The geometry and arithmetic of cubic hypersurfaces

Lecture notes

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[1] Last updated: November 17, 2023.

[1] This is an incomplete, preliminary version of my lecture notes on cubic hypersurfaces. These notes will be updated weakly, see `https://olivierfortman.github.io`. For comments on the text, please write me an e-mail (*degaayfortman@math.uni-hannover.de*).

These are lectures notes for a course given at the Institute of Algebraic Geometry in Hannover between October 2023 and February 2024. The goal of these lectures was to give an introduction to the theory of cubic hypersurfaces. In these notes, I will treat geometric as well as arithmetic aspects of the theory. Some of the topics covered:

(1) Topology of hypersurfaces.

(2) Hodge theory of cubic hypersurfaces.

(3) Lines on cubic hypersurfaces.

(4) Two-dimensional birational geometry, intersection theory, deformation theory.

(5) Cubic surfaces and cubic threefolds.

(6) Moduli spaces, algebraic stacks; period domains and period mappings.

(7) Étale cohomology and cubic hypersurfaces over finite fields.

Should you have any questions, or comments on these notes, do not hesitate to send me an e-mail[1].

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Chapter 1

Introduction

One of the goals of algebraic geometry is to study zero sets of systems of homogenous polynomials in multiple variables with coefficients in a field k. To do so, one is led to investigate the geometry of *algebraic varieties* over k. Among the simplest ways to obtain examples of an algebraic variety, is to consider a degree d hypersurface

$$X = Z(F) = \{F = 0\} \subset \mathbb{P}^m_k, \quad F \in k[x_0, \dots, x_m]_d, \quad d \in \mathbb{Z}_{\geq 1}.$$

It turns out that, although their definition is simple, hypersurfaces $X \subset \mathbb{P}^m_k$ are in general difficult objects to study.

To facilitate the study of hypersurfaces in $\mathbb{P}^m$, one can restrict to the *smooth hypersurfaces*, i.e. those for which the equation $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_m = 0$ has no solution in $\mathbb{P}^m(\bar{k})$. If $d = 1$ then $X \cong \mathbb{P}^m$ is a hyperplane. If $d = 2$ then X is a smooth quadric, which implies that F is projectively equivalent to $x_0^2 + \cdots + x_m^2 = 0$. When $d \geq 3$, degree d hypersurfaces in $\mathbb{P}^m$ for $m \geq 2$ come in positive dimensional families, and their investigation starts to become more complicated.

When $d = 3$, one enters the realm of *smooth cubic hypersurfaces*. For each value of $n = \dim(X)$, the class of cubic hypersurfaces of dimension n is very rich; however, only for small n, the theory is fairly well understood. When $n = \dim(X) = 1$ and X is equipped with a rational point $e \in X(k)$, then $E = (X, e)$ is called an *elliptic curve*. The fundamental theorem in the theory of elliptic curves says that there exists an algebraic group law $E \times E \to E$ in this case, turning E into a one-dimensional smooth projective group variety. If $n = \dim(X) = 2$, then $X = S$ is a *cubic surface*, and $S_{\bar{k}}$ turns out to contain exactly 27 lines over $\bar{k}$. In higher dimensions, cubic hypersurfaces provide a rich class of objects to test important conjectures in algebraic geometry on; think of the Hodge and Tate conjectures. Another example is provided by the Weil conjectures, that were proven for cubic threefolds before they were proven in general.

In the theory of cubic hypersurfaces, many beautiful areas in mathematics interact with one another, such as arithmetic geometry, algebraic topology, étale cohomology, Hodge theory and moduli theory. Open questions concern cycle class conjectures and rationality questions. The goal of these lectures is to dive into these theories, and use the developed techniques to study the geometry and arithmetic of cubic hypersurfaces.

Chapter 2

Preliminaries

To follow this course, it is useful, but not strictly necessary, to be familiar with the basic theory of schemes (as in [Mum88] or [Har77, Ch. I, §1-2]) and sheaf cohomology (see e.g. [Har77, Ch. II, §1-4]). In any case, the reader should have followed a first course in algebraic geometry.

Throughout the course, we will make use of some classical, fundamental results in algebraic geometry, without providing a proof. We collect these results this section [or in the appendix, to be added later]. Apart from this, we aim to make the body of the text will be as self-contained as possible; in particular, we try to avoid presenting a theorem without providing at least a sketch of its proof.

Chapter 3

Topology and differential forms

3.1 Lecture 1: Kähler differentials on hypersurfaces

Let k be a field. Let $n \in \mathbb{Z}_{\geq 0}$ and $m = n + 1$. We define

$$\mathbb{P} = \mathbb{P}^m = \mathbb{P}^{n+1}_k. \tag{3.1}$$

Before we start to study algebraic differential forms on hypersurfaces $X \subset \mathbb{P}^m_k$, we study them on the projective space $\mathbb{P}^m$ itself. To do so, we shall need some generalizations to the theory of vector bundles on schemes (or, more generally, ringed spaces) of classical linear algebra statements.

3.1.1 Linear algebra constructions on ringed spaces

The goal of this section is to prove two basic lemmas.

Lemma 3.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) If $0 \to E \to F \to L \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^p E \to \bigwedge^p F \to \bigwedge^{p-1} E \otimes L \to 0.$$

(2) Similarly, if $0 \to L \to E \to F \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for each $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^{p-1} F \otimes L \to \bigwedge^p E \to \bigwedge^p F \to 0.$$

(3) Let E be a vector bundle and L a line bundle on X. Let $a > 0$ be an integer. There is a canonical isomorphism

$$\bigwedge^a (E \otimes L) = \left(\bigwedge^a E\right) \otimes L^{\otimes a}.$$

Proof. 1. Let Q be the cokernel of $\wedge^p E \to \wedge^p F$. Wedge the original sequence with $\wedge^{p-1}E$, and consider the canonical morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^{p-1} E \otimes E \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes F \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigwedge^{p} E \arrow[r] & \bigwedge^{p} F \arrow[r] & Q \arrow[r] & 0.
\end{tikzcd}$$

It suffices to show that the so-constructed natural map $\wedge^{p-1}E \otimes L \to Q$ is an isomorphism. For this, we may assume that $F = E \oplus L$. In this case, $\wedge^p F = \wedge^p(E \oplus L) = \oplus_{i+j=p} \wedge^i E \otimes \wedge^j L = (\wedge^{p-1}E \otimes L) \oplus \wedge^p E$, and hence $\wedge^p F / \wedge^p E = \wedge^{p-1}E \otimes L$.

2. Dualize the exact sequence $0 \to L \to E \to F \to 0$, use item 1, and then dualize the result.

3. The map

$$(E \otimes F)^{\otimes a} \to \left(\bigwedge^a E\right) \otimes L^{\otimes a}, \quad e_1 \otimes f_1 \otimes \cdots \otimes e_a \otimes f_a \mapsto (e_1 \wedge \cdots \wedge e_a) \otimes (f_1 \otimes \cdots \otimes f_a),$$

factors through a map

$$\bigwedge^a (E \otimes L) \to \left(\bigwedge^a E\right) \otimes L^{\otimes a},$$

which is an isomorphism (this can be verified on stalks, where this is clear). ∎

Lemma 3.1.2. *Let E and F be vector bundles on a ringed space $(X, \mathcal{O}_X)$. For each integer $k \geq 0$, we have a canonical isomorphism*

$$\bigwedge^k (E \oplus F) = \bigoplus_{p+q=k} \left(\bigwedge^p E\right) \otimes \left(\bigwedge^q F\right).$$

Proof. Let R be a commutative ring. Then $\wedge(-)$ is a functor from R-modules to graded-commutative R-algebras which is left adjoint to the functor which takes the degree one part. Because it is left adjoint, it preserves colimits, and in particular coproducts. Therefore, for two R-modules M and N, we have a canonical isomorphism of graded R-algebras $\wedge(M \oplus N) = \wedge(M) \otimes \wedge(N)$. Now sheafify to get the result. ∎

3.1.2 Bott vanishing

Let k be a field and define $\mathbb{P}$ as in (3.1).

Lemma 3.1.3. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\mathbb{P} = \mathbb{P}^m$. For each $p \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$, there is a canonical exact sequence*

$$0 \to \Omega^p_{\mathbb{P}}(k) \to \mathcal{O}_{\mathbb{P}}^{\oplus \binom{m+1}{p}}(k-p) \to \Omega^{p-1}(k) \to 0. \tag{3.2}$$

Proof. Consider the *Euler sequence*, which is the exact sequence

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}} \to 0. \tag{3.3}$$

It yields

$$0 \to \Omega_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}^{\oplus(m+1)} \to \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

By item 1 in Lemma 3.1.1, this yields an exact sequence

$$0 \to \bigwedge^{p}(\Omega_{\mathbb{P}}(1)) \to \bigwedge^{p}(\mathcal{O}_{\mathbb{P}}^{\oplus(m+1)}) \to \bigwedge^{p-1}(\Omega_{\mathbb{P}}(1)) \to 0.$$

By item 3 in Lemma 3.1.1, we obtain:

$$\bigwedge^{p}(\Omega_{\mathbb{P}}(1)) = \left(\bigwedge^{p}\Omega_{\mathbb{P}}\right) \otimes \mathcal{O}(p) = \Omega^p(p).$$

The lemma follows. □

Lemma 3.1.4. *Let X be a projective variety of dimension n over k, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let E a vector bundle of rank r on X. For $p \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, there is a canonical isomorphism*

$$\left(\left(\bigwedge^{p} E\right)(k)\right)^* = \left(\bigwedge^{r} E\right)^* \otimes \left(\bigwedge^{r-p} E\right)(-k).$$

Proof. We have

$$\left(\left(\bigwedge^{p} E\right) \otimes \mathcal{O}_X(k)\right)^* = \left(\bigwedge^{p} E\right)^* \otimes \mathcal{O}_X(-k).$$

Hence, it suffices to prove the lemma in the case $k = 0$. Consider the natural map

$$\bigwedge^{p} E \to \mathrm{Hom}\left(\bigwedge^{p-r} E, \bigwedge^{r} E\right) = \mathrm{Hom}\left(\bigwedge^{p-r} E, \mathcal{O}_X\right) \otimes \bigwedge^{r} E = \left(\bigwedge^{p-r} E\right)^* \otimes \bigwedge^{r} E.$$

We claim that this map is an isomorphism. This may be checked locally, in which case it is clear. As $(\wedge^p E)^* = \wedge^p E^*$, the lemma follows by duality. □

Corollary 3.1.5. *Let X be a smooth projective variety of dimension n over k, with ample line bundle $\mathcal{O}_X(1)$. For $k \in \mathbb{Z}$, there are canonical isomorphisms*

$$(\Omega_X^p(k))^* \cong \omega_X^* \otimes \Omega_X^{r-p}(-k) \quad \textit{and} \quad H^q(X, \Omega_X^p(k)) \cong H^{n-q}(X, \Omega_X^{n-p}(-k))^\vee. \tag{3.4}$$

In particular, $h^q(X, \Omega^p(k)) = h^{n-q}(X, \Omega_X^{n-p}(-k))$ *for each* $k \in \mathbb{Z}$.

Proof. Lemma 3.1.4 shows that

$$((\wedge^p \Omega_X)(k))^* = (\wedge^n \Omega_X)^* \otimes (\wedge^{n-p}\Omega_X)(-k) = \omega_X^* \otimes \Omega_X^{n-p}(-k).$$

By Serre duality [reference], we obtain:

$$\begin{aligned} H^q(X, \Omega_X^p(k)) &= H^{n-q}(X, \omega_X \otimes (\Omega_X^p(k))^*)^\vee \\ &= H^q(X, \omega_X \otimes \omega_X^* \otimes \Omega_X^{n-p}(-k))^\vee = H^q(X, \Omega^{n-p}(-k))^\vee. \end{aligned}$$

The last statement follows readily from (3.4). □

Theorem 3.1.6 (Bott vanishing). *Consider the projective space $\mathbb{P} = \mathbb{P}_k^m$ of dimension $m > 0$ over k. Then $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in each of the following cases:*

(a) $p \neq q$ and $0 < q < m$;

(b) $p = q > 0$ and $k \neq 0$, and $k > 0$ if $p = q = m$;

(c) $q = 0$ and $k \leq p$, and $k < 0$ if $p = 0$;

(d) $q = m$ and $k \geq p - m$, and $k > 0$ if $p = m$.

Proof. We assume that we are in one of the cases (a) – (d); our goal is to prove that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$. By Serre duality, see Corollary 3.1.5, we may assume that $q \geq p$. We proceed by induction on p.

First, assume that $p = 0$. In this case, either $q = 0$ in which case $k < 0$ hence $H^q(\mathbb{P}, \mathcal{O}(k)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) = 0$, or $m > q > 0$ in which case $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$, or $m = q$ in which case $k \geq p - m = -m$ hence again $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$. We conclude that the assertion holds if $p = 0$.

Next, assume that $p > 0$. Then $q \geq p > 0$. Sequence (3.2) gives us a long exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(\mathcal{O}(k-p)^{\oplus \binom{m+1}{p}}) \to H^{q-1}(\Omega^{p-1}(k)) \to H^q(\Omega^p(k)) \to H^q(\mathcal{O}(k-p)^{\oplus \binom{m+1}{p}}) \\ \to H^q(\Omega^{p-1}(k)) \to H^{q+1}(\Omega^p(k)) \to \cdots \end{aligned} \tag{3.5}$$

We claim that $H^q\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-p)^{\oplus \binom{m+1}{p}}\right) = 0$. Indeed, this follows from the fact that $q > 0$, and $k - p \geq -m$ if $q = m$. Therefore, using the exact sequence (3.5), we conclude that the canonical map

$$H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) \to H^q(\mathbb{P}, \Omega^p(k)) \tag{3.6}$$

is surjective.

We claim that we may assume that $q > p$. To see this, suppose that $q = p$. If $q = p \geq 2$, then the induction hypothesis implies that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$ (since $k \neq 0$), hence by the surjection (3.6), we have $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in this case. Thus, suppose that $p = q = 1$. In this case, we have $k \neq 0$, and we want to show that $H^1(\mathbb{P}, \Omega^1(k)) = H^{m-1}(\mathbb{P}, \Omega^{m-1}(-k))^\vee = 0$.

To prove this, we proceed by induction on m. Suppose first that $m = 1 = p = q$. Then $k > 0$, and hence $H^1(\Omega^1(k)) = H^{m-1}(\Omega^{m-1}(-k))^\vee = H^0(\Omega^0(-k))^\vee = 0$. Next, assume $m \geq 2$. Then there are two cases to distinguish: $k > 0$ and $k < 0$. If $k < 0$, then the surjection (3.6) implies that $H^1(\Omega^1(k)) = 0$. Thus, assume that $k > 0$. We need to show that $H^{m-1}(\Omega^{m-1}(-k)) = 0$ for $k > 0$. We obtain a long exact sequence

$$\cdots \to H^{m-2}(\Omega^{m-2}(-k)) \to H^{m-1}(\Omega^{m-1}(-k)) \to H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}}) \to \cdots$$

The group $H^{m-2}(\Omega^{m-2}(-k))$ is zero by induction, and $H^{m-1}(\mathcal{O}(k-m)^{\binom{m+1}{m}})$ vanishes as well, as $m \geq 2$. Therefore, $H^{m-1}(\Omega^{m-1}(-k)) = 0$ as desired.

By the above claim, we may assume $q > p \geq 1$. We can then apply the induction hypothesis (recall that we are still arguing by induction on p) to see that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$. Indeed, we have $0 < q - 1 < m$. Therefore, the surjection (3.6) implies that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$, and we are done. □

Exercise 3.1.7. Show that the non-zero twisted Hodge numbers $h^q(\Omega^p(k))$ are:

(a) $h^p(\Omega^p) = 1$,

(b) $h^0(\Omega^p(k)) = \binom{m+k-p}{k} \cdot \binom{k-1}{p}$ if $k > p$,

(c) $h^m(\Omega^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{m-p}$ if $k < p - m$.

Exercise 3.1.8. Consider the projective space $\mathbb{P} = \mathbb{P}^m_{\mathbb{C}}$ of dimension m over $\mathbb{C}$. By Theorem 3.1.6, we have $H^0(\mathbb{P}, \Omega^p_{\mathbb{P}}) = 0$ for $p > 0$. Show directly that there are no non-zero holomorphic one-forms on $\mathbb{P}^1(\mathbb{C})$.

3.1.3 Kähler differentials on hypersurfaces

Lemma 3.1.9. *Let $X \subset \mathbb{P}$ be a smooth hypersurface of degree $d > 0$. For each $k \in \mathbb{Z}$, there are canonical exact sequences*

$$0 \to \Omega^p_{\mathbb{P}}(k-d) \to \Omega^p_{\mathbb{P}}(k) \to \Omega^p_{\mathbb{P}}|_X(k) \to 0, \tag{3.7}$$
$$0 \to \mathcal{O}_X(k-d) \to \Omega_{\mathbb{P}}|_X(k) \to \Omega_X(k) \to 0, \tag{3.8}$$
$$0 \to \Omega^{p-1}_X(k-d) \to \Omega^p_{\mathbb{P}}|_X(k) \to \Omega^p_X(k) \to 0. \tag{3.9}$$

Proof. It suffices to take $k = 0$.

To prove (3.7), we may take $p = 0$. In this case, the result follows from the following exact sequence, where i denotes the inclusion $X \hookrightarrow \mathbb{P}$:

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0. \tag{3.10}$$

One obtains (3.10) via the identification $\mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{I}_X$, where the latter denotes the ideal sheaf of $X \subset \mathbb{P}$, resulting from the isomorphisms $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}}(-X)$ (see

[Har77, II, Proposition 6.18]) and $\mathcal{O}_{\mathbb{P}}(X) \cong \mathcal{O}_{\mathbb{P}}(d)$ (which holds because $\deg(X) = d$). Note by the way that (3.10) corresponds to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \xrightarrow{1 \mapsto F} \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_X(d) \to 0,$$

where $F \in \mathcal{O}_{\mathbb{P}}(d) = k[x_0, \dots, x_{n_1}]_d$ is a polynomial that defines X.

To obtain the exact sequence (3.8), one combines the conormal exact sequence

$$0 \to \mathcal{N}^{\vee}_{Z/Y} \to \Omega_Y|_Z \to \Omega_Z \to 0$$

for any smooth hypersurface $i\colon Z \hookrightarrow Y$ in a smooth variety Y, where $\mathcal{N}^{\vee}_{Z/Y}$ is a sheaf on Z such that $i_*\mathcal{N}^{\vee}_{Z/Y} \cong I/I^2$ (see [Har77, II, Theorem 8.17]), and the canonical isomorphism

$$\mathcal{N}^{\vee}_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d) = \mathcal{O}_X(-d). \tag{3.11}$$

The second isomorphism in (3.11) being clear, it suffices to prove $\mathcal{N}^{\vee}_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d)$. This is again a general statement: if $i\colon Z \to Y$ is a closed immersion of schemes, then $i^*\mathcal{I}_Z$ has the property that $i_*i^*\mathcal{I}_Z = \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{I}_Z = \mathcal{I}_Z/\mathcal{I}_Z^2$ (to see this, reduce to the case where Y affine, where this is clear).

Finally, note that (3.9) follows from (3.8) together with Lemma 3.1.1. □

Proposition 3.1.10. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$ with canonical bundle ω_X. Then $\omega_X \cong \mathcal{O}_X(d-n-2)$. In particular,*

(1) ω_X is ample if $d > n+2$;

(2) $\omega_X \cong \mathcal{O}_X$ if $d = n+2$;

(3) ω_X^* is ample if $d < n+2$.

Proof. Consider sequence (3.9) with $p = n+1 = m$ and $k = d$. This gives

$$\omega_X \cong \omega_{\mathbb{P}}|_X(d) \cong \mathcal{O}_{\mathbb{P}}(-m-1)|_X \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(d-m-1).$$

The remaining statement follow directly. □

We proceed to prove:

Theorem 3.1.11. *Let $X \subset \mathbb{P} = \mathbb{P}^m = \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$. Then the following holds.*

(1) Let $k \in \mathbb{Z}$ with $k < d$. The natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p(k))$$

is bijective for $p+q < n$ and injective for $p+q \leq n$.

(2) We have

$$H^q(X, \Omega^p(k-d)) = 0 \quad \textit{for} \quad p+q < n \quad \textit{and} \quad k < d. \tag{3.12}$$

(3) We have $H^q(X, \Omega^p(k)) = 0$ for $(p+q<n, k<0)$ and for $(p+q>n, k>0)$.

Proof. Throughout the proof, we will use Theorem 3.1.6 without mention. We first prove 1 and 2 by induction on p. Therefore, assume that $k<d$.

Suppose first that $p=0$. Then (3.7) yields the following exact sequence:

$$H^q(\mathcal{O}_{\mathbb{P}}(k-d)) \longrightarrow H^q(\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^q(\mathcal{O}_X(k)) \longrightarrow H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d))$$

For $q \leq n < m$, we have $H^q(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ because $k-d<0$. Thus, $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is injective for $q \leq n$ and $k-d<0$. Moreover, if $q<n$ then $q+1 \leq n < m$, hence $H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ for $q<n$ and $k-d<0$. This implies that $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is bijective $q<n$ and $k-d<0$. In particular,

$$H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-d)) = H^q(X, \mathcal{O}_X(k-d)) = 0 \qquad \text{for} \quad (q<n, k-d<0).$$

This proves that 1 and 2 hold whenever $p=0$.

Next, let $p>0$. Notice that in this case, $p+q \leq n$ implies $q<n$. Similarly, $p+q<n$ implies $q<n-1$. Notice also that (3.8) and (3.9) yield the following diagram, in which the rows are exact:

$$\begin{array}{ccccccc} H^q(\Omega^p_{\mathbb{P}}(k-d)) \longrightarrow H^q(\Omega^p_{\mathbb{P}}(k)) \xrightarrow{f(p,q)} & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \to H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) & \\ & \| & \\ H^q(\Omega^{p-1}_X(k-d)) \to & H^q(\Omega^p_{\mathbb{P}}|_X(k)) & \xrightarrow{g(p,q)} H^q(\Omega^p_X(k)) \longrightarrow H^{q+1}(\Omega^{p-1}_X(k-d)). \end{array}$$

If $p+q \leq n < m$, then $q<m$ hence $H^q(\Omega^p_{\mathbb{P}}(k-d)) = 0$ as $k-d<0$. This implies that $f(p,q)$ is injective if $p+q \leq n$. Moreover, if $p+q \leq n < m$ then $(p-1)+q<n$, hence $H^q(\Omega^{p-1}_X(k-d)) = 0$ by the induction hypothesis, as $k-d<0$. Therefore, the maps $f(p,q)$ and $g(p,q)$ in the above diagram are both injective if $p>0$ and $p+q \leq n$. This implies that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is injective for all $p, q \geq 0$ such that $p+q \leq n$.

Still assume $p>0$. If $p+q<n$, then $q<n$, hence $q+1<n+1=m$. Therefore, $H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) = 0$ as $k-d<0$. Moreover, if $p+q<n$ then $(p-1)+(q+1)<n$, hence $H^{q+1}(X, \Omega^{p-1}_X(k-d)) = 0$ by induction, as $k-d<0$. Therefore, the maps $f(p,q)$ and $g(p,q)$ in the above diagram are both bijective if $p>0$ and $p+q<n$. We conclude that the natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p_X(k))$$

is bijective for all $p, q \geq 0$ such that $p+q<n$.

Continue to assume that $k<d$. Let $p, q \geq 0$ such that $p+q<n$. By what we have already proved, we have $H^q(X, \Omega^p_X(k-d)) = H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k-d))$, and this is zero because $q<m$ and $k-d<0$.

It remains to prove assertion 3. Notice that (3.12) implies $H^q(X, \Omega_X^p(k)) = 0$ for $(p+q < n, k < 0)$. This also implies, via Corollary 3.1.5, that

$$H^q(X, \Omega^p(k)) \cong H^{n-q}(X, \Omega^{n-p}(-k))^\vee = 0 \quad \text{if} \quad (p+q > n, k > 0).$$

This finishes the proof of the theorem. □

Corollary 3.1.12. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth hypersurface of degree d . If $n > 2$, then* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$. *Similarly, for $n = 2$ and $d \leq 3$, one has* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$.

Proof. Consider the *exponential exact sequence* of abelian sheaves on $X(\mathbb{C}) = X^{an}$:

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2i\pi} \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}_{X^{an}}^* \to 0.$$

Taking cohomology gives an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X). \tag{3.13}$$

As $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, see Exercise 3.1.14 below, it suffices to prove the following:

Claim 3.1.13. *If $n > 2$ or $n = 2$ and $d \leq 3$, then $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

On the one hand, by Theorem 3.1.6, we have $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = H^2(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = 0$ for $n > 1$. On the other hand, by Theorem 3.1.11, we see that if $n > 1$, then $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^1(X, \mathcal{O}_X)$ and if $n > 2$ then $H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^2(X, \mathcal{O}_X)$. Therefore, for $n > 1$, we have $H^1(X, \mathcal{O}_X) = 0$ and for $n > 2$, we have $H^2(X, \mathcal{O}_X) = 0$.

By Corollary 3.1.5, we have $h^i(X, \mathcal{O}_X) = h^{n-i}(X, \omega_X))$, and by Proposition 3.1.10, we have $h^{n-i}(X, \omega_X) = h^{n-i}(X, \mathcal{O}_X(d-(n+2))$. Thus, for $n = 2$, this gives

$$h^i(X, \mathcal{O}_X) = h^{2-i}(X, \mathcal{O}_X(d-4)) = 0 \quad \text{for} \quad i \in \{1, 2\} \quad \text{and} \quad d \leq 3.$$

This proves the claim, and thereby the corollary. □

Exercise 3.1.14. Sketch a proof of the fact that, for a locally ringed space $(X, \mathcal{O}_X)$, there is a natural isomorphism $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. Use this to conclude that if X is a smooth projective variety over $\mathbb{C}$, then $H^1(X, \mathcal{O}_X^*) = H^1(X^{an}, \mathcal{O}_{X^{an}}^*)$. Give an example of a sheaf $\mathcal{F}$ on a smooth projective variety X over $\mathbb{C}$ such that the natural map $H^1(X, \mathcal{F}) \to H^1(X^{an}, \mathcal{F}^{an})$ is not an isomorphism.

Exercise 3.1.15. Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth hypersurface of dimension n and degree d. Provide all (n, d) for which the homomorphism $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Analyze the group which measures the possible failure of the injectivity of c_1.

Exercise 3.1.16. Consider a smooth hypersurface $S \subset \mathbb{P}_{\mathbb{C}}^3$. Let $C \subset S$ be a curve contained in S. Prove that

$$[C] = c_1(\mathcal{O}_S(k)) \in H^2(S, \mathbb{Z})$$

if and only if there exists a hypersurface $Y \subset \mathbb{P}_{\mathbb{C}}^3$ of degree k such that $C = Y \cap S$.

3.2 Lecture 2: Lefschetz hyperplane theorem

To prove the Lefschetz hyperplane theorem, we will need some Morse theory. Let M be a smooth manifold of dimension n. Let $f\colon M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Then $0 \in M$ is called a *critical point* if $(df)_0 = 0$ as maps $T_0M \to T_{f(0)}\mathbb{R}$; in this case $f(0)$ is called a *critical value*. Consider the bilinear map

$$\mathrm{Hess}(f)_0 = (d^2 f)_0 \colon T_0M \times T_0M \to \mathbb{R}$$

defined as follows. Choose coordinates $x_1, \ldots, x_n$ on M centred around 0, and put

$$\mathrm{Hess}(f)_0 \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Lemma 3.2.1. *Show that the function $Hess(f)_0$ does not depend on the choice of coordinates around 0. Show that $Hess(f)_0$ defines a symmetric bilinear form on T_0M.*

Proof. Exercise. □

We say that a critical point $0 \in M$ is *non-degenerate* if $\mathrm{Hess}(f)_0$ is non-degenerate. By Lemma 3.2.1, if $0 \in M$ is a non-degenerate critical point, then $\mathrm{Hess}(f)_0$ defines a non-degenerate quadratic form, which can be diagonalized; define $\lambda_0(f)$ as the number of negative eigenvalues in this case. The Morse lemma, see [Mil63, Lemma 2.2], states that in suitable local coordinates $x_1, \ldots, x_n$ around a non-degenerate critical point $0 \in M$ of $f\colon M \to \mathbb{R}$, the function f can be written as the quadratic function

$$f(x) = f(0) - \sum_{i=1}^{\lambda_0(f)} x_i^2 + \sum_{i=\lambda_0(f)+1}^{n} x_i^2.$$

In particular, non-degenerate critical points (resp. values) are isolated in M (resp. $\mathbb{R}$).

