The geometry and arithmetic of cubic hypersurfaces

Lecture notes

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[1]This is an incomplete, preliminary version of my lecture notes on cubic hypersurfaces. These notes will be updated weakly, see https://olivierfortman.github.io. For comments on the text, please write me an e-mail (*degaayfortman@math.uni-hannover.de*).

Goal of these notes. These are lectures notes for a course given at the Institute of Algebraic Geometry in Hannover between October 2023 and February 2024. The goal of these lectures was to give an introduction to the theory of cubic hypersurfaces. In these notes, I will treat geometric as well as arithmetic aspects of the theory. The idea is to introduce general topics in algebraic geometry, and then apply them to cubic hypersurfaces. Among the topics to be studied in some detail, are the following:

(1) Topology of hypersurfaces.

(2) Hodge theory of hypersurfaces.

(3) Formal algebraic geometry.

(4) Deformation theory.

(5) Intersection theory.

(6) Étale cohomology.

(7) Cubic surfaces.

Should you have any questions, or comments on these notes, do not hesitate to send me an e-mail[1].

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Chapter 1

Introduction

Algebraic geometry concerns the study of zero sets of systems of homogenous polynomials in multiple variables with coefficients in a field k. To do so, one investigates the geometry of *algebraic varieties* over k. Among the simplest ways to obtain examples of an algebraic variety is to consider a positive integer d and a degree d hypersurface

$$X = Z(F) = \{F = 0\} \subset \mathbb{P}^{n+1}_k, \quad F \in k[x_0, \dots, x_m]_d, \quad d \in \mathbb{Z}_{\geq 1}.$$

Although their definition is simple, hypersurfaces $X \subset \mathbb{P}^{n+1}_k$ are in general difficult objects to study. To facilitate the study of hypersurfaces in $\mathbb{P}^{n+1}$, one can restrict to the *smooth hypersurfaces*, i.e. those for which the equation

$$F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_{n+1} = 0$$

has no solution in $\mathbb{P}^{n+1}(\bar{k})$. If $d = 1$ then $X \cong \mathbb{P}^{n+1}$ is a hyperplane. If $d = 2$ then X is a smooth quadric, which implies that F is projectively equivalent to $x_0^2 + \cdots + x_{n+1}^2 = 0$. When $d \geq 3$, degree d hypersurfaces in $\mathbb{P}^{n+1}$ for $n \geq 1$ come in positive dimensional families, and their investigation starts to become more complicated.

When $d = 3$, one enters the realm of *smooth cubic hypersurfaces*. For each value of $n = \dim(X)$, the class of cubic hypersurfaces of dimension n is very rich; however, only for small n, the theory is fairly well understood. When $n = \dim(X) = 1$ and X is equipped with a rational point $e \in X(k)$, then $E = (X, e)$ is called an *elliptic curve*. The fundamental theorem in the theory of elliptic curves says that there exists an algebraic group law $E \times E \to E$ in this case, turning E into a one-dimensional smooth projective group variety. If $n = \dim(X) = 2$, then $X = S$ is a *cubic surface*, and $S_{\bar{k}}$ turns out to contain exactly 27 lines over $\bar{k}$. In higher dimensions, cubic hypersurfaces provide a rich class of objects to test important conjectures in algebraic geometry on; think of the Hodge and Tate conjectures. Another example is provided by the Weil conjectures, that were proven for cubic threefolds before they were proven in general.

In the theory of cubic hypersurfaces, many beautiful areas in mathematics interact with one another, such as arithmetic geometry, algebraic topology, étale cohomology, Hodge theory and moduli theory. Open questions concern cycle class conjectures and rationality questions. The goal of these lectures is to dive into these theories, and use the developed techniques to study the geometry and arithmetic of cubic hypersurfaces.

Chapter 2

Topology and differential forms

2.1 Lecture 1: Kähler differentials on hypersurfaces

Let k be a field. Let $n \in \mathbb{Z}_{\geq 0}$. We define

$$\mathbb{P} = \mathbb{P}^{n+1} = \mathbb{P}_k^{n+1}. \tag{2.1}$$

Before we start to study algebraic differential forms on hypersurfaces $X \subset \mathbb{P}_k^{n+1}$, we study them on the projective space $\mathbb{P}^{n+1}$ itself. To do so, we shall need some generalizations to the theory of vector bundles on schemes (or, more generally, ringed spaces) of classical linear algebra statements.

2.1.1 Linear algebra constructions on ringed spaces

The goal of this section is to prove two basic lemmas.

Lemma 2.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) If $0 \to E \to F \to L \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^p E \to \bigwedge^p F \to \bigwedge^{p-1} E \otimes L \to 0.$$

(2) Similarly, if $0 \to L \to E \to F \to 0$ is an exact sequence of vector bundles such that L is a line bundle, then for each $p \in \mathbb{Z}_{\geq 1}$, there is a canonical exact sequence

$$0 \to \bigwedge^{p-1} F \otimes L \to \bigwedge^p E \to \bigwedge^p F \to 0.$$

(3) Let E be a vector bundle and L a line bundle on X. Let $a > 0$ be an integer. There is a canonical isomorphism

$$\bigwedge^a (E \otimes L) = \left(\bigwedge^a E\right) \otimes L^{\otimes a}.$$

Proof. 1. Let Q be the cokernel of $\wedge^p E \to \wedge^p F$. Wedge the original sequence with $\wedge^{p-1} E$, and consider the canonical morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^{p-1} E \otimes E \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes F \arrow[r] \arrow[d] & \bigwedge^{p-1} E \otimes L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigwedge^{p} E \arrow[r] & \bigwedge^{p} F \arrow[r] & Q \arrow[r] & 0. \end{tikzcd}$$

It suffices to show that the so-constructed natural map $\wedge^{p-1} E \otimes L \to Q$ is an isomorphism. For this, we may assume that $F = E \oplus L$. In this case, $\wedge^p F = \wedge^p (E \oplus L) = \oplus_{i+j=p} \wedge^i E \otimes \wedge^j L = (\wedge^{p-1} E \otimes L) \oplus \wedge^p E$, and hence $\wedge^p F / \wedge^p E = \wedge^{p-1} E \otimes L$.

2. Dualize the exact sequence $0 \to L \to E \to F \to 0$, use item 1, and then dualize the result.

3. The map

$$(E \otimes F)^{\otimes a} \to \left(\bigwedge^a E \right) \otimes L^{\otimes a}, \quad e_1 \otimes f_1 \otimes \cdots \otimes e_a \otimes f_a \mapsto (e_1 \wedge \cdots \wedge e_a) \otimes (f_1 \otimes \cdots \otimes f_a) \,,$$

factors through a map

$$\bigwedge^a (E \otimes L) \to \left(\bigwedge^a E \right) \otimes L^{\otimes a},$$

which is an isomorphism (this can be verified on stalks, where this is clear). $\square$

Lemma 2.1.2. *Let E and F be vector bundles on a ringed space $(X, \mathcal{O}_X)$. For each integer $k \geq 0$, we have a canonical isomorphism*

$$\bigwedge^k (E \oplus F) = \bigoplus_{p+q=k} \left(\bigwedge^p E \right) \otimes \left(\bigwedge^q F \right).$$

Proof. Let R be a commutative ring. Then $\wedge(-)$ is a functor from R-modules to graded-commutative R-algebras which is left adjoint to the functor which takes the degree one part. Because it is left adjoint, it preserves colimits, and in particular coproducts. Therefore, for two R-modules M and N, we have a canonical isomorphism of graded R-algebras $\wedge(M \oplus N) = \wedge(M) \otimes \wedge(N)$. Now sheafify to get the result. $\square$

Lemma 2.1.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let p and m be non-negative integers with $p \leq m$. Then $\wedge^p (\mathcal{O}_X^{\oplus m}) = \mathcal{O}_X^{\oplus \binom{m}{p}}$.*

Proof. Clear. $\square$

2.1.2 Bott vanishing

Let k be a field and define $\mathbb{P}$ as in (2.1).

