}$$

The natural totally geodesic embedding $\nu\colon \mathbb{C}H^2 \hookrightarrow \mathbb{C}H^n$ induces a totally geodesic embedding

$$\nu\colon \quad \widetilde{Y}_2^{\#} := \coprod_{i=0}^{2} \mathbb{R}H^2_{\alpha_i} \hookrightarrow \coprod_{i=0}^{2} \mathbb{R}H^n_{\beta_i} \subset \coprod_{\alpha \in P\mathscr{A}_n} \mathbb{R}H^n_\alpha =: \widetilde{Y}_n. \tag{23}$$

As the composition $\mathrm{SO}(2,1) \to \mathrm{O}(2,1) \to \mathrm{PO}(2,1) = \mathrm{Isom}(\mathbb{R}H^2)$ is an isomorphism, there are natural embeddings $\mathrm{PO}(2,1) \hookrightarrow \mathrm{PO}(n,1)$ and $\mathrm{PO}(\Lambda_2^{\alpha_i})(\mathbb{Z}) \hookrightarrow \mathrm{PO}(\Lambda_n^{\beta_i})(\mathbb{Z})$ for $i = 0, 1, 2$. Moreover, as

$$P\Gamma(2)_{\alpha_i} = \mathrm{Stab}_{P\Gamma(2)}\left(\mathbb{R}H^2_{\alpha_i}\right) = \mathrm{PO}(\Lambda_2^{\alpha_i})(\mathbb{Z})$$

by [ACT06, (5.1)], one can use the equality (8) to observe that (22) induces, for each $i \in \{0,1,2\}$, an embedding $j\colon P\Gamma(2)_{\alpha_i} \hookrightarrow P\Gamma(n)_{\beta_i}$. The map $\nu\colon \widetilde{Y}_2^{\#} \to \widetilde{Y}_n$ defined in (23) is equivariant with respect to these embeddings j.

By Lemma 7.2, the classes of β_0, β_1 and β_2 in $P\mathscr{A}_n$ are pairwise not $P\Gamma(n)$-conjugate, so that the induced map of hyperbolic orbifolds

$$O_2 := \coprod_{i=0}^{2} P\Gamma(2)_{\alpha_i} \setminus \left(\mathbb{R}H^2_{\alpha_i} - \mathscr{H}_2\right) \to \coprod_{\alpha \in C\mathscr{A}_n} P\Gamma(n)_\alpha \setminus \left(\mathbb{R}H^n_\alpha - \mathscr{H}_n\right) =: O_n \tag{24}$$

induces an injective map on sets of connected components $\pi_0(O_2) \to \pi_0(O_n)$.

Lemma 7.3. *For each $n \geq 2$, there exists a natural map $\iota\colon X(\Lambda_2) \to X(\Lambda_n)$.*

Proof. This follows readily from the following assertions:

1. The map $\nu\colon \widetilde{Y}_2^{\#} \to \widetilde{Y}_n$ defined in (23) is compatible with the equivalence relations on both sides (see Definition 3.9).

2. Let $Y(\Lambda_2)^{\#} \subset Y(\Lambda_2)$ be the image of $\widetilde{Y}_2^{\#}$ in $Y(\Lambda_2)$ under the canonical map $\widetilde{Y}_2 \to Y(\Lambda_2)$. The resulting map of topological spaces

$$\phi\colon Y(\Lambda_2)^{\#} \to Y(\Lambda_n) \tag{25}$$

is equivariant with respect to the morphism $j\colon \mathrm{Stab}_{\Gamma(2)}(\widetilde{Y}_2^{\#}) \to \Gamma(n)$ induced by (22). Thus, ϕ descends to a map of topological spaces

$$\iota\colon X(\Lambda_2) = \Gamma(2) \setminus Y(\Lambda_2) = \mathrm{Stab}_{\Gamma(2)}(\widetilde{Y}_2^{\#}) \setminus Y(\Lambda_2)^{\#} \to \Gamma(n) \setminus Y(\Lambda_n) = X(\Lambda_n),$$

where the second equality follows from Lemma 7.1. □

7.3 Orbifold maps between glued spaces

Proposition 7.4. *The map ι defined in Lemma 7.3 is a map of hyperbolic orbifolds.*