We call f a *Morse function* if $f^{-1}(-\infty, a] \subset M$ is compact for each $a \in \mathbb{R}$, and f each critical point of f is non-degenerate. If f is a Morse function, then f is proper and its fibres $M_a = f^{-1}(a)$ are compact. Moreover, each critical value corresponds to a finite number of critical points, and the set of critical values is discrete in $\mathbb{R}$. In particular, for each $a \in \mathbb{R}$, there exist only finitely many critical values in $(-\infty, a] \subset \mathbb{R}$.

The basic theorem of Morse theory [Mil63, Theorem 3.5] says that if $f\colon M \to \mathbb{R}$ is a Morse function, then M has the homotopy type of a CW complex with one cell of dimension $\lambda_p(f)$ for each critical point $p \in M$.

Assume $M \subset \mathbb{R}^N$ is a closed submanifold of dimension n. By [Mil63, Theorem 6.6], for almost all (all but a set of measure 0) points $p \in \mathbb{R}$, the distance function

$$\varphi_p \colon M \to \mathbb{R}, \quad \varphi_p(x) = \|x - p\|^2$$

is a Morse function. We are now ready to prove:

Theorem 3.2.2 (Andreotti–Frankel [AF59])**.** *A closed n-dimensional complex submanifold $X \subset \mathbb{C}^r$ has the homotopy type of a CW complex of dimension $\leq n$.*

Proof. Let $c \in \mathbb{C}^r$ be a point such that the distance function $\varphi_c \colon X \to \mathbb{R}$ has only non-degenerate critical points.

Claim $(*)$. Let $p \in X$ be a critical point of $\varphi_c \colon X \to \mathbb{R}$. Then $\lambda_p(\varphi_c) \leq n$.

Before we prove Claim $(*)$, we will show that it implies the theorem. Indeed, by the basic theorem of Morse theory, X has the homotopy type of a CW complex with one cell of dimension $\lambda_p(\varphi_c)$ for each critical point $p \in M$ of φ_c. By Claim $(*)$, we have $\lambda_p(\varphi_c) \leq n$ for each critical point $p \in M$. Hence X has the homotopy type of a CW complex with one cell of dimension $\leq n$ for each critical point $p \in M$ of φ_c.

It remains to prove Claim $(*)$. We need:

Claim 3.2.3. *There exist local holomorphic coordinates on* $\mathbb{C}^r$ *such that* $p = 0$, $c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ *with* 1 *in the* $(n+1)$*-st position, and such that there exist open neighborhoods* $0 \in V_1 \subset \mathbb{C}^n$ *and* $0 \in V_2 \subset \mathbb{C}^{r-n}$ *and a holomorphic function*

$$\mathbb{C}^n \supset V_1 \xrightarrow{f} V_2 \subset \mathbb{C}^{r-n}$$

with $M \cap (V_1 \times V_2) = \mathrm{Graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^{r-n}$, *and such that* $df_0 = 0$.

Proof of Claim 3.2.3. Applying a suitable change of coordinates of $\mathbb{C}^r$, we may assume that $p = 0 \in M \subset \mathbb{C}^r$. As $M \subset \mathbb{C}^r$ is a closed submanifold, there exists an open subset $U \subset \mathbb{C}^r$ containing $p = 0$, and holomorphic functions $g_1, \ldots, g_m \colon U \to \mathbb{C}$ such that $X \cap U = \{g_1 = \cdots = g_m = 0\} \subset \mathbb{C}^r$. This gives a holomorphic function $g = (g_1, \ldots, g_m) \colon U \to \mathbb{C}^m$ such that $X \cap U = g^{-1}(0) = \{g = 0\} \subset U$. Thus, the fibre $g^{-1}(0)$ is smooth, which implies that g has maximal rank at each point of $X = g^{-1}(0)$. Applying the implicit function theorem, we obtain a holomorphic function $f \colon V_1 \to V_2 \subset \mathbb{C}^{r-n}$ defined on a open neighborhood $V_1 \subset \mathbb{C}^n$ of 0, such that $f(0) = 0$, $V_1 \times V_2 \subset U$ and such that

$$X \cap V_1 \times V_2 = \{(x, f(x)) \mid x \in V_1\} \subset V_1 \times V_2 \subset \mathbb{C}^n \times \mathbb{C}^{r-n} = \mathbb{C}^r.$$

Now $0 \neq c \in \mathbb{C}^r$, hence c defines a basis element of $\mathbb{C}^r$, so that there exists a matrix $\alpha \in \mathrm{GL}_r(\mathbb{C})$ with $\alpha \cdot c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $(n+1)$-st position. As α is linear, we have $\alpha \cdot 0 = 0$. Finally, we claim that $df_0 = 0$. This follows from the fact that $\varphi_c \colon X \to \mathbb{R}$ is a distance function, with critical point $p = 0$. In other words, $(d\varphi_c)_0 = 0$, because $\varphi_c(x, f(x)) = \|(x, f(x)) - (0, 0, \ldots, 0, 1, 0, \ldots, 0)\|^2$. □

As $|z-1|^2 = |x+iy-1|^2 = (x-1)^2+y^2 = (x^2+y^2)+(1-2x) = |z|^2+(1-2\cdot\Re(z))$, the distance function is now given by the formula

$$\varphi_c(z) = 1 - 2 \cdot \Re(f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2 . \tag{3.14}$$

As $\mathrm{ord}_0(f_i) \geq 2$ for all i, the last sum in (3.14) does not contribute to $\mathrm{Hess}(\varphi_c)_0$. Write

$$f_1(z) = Q(z) + \text{terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \ldots, z_n$. We obtain:

$$\mathrm{Hess}(\varphi_c)_0 = -2 \cdot \mathrm{Hess}\left(\Re(Q(z))\right)_0 + 2 \cdot \mathrm{Id}.$$

We claim that $\mathrm{Hess}(\Re(Q(z)))_0$ has at most n positive and at most n negative eigenvalues. Indeed, after a change of coordinates $z \mapsto w$, we can write

$$Q(w) = w_1^2 + \cdots + w_s^2, \qquad s \leq n;$$

writing $w_j = x_j + i \cdot y_j$, we obtain

$$\Re\left(Q(w)\right) = (x_1^2 - y_1^2) + \cdots + (x_s^2 - y_s^2).$$

This finishes the proof of Claim $(*)$, and thereby the proof of Theorem 3.2.2. $\square$

As a corollary, we obtain:

Theorem 3.2.4. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$. Let $Y = X \cap H$ be a hyperplane section such that $U := X \setminus Y$ is smooth of dimension n and let $j \colon Y \hookrightarrow X$ denote the canonical inclusion. The restriction map*

$$j^* \colon H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

Proof. For the proof, we need the following:

Claim 3.2.5. *We have a natural isomorphism $H^i(X, Y, \mathbb{Z}) \cong H_{2n-i}(U, \mathbb{Z})$.*

Assuming the claim, we obtain a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{i+1}(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{2n-i}(U,\mathbb{Z}) \arrow[r] & H^i(X,\mathbb{Z}) \arrow[r] & H^i(Y,\mathbb{Z}) \arrow[r] & H_{2n-i-1}(U,\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, to prove the theorem, we must show that $H_{2n-i}(U, \mathbb{Z}) = 0$ for $i \leq n-1$. As $i \leq n-1$ if and only if $2n - i \geq 2n - n + 1 = n+1$, we need to prove that $H_i(U, \mathbb{Z}) = 0$ for $i \geq n+1$. Note that $U = X \setminus Y \subset \mathbb{P}^N \setminus H \cong \mathbb{A}^N_{\mathbb{C}}$ defines a closed submanifold $U(\mathbb{C}) \subset \mathbb{C}^N$ of dimension n. By Theorem 3.2.2, $U(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\leq n$. In particular, $H_i(U, \mathbb{Z}) = 0$ for $i \geq n+1$, and Theorem 3.2.4 follows.

It remains to prove Claim 3.2.5; for this, we follow the exposition in [Voi02, page 306]. We admit the fact that Y admits a fundamental system of open neighborhoods $Y \subset Y_k \subset X$ that admit a deformation retract onto Y. It follows that the natural map

$$\varinjlim H^i(X, Y_k, \mathbb{Z}) \to H^i(X, Y, \mathbb{Z})$$

is an isomorphism. By excision, we have

$$H^i(X, Y_k, \mathbb{Z}) \cong H^i(U, U \cap Y_k, \mathbb{Z}).$$

If $K \subset U$ is a compact subset such that K is the deformation retract of an open subset $K \subset V \subset U$, then $H^i(U, U \setminus K, \mathbb{Z}) \cong H_{2n-i}(K, \mathbb{Z})$ (this is a refined version of Poincaré duality, see [Spa81, Section 6.2]). Applying this to

$$K_k := U \setminus (Y_k \cap U) = X \setminus Y_k,$$

which is a closed, hence compact, subset of X which admits a deformation retract of $X \setminus Y = U \supset K_k$, we obtain

$$H^i(U, Y_k \cap U, \mathbb{Z}) = H^i(U, U \setminus K_k, \mathbb{Z}) \cong H_{2n-i}(K_k, \mathbb{Z}).$$

As every singular chain on U is contained in one of the compact subsets $K_k \subset U$, the natural map $\varinjlim_k H_{2n-i}(K_k, \mathbb{Z}) \to H_{2n-i}(U, \mathbb{Z})$ is an isomorphism, and hence

$$H^i(X, Y, \mathbb{Z}) \cong \varinjlim H^i(U, U \cap Y_k, \mathbb{Z}) \cong \varinjlim H_{2n-i}(K_k, \mathbb{Z}) \cong H_{2n-i}(U, \mathbb{Z}),$$

proving Claim 3.2.5. □

Remark 3.2.6. For a compact oriented n-dimensional manifold M, and a closed submanifold $N \subset M$, cup-product with the fundamental class defines an isomorphism $H^i(M, N, \mathbb{Z}) \cong H_{n-i}(M \setminus N, \mathbb{Z})$. This is relative Poincaré duality, cf. [Dol95, Section 7]. In particular, if $X \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth projective variety of dimension n and $Y = X \cap H$ a smooth hyperplane section, then it readily follows that $H^i(X, Y, \mathbb{Z}) \cong H_{2n-i}(X \setminus Y, \mathbb{Z})$.

Corollary 3.2.7. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a hypersurface.*

(1) The restriction map $H^i(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < n$. In particular, $H^i(X, \mathbb{Z}) = 0$ for i odd and $i < n$, and $H^{2i}(X, \mathbb{Z}) = \mathbb{Z} \cdot h^i$ for $2i < n$.

(2) Suppose X is smooth. Then $H^i(X, \mathbb{Z}) = 0$ for $i > n$ odd, and for each $j > n$ there is a unique $\alpha_{2j} \in H^{2j}(X, \mathbb{Z})$ such that $H^{2j}(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha_{2j}$ and $\alpha_{2j} \cup h^{n-j} = 1$.

Proof. Let d be the degree of X, and consider the d-th Veronese embedding $\mathbb{P}^{n+1}_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$. Then $X = \mathbb{P}^{n+1}_{\mathbb{C}} \cap H$ for a hyperplane $H \subset \mathbb{P}^N_{\mathbb{C}}$. Apply Theorem 3.2.4 to obtain the first assertion. The second assertion follows from the first via Poincaré duality. □

Corollary 3.2.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then the restriction maps $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ and $\mathrm{Pic}(\mathbb{P}^{n+1}) \to \mathrm{Pic}(X)$ are isomorphisms. In particular, $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.*

Proof. The fact $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism is immediate from Theorem 3.2.4. From this, together with the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathbb{P}^{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[d] \\
H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \arrow[r] & H^2(X, \mathbb{Z}),
\end{tikzcd}$$

we deduce that $\mathrm{Pic}(\mathbb{P}^{n+1}) \to \mathrm{Pic}(X)$ is also an isomorphism, as the restriction map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism by Corollary 3.1.12. □

Corollary 3.2.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p + q > n$ and $L \in \operatorname{Pic}(X)$ ample.* $\square$

Remark 3.2.10. Later we will see that Corollary 3.2.9 remains valid for hypersurfaces over arbitrary fields k. Namely, if $X \subset \mathbb{P}^{n+1}_k$ is a smooth hypersurface of degree d, and if $n > 2$, then $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. See [refer to future section].

Exercise 3.2.11. Provide the equation of a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$ such that $\operatorname{Pic}(S) \ncong \mathbb{Z}$. See also Exercise 3.1.16. Define $V = H^0(\mathbb{P}^3, \mathcal{O}(d))$ and let $\mathbb{P}(V)$ be its projectivization. Let $\mathbb{P}(V)_0 \subset \mathbb{P}(V)$ be the locus of classes $[F] \in \mathbb{P}(V)$ such that $S_F \coloneqq \{F = 0\}$ is smooth. Show that $\mathbb{P}(V)_0$ is Zariski open in the projective space $\mathbb{P}(V)$. Show that the locus of $[F] \in \mathbb{P}(V)_0$ such that $\operatorname{Pic}(S_F) \ncong \mathbb{Z}$ is a countable union $\mathscr{H} = \cup_n Z_n$ of closed algebraic subvarieties $Z_n \subset \mathbb{P}(V_0)$. Show that $\mathscr{H} \neq \mathbb{P}(V_0)$.

Exercise 3.2.12. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n. Suppose that $n \geq 2$. Show that X is simply connected.

Exercise 3.2.13. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of degree $d = 3$. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of arbitrary degree $d \geq 4$.

Exercise 3.2.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of dimension $n \geq 2$. Let $C \subset X \subset \mathbb{P}^{n+1}$ be a smooth curve contained in X, and consider the Gysin map

$$\varphi \colon \mathbb{Z} = H^0(C, \mathbb{Z}) \cong H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \cong H^{2n-2}(X, \mathbb{Z}).$$

Define $[C] = \varphi(1) \in H^{2n-2}(X, \mathbb{Z})$. Consider the class $\alpha_{2n-2} \in H^{2n-2}(X, \mathbb{Z})$, see Corollary 3.2.7. Show that $[C] = \alpha_{2n-2}$ if and only if C intersects a general hyperplane $H \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ in a unique point with multiplicity one. Given equations for a cubic surface $X = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and a curve $C = \{F = G = 0\} \subset X \subset \mathbb{P}^3_{\mathbb{C}}$ such that $[C] = \alpha_2$.

3.3 Lecture 3: Betti numbers of hypersurfaces

Convention 3.3.1. We assume all topological manifolds to be second-countable and Hausdorff. In particular, they are paracompact and admit partitions of unity subordinate to any open cover.

3.3.1 Chern classes in topology

Let X be a topological manifold. Let $E \to X$ be a complex vector bundle of rank r. We would like to define the *Chern classes*

$$c_i(E) \in E^{2i}(X, \mathbb{Z}), \quad 1 \leq i \leq r$$

of X. We put $c_0(E) = 1$ and $c_i(E) = 0$ for $i > r = \operatorname{rank}(E)$, and introduce the *Chern polynomial*

$$c(E) = \sum_{i=0}^{r} c_i(E) \cdot t^i \in H^\bullet(X, \mathbb{Z})[t]$$

whose coefficients we shall now define. Consider the exponential exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2i\pi} \mathscr{C}^0 \xrightarrow{\exp} (\mathscr{C}^0)^* \to 0, \tag{3.15}$$

where $\mathscr{C}^0$ is the sheaf of continuous complex-valued functions on X, and $(\mathscr{C}^0)^*$ the subsheaf of invertible functions. The sequence (3.15) defines a morphism

$$c_1 \colon \{\text{complex line bundles } L \text{ on } X\} /_{\cong} = H^1(X, (\mathscr{C}^0)^*) \to H^2(X, \mathbb{Z}). \tag{3.16}$$

In particular, if E is a vector bundle of rank $r = 1$ on X, we obtain an element $c_1(E) \in H^2(X, \mathbb{Z})$ such that $c_1(E) = c_1(E')$ if $E \cong E'$ as vector bundles on X.

Lemma 3.3.2. *Let X be a topological space and $E \to X$ a vector bundle of rank r on X. Let $\psi \colon \mathbb{P}(E) \to X$ be the associated projective bundle. Let $S \subset \psi^* E$ be the tautological line bundle, and define $h = c_1(S^*) \in H^2(\mathbb{P}(E), \mathbb{Z})$. Then $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(X, \mathbb{Z})$, with basis $1, h, \ldots, h^{r-1}$.*

Proof. This follows from the Leray–Hirsch theorem (see [Hat02, Theorem 4D.1]). ☐

Lemma 3.3.3. *Let X be a topological manifold. Let $E \to X$ be a complex vector bundle on X. Then E admits a hermitian metric $E \times E \to \mathbb{C}$.*

Proof. Exercise. ☐

Theorem 3.3.4. *Let X be a topological manifold, and let $K(X)$ be the set of isomorphism classes of complex vector bundles of finite rank on X. There exists a unique function*

$$c_t \colon VB(X) \to H^\bullet(X, \mathbb{Z})[t], \quad E \mapsto c_t(E) = \sum_i c_i(E) \cdot t^i,$$

such that $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $E \in VB(X)$, $c_0(E) = 1$, $c_i(E) = 0$ for $i > r = \operatorname{rank}(E)$, and such that the following conditions are satisfied:

1. *(Compatibility with (3.16).) If $r = \operatorname{rank}(E) = 1$, then $c_t(E) = 1 + c_1(E) \cdot t$.*
2. *(Functoriality.) If $\phi \colon Y \to X$ is continuous, then $c_t(\phi^*(E)) = \phi^*(c_t(E))$ for $E \in VB(X)$, where $\phi^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet(Y, \mathbb{Z})$ is the pull-back of ϕ.*
3. *(Turning exact sequences into products.) If $0 \to F \to E \to G \to 0$ is an exact sequence, then $c_t(E) = c_t(F) \cdot c_t(G)$.*

Proof of uniqueness. This follows readily from the following:

Claim 3.3.5. *Let $E \to X$ be a complex vector bundle. There exists a topological manifold Y and a continuous map $\phi \colon Y \to X$ such that $\phi^* \colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ is injective for each i, and such that $\phi^* E$ is a direct sum of line bundles.*

To prove the claim, consider the projective bundle $\psi\colon \mathbb{P}(E) \to X$. The morphism $\psi^*\colon H^*(X,\mathbb{Z}) \to H^*(\mathbb{P}(E),\mathbb{Z})$ turns $H^*(\mathbb{P}(E),\mathbb{Z})$ into a free module over $H^*(X,\mathbb{Z})$, see Lemma 3.3.2. In particular, ψ^* is injective. Consider the tautological line bundle $S \subset \psi^*(E)$; it has fibre $S_x = \Delta_x \subset E_x$ above the point $x = [\Delta_x] \in \mathbb{P}(E_x)$ corresponding to a line $\Delta_x \subset E_x$. Put a hermitian metric h on $\psi^*(E)$ (cf. Lemma 3.3.3) and define Q as the orthogonal complement of S with respect to h; then $\psi^*(E) \cong S \oplus Q$. By induction on the rank of E, the claim follows.

To see why uniqueness follows, let $\phi\colon Y \to X$ as in the claim. We obtain an isomorphism $\phi^*(E) \cong L_1 \oplus \cdots \oplus L_n$ for some line bundles L_i on Y. Suppose that

$$c_t(E) = 1 + c_1(E)\cdot t + c_2(E)\cdot t^2 + \cdots + c_r(E)\cdot t^r = \sum_{i=0}^r c_i(E)\cdot t^i.$$

Then

$$\sum_{i=0}^r \phi^*\left(c_i(E)\right)\cdot t^i = \phi^*\left(c_t(E)\right) = c_t\left(\phi^*(E)\right) = c_t\left(L_1 \oplus \cdots \oplus L_n\right) = \prod_{i=1}^n \left(1 + c_1(L_i)\cdot t\right).$$

Thus, if c_t' is another map $VB(X) \to H^\bullet(X,\mathbb{Z})[t]$ with the desired properties, then $\phi^*(c_i'(E)) = \phi^*(c_i(E))$ for each i; as ϕ^* is injective, we get $c_i(E) = c_i'(E)$ for each i. $\square$

Proof of existence. Let $\psi\colon \mathbb{P}(E) \to X$ be the projective bundle associated to E, and let $S \subset \psi^*(E)$ be the tautological line bundle. Define $h = c_1(S^*) \in H^2(\mathbb{P}(E),\mathbb{Z})$. By Lemma 3.3.2, $H^*(\mathbb{P}(E),\mathbb{Z})$ is free as a module over $H^*(X,\mathbb{Z})$, and the elements $1, h, \dots, h^{r-1}$ form a basis for $H^*(\mathbb{P}(E),\mathbb{Z})$ over $H^*(X,\mathbb{Z})$. Therefore, there are elements $a_i \in H^{2i}(X,\mathbb{Z})$ such that

$$h^r + \psi^*(a_1)\cdot h^{r-1} + \cdots + \psi^*(a_{r-1})\cdot h + \phi^*(a_r) = 0 \qquad \text{in} \qquad H^{2r}(\mathbb{P}(E),\mathbb{Z}).$$

We put $c_0(E) = 1$, $c_i(E) = a_i$ for $1 \leq i \leq r$, and $c_i(E) = 0$ for $i > r$. We leave it to the reader to verify that conditions (1) – (3) are satisfied. $\square$

Exercise 3.3.6. Let X be a topological manifold. Show that $H^1(X,\mathscr{C}^0) = H^2(X,\mathscr{C}^0) = 0$. Conclude that the morphism $c_1\colon H^1(X,(\mathscr{C}^0)^*) \to H^2(X,\mathbb{Z})$ is an isomorphism.

3.3.2 Hirzebruch–Riemann–Roch theorem

Let E be a vector bundle on a topological manifold X. Let $a_1,\dots,a_r$ be the *formal Chern roots* of E. To be precise, we define them as formal symbols via the following formula:

$$c_t(E) = \sum_{i=0}^r c_i(E)\cdot t^i = \prod_{i=1}^r (1 + a_i\cdot t^i). \tag{3.17}$$

Thus, this means that the a_i are variables, subject to the following relations:

$$c_1(E) = \sum_{i=1}^r a_i, \quad c_2(E) = \sum_{1\leq i<j\leq r} a_i\cdot a_j, \quad \dots \quad , \quad c_r(E) = \prod_{i=1}^r a_i. \tag{3.18}$$

Define the *exponential Chern character* of E as the formal power series

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i}, \quad \text{where} \quad e^{a_i} = 1 + a_i + \frac{1}{2}a_i^2 + \cdots . \tag{3.19}$$

Similarly, define the *total Todd class* of E as the following formal power series, where the B_k are the Bernoulli numbers:

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}}, \quad \text{where} \quad \begin{aligned} \frac{a_i}{1-e^{-a_i}} &= 1 + \frac{1}{2}a_i + \frac{1}{12}a_i^2 - \frac{1}{720}a_i^4 + \cdots \\ &= 1 + \frac{a_i}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{B_k}{(2k)!}t^{2k}. \end{aligned} \tag{3.20}$$

Lemma 3.3.7. *Let X be a topological manifold. Then (3.19) and (3.20) can be expressed as polynomials in the $c_i(E)$ with rational coefficients.*

Proof. Exercise. □

Let X be a topological manifold, and let E be a complex vector bundle on X. Define, for each i, the *i-th Chern character* and the *i-th Todd class* of E via the formulae

$$\begin{aligned} \mathrm{td}(E) &= \mathrm{td}_0(E) + \mathrm{td}_1(E) + \cdots , & \mathrm{td}_i(E) &\in H^{2i}(X,\mathbb{Q}) \\ \mathrm{ch}(E) &= \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \ldots , & \mathrm{ch}_i(E) &\in H^{2i}(X,\mathbb{Q}). \end{aligned}$$

For a complex manifold X of dimension n, with holomorphic tangent bundle $\mathcal{T}_X$, define the following invariants:

$$c_i(X) = c_i(\mathcal{T}_X), \qquad \mathrm{ch}_i(X) = \mathrm{ch}_i(\mathcal{T}_X) \qquad \text{and} \qquad \mathrm{td}_i(X) = \mathrm{td}_i(\mathcal{T}_X).$$

Moreover, if E is a holomorphic vector bundle on X, we put

$$\chi(X,E) = \sum_{i=0}^{n}(-1)^i \dim H^i(X,E).$$

We then have the following fundamental result, whose proof we will omit.

Theorem 3.3.8 (Hirzebruch–Riemann–Roch)**.** *Let E be a holomorphic vector bundle on a compact complex manifold X of dimension n. Consider the degree $2n$-component of $\mathrm{ch}(E)\cdot\mathrm{td}(X)$, defined as $(\mathrm{ch}(E)\cdot\mathrm{td}(X))_{2n} = \sum_{i=0}^{n}\mathrm{ch}_i(E)\mathrm{td}_{n-i}(X)$. Then*

$$\chi(X,E) = \int_X (\mathrm{ch}(E)\cdot\mathrm{td}(X))_{2n}\,.$$

Proof. See [BS58]. □

Exercise 3.3.9. Let E be a vector bundle Let E and F be vector bundles on a topological space X. Show that $\mathrm{ch}(E\oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$, and that $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\cdot\mathrm{ch}(F)$. Show also that $c_i(E^\vee) = (-1)^i\cdot c_i(E)$ for each i.

Exercise 3.3.10. Let E be a holomorphic vector bundle on a complex compact manifold X. Deduce from Theorem 3.3.8 that $\chi(X,E)$ is independent of the holomorphic structure of E. In other words, prove that $\chi(X,E)$ depends only on the structure of E as a complex topological vector bundle.

3.3.3 Gauss–Bonnet formula

Let X be a compact complex manifold of dimension n. For integers $p, q \geq 0$, define $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$. The *Hirzebruch* χ_y*-genus* is the polynomial

$$\chi_y(X) = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) \cdot y^p. \tag{3.21}$$

The *Euler number* of X is defined as $e(X) = \sum_i (-1)^i b_i(X)$, where $b_i(X)$ is the i-th Betti number $b_i(X) = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ of X.