Lemma 2.1.4. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\mathbb{P} = \mathbb{P}^{n+1}$. For each $p \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$, there is a canonical exact sequence*

$$0 \to \Omega^p_{\mathbb{P}}(k) \to \mathcal{O}_{\mathbb{P}}^{\oplus \binom{n+2}{p}}(k-p) \to \Omega^{p-1}(k) \to 0. \tag{2.2}$$

Proof. Consider the *Euler sequence*, which is the exact sequence

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus(n+2)} \to \mathcal{O}_{\mathbb{P}} \to 0. \tag{2.3}$$

It yields

$$0 \to \Omega_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}^{\oplus(n+2)} \to \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

By item 1 in Lemma 2.1.1, this yields an exact sequence

$$0 \to \bigwedge^p(\Omega_{\mathbb{P}}(1)) \to \bigwedge^p(\mathcal{O}_{\mathbb{P}}^{\oplus(n+2)}) \to \bigwedge^{p-1}(\Omega_{\mathbb{P}}(1)) \otimes \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

By item 3 in Lemma 2.1.1, we obtain:

$$\bigwedge^p(\Omega_{\mathbb{P}}(1)) = \left(\bigwedge^p \Omega_{\mathbb{P}}\right) \otimes \mathcal{O}(p) = \Omega^p(p).$$

By Lemma 2.1.3, we have $\bigwedge^p(\mathcal{O}_{\mathbb{P}}^{\oplus(n+2)}) = \mathcal{O}_{\mathbb{P}}^{\oplus\binom{n+2}{p}}$. Hence, we obtain an exact sequence

$$0 \to \Omega^p_{\mathbb{P}}(p) \to \mathcal{O}_{\mathbb{P}}^{\oplus\binom{n+2}{p}} \to \Omega^{p-1}_{\mathbb{P}}(p-1) \otimes \mathcal{O}_{\mathbb{P}}(1) = \Omega^{p-1}_{\mathbb{P}}(p) \to 0.$$

The lemma follows by tensoring this sequence with $\mathcal{O}_{\mathbb{P}}(k-p)$. $\square$

Lemma 2.1.5. *Let X be a projective variety of dimension n over k, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let E a vector bundle of rank r on X. For $p \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, there is a canonical isomorphism*

$$\left(\left(\bigwedge^p E\right)(k)\right)^* = \left(\bigwedge^r E\right)^* \otimes \left(\bigwedge^{r-p} E\right)(-k).$$

Proof. We have

$$\left(\left(\bigwedge^p E\right) \otimes \mathcal{O}_X(k)\right)^* = \left(\bigwedge^p E\right)^* \otimes \mathcal{O}_X(-k).$$

Hence, it suffices to prove the lemma in the case $k = 0$. Consider the natural map

$$\bigwedge^p E \to \mathrm{Hom}\left(\bigwedge^{p-r} E, \bigwedge^r E\right) = \mathrm{Hom}\left(\bigwedge^{p-r} E, \mathcal{O}_X\right) \otimes \bigwedge^r E = \left(\bigwedge^{p-r} E\right)^* \otimes \bigwedge^r E.$$

We claim that this map is an isomorphism. This may be checked locally, in which case it is clear. As $(\wedge^p E)^* = \wedge^p E^*$, the lemma follows by duality. $\square$

Theorem 2.1.6 (Serre duality). *Let X be a smooth projective variety of dimension n over a field k, with canonical bundle ω_X, and let $i \geq 0$ be an integer. Then for any coherent sheaf $\mathcal{F}$ on X, there is a canonical isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) = H^{n-i}(X, \mathcal{F})^*.$$

In particular, if $\mathcal{F}$ is locally free, then

$$H^i(X, \mathcal{F}^\vee \otimes \omega) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X) = \mathrm{Ext}^i(\mathcal{F}, \omega_X) = H^{n-i}(X, \mathcal{F})^*.$$

Proof. See [Kle80] for the first assertion. The second assertion follows readily. □

Corollary 2.1.7. *Let X be a smooth projective variety of dimension n over k, with ample line bundle $\mathcal{O}_X(1)$. For $k \in \mathbb{Z}$, there are canonical isomorphisms*

$$(\Omega^p_X(k))^* \cong \omega^*_X \otimes \Omega^{r-p}_X(-k) \quad \text{and} \quad H^q(X, \Omega^p_X(k)) \cong H^{n-q}(X, \Omega^{n-p}_X(-k))^\vee. \tag{2.4}$$

In particular, $h^q(X, \Omega^p(k)) = h^{n-q}(X, \Omega^{n-p}_X(-k))$ for each $k \in \mathbb{Z}$.

Proof. Lemma 2.1.5 shows that

$$((\wedge^p \Omega_X)\,(k))^* = (\wedge^n \Omega_X)^* \otimes (\wedge^{n-p}\Omega_X)(-k) = \omega^*_X \otimes \Omega^{n-p}_X(-k).$$

By Serre duality, see Theorem 2.1.6, we obtain:

$$\begin{aligned} H^q(X, \Omega^p_X(k)) &= H^{n-q}(X, \omega_X \otimes (\Omega^p_X(k))^*)^\vee \\ &= H^q(X, \omega_X \otimes \omega^*_X \otimes \Omega^{n-p}_X(-k))^\vee = H^q(X, \Omega^{n-p}(-k))^\vee. \end{aligned}$$

The last statement follows readily from (2.4). □

Theorem 2.1.8 (Bott vanishing). *Consider the projective space $\mathbb{P} = \mathbb{P}^{n+1}_k$ of dimension $m > 0$ over k. Then $H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) = 0$ in each of the following cases:*

(a) $p \neq q$ and $0 < q < m$;

(b) $p = q > 0$ and $k \neq 0$, and $k > 0$ if $p = q = m$;

(c) $q = 0$ and $k \leq p$, and $k < 0$ if $p = 0$;

(d) $q = m$ and $k \geq p - m$, and $k > 0$ if $p = m$.

Proof. We assume that we are in one of the cases (a) – (d); our goal is to prove that $H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) = 0$. By Serre duality, see Corollary 2.1.7, we may assume that $q \geq p$. We proceed by induction on p.

First, assume that $p = 0$. In this case, either $q = 0$ in which case $k < 0$ hence $H^q(\mathbb{P}, \mathcal{O}(k)) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) = 0$, or $m > q > 0$ in which case $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$, or $m = q$ in which case $k \geq p - m = -m$ hence again $H^q(\mathbb{P}, \mathcal{O}(k)) = 0$. We conclude that the assertion holds if $p = 0$.

Next, assume that $p > 0$. Then $q \geq p > 0$. Sequence (2.2) gives us a long exact sequence

$$\cdots \to H^{q-1}(\mathcal{O}(k-p)^{\oplus\binom{n+2}{p}}) \to H^{q-1}(\Omega^{p-1}(k)) \to H^q(\Omega^p(k)) \to H^q(\mathcal{O}(k-p)^{\oplus\binom{n+2}{p}}) \\ \to H^q(\Omega^{p-1}(k)) \to H^{q+1}(\Omega^p(k)) \to \cdots \tag{2.5}$$

We claim that $H^q\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-p)^{\oplus\binom{n+2}{p}}\right) = 0$. Indeed, this follows from the fact that $q > 0$, and $k - p \geq -m$ if $q = m$. Therefore, using the exact sequence (2.5), we conclude that the canonical map

$$H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) \to H^q(\mathbb{P}, \Omega^p(k)) \tag{2.6}$$

is surjective.

We claim that we may assume that $q > p$. To see this, suppose that $q = p$. If $q = p \geq 2$, then the induction hypothesis implies that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$ (since $k \neq 0$), hence by the surjection (2.6), we have $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ in this case. Thus, suppose that $p = q = 1$. In this case, we have $k \neq 0$, and we want to show that $H^1(\mathbb{P}, \Omega^1(k)) = H^{m-1}(\mathbb{P}, \Omega^{m-1}(-k))^\vee = 0$.

To prove this, we proceed by induction on m. Suppose first that $m = 1 = p = q$. Then $k > 0$, and hence $H^1(\Omega^1(k)) = H^{m-1}(\Omega^{m-1}(-k))^\vee = H^0(\Omega^0(-k))^\vee = 0$. Next, assume $m \geq 2$. Then there are two cases to distinguish: $k > 0$ and $k < 0$. If $k < 0$, then the surjection (2.6) implies that $H^1(\Omega^1(k)) = 0$. Thus, assume that $k > 0$. We need to show that $H^{m-1}(\Omega^{m-1}(-k)) = 0$ for $k > 0$. We obtain a long exact sequence

$$\cdots \to H^{m-2}(\Omega^{m-2}(-k)) \to H^{m-1}(\Omega^{m-1}(-k)) \to H^{m-1}(\mathcal{O}(k-m)^{\binom{n+2}{m}}) \to \cdots$$

The group $H^{m-2}(\Omega^{m-2}(-k))$ is zero by induction, and $H^{m-1}(\mathcal{O}(k-m)^{\binom{n+2}{m}})$ vanishes as well, as $m \geq 2$. Therefore, $H^{m-1}(\Omega^{m-1}(-k)) = 0$ as desired.

By the above claim, we may assume $q > p \geq 1$. We can then apply the induction hypothesis (recall that we are still arguing by induction on p) to see that $H^{q-1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p-1}(k)) = 0$. Indeed, we have $0 < q - 1 < m$. Therefore, the surjection (2.6) implies that $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$, and we are done. $\square$

Exercise 2.1.9. Show that the non-zero twisted Hodge numbers $h^q(\Omega^p(k))$ are:

(a) $h^p(\Omega^p) = 1$,

(b) $h^0(\Omega^p(k)) = \binom{m+k-p}{k} \cdot \binom{k-1}{p}$ if $k > p$,

(c) $h^m(\Omega^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{m-p}$ if $k < p - m$.