Proof. Let $\bar{f} \in X(\Lambda_2)$ and lift $\bar{f}$ to an element $f \in Y(\Lambda_2)^{\#} \subset Y(\Lambda_2)$ (this is possible by Lemma 7.1). In turn, we can lift f to an element $(x, \alpha_{j_0}) \in \widetilde{Y}_2^{\#}$ for some $j_0 \in \{0,1,2\}$ with α_{j_0} as in (19)). To prove that $\iota\colon X(\Lambda_2) \to X(\Lambda_n)$ is an orbifold map at $\bar{f}$, there are three cases to consider:

Case I: *The element $f \in Y(\Lambda_2)$ has zero nodes* (see Definition 3.1). In this case, $\bar{f}$ lies in $O_2 \subset X(\Lambda_2)$ and the map $O_2 \to O_n$ of (24) is a morphism of orbifolds.

Case II: *The element $f \in Y(\Lambda_2)$ has one real node.* Let the node be defined by a short root $r \in \mathscr{R}_2$ such that $x \in H_r$. Consider the image $g = \phi(f) \in Y(\Lambda_n)$ of f in $Y(\Lambda_n)$ by the map $\phi\colon Y(\Lambda_2)^{\#} \to Y(\Lambda_n)$ defined in (25). It admits the lift $(\nu(x), \beta_{j_0}) \in \widetilde{Y}_n$. For $p \in \{2, \dots, n-1\}$, let $r_p = (0, 0, 0, \dots, 1, \dots, 0) \in \mathscr{R}_n$, where the 1 is on the $(p+1)$-th coordinate. Note that $\nu(x)$ has $n-1$ nodes. Define $r_1 = \Psi(r) \in \mathscr{R}_n$, so that

$$\nu(x) \in \bigcap_{i=1}^{n-1} H_{r_i} \subset \mathbb{C}H^n, \quad \text{and} \quad \mathscr{S} = \{r_1, r_2, \dots, r_{n-1}\} \subset \mathscr{R}_n$$

is a set of short roots of maximal cardinality such that $\nu(x) \in H_t$ for all $t \in \mathscr{S}$.

Lemma 7.5. *Consider $x \in \mathbb{C}H^2$, $\nu(x) \in \mathbb{C}H^n$, $r \in \mathscr{R}_2$ and $\Psi(r) \in \mathscr{R}_n$ as above. There are isometries $\kappa_2\colon \mathbb{C}H^2 \xrightarrow{\sim} \mathbb{B}^2(\mathbb{C})$ and $\kappa_n\colon \mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$ identifying ν with the canonical embedding $\mathbb{B}^2(\mathbb{C}) \hookrightarrow \mathbb{B}^n(\mathbb{C})$, the point x (resp. $\nu(x)$) with the origin, ϕ_r (resp. ϕ_{r_1}, resp. ϕ_{r_i} for $i \geq 2$) with the respective maps $(t_1, t_2) \mapsto (\zeta_6 \cdot t_1, t_2)$, $(t_1, \dots, t_n) \mapsto (\zeta_6 \cdot t_1, \dots, t_n)$ and $(t_1, \dots, t_n) \mapsto (t_1, \dots, \zeta_6 \cdot t_{i+1}, \dots, t_n)$, and α_{j_0} and β_{j_0} with the maps $(t_1, t_2) \mapsto (\bar{t}_1, \bar{t}_2)$ and $(t_1, \dots, t_n) \mapsto (\bar{t}_1, \dots, \bar{t}_n)$.*

Proof of Lemma 7.5. Let V_n be the $n+1$-dimensional hermitian space $(\Lambda \otimes \mathbb{C}, (h_n)_{\mathbb{C}})$. Let $x \in V_2$ be a lift of $x \in \mathbb{C}H^2$. Via the natural embedding $V_2 \to V_n$, we can consider x and r an elements of V_n. Then define

$$T_2 = \langle x \rangle \oplus \langle r \rangle \subset V_2, \quad R_2 = T_2^{\perp} \subset V_2, \quad T_n = \langle x \rangle \oplus \langle r \rangle \subset V_n, \quad R_n = T_n^{\perp} \subset V_n.$$

Choose suitable bases for R_2 and R_n scale appropriately as in Lemma 4.9. $\square$