Corollary 3.3.11. *Let X be a compact Kähler manifold. Then $\chi_{y=-1}(X) = e(X)$.*

Proof. This will follow from Hodge theory, see Section ... Indeed, Hodge theory shows that $b_i(X) = \sum_{p=0}^{i} h^{p,i-p}(X)$. Therefore,

$$\chi_{y=-1}(X) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i \sum_{p+q=i} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i b_i(X),$$

proving the corollary. □

Corollary 3.3.12. *Let X be a compact complex manifold. Let $\gamma_1, \ldots, \gamma_n$ be the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$ of X, see (3.17). Then*

$$\chi_y(X) = \int_X \prod_{i=1}^{n} \left(1 + y \cdot e^{-\gamma_i}\right) \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

Proof. Exercise. □

Proposition 3.3.13. *Let X be a compact Kähler manifold of dimension n. Then*

$$e(X) = \int_X c_n(X).$$

Proof. By Corollary 3.3.11, we have $e(X) = \chi_{y=-1}(X)$, and by Corollary 3.3.12, we have $\chi_{y=-1}(X) = \int_X \prod_i \gamma_i$, where $\gamma_1, \ldots, \gamma_n$ are the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$. The proposition follows, as $\prod_i \gamma_i = c_n(X)$ by (3.18). □

3.3.4 Betti cohomology of smooth hypersurfaces

Let $X \subset \mathbb{P} = \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth complex hypersurface. Our goal now is to compute the middle Betti number $b_n(X) = \dim_{\mathbb{Q}} H^n(X, \mathbb{Q})$. Define the *Euler number of* X as follows:

$$e(X) = \sum_{i=0}^{2n} (-1)^i b_i(X) = \sum_{i=0, i \neq n}^{2n} (-1)^i b_i(X) + (-1)^n b_n(X).$$

Lemma 3.3.14. *Let $X \subset \mathbb{P} = \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth complex hypersurface. Then*

$$e(X) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n). \tag{3.22}$$

Proof. By Corollary 3.2.7, we have $b_i(X) = 0$ for $i \neq n$ odd and $b_i(X) = 1$ for $i \neq n$ even. Hence

$$\begin{aligned} e(X) &= \sum_{i=0,2i\neq n}^{n} (-1)^{2i} b_{2i}(X) + \sum_{i=1,2i-1\neq n}^{n} (-1)^{2i-1} b_i(X) + (-1)^n b_n(X) \\ &= \left(\sum_{i=0,2i\neq n}^{n} 1 \right) + (-1)^n b_n(X) \\ &= \begin{cases} n + b_n(X) & \text{if } n \equiv 0(2), \\ n + 1 - b_n(X) & \text{if } n \equiv 1(2). \end{cases} \end{aligned}$$

This proves what we want. □

Proposition 3.3.15. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d and dimension $n \geq 0$. Let $b_n(X)$ be the n-th Betti number of X. Then*

$$b_n(X) = \frac{(-1)^n}{2d} \cdot \left(2 \cdot (1-d)^{n+2} + 3 \cdot d + (-1)^n \cdot d - 2\right). \tag{3.23}$$

Proof. See Section 3.3.1 above for an introduction to Chern classes. By Proposition 3.3.13, we have

$$e(X) = \int_X c_n(X), \quad \text{where} \quad c_n(X) = c_n(\mathcal{T}_X) \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Notice that sequence (3.3) yields an exact sequence

$$0 \to \Omega_{\mathbb{P}}|_X \to \mathcal{O}_X(-1)^{n+2} \to \mathcal{O}_X \to 0,$$

which, after dualizing, gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+2} \to \mathcal{T}_{\mathbb{P}}|_X \to 0.$$

We also consider the sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}|_X \to \mathcal{O}_X(d) \to 0,$$

that follows by dualizing (3.8). By item (3) in Theorem 3.3.4, we obtain

$$\begin{aligned} c(X) &= \sum_i c_i(X) = \sum_i c_i(\mathcal{T}_X) = c(\mathcal{T}_X) = c(\mathcal{T}_{\mathbb{P}}|_X) \cdot c(\mathcal{O}_X(d))^{-1} \\ &= c\left(\mathcal{O}_X(1)^{\oplus(n+2)}\right) \cdot c(\mathcal{O}_X(d))^{-1} = \frac{(1+h)^{n+2}}{(1+dh)}, \qquad h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z}). \end{aligned}$$

We now have the following:

Claim 3.3.16. *Let h be a variable, and consider the ring $R = \mathbb{Q}[h]/(h^{n+1})$. Then $(1+dh)$ is invertible in R hence $(1+dh)^{-1} \cdot (1+h)^{n+2}$ is a well-defined element in $\mathbb{Q}[h]/(h^{n+1})$. Moreover, its coefficient before h^n is $(1/d^2) \cdot ((1-d)^{n+2} + d \cdot (n+2) - 1)$.*

Proof. Exercise. $\square$

By combining $\deg(h) = \int_X h^n = d$, equality (3.22) and Claim 3.3.16, we obtain:

$$e(X) = \frac{1}{d} \cdot \left((1-d)^{n+2} + d \cdot (n+2) - 1\right) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n) .$$

In particular,

$$(-1)^n \cdot b_n(X) = \frac{2(1-d)^{n+2} + 2d \cdot (n+2) - 2 - 2nd - d + (-1)^n \cdot d}{d},$$

and equality (3.23) follows. $\square$

Corollary 3.3.17. *The n-th Betti number $b_n(X_3)$ of a smooth cubic hypersurface $X_3 \subset \mathbb{P}^n_{\mathbb{C}}$ of dimension $n \geq 0$ is given by the following formula:*

$$b_n(X_3) = \frac{1}{6} \cdot \left(2^{n+3} + (-1)^n \cdot 7 + 3\right) .$$

For instance, $b_0(X_3) = 3$, $b_1(X_3) = 2$, $b_2(X_3) = 7$ and $b_3(X_3) = 10$. $\square$

Exercise 3.3.18. For a smooth projective variety X over $\mathbb{C}$, define $h^{p,q}(X) = h^q(X, \Omega^p_X)$. Calculate all the values $h^{p,q}(X)$ with $p + q = 3$ for a smooth cubic threefold $X \subset \mathbb{P}^4_{\mathbb{C}}$, and all the $h^{p,q}(X)$ with $p + q = 4$, for a smooth cubic fourfold $X \subset \mathbb{P}^5_{\mathbb{C}}$.

3.4 Lecture 4: Intersection form on middle cohomology

Let X be a compact complex manifold of dimension n. Poincaré duality provides canonical isomorphisms $H^i(X, \mathbb{Z}) \cong H_{2n-i}(X, \mathbb{Z})$. Moreover, the universal coefficient theorem provides a canonical isomorphism $H^i(X, \mathbb{Z})/(\mathrm{tors}) \cong \mathrm{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z})$. Combining the two assertions, one sees that the cap product pairing

$$H_i(X, \mathbb{Z})/(\mathrm{tors}) \otimes H_{2n-i}(X, \mathbb{Z})/(\mathrm{tors}) \to H_0(X, \mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing. Dually, the cup product pairing

$$H^i(X, \mathbb{Z})/(\mathrm{tors}) \otimes H^{2n-i}(X, \mathbb{Z})/(\mathrm{tors}) \to H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing.

Lemma 3.4.1. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 0$. Then $H^n(X, \mathbb{Z})$ is torsion-free.*

Proof. For $n = 0$, the claim is trivial, so we may assume $n \geq 1$. The universal coefficient theorem gives then an exact sequence

$$0 = \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^n(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Here, $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) = 0$ because $H_{n-1}(X,\mathbb{Z}),\mathbb{Z})$ is trivial or isomorphic to $\mathbb{Z}$, see Corollary 3.2.7. As $\mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z})$ is torsion-free, the lemma follows. $\square$

In particular, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, we obtain a perfect pairing

$$\cup\colon H^n(X,\mathbb{Z}) \otimes H^n(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z}) = \mathbb{Z}. \tag{3.24}$$

Recall that, for $\alpha, \beta \in H^n(X,\mathbb{Z})/(\mathrm{tors})$, we have $\alpha \cup \beta = (-1)^n \cdot \beta \cup \alpha$. This implies that (3.24) is symmetric if n is even, and alternating if n is odd. The goal of this section is to study (3.24) in case $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth cubic hypersurface of dimension n.

3.4.1 Odd-dimensional cubic hypersurfaces

It turns out that if X is an odd-dimensional hypersurface, the intersection form on $H^n(X,\mathbb{Z})$ is quite easily calculated, as follows from the following lemma.

Lemma 3.4.2. *Let Λ be a free $\mathbb{Z}$-module of rank $n > 0$ and let*

$$E\colon \Lambda \otimes \Lambda \to \mathbb{Z} \tag{3.25}$$

be an alternating bilinear form on $\mathbb{Z}$, defining a perfect pairing. Then $n = 2g$ and there exists a basis $\{e_1, \dots, e_g; f_1, \dots f_g\}$ for Λ such that $E(e_i, e_j) = E(f_i, f_j) = 0$ for all i, j, and such that $E(e_i, f_i) = 1$ for all i and $E(e_i, f_j) = 0$ if $i \neq j$.

Proof. Notice that $n = \mathrm{rank}(\Lambda) \geq 2$, for if $n = 1$ then $E(x, y) = 0$ for each $x, y \in \Lambda$. Suppose first that $n = 2$. Let $\{x, y\} \subset \Lambda$ be a basis for Λ. Let $M = (m_{ij})$ be the intersection matrix of E with respect to this basis. We have $m_{11} = E(x, x) = 0$, $m_{12} = E(x, y)$, $m_{21} = -E(x, y)$ and $m_{22} = E(y, y) = 0$. Thus, the determinant of M equals $E(x, y)^2$, which must be invertible in $\mathbb{Z}$. Hence $E(x, y) = \pm 1$, and the result follows.

Next, assume $n \geq 3$. Let $y \in \Lambda$ and $W \subset \Lambda$ such that

$$\mathbb{Z} \cdot y \oplus W = \Lambda.$$

Define a linear map $f\colon \Lambda \to \mathbb{Z}$ by putting $f(y) = 1$ and $f(w) = 0$ for each $w \in W$, and extending linearly. As the pairing (3.25) is perfect, there exists an element $x \in \Lambda$ such that $E(x, -) = f$ as linear maps $\Lambda \to \mathbb{Z}$. This implies that $E(x, y) = 1$ and $E(x, w) = 0$ for each $w \in W$. Let $P = \langle x, y\rangle^{\perp}$ be the orthogonal complement of $\langle x, y\rangle$ in Λ with respect to E. We claim that

$$\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y \oplus P = \Lambda. \tag{3.26}$$

To prove this, let $\lambda \in \Lambda$. We must show that there exist unique $a, b \in \mathbb{Z}$ such that $\lambda - a \cdot x - b \cdot y \in P$. That is, we need to show there exist unique $a, b \in \mathbb{Z}$ such that

$$E(x, \lambda - a \cdot x - b \cdot y) = E(x, \lambda) + b = 0,$$
$$E(y, \lambda - a \cdot x - b \cdot y) = E(y, \lambda) - a = 0.$$

We may simply put $b = -E(x, \lambda)$ and $a = E(y, \lambda)$. Decomposition (3.26) follows.

To finish the proof, we would like to show that the restriction of E to $P \otimes P$ defines a perfect pairing, i.e. a unimodular alternating bilinear form. To see this, observe that by choosing a basis $\{p_1, \dots, p-2\}$ for P, the form E becomes associated to a $(n-2) \times (n-2)$-matrix $M_P := (E(p_i, p_j))$. Similarly, one attaches a matrix $M_{x,y}$ to the pairing that E defines on $\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y$. The basis $\{x, y, p_1, \dots, p_{n-2}\}$ for Λ then associates a matrix M_Λ to E, and we have

$$\det(M_P) \cdot \det(M_{x,y}) = \det(M) = \pm 1,$$

where the last equality holds because E is unimodular. Therefore, $\det(M_P) = \pm 1$, hence the restriction of E to $P \otimes P$ is unimodular. The lemma follows by induction on the rank n of Λ. $\square$

Corollary 3.4.3. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be an odd-dimensional smooth cubic hypersurface. Then $H^n(X, \mathbb{Z})$ is free of rank $b_n(X) = 2m$ over $\mathbb{Z}$, and admits a basis $\{\gamma_1, \dots, \gamma_{2m}\}$ with respect to which the intersection matrix of the pairing $H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$ has the following form, where $\mathrm{Id} \in \mathrm{GL}_m(\mathbb{Z})$ denotes the identity matrix:*

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Proof. Torsion-freeness follows from Lemma 3.4.1. As the dimension of X is odd, (3.24) is a unimodular, alternating bilinear form, and we can apply Lemma 3.4.2. $\square$

3.4.2 Even-dimensional cubic hypersurfaces

We are going to use the following result, without providing a proof:

Proposition 3.4.4. *If a smooth projective variety X over $\mathbb{C}$ (or, more generally, a compact Kähler manifold) has even dimension $2m$, and if $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$, then the intersection pairing on $H^n(X, \mathbb{R})$ has signature*

$$\mathrm{sgn}(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X).$$

Proof. See [Voi02, Théorème 6.33] or [Huy05, Corollary 3.3.18]. $\square$

Corollary 3.4.5. *Let X be a smooth projective variety of dimension $n = 2m$ over $\mathbb{C}$. Consider the Hirzebruch χ_y-genus $\chi_y(X)$, see (3.21). Then $\chi_{y=1}(X) = \tau(X)$.* $\square$

We shall also need the following result, whose proof we omit:

Theorem 3.4.6 (Hirzebruch). *Let $X_n \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a sequence of smooth hypersurfaces of degree d. For each n, let $\chi_y(X_n)$ be the Hirzebruch χ_y-genus of X_n, cf. (3.21). Then*

$$\sum_{n=0}^{\infty} \chi_y(X_n) z^{n+1} = \frac{1}{(1+yz)(1-z)} \cdot \frac{(1+yz)^d - (1-z)^d}{(1+yz)^d + y(1-z)^d}. \tag{3.27}$$

Proof. See [Hir95, Theorem 22.1.1]. □

Notice that, by Proposition 3.4.4, for $y = 1$ and $d = 3$ we can rewrite (3.27) as

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(X_n) z^{n+1} &= \frac{1}{(1+z)(1-z)} \cdot \frac{(1+z)^3 - (1-z)^3}{(1+z)^3 + (1-z)^3} \\ &= (-1) \cdot \frac{z^3 + 3z}{3z^4 - 2z^2 - 1} \\ &= z \cdot \frac{3 + z^2}{(1+3z^2) \cdot (1-z^2)}.\end{aligned}$$

Lemma 3.4.7. *Consider the power series expansion*

$$z \cdot \frac{3 + z^2}{(1+3z^2) \cdot (1-z^2)} = z \cdot \sum_{i=0}^{\infty} a_i \cdot z^i.$$

Then $a_{2m} = (-1)^m \cdot 2 \cdot 3^m + 1$ for each $m \geq 0$.

Proof. Exercise. □

Combining the above, we obtain:

Proposition 3.4.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. Let $\tau(X)$ be the signature of the pairing $H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to \mathbb{R}$. Then $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.* □

Let Λ be a *lattice*, i.e. a free $\mathbb{Z}$-module equipped with a symmetric bilinear form $(\cdot\,,\cdot)$. We say that Λ is unimodular when the pairing is perfect, i.e. when the determinant of any intersection matrix is ± 1. We say that a unimodular lattice is *even* if $(\alpha, \alpha) \equiv 0 \bmod 2$ for all $\alpha \in \Lambda$; otherwise, we say that Λ is *odd*. For example, the rank one lattice $\mathbb{Z}(a)$ with $(1, 1) = a$ is odd if and only if a is odd.

If Λ is unimodular, odd, and indefinite, then for some positive integers r, s, we have

$$\Lambda \cong I_{r,s} := \mathbb{Z}(1)^{\oplus r} \oplus \mathbb{Z}(-1)^{\oplus s}. \tag{3.28}$$

For this, see for example [Ser73, Chapter V, Theorem 4].

Theorem 3.4.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. The intersection form on $H^n(X,\mathbb{Z})$ turns $H^n(X,\mathbb{Z})$ into a unimodular lattice, and there exists an isomorphism of lattices*

$$H^n(X,\mathbb{Z}) \cong \mathbb{Z}(1)^{\oplus b_n^+} \oplus \mathbb{Z}(-1)^{\oplus b_n^-} = I_{b_n^+, b_n^-}. \tag{3.29}$$

Here, $b_n^+ := b_n^+(X)$ is defined as the number of positive eigenvalues of an intersection matrix of the associated form on $H^n(X,\mathbb{R})$, and $b_n^- = b_n^-(X) = b_n(X) - b_n^+(X)$. The two integers b_n^+ and b_n^- can be calculated from the two equalities $b_n^+ + b_n^- = b_n(X) = (1/6)\cdot(2^{n+3} + (-1)^n \cdot 7 + 3)$ and $b_n^+ - b_n^- = \tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.

Proof. We prove that $H^n(X,\mathbb{Z})$ is odd. This is easy: the class $h^m = c_1(\mathcal{O}_X(1))$ in $H^n(X,\mathbb{Z})$ satisfies $(h^m, h^m) = \int_X h^n = d$. Moreover, it was shown in Corollary 3.3.17 that we have $b_n(X) = (1/6)\cdot(2^{n+3} + (-1)^n \cdot 7 + 3)$, and the fact that $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$ follows from Proposition 3.4.8. In particular, $b_n(X) \neq \pm\tau(X)$, hence $H^n(X,\mathbb{Z})$ is indefinite. The isomorphism (3.29) follows then by the above-mentioned classification of odd indefinite unimodular lattices. □

3.4.3 Cubic surfaces

Proposition 3.4.10. *Let X be a compact complex manifold of dimension two. Let L be a line bundle on X. Then*

$$\chi(X,\mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} \qquad \textit{and}$$
$$\chi(X,L) = \int_X \frac{c_1(L)^2 + c_1(L)\cdot c_1(X)}{2} + \chi(X,\mathcal{O}_X).$$

Proof. One calculates the value of $\mathrm{td}(X)$, which is $\mathrm{td}(X) = 1 + c_1(X)/2 + c_1(X)^2/12 + c_2(X)/12$. Moreover, $\mathrm{ch}(L) = e^{c_1(L)}$, and the result follows from Theorem 3.3.8. □

Lemma 3.4.11. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface, and let L be a line bundle on X. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X,\mathbb{Z})$. Then*

$$\chi(X,L) = \frac{(L,L) + (L,h)}{2} + 1.$$

Proof. Let X be a smooth cubic hypersurface. Then

$$c(X) = \frac{(1+h)^{n+2}}{1+3h} = \left(1 - 3h + (3h)^2 \pm \cdots\right)\cdot \sum_{i=0}^{n} \binom{n+2}{i} h^i.$$

Hence, $c(X) = (1 - 3h + (3h)^2 \pm \cdots)\cdot\left(1 + (n+2)\cdot h + \binom{n+2}{2}\cdot h^2 + \cdots\right)$, which gives

$$c_1(X) = (n+2)\cdot h - 3h = (n-1)\cdot h$$
$$c_2(X) = \left(9 - 3\cdot(n+2) + \binom{n+2}{2}\right)\cdot h^2.$$

For $n = 2$, this becomes $c_1(X) = h \in H^2(X, \mathbb{Z})$ and $c_2(X) = 3 \cdot h^2$. Together with Proposition 3.4.10, this implies that $\chi(X, L) = (1/2) \cdot ((L, L) + (L, h)) + \chi(X, \mathcal{O}_X)$.

It remains to show that

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 \tag{3.30}$$

This follows immediately from the fact that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ by Claim 3.1.13. Alternatively, we can use the fact that $c_1(X)^2 + c_2(X) = h^2 + 3h^2 = 4h^2$; applying Proposition 3.4.10 yields

$$\chi(X, \mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} = \int_X \frac{4h^2}{12} = 1.$$

This proves (3.30), and hence the lemma. □

We can now study the intersection form on $H^2(X, \mathbb{Z})$ for a smooth cubic surface $X \subset \mathbb{P}^3_{\mathbb{C}}$.

Lemma 3.4.12. *Let X be a smooth cubic surface. Then $H^2(X, \mathbb{Z}) \cong I_{1,6}$.*

Proof. We have $H^2(X, \mathbb{Z}) \cong I_{r,s}$ for some $r, s \in \mathbb{Z}_{\geq 1}$ by Theorem 3.4.9. We need to prove $r = 1$ and $s = 6$. This follows, as $\tau(X) = -5$, see Theorem 3.4.9. □

Let Λ be an odd unimodular lattice. A primitive vector $\alpha \in \Lambda$ is called *characteristic* if $(\alpha, \beta) \equiv (\beta, \beta) \bmod 2$ for all $\beta \in \Lambda$.

Consider the lattice $I_{1,6}$, see (3.28). Let $\alpha = (3, 1, 1, 1, 1, 1, 1)$ and define $e_1 = (0, 1, -1, 0, 0, 0, 0)$. Similarly, define $e_2 = (0, 0, 1, -1, 0, 0, 0)$, $e_3 = (0, 0, 0, 1, -1, 0, 0)$, $e_4 = (1, 0, 0, 0, 1, 1, 1)$, $e_5 = (0, 0, 0, 0, 1, -1, 0)$ and $(e_7 = (0, 0, 0, 0, 1, -1)$.

Lemma 3.4.13. *The element $\alpha \in I_{1,6}$ is characteristic. Moreover, the elements e_i with $i \in \{1, 2, 3, 4, 5, 7\}$ span the lattice $\alpha^\perp$, and their intersection matrix is an intersection matrix for the lattice $E_6(-1)$. In particular, $\alpha^\perp \cong E_6(-1)$.*

Proof. Exercise. □

Theorem 3.4.14. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$, and consider the sublattice $H^2(X, \mathbb{Z})_{prim} := \langle h \rangle^\perp$ of $H^2(X, \mathbb{Z})$. The lattice $H^2(X, \mathbb{Z})_{prim}$ is isomorphic to $E_6(-1)$.*

Proof. We claim that $h \in H^2(X, \mathbb{Z})$ is characteristic. Indeed, as $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ by Corollary 3.1.12, it suffices to show that $(L, L) \equiv (L, h) \bmod 2$ for every $L \in \mathrm{Pic}(X)$, which follows from Lemma 3.4.11. We then apply a general result for unimodular lattices: two primitive vectors $x, y \in \Lambda$ are in the same $O(\Lambda)$ orbit if and only if $(x, x) = (y, y)$ and either both are characteristic or both are not. As $\alpha = (3, 1, 1, 1, 1, 1, 1) \in I_{1,6}$ is characteristic by Lemma 3.4.13, and as $h \in H^2(X, \mathbb{Z})$ is characteristic by the above, it follows that α and the image of h in $I_{1,6}$ are in the same $O(I_{1,6})$-orbit. In particular, $H^2(X, \mathbb{Z})_{prim} = \langle h \rangle^\perp \cong \langle \alpha \rangle^\perp$, which is isomorphic to $E_6(-1)$, see Lemma 3.4.13. □

Chapter 4

Hodge theory: Part I

4.1 Lecture 5: Hodge decomposition theorem (statement)

4.1.1 Abstract Hodge structures

Definition 4.1.1. Let $k \in \mathbb{Z}_{\geq 0}$. An *integral Hodge structure of weight k* consists of a finitely generated abelian group $V_{\mathbb{Z}}$ and a decomposition of $V_{\mathbb{C}}$ into complex vector subspaces

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}, \tag{4.1}$$

such that $V^{p,q} = \overline{V^{q,p}}$. Here, $x \mapsto \bar{x}$ is the anti-linear $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on $V_{\mathbb{C}}$.

Let $V_{\mathbb{Z}}$ be a Hodge structure of weight k. Define the *Hodge filtration* $F^{\bullet}V_{\mathbb{C}}$ as the filtration

$$F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r}. \tag{4.2}$$

This is a decreasing filtration on $V_{\mathbb{C}}$ and satisfies the property that

$$F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} = V_{\mathbb{C}}.$$

One retrieves the Hodge decomposition (4.1) as follows:

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

Definition 4.1.2. Let $V_{\mathbb{Z}}$ be an integral Hodge structure of weight k. The *Weil operator* is the automorphism $C \colon V_{\mathbb{C}} \xrightarrow{\sim} V_{\mathbb{C}}$ defined by $C \cdot v = i^{p-q} v$ for $v \in V^{p,q}$. A *polarization* of $V_{\mathbb{Z}}$ is a bilinear form $Q \colon V \otimes V \to \mathbb{Z}$ which is $(-1)^k$-symmetric and such that, for the $\mathbb{C}$-bilinear extension $Q_{\mathbb{C}}$ of Q to $V_{\mathbb{C}}$, one has:

(1) The orthogonal complement of F^p is F^{k-p+1};

(2) The hermitian form $(u, v) \mapsto Q_{\mathbb{C}}(C \cdot u, \bar{v})$ is positive definite.

Remark that as the Weil operator is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, it descends to an automorphism $C\colon V_{\mathbb{R}} \xrightarrow{\sim} V_{\mathbb{R}}$, where $V_{\mathbb{R}} = V \otimes \mathbb{R}$. Moreover, the above definitions readily extend to subrings $R \subset \mathbb{R}$ other than $\mathbb{Z}$. In particular, on defines (polarized) *rational* and *real Hodge structures* in a similar way, replacing $\mathbb{Z}$ by $\mathbb{Q}$ or $\mathbb{R}$ respectively, in the definitions above. A Hodge structure is *polarizable* if it admits a polarization. The category of polarizable rational Hodge structures is abelian and semi-simple.

Example 4.1.3. Let C be a compact Riemann surface of genus $g \geq 1$. The exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$$

gives rise to a surjection

$$H^1(C, \mathbb{C}) = H^1(C, \mathbb{Z}) \otimes \mathbb{Z} \to H^1(C, \mathcal{O}_C)$$

whose kernel is the subspace $H^0(C, \Omega_C)$ of holomorphic one-forms on C. Indeed, the exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_C \to \Omega_C \to 0$$

induces a long exact sequence

$$\mathbb{C} = H^0(C, \mathcal{O}_C) \xrightarrow{0} H^0(C, \Omega_C) \hookrightarrow H^1(C, \mathbb{C}) \twoheadrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{0} H^1(C, \Omega_C) \xrightarrow{\sim} H^2(C, \mathbb{C}).$$

Consider the complex conjugate subspace $\overline{H^0(C, \Omega_C)}$ of $H^0(C, \Omega_C)$ in $H^1(C, \mathbb{C})$. As $H^0(C, \Omega_C) \cap \overline{H^0(C, \Omega_C)} = 0$ and $\dim H^1(C, \mathbb{C}) = 2g = 2 \cdot \dim H^0(C, \Omega_C)$, we have

$$H^1(C, \mathbb{C}) = H^0(C, \Omega_C) \oplus \overline{H^0(C, \Omega_C)}.$$

Therefore, the projection $H^1(C, \mathbb{C}) \to H^1(C, \mathcal{O}_C)$ induces a canonical isomorphism

$$\overline{H^0(C, \Omega_C)} = H^1(C, \mathcal{O}_C).$$

Finally, consider the pairing

$$H\colon H^1(C, \mathbb{C}) \times H^1(C, \mathbb{C}) \to \mathbb{C}, \quad H(\alpha, \beta) = i \cdot Q(\alpha, \bar{\beta}) = i \cdot \int_C \alpha \wedge \overline{\beta}.$$

Then $H(\alpha, \alpha) > 0$ for $\alpha \in H^0(C, \Omega_C) \subset H^1(C, \mathbb{C})$ non-zero.

The goal of Lectures 5 and 6 is to generalize the above example by proving the following:

Theorem 4.1.4 (Hodge)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Then for each integer $k \geq 0$, the singular cohomology group $H^k(X, \mathbb{Z})$ admits an integral Hodge structure of weight k in a canonical way, and $H^k(X, \mathbb{Z})_{prim}$ admits a sub-Hodge structure of $H^k(X, \mathbb{Z})$ which has a canonical polarization. Moreover, associating a weight k integral Hodge structure to a smooth projective variety X is contravariantly functorial in X, as well as compatible with cup-products and Gysin homomorphisms.*

4.1.2 Algebraic De Rham complex

We remark that although the above theorem only makes sense for varieties over $\mathbb{C}$, the *Hodge filtration* has a meaning in much larger generality. Namely, for a smooth projective variety X over a field k, one can consider the *algebraic De Rham complex*

$$\Omega_X^\bullet \coloneqq \left(0 \to \mathcal{O}_X \to \Omega_X \to \Omega_X^2 \to \cdots \to \Omega_X^{\dim X} \to 0\right), \tag{4.3}$$

as well as, for each integer $p \geq 0$, the sub-complex

$$\Omega_X^\bullet \supset F^p\Omega_X^\bullet = \left(0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^{\dim X} \to 0\right).$$

We may then define

$$\begin{aligned} H^k_{dR}(X/k) &= H^k\left(R\Gamma(X, \Omega_X^\bullet)\right) \qquad \text{and} \\ F^pH^k_{dR}(X/k) &= \operatorname{Im}\left(H^k\left(R\Gamma(X, F^p\Omega_X^\bullet)\right) \to H^k_{dR}(X/k)\right). \end{aligned} \tag{4.4}$$

If $k = \mathbb{C}$, then $H^k_{dR}(X/\mathbb{C}) = H^k(X, \mathbb{C})$. Indeed, one has $(\Omega_X^\bullet)^{an} = \Omega_{X^{an}}^\bullet$ by Serre's GAGA theorem, and this complexification $\Omega_{X^{an}}^\bullet$ of (4.3) provides a resolution of the constant sheaf $\mathbb{C}$ on X. The filtration $F^\bullet$ on $H^k(X, \mathbb{C})$ induced by (4.4) is exactly the Hodge filtration (4.2) associated to the Hodge structure on $H^k(X, \mathbb{Z})$ provided by Theorem 4.1.4.