Exercise 2.1.10. Consider the projective space $\mathbb{P} = \mathbb{P}_{\mathbb{C}}^{n+1}$ of dimension m over $\mathbb{C}$. By Theorem 2.1.8, we have $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^p) = 0$ for $p > 0$. Show directly that there are no non-zero holomorphic one-forms on $\mathbb{P}^1(\mathbb{C})$.

2.1.3 Kähler differentials on hypersurfaces

Lemma 2.1.11. *Let $X \subset \mathbb{P}$ be a smooth hypersurface of degree $d > 0$. For each $k \in \mathbb{Z}$, there are canonical exact sequences*

$$0 \to \Omega^p_{\mathbb{P}}(k-d) \to \Omega^p_{\mathbb{P}}(k) \to \Omega^p_{\mathbb{P}}|_X(k) \to 0, \tag{2.7}$$
$$0 \to \mathcal{O}_X(k-d) \to \Omega_{\mathbb{P}}|_X(k) \to \Omega_X(k) \to 0, \tag{2.8}$$
$$0 \to \Omega^{p-1}_X(k-d) \to \Omega^p_{\mathbb{P}}|_X(k) \to \Omega^p_X(k) \to 0. \tag{2.9}$$

Proof. It suffices to take $k = 0$.

To prove (2.7), we may take $p = 0$. In this case, the result follows from the following exact sequence, where i denotes the inclusion $X \hookrightarrow \mathbb{P}$:

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0. \tag{2.10}$$

One obtains (2.10) via the identification $\mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{I}_X$, where the latter denotes the ideal sheaf of $X \subset \mathbb{P}$, resulting from the isomorphisms $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}}(-X)$ (see [Har77, II, Proposition 6.18]) and $\mathcal{O}_{\mathbb{P}}(X) \cong \mathcal{O}_{\mathbb{P}}(d)$ (which holds because $\deg(X) = d$). Note by the way that (2.10) corresponds to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \xrightarrow{1 \mapsto F} \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_X(d) \to 0,$$

where $F \in \mathcal{O}_{\mathbb{P}}(d) = k[x_0, \ldots, x_{n_1}]_d$ is a polynomial that defines X.

To obtain the exact sequence (2.8), one combines the conormal exact sequence

$$0 \to \mathcal{N}^\vee_{Z/Y} \to \Omega_Y|_Z \to \Omega_Z \to 0$$

for any smooth hypersurface $i\colon Z \hookrightarrow Y$ in a smooth variety Y, where $\mathcal{N}^\vee_{Z/Y}$ is a sheaf on Z such that $i_*\mathcal{N}^\vee_{Z/Y} \cong I/I^2$ (see [Har77, II, Theorem 8.17]), and the canonical isomorphism

$$\mathcal{N}^\vee_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d) = \mathcal{O}_X(-d). \tag{2.11}$$

The second isomorphism in (2.11) being clear, it suffices to prove $\mathcal{N}^\vee_{X/\mathbb{P}} = i^*\mathcal{O}_{\mathbb{P}}(-d)$. This is again a general statement: if $i\colon Z \to Y$ is a closed immersion of schemes, then $i^*\mathcal{I}_Z$ has the property that $i_*i^*\mathcal{I}_Z = \mathcal{I}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_Y/\mathcal{I}_Z = \mathcal{I}_Z/\mathcal{I}_Z^2$ (to see this, reduce to the case where Y affine, where this is clear).

Finally, note that (2.9) follows from (2.8) together with Lemma 2.1.1. $\square$

Proposition 2.1.12. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$ with canonical bundle ω_X. Then $\omega_X \cong \mathcal{O}_X(d-n-2)$. In particular,*

(1) ω_X is ample if $d > n+2$;

(2) $\omega_X \cong \mathcal{O}_X$ if $d = n+2$;

(3) ω_X^* is ample if $d < n+2$.

Proof. Consider sequence (2.9) with $p = n + 1 = m$ and $k = d$. This gives

$$\omega_X \cong \omega_{\mathbb{P}}|_X(d) \cong \mathcal{O}_{\mathbb{P}}(-m-1)|_X \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(d-m-1).$$

The remaining statement follow directly. □

We proceed to prove:

Theorem 2.1.13. *Let $X \subset \mathbb{P} = \mathbb{P}^{n+1} = \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > 0$. Then the following holds.*

(1) Let $k \in \mathbb{Z}$ with $k < d$. The natural map

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(k)) \to H^q(X, \Omega^p(k))$$

is bijective for $p + q < n$ and injective for $p + q \leq n$.

(2) We have

$$H^q(X, \Omega^p(k-d)) = 0 \quad \textit{for} \quad p + q < n \quad \textit{and} \quad k < d. \tag{2.12}$$

(3) We have $H^q(X, \Omega^p(k)) = 0$ for $(p + q < n, k < 0)$ and for $(p + q > n, k > 0)$.

Proof. Throughout the proof, we will use Theorem 2.1.8 without mention. We first prove 1 and 2 by induction on p. Therefore, assume that $k < d$.

Suppose first that $p = 0$. Then (2.7) yields the following exact sequence:

$$H^q(\mathcal{O}_{\mathbb{P}}(k-d)) \longrightarrow H^q(\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^q(\mathcal{O}_X(k)) \longrightarrow H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d))$$

For $q \leq n < m$, we have $H^q(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ because $k - d < 0$. Thus, $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is injective for $q \leq n$ and $k - d < 0$. Moreover, if $q < n$ then $q+1 \leq n < m$, hence $H^{q+1}(\mathcal{O}_{\mathbb{P}}(k-d)) = 0$ for $q < n$ and $k - d < 0$. This implies that $H^q(\mathcal{O}_{\mathbb{P}}(k)) \to H^q(\mathcal{O}_X(k))$ is bijective $q < n$ and $k - d < 0$. In particular,

$$H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k-d)) = H^q(X, \mathcal{O}_X(k-d)) = 0 \qquad \text{for} \quad (q < n, k-d < 0).$$

This proves that 1 and 2 hold whenever $p = 0$.

Next, let $p > 0$. Notice that in this case, $p + q \leq n$ implies $q < n$. Similarly, $p + q < n$ implies $q < n - 1$. Notice also that (2.8) and (2.9) yield the following diagram, in which the rows are exact:

$$\begin{tikzcd}
H^q(\Omega^p_{\mathbb{P}}(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}(k)) \arrow[r, "f(p,q)"] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r] \arrow[d, equal] & H^{q+1}(\Omega^p_{\mathbb{P}}(k-d)) & \\
& H^q(\Omega^{p-1}_X(k-d)) \arrow[r] & H^q(\Omega^p_{\mathbb{P}}|_X(k)) \arrow[r, "g(p,q)"] & H^q(\Omega^p_X(k)) \arrow[r] & H^{q+1}(\Omega^{p-1}_X(k-d)).
\end{tikzcd}$$

If $p+q \leq n < m$, then $q < m$ hence $H^q(\Omega^p_{\mathbb{P}}(k-d)) = 0$ as $k-d < 0$. This implies that $f(p,q)$ is injective if $p+q \leq n$. Moreover, if $p+q \leq n < m$ then $(p-1)+q < n$, hence

$H^q(\Omega_X^{p-1}(k-d)) = 0$ by the induction hypothesis, as $k-d<0$. Therefore, the maps $f(p,q)$ and $g(p,q)$ in the above diagram are both injective if $p>0$ and $p+q \leq n$. This implies that the natural map

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) \to H^q(X, \Omega_X^p(k))$$

is injective for all $p,q \geq 0$ such that $p+q \leq n$.

Still assume $p>0$. If $p+q<n$, then $q<n$, hence $q+1<n+1=m$. Therefore, $H^{q+1}(\Omega_{\mathbb{P}}^p(k-d)) = 0$ as $k-d<0$. Moreover, if $p+q<n$ then $(p-1)+(q+1)<n$, hence $H^{q+1}(X, \Omega_X^{p-1}(k-d)) = 0$ by induction, as $k-d<0$. Therefore, the maps $f(p,q)$ and $g(p,q)$ in the above diagram are both bijective if $p>0$ and $p+q<n$. We conclude that the natural map

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) \to H^q(X, \Omega_X^p(k))$$

is bijective for all $p,q \geq 0$ such that $p+q<n$.

Continue to assume that $k<d$. Let $p,q \geq 0$ such that $p+q<n$. By what we have already proved, we have $H^q(X, \Omega_X^p(k-d)) = H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k-d))$, and this is zero because $q<m$ and $k-d<0$.