We continue with the proof of Proposition 7.4. Consider the coordinates $\mathbb{C}H^2 \xrightarrow{\sim} \mathbb{B}^2(\mathbb{C})$ of Lemma 7.5: they make the involution α_{j_0} correspond to $(t_1, t_2) \mapsto (\bar{t}_1, \bar{t}_2)$, and any $\xi \in P\mathscr{A}_2$ with $(x, \alpha) \sim (x, \xi)$ to an involution of the form $(t_1, t_2) \mapsto (\zeta_6^i \cdot \bar{t}_1, \bar{t}_2)$. Thus if the ξ_i for $i = 1, \dots, 6$ are the six anti-unitary involutions such that $(x, \alpha) \sim (x, \xi_i)$, then for the set Y_f defined in (11), one has $Y_f \cong \bigcup_{i=1}^6 \mathbb{R}H^2_{\xi_i} \cong \left\{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) \mid t_1^6, t_2 \in \mathbb{R}\right\}$. Write $i = (i_1, \dots, i_{n-1})$ and $\chi_i = \prod_{p=1}^{n-1} \phi_{r_p}^{i_p} \circ \beta_{j_0}$. In the coordinates of Lemma 7.5, we have

$$Y_g \cong \bigcup_{i_1, \dots, i_{n-1}=1}^{6} \mathbb{R}H^n_{\chi_i} \cong \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_1^6, t_2, t_3^6, \dots, t_n^6 \in \mathbb{R}\right\}.$$

Next, define $K_{f,\epsilon} = \left\{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) \mid i^{-\epsilon} \cdot t_1 \in \mathbb{R}_{\geq 0}, t_2 \in \mathbb{R}\right\}$ and

$$K_{g,\epsilon_1,\dots,\epsilon_{n-1}} = \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid i^{\epsilon_1} t_1, i^{-\epsilon_2} t_3, \dots, i^{-\epsilon_{n-1}} t_n \in \mathbb{R}_{\geq 0}, t_2 \in \mathbb{R}\right\}.$$

With respect to the natural embedding $\nu\colon \mathbb{C}H^2 \to \mathbb{C}H^n$, one has for each $i = 1, \dots, 6$, that $\nu\left(\mathbb{R}H^n_{\xi_i}\right) \subset \mathbb{R}H^n_{\chi_{i,6,\dots,6}}$. Therefore, the maps defined above fit together in the following commutative diagram:

$$\begin{tikzcd}
B_f \setminus Y_f \arrow[r, hook] \arrow[d, "\wr"] & B_g \setminus Y_g \arrow[d, "\wr"] \\
B_f \setminus \bigcup_{i=1}^{6} \mathbb{R}H^n_{\xi_i} \arrow[r, hook] \arrow[d, "\wr"] & B_g \setminus \bigcup_{i_1,\ldots,i_{n-1}=1}^{6} \mathbb{R}H^2_{\chi_i} \arrow[d, "\wr"] \\
\langle\zeta_6\rangle \setminus \{(t_1,t_2) \in \mathbb{B}^2(\mathbb{C}) \mid t_1^6, t_2 \in \mathbb{R}\} \arrow[r, hook] \arrow[d, "\wr"] & \langle\zeta_6\rangle^{n-1} \setminus \{(t_1,\ldots,t_n) \mid t_1^6, t_2, t_3^6, \ldots, t_n^6 \in \mathbb{R}\} \arrow[d, "\wr"] \\
\bigcup_{\epsilon\in\{0,1\}} K_{f,\epsilon} \arrow[r, hook] \arrow[d, "\wr"] & \bigcup_{\epsilon_1,\ldots,\epsilon_{n-1}=0}^{1} K_{g,\epsilon_1,\ldots,\epsilon_{n-1}} \arrow[d, "\wr"] \\
\mathbb{B}^2(\mathbb{R}) \arrow[r, hook] & \mathbb{B}^n(\mathbb{R}).
\end{tikzcd} \tag{26}$$

Since the map $\phi\colon Y(\Lambda_2)^{\#} \to Y(\Lambda_n)$ of (25) is equivariant with respect to $\Gamma(2) \hookrightarrow \Gamma(n)$, we obtain a map $\mathrm{Stab}_{\Gamma(2)}(f) \hookrightarrow \mathrm{Stab}_{\Gamma(n)}(g)$, and consequently a map

$$A_f/B_f = \mathrm{Stab}_{P\Gamma(2)}(f)/B_f \to \mathrm{Stab}_{P\Gamma(n)}(g)/B_g = A_g/B_g. \tag{27}$$