There are two crucial differences between the complex case and the general case. First of all, even though for any smooth projective variety X over a field k, there is a canonical spectral sequence, de *Hodge to De Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}_{dR}(X/k) \tag{4.5}$$

with corresponding filtration on $H^k_{dR}(X/k)$ given by (4.4), this spectral sequence does (in contrast to the case $k = \mathbb{C}$) not always degenerate. Secondly, even if (4.5) degenerates for a certain smooth projective variety X over k, there is on the one hand no natural analogue of complex conjugation on $H^k_{dR}(X/k)$ if $k \neq \mathbb{C}$, and on the other in general no natural inclusion of $H^q(X, \Omega^p)$ into $H^{p+q}_{dR}(X/k)$.

4.1.3 Hodge star operator

Let X be a differentiable manifold, provided with a (smooth) Riemannian metric g. Suppose that X is oriented and compact, and let Vol be the volume form of X relative to g. This means that $\mathrm{Vol} \in A^n(X)$ is a smooth n-form, where $n = \dim(X)$, which is everywhere non-zero and such that for each $x \in X$, $\mathrm{Vol}(x) \in \Omega^n_{X,x}$ is the unique n-form which is positive on each oriented basis of $T_{X,x}$ and of norm one with respect to the induced metric on $\Omega^n_{X,x}$.

Observe that g induces a metric $(\ ,\ )_x$ on each vector space $\Omega^k_{X,x}$. For $\alpha, \beta \in A^k(X)$, one obtains a smooth function $(\alpha, \beta)\colon X \to \mathbb{R}$ sending x to $(\alpha, \beta)(x) = (\alpha_x, \beta_x)_x$. Define the *$L^2$-metric* on the space of real differentiable k-forms as follows:

$$(\ ,\ )_{L^2}\colon A^k(X) \times A^k(X) \to \mathbb{R}, \quad (\alpha, \beta)_{L^2} = \int_X (\alpha, \beta)\mathrm{Vol}.$$

For $x \in X$, consider the canonical isomorphism $\mathrm{Vol}(x)\colon \wedge^n \Omega_{X,x} \to \mathbb{R}$ provided by the volume form. We have a natural composition of isomorphisms

$$\bigwedge^{n-k} \Omega_{X,x} \xrightarrow[\sim]{p} \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \bigwedge^n \Omega_{X,x}\right) \xrightarrow[\sim]{\mathrm{Vol}(x)} \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \mathbb{R}\right).$$

Moreover, the metric $(\ ,\ )_x$ provides an isomorphism

$$m\colon \bigwedge^{k} \Omega_{X,x} \xrightarrow{\sim} \mathrm{Hom}\left(\bigwedge^{k} \Omega_{X,x}, \mathbb{R}\right).$$

Definition 4.1.5. Let X be an oriented compact Riemannian manifold. Define

$$\star_x\colon \bigwedge^{k} \Omega_{X,x} \xrightarrow{\sim} \bigwedge^{n-k} \Omega_{X,x} \qquad \text{as the isomorphism} \qquad p^{-1} \circ m.$$

Similarly, denote by $\star$ the induced isomorphism of vector bundles, respectively spaces of global sections:

$$\star\colon \Omega_X^k \xrightarrow{\sim} \Omega_X^{n-k}, \qquad \text{respectively} \qquad \star\colon A^k(X) \xrightarrow{\sim} A^{n-k}(X).$$

We call $\star\colon A^k(X) \xrightarrow{\sim} A^{n-k}(X)$ the *Hodge star operator*. We extend $\star$ by $\mathbb{C}$-linearity to an isomorphism $\star\colon A_{\mathbb{C}}^k(X) \xrightarrow{\sim} A_{\mathbb{C}}^{n-k}(X)$ of spaces of complex differential forms on X.

Lemma 4.1.6. *Let X be an oriented compact Riemannian manifold. We have*

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star\beta \qquad \forall\, \alpha, \beta \in A^k(X).$$

Proof. It suffices to show that for each $x \in X$, we have $(\alpha_x, \beta_x)_x \mathrm{Vol}_x = \alpha_x \wedge \star\beta_x$. By construction, the following diagram commutes:

$$\begin{tikzcd}
\bigwedge^k \Omega_{X,x} \arrow[r, "m"] \arrow[d, "\star"] & \mathrm{Hom}(\bigwedge^k \Omega_{X,x}, \mathbb{R}) \\
\bigwedge^{n-k} \Omega_{X,x} \arrow[r, "p"] & \mathrm{Hom}(\bigwedge^k \Omega_{X,x}, \bigwedge^n \Omega_{X,x}). \arrow[u, "\mathrm{Vol}(x)"]
\end{tikzcd}$$

The equality $(\alpha_x, \beta_x)_x \mathrm{Vol}_x = \alpha_x \wedge \star\beta_x$ follows from this. $\square$

Lemma 4.1.7. *Let X be an oriented compact Riemannian manifold. Consider the composition $\star^2 = \star \circ \star$. Then $\star^2 = (-1)^{k(n-k)}$ as maps $A^k(X) \to A^k(X)$.*

Proof. Indeed, for every $\alpha, \beta \in A^k(X)$, we have

$$\alpha_x \wedge \star\beta_x = (\alpha_x, \beta_x)\mathrm{Vol}_x = (\star\alpha_x, \star\beta_x)\mathrm{Vol}_x = \star\beta_x \wedge \star\star\,\alpha_x = (-1)^{k(n-k)} \star\star\alpha_x \wedge \star\beta_x.$$

As this holds for every $\beta_x \in A^k(X)$, we have $(-1)^{k(n-k)} \star\star\alpha_x = \alpha_x$ as desired. $\square$

Define an operator d^* as

$$d^*\colon A^k(X) \to A^{k-1}(X), \quad d^* = (-1)^k \star^{-1} d \star .$$

Lemma 4.1.8. *Let X be an oriented compact Riemannian manifold. Let $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in A^{k-1}(X)$ and $\beta \in A^k(X)$. Then*

$$(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}.$$

Proof. On the one hand, we have

$$(d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge \star\beta = \int_X d(\alpha \wedge \star\beta) - \int_X (-1)^{k-1}\alpha \wedge d \star \beta = -\int_X (-1)^{k-1}\alpha \wedge d \star \beta.$$

On the other hand, we have

$$(\alpha, d^*\beta)_{L^2} = (-1)^k \int_X \alpha \wedge d \star \beta.$$

We are done. □

Corollary 4.1.9. *Let X be an oriented compact Riemannian manifold. Let $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in A^k(X)$ and $\beta \in A^{k-1}(X)$. Then*

$$(d^*\alpha, \beta)_{L^2} = (\alpha, d\beta)_{L^2}.$$

Proof. Let $n = \dim(X)$. Using Lemma 4.1.8 and the fact that $\star$ preserves the L^2-metric, we get

$$\begin{aligned}
(d^*\alpha, \beta)_{L^2} &= \left((-1)^k \star^{-1} d \star \alpha, \beta\right)_{L^2} \\
&= (-1)^k \cdot \left(\star \star^{-1} d \star \alpha, \star\beta\right)_{L^2} \\
&= (-1)^k \cdot (d \star \alpha, \star\beta)_{L^2} \\
&= (-1)^k \cdot (\star\alpha, d^* \star \beta)_{L^2} \\
&= (-1)^k \cdot \left(\star\alpha, (-1)^{n-k+1} \star^{-1} d \star \star\beta\right)_{L^2} \\
&= (-1)^k \cdot (-1)^{n-k+1} \cdot (-1)^{(k-1)(n-k+1)} \left(\star\alpha, \star^{-1} d\beta\right)_{L^2} \\
&= (-1)^k \cdot (-1)^{(n-k+1)k} \left(\star \star \alpha, d\beta\right)_{L^2} \\
&= (-1)^k \cdot (-1)^{(n-k+1)k} \cdot (-1)^{k(n-k)} \cdot (\alpha, d\beta)_{L^2} \\
&= (-1)^k \cdot (-1)^{k(n-k+n-k+1)} \cdot (\alpha, d\beta)_{L^2} \\
&= (\alpha, d\beta)_{L^2} .
\end{aligned}$$

This proves the corollary. □

Let X be an oriented and compact Riemannian manifold. Let $x \in X$ and consider the metric $(\ ,\ )_x\colon \Omega^k_{X,x} \times \Omega^k_{X,x} \to \mathbb{R}$. We can extend it $\mathbb{C}$-bilinearly to obtain a $\mathbb{C}$-bilinear form

$$(\ ,\ )_x\colon \Omega^k_{X,x,\mathbb{C}} \times \Omega^k_{X,x,\mathbb{C}} \to \mathbb{C}$$

and hence an $\mathbb{R}$-bilinear form

$$\langle\ ,\ \rangle_x \colon \Omega^k_{X,x,\mathbb{C}} \times \Omega^k_{X,x,\mathbb{C}} \to \mathbb{C}, \quad \langle \alpha_x, \beta_x \rangle_x = \left(\alpha_x, \overline{\beta}_x\right)_x. \tag{4.6}$$

Let $\alpha_x = \sum_i \lambda_i u_i \in \Omega^k_{X,x,\mathbb{C}}$ with $\lambda_i \in \mathbb{C}$ and $u_i \in \Omega^k_{X,x}$. Similarly, let $\beta_x = \sum_j \mu_j v_j \in \Omega^k_{X,x,\mathbb{C}}$ with $\mu_j \in \mathbb{C}$ and $v_j \in \Omega^k_{X,x}$. Then

$$\langle \alpha_x, \beta_x \rangle_x = \sum_{i,j} \lambda_i \cdot \overline{\mu}_j \cdot (u_i, v_j)_x = \overline{\sum_{j,i} \mu_j \cdot \overline{\lambda}_i \cdot (v_j, u_i)_x} = \overline{\langle \beta_x, \alpha_x \rangle_x}.$$

Next, let $\{e_1, \ldots, e_r\} \subset \Omega^k_{X,x}$ be an orthonormal basis for $(\ ,\ )_x$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, and define $\alpha = \sum_{i=1}^r \lambda_i e_i$. Then

$$\langle \alpha_x, \alpha_x \rangle_x = \sum_{i,j} \lambda_i \cdot \overline{\lambda}_j \cdot (e_i, e_j)_x = \sum_{i=1}^r |\lambda_i|^2.$$

We conclude that (4.6) is a positive definite hermitian form, i.e. a hermitian metric.

Notice that, for $x \in X$, the hermitian metric $\langle\ ,\rangle_x$ satisfies the property that

$$\langle \alpha_x, \beta_x \rangle_x \mathrm{Vol}_x = \alpha_x \wedge \overline{\star\beta_x}, \qquad \alpha_x, \beta_x \in \Omega^k_{X,x,\mathbb{C}}.$$

For $\alpha, \beta \in A^k_{\mathbb{C}}(X)$, the function $\langle \alpha, \beta \rangle \colon X \to \mathbb{C}$ defined as $\langle \alpha, \beta \rangle(x) = \langle \alpha_x, \beta_x \rangle_x$ is smooth, and we obtain a metric, the *Hermitian L^2-metric*, on the space of complex differentiable forms:

$$\langle\ ,\ \rangle_{L^2} \colon A^k_{\mathbb{C}}(X) \times A^k_{\mathbb{C}}(X) \to \mathbb{C}, \quad \langle \alpha, \beta \rangle_{L^2} = \int_X \langle \alpha, \beta \rangle \mathrm{Vol} = \int_X \alpha \wedge \overline{\star\beta} = \left(\alpha, \overline{\beta}\right)_{L^2}.$$

4.2 Lecture 6: Hodge decomposition theorem (proof)

4.2.1 *Complex differentiable forms*

Let X be an n-dimensional complex manifold. For $k \geq 0$, let $A^k_{\mathbb{C}}(X)$ be the space of complex differentiable forms on X, and consider the differential

$$d \colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X).$$

It decomposes as $d = \partial + \bar{\partial}$. To explain this, define $\Omega^{p,q}_X = \wedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$. Then, by Lemma 3.1.2, we have:

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega_{X,\mathbb{C}} = \bigwedge^k \left(\Omega^{1,0}_X \oplus \Omega^{0,1}_X\right) = \bigoplus_{p+q=k} \bigwedge^p \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X = \bigoplus_{p+q=k} \Omega^{p,q}_X.$$

Let $f \colon X \to \mathbb{C}$ be a complex differentiable function on X. In local coordinates $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, we can write

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i =: \partial f + \bar{\partial} f. \tag{4.7}$$

We conclude that, for our $f \in A^0_{\mathbb{C}}(X)$, we have

$$df = \partial f + \bar{\partial} f \tag{4.8}$$

for unique $\partial f \in A^{1,0}(X)$ and $\bar{\partial} f \in A^{0,1}(X)$, where $A^{p,q}(X)$ is the space of global sections of the bundle $\Omega^{p,q}_X$.

More generally, let $\alpha \in A^{p,q}(X)$ be a global section of $\Omega^{p,q}_X$. Then locally, α is of the form $\sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$ with $\alpha_{I,J}$ of type $(0,0)$. Consequently, $d\alpha$ can locally be written as

$$d\left(\sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J.$$

Now by (4.7), we have $d\alpha_{I,J} = \partial\alpha_{I,J} + \bar{\partial}\alpha_{I,J}$. Remark that $\sum_{I,J} \partial\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$ is a form of type $(p+1,q)$. Similarly, $\sum_{I,J} \bar{\partial}\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$ is a form of type $(p,q+1)$. We conclude:

Lemma 4.2.1. *Let X be a complex manifold of dimension n. There are unique operators ∂ and $\bar{\partial}$ on $A^k_{\mathbb{C}}(X)$ such that $\partial(A^{p,q}(X)) \subset A^{p+1,q}(X)$ and $\bar{\partial} A^{p,q}(X) \subset A^{p,q+1}(X)$ and such that the differential $d\colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X)$ decomposes as $d = \partial + \bar{\partial}$.* □

4.2.2 Hermitian manifolds

Let X be an n-dimensional compact *hermitian manifold*. Thus, X is a complex manifold of dimension n equipped with a Riemannian metric g that preserves the almost complex structure $I\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ on the real tangent bundle $T_{X,\mathbb{R}}$ of X.

Lemma 4.2.2. *The operators $\partial^* := - \star \bar{\partial} \star$ and $\bar{\partial}^* = - \star \partial \star$ are adjoints of ∂ and $\bar{\partial}$ respectively, for the hermitian metric $\langle\ ,\ \rangle_{L^2}$ on the space of complex differential forms.*

Proof. We prove the result only for $\bar{\partial}$; the other case is similar. Let $k \geq 1$. For $u, v \in A^k_{\mathbb{C}}(X)$, we have $\langle u, v\rangle_{L^2} = \int_X u \wedge \overline{\star v}$. In particular, for $\alpha \in A^{k-1}_{\mathbb{C}}(X)$ and $\beta \in A^k_{\mathbb{C}}(X)$, we have

$$\langle \bar{\partial}\alpha, \beta\rangle_{L^2} = \int_X \bar{\partial}\alpha \wedge \overline{\star\beta}.$$

As $\int_X \bar{\partial}\phi = 0$ for every $\phi \in A^{2n-1}_{\mathbb{C}}(X)$, we get (via the Leibniz formula) that

$$\langle \bar{\partial}\alpha, \beta\rangle_{L^2} = \int_X \bar{\partial}\alpha \wedge \overline{\star\beta} = -\int_X (-1)^{k-1}\alpha \wedge \overline{\partial\star\beta} = -\int_X (-1)^{k-1}\alpha \wedge \overline{\star\star^{-1}\partial\star\beta}.$$

Moreover, $\star^{-1}\partial\star\beta = (-1)^{k-1}\star\partial\star\beta$ because $\deg(\partial\star\beta) = 2n-k+1 = 2n - \deg(\alpha)$. Therefore,

$$-\int_X (-1)^{k-1}\alpha \wedge \overline{\star\star^{-1}\partial\star\beta} = -\int_X \alpha \wedge \overline{\star\star\partial\star\beta} = (\alpha, \bar{\partial}^*\beta)_{L^2}$$

and the result follows. □

Definition 4.2.3. Let (X, g) be an oriented compact Riemannian manifold. Define $\Delta_d = dd^* + d^*d$. If X has a complex structure compatible with g, let $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. We say that a form $\alpha \in A^k_{\mathbb{C}}(X)$ is Δ_d*-harmonic* if $\Delta_d(\alpha) = 0$.

Lemma 4.2.4. *Let (X, g) be an oriented compact Riemannian manifold, and consider a complex differentiable k-form $\alpha \in A^k_{\mathbb{C}}(X)$. We have*

$$(\alpha, \Delta_d\alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2}.$$

Proof. Indeed, by Lemma 4.1.8 and Corollary 4.1.9, we have

$$(\alpha, \Delta_d\alpha)_{L^2} = (\alpha, dd^*\alpha + d^*d\alpha)_{L^2} = (\alpha, dd^*\alpha)_{L^2} + (\alpha, d^*d\alpha)_{L^2} = (d^*\alpha, d^*\alpha)_{L^2} + (d\alpha, d\alpha)_{L^2}.$$

This proves the lemma. □

Corollary 4.2.5. *Let X be an oriented compact Riemannian manifold. For each integer $k \in \mathbb{Z}_{\geq 1}$, we have* $\mathrm{Ker}(\Delta_d) = \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*) \subset A^k(X)$.

Proof. The inclusion $\mathrm{Ker}(\Delta_d) \supset \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$ being clear, we claim that any $\alpha \in \mathrm{Ker}(\Delta_d)$ is killed by d and by d^*. By Lemma 4.2.4,

$$0 = (\alpha, \Delta_d\alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2}.$$

As $(\ ,\ )_{L^2}$ is positive definite, this implies that $d\alpha$ and $d^*\alpha$ must be zero. □

Theorem 4.2.6. *Let (X, g) be an oriented compact Riemannian manifold. For $k \geq 0$, consider the Laplacian $\Delta_d\colon A^k(X) \to A^k(X)$ and its kernel $\mathscr{H}^k = \mathrm{Ker}(\Delta_d)$. We have*

$$A^k(X) = \mathscr{H}^k \oplus \Delta_d\left(A^k(X)\right).$$

Proof. This follows from [Voi02, Corollaire 5.20] and [Voi02, Théorème 5.22]. □

Theorem 4.2.7. *Let (X, g) be an oriented compact Riemannian manifold. Any k-form $\alpha \in \mathrm{Ker}(\Delta_d) \subset A^k(X)$ is closed. Moreover, the linear map*

$$\begin{aligned} \mathrm{Ker}(\Delta_d) = \{\Delta_d\text{-harmonic } k\text{-forms on } X\} = \mathscr{H}^k &\to H^k_{dR}(X, \mathbb{R}) = H^k(X, \mathbb{R}), \\ \alpha &\mapsto [\alpha], \end{aligned} \tag{4.9}$$

that sends a harmonic form to its De Rham cohomology class, is an isomorphism.

Proof. The injectivity of (4.9) can be seen as follows. Let $\beta \in \mathscr{H}^k$ and suppose that $[\beta] = 0$. Then $\beta = d\alpha$ for some $k-1$-form α on X. Moreover, as $\Delta_d(\beta) = 0$, we have $d^*(\beta) = 0$ by Corollary 4.2.5. Hence $d^*d(\alpha) = 0$. But then, by Lemma 4.1.8, we obtain

$$0 = (\alpha, d^*d(\alpha))_{L^2} = (d\alpha, d\alpha)_{L^2}$$

which implies that $d\alpha = \beta = 0$. Thus, (4.9) is injective.

As for the surjectivity of (4.9), let $\beta \in A^k(X)$ be a closed form. By Theorem 4.2.7, we can write $\beta = \alpha + \Delta_d\gamma$ for a harmonic form α. Thus,

$$\beta = \alpha + dd^*\gamma + d^*d\gamma.$$

As β is closed by assumption, and as α is closed by Corollary 4.2.5, we have $dd^*(d\gamma) = 0$. Hence, by Corollary 4.1.9, we have

$$0 = (d\gamma, dd^*(d\gamma))_{L^2} = (d^*d\gamma, d^*d\gamma)_{L^2}$$

which implies that $d^*d\gamma = 0$. Therefore, we have $\beta = \alpha + dd^*\gamma$, and we deduce that $[\beta] = [\alpha] \in H^k(X, \mathbb{R})$. The k-form α is harmonic, and we are done. □

4.2.3 Kähler manifolds

Lemma 4.2.8. *Let V be a complex vector space of finite dimension. Consider the sets S_1, S_2 and S_3 defined as follows:*

$S_1 =$ *The set of hermitian forms $h\colon V \times V \to \mathbb{C}$.*

$S_2 =$ *The set of symmetric $\mathbb{R}$-bilinear forms $g\colon V \times V \to \mathbb{R}$ such that $g(i \cdot u, i \cdot v) = g(u, v)$ for each $u, v \in V$.*

$S_3 =$ *The set of anti-symmetric $\mathbb{R}$-bilinear forms $\omega\colon V \times V \to \mathbb{R}$ such that the $\mathbb{C}$-bilinear extension $\omega_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is zero on $V^{1,0} \times V^{1,0}$ and on $V^{0,1} \times V^{0,1}$.*

Let $h \in S_1$. The function $-\Im h\colon (u, v) \mapsto -\Im(h(u, v))$ defines an element $\omega \in S_3$. Moreover, for $\omega \in S_3$, the function $g(u, v) = \omega(u, i \cdot v)$ defines an element $g \in S_2$. This construction defines bijections $S_1 \cong S_2 \cong S_3$.

Proof. Exercise. □

Lemma 4.2.9. *Let V be a finite dimensional complex vector space and $h\colon V \times V \to \mathbb{C}$ and $g\colon V \times V \to \mathbb{R}$ be a hermitian (resp. symmetric $\mathbb{R}$-bilinear) form such that g and h correspond to each other via the bijection in Lemma 4.2.8. Then h is positive definite as a hermitian form if and only if g is positive definite as a symmetric bilinear form.*

Proof. Exercise. □

Definition 4.2.10. We call an anti-symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$ *of type* $(1, 1)$ if it satisfies property (3) above. We say that ω is *positive* if the hermitian form $h\colon V \times V \to \mathbb{C}$ is positive definite.

Let X be a hermitian manifold. Let g be the Riemannian metric of X. As g is compatible with the almost complex structure of X, it yields a hermitian metric on the tangent bundle $T_{X,\mathbb{R}}$, see Lemmas 4.2.8 and 4.2.9. In other words, for every $x \in X$, the real tangent bundle $T_{X,x,\mathbb{R}}$ with its natural complex structure $I\colon T_{X,x,\mathbb{R}} \to T_{X,x,\mathbb{R}}$ has a hermitian metric h_x, and these metrics vary differentiably with x.

Definition 4.2.11. We say that the hermitian metric h on $T_{X,\mathbb{R}}$ is *Kähler* if the real differentiable two-form

$$\omega = -\Im(h) \in A^{1,1}(X) \cap A^2_{\mathbb{R}}(X)$$

is closed. If this is the case, we call (X, ω) a *Kähler manifold*.

Theorem 4.2.12. *Let (X, ω) be a Kähler manifold. Let $\Delta_d, \Delta_\partial, \Delta_{\bar\partial}$ the Laplacians associated to the respective operators $d, \partial, \bar\partial$. Then $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.*

Proof. See [Voi02, Théorème 6.7]. □

Corollary 4.2.13. *Let (X, ω) be a Kähler manifold. Then $\Delta_d(A^{p,q}(X)) \subset A^{p,q}(X)$.*

Proof. Let $\alpha \in A^{p,q}(X)$. Then $\Delta_\partial(\alpha) = \partial^*\partial(\alpha)+\partial\partial^*(\alpha) \in A^{p,q}(X)$. The result follows because of Theorem 4.2.12. $\square$

Corollary 4.2.14. *Let (X, ω) be a Kähler manifold. Let $\alpha \in A^k_{\mathbb{C}}(X)$. Define*

$$\mathscr{H}_{\mathbb{C}} = \operatorname{Ker}\left(\Delta_d \colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X)\right), \quad \mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} \cap A^{p,q}(X) \subset \mathscr{H}^k_{\mathbb{C}}.$$

Thus, $\mathscr{H}^{p,q} \subset \mathscr{H}^k_{\mathbb{C}}$ is the space of Δ_d-harmonic forms of type (p,q).

(1) If α is harmonic, then each of its components $\alpha^{p,q} \in A^{p,q}(X)$ is harmonic.

(2) There is a canonical decomposition

$$\mathscr{H}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathscr{H}^{p,q}. \tag{4.10}$$

Proof. 1. Indeed, the relation $0 = \Delta_d(\alpha) = \sum_{p+q=k} \Delta_d(\alpha^{p,q})$ implies that $\Delta_d(\alpha^{p,q}) = 0$ by Corollary 4.2.13. 2. This is immediate from item 1. $\square$

Lemma 4.2.15. *Let (X, ω) be a Kähler manifold. Let $\mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} \cap A^{p,q}(X)$ be the space of Δ_d-harmonic forms of type (p,q), $p+q=k$. Let $K^{p,q} \subset H^k(X,\mathbb{C})$ be the space of degree k cohomology classes $[\alpha]$ that admit a closed representative $\alpha' \in [\alpha]$ such that $\alpha' \in A^{p,q}(X)$. The image of the natural map*

$$\mathscr{H}^{p,q} \to H^k(X,\mathbb{C}) \tag{4.11}$$

equals exactly $K^{p,q}$.

Proof. Let $H^{p,q}(X)$ be the image of (4.11). As the elements of $\mathscr{H}^{p,q} \subset A^k_{\mathbb{C}}(X)$ are closed of type (p,q), we have $H^{p,q}(X) \subset K^{p,q}$. Conversely, let $[\omega] \in K^{p,q} \subset H^k(X,\mathbb{C})$ with $\omega \in A^{p,q}(X)$ such that $d\omega = 0$. By Theorem 4.2.7, we can uniquely write

$$\omega = \alpha + \Delta_d\beta,$$

with $\Delta_d\alpha = 0$ and $\beta \in A^k_{\mathbb{C}}(X)$. By looking at the components of type (p,q) with respect to (4.10), it follows from Corollary 4.2.13 that we can write

$$\omega = \omega^{p,q} = \alpha^{p,q} + (\Delta_d\beta)^{p,q} = \alpha^{p,q} + \Delta_d\beta^{p,q}, \quad \alpha^{p,q} \in A^{p,q}(X), \quad \beta^{p,q} \in A^{p,q}(X),$$

where $\alpha^{p,q}$ is harmonic. As ω and $\alpha^{p,q}$ are closed, we have that

$$\Delta_d\beta^{p,q} = dd^*\beta^{p,q} + d^*d\beta^{p,q}$$

is closed, hence $dd^*(d\beta^{p,q}) = 0$, which implies (via Corollary 4.1.9) that $d\beta^{p,q} = 0$. Therefore, $\Delta_d\beta^{p,q} = dd^*\beta^{p,q}$ is exact, hence

$$[\omega] = [\alpha^{p,q}] \in H^k(X,\mathbb{C}).$$

It follows that $[\omega]$ can be represented by a harmonic form of type (p,q), that is, we have $[\omega] \in H^{p,q}(X)$. Thus, $K^{p,q} \subset H^{p,q}(X)$ and we win. $\square$

We proceed to show that any smooth projective variety is naturally a Kähler manifold. To do so, we need to show how to associate a closed real two-form of type $(1,1)$ to any pair (L,h), where L is a hermitian line bundle on a complex manifold X and h a hermitian metric on L. Let $\{U_i\}_{i\in I}$ be an open cover of X that trivializes L. For each i, we get a nowhere vanishing holomorphic section $\sigma_i\colon U_i \to L$. Let $i,j \in J$ with $U_{ij} = U_i \cap U_j \neq \emptyset$. There exists a holomorphic function $g_{ij}\colon U_{ij} \to \mathbb{C}^*$ such that

$$\sigma_i = g_{ij} \cdot \sigma_j.$$

Having fixed the above trivialization of L, for each $x \in X$, the hermitian metric h_x on L_x is determined by a non-zero element in $\mathbb{C}$. Consider the function

$$h_i\colon U_i \to \mathbb{R}, \quad z \mapsto h(\sigma_i(z), \sigma_i(z)).$$

Then $h_i(z) > 0$ for $z \in U_i$, and on $U_{ij} = U_i \cap U_j$, we have

$$h_i(z) = h(\sigma_i(z), \sigma_i(z)) = h\left(g_{ij}(z)\cdot\sigma_j(z), g_{ij}(z)\cdot\sigma_j(z)\right) = |g_{ij}(z)|^2 \cdot h_j(z).$$

We obtain differentiable two-forms

$$\omega_i = \frac{1}{2i\pi}\partial\bar{\partial}\log h_i \in A^2(U_i), \qquad i \in I.$$

Notice that, on U_{ij}, we have

$$\omega_i|_{U_{ij}} = \frac{1}{2i\pi}\partial\bar{\partial}\log h_i = \frac{1}{2i\pi}\partial\bar{\partial}\log\left(|g_{ij}|^2\cdot h_j\right) = \frac{1}{2i\pi}\partial\bar{\partial}\log|g_{ij}|^2 + \omega_j|_{U_{ij}}.$$

As

$$\frac{1}{2i\pi}\partial\bar{\partial}\log|g_{ij}|^2 = 0$$

we have that ω_i and ω_j coincide on U_{ij}. Therefore, there exists a unique two-form

$$\omega \in A^2(X)$$

such that $\omega|_{U_i} = \omega_i$ for each $i \in I$. Notice that:

(1) The two-form $\omega \in A^2(X)$ is closed. Indeed, $\omega_i \in A^2(U_i)$ is exact.