It remains to prove assertion 3. Notice that (2.12) implies $H^q(X, \Omega_X^p(k)) = 0$ for $(p+q<n, k<0)$. This also implies, via Corollary 2.1.7, that

$$H^q(X, \Omega^p(k)) \cong H^{n-q}(X, \Omega^{n-p}(-k))^\vee = 0 \quad \text{if} \quad (p+q>n, k>0).$$

This finishes the proof of the theorem. ☐

Corollary 2.1.14. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth hypersurface of degree d . If $n>2$, then* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$. *Similarly, for $n=2$ and $d \leq 3$, one has* $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$.

Proof. Consider the *exponential exact sequence* of abelian sheaves on $X(\mathbb{C}) = X^{an}$:

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 2i\pi} \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}_{X^{an}}^* \to 0.$$

Taking cohomology gives an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X). \tag{2.13}$$

As $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, see Exercise 2.1.16 below, it suffices to prove the following:

Claim 2.1.15. *If $n>2$ or $n=2$ and $d \leq 3$, then $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

On the one hand, by Theorem 2.1.8, we have $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = H^2(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) = 0$ for $n>1$. On the other hand, by Theorem 2.1.13, we see that if $n>1$, then $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^1(X, \mathcal{O}_X)$ and if $n>2$ then $H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = H^2(X, \mathcal{O}_X)$. Therefore, for $n>1$, we have $H^1(X, \mathcal{O}_X) = 0$ and for $n>2$, we have $H^2(X, \mathcal{O}_X) = 0$.

By Corollary 2.1.7, we have $h^i(X, \mathcal{O}_X) = h^{n-i}(X, \omega_X))$, and by Proposition 2.1.12, we have $h^{n-i}(X, \omega_X) = h^{n-i}(X, \mathcal{O}_X(d-(n+2))$. Thus, for $n=2$, this gives

$$h^i(X, \mathcal{O}_X) = h^{2-i}(X, \mathcal{O}_X(d-4)) = 0 \quad \text{for} \quad i \in \{1,2\} \quad \text{and} \quad d \leq 3.$$

This proves the claim, and thereby the corollary. ☐

Exercise 2.1.16. Sketch a proof of the fact that, for a locally ringed space $(X, \mathcal{O}_X)$, there is a natural isomorphism $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. Use this to conclude that if X is a smooth projective variety over $\mathbb{C}$, then $H^1(X, \mathcal{O}_X^*) = H^1(X^{an}, \mathcal{O}_{X^{an}}^*)$. Give an example of a sheaf $\mathcal{F}$ on a smooth projective variety X over $\mathbb{C}$ such that the natural map $H^1(X, \mathcal{F}) \to H^1(X^{an}, \mathcal{F}^{an})$ is not an isomorphism.

Exercise 2.1.17. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n and degree d. Provide all (n, d) for which the homomorphism $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Analyze the group which measures the possible failure of the injectivity of c_1.

Exercise 2.1.18. Consider a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$. Let $C \subset S$ be a curve contained in S. Prove that

$$[C] = c_1(\mathcal{O}_S(k)) \in H^2(S, \mathbb{Z})$$

if and only if there exists a hypersurface $Y \subset \mathbb{P}^3_{\mathbb{C}}$ of degree k such that $C = Y \cap S$.

2.2 Lecture 2: Lefschetz hyperplane theorem

To prove the Lefschetz hyperplane theorem, we will need some Morse theory. Let M be a smooth manifold of dimension n. Let $f \colon M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Then $0 \in M$ is called a *critical point* if $(df)_0 = 0$ as maps $T_0M \to T_{f(0)}\mathbb{R}$; in this case $f(0)$ is called a *critical value*. Consider the bilinear map

$$\mathrm{Hess}(f)_0 = (d^2 f)_0 \colon T_0M \times T_0M \to \mathbb{R}$$

defined as follows. Choose coordinates $x_1, \ldots, x_n$ on M centred around 0, and put

$$\mathrm{Hess}(f)_0 \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Lemma 2.2.1. *Show that the function $\mathrm{Hess}(f)_0$ does not depend on the choice of coordinates around 0. Show that $\mathrm{Hess}(f)_0$ defines a symmetric bilinear form on T_0M.*

Proof. Exercise. □

We say that a critical point $0 \in M$ is *non-degenerate* if $\mathrm{Hess}(f)_0$ is non-degenerate. By Lemma 2.2.1, if $0 \in M$ is a non-degenerate critical point, then $\mathrm{Hess}(f)_0$ defines a non-degenerate quadratic form, which can be diagonalized; define $\lambda_0(f)$ as the number of negative eigenvalues in this case. The Morse lemma, see [Mil63, Lemma 2.2], states that in suitable local coordinates $x_1, \ldots, x_n$ around a non-degenerate critical point $0 \in M$ of $f \colon M \to \mathbb{R}$, the function f can be written as the quadratic function

$$f(x) = f(0) - \sum_{i=1}^{\lambda_0(f)} x_i^2 + \sum_{i=\lambda_0(f)+1}^{n} x_i^2.$$

In particular, non-degenerate critical points (resp. values) are isolated in M (resp. $\mathbb{R}$).

We call f a *Morse function* if $f^{-1}(-\infty, a] \subset M$ is compact for each $a \in \mathbb{R}$, and f each critical point of f is non-degenerate. If f is a Morse function, then f is proper and its fibres $M_a = f^{-1}(a)$ are compact. Moreover, each critical value corresponds to a finite number of critical points, and the set of critical values is discrete in $\mathbb{R}$. In particular, for each $a \in \mathbb{R}$, there exist only finitely many critical values in $(-\infty, a] \subset \mathbb{R}$.

The basic theorem of Morse theory [Mil63, Theorem 3.5] says that if $f\colon M \to \mathbb{R}$ is a Morse function, then M has the homotopy type of a CW complex with one cell of dimension $\lambda_p(f)$ for each critical point $p \in M$.

Assume $M \subset \mathbb{R}^N$ is a closed submanifold of dimension n. By [Mil63, Theorem 6.6], for almost all (all but a set of measure 0) points $p \in \mathbb{R}$, the distance function

$$\varphi_p\colon M \to \mathbb{R}, \quad \varphi_p(x) = \|x - p\|^2$$

is a Morse function. We are now ready to prove:

Theorem 2.2.2 (Andreotti–Frankel [AF59])**.** *A closed n-dimensional complex submanifold $X \subset \mathbb{C}^r$ has the homotopy type of a CW complex of dimension $\leq n$.*

Proof. Let $c \in \mathbb{C}^r$ be a point such that the distance function $\varphi_c\colon X \to \mathbb{R}$ has only non-degenerate critical points.

Claim $(*)$. Let $p \in X$ be a critical point of $\varphi_c\colon X \to \mathbb{R}$. Then $\lambda_p(\varphi_c) \leq n$.

Before we prove Claim $(*)$, we will show that it implies the theorem. Indeed, by the basic theorem of Morse theory, X has the homotopy type of a CW complex with one cell of dimension $\lambda_p(\varphi_c)$ for each critical point $p \in M$ of φ_c. By Claim $(*)$, we have $\lambda_p(\varphi_c) \leq n$ for each critical point $p \in M$. Hence X has the homotopy type of a CW complex with one cell of dimension $\leq n$ for each critical point $p \in M$ of φ_c.

It remains to prove Claim $(*)$. We need:

Claim 2.2.3. *There exist local holomorphic coordinates on $\mathbb{C}^r$ such that $p = 0$, $c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(n+1)$-st position, and such that there exist open neighborhoods $0 \in V_1 \subset \mathbb{C}^n$ and $0 \in V_2 \subset \mathbb{C}^{r-n}$ and a holomorphic function*

$$\mathbb{C}^n \supset V_1 \xrightarrow{f} V_2 \subset \mathbb{C}^{r-n}$$

with $M \cap (V_1 \times V_2) = \mathrm{Graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^{r-n}$, and such that $df_0 = 0$.