The embedding $B_f \setminus Y_f \hookrightarrow B_g \setminus Y_g$ is equivariant with respect to (27). As in the proof of Theorem 4.1.2, we choose an A_f (resp. A_g)-equivariant open neighbourhood $f \in V_f \subset Y_f$ (resp. $g \in V_g \subset Y_g$) such that $A_f \setminus V_f \subset P\Gamma_f \setminus Y(\Lambda_2)$ (resp. such that $A_g \setminus V_g \subset P\Gamma_g \setminus Y(\Lambda_n)$). Using the above diagram (26), we get open neighbourhoods $W_f \subset \mathbb{B}^2(\mathbb{R})$ and $W_g \subset \mathbb{B}^n(\mathbb{R})$, acted upon by A_f/B_f and A_g/B_g respectively, such that $A_f \setminus V_f = (A_f/B_f) \setminus W_f$ and $A_g \setminus V_g = (A_g/B_g) \setminus W_g$, and such that there exists a totally geodesic embedding

$$\mathbb{B}^2(\mathbb{R}) \supset W_f \overset{\rho}{\hookrightarrow} W_g \subset \mathbb{B}^n(\mathbb{R}) \tag{28}$$

which is equivariant for (27) and makes the following diagram commute:

$$\begin{tikzcd}
W_f \arrow[r] \arrow[d, hook, "\rho"] & (A_f/B_f) \setminus W_f \arrow[r] \arrow[d, "\rho"] & P\Gamma(2) \setminus Y(\Lambda_2) \arrow[d, "\iota"] \\
W_g \arrow[r] & (A_g/B_g) \setminus W_g \arrow[r] & P\Gamma(n) \setminus Y(\Lambda_n).
\end{tikzcd}$$

Case III: *The element $f \in Y(\Lambda_2)$ has two real nodes.* This case is similar to Case II and we leave the proof to the reader.

Case IV: *The element $f \in Y(\Lambda_2)$ has a pair of complex conjugate nodes and no real nodes.* Let the two nodes be defined by short roots $r, t \in \mathscr{R}_2$ such that $x \in H_r \cap H_t$. Consider the images $g = \phi(f) \in Y(\Lambda_n)$ of f in $Y(\Lambda_n)$ by the map $\phi\colon Y(\Lambda_2)^{\#} \to Y(\Lambda_n)$ defined in (25). It admits the lift $(\nu(x), \beta_{j_0}) \in \widetilde{Y}_n$. For $p \in \{3,\ldots,n-1\}$, let $r_p = (0,0,0,\ldots,1,\ldots,0) \in \mathscr{R}_n$, where the 1 is on the $(p+1)$-th coordinate. Note that $\nu(x)$ has n nodes: two complex conjugate nodes and $n-2$ real nodes. Define $r_1 = \Psi(r) \in \mathscr{R}_n$ and $r_2 = \Psi(t) \in \mathscr{R}_n$. This gives $\nu(x) \in \cap_{i=1}^n H_{r_i} \subset \mathbb{C}H^n$, and

$\mathscr{S} = \{r_1, r_2, \dots, r_n\} \subset \mathscr{R}_n$ is a set of short roots of maximal cardinality such that $\nu(x) \in H_{r'}$ for each $r' \in \mathscr{S}$. As in the proof of Proposition 4.13.4 (see also Case II above), we have that

$$Y_f = B_f \setminus Y_f \cong \left\{(t_1, t_2) \in \mathbb{B}^2(\mathbb{C}) : t_2^6 = \bar{t}_1^6\right\} = \bigcup_{i=1}^{6} \mathbb{B}^2(\mathbb{R})_{\xi_i},$$

where $\xi_i \colon \mathbb{B}^2(\mathbb{C}) \to \mathbb{B}^2(\mathbb{C})$ is defined as $\xi_i(t_1, t_2) = (\bar{t}_2\zeta^i, \bar{t}_1\zeta^i)$ for $i \in \mathbb{Z}/6$. Similarly, we can describe $B_g \setminus Y_g$ as follows: we have

$$Y_g \cong \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^6 = \bar{t}_1^6, t_3^6, \dots, t_n^6 \in \mathbb{R}\right\}, \qquad \text{and}$$