(2) The two-form ω lies in $A^{1,1}(X) \subset A^2_{\mathbb{C}}(X)$, i.e. ω is of type $(1,1)$.

We have proved:

Lemma 4.2.16. *Let X be a complex manifold. The above construction allows one to associate a closed two-form $\omega \in A^2(X)$ of type $(1,1)$ to any pair (L,h) where L is a line bundle on X and h a hermitian metric on L.* □

Exercise 4.2.17. Show that the construction $(L,h) \mapsto \omega$, where L is a line bundle and h a hermitian metric on L, does not depend on the trivialization $\{U_i\}_{i\in I}$ for L.

Lemma 4.2.18. *Let X be a smooth projective variety. Then X defines a Kähler manifold (X, ω) in a natural way.*

Proof. As X admits a closed embedding into projective space, it suffices to prove this for the projective space $\mathbb{P}^n(\mathbb{C})$. Consider the tautological line bundle

$$S = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-1) \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}.$$

Let h be the standard hermitian metric on $\mathbb{C}^{n+1}$. It induces a hermitian metric on the holomorphic vector bundle

$$\mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1} \to \mathbb{P}^n(\mathbb{C})$$

and hence, by restriction, one on S. Let h^* be the induced hermitian metric on $S^* = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. By Lemma 4.2.16, we obtain a closed two-form

$$\omega \in A^2(\mathbb{P}^n(\mathbb{C})) \cap A^{1,1}(\mathbb{P}^n(\mathbb{C})). \tag{4.12}$$

It remains to prove that ω is positive, in the sense of Definition 4.2.10. We leave this as an exercise for the reader. □

Exercise 4.2.19. Prove that the two-form (4.12) is positive.

Proof of Theorem 4.1.4. Let X be a smooth projective variety. By Lemma 4.2.18, the variety X defines a Kähler manifold (X, ω) in a natural way. Moreover, by Theorem 4.2.7 and Corollary 4.2.14, we have canonical isomorphisms

$$\bigoplus_{p+q=k} \mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} = H^k(X, \mathbb{C}).$$

Define $H^{p,q}(X)$ as the image of $\mathscr{H}^{p,q}$ in $H^k(X, \mathbb{C})$ under $\mathscr{H}^k_{\mathbb{C}} \xrightarrow{\sim} H^k(X, \mathbb{C})$. It remains to show that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. This follows from Lemma 4.2.15, which shows that $H^{p,q}(X) = K^{p,q}$, where $K^{p,q} \subset H^k(X, \mathbb{C})$ is the space of De Rham cohomology classes $[\alpha]$ that admit a closed representative $\alpha' \in [\alpha]$ of type (p, q). □

Proposition 4.2.20. *Let X be a smooth projective variety over $\mathbb{C}$. For each $p, q \geq 0$, there is a canonical isomorphism $H^{p,q}(X) = H^q(X, \Omega^p_X)$.*

Proof. Let $n = \dim(X)$. The operator $\bar{\partial}$ induces a complex of sheaves

$$0 \to \Omega^p_X \to \Omega^{p,0}_X \to \Omega^{p,1}_X \to \cdots \to \Omega^{p,q}_X \to \cdots \to \Omega^{p,n}_X \to 0,$$

and this complex is exact, see [Voi02, Proposition 4.19]. In other words, the natural map of complexes

$$\Omega^p_X \to (\Omega^{p,\bullet}_X)$$

defines a resolution of Ω^p_X, and this resolution is in fact acyclic. As $\Gamma(X, \Omega^{p,q}_X) = A^{p,q}(X)$ by definition, we obtain a canonical isomorphism

$$H^q(X, \Omega^p_X) = \frac{\operatorname{Ker}\left(A^{p,q}(X) \xrightarrow{\bar{\partial}} A^{p,q+1}(X)\right)}{\operatorname{Im}\left(A^{p,q-1}(X) \xrightarrow{\bar{\partial}} A^{p,q}(X)\right)}.$$

Moreover, if we put a Kähler metric on X, there is a canonical isomorphism

$$H^{p,q}(X) = \mathrm{Ker}\,(\Delta_d) \cap A^{p,q}(X) = \mathrm{Ker}(\Delta_{\bar\partial}) \cap A^{p,q}(X),$$

and the natural map

$$\mathrm{Ker}(\Delta_{\bar\partial}) \cap A^{p,q}(X) \to \frac{\mathrm{Ker}\left(A^{p,q}(X) \xrightarrow{\bar\partial} A^{p,q+1}(X)\right)}{\mathrm{Im}\left(A^{p,q-1}(X) \xrightarrow{\bar\partial} A^{p,q}(X)\right)} \tag{4.13}$$

is an isomorphism. Indeed, if $\alpha = \partial\beta$ is of type (p,q) and $\bar\partial\alpha = \bar\partial^*\alpha = 0$, then $\bar\partial^*\partial\beta = 0$, which by the adjoint property of $\bar\partial^*$ with respect to $\bar\partial$ (see Lemma 4.2.2) implies that $\partial\beta = \alpha = 0$. This proves the injectivity of (4.13). For the surjectivity of (4.13), see [Voi02, Théorème 5.24].

It remains to verify that the so-constructed isomorphism $H^q(X, \Omega_X^p) \cong H^{p,q}(X)$ is truly canonical, i.e. does not depend on the Kähler metric that we chose to define it. Recall the Hodge to De Rham spectral sequence, see Section 4.1.2:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}(X, \mathbb{C}). \tag{4.14}$$

As we have $\dim H^k(X,\mathbb{C}) = \sum_{p+q=k} H^q(X, \Omega_X^p)$ by Theorem 4.1.4, the spectral sequence (4.14) degenerates. Therefore, there are canonical isomorphisms

$$F^pH^k(X,\mathbb{C})/F^{p+q}(X,\mathbb{C}) = E_\infty^{p,q} = E_1^{p,q} = H^q(X, \Omega_X^p).$$

Finally, the filtration F^p on $H^k(X,\mathbb{C})$ induced by (4.14) is exactly the Hodge filtration (4.2) attached to the Hodge structure on $H^k(X,\mathbb{Z})$ that Theorem 4.1.4 provides, as follows from [Voi02, Proposition 7.5]. In particular, we have

$$F^pH^k(X,\mathbb{C})/F^{p+1}H^k(X,\mathbb{C}) = H^{p,q}(X).$$

This finishes the proof of the proposition. □

4.2.4 *Example: complex elliptic curves*

Definition 4.2.21. (1) A *complex elliptic curve* is a smooth cubic $E \subset \mathbb{P}^2_{\mathbb{C}}$ equipped with a point $\mathcal{O} \in E(\mathbb{C})$. If $(E_1, \mathcal{O}_1)$ and $(E_2, \mathcal{O}_2)$ are complex elliptic curves, then a *morphism of elliptic curves* $(E_1, \mathcal{O}_1) \to (E_2, \mathcal{O}_2)$ is a morphism of varieties $\phi\colon E_1 \to E_2$ such that $\phi(\mathcal{O}_1) = \mathcal{O}_2$. In this way, elliptic curves form a category.

(2) A *complex torus* is the quotient of a finite dimensional complex vector space $V \cong \mathbb{C}^n$ by a discrete subgroup $\Lambda \subset V$ with $\Lambda \otimes \mathbb{R} = V$. A *morphism of complex tori* is a holomorphic group homomorphism. Thus, complex tori form a category.

Proposition 4.2.22. *There are three compatible functors as in the following diagram:*

$$\begin{tikzcd}
 & (\textit{Elliptic curves } E_{/\mathbb{C}}) \arrow[dl, bend right] \arrow[dr, bend left] & \\
(\textit{One-dimensional complex tori}) & & (\textit{Weight one Hodge structures on } \mathbb{Z}^2) \arrow[ll]
\end{tikzcd}$$

These three functors are equivalences of categories.

Remark 4.2.23. It follows from Proposition 4.2.22 that complex elliptic curves are algebraic groups in a natural way, where an *algebraic group* is an algebraic variety X of finite type over a field k which is a group object in the category of schemes over k. The fact that complex elliptic curves E are algebraic groups can be proven directly, by constructing an algebraic group law $E \times E \to E$ explicitly using the defining equation for E in $\mathbb{P}^2_{\mathbb{C}}$. This can be done for smooth cubics $E \subset \mathbb{P}^2_k$ over any field k, as long as $E(k) \neq \emptyset$, leading to the notion of *elliptic curve over* k. For more on this, see [Sil09].

Proof of Proposition 4.2.22. Let E be a complex elliptic curve. By Theorem 4.1.4 (or by Example 4.1.3), there is a natural Hodge structure of weight one on $H^1(E,\mathbb{Z}) \cong \mathbb{Z}^2$, which defines the functor on the right. Next, let $V_{\mathbb{Z}}$ be any weight one Hodge structure on $\mathbb{Z}^2$. The composition

$$V_{\mathbb{R}} \to V_{\mathbb{C}} \to V^{0,1}$$

is an isomorphism, hence the composition $V_{\mathbb{Z}} \to V_{\mathbb{C}} \to V^{0,1}$ is an embedding, and

$$X = V^{0,1}/V_{\mathbb{Z}}$$

is a complex torus of dimension one. These two constructions are functorial, and compatible with the functor that associates the complex torus $X = H^1(E,\mathcal{O}_E)/H^1(E,\mathbb{Z})$ to an elliptic curve E over $\mathbb{C}$. It remains to show that:

($\star$) Any one-dimensional complex torus V/Λ is isomorphic to $H^1(E,\mathcal{O}_E)/H^1(E,\mathbb{Z})$ for a smooth complex elliptic curve $E \subset \mathbb{P}^2_{\mathbb{C}}$.

($\star\star$) If E_1 and E_2 are complex elliptic curves, and X_1 and X_2 the associated complex tori, then any holomorphic group homomorphism $X_1 \to X_2$ is induced by a unique morphism of algebraic groups $E_1 \to E_2$.

In fact, we claim that ($\star\star$) follows from ($\star$). Indeed, if ($\star$) holds, then any one-dimensional complex torus is projective, hence any holomorphic map between two one-dimensional complex tori is uniquely algebraizable by the GAGA principle.

To prove ($\star$), we may assume that $V = \mathbb{C}$, so that Λ is a lattice in $\mathbb{C}$. Then

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z} \cdot \omega \qquad \text{for some } \omega \in \mathbb{C} \text{ with } \Im(\omega) > 0.$$

Consider the meromorphic function

$$\wp\colon \mathbb{C} \to \mathbb{C}, \quad \wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z-n-m\cdot\omega)^2} - \frac{1}{(n+m\cdot\omega)^2} \right).$$

Notice that $\wp$ is periodic with respect to Λ, and that the poles of $\wp$ are given by $z = n + m\cdot\omega$ for $(n,m) \in \mathbb{Z}^2$. The function

$$\mathbb{C} \to \mathbb{P}^2(\mathbb{C}), \quad z \mapsto [\wp(z)\colon \wp'(z)\colon 1]$$

defines a holomorphic and Λ-periodic function, hence induces a morphism

$$X = \mathbb{C}/\Lambda \to \mathbb{P}^2(\mathbb{C}). \tag{4.15}$$

In fact, (4.15) is a closed embedding. To determine its image, define, for $k \in \mathbb{Z}_{\geq 2}$,

$$G_{2k}(\Lambda) = \sum_{x \in \Lambda - \{(0,0)\}} x^{-2k}.$$

Let $g_2(\Lambda) = 60 \cdot G_4(\Lambda)$ and $g_3(\Lambda) = 140 \cdot G_6(\Lambda)$. Then one has:

$$\wp'(z)^2 = 4 \cdot \wp(z)^3 - g_2(\Lambda) \cdot \rho(z) - g_3(\Lambda), \qquad z \in \mathbb{C} \setminus \Lambda. \tag{4.16}$$

Therefore, the closed embedding (4.15) identifies $X = \mathbb{C}/\Lambda$ with the plane cubic curve $E \subset \mathbb{P}^2(\mathbb{C})$ of affine equation $y^2 = 4x^3 - g_2(\Lambda) \cdot x - g_3(\Lambda)$. □

Exercise 4.2.24. Prove that (4.16) holds.

Exercise 4.2.25. Let $V_\mathbb{R}$ be a finite dimensional real vector space, and let $k \in \mathbb{Z}_{\geq 0}$. Define $V_\mathbb{C} = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. Prove that to give a Hodge structure of weight k on $V_\mathbb{R}$ is to give a continuous homomorphism

$$\rho \colon \mathbb{C}^* \to \mathrm{GL}(V_\mathbb{C})$$

such that

$$\rho(t) = t^k \cdot \mathrm{Id} \quad \text{and} \quad \overline{\rho(z)} = \rho(\overline{z}) \qquad \forall t \in \mathbb{R}^*, z \in \mathbb{C}^*.$$

Exercise 4.2.26. Let X be a smooth projective variety of dimension $n \geq 1$. Let k be an integer with $0 \leq k \leq n$, and define

$$\mathrm{Hdg}^{2k}(X, \mathbb{Z}) = \left\{ \alpha \in H^{2k}(X, \mathbb{Z}) \colon \text{the image } \alpha_\mathbb{C} \text{ of } \alpha \text{ in } H^{2k}(X, \mathbb{C}) \text{ lies in } H^{k,k}(X) \right\}.$$

Let $Z \subset X$ be a smooth closed subvariety of codimension k. Define φ as the composition

$$\mathbb{Z} = H^0(Z, \mathbb{Z}) = H_{2n-2k}(Z, \mathbb{Z}) \to H_{2n-2k}(X, \mathbb{Z}) = H^{2k}(X, \mathbb{Z}),$$

and put $[Z] = \varphi(1) \in H^{2k}(X, \mathbb{Z})$. Prove that $[Z] \in \mathrm{Hdg}^{2k}(X, \mathbb{Z})$.

Chapter 5

Formal algebraic geometry and line bundles on hypersurfaces

The goal of this chapter is to prove the following theorem, due to Grothendieck [Gro05].

Theorem 5.0.1. *Let k be a field and let $X \subset \mathbb{P}_k^{n+1}$ be a hypersurface of dimension $n \geq 3$. Let $\mathcal{O}_X(1)$ be the pull-back of $\mathcal{O}_{\mathbb{P}_k^n}(1)$. Then* $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.

To prove this theorem, we need some tools that lie outside the scope of scheme theory. For as is sometimes the case in algebraic geometry, the category of schemes is not big enough to carry out certain constructions. Think of the proof of the above theorem in the case $k = \mathbb{C}$: one needed complex analytic functions; algebraic functions were not enough.

5.1 Lecture 7: Formal algebraic geometry

To prove Theorem 5.0.1, we need to have some tools of formal algebraic geometry to our disposal.

5.1.1 *Adic completion of a local ring*

Let R be a local ring with maximal ideal $\mathfrak{m}$. The powers of $\mathfrak{m}$ define a topology on R, called the $\mathfrak{m}$-*adic topology*. By definition, the ideals $\mathfrak{m}^n$, $n \geq 1$ define a fundamental system of open neighbourhoods around $0 \in R$. This topology is induced by the pseudometric d on R that is determined by its property that, for each $x \in R$, one has $d(x, 0) = 2^{-n}$ if $n = \max(k \mid x \in \mathfrak{m}^k)$ exists, and $d(x, 0) = 0$ otherwise. One defines $d(x, y) = d(x - y, 0)$.

Let $x, y, z \in R$. Suppose that $x - y \in \mathfrak{m}^{k_1}$ and $y - z \in \mathfrak{m}^{k_2}$, and that $k_1 \geq k_2$. Then

$$x - z = x - y + y - z \in \mathfrak{m}^{k_1} + \mathfrak{m}^{k_2} = \mathfrak{m}^{k_1}\left(1 + \mathfrak{m}^{k_1 - k_2}\right) = \mathfrak{m}^{k_1}.$$

It follows that d indeed defines a pseudometric on R. The *completion* $\widehat{R}$ of R is the completion of R with respect to its $\mathfrak{m}$-adic metric. Alternatively, one can define $\widehat{R}$ as the inverse limit

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

The basic properties of completion are:

Theorem 5.1.1. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $\widehat{R}$ be its completion.*

(1) The ring $\widehat{R}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$, and there is a natural injective homomorphism $R \to \widehat{R}$.

(2) If M is a finitely generated R module, then its completion $\widehat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \widehat{A} = \varprojlim_n M/\mathfrak{m}^n M$.

(3) The dimension of R equals the dimension of $\widehat{R}$.

(4) The local ring R is regular if and only if the local ring $\widehat{R}$ is regular.

Proof. See [Atiyah–Macdonald][Ch.10 & 11]. □

5.1.2 Adic completion of an arbitrary ring

Let A be a ring and let $I \subset A$ be an ideal. For $n \leq m$, there is a natural surjective homomorphism

$$A/I^m \to A/I^n,$$

and the collection of these maps turns (A/I^n) into an inverse system of topological rings, where each A/I^n is equipped with the discrete topology. The inverse limit

$$\widehat{A} \coloneqq \varprojlim A/I^n$$

exists in the category of topological rings, and we call it the *completion of A with respect to I* or the *I-adic completion of A*. Similarly, if M is an A-module, we define a topological $\widehat{A}$-module

$$\widehat{M} = \varprojlim M/I^n M$$

and call it the *I-adic completion of M*. The limit is taken in the category of topological abelian groups, where each $M/I^n M$ is equipped with the discrete topology. Notice that $\widehat{M}$ has indeed a natural $\widehat{A}$-module structure. The most important properties of this construction are summarized as follows. For proofs, see [Atiyah–Macdonald] and [Bourbaki].

Theorem 5.1.2. *Let A be a noetherian ring and let I be an ideal of A. Then:*

(1) The $\widehat{A}$-module $\widehat{I} = \varprojlim I/I^n$ defines an ideal of $\widehat{A}^n$. Moreover, $\widehat{I}^n = I^n\widehat{A}$ and $\widehat{A}/\widehat{I}^n \cong A/I^n$ for all n.

(2) If M is a finitely generated A module, then $\widehat{M} \cong M \otimes_A \widehat{A}$.

(3) *The functor*

$$M \mapsto \widehat{M}$$

is an exact functor on the category of finitely generated A-modules.

(4) *The topological ring $\widehat{A}$ is noetherian, and flat over A.*

(5) *Let (M_n) be an inverse system, where each M_n is a finitely generated A/I^n module. Suppose that each transition map*

$$\varphi_{m,n} \colon M_m \to M_n, \quad n \leq m,$$

is surjective, with kernel $\mathrm{Ker}(\varphi_{m,n}) = I^n M_m$. *Then $M = \varprojlim M_n$ is a finitely generated $\widehat{A}$-module, and for each n, one has $M_n \cong M/I^n M$.*

5.1.3 Adic noetherian rings

Definition 5.1.3. (1) An *adic noetherian ring* is a noetherian ring A equipped with a topology having the following property: there exists a fundamental system of neighbourhoods of zero in A consisting of the powers I^n $(n > 0)$ of an ideal I and A is separated and complete for this topology. In other words, A is the projective limit of the discrete rings $A_n = A/I^{n+1}$ $(n \geq 0)$.

(2) An *ideal of definition* of A is an ideal I which has the above property. Equivalently, I is an ideal of A which is open and whose powers tend to zero.

(3) If I is an ideal of definition, one says that A is *I-adic*, the topology is called the *I-adic topology*, and the filtration of A by the powers of I is called the *I-adic filtration.*

Remarks 5.1.4. Let $I \subset A$ be an ideal of definition in an adic ring A. Let $J \subset A$ be any ideal.

(1) J is an ideal of definition if and only if there exist integers $p, q > 0$ such that $J^q \subset I^p \subset J$.

(2) If I and J are ideals of a ring A such that $A = \varprojlim A/I^n = \varprojlim A/J^n$, then the topologies that I and J define on A are the same.

(3) The ideal I^n $(n \geq 1)$ is an ideal of definition for each $n \geq 1$.

Let A be an adic noetherian ring, with ideal of definition $I \subset A$. Let

$$u_n \colon A \to A/I^n =: A_n$$

be the canonical morphism, and for $m \geq n$, let $u_{m,n} \colon A_m \to A_n$ be the canonical morphism. Let $S \subset A$ be a multiplicative subset of A, and define $S_n = u_n(S)$. The maps $u_{m,n}$ define natural maps

$$S_m^{-1} A_m \to S_n^{-1} A_n,$$

for which these rings form an inverse system $(S_n^{-1}A_n)$. Define $A\{S^{-1}\}$ as the projective limit of this system.

Proposition 5.1.5. *The topological ring $A\{S^{-1}\}$ is topologically isomorphic to the projective limit $\varprojlim_n S^{-1}A/S^{-1}I^n$.*

Proof. Let $v_n\colon S^{-1}A \to S_n^{-1}A_n$ be the canonical morphism. Then the kernel of v_n is $S^{-1}I_n$ and v_n is surjective. In other words, we have an exact sequence

$$0 \to S^{-1}I_n \to S^{-1}A \to S_n^{-1}A_n \to 0.$$

Let $B = \varprojlim_n S^{-1}A/S^{-1}I_n$. Then, we have surjective morphisms

$$B \to B/\left(S^{-1}I_nB\right) = S^{-1}A/S^{-1}I_n = S_n^{-1}A_n,$$

and hence a continuous morphism

$$\varphi\colon B \to \varprojlim_n \left(B/\left(S^{-1}I_nB\right)\right) = \varprojlim_n S_n^{-1}A_n = A\left\{S^{-1}\right\}.$$

As B is separated and complete, φ is an isomorphism. $\square$

Let A be an I-adic noetherian ring, and $S \subset A$ be a multiplicative subset. Consider the natural map

$$A \to S^{-1}A, \tag{5.1}$$

and observe that the inverse image of the ideal $S^{-1}I_n \subset S^{-1}A$ in A contains I_n. Hence (5.1) is continuous. As $S^{-1}A \to A\{S^{-1}\}$ is continuous as well, we obtain a continuous morphism

$$A \to A\left\{S^{-1}\right\}.$$

Remark 5.1.6. Let A be an I-adic noetherian ring and $S \subset A$ a multiplicative subset. Then up to isomorphism, $A\{S^{-1}\}$ does not depend on the ideal of definition I. Moreover, The pair $(A\{S^{-1}\}, A \to A\{S^{-1}\})$ is characterized by the following universal property: let B be an adic noetherian ring and let $u\colon A \to B$ be a continuous morphism such that $u(S)$ is contained in the set of invertible elements of B. Then u factors uniquely as

$$A \to A\left\{S^{-1}\right\} \xrightarrow{u'} B,$$

where u' is continuous.

Let A be an adic noetherian ring with ideal of definition $I \subset A$, and let $\mathfrak{a}$ be an open ideal of A. Then $I^n \subset \mathfrak{a}$ for almost all $n \geq 1$. In particular, up to replacing I by $J = I^n$ for $n \gg 0$, we may assume that I is an ideal of definition such that $I^n \subset \mathfrak{a}$ for each $n \geq 1$ (see Remarks 5.1.4). In particular, $S^{-1}I^n \subset S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$ for all $n > 0$, hence $S^{-1}\mathfrak{a}$ is an open ideal of $S^{-1}A$. We denote by $\mathfrak{a}\{S^{-1}\}$ its $S^{-1}I$-adic completion. Then $\mathfrak{a}\{S^{-1}\}$ is an open ideal of $A\{S^{-1}\}$. Moreover, there is a canonical isomorphism

$$A\left\{S^{-1}\right\}/\mathfrak{a}\left\{S^{-1}\right\} = S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a}). \tag{5.2}$$

Remark 5.1.7. Let A be an adic noetherian ring. Let $f \in A$ and let S_f be the multiplicative subset of the f^n $(n \geq 0)$ in A. Define $A_{\{f\}} = A\left\{S_f^{-1}\right\}$. For an open ideal $\mathfrak{a} \subset A$, we write $\mathfrak{a}_{\{f\}} = \mathfrak{a}\{S_f^{-1}\}$. If $g \in A$, then we obtain a canonical continuous morphism $A_{\{f\}} \to A_{\{fg\}}$. Let $S \subset A$ be a multiplicative system. We obtain an inductive system $A_{\{f\}}$, and define

$$A_{\{S\}} = \varinjlim_{f \in S} A_{\{f\}}.$$

For every $f \in S$, there is a canonical morphism $A_{\{f\}} \to A\left\{S^{-1}\right\}$, and these morphisms form an inductive system. In particular, they define a canonical morphism

$$A_{\{S\}} \to A\left\{S^{-1}\right\}.$$

One can show that, with respect to this morphism, $A\left\{S^{-1}\right\}$ is a flat module over $A_{\{S\}}$.

5.1.4 Sheaves of topological rings and modules

We start with two basic lemmas.

Lemma 5.1.8. *Let X be a topological space and let $\mathfrak{C}$ be the category of sheaves of topological abelian groups on X. Then inverse limits exist in $\mathfrak{C}$. Furthermore, for each open $U \subset X$, consider the functor*

$$\mathcal{F} \mapsto \mathcal{F}(U)$$

from topological abelian sheaves on X to topological abelian groups. This functor commutes with projective limits.

Proof. Let $(\mathcal{F}_n)$ be an inverse system of topological abelian sheaves. Let $\mathcal{F}$ be the presheaf defined by $\mathcal{F}(U) = \varprojlim \mathcal{F}_n(U)$ for $U \subset X$ open. Then $\mathcal{F}$ is a sheaf of topological abelian groups. Moreover, $\mathcal{F}$ is the inductive limit of the system $(\mathcal{F}_n)$. $\square$

Assume now that the topological space X has a basis consisting of quasi-compact opens. Given a sheaf $\mathcal{F}$ of sets, groups, rings, modules over a ring, one can endow $\mathcal{F}$ with the structure of a sheaf of topological spaces, topological groups, topological rings, topological modules. Namely, if $U \subset X$ is quasi-compact open, we endow $\mathcal{F}(U)$ with the discrete topology. If $U \subset X$ is arbitrary, then we choose an open covering $U = \cup_i U_i$ by quasi-compact opens U_i. As $\mathcal{F}$ is a sheaf, there is an equalizer diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}),$$

which is an equalizer diagram in the category of topological spaces, topological groups, topological rings, topological modules. In particular, the first map identifies $\mathcal{F}(U)$ with a subspace of $\prod_{i \in I} \mathcal{F}(U_i)$, which is endowed with the product topology. A sheaf of topological spaces, topological groups, topological rings, topological modules is *pseudo-discrete* if the topology on $\mathcal{F}(U)$ is discrete for every quasi-compact open $U \subset X$. Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological spaces (similarly for groups, rings, modules).