Proof of Claim 2.2.3. Applying a suitable change of coordinates of $\mathbb{C}^r$, we may assume that $p = 0 \in M \subset \mathbb{C}^r$. As $M \subset \mathbb{C}^r$ is a closed submanifold, there exists an open subset $U \subset \mathbb{C}^r$ containing $p = 0$, and holomorphic functions $g_1, \ldots, g_m\colon U \to \mathbb{C}$ such that $X \cap U = \{g_1 = \cdots = g_m = 0\} \subset \mathbb{C}^r$. This gives a holomorphic function $g = (g_1, \ldots, g_m)\colon U \to \mathbb{C}^m$ such that $X \cap U = g^{-1}(0) = \{g = 0\} \subset U$. Thus, the fibre $g^{-1}(0)$ is smooth, which implies that g has maximal rank at each point of $X = g^{-1}(0)$. Applying the implicit function theorem, we obtain a holomorphic function $f\colon V_1 \to V_2 \subset \mathbb{C}^{r-n}$ defined on a open neighborhood $V_1 \subset \mathbb{C}^n$ of 0, such that $f(0) = 0$, $V_1 \times V_2 \subset U$ and such that

$$X \cap V_1 \times V_2 = \{(x, f(x)) \mid x \in V_1\} \subset V_1 \times V_2 \subset \mathbb{C}^n \times \mathbb{C}^{r-n} = \mathbb{C}^r.$$

Now $0 \neq c \in \mathbb{C}^r$, hence c defines a basis element of $\mathbb{C}^r$, so that there exists a matrix $\alpha \in \mathrm{GL}_r(\mathbb{C})$ with $\alpha \cdot c = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $(n+1)$-st position. As α is linear, we have $\alpha \cdot 0 = 0$. Finally, we claim that $df_0 = 0$. This follows from the fact that $\varphi_c \colon X \to \mathbb{R}$ is a distance function, with critical point $p = 0$. In other words, $(d\varphi_c)_0 = 0$, because $\varphi_c(x, f(x)) = \|(x, f(x)) - (0, 0, \ldots, 0, 1, 0, \ldots, 0)\|^2$. □

As $|z-1|^2 = |x+iy-1|^2 = (x-1)^2 + y^2 = (x^2+y^2) + (1-2x) = |z|^2 + (1 - 2 \cdot \Re(z))$, the distance function is now given by the formula

$$\varphi_c(z) = 1 - 2 \cdot \Re(f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2 . \tag{2.14}$$

As $\mathrm{ord}_0(f_i) \geq 2$ for all i, the last sum in (2.14) does not contribute to $\mathrm{Hess}(\varphi_c)_0$. Write

$$f_1(z) = Q(z) + \text{terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \ldots, z_n$. We obtain:

$$\mathrm{Hess}(\varphi_c)_0 = -2 \cdot \mathrm{Hess}\left(\Re(Q(z))\right)_0 + 2 \cdot \mathrm{Id}.$$

We claim that $\mathrm{Hess}(\Re(Q(z)))_0$ has at most n positive and at most n negative eigenvalues. Indeed, after a change of coordinates $z \mapsto w$, we can write

$$Q(w) = w_1^2 + \cdots + w_s^2, \qquad s \leq n;$$

writing $w_j = x_j + i \cdot y_j$, we obtain

$$\Re\left(Q(w)\right) = (x_1^2 - y_1^2) + \cdots + (x_s^2 - y_s^2).$$

This finishes the proof of Claim $(*)$, and thereby the proof of Theorem 2.2.2. □

As a corollary, we obtain:

Theorem 2.2.4. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$. Let $Y = X \cap H$ be a hyperplane section such that $U := X \setminus Y$ is smooth of dimension n and let $j \colon Y \hookrightarrow X$ denote the canonical inclusion. The restriction map*

$$j^* \colon H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

Proof. For the proof, we need the following:

Claim 2.2.5. *We have a natural isomorphism $H^i(X, Y, \mathbb{Z}) \cong H_{2n-i}(U, \mathbb{Z})$.*

Assuming the claim, we obtain a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i(Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{i+1}(X,Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{2n-i}(U,\mathbb{Z}) \arrow[r] & H^i(X,\mathbb{Z}) \arrow[r] & H^i(Y,\mathbb{Z}) \arrow[r] & H_{2n-i-1}(U,\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, to prove the theorem, we must show that $H_{2n-i}(U,\mathbb{Z})=0$ for $i\le n-1$. As $i\le n-1$ if and only if $2n-i\ge 2n-n+1=n+1$, we need to prove that $H_i(U,\mathbb{Z})=0$ for $i\ge n+1$. Note that $U=X\setminus Y\subset\mathbb{P}^N\setminus H\cong\mathbb{A}^N_{\mathbb{C}}$ defines a closed submanifold $U(\mathbb{C})\subset\mathbb{C}^N$ of dimension n. By Theorem 2.2.2, $U(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\le n$. In particular, $H_i(U,\mathbb{Z})=0$ for $i\ge n+1$, and Theorem 2.2.4 follows.

It remains to prove Claim 2.2.5; for this, we follow the exposition in [Voi02, page 306]. We admit the fact that Y admits a fundamental system of open neighborhoods $Y\subset Y_k\subset X$ that admit a deformation retract onto Y. It follows that the natural map

$$\varinjlim H^i(X,Y_k,\mathbb{Z})\to H^i(X,Y,\mathbb{Z})$$

is an isomorphism. By excision, we have

$$H^i(X,Y_k,\mathbb{Z})\cong H^i(U,U\cap Y_k,\mathbb{Z}).$$

If $K\subset U$ is a compact subset such that K is the deformation retract of an open subset $K\subset V\subset U$, then $H^i(U,U\setminus K,\mathbb{Z})\cong H_{2n-i}(K,\mathbb{Z})$ (this is a refined version of Poincaré duality, see [Spa81, Section 6.2]). Applying this to

$$K_k:=U\setminus(Y_k\cap U)=X\setminus Y_k,$$

which is a closed, hence compact, subset of X which admits a deformation retract of $X\setminus Y=U\supset K_k$, we obtain

$$H^i(U,Y_k\cap U,\mathbb{Z})=H^i(U,U\setminus K_k,\mathbb{Z})\cong H_{2n-i}(K_k,\mathbb{Z}).$$

As every singular chain on U is contained in one of the compact subsets $K_k\subset U$, the natural map $\varinjlim_k H_{2n-i}(K_k,\mathbb{Z})\to H_{2n-i}(U,\mathbb{Z})$ is an isomorphism, and hence

$$H^i(X,Y,\mathbb{Z})\cong\varinjlim H^i(U,U\cap Y_k,\mathbb{Z})\cong\varinjlim H_{2n-i}(K_k,\mathbb{Z})\cong H_{2n-i}(U,\mathbb{Z}),$$

proving Claim 2.2.5. □

Remark 2.2.6. For a compact oriented n-dimensional manifold M, and a closed submanifold $N\subset M$, cup-product with the fundamental class defines an isomorphism $H^i(M,N,\mathbb{Z})\cong H_{n-i}(M\setminus N,\mathbb{Z})$. This is relative Poincaré duality, cf. [Dol95, Section 7]. In particular, if $X\subset\mathbb{P}^N_{\mathbb{C}}$ is a smooth projective variety of dimension n and $Y=X\cap H$ a smooth hyperplane section, then it readily follows that $H^i(X,Y,\mathbb{Z})\cong H_{2n-i}(X\setminus Y,\mathbb{Z})$.

Corollary 2.2.7. *Let $X\subset\mathbb{P}^{n+1}_{\mathbb{C}}$ be a hypersurface.*

(1) The restriction map $H^i(\mathbb{P}^{n+1},\mathbb{Z})\to H^i(X,\mathbb{Z})$ is an isomorphism for $i<n$. In particular, $H^i(X,\mathbb{Z})=0$ for i odd and $i<n$, and $H^{2i}(X,\mathbb{Z})=\mathbb{Z}\cdot h^i$ for $2i<n$.

(2) Suppose X is smooth. Then $H^i(X,\mathbb{Z})=0$ for $i>n$ odd, and for each $j>n$ there is a unique $\alpha_{2j}\in H^{2j}(X,\mathbb{Z})$ such that $H^{2j}(X,\mathbb{Z})=\mathbb{Z}\cdot\alpha_{2j}$ and $\alpha_{2j}\cup h^{n-j}=1$.

Proof. Let d be the degree of X, and consider the d-th Veronese embedding $\mathbb{P}^{n+1}_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$. Then $X = \mathbb{P}^{n+1}_{\mathbb{C}} \cap H$ for a hyperplane $H \subset \mathbb{P}^N_{\mathbb{C}}$. Apply Theorem 2.2.4 to obtain the first assertion. The second assertion follows from the first via Poincaré duality. $\square$

Corollary 2.2.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then the restriction maps $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ and $\operatorname{Pic}(\mathbb{P}^{n+1}) \to \operatorname{Pic}(X)$ are isomorphisms. In particular, $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.*

Proof. The fact $H^2(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isomorphism is immediate from Theorem 2.2.4. From this, together with the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic}(\mathbb{P}^{n+1}) & \longrightarrow & \operatorname{Pic}(X) \\ \downarrow & & \downarrow \\ H^2(\mathbb{P}^{n+1}, \mathbb{Z}) & \longrightarrow & H^2(X, \mathbb{Z}), \end{array}$$

we deduce that $\operatorname{Pic}(\mathbb{P}^{n+1}) \to \operatorname{Pic}(X)$ is also an isomorphism, as the restriction map $\operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism by Corollary 2.1.14. $\square$

Corollary 2.2.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 3$. Then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p + q > n$ and $L \in \operatorname{Pic}(X)$ ample.* $\square$

Remark 2.2.10. Later we will see that Corollary 2.2.9 remains valid for hypersurfaces over arbitrary fields k. Namely, if $X \subset \mathbb{P}^{n+1}_k$ is a smooth hypersurface of degree d, and if $n > 2$, then $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. See Theorem 4.3.14 in Chapter 4.