$$B_g \setminus Y_g \cong \left\{(t_1, \dots, t_n) \in \mathbb{B}^n(\mathbb{C}) \mid t_2^6 = \bar{t}_1^6, t_3, \dots, t_n \in \mathbb{R}\right\} = \bigcup_{i=1}^{6} \mathbb{B}^n(\mathbb{R})_{\xi_i},$$

where $\xi_i \colon \mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C})$ is defined as $\xi_i(t_1, \dots, t_n) = (\bar{t}_2\zeta^i, \bar{t}_1\zeta^i, \bar{t}_3, \dots, \bar{t}_n)$ for $i \in \mathbb{Z}/6$. By Proposition 4.13.5, the group A_f (resp. A_g) acts transitively on the copies of $\mathbb{B}^2(\mathbb{R})$ (resp. $\mathbb{B}^n(\mathbb{R})$). Therefore, if we define $\Gamma_f = \mathrm{Stab}_{A_f/B_f}(\mathbb{B}^2_{\xi_1})$ and $\Gamma_g = \mathrm{Stab}_{A_g/B_g}(\mathbb{B}^n_{\xi_1})$ as the stabilizer of $\mathbb{B}^2(\mathbb{R})_{\xi_1}$ (resp. $\mathbb{B}^2(\mathbb{R})_{\xi_1}$), then we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{B}^2(\mathbb{R})_{\xi_1} & \longrightarrow & \mathbb{B}^n(\mathbb{R})_{\xi_1} \\ \downarrow & & \downarrow \\ \Gamma_f \setminus \mathbb{B}^2(\mathbb{R})_{\xi_1} & \longrightarrow & \Gamma_g \setminus \mathbb{B}^n(\mathbb{R})_{\xi_1}. \end{array}$$

By definition of the orbifold structure of $X(\Lambda_2)$ and $X(\Lambda_n)$ (see the proof of Theorem 4.1.2), this shows that $\iota \colon X(\Lambda_2) \to X(\Lambda_n)$ is a morphism of orbifolds at $\bar{f} \in X(\Lambda_2)$. This finishes the proof of Proposition 7.4. $\square$

8 Totally geodesic immersions and non-arithmeticity

We continue with the notation of Section 7. The goal of Section 8 is twofold: we first prove that, for each $n \in \mathbb{Z}_{\geq 2}$, the map ι defined in Lemma 7.3 defines a totally geodesic immersion of $X(\Lambda_2)$ into the connected component $X(\Lambda_n)^+$ of $X(\Lambda_n)$. Next, we show that the lattice Γ_n^+ underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$ is non-arithmetic, see Theorem 8.7.

Definition 8.1 (c.f. §2.3.1 in [Bel+21])**.** Let $i \colon \Gamma\backslash\mathbb{R}H^n \to \Lambda\backslash\mathbb{R}H^m$ be a map of hyperbolic orbifolds of finite volume. We say that i is a *totally geodesic immersion* if for any of the lifts $\widetilde{i} \colon \mathbb{R}H^n \to \mathbb{R}H^m$ of i, there exists a totally geodesic subspace $U \subset \mathbb{R}H^m$ such that $\widetilde{i}$ factors through an isometry $\mathbb{R}H^n \xrightarrow{\sim} U \subset \mathbb{R}H^m$. If the map i is moreover an embedding, then we call it a *totally geodesic embedding.*

Lemma 8.2. *Let $\phi \colon \mathbb{R}H^n \to \mathbb{R}H^m$ be a smooth map. Suppose that for each $x \in \mathbb{R}H^n$ there exist connected open neighborhoods $x \in U \subset \mathbb{R}H^n$, $y = \phi(x) \in V \subset \mathbb{R}H^m$, such that $\phi(U) \subset V$, and such that the induced map $\phi_{U,V} = \phi|_U \colon U \to V$ is a totally geodesic embedding. Then ϕ is a totally geodesic closed embedding.*

Proof. It is clear that ϕ preserves geodesics. Thus, for $x \in \mathbb{R}H^n$, ϕ commutes with $d\phi_x$ and the two exponentials $T_x\mathbb{R}H^n \xrightarrow{\sim} \mathbb{R}H^n$ and $T_{\phi(x)}\mathbb{R}H^m \xrightarrow{\sim} \mathbb{R}H^m$. In particular, $\phi = \exp_{\phi(x)} \circ d\phi_x \circ \exp_x^{-1}$ is a totally geodesic closed embedding. $\square$