5.1.5 Noetherian formal schemes as locally topologically ringed spaces

To an I-adic noetherian ring A, one can define a topologically ringed space

$$\mathrm{Spf}(A) \tag{5.3}$$

as follows. For $n \in \mathbb{Z}_{\geq 0}$, let $X_n = \mathrm{Spec}(A_n)$ ($A_n = A/I^{n+1}$), and put $Y = X_0 = \mathrm{Spec}(A/I)$. These schemes form an increasing sequence of closed subschemes of $\mathrm{Spec}(A)$,

$$Y = \mathrm{Spec}(A_0) \to X_1 \to \cdots \to X_n \to \cdots .$$

The schemes X_n all have the same underlying space $|\mathrm{Spf}(A)|$, called the *formal spectrum* of A.

Proposition 5.1.9. *Let I be an ideal of definition of an adic noetherian ring A. Then I is contained the radical* $\mathrm{rad}(A) = \cap_{\mathfrak{m} \text{ maximal}} \mathfrak{m} \subset A$ *of A.*

Proof. See [EGAI, §0, 7.1.10]. □

Let $\mathfrak{m} \subset A$ be a maximal ideal, corresponding to a point $x \in \mathrm{Spec}(A)$. It follows from Proposition 5.1.9 that $I \subset \mathfrak{m}$, i.e., that $x \in \mathrm{Spec}(A/I)$. We conclude that $\mathrm{Spf}(A)$, considered as a closed subset of $\mathrm{Spec}(A)$, contains all the closed points of A.

Lemma 5.1.10. *Let A be an I-adic noetherian ring. Every open subset of* $\mathrm{Spec}(A)$ *containing* $\mathrm{Spf}(A)$ *is equal to* $\mathrm{Spec}(A)$.

Proof. Let $U \subset \mathrm{Spec}(A)$ be an open subset containing $\mathrm{Spf}(A)$. Then the complement Z of U in $\mathrm{Spec}(A)$ is of the form $\mathrm{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. In particular, if $\mathfrak{a} \neq A$, then Z contains a closed point. As this is impossible, we conclude that $\mathfrak{a} = A$ hence $U = \mathrm{Spec}(A)$. □

Let us now define the sheaf of rings $\mathcal{O}_{\mathrm{Spf}(A)}$ of $\mathrm{Spf}(A)$ as the inverse limit

$$\mathcal{O}_{\mathrm{Spf}(A)} = \varprojlim_n \mathcal{O}'_{X_n}$$

of the pseudo-discrete sheaves $\mathcal{O}'_{X_n}$ on $\mathrm{Spf}(A)$, equipped with the natural topology such that on any open subset U of $\mathrm{Spf}(A)$, we have

$$\mathcal{O}_{\mathrm{Spf}(A)}(U) = \varprojlim_n \mathcal{O}'_{X_n}(U)$$

in the category of topological rings. Here, $\mathcal{O}'_{X_n}$ is the pseudo-discrete sheaf of topological rings associated to $\mathcal{O}_{X_n}$, see Section 5.1.4 (in particular, $\mathcal{O}_{X_n}(U)$ has the discrete topology for each n and each quasi-compact open U). Thus,

$$\mathcal{O}_{\mathrm{Spf}(A)}(\mathrm{Spf}(A)) = \varprojlim_n \mathcal{O}_{X_n}(X_n) = \varprojlim_n A_n = \widehat{A}. \tag{5.4}$$

For $f \in A$, let $D(f) = \mathrm{Spec}(A_f) \subset \mathrm{Spec}(A)$, and define

$$\mathcal{D}(f) = \mathrm{Spec}(A_f) \cap \mathrm{Spf}(A) = \left\{\mathfrak{p} \in \mathrm{Spf}(A) \mid \bar{f}(\mathfrak{p}) \neq 0\right\},$$

where $\bar{f} \in A/I$ is the image of f in $A_0 = A/I$.

Lemma 5.1.11. *Let A be an adic noetherian ring. Let $f \in A$ and write $\mathcal{D}(f) = D(f) \cap \mathrm{Spf}(A)$. Then the topologically ringed space $(\mathcal{D}(f), \mathcal{O}_{\mathrm{Spf}(A)}|_{\mathcal{D}(f)})$ is isomorphic to the formal spectrum*

$$\mathrm{Spf}(\widehat{A_f}) = \mathrm{Spf}(A_{\{f\}}).$$

Proof. Let I be an ideal of definition, and define $A_n = A/I^{n+1}$ and $X_n = \mathrm{Spec}(A_n)$. As topological spaces $\mathrm{Spf}(A) \cap D(f)$ is identified with $\mathrm{Spec}(A/I) \cap D(f) = \mathrm{Spec}(A_f/I_f)$. By (5.2), we have $A_{\{f\}}/I_{\{f\}} = A_f/I_f$. Hence, as topological spaces, $\mathrm{Spf}(A_{\{f\}})$ is identified with $\mathrm{Spec}(A_f/I_f)$, and thus with $\mathrm{Spec}(A) \cap D(f) = \mathcal{D}(f)$.

As for global sections of the structure sheaf, we indeed have

$$\begin{aligned}\mathcal{O}_{\mathrm{Spf}(A)}(\mathcal{D}(f)) &= \varprojlim_n \mathcal{O}_{X_n}(\mathrm{Spec}(A_f) \cap \mathrm{Spf}(A)) = \varprojlim_n \mathcal{O}_{X_n}\left(\mathrm{Spec}(A_f) \cap \mathrm{Spec}(A/I^n)\right) \\ &= \varprojlim_n \mathcal{O}_{X_n}\left(\mathrm{Spec}(A_f/I_f^n)\right) = \varprojlim_n A_f/I_f^n = \widehat{A_f}.\end{aligned}$$

More generally, let U' be a quasi-compact open of $\mathrm{Spec}(A)$ contained in $D(f)$ and define $U = U' \cap \mathrm{Spf}(A) \subset \mathcal{D}(f)$. Then $\Gamma(U, \mathcal{O}_{X_n})$ is canonically identified with the module of sections of the structure sheaf of $\mathrm{Spec}(A_f/I_f^n)$, hence, if we define $\mathfrak{Y} = \mathrm{Spf}(A_f)$, then

$$\Gamma(U, \mathcal{O}_{\mathfrak{Y}}) = \varprojlim_n \mathcal{O}_{\mathrm{Spec}(A_f/I_f^n)}(U) = \varprojlim_n \Gamma(U, \mathcal{O}_{X_n}) = \mathcal{O}_{\mathrm{Spf}(A)}(U).$$

Thus, the topologically ringed spaces $\mathcal{D}(f)$ and $\mathfrak{Y} = \mathrm{Spf}(A_f)$ are isomorphic. □

Let A be an adic noetherian ring. Let $x = \{\mathfrak{p}\} \in \mathrm{Spf}(A)$, where $\mathfrak{p} \subset A/I$ is a prime ideal. Then

$$\mathcal{O}_{\mathrm{Spf}(A),x} = \varinjlim_{f \in A | \bar{f}(\mathfrak{p}) \neq 0} \mathcal{O}_{\mathrm{Spf}(A)}(\mathcal{D}(f)) = \varinjlim_{f \in A | \bar{f}(\mathfrak{p}) \neq 0} A_{\{f\}}.$$

This is a noetherian, local, topological ring, but not complete in general. We have:

Lemma 5.1.12. *Let A be an adic noetherian ring.*

(1) The topologically ringed space $\mathrm{Spf}(A)$ *is depends only on A as a topological ring: it does not depend on an ideal of definition.*

(2) The space $\mathrm{Spf}(A)$ *is the subspace of* $\mathrm{Spec}(A)$ *consisting of open prime ideals.*

(3) Moreover, $\mathcal{O}_{\mathrm{Spf}(A)}$ is the inverse limit of the sheaves $(A/J)^\sim$, where J runs through the ideals of definition of A.

Proof. The statements follow directly from the above discussion. Let us be more elaborate about the argument that $\mathrm{Spf}(A)$ is the subspace of $\mathrm{Spec}(A)$ consisting of open prime ideals. Indeed, a prime ideal $\mathfrak{p}$ is open if and only if $I^n \subset \mathfrak{p}$ for some integer $n \geq 1$, which, as $\mathfrak{p}$ is prime, is equivalent to the condition that $I \subset \mathfrak{p}$. □

Let X and Y be locally topologically ringed spaces. A *morphism* of locally topologically ringed spaces $f\colon X \to Y$ is a morphism as locally ringed spaces such that for each open $U \subset Y$, the induced morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous.

Definition 5.1.13. An *affine noetherian formal scheme* is a topologically ringed space isomorphic to one of the form (5.3). A *locally noetherian formal scheme* is a topologically ringed space such that any point has an open neighbourhood which is an affine noetherian formal scheme. It is called *noetherian* if the underlying space is noetherian. A *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ between locally noetherian formal schemes is a morphism as locally topologically ringed spaces.

Let A and B be adic rings, and define $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Let $\varphi\colon A \to B$ be a continuous homomorphism of rings. As the inverse image in A of every open prime ideal of B is an open prime ideal of A, the continuous map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ restricts to a continuous map ${}^a\varphi\colon \mathrm{Spf}(B) \to \mathrm{Spf}(A)$. Moreover, for every $f \in A$, we have a canonical morphism

$$\Gamma(\mathcal{D}(f), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} \to B_{\{\varphi(f)\}} = \Gamma(D(\varphi(f)), \mathcal{O}_{\mathrm{Spf}(B)}).$$

As the right compatibility conditions are satisfied, these maps define a continuous morphism of sheaves of topological rings

$$\widetilde{\varphi}\colon \mathcal{O}_{\mathfrak{Y}} \to^a \varphi_* \mathcal{O}_{\mathfrak{X}}.$$

This yields a morphism of topologically ringed spaces

$$\Phi = ({}^a\varphi, \widetilde{\varphi})\colon \mathfrak{X} = \mathrm{Spf}(A) \to \mathrm{Spf}(B) = \mathfrak{Y}.$$

Proposition 5.1.14. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes such that $\mathcal{Y} = \mathrm{Spf}(A)$ is a noetherian affine formal scheme. Then, there is a canonical bijection*

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}_{\mathrm{cont}}(A, \mathcal{O}_{\mathcal{X}}(\mathcal{X})).$$

Here, $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ denotes the set of morphisms of locally noetherian formal schemes $\mathcal{X} \to \mathcal{Y}$, see Definition 5.1.13.

Proof. We first treat the case where $\mathcal{X} = \mathrm{Spf}(B)$ is formally affine. Let $\varphi\colon A \to B$ be a continuous morphism of rings. We need to show that the induced morphism of topologically ringed spaces

$$\Phi = ({}^a\varphi, \widetilde{\varphi})\colon \mathfrak{X} \to \mathfrak{Y} \tag{5.5}$$

is a morphism of locally topologically ringed spaces. Let $x = \mathfrak{p} \in \mathrm{Spf}(B)$ and $y = \mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spf}(A)$. Let $f \in A$ such that $f \notin \mathfrak{q}$. Then $\varphi(f) \notin \mathfrak{p}$, hence we obtain a morphism

$$A_{\{f\}} \to B_{\{\varphi(f)\}},$$

that maps $\mathfrak{q}_{\{f\}}$ into $\mathfrak{p}_{\{\varphi(f)\}}$. Hence, the induced map

$$\mathcal{O}_{\mathfrak{Y},y} = \varinjlim_{f \notin \mathfrak{q}} A_{\{f\}} \to \varinjlim_{g \notin \mathfrak{p}} B_{\{g\}} = \mathcal{O}_{\mathfrak{X},x}$$

is a morphism of local rings. We conclude that (5.5) is a morphism of locally topologically ringed spaces.

Consider a morphism of locally topologically ringed spaces $(\psi, \theta)\colon \mathrm{Spf}(B) \to \mathrm{Spf}(A)$, where ψ is a continuous map of topological spaces and $\theta\colon \mathcal{O}_{\mathrm{Spf}(A)} \to \psi_*\mathcal{O}_{\mathrm{Spf}(B)}$ a continuous map of sheaves of topological rings. By (5.4), we obtain a continuous morphism

$$\varphi\colon A = \Gamma(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) \to \Gamma(\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}) = B.$$

It is readily checked that ${}^a\varphi = \psi$, and that $\widetilde{\varphi} = \theta$. This finished the proof of the proposition in the case where $\mathfrak{X}$ is formally affine.

The proof general case is similar to the proof of the analogous statement for affine schemes, see [EGA I, (2.2.4)]. $\square$

Remark 5.1.15. To conclude the section, we make the trivial but important remark that every noetherian affine scheme $X = \mathrm{Spec}(A)$ can be viewed in one and only one way as an affine formal scheme, by considering A as a discrete topological ring. Equivalently, (0) is an ideal of definition for A. As such, the topological rings $\Gamma(U, \mathcal{O}_X)$ are discrete for quasi-compact opens $U \subset X$ (but not in general for arbitrary open subsets of X). In exactly the same way, we can start with any locally noetherian scheme $(X, \mathcal{O}_X)$, and associate to it $(X, \mathcal{O}'_X)$, where $\mathcal{O}'_X$ is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is $\mathcal{O}_X$ (see Section 5.1.4). This construction is compatible with morphisms and defines a functor

$$(\text{Locally noetherian schemes}) \to (\text{Locally noetherian formal schemes})\,.$$

It is straightforward to show that this functor is fully faithful.

5.2 Lecture 8: Algebraizing coherent sheaves on formal schemes

5.2.1 *Coherent sheaves of affine noetherian formal schemes*

Let $\mathcal{X} = \mathrm{Spf}(A)$ be an affine noetherian formal scheme, and let $I \subset A$ be an ideal of definition for A. Let M be an A-module of finite type. With M is associated a coherent module $\tilde{M}$ on $X = \mathrm{Spec}(A)$. In an analogous way, one associates with M a module M^Δ on $\mathrm{Spf}(A)$, defined as follows. For $n \in \mathbb{Z}_{\geq 0}$, let $X_n = \mathrm{Spec}(A/I^{n+1})$, and put

$$M^\Delta = \varprojlim_n \tilde{M}_n, \quad M_n = M/I^{n+1}M.$$

Then M^Δ does not depend on the choice of I, and the functor $M \mapsto M^\Delta$ is exact (on the category of finite A-modules). Moreover,

$$\Gamma(\mathcal{X}, M^\Delta) = M,$$

and the formation of M^Δ commutes with tensor products and internal $\mathcal{H}om$. Let $X = \mathrm{Spec}(A)$ and let

$$i\colon \mathcal{X} \to X$$

be the natural morphism of locally ringed spaces; it is defined by the inclusion on the underlying topological spaces and the canonical map $\mathcal{O}_X \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings. Then, since M is of finite type, Theorem 5.1.2 implies that

$$M^\Delta = i^*\tilde{M}.$$

Since, for any $f \in A$, $A_{\{f\}}$ is adic noetherian, it follows that $\mathcal{O}_{\mathrm{Spf}(A)}$ is a coherent sheaf of rings, that M^Δ is coherent, and that the coherent modules on $\mathcal{X}$ are exactly those of the form M^Δ for M of finite type over A.

5.2.2 *Formal schemes as inductive limits of nilpotent thickenings*

A *thickening* is a closed immersion of schemes $X \to X'$ whose ideal I is a nilideal; the schemes X and X' then have the same underlying topological space. If X' is noetherian, the same holds for X and I is nilpotent; conversely, if X is noetherian and I/I^2 is a coherent $\mathcal{O}_X$-module, then X' is noetherian [EGA I, Ch. 0, 7.2.6, 10.6.4]. If X' is noetherian, X' is affine if and only if X is [EGA I, 6.1.7]. We say that a thickening is *of order* n if $I^{n+1} = 0$.

Let $\mathcal{X}$ be a locally noetherian formal scheme. Then $\mathcal{O}_{\mathcal{X}}$ is a coherent sheaf of rings (see Section 5.2.1 above). Moreover, the coherent modules on $\mathcal{X}$ are exactly the modules which are of finite presentation, or equivalently, which on any affine open $U = \mathrm{Spf}(A)$ are of the form M^Δ for an A-module M of finite type.

Definition 5.2.1. Let $\mathcal{X}$ be a locally noetherian scheme. An *ideal of definition* of $\mathcal{X}$ is a coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$ such that, for any $x \in \mathcal{X}$, there exists an affine neighbourhood $U = \mathrm{Spf}(A)$ of x such that $\mathcal{I}|_U$ is of the form I^Δ for an ideal of definition I of A.

Let $\mathcal{X}$ be a locally noetherian formal scheme.

Lemma 5.2.2. *A coherent ideal $\mathcal{I}$ is an ideal of definition if and only if the ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is a scheme having $\mathcal{X}$ as an underlying space.*

Ideals of definition of $\mathcal{X}$ exist, and there is a largest one, $\mathcal{T} = \mathcal{T}_{\mathcal{X}}$, which is the unique ideal of definition $\mathcal{I}$ such that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is reduced. If $U = \mathrm{Spf}(A)$ is an affine open subset, then $\mathcal{T}|_U = T^\Delta$, where T is the ideal of elements $a \in A$ whose image in A/I are nilpotent. If $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$, then so is any power $\mathcal{I}^{n+1}$, $n \geq 0$. If $\mathcal{X}$ is noetherian, and $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$ and $\mathcal{J}$ is any coherent ideal, then $\mathcal{J}$ is an ideal of definition if and only if there exist positive integers p, q such that $J^q \subset I^p \subset J$.

Fix an ideal of definition $\mathcal{I}$ of $\mathcal{X}$. For $n \in \mathbb{Z}_{\geq 1}$, put

$$X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}),$$

which is a locally noetherian scheme. Moreover, we have an increasing chain of thickenings

$$X_\bullet = (X_0 \to X_1 \to \cdots \to X_n \to \cdots \to \mathcal{X}. \tag{5.6}$$

Moreover,

$$\mathcal{X} = \varinjlim_n X_n,$$

where the colimit is taken in the category of locally noetherian formal schemes (cf. Remark 5.1.15). Indeed, the underlying topological spaces of the X_n are all equal to the underlying space of $\mathcal{X}$, and

$$\mathcal{O}_{\mathcal{X}} = \varprojlim_n \mathcal{O}'_{X_n}$$

as topological rings, where $\mathcal{O}'_{X_n}$ is the pseudo-discrete sheaf of topological rings attached to the sheaf of rings $\mathcal{O}_{X_n}$ as in Section 5.1.4.

Proposition 5.2.3. *Let $X_\bullet = (X_0 \to X_1 \to \cdots)$ be a sequence of ringed spaces such that*

(1) X_0 is a locally noetherian scheme,

(2) the underlying maps of topological spaces are homeomorphisms, and with respect to them, the maps $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ are surjective,

(3) if $J_n = \mathrm{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0})$, then for $m \leq n$, one has $\mathrm{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}) = J_n^{m+1}$,

(4) J_1 is a coherent $\mathcal{O}_{X_0}$-module.

Then, the topologically ringed space

$$\mathcal{X} = \varinjlim X_i = (X_0, \varprojlim \mathcal{O}_{X_n})$$

is a locally noetherian formal scheme. Moreover, if $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_0}) = \varprojlim J_n$, then $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$ and $\mathcal{I}^{n+1} = \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n})$.

Proof. See [EGA I, 10.6.3 - 10.6.5]. □

5.2.3 Coherent sheaves on formal schemes and adic morphisms

Let $\mathcal{X}$ be a locally noetherian formal scheme. Let $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$, and consider the corresponding chain of thickenings $X_\bullet$ (see (5.6)). For $m \leq n$, denote by

$$u_{m,n} \colon X_m \to X_n, \quad u_n \colon X_n \to \mathcal{X}$$

the canonical morphisms.

Lemma 5.2.4. *Let (F_n) be an inverse system of sheaves of abelian groups on X_0, such that F_n is an $\mathcal{O}_{X_n}$-module for each n and the transition maps $f_{n,m} \colon F_n \to F_m$ are $\mathcal{O}_{X_n}$-linear (with respect to $\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}$). Call (F_n)* coherent *if each F_n is coherent and the transition maps $f_{n,m}$ induce isomorphisms $u^*_{m,n} F_n \cong F_m$. Then the functor*

$$\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(X_\bullet), \quad E \mapsto (u_n^* E), \tag{5.7}$$

from the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules to the category of coherent inverse systems on $X_\bullet$, is an equivalence of categories.

Proof. Exercise. □

Proposition 5.2.5. *Consider the notation in Lemma 5.2.4. Let $r \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathcal{X}}$-module. Then $\mathcal{E}$ is locally free of rank r over $\mathcal{O}_{\mathcal{X}}$ if and only if $u_n^*(\mathcal{E})$ is locally free of rank r over $\mathcal{O}_{X_n}$ for each $n \geq 0$.*

Proof. See the flatness criterion of [Bourbaki61, III, §5, Theorem 1]. □

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes, and let $\mathcal{J}$ be an ideal of definition of $\mathcal{Y}$. Since $\mathcal{J} \subset \mathcal{T}_Y$, the ideal $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is contained in $\mathcal{T}_X$. Fix an ideal of definition $\mathcal{I}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. Let $X_\bullet$ and $Y_\bullet$ be the chains of thickenings induced by $\mathcal{I}$ and $\mathcal{J}$ respectively. Then f induces a morphism of inductive systems

$$f_\bullet\colon X_\bullet \to Y_\bullet.$$

Moreover, one retrieves $f\colon \mathcal{X} \to \mathcal{Y}$ as the colimit $f = \varinjlim f_n$.

Lemma 5.2.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes. The above construction yields a bijection between the set of morphisms from $f\colon \mathcal{X} \to \mathcal{Y}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ and the set of morphisms of inductive systems $f_\bullet\colon X_\bullet \to Y_\bullet$.*

Proof. See [EGA I, 10.6.8]. □

Definition 5.2.7. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes.

(1) If, for some ideal of definition $\mathcal{J}$ of $\mathcal{Y}$, the ideal $\mathcal{I} = f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$, then we say that f is an *adic morphism*.

(2) If f is adic, then f is *flat* if for every $x \in \mathcal{X}$, the stalk $\mathcal{O}_{\mathcal{X},x}$ is flat over $\mathcal{O}_{\mathcal{Y},f(x)}$.

The flatness of an adic morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is equivalent to the condition that $\mathcal{O}_{X_n}$ is flat over $\mathcal{O}_{Y_n}$ for every n (this is a consequence of [Bourbaki61, III, §5, th. 2, prop. 2]. Here, $\mathcal{O}_{X_n} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}$ and $\mathcal{O}_{Y_n} = \mathcal{O}_{\mathcal{Y}}/\mathcal{J}^{n+1}$ for ideals of definition $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{X}$ and $\mathcal{Y}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} = \mathcal{I}$. In other words, $f\colon \mathcal{X} \to \mathcal{Y}$ is flat if and only if it is adic and $f_n\colon X_n \to Y_n$ is flat for each n.

Lemma 5.2.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes, and let $\mathcal{J}$ be an ideal of definition of $\mathcal{Y}$. Let $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. Consider the inductive systems $X_\bullet$ and $Y_\bullet$ defined by $\mathcal{I}$ and $\mathcal{J}$. Under the correspondence $f \mapsto f_\bullet$ of Lemma 5.2.6, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is adic if and only if the commutative diagram*

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "f_m"] & X_n \arrow[d, "f_n"] \\ Y_m \arrow[r] & Y_n, \end{tikzcd}$$

is cartesian for every n, m.

Proof. Exercise. □

5.2.4 *Formal completions along closed subschemes*

Let X be a locally noetherian scheme, and let X' be a closed subset of the underlying space $|X|$ of X. Choose a coherent ideal I of $\mathcal{O}_X$ such that the closed subscheme of X defined by I has X' as underlying space. Consider the inductive system of locally noetherian schemes, all having X' as underlying space,

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots ,$$

where X_n is the closed subscheme of X defined by I^{n+1}. By Proposition 5.2.3, the topologically ringed space

$$\varinjlim_n X_n$$

is a locally noetherian formal scheme, having X' as underlying space.

Definition 5.2.9. Let X be a locally noetherian scheme, and $X' \subset |X|$ a closed subspace. Define

$$X_{/X'} = \varinjlim X_n = (X', \varprojlim \mathcal{O}_X/I^{n+1}).$$

This locally noetherian formal scheme is called the *formal completion of X along X'*. When no confusion can arise, we write $\widehat{X} = X_{/X'}$.

Lemma 5.2.10. *Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. Choose a coherent ideal I of $\mathcal{O}_X$ such that the closed subscheme defined by I has X' as underlying space.*

(1) The locally noetherian formal scheme $X_{/X'}$ does not depend on the choice of I.

(2) If $X = \operatorname{Spec}(A)$ is affine and $I = \tilde{J}$, then $\widehat{X} = \operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is the completion of A with respect to the J-adic topology (see Section 5.1.2).

Proof. Exercise. $\square$

Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. The closed immersions $i_n \colon X_n \to X$ define a morphism of ringed spaces

$$i = i_X \colon \widehat{X} \to X. \tag{5.8}$$

Definition 5.2.11. Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. For a coherent sheaf F on X, define $F_{/X'} = \varprojlim_n i_n^*(F)$. Sometimes, if no confusion can arise, we shall write $\widehat{F} = F_{/X'}$.

Lemma 5.2.12. *Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset.*

(1) The morphism $i \colon \widehat{X} \to X$ defined in (5.8) is flat.

(2) For any coherent sheaf F on X, the natural map

$$i^*F \to F_{/X'} = \varprojlim_n i_n^*(F)$$

is an isomorphism.

(3) *Let* $X = \operatorname{Spec}(A)$*, and let* I *be an ideal that defines the closed subset* $X' \subset X$*. Let* $F = \tilde{M}$*, with* M *an* A*-module of finite type. Then* $F_{/X'} = \widehat{M}^{\Delta}$*. Here,* $\widehat{M} = \varprojlim M_n$ *with* $M_n = M/I^{n+1}M$*, see Section 5.1.2.*

Proof. The assertions follow from Theorem 5.1.2. □

Finally, let $f\colon X \to Y$ be a morphism of locally noetherian schemes. Let X' (resp. Y') be a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Choose coherent ideals $J \subset \mathcal{O}_X$ and $K \subset \mathcal{O}_Y$ defining closed subschemes with underlying spaces X' and Y' respectively, and such that $f^*(K)\mathcal{O}_X \subset J$. Then, f induces a morphism of inductive systems

$$f_\bullet\colon X_\bullet \to Y_\bullet,$$

where X_n (resp. Y_n) is the closed subset of X (resp. Y) defined by J^{n+1} (resp. K^{n+1}). By Lemma 5.2.6, we obtain a morphism of locally noetherian formal schemes

$$\widehat{f} = f_{/X'}\colon \widehat{X} = X_{/X'} \to Y_{/Y'} = \widehat{Y}, \tag{5.9}$$

which does not depend on the choices of J and K.

Definition 5.2.13. Let $f\colon X \to Y$ be a morphism of locally noetherian formal schemes. Let X' and Y' be closed subsets of X and Y such that $f(X') \subset Y'$. The morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ defined in (5.9) is called the *extension* of f to the completions.