Exercise 2.2.11. Provide the equation of a smooth hypersurface $S \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d \geq 4$ such that $\operatorname{Pic}(S) \ncong \mathbb{Z}$. See also Exercise 2.1.18. Define $V = H^0(\mathbb{P}^3, \mathcal{O}(d))$ and let $\mathbb{P}(V)$ be its projectivization. Let $\mathbb{P}(V)_0 \subset \mathbb{P}(V)$ be the locus of classes $[F] \in \mathbb{P}(V)$ such that $S_F \coloneqq \{F = 0\}$ is smooth. Show that $\mathbb{P}(V)_0$ is Zariski open in the projective space $\mathbb{P}(V)$. Show that the locus of $[F] \in \mathbb{P}(V)_0$ such that $\operatorname{Pic}(S_F) \ncong \mathbb{Z}$ is a countable union $\mathscr{H} = \cup_n Z_n$ of closed algebraic subvarieties $Z_n \subset \mathbb{P}(V_0)$. Show that $\mathscr{H} \neq \mathbb{P}(V_0)$.

Exercise 2.2.12. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension n. Suppose that $n \geq 2$. Show that X is simply connected.

Exercise 2.2.13. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of degree $d = 3$. Describe the fundamental group $\pi_1(X)$ of X when $X \subset \mathbb{P}^2_{\mathbb{C}}$ is a smooth plane curve of arbitrary degree $d \geq 4$.

Exercise 2.2.14. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of dimension $n \geq 2$. Let $C \subset X \subset \mathbb{P}^{n+1}$ be a smooth curve contained in X, and consider the Gysin map

$$\varphi \colon \mathbb{Z} = H^0(C, \mathbb{Z}) \cong H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \cong H^{2n-2}(X, \mathbb{Z}).$$

Define $[C] = \varphi(1) \in H^{2n-2}(X, \mathbb{Z})$. Consider the class $\alpha_{2n-2} \in H^{2n-2}(X, \mathbb{Z})$, see Corollary 2.2.7. Show that $[C] = \alpha_{2n-2}$ if and only if C intersects a general hyperplane $H \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ in a unique point with multiplicity one. Given equations for a cubic surface $X = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and a curve $C = \{F = G = 0\} \subset X \subset \mathbb{P}^3_{\mathbb{C}}$ such that $[C] = \alpha_2$.

2.3 Lecture 3: Betti numbers of hypersurfaces

Convention 2.3.1. We assume all topological manifolds to be second-countable and Hausdorff. In particular, they are paracompact and admit partitions of unity subordinate to any open cover.

2.3.1 Chern classes in topology

Let X be a topological manifold. Let $E \to X$ be a complex vector bundle of rank r. We would like to define the *Chern classes*

$$c_i(E) \in H^{2i}(X, \mathbb{Z}), \quad 1 \leq i \leq r$$

of X. We put $c_0(E) = 1$ and $c_i(E) = 0$ for $i > r = \operatorname{rank}(E)$, and introduce the *Chern polynomial*

$$c(E) = \sum_{i=0}^{r} c_i(E) \cdot t^i \in H^\bullet(X, \mathbb{Z})[t]$$

whose coefficients we shall now define. Consider the exponential exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2i\pi} \mathscr{C}^0 \xrightarrow{\exp} (\mathscr{C}^0)^* \to 0, \tag{2.15}$$

where $\mathscr{C}^0$ is the sheaf of continuous complex-valued functions on X, and $(\mathscr{C}^0)^*$ the subsheaf of invertible functions. The sequence (2.15) defines a morphism

$$c_1 \colon \{\text{complex line bundles } L \text{ on } X\} /_{\cong} = H^1(X, (\mathscr{C}^0)^*) \to H^2(X, \mathbb{Z}). \tag{2.16}$$

In particular, if E is a vector bundle of rank $r = 1$ on X, we obtain an element $c_1(E) \in H^2(X, \mathbb{Z})$ such that $c_1(E) = c_1(E')$ if $E \cong E'$ as vector bundles on X.

Lemma 2.3.2. *Let X be a topological space and $E \to X$ a vector bundle of rank r on X. Let $\psi \colon \mathbb{P}(E) \to X$ be the associated projective bundle. Let $S \subset \psi^* E$ be the tautological line bundle, and define $h = c_1(S^*) \in H^2(\mathbb{P}(E), \mathbb{Z})$. Then $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(X, \mathbb{Z})$, with basis $1, h, \ldots, h^{r-1}$.*

Proof. This follows from the Leray–Hirsch theorem (see [Hat02, Theorem 4D.1]). □

Lemma 2.3.3. *Let X be a topological manifold. Let $E \to X$ be a complex vector bundle on X. Then E admits a hermitian metric $E \times E \to \mathbb{C}$.*

Proof. Exercise. □

Theorem 2.3.4. *Let X be a topological manifold, and let $K(X)$ be the set of isomorphism classes of complex vector bundles of finite rank on X. There exists a unique function*

$$c_t \colon VB(X) \to H^\bullet(X, \mathbb{Z})[t], \quad E \mapsto c_t(E) = \sum_i c_i(E) \cdot t^i,$$

such that $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $E \in VB(X)$, $c_0(E) = 1$, $c_i(E) = 0$ for $i > r =$ $\operatorname{rank}(E)$*, and such that the following conditions are satisfied:*

(1) *(Compatibility with (2.16).) If $r = \operatorname{rank}(E) = 1$, then $c_t(E) = 1 + c_1(E) \cdot t$.*

(2) *(Functoriality.) If $\phi\colon Y \to X$ is continuous, then $c_t(\phi^*(E)) = \phi^*(c_t(E))$ for $E \in VB(X)$, where $\phi^*\colon H^\bullet(X,\mathbb{Z}) \to H^\bullet(Y,\mathbb{Z})$ is the pull-back of ϕ.*

(3) *(Turning exact sequences into products.) If $0 \to F \to E \to G \to 0$ is an exact sequence, then $c_t(E) = c_t(F) \cdot c_t(G)$.*

Proof of uniqueness. This follows readily from the following:

Claim 2.3.5. *Let $E \to X$ be a complex vector bundle. There exists a topological manifold Y and a continuous map $\phi\colon Y \to X$ such that $\phi^*\colon H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is injective for each i, and such that $\phi^* E$ is a direct sum of line bundles.*

To prove the claim, consider the projective bundle $\psi\colon \mathbb{P}(E) \to X$. The morphism $\psi^*\colon H^*(X,\mathbb{Z}) \to H^*(\mathbb{P}(E),\mathbb{Z})$ turns $H^*(\mathbb{P}(E),\mathbb{Z})$ into a free module over $H^*(X,\mathbb{Z})$, see Lemma 2.3.2. In particular, ψ^* is injective. Consider the tautological line bundle $S \subset \psi^*(E)$; it has fibre $S_x = \Delta_x \subset E_x$ above the point $x = [\Delta_x] \in \mathbb{P}(E_x)$ corresponding to a line $\Delta_x \subset E_x$. Put a hermitian metric h on $\psi^*(E)$ (cf. Lemma 2.3.3) and define Q as the orthogonal complement of S with respect to h; then $\psi^*(E) \cong S \oplus Q$. By induction on the rank of E, the claim follows.

To see why uniqueness follows, let $\phi\colon Y \to X$ as in the claim. We obtain an isomorphism $\phi^*(E) \cong L_1 \oplus \cdots \oplus L_n$ for some line bundles L_i on Y. Suppose that

$$c_t(E) = 1 + c_1(E) \cdot t + c_2(E) \cdot t^2 + \cdots + c_r(E) \cdot t^r = \sum_{i=0}^{r} c_i(E) \cdot t^i.$$

Then

$$\sum_{i=0}^{r} \phi^*\left(c_i(E)\right) \cdot t^i = \phi^*\left(c_t(E)\right) = c_t\left(\phi^*(E)\right) = c_t\left(L_1 \oplus \cdots \oplus L_n\right) = \prod_{i=1}^{n} \left(1 + c_1(L_i) \cdot t\right).$$

Thus, if c'_t is another map $VB(X) \to H^\bullet(X,\mathbb{Z})[t]$ with the desired properties, then $\phi^*(c'_i(E)) = \phi^*(c_i(E))$ for each i; as ϕ^* is injective, we get $c_i(E) = c'_i(E)$ for each i. □

Proof of existence. Let $\psi\colon \mathbb{P}(E) \to X$ be the projective bundle associated to E, and let $S \subset \psi^*(E)$ be the tautological line bundle. Define $h = c_1(S^*) \in H^2(\mathbb{P}(E),\mathbb{Z})$. By Lemma 2.3.2, $H^*(\mathbb{P}(E),\mathbb{Z})$ is free as a module over $H^*(X,\mathbb{Z})$, and the elements $1, h, \dots, h^{r-1}$ form a basis for $H^*(\mathbb{P}(E),\mathbb{Z})$ over $H^*(X,\mathbb{Z})$. Therefore, there are elements $a_i \in H^{2i}(X,\mathbb{Z})$ such that

$$h^r + \psi^*(a_1) \cdot h^{r-1} + \cdots + \psi^*(a_{r-1}) \cdot h + \phi^*(a_r) = 0 \qquad \text{in} \qquad H^{2r}(\mathbb{P}(E),\mathbb{Z}).$$

We put $c_0(E) = 1$, $c_i(E) = a_i$ for $1 \le i \le r$, and $c_i(E) = 0$ for $i > r$. We leave it to the reader to verify that conditions (1) – (3) are satisfied. □

Exercise 2.3.6. Let X be a topological manifold. Show that $H^1(X,\mathscr{C}^0) = H^2(X,\mathscr{C}^0) = 0$. Conclude that the morphism $c_1\colon H^1(X,(\mathscr{C}^0)^*) \to H^2(X,\mathbb{Z})$ is an isomorphism.