Lemma 8.3. *Let* $\Gamma \subset \mathrm{PO}(n,1)$ *and* $\Lambda \subset \mathrm{PO}(m,1)$ *be lattices. Let*

$$i\colon \Gamma \setminus \mathbb{R}H^n \to \Lambda \setminus \mathbb{R}H^m$$

be a morphism of hyperbolic orbifolds, locally given by a map $\Gamma_x \backslash U \to \Lambda_y \backslash V$ *induced by a totally geodesic immersion* $U \to V$ *of connected opens* $U \subset \mathbb{R}H^n, V \subset \mathbb{R}H^m$*, equivariant for a homomorphism of finite groups* $\mathrm{PO}(n,1) \supset \Gamma_x \to \Lambda_y \subset \mathrm{PO}(m,1)$*. Then any lift* $\mathbb{R}H^n \to \mathbb{R}H^m$ *of* i *is a totally geodesic embedding. In particular, the map* i *is a totally geodesic embedding of hyperbolic orbifolds.*

Proof. This follows directly from Lemma 8.2 and Definition 8.1. □

Definition 8.4. Consider the map $\iota\colon X(\Lambda_2) \to X(\Lambda_n)$, see Proposition 7.4. Recall that $X(\Lambda_2)$ is connected (Lemma 7.1); let $X(\Lambda_n)^+ \subset X(\Lambda_n)$ be the connected component containing $\iota(X(\Lambda_2))$. Denote by $\Gamma_n^+ \subset \mathrm{PO}(n,1)$ the lattice underlying the complete hyperbolic orbifold $X(\Lambda_n)^+$, see Theorem 4.1.

We are now in position to prove the following theorem.

Theorem 8.5. *For each* $n \geq 2$*, there exists a canonical totally geodesic immersion of complete connected hyperbolic orbifolds* $\iota\colon X(\Lambda_2) \to X(\Lambda_n)^+$*.*

Proof. Consider the map $\iota\colon X(\Lambda_2) \to X(\Lambda_n)^+$ defined in Lemma 7.3, where $X(\Lambda_n)^+$ is as in Definition 8.4. By Proposition 7.4, ι is a morphism of hyperbolic orbifolds. Moreover, locally ι lifts to a totally geodesic embedding $\rho\colon W_f \hookrightarrow W_g$ as in the proof of Proposition 7.4, for some connected open subsets $W_f \subset \mathbb{B}^2(\mathbb{R})$ and $W_g \subset \mathbb{B}^n(\mathbb{R})$, equivariant for the morphism of stabilizer groups $A_f/B_f \to A_g/B_g$. By Lemma 8.3, this implies that ι is a totally geodesic embedding of hyperbolic orbifolds. □

Consider $\Gamma_n^+ \subset \mathrm{PO}(n,1)$, see Definition 8.4. We will prove that Γ_n^+ is non-arithmetic. The key is to combine Theorem 8.5 with the following result.

Theorem 8.6 (Bergeron–Clozel)**.** *Let* $\Gamma\backslash\mathbb{R}H^n \to \Lambda\backslash\mathbb{R}H^m$ *be a totally geodesic immersion of real hyperbolic orbifolds of finite volume, c.f. Definition 8.1. If the lattice* $\Lambda \subset \mathrm{PO}(m,1)$ *is arithmetic, then* $\Gamma \subset \mathrm{PO}(n,1)$ *is arithmetic as well.*

Proof. See [BC05, Proposition 15.2.2] (compare also [Bel+21, Theorem 1.4]). □

Theorem 8.7. *For each* $n \in \mathbb{Z}_{\geq 2}$*, the lattice* $\Gamma_n^+ \subset \mathrm{PO}(n,1)$ *is non-arithmetic.*

Proof. By Theorem 8.5, there is a proper totally geodesic immersion of hyperbolic orbifolds $\iota\colon X(\Lambda_2) \to X(\Lambda_n)^+$. By Theorem 8.6, this reduces us to the case $n = 2$. The lattice $\Gamma_2^+ \subset \mathrm{PO}(2,1)$ is non-arithmetic by [ACT06, Section 5]. □

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