The reason for the terminology is that the extension $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ makes the following diagram commute:

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "i_X"] & \widehat{Y} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

We conclude the section with the following:

Lemma 5.2.14. *Let* $f\colon X \to Y$ *be a morphism of locally noetherian formal schemes. Let* Y' *be a closed subset of* Y*, and define* $X' = f^{-1}(Y')$*. The extension* $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ *of* f *is an adic morphism of locally noetherian formal schemes.*

Proof. All the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"] & X \arrow[d, "f"] \\ Y_n \arrow[r] & Y \end{tikzcd} \tag{5.10}$$

are cartesian. Let $m \leq n$ and consider the square

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "f_m"] & X_n \arrow[d, "f_n"] \\ Y_m \arrow[r] & Y_n. \end{tikzcd}$$

This square is cartesian because (5.10) is cartesian for each $n \geq 0$. The result follows from this, because of Lemma 5.2.8. □

Remark 5.2.15. Let X be a locally noetherian scheme and let $X' \subset X$ be a closed subscheme. Choose an ideal $\mathcal{I} \subset \mathcal{O}_X$ that cuts out the closed subset X' inside X, and let $X_n = (X', \mathcal{O}_X/\mathcal{I}^{n+1})$. Then we have:

(1) closed immersions $X_n \hookrightarrow X$; and

(2) a morphism of ringed spaces $\widehat{X} \to X$.

These fit together in the following commutative diagram:

$$\begin{tikzcd}
 & & & X & & \widehat{X} \arrow[ll] \\
\cdots \arrow[r, hook] & X_n \arrow[r, hook] \arrow[urr, hook] \arrow[urrrr] & X_{n+1} \arrow[r, hook] \arrow[ur, hook] \arrow[urrr] & \cdots .
\end{tikzcd}$$

5.2.5 *Grothendieck's existence theorem: Algebraizing coherent sheaves*

Grothendieck's existence theorem is the following powerful result.

Theorem 5.2.16 (Grothendieck)**.** *Let A be an adic noetherian ring with ideal of definition I, and define $Y = \mathrm{Spec}(A)$, $Y_n = \mathrm{Spec}(A_n)$, $A_n = A/I^{n+1}$. Let $\widehat{Y} = \mathrm{Spf}(A)$. Let X be a noetherian scheme, separated and of finite type over $Y = \mathrm{Spec}(A)$. Then the functor*

$$\mathrm{Coh}_Y(X) \to \mathrm{Coh}_{\widehat{Y}}(\widehat{X})$$

from the category of coherent sheaves with proper support over Y to the category of coherent sheaves on $\widehat{X}$ with proper support over $\widehat{Y}$, is an equivalence.

Two important ingredients in the proof of Theorem 5.2.16 are the following results, that we state without proof.

Theorem 5.2.17. *Let $f\colon X \to Y$ be a finite type morphism of noetherian schemes, Y' a closed subset of Y, $X' = f^{-1}(Y')$. Let $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ be the extension of f to the formal completions. Let F be a coherent sheaf on X whose support is proper over Y. Then, for all q, the canonical maps*

$$(R^q f_* F)^\wedge \to R^q \widehat{f}_* \widehat{F} \qquad \text{and} \qquad R^q \widehat{f}_* \widehat{F} \to \varprojlim R^q (f_n)_* F_n$$

are topological isomorphism.

Proof. See [Fan+05, Theorem 8.2.2]. □

Theorem 5.2.18. *Let A be a noetherian ring and I an ideal. Let $f\colon X \to Y = \mathrm{Spec}(A)$ be a morphism of finite type and $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ be its completion along $V(I) \subset Y$ and $f^{-1}(V(I)) \subset X$. Let F and G be coherent sheaves on X whose supports have an intersection which is proper over Y. Then, for all $r \in \mathbb{Z}$, we have that $\mathrm{Ext}^r(F, G)$ is an A-module of finite type, and the natural map $\mathrm{Ext}^r(F, G) \to \mathrm{Ext}^r(\widehat{F}, \widehat{G})$ induces an isomorphism*

$$\mathrm{Ext}^r(F, G)^\wedge \xrightarrow{\sim} \mathrm{Ext}^r(\widehat{F}, \widehat{G}).$$

Proof. See [Fan+05, Corollary 8.2.9] or [EGAIII 1, 4.5.1]. □

Before we start with the proof of Theorem 5.2.16, we introduce:

Definition 5.2.19. Let X be a locally noetherian formal scheme, let $X' \subset |X|$ be a closed subset and let $\widehat{X}$ be the completion of X along X'. A coherent sheaf $\mathcal{F} \in \mathrm{Coh}(\widehat{X})$ is called *algebraizable* if it lies in the essential image of the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(\widehat{X})$ (see Definition 5.2.11). In other words, a coherent sheaf $\mathcal{F}$ on $\widehat{X}$ is algebraizable if there exists a coherent sheaf F on X such that $\mathcal{F} \cong \widehat{F}$.

Proof of Theorem 5.2.16. We prove the theorem in four steps:

Step 1. Fully faithfulness. Let F and G be coherent sheaves on X with proper supports over Y. By Theorem 5.2.18, $\mathrm{Hom}(F, G)$ is an A-module of finite type. Hence, it is separated and complete for the I-adic topology. Therefore, by Theorem 5.2.18 again, the canonical map

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(\widehat{F}, \widehat{G})$$

is an isomorphism. This proves that $(-)^\wedge$ is fully faithful.

Step 2. Reduction to the case where $X \to Y$ is quasi-projective. This follows from:

Lemma 5.2.20. *Assume that the theorem holds for quasi-projective morphisms. Assume moreover that for every closed subscheme T of X whose underlying space is strictly contained in that of X, all coherent sheaves on $\widehat{T}$ whose support is proper over $\widehat{Y}$ are algebraizable. Let E be a coherent sheaf on $\widehat{X}$ whose support is proper over $\widehat{Y}$. Then E is algebraizable.*

Proof of Lemma 5.2.20. We proceed in steps. By Chow's lemma, there are morphisms

$$Z \xrightarrow{g} X \xrightarrow{f} Y$$

such that g is projective and surjective, fg is quasi-projective, and there exists an open immersion $j : U \to X$ with U nonempty, such that g induces an isomorphism over U.

Let $T = X - U$ with the reduced scheme structure. As we assume that the theorem holds for quasi-projective morphisms, we may assume that $T \neq X$. Let J be the ideal of T in X. Let E be a coherent sheaf on $\widehat{X}$ whose support is proper over $\widehat{Y}$. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow E \longrightarrow \widehat{g}_*\widehat{g}^*E \longrightarrow C \longrightarrow 0. \tag{5.11}$$

We claim that $\widehat{g}_*\widehat{g}^*E$ is algebraizable. Indeed, as we assume that the theorem holds for quasi-projective morphisms, $\widehat{g}^*E$ is algebraizable. Then $\widehat{g}_*\widehat{g}^*E$ is also algebraizable, see Theorem 5.2.17.

Next, observe that K and C are killed by a positive power $\widehat{J}^N$ of $\widehat{J}$. Therefore, these sheaves can be viewed as coherent sheaves on $\widehat{T'}$, where T' is the thickening of

T defined by J^N. In particular, by the induction hypothesis, K and C, as coherent sheaves on $\widehat{T}'$, are algebraizable.

Let $\mathscr{C}$ be the category of algebraizable coherent sheaves on $\widehat{X}$ whose support is proper over Y. We claim that $\mathscr{C}$ is closed under kernels, cokernels and extensions. As for kernels and cokernels, this follows from the exactness of $(-)^\wedge$ on coherent sheaves: if $f\colon F \to G$ is a morphism of coherent sheaves on X with proper support over Y, with kernel A and cokernel B, then $\widehat{A} = \mathrm{Ker}(\widehat{f})$ and $\widehat{B} = \mathrm{Coker}(\widehat{f})$.

We are in the situation that K, $\widehat{g}_*\widehat{g}^*E$, and C in (5.11) are algebraizable coherent sheaves on $\widehat{X}$. As $\mathscr{C}$ is closed under kernels, cokernels and extensions, it follows that E is algebraizable. $\square$

Step 2 follows by noetherian induction and Lemma 5.2.20.

Step 3. Reduction to the case where $X \to Y$ is projective. By Step 2, we may assume that $f\colon X \to Y$ is quasi-projective. Thus, we have a factorization of $f\colon X \to Y$ into an open immersion $j\colon X \to Z$ and a projective morphism $g\colon Z \to Y$. Let $\mathcal{E}$ be a coherent sheaf on $\widehat{X}$ whose support T' is proper over $\widehat{Y}$. Suppose that the theorem holds for projective morphisms. We claim that $\mathcal{E}$ is algebraizable. To see this, let $\mathcal{F} = \widehat{j}_!\mathcal{E}$ be the extension by zero of $\mathcal{E}$ on $\widehat{Z}$. Then $\mathcal{F}$ is coherent and has proper support over $\widehat{Y}$. Thus, by the assumption that the theorem holds for projective morphisms, we get that $\mathcal{F} = \widehat{F}$ for a coherent sheaf F on Z.

Lemma 5.2.21. *Let R be a noetherian ring and $I \subset R$ an ideal. Let M be a finite R-module. Let $\widehat{M} = \varprojlim M/I^{n+1}M$ and $\widehat{R} = \varprojlim R/I^{n+1}$. Let $\mathfrak{p}$ be a prime ideal $\widehat{R}$; abusing notation, let $\mathfrak{p}$ be its inverse image in R. Then $(\widehat{M})_\mathfrak{p} = 0$ if and only if $M_\mathfrak{p} = 0$.*

Proof. By Nakayama's lemma, we have

$$(\widehat{M})_\mathfrak{p} = 0 \qquad \text{if and only if} \qquad (\widehat{M})_\mathfrak{p} \otimes_{(\widehat{R})_\mathfrak{p}} (\widehat{R})_\mathfrak{p}/\mathfrak{p}(\widehat{R})_\mathfrak{p} = 0.$$

Moreover,

$$(\widehat{M})_\mathfrak{p} \otimes_{(\widehat{R})_\mathfrak{p}} (\widehat{R})_\mathfrak{p}/\mathfrak{p}(\widehat{R})_\mathfrak{p} = (\widehat{M})_\mathfrak{p}/\mathfrak{p}(\widehat{M})_\mathfrak{p} = \left(\widehat{M}/\mathfrak{p}\widehat{M}\right)_\mathfrak{p}.$$

Now note that $I \subset \mathfrak{p}$, hence $\widehat{I} \subset \widehat{\mathfrak{p}}$. This yields

$$\widehat{M}/\mathfrak{p}\widehat{M} = (\widehat{M}/\widehat{IM})/(\widehat{\mathfrak{p}M}/\widehat{IM}) = (M/IM)/(\mathfrak{p}M/IM) = M/\mathfrak{p}M.$$

Hence,

$$(\widehat{M})_\mathfrak{p} \otimes_{(\widehat{R})_\mathfrak{p}} (\widehat{R})_\mathfrak{p}/\mathfrak{p}(\widehat{R})_\mathfrak{p} = (M/\mathfrak{p}M)_\mathfrak{p} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}.$$

This is zero if and only if $M_\mathfrak{p} = 0$. $\square$

Let T be the support of F. We claim that $T \subset X$. To see this, notice that $T' = \mathrm{Supp}(\mathcal{E}) \subset |\widehat{X}| \subset |X|$ and that

$$T' = \mathrm{Supp}(\mathcal{E}) = \mathrm{Supp}(\mathcal{F}) = \widehat{T}, \tag{5.12}$$

where the last equality follows from Lemma 5.2.21. This gives

$$\widehat{T} = T' \subset |\widehat{X}| \cap |\widehat{T}| \subset |X| \cap |T|.$$

Moreover, $X \cap T$ is open in T (because X is open in Z). As it contains $\widehat{T}$, we must have $X \cap T = T$ (see Lemma 5.1.10). This gives $T \subset X$, proving the claim.

Consequently, $F = j_! j^* F$, and hence

$$\widehat{j}_! \mathcal{E} = \mathcal{F} = \widehat{F} = \widehat{j}_! \widehat{j}^* \widehat{F}.$$

This implies that $\mathcal{E} = \widehat{j}^* \widehat{F} = (j^* F)^\wedge$ is algebraizable.

Step 4. Projective case.

Lemma 5.2.22. *Let $\mathcal{X}$ be a $\widehat{Y}$-adic formal scheme such that $X_0 = \mathcal{X} \times_Y Y_0$ is proper, and let L be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that $L_0 = L \otimes \mathcal{O}_{X_0}$ is ample. Let E be a coherent sheaf on $\mathcal{X}$.*

(1) Then, there exist non-negative integers m, r and a surjective morphism

$$(L^{\otimes -m})^{\oplus r} \to E.$$

(2) There exists an integer $n_0 \geq 1$ such that $\Gamma(\mathcal{X}, E(n)) \to \Gamma(X_0, E_0(n))$ is surjective for all $n \geq n_0$, where $E(n) = E \otimes L^{\otimes n}$ for $n \geq 0$.

We will now use the lemma. Assume $f\colon X \to Y$ is projective. Let L be an ample line bundle on X. If M is an $\mathcal{O}_X$-module, and $r \in \mathbb{Z}$, write $M(n) = M \otimes L^{\otimes n}$, and similarly for $\mathcal{O}_{\widehat{X}}$-modules. Let E be a coherent sheaf on $\widehat{X}$. By the lemma, we can find an exact sequence

$$\mathcal{O}(-m_1)^{r_1} \to \mathcal{O}(-m_0)^{r_0} \to E \to 0.$$

By the theorem, there exists a unique morphism $\mathcal{O}(-m_1)^{r_1} \to \mathcal{O}(-m_0)^{r_0}$ that completes to the above one. Define F to be the cokernel. By the fact that $(-)^\wedge$ is exact on coherent modules, we get $E = \widehat{F}$, and we are done.

The proof of Lemma 5.2.22 is left as an exercise for the reader. □

5.3 Lecture 9: Line bundles on hypersurfaces

The goal of this section is to study line bundles on hypersurfaces $X \subset \mathbb{P}^{n+1}_k$ of dimension $n \geq 3$ over arbitrary fields k. Before doing so, we provide applications of the existence theorem.

5.3.1 First applications of Grothendieck's existence theorem

Let $\mathcal{X}$ be a locally noetherian formal scheme, and let $\mathcal{A} \subset \mathcal{O}_\mathcal{X}$ be a coherent ideal sheaf. Let $\mathcal{Y}$ be the topologically ringed space with underlying space the support of $\mathcal{O}_\mathcal{X}/\mathcal{A}$ and sheaf of rings $\mathcal{O}_\mathcal{Y} = i^{-1}\mathcal{O}_\mathcal{X}/\mathcal{A}$ (with i the inclusion $\mathcal{Y} \to \mathcal{X}$). Then $\mathcal{Y}$ is a locally noetherian formal scheme, adic over $\mathcal{X}$, and is called the *closed formal subscheme* of $\mathcal{X}$ defined by $\mathcal{A}$. If $\mathcal{X} = \operatorname{Spf}(A)$ is affine, for some adic ring A, then there is an ideal $\mathfrak{a} \subset A$ such that $\mathcal{Y} = \operatorname{Spf}(A/\mathfrak{a})$.

Corollary 5.3.1. *Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme, separated of finite type over Y. Then $Z \mapsto \widehat{Z}$ defines a bijection from the set of closed subschemes of X which are proper over Y to the set of closed formal subschemes of $\widehat{X}$ which are proper over $\widehat{Y}$.*

Proof. Let I and J be coherent ideal sheaves in $\mathcal{O}_X$, and suppose that $\widehat{I} = \widehat{J} \subset \mathcal{O}_{\widehat{X}}$ and that the quotients $\mathcal{O}_X/I$ and $\mathcal{O}_X/J$ have proper support over Y. We get a canonical isomorphism $\mathcal{O}_{\widehat{X}}/\widehat{I} \xrightarrow{\sim} \mathcal{O}_{\widehat{X}}/\widehat{J}$ compatible with the projections. Thus, by Theorem 5.2.16, we get a canonical isomorphism $\mathcal{O}_X/I \xrightarrow{\sim} \mathcal{O}_X/J$ compatible with the projections $\mathcal{O}_X \to \mathcal{O}_X/I$ and $\mathcal{O}_X \to \mathcal{O}_X/J$. This gives $I = J \subset \mathcal{O}_X$.

To prove surjectivity, let $\mathcal{Z}$ be a closed formal subscheme of $\widehat{X}$ which is proper over $\widehat{Y}$. By Theorem 5.2.16, there exists an $\mathcal{O}_X$-module F, unique up to isomorphism, such that $\widehat{F} \cong \mathcal{O}_\mathcal{Z}$. We need to algebraize the quotient map $u\colon \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Z}$. Notice that the support of F is proper over Y. In particular, $\mathcal{O}_X$ and F are two coherent sheaves on X whose supports have intersection which is proper over Y. Thus, we can apply Theorem 5.2.18. We find that $\operatorname{Hom}(\mathcal{O}_X, F) = \operatorname{Hom}(\mathcal{O}_{\widehat{X}}, \widehat{F})$. This gives a unique map $v\colon \mathcal{O}_X \to F$ such that $\widehat{v} = u$. Since $v_0 = u_0$ is surjective, so is v (Nakayama's lemma), hence $F = \mathcal{O}_Z$ for a closed subscheme $Z \subset X$ which is proper over Y, and $\widehat{Z} = \mathcal{Z}$. □

Let $\mathcal{X} = \varinjlim X_n$ be a locally noetherian formal scheme, where X_n is a locally noetherian scheme for each $n \geq 0$. A morphism of locally noetherian formal schemes $f\colon \mathcal{Z} \to \mathcal{X}$ is called *finite* if f is adic and $f_0\colon X_0 \to Z_0$ is finite. If f is finite, then $f_*\mathcal{O}_\mathcal{Z}$ is a finite $\mathcal{O}_\mathcal{X}$-algebra. Conversely, every finite $\mathcal{O}_\mathcal{X}$-algebra arises in this way.

Corollary 5.3.2. *Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme, separated of finite type over Y. Then $Z \mapsto \widehat{Z}$ is an equivalence of the category of finite X-schemes which are proper over Y to the category of finite $\widehat{X}$-formal schemes which are proper over $\widehat{Y}$.*

Proof. Let A and B be finite $\mathcal{O}_X$-algebras with proper supports over Y, and $u\colon A \to B$ a map of $\mathcal{O}_X$-modules such that $\widehat{u}\colon \widehat{A} \to \widehat{B}$ is a map of $\mathcal{O}_{\widehat{X}}$-algebras, then u preserves the $\mathcal{O}_X$-algebra structures by the fully faithfulness part of Theorem 5.2.16. Hence $Z \mapsto \widehat{Z}$ is fully faithful. Let $\mathcal{A}$ be a finite $\mathcal{O}_{\widehat{X}}$-algebra with proper support over $\widehat{Y}$, and let A be a finite $\mathcal{O}_X$-module with $\widehat{A} = \mathcal{A}$. By Theorem 5.2.16, the maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\mathcal{O}_{\widehat{X}} \to \mathcal{A}$ that give $\mathcal{A}$ the structure of an $\mathcal{O}_{\widehat{X}}$-algebra, uniquely algebraize to give maps that turn A into an $\mathcal{O}_X$-algebra, such that $\widehat{A} = \mathcal{A}$ as $\mathcal{O}_{\widehat{X}}$-algebras. □

Corollary 5.3.3. *Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme which is proper over Y. Let Z be a noetherian scheme, separated and of finite type over Y. Then, the map*

$$\operatorname{Hom}_Y(X, Z) \to \operatorname{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z})$$

is bijective.

Proof. We have a canonical isomorphism

$$(X \times_Y Z)^\wedge = \widehat{X} \times_Y \widehat{Z},$$

see [EGA I, Proposition 10.9.7]. Under this isomorphism, the completion $(\Gamma_f)^\wedge$ of the graph Γ_f of a Y-morphism $f\colon X \to Z$ identifies with the graph $\Gamma_{\widehat{f}}$ of the completion $\widehat{f}\colon \widehat{X} \to \widehat{Z}$ of f, see [EGA I, Corollary 10.9.8]. Let f and g be morphisms $X \to Z$ over Y such that $\widehat{f} = \widehat{g}$. Then the completions of Γ_f and Γ_g are the same formal subschemes of $(X \times_Y Z)^\wedge$, hence, as Γ_f and Γ_g are proper over Y, Γ_f and Γ_g agree as subschemes of $X \times_Y Z$ (see Corollary 5.3.1). Similarly, by algebraizing the graph of a given morphism $f\colon \widehat{X} \to \widehat{Z}$ over $\widehat{Y}$, one algebraizes the morphism f. □

Remark 5.3.4. Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme which is not necessarily proper over Y. Then the conclusion of Corollary 5.3.3 is no longer valid in general. Namely, if $X = Z = \operatorname{Spec} A[t]$, then $\widehat{X} = \widehat{Z} = \operatorname{Spf}(A\{t\})$ and

$$\operatorname{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z}) = \operatorname{Hom}_{A\text{-cont}}(A\{t\}, A\{t\}) = A\{t\},$$

see Proposition 5.1.14, whereas $\operatorname{Hom}_Y(X, Z) = A[t]$. Here, $A\{t\}$ is the ring of restricted formal power series $\sum_n t^n$, which are those power series such that a_n tends to zero for the I-adic topology as n tends to infinity.

5.3.2 Algebraizing projective formal schemes

Before we come to the main theorem of this section, we provide two lemmas.

Lemma 5.3.5. *Let $\mathcal{X}$ be a locally noetherian formal scheme and let $f\colon \mathcal{X} \to \widehat{Y}$ be an adic morphism of locally noetherian formal schemes. Then for each $n \geq 0$, the fibre product $X_n = \mathcal{X} \times_Y Y_n$ is a locally noetherian scheme. Moreover, $\mathcal{X} = \varinjlim X_n$.*

Proof. Consider the ideal of definition $\mathcal{I} = I^\Delta$ of $\widehat{Y} = \operatorname{Spf}(A)$. Since f is adic, we know that $\mathcal{J} = f^*(\mathcal{I})\mathcal{O}_\mathcal{X}$ is an ideal of definition for $\mathcal{X}$, hence the same holds for $\mathcal{J}^n$ ($n \geq 0$). In particular, the ringed space $(\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{J}^n)$ is a scheme, see Lemma 5.2.2. The other assertion is clear. □

Lemma 5.3.6. *Let $f\colon \mathcal{Z} \to \mathcal{X} = \varinjlim X_n$ be a morphism of locally noetherian formal schemes. The following are equivalent.*

(1) f is finite, in other words, f is adic and $f_0\colon \mathcal{Z} \times_{\mathcal{X}} X_0 \to X_0$ is finite.

(2) f is locally of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ for A an adic noetherian ring with ideal of definition A and B a finite IB-adic A-algebra.

(3) f is adic and each $f_n\colon \mathcal{Z} \times_{\mathcal{X}} X_n \to X_n$ is finite.

Proof. See [EGA I, Ch. 0, 7.2.9]. □

We proceed to give a profound application of Theorem 5.2.16. Let A be an adic noetherian ring, I an ideal of definition of A, $Y = \mathrm{Spec}(A)$, $Y_n = \mathrm{Spec}(A/I^{n+1})$ and

$$\widehat{Y} = \varinjlim Y_n = \mathrm{Spf}(A).$$

Let X be a locally noetherian scheme over Y, and define $X_n = X \times_Y Y_n$. Then the *I-adic completion of X*,

$$\widehat{X} = \varinjlim X_n,$$

is a locally noetherian formal scheme over $\widehat{Y}$. One can ask: which locally noetherian formal schemes $\mathcal{X}$ over $\widehat{Y}$ arise in this way? This leads to the following definition.

Definition 5.3.7. Let $\mathcal{X}$ be a locally noetherian formal scheme and $f\colon \mathcal{X} \to \widehat{Y}$ an adic morphism of locally noetherian formal schemes.

(1) Let $\mathcal{X}$ be a locally noetherian adic formal scheme over $\widehat{Y}$. Define $X_n = \mathcal{X} \times_Y Y_n$ ($n \geq 0$); this is a scheme by Lemma 5.3.5. The morphism

$$\mathcal{F}\colon \mathcal{X} \to \widehat{Y}$$

is called *proper* if the induced morphism $\mathcal{F}_0\colon X_0 \to Y_0$ is proper.

(2) The $\widehat{Y}$-adic formal scheme $\mathcal{X}$ is *algebraizable* if one of the following equivalent conditions is satisfied.

   (a) There exists a locally noetherian scheme X over Y and a $\widehat{Y}$-isomorphism $\mathcal{X} \cong X_{/X'}$, where X' denotes the inverse image of $V(I) = \mathrm{Spec}(A/I)$ in X.

   (b) There exists a locally noetherian Y-scheme $X \to Y$ such that, if $X_n = X \times_Y Y_n$ and $\widehat{X} = \varinjlim X_n$, then there is an isomorphism $\widehat{X} \cong \mathcal{X}$ over $\widehat{Y}$.

The answer to the above question is then as follows.

Theorem 5.3.8. *Let A be an adic noetherian ring with ideal of definition $I \subset A$ and define $Y = \mathrm{Spec}(A)$ and $Y_n = \mathrm{Spec}(A/I^{n+1})$. Let $\mathcal{X}$ be a proper, adic formal scheme over $\widehat{Y}$, and define $X_n = \mathcal{X} \times_Y Y_n$, so that X_n is a scheme (cf. Lemma 5.3.5) and $\mathcal{X} = \varinjlim X_n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that*

$$L_0 = \mathcal{L} \otimes \mathcal{O}_{X_0} = \mathcal{L}/I\mathcal{L}$$

is ample on X_0, and so X_0 is projective over Y_0. The following assertions are true.

(1) The noetherian $\widehat{Y}$-adic formal scheme $\mathcal{X}$ is algebraizable.

(2) Let X be a Y-scheme with $\widehat{X} = \mathcal{X}$ over $\widehat{Y}$. There is a line bundle M on X, unique up to isomorphism, with $\mathcal{L} = \widehat{M}$. The line bundle M is ample, so that the morphism $X \to Y$ is projective.

Proof. By Lemma 5.2.22, there exists an integer $n \gg 0$ such that

(1) There exists a closed immersion $i_0 \colon X_0 \to P_0 = \mathbb{P}^r_{Y_0}$ such that $L_0^{\otimes n} = i_0^* \mathcal{O}_{P_0}(1)$.

(2) The map $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes n}) \to \Gamma(X_0, L_0^{\otimes n})$ is surjective.

The map $i_0 \colon X_0 \to P_0$ corresponds to an epimorphism $u_0 \colon \mathcal{O}_{X_0}^{r+1} \to L_0^{\otimes n}$ which we can lift to an $\mathcal{O}_{\mathcal{X}}$-linear map

$$u \colon \mathcal{O}_{\mathcal{X}}^{r+1} \to L^{\otimes n}.$$

By Nakayama's lemma, each $u_p \colon \mathcal{O}_{X_p}^{r+1} \to L_p^{\otimes n}$ is surjective, hence corresponds to a morphism $i_p \colon X_p \to P_p = \mathbb{P}^r_{Y_p}$ of Y_p-schemes such that $L_p^{\otimes n} = i_p^* \mathcal{O}_{P_p}(1)$. These closed immersions form an inductive system $i_\bullet \colon X_\bullet \to P_\bullet$, hence they define a morphism of formal schemes

$$i \colon \mathcal{X} \to \widehat{P}$$

such that $\mathcal{L}^{\otimes n} = i^* \mathcal{O}_{\widehat{P}}(A)$, where $\widehat{P}$ is the completion of $P = \mathbb{P}^r_Y$ over $Y = \operatorname{Spec}(A)$. As $i \colon \mathcal{X} \to \mathbb{P}^r_{\widehat{Y}}$ is an adic morphism such that i_0 is finite, it follows that i_p is finite for each $p \geq 0$, see Lemma 5.3.6. By Nakayama's lemma, it follows that i_p is a closed immersion for each p. Consequently, $i \colon \mathcal{X} \to \widehat{P}$ is a closed immersion of formal schemes. By Corollary 5.3.1, there exists a unique closed subscheme

$$j \colon X \to P = \mathbb{P}^r_Y$$

such that $\widehat{X} = \mathcal{X}$ as subschemes of $\mathbb{P}^r_Y$. Moreover, by Theorem 5.2.16, there exists a line bundle M on X, unique up to isomorphism, such that $\mathcal{L} \cong \widehat{M}$. Since $\mathcal{L}^{\otimes n} = i^* \mathcal{O}_{\widehat{P}}(1)$ and $(M^{\otimes n})^\wedge = \widehat{M}^{\otimes n}$, we get $(M^{\otimes n})^\wedge = (j^* \mathcal{O}_P(1))^\wedge$. Hence, by Theorem 5.2.16, we get $M^{\otimes n} = j^* \mathcal{O}_P(1)$, and therefore M is ample. □

5.3.3 More on algebraization in formal geometry

We state without proof the following powerful theorems.