2.3.2 Hirzebruch–Riemann–Roch theorem

Let E be a vector bundle on a topological manifold X. Let $a_1, \ldots, a_r$ be the *formal Chern roots* of E. To be precise, we define them as formal symbols via the following formula:

$$c_t(E) = \sum_{i=0}^{r} c_i(E) \cdot t^i = \prod_{i=1}^{r} (1 + a_i \cdot t). \tag{2.17}$$

Thus, this means that the a_i are formal variables, subject to the following relations:

$$c_1(E) = \sum_{i=1}^{r} a_i, \quad c_2(E) = \sum_{1 \le i < j \le r} a_i \cdot a_j, \quad \ldots \quad , \quad c_r(E) = \prod_{i=1}^{r} a_i. \tag{2.18}$$

Define the *exponential Chern character* of E as the formal power series

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i}, \quad \text{where} \quad e^{a_i} = 1 + a_i + \frac{1}{2} a_i^2 + \cdots . \tag{2.19}$$

Similarly, define the *total Todd class* of E as the following formal power series, where the B_k are the Bernoulli numbers:

$$\begin{aligned} \mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}}, \quad \text{where} \quad \frac{a_i}{1 - e^{-a_i}} &= 1 + \frac{1}{2} a_i + \frac{1}{12} a_i^2 - \frac{1}{720} a_i^4 + \cdots \\ &= 1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_k}{(2k)!} t^{2k}. \end{aligned} \tag{2.20}$$

Lemma 2.3.7. *Let X be a topological manifold. Then (2.19) and (2.20) can be expressed as polynomials in the $c_i(E)$ with rational coefficients.*

Proof. Exercise. □

Let X be a topological manifold, and let E be a complex vector bundle on X. Define, for each i, the *i-th Chern character* and the *i-th Todd class* of E via the formulae

$$\begin{aligned} \mathrm{td}(E) &= \mathrm{td}_0(E) + \mathrm{td}_1(E) + \cdots , & \mathrm{td}_i(E) &\in H^{2i}(X, \mathbb{Q}) \\ \mathrm{ch}(E) &= \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \ldots , & \mathrm{ch}_i(E) &\in H^{2i}(X, \mathbb{Q}). \end{aligned}$$

For a complex manifold X of dimension n, with holomorphic tangent bundle $\mathcal{T}_X$, define the following invariants:

$$c_i(X) = c_i(\mathcal{T}_X), \qquad \mathrm{ch}_i(X) = \mathrm{ch}_i(\mathcal{T}_X) \qquad \text{and} \qquad \mathrm{td}_i(X) = \mathrm{td}_i(\mathcal{T}_X).$$

Moreover, if E is a holomorphic vector bundle on X, we put

$$\chi(X, E) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, E).$$

We then have the following fundamental result, whose proof we will omit.

Theorem 2.3.8 (Hirzebruch–Riemann–Roch)**.** *Let E be a holomorphic vector bundle on a compact complex manifold X of dimension n. Consider the degree $2n$-component of* $\mathrm{ch}(E) \cdot \mathrm{td}(X)$*, defined as* $(\mathrm{ch}(E) \cdot \mathrm{td}(X))_{2n} = \sum_{i=0}^{n} \mathrm{ch}_i(E)\mathrm{td}_{n-i}(X)$*. Then*

$$\chi(X, E) = \int_X (\mathrm{ch}(E) \cdot \mathrm{td}(X))_{2n} .$$

Proof. See [BS58]. □

Exercise 2.3.9. Let E be a vector bundle Let E and F be vector bundles on a topological space X. Show that $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$, and that $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$. Show also that $c_i(E^\vee) = (-1)^i \cdot c_i(E)$ for each i.

Exercise 2.3.10. Let E be a holomorphic vector bundle on a complex compact manifold X. Deduce from Theorem 2.3.8 that $\chi(X, E)$ is independent of the holomorphic structure of E. In other words, prove that $\chi(X, E)$ depends only on the structure of E as a complex topological vector bundle.

2.3.3 Gauss–Bonnet formula

Let X be a compact complex manifold of dimension n. For integers $p, q \geq 0$, define $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$. The *Hirzebruch χ_y-genus* is the polynomial

$$\chi_y(X) = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) \cdot y^p. \tag{2.21}$$

Define the *Euler number of X* as follows:

$$e(X) = \sum_{i=0}^{2n} (-1)^i b_i(X) = \sum_{i=0, i \neq n}^{2n} (-1)^i b_i(X) + (-1)^n b_n(X). \tag{2.22}$$

Here, $b_i(X)$ is the i-th Betti number $b_i(X) = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ of X.

Corollary 2.3.11. *Let X be a compact Kähler manifold. Then $\chi_{y=-1}(X) = e(X)$.*

Proof. This will follow from Hodge theory, see Chapter 3. Indeed, Hodge theory shows that $b_i(X) = \sum_{p=0}^{i} h^{p,i-p}(X)$, see Theorem 3.1.4 and Proposition 3.2.20. Therefore,

$$\chi_{y=-1}(X) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i \sum_{p+q=i} h^{p,q}(X) = \sum_{i=0}^{n} (-1)^i b_i(X),$$

proving the corollary. □

Corollary 2.3.12. *Let X be a compact complex manifold. Let $\gamma_1, \ldots, \gamma_n$ be the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$ of X, see (2.17). Then*

$$\chi_y(X) = \int_X \prod_{i=1}^{n} \left(1 + y \cdot e^{-\gamma_i}\right) \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

Proof. Exercise. □

Proposition 2.3.13. *Let X be a compact Kähler manifold of dimension n. Then*

$$e(X) = \int_X c_n(X).$$

Proof. By Corollary 2.3.11, we have $e(X) = \chi_{y=-1}(X)$, and by Corollary 2.3.12, we have $\chi_{y=-1}(X) = \int_X \prod_i \gamma_i$, where $\gamma_1, \dots, \gamma_n$ are the formal Chern roots of the holomorphic tangent bundle $\mathcal{T}_X$. The proposition follows, as $\prod_i \gamma_i = c_n(X)$ by (2.18). □

2.3.4 Betti cohomology of smooth hypersurfaces

Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth complex hypersurface. Our next goal is to compute the middle Betti number $b_n(X) = \dim_{\mathbb{Q}} H^n(X, \mathbb{Q})$. Consider the Euler number $e(X)$ of X defined in (2.22).

Lemma 2.3.14. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth complex hypersurface. Then*

$$e(X) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n)\,. \tag{2.23}$$

Proof. By Corollary 2.2.7, we have $b_i(X) = 0$ for $i \neq n$ odd and $b_i(X) = 1$ for $i \neq n$ even. Hence

$$\begin{aligned}
e(X) &= \sum_{i=0, 2i\neq n}^{n} (-1)^{2i} b_{2i}(X) + \sum_{i=1, 2i-1\neq n}^{n} (-1)^{2i-1} b_i(X) + (-1)^n b_n(X) \\
&= \left(\sum_{i=0, 2i\neq n}^{n} 1 \right) + (-1)^n b_n(X) \\
&= \begin{cases} n + b_n(X) & \text{if } n \equiv 0(2), \\ n + 1 - b_n(X) & \text{if } n \equiv 1(2). \end{cases}
\end{aligned}$$

This proves what we want. □

Proposition 2.3.15. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d and dimension $n \geq 0$. Let $b_n(X)$ be the n-th Betti number of X. Then*

$$b_n(X) = \frac{(-1)^n}{2d} \cdot \left(2 \cdot (1-d)^{n+2} + 3 \cdot d + (-1)^n \cdot d - 2\right). \tag{2.24}$$