Theorem 5.3.9. *Let $f \colon X \to S$ be a projective morphism of schemes, with S noetherian. Let $\mathcal{O}_X(1)$ be an invertible sheaf on X, ample relative to S. Let X_0 be the scheme of zeros of a section t of $\mathcal{O}_X(1)$, and let $\widehat{X}$ be the formal completion of X along X_0. Let $\mathfrak{F}$ be a coherent module on $\widehat{X}$, and let F_0 be the induced coherent module on X_0. Suppose moreover that:*

(1) $\mathfrak{F}$ is flat over S.

(2) For each $s \in S$, the section t_s of $\mathcal{O}_{X_s}(1)$ is $\mathfrak{F}_s$-regular.

(3) *For each $s \in S$, F_{0s} is of depth ≥ 2 at the closed points of X_{0s}.*

Assume S admits an ample invertible sheaf. There exists a coherent sheaf F on X and an isomorphism between $\widehat{F}$, the formal completion of F, and $\mathfrak{F}$.

Theorem 5.3.10. *Let $f\colon X \to S$ be a projective morphism of schemes, with S noetherian. Let $\mathcal{O}_X(1)$ be an invertible sheaf on X, ample relative to S, let Y be the scheme of zeros of a section t of $\mathcal{O}_X(1)$, let J be the ideal that defines $Y \subset X$, let X_n be the subscheme of X defined by J^{n+1}, $\widehat{X}$ the formal completion of X along Y, $\widehat{f}\colon \widehat{X} \to S$ the composition $\widehat{X} \to X \to S$, F a coherent module on X, flat relative to S. We suppose moreover that for each $s \in S$, the coherent $\mathcal{O}_{X_s}$-module F_s is of depth $> n$ at the closed points of X_s, and that t is F-regular. Then the following holds.*

(1) *The canonical morphism*
$$R^i f_*(F) \to R^i \widehat{f}_*(\widehat{F})$$
is an isomorphism for $i < n$, and a monomorphism for $i = n$.

(2) *The canonical morphism*
$$R^i \widehat{f}_*(\widehat{F}) \to \varprojlim R^i f_*(F_m)$$
is an isomorphism for $i \leq n$.

Corollary 5.3.11. *Assume the conditions in Theorem 5.3.10. Suppose that S is affine. Then:*

(1) *The canonical morphism*
$$H^i(X, F) \to H^i(\widehat{X}, \widehat{F})$$
is an isomorphism for $i < n$ and injective for $i = n$.

(2) *The canonical morphism*
$$H^i(\widehat{X}, \widehat{F}) \to \varprojlim H^i(X_m, F_m)$$
is an isomorphism for $i \leq n$.

5.3.4 The Picard group of a hypersurface

We begin with:

Lemma 5.3.12. *Let k be a field and P a smooth variety of dimension n over k. Let $X \subset P$ be a subvariety of dimension $r \geq 1$. Let $U \subset P$ be an open neighbourhood of X in P. Then* $\operatorname{codim}(P - U, P) \geq 2$.

Proof. Suppose that $U \neq P$ and let Z be an irreducible component of $P \setminus U$, endowed with its reduced subscheme structure. Consider the varieties $X \subset P$ and $Z \subset P$. We have $\dim(X) = n$; suppose that $s = \dim(Z)$. By [Fulton, §8.2, p. 137], each irreducible component of $X \cap Z$ has dimension at least $r + s - n$. As we know that $X \cap Z = \emptyset$, we must have $r + s - n < 0$. Therefore, $s < n - r \leq n - 1$ because $r \geq 1$. $\square$

Proposition 5.3.13. *Let X be a smooth variety over a field k. Let $Y \subset X$ be a closed subset of codimension ≥ 2. The restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(X \setminus Y)$ is an isomorphism.*

Proof. See [Gro05, Exposé XI, Corollaire 3.8]. $\square$

Theorem 5.3.14. *Let $X_0 \subset X = \mathbb{P}_k^{n+1}$ be a hypersurface of dimension ≥ 3. Then $\operatorname{Pic}(X_0) = \mathbb{Z} \cdot \mathcal{O}_{X_0}(1)$.*

Proof. We proceed in steps:

(1) First, we prove that $\operatorname{Pic}(X_0) = \operatorname{Pic}(\widehat{X})$. This follows from the exact sequences

$$0 \to I^{n+1}/I^{n+2} \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0, \tag{5.13}$$

where $I^{n+1}/I^{n+2} \to \mathcal{O}^*_{X_{n+1}}$ is the map sending x to $1 + x$. Indeed, the sequence (5.13) induces for $n \geq 0$ an exact sequence

$$H^1(X_0, I^{n+1}/I^{n+2}) \to H^1(X_{n+1}, \mathcal{O}^*_{X_{n+1}}) \to H^1(X_n, \mathcal{O}^*_{X_n}) \to H^2(X_0, I^{n+1}/I^{n+2}).$$

Moreover,

$$H^i(X_0, I^{n+1}/I^{n_2}) = H^i(X_0, \mathcal{O}_{X_0}(-(n+1)d)) = 0, \qquad i = 1, 2.$$

Consequently, as $H^1(X_n, \mathcal{O}^*_{X_n}) = \operatorname{Pic}(X_n)$, we get $\operatorname{Pic}(X_{n+1}) = \operatorname{Pic}(X_n)$ for $n \geq 0$. Hence $\operatorname{Pic}(X_0) = \operatorname{Pic}(\widehat{X})$ as desired.

(2) Next, we need to show that $\operatorname{Pic}(\widehat{X}) = \operatorname{Pic}(X)$:

(a) Let $\mathcal{L}$ be an invertible sheaf on $\widehat{X}$. By Theorem 5.3.9, there exists a coherent sheaf F on X such that $\widehat{F} \cong \mathcal{L}$. Now $\mathcal{L} = \widehat{F}$ is locally free of rank one on $\widehat{X}$, hence (by Nakayama), there exists an open neighbourhood $U \supset X_0$ such that $F|_U$ is locally free of rank one on U. Thus $F|_U \in \operatorname{Pic}(U)$.

(b) Next, observe that since U is an open neighbourhood of X_0, we have $\operatorname{codim}(X - U, X) \geq 2$, see Lemma 5.3.12. Hence $\operatorname{Pic}(X) = \operatorname{Pic}(U)$ by Proposition 5.3.13. Thus, we can extend $F|_U$ to a line bundle M on X. We get that the map $\operatorname{Pic}(X) \to \operatorname{Pic}(\widehat{X})$ is surjective.

(c) It remains to prove that the map $\operatorname{Pic}(X) \to \operatorname{Pic}(\widehat{X})$ is injective. For this, let L_1 and L_2 be line bundles on X and suppose that there is a morphism $f\colon \widehat{L}_1 \to L_2$. This gives a section

$$s(f) \in \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)).$$

Consider the natural map $\mathscr{H}om(L_1, L_2)^\wedge \to \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)$; we claim that

$$\varprojlim_m \Gamma(X_m, \mathscr{H}om(L_1, L_2)|_{X_m}) = \Gamma(\widehat{X}, \varprojlim_m \mathscr{H}om(L_1, L_2)|_{X_m}) = \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)).$$

Indeed, this follows from item (2) in Corollary 5.3.11. Combining this with item (1) in Corollary 5.3.11, we get that the natural map

$$\Gamma(X, \mathscr{H}om(L_1, L_2)) \to \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2))$$

is an isomorphism. Hence there exists a unique section $\sigma(g) \in \Gamma(X, \mathscr{H}om(L_1, L_2))$, corresponding to a morphism $L_1 \to L_2$, that induces the section $s(f)$. In other words, $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(\widehat{L}_1, \widehat{L}_2)$. We conclude that, in particular, if $\widehat{L} \cong \mathcal{O}_{\widehat{X}}$ for some line bundle L on X, then $L \cong \mathcal{O}_X$.

This finishes the proof of Theorem 5.3.14. □

Chapter 6

Hodge theory: Part II

6.1 Variation of Hodge structure

The goal of this lecture will be to explain the basic concepts and theorems in the theory of variations of Hodge structure. We will restrict ourselves to providing definitions and statements, and refer to [Voi02] for the proofs of the big theorems. Almost all of these fundamental theorems are due to Griffiths.

6.1.1 *Family of deformations*

Let $\mathscr{X}$ and B be complex manifolds. Let $\phi\colon \mathscr{X} \to B$ be a holomorphic map. We let $X_t = \phi^{-1}(t)$ be the fibre of ϕ over $t \in B$.

Definition 6.1.1. If B is connected with reference point $0 \in B$, and if ϕ is a proper submersion, then each fibre X_t is naturally a compact complex manifold, and we call $\mathscr{X}$ a *family of deformations* of the fibre X_0.

Proposition 6.1.2 (Ehresmann)**.** *Let $\phi\colon \mathscr{X} \to B$ be a proper submersion of differentiable manifolds. Let $0 \in B$ and assume that B is contractible. There is an isomorphism $T\colon \mathscr{X} \cong X_0 \times B$ of differentiable manifolds over B.*

6.1.2 *Gauss-Manin connection*

Definition 6.1.3. Let B be a topological space and G a group. A *local system* on B is a sheaf of abelian groups F on B such that F is locally isomorphic to the constant sheaf attached to G.

Let B be a differentiable manifold, and let H be a local system of $\mathbb{R}$-vector spaces on B. We can then consider the sheaf of free $\mathscr{C}^\infty$-modules

$$\mathscr{H} = H \otimes_{\mathbb{R}} \mathscr{C}^\infty.$$

Definition 6.1.4. Let B be a differentiable manifold and H a local system of $\mathbb{R}$-vector spaces on B. The associated connection $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ is defined as follows.

For $\sigma \in \mathscr{H}$, we can locally write $\sigma = \sum_i \sigma_i \otimes \alpha_i$ for $\sigma_i \in H$ and $\alpha_i \in \mathscr{C}^\infty$. Then, we put

$$\nabla\sigma = \sum_i \sigma_i \otimes d\alpha_i \in \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B.$$

Notice that, if $\sigma \in \mathscr{H}$ and $t \in \mathscr{C}^\infty$, then

$$\begin{aligned}\nabla(t \cdot \sigma) &= \nabla(\sigma_i \otimes t\alpha_i) = \sum_i \sigma_i \otimes d(t\alpha_i) = \sum_i \sigma_i \otimes (td\alpha_i + \alpha_i dt) \\ &= t\nabla(\sigma) + \sigma \otimes dt.\end{aligned}$$

Now let $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ be any connection. Then ∇ induces a map

$$\nabla\colon \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B \to \mathscr{H} \otimes \bigwedge^2 \Omega_B, \quad \nabla(\sigma \otimes \alpha) = \nabla\sigma \wedge \alpha + \sigma \otimes d\alpha.$$

Definition 6.1.5. The *curvature*

$$\Theta\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \bigwedge^2 \Omega_B$$

of ∇ is defined as $\Theta = \nabla \circ \nabla$. The connexion $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ is *flat* if $\Theta = 0$.

Lemma 6.1.6. *Let H be a local system of $\mathbb{R}$-vector spaces on a differentiable manifold B. Let $\mathscr{H} = H \otimes \mathscr{C}^\infty$ and let $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ be the connection constructed above. Then the curvature of ∇ is zero.*

Proof. Let σ be a local section of H and α a local section of $\mathscr{C}^\infty$. Then

$$\nabla(\sigma \otimes \alpha) = \sigma \otimes d\alpha \qquad \text{and} \qquad \nabla(\sigma) = 0.$$

Hence,

$$\Theta(\sigma \otimes \alpha) = \nabla(\sigma \otimes d\alpha) = \nabla\sigma \wedge d\alpha + \sigma \otimes d^2\alpha = 0.$$

This proves the lemma. □

Proposition 6.1.7. *Let B be a differentiable manifold. Then $H \mapsto (H \otimes_{\mathbb{R}} \mathscr{C}^\infty, \nabla)$ defines a bijection between the set of isomorphism classes of local systems of $\mathbb{R}$-vector spaces and the set of isomorphism classes of differentiable bundles with flat connexion.*

Proof. See [Voi02]. □

Let $\pi\colon \mathscr{X} \to B$ be a proper submersion of differentiable manifolds. Let A be a subring of $\mathbb{C}$. By Ehresmann's lemma (see Proposition 6.1.2), the sheaves

$$R^k\pi_*A$$

are local systems of A-modules on B. Defining $\mathscr{H}^k = R^k\pi_*\mathbb{R}$, the flat connection

$$\nabla\colon \mathscr{H}^k \to \mathscr{H}^k \otimes_{\mathscr{C}^\infty} \Omega_B$$

associated to the local system $R^k\pi_*\mathbb{R}$ is called the *Gauss-Manin connection*.

6.1.3 Semi-continuity of Hodge numbers

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds.

Theorem 6.1.8. *Let $\mathcal{F}$ be a holomorphic vector bundle on $\mathscr{X}$. The function $B \to \mathbb{Z}$ defined as $b \mapsto \dim H^q(X_b, \mathcal{F}|_{X_b})$ is upper semi-continuous.*

Proof. See [Voi02, Théorème 9.15]. □

Corollary 6.1.9. *The function $b \mapsto h^{p,q}(X_b)$ is upper semi-continuous.* □

Proposition 6.1.10. *Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds. Assume that the fibre X_0 above $0 \in B$ is a Kähler manifold. Then, in a neighbourhood of 0, we have $h^{p,q}(X_b) = h^{p,q}(X_0)$.*

Proof. Consider the spectral sequence

$$E_1^{p,q} = H^q(X_b, \Omega^p_{X_b}) \implies H^{p+q}(X_b, \mathbb{C}).$$

We have $h^{p,q}(X_b) \leq h^{p,q}(X_0)$ by Corollary 6.1.9. Moreover, $\dim E_\infty^{p,q} \leq \dim E_1^{p,q}$, where

$$E_\infty^{p,q} = F^p H^{p+q}(X_b)/F^{p+1}H^{p+q}(X_b).$$

We get

$$b_k = \sum_{p+q=k} \dim E_\infty^{p,q}(X_b) \leq \sum_{p+q=k} \dim E_1^{p,q}(X_b) = \sum_{p,q} h^{p,q}(X_b) \leq \sum_{p,q} h^{p,q}(X_0) = b_k.$$

The proposition follows. □

6.1.4 Period maps

Let X be a Kähler manifold and $\phi\colon \mathscr{X} \to B$ a proper holomorphic submersion, with B a connected complex manifold, $0 \in B$ a base point, and X_b Kähler for $b \in B$. Suppose that the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$ (which we can always obtain up to shrinking B around $0 \in B$). Define

$$b^{p,k} = \dim F^p H^k(X_0, \mathbb{C}).$$

Then $\dim F^p H^k(X_b, \mathbb{C}) = b^{p,k}$ for each $b \in B$. Suppose also that B is contractible (which we can achieve by shrinking B further around 0). Then, there is a canonical isomorphism $H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C})$ for $b \in B$, given by the composition

$$H^k(X_b, \mathbb{C}) \xleftarrow{\sim} H^k(\mathscr{X}, \mathbb{C}) \xrightarrow{\sim} H^k(X_0, \mathbb{C}).$$

We can then define the *period map* as follows:

$$\mathscr{P}^{p,k}\colon B \to \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C})), \quad b \mapsto F^p H^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C}).$$

Theorem 6.1.11 (Griffiths)**.** *The period map $\mathscr{P}^{p,k}$ is holomorphic for p, k with $p \leq k$.*

Proof. See [Voi02, Théorème 10.9]. □

Define $V = H^k(X_0, \mathbb{C})$ and $G = \mathrm{Grass}(b^{p,k}, V) = \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C}))$ and let $W \subset H^k(X_0, \mathbb{C})$ be a $b^{p,k}$-dimensional subspace. Let $[W] \in G$ be the corresponding point. Recall that there is a canonical isomorphism

$$T_{[W]}G = \mathrm{Hom}(W, V/W).$$

In particular,

$$T_{[F^pH^k(X_b)}G = \mathrm{Hom}(F^pH^k(X_b), H^k(X_b)/F^pH^k(X_b)).$$

Theorem 6.1.12 (Griffiths)**.** *Let $b \in B$. Consider the differential*

$$d\mathscr{P}^{p,k} \colon T_{B,b} \to T_{[F^pH^k(X_b)],G} = \mathrm{Hom}(F^pH^k(X_b), H^k(X_b)/F^pH^k(X_b))$$

of the period map $\mathscr{P}^{p,k} \colon B \to G$ at the point $b \in B$. Then the image of $d\mathscr{P}^{p,k}$ is contained in

$$\mathrm{Hom}(F^pH^k(X_b), F^{p-1}H^k(X_b)/F^pH^k(X_b)).$$

Proof. See [Voi02, Proposition 10.12]. □

Let $b \in B$. Then we have the Hodge filtration

$$0 = F^{k+1}H^k(X_b) \subset F^kH^k(X_b) \subset \cdots \subset F^pH^k(X_b) \subset \cdots \subset F^0H^k(X_b) = H^k(X_0, \mathbb{C}).$$

Hence, if we let

$$F_{b^{\bullet,k}}(H^k(X_0, \mathbb{C})) = \mathrm{Flag}(b^{k,k}, b^{k-1,k}, \cdots b^{1,k}, H^k(X_0, \mathbb{C}))$$

be the flag variety attached to the numbers $b^{k,k} \leq b^{k-1,k} \leq \cdots \leq b^{1,k}$, then we get a *period map*

$$\begin{aligned} &B \to F_{b^{\bullet,k}}(H^k(X_0, \mathbb{C})), \\ &b \mapsto \left(F^kH^k(X_b) \subset F^{k-1}H^k(X_b) \subset \cdots \subset F^1H^k(X_b) \subset H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C})\right). \end{aligned}$$

By Theorem 6.1.11, this map is holomorphic. Moreover, note that for $b \in B$, we have

$$F^pH^k(X_b) \oplus \overline{F^{k-p+1}H^k(X_b)} = H^k(X_b, \mathbb{C}).$$

This condition defines an open subset

$$\mathscr{D} \subset F_{b^{\bullet,k}}(H^k(X_0, \mathbb{C})),$$

called a (unpolarized) *period domain.*

Now suppose that there exists a class $\omega \in H^2(\mathscr{X}, \mathbb{Z})$ such that $\omega|_{X_b}$ is Kähler for every $b \in B$. Then one obtains a non-degenerate bilinear form

$$Q \colon H^k(X_b, \mathbb{Z}) \times H^k(X_b, \mathbb{Z}) \to \mathbb{Z}, \quad Q(\alpha, \beta) = \langle L^{n-k}\alpha, \beta\rangle = \int_X \omega^{n-k} \wedge \alpha \wedge \beta,$$

and one has:

(1) $F^pH^k(X_b,\mathbb{C}) = F^{k-p+1}H^k(X_b,\mathbb{C})^\perp$.

(2) $F^pH^k(X_b) \oplus \overline{F^{k-p+1}H^k(X_b,\mathbb{C})} = H^k(X_b,\mathbb{C})$.

(3) The hermitian form $(u,v) \mapsto Q(Cu,\bar{v})$ is positive definite on $H^k(X_b,\mathbb{C})_{prim}$.

Here, we recall that

$$H^k(X_b,\mathbb{C})_{prim} = \mathrm{Ker}L^{n-k+1} = \left\{\alpha \in H^k(X_b,\mathbb{C}) \mid \omega^{n-k+1} \cup \alpha = 0 \in H^{2n-k+2}(X_b,\mathbb{C})\right\}.$$

This leads us to define

$$\mathscr{D} \subset F_{b^\bullet,k}(H^k(X_0,\mathbb{C}))$$

as the subset of flags $F^k \subset \cdots \subset F^1 \subset H^k(X_0,\mathbb{C})$ satisfying (1), (2) and (3). We call $\mathscr{D}$ the *polarized period domain*. The first condition is closed (in fact already in the Zariski topology). The second and third conditions are open conditions (in the euclidean topology) on the set of filtrations satisfying the first condition.

All in all, we obtain a holomorphic period map

$$\mathscr{P}\colon B \to \mathscr{D}.$$

6.1.5 The Kodaira–Spencer map

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds. Let $b \in B$ and consider the following exact sequence of holomorphic vector bundles on X_b:

$$0 \to T_{X_b} \to T_{\mathscr{X}}|_{X_b} \to \phi^*(T_B)|_{X_b} \to 0.$$

Notice that $\phi^*(T_B)|_{X_b} = T_{B,b} \times X_b$ as holomorphic vector bundles on X_b, hence we get an exact sequence

$$0 \to T_{X_b} \to T_{\mathscr{X}}|_{X_b} \to T_{B,b} \times X_b \to 0.$$

In particular, taking cohomology yields a morphism

$$\rho\colon T_{B,b} \to H^1(X_b, T_{X_b})$$

called the *Kodaira–Spencer map*.

6.1.6 Variations of Hodge structure

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds whose fibres are Kähler. For $k \geq 0$, let $\mathscr{H}^k = R^k\phi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$, equipped with its Gauss–Manin connection

$$\nabla\colon \mathscr{H}^k \to \mathscr{H}^k \otimes_{\mathcal{O}_B} \Omega_B.$$

We assume that B is contractible and that the Hodge numbers are constant on B. We know that the period map

$$\mathscr{P}^{p,q}\colon B \to G = \mathrm{Grass}(b^{p,k}, H^k(X_0,\mathbb{C}))$$

is holomorphic (see Theorem 6.1.11). In particular, there exists a holomorphic subbundle

$$F^p\mathscr{H}^k \subset \mathscr{H}^k$$

such that $F^p\mathscr{H}^k_b = F^pH^k(X_b, \mathbb{C})$ for $b \in B$. The bundles $F^p\mathscr{H}^k$ are called the *Hodge subbundles* of $\mathscr{H}^k$. We define

$$\mathscr{H}^{p,q} = F^p\mathscr{H}^k / F^{p+1}\mathscr{H}^k,$$

so that $\mathscr{H}^{p,q}_b = H^q(X_b, \Omega^p_{X_b})$ for $p + q = k$.

Theorem 6.1.13 (Griffiths)**.** *We have*

$$\nabla F^p\mathscr{H}^k \subset F^{p-1}\mathscr{H}^k \otimes \Omega_B.$$

Proof. See [Voi02, Proposition 10.18]. □

As a corollary, we get an induced map

$$\overline{\nabla}^{p,q} \colon \mathscr{H}^{p,q} \to \mathscr{H}^{p-1,q+1} \otimes \Omega_B.$$

Proposition 6.1.14. *The differential*

$$d\mathscr{P}^{p,k}_b \colon T_{B,b} \to T_{G,F^pH^k(X_b)} = \mathrm{Hom}(F^pH^k(X_b), H^k(X_b)/F^pH^k(XX_b))$$

is the map induced by adjunction from the map

$$\overline{\nabla}^p_b \colon F^pH^k(X_b) \to H^k(X_b)/F^pH^k(X_b) \otimes \Omega_{B,b},$$

which is the map induced by the composition

$$F^p\mathscr{H}^k \xrightarrow{\nabla} \mathscr{H}^k \otimes \Omega_B \to (\mathscr{H}^k/F^p\mathscr{H}^k) \otimes \Omega_B.$$

Proof. See [Voi02, Lemme 10.19]. □

Combining the above results, we get a map

$$d\mathscr{P}^{p,k}_b \colon T_{B,b} \to \mathrm{Hom}(F^pH^k(X_b)/F^{p+1}H^k(X_b), F^{p-1}H^k(X_b)/F^pH^k(X_b)),$$

induced by adjunction from the map

$$\overline{\nabla}^{p,q}_b \colon F^pH^k(X_b)/F^{p+1}H^k(X_b) \to F^{p-1}H^k(X_b)/F^pH^k(X_b) \otimes \Omega_{B,b}.$$

This yields an element

$$d\mathscr{P}^{p,k}_b(u) \in \mathrm{Hom}(H^q(X_b, \Omega^p_{X_b}), H^{q+1}(X_b, \Omega^{p-1}_{X_b})), \quad \text{for all} \quad u \in T_{B,b}.$$

Theorem 6.1.15 (Griffiths)**.** *Let $b \in B$ and $u \in T_{B,b}$. Then, the linear map*

$$d\mathscr{P}^{p,k}(u) \colon H^q(X_b, \Omega^p_{X_b}) \to H^{q+1}(X_b, \Omega^{p-1}_{X_b})$$

is the cup-product with the Kodaira–Spencer class $\rho(u) \in H^1(X_b, T_{X_b})$ composed with the contraction induced by the natural map $\Omega^p_{X_b} \otimes T_{X_b} \to \Omega^{p-1}_{X_b}$. In other words, the following diagram commutes for all $u \in T_{B,b}$:

$$\begin{tikzcd}
H^q(X_b, \Omega^p_{X_b}) \arrow[d, "\cup\rho(u)"] \arrow[dr, bend left, "d\mathscr{P}^{p,k}(u)"] & \\
H^{q+1}(X_b, \Omega^p_{X_b} \otimes T_{X_b}) \arrow[r] & H^{q+1}(X_b, \Omega^{p-1}_{X_b}).
\end{tikzcd}$$

Proof. See [Voi02, Théorème 10.21]. □

Remark 6.1.16. Another way to formulate the theorem is to say that the following diagram commutes for all $b \in B$:

$$\begin{tikzcd}
T_{B,b} \arrow[d, "\rho"] \arrow[dr, bend left, "d\mathscr{P}^{p,k}"] & \\
H^1(X_b, T_{X_b}) \arrow[r] & \mathrm{Hom}(H^q(X_b, \Omega^p_{X_b}), H^{q+1}(X_b, \Omega^{p-1}_{X_b})).
\end{tikzcd}$$

Finally, consider the flag variety

$$F = F_{b^{\bullet,k}}(H^k(X_0, \mathbb{C})) = \left\{ F^k \subset F^{k-1} \subset \cdots \subset F^1 \subset H^k(X_0, \mathbb{C}) \mid \dim F^p = b^{p,k} \right\},$$

and the period map

$$\begin{aligned} \mathscr{P}^k &\colon B \to F, \\ &b \mapsto [F^\bullet H^k(X_b) \subset H^k(X_0)] = \left[F^k H^k(X_b) \subset \cdots \subset F^1 H^k(X_b) \subset H^k(X_b) = H^k(X_0) \right]. \end{aligned}$$

If we define

$$G_{b^{p,k}} = \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C})),$$

then there is a natural map

$$\iota \colon F \to G_{b^{k,k}} \times G_{b^{k-1,k}} \times \cdots \times G_{b^{1,k}},$$

and this map is a closed immersion. In particular, for $b \in B$, and $\mathscr{P}$ we have

$$\begin{aligned} T_{F, \mathscr{P}^k(b)} &\subset \left(\text{tangent space of } \prod_p G_{b^{p,k}} \text{ at } \iota(\mathscr{P}^k(b)) \right) \\ &= \bigoplus_p \mathrm{Hom}\left(F^p H^k(X_b), H^k(X_b)/F^p H^k(X_b) \right). \end{aligned}$$

Furthermore, by Theorem 6.1.12, the map

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(F^pH^k(X_b), H^k(X_b)/F^pH^k(X_b)\right)$$

factors through a map which we still denote by $d\mathscr{P}_b^k$:

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(F^pH^k(X_b)/F^{p+1}H^k(X_b), F^{p-1}H^k(X_b)/F^pH^k(X_b)\right).$$

As $F^pH^k(X_b)/F^{p+1}H^k(X_b) = H^{p,k-p}(X_b)$, we can, for $b \in B$, view $d\mathscr{P}_b^k$ as a map

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(H^{p,k-p}(X_b), H^{p-1,k-p+1}(X_b)\right). \tag{6.1}$$

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