Proof. See Section 2.3.1 above for an introduction to Chern classes. By Proposition 2.3.13, we have

$$e(X) = \int_X c_n(X), \quad \text{where} \quad c_n(X) = c_n(\mathcal{T}_X) \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Notice that sequence (2.3) yields an exact sequence

$$0 \to \Omega_{\mathbb{P}}|_X \to \mathcal{O}_X(-1)^{n+2} \to \mathcal{O}_X \to 0,$$

which, after dualizing, gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+2} \to \mathcal{T}_{\mathbb{P}}|_X \to 0.$$

We also consider the sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}|_X \to \mathcal{O}_X(d) \to 0,$$

that follows by dualizing (2.8). By item (3) in Theorem 2.3.4, we obtain

$$\begin{aligned} c(X) &= \sum_i c_i(X) = \sum_i c_i(\mathcal{T}_X) = c(\mathcal{T}_X) = c(\mathcal{T}_{\mathbb{P}}|_X) \cdot c(\mathcal{O}_X(d))^{-1} \\ &= c\left(\mathcal{O}_X(1)^{\oplus(n+2)}\right) \cdot c(\mathcal{O}_X(d))^{-1} = \frac{(1+h)^{n+2}}{(1+dh)}, \qquad h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z}). \end{aligned}$$

We now have the following:

Claim 2.3.16. *Let h be a variable, and consider the ring $R = \mathbb{Q}[h]/(h^{n+1})$. Then $(1+dh)$ is invertible in R hence $(1+dh)^{-1} \cdot (1+h)^{n+2}$ is a well-defined element in $\mathbb{Q}[h]/(h^{n+1})$. Moreover, its coefficient before h^n is $(1/d^2) \cdot ((1-d)^{n+2} + d \cdot (n+2) - 1)$.*

Proof. Exercise. □

By combining $\deg(h) = \int_X h^n = d$, equality (2.23) and Claim 2.3.16, we obtain:

$$e(X) = \frac{1}{d} \cdot \left((1-d)^{n+2} + d \cdot (n+2) - 1\right) = n + (-1)^n \cdot b_n(X) + \frac{1}{2} \cdot (1 - (-1)^n).$$

In particular,

$$(-1)^n \cdot b_n(X) = \frac{2(1-d)^{n+2} + 2d \cdot (n+2) - 2 - 2nd - d + (-1)^n \cdot d}{d},$$

and equality (2.24) follows. □

Corollary 2.3.17. *The n-th Betti number $b_n(X_3)$ of a smooth cubic hypersurface $X_3 \subset \mathbb{P}^n_{\mathbb{C}}$ of dimension $n \geq 0$ is given by the following formula:*

$$b_n(X_3) = \frac{1}{6} \cdot \left(2^{n+3} + (-1)^n \cdot 7 + 3\right).$$

For instance, $b_0(X_3) = 3$, $b_1(X_3) = 2$, $b_2(X_3) = 7$ and $b_3(X_3) = 10$. □

Exercise 2.3.18. For a smooth projective variety X over $\mathbb{C}$, define $h^{p,q}(X) = h^q(X, \Omega^p_X)$. Calculate all the values $h^{p,q}(X)$ with $p + q = 3$ for a smooth cubic threefold $X \subset \mathbb{P}^4_{\mathbb{C}}$, and all the $h^{p,q}(X)$ with $p + q = 4$, for a smooth cubic fourfold $X \subset \mathbb{P}^5_{\mathbb{C}}$.

2.4 Lecture 4: Intersection form on middle cohomology

Let X be a compact complex manifold of dimension n. Poincaré duality provides canonical isomorphisms $H^i(X,\mathbb{Z}) \cong H_{2n-i}(X,\mathbb{Z})$. Moreover, the universal coefficient theorem provides a canonical isomorphism $H^i(X,\mathbb{Z})/(\mathrm{tors}) \cong \mathrm{Hom}(H_i(X,\mathbb{Z}),\mathbb{Z})$. Combining the two assertions, one sees that the cap product pairing

$$H_i(X,\mathbb{Z})/(\mathrm{tors}) \otimes H_{2n-i}(X,\mathbb{Z})/(\mathrm{tors}) \to H_0(X,\mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing. Dually, the cup product pairing

$$H^i(X,\mathbb{Z})/(\mathrm{tors}) \otimes H^{2n-i}(X,\mathbb{Z})/(\mathrm{tors}) \to H^{2n}(X,\mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing.

Lemma 2.4.1. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of dimension $n \geq 0$. Then $H^n(X,\mathbb{Z})$ is torsion-free.*

Proof. For $n = 0$, the claim is trivial, so we may assume $n \geq 1$. The universal coefficient theorem gives then an exact sequence

$$0 = \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) \to H^n(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Here, $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),\mathbb{Z}) = 0$ because $H_{n-1}(X,\mathbb{Z}),\mathbb{Z})$ is trivial or isomorphic to $\mathbb{Z}$, see Corollary 2.2.7. As $\mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}),\mathbb{Z})$ is torsion-free, the lemma follows. □

In particular, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, we obtain a perfect pairing

$$\cup\colon H^n(X,\mathbb{Z}) \otimes H^n(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z}) = \mathbb{Z}. \tag{2.25}$$

Recall that, for $\alpha, \beta \in H^n(X,\mathbb{Z})/(\mathrm{tors})$, we have $\alpha \cup \beta = (-1)^n \cdot \beta \cup \alpha$. This implies that (2.25) is symmetric if n is even, and alternating if n is odd. The goal of this section is to study (2.25) in case $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth cubic hypersurface of dimension n.

2.4.1 Odd-dimensional cubic hypersurfaces

It turns out that if X is an odd-dimensional hypersurface, the intersection form on $H^n(X,\mathbb{Z})$ is quite easily calculated, as follows from the following lemma.

Lemma 2.4.2. *Let Λ be a free $\mathbb{Z}$-module of rank $n > 0$ and let*

$$E\colon \Lambda \otimes \Lambda \to \mathbb{Z} \tag{2.26}$$

be an alternating bilinear form on $\mathbb{Z}$, defining a perfect pairing. Then $n = 2g$ and there exists a basis $\{e_1, \dots, e_g; f_1, \dots f_g\}$ for Λ such that $E(e_i, e_j) = E(f_i, f_j) = 0$ for all i, j, and such that $E(e_i, f_i) = 1$ for all i and $E(e_i, f_j) = 0$ if $i \neq j$.

Proof. Notice that $n = \mathrm{rank}(\Lambda) \geq 2$, for if $n = 1$ then $E(x, y) = 0$ for each $x, y \in \Lambda$. Suppose first that $n = 2$. Let $\{x, y\} \subset \Lambda$ be a basis for Λ. Let $M = (m_{ij})$ be the intersection matrix of E with respect to this basis. We have $m_{11} = E(x, x) = 0$, $m_{12} = E(x, y)$, $m_{21} = -E(x, y)$ and $m_{22} = E(y, y) = 0$. Thus, the determinant of M equals $E(x, y)^2$, which must be invertible in $\mathbb{Z}$. Hence $E(x, y) = \pm 1$, and the result follows.

Next, assume $n \geq 3$. Let $y \in \Lambda$ and $W \subset \Lambda$ such that

$$\mathbb{Z} \cdot y \oplus W = \Lambda.$$

Define a linear map $f \colon \Lambda \to \mathbb{Z}$ by putting $f(y) = 1$ and $f(w) = 0$ for each $w \in W$, and extending linearly. As the pairing (2.26) is perfect, there exists an element $x \in \Lambda$ such that $E(x, -) = f$ as linear maps $\Lambda \to \mathbb{Z}$. This implies that $E(x, y) = 1$ and $E(x, w) = 0$ for each $w \in W$. Let $P = \langle x, y\rangle^{\perp}$ be the orthogonal complement of $\langle x, y\rangle$ in Λ with respect to E. We claim that

$$\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y \oplus P = \Lambda. \tag{2.27}$$

To prove this, let $\lambda \in \Lambda$. We must show that there exist unique $a, b \in \mathbb{Z}$ such that $\lambda - a \cdot x - b \cdot y \in P$. That is, we need to show there exist unique $a, b \in \mathbb{Z}$ such that

$$\begin{aligned} E\left(x, \lambda - a \cdot x - b \cdot y\right) &= E(x, \lambda) + b = 0, \\ E\left(y, \lambda - a \cdot x - b \cdot y\right) &= E(y, \lambda) - a = 0. \end{aligned}$$

We may simply put $b = -E(x, \lambda)$ and $a = E(y, \lambda)$. Decomposition (2.27) follows.

To finish the proof, we would like to show that the restriction of E to $P \otimes P$ defines a perfect pairing, i.e. a unimodular alternating bilinear form. To see this, observe that by choosing a basis $\{p_1, \dots, p-2\}$ for P, the form E becomes associated to a $(n-2) \times (n-2)$-matrix $M_P := (E(p_i, p_j))$. Similarly, one attaches a matrix $M_{x,y}$ to the pairing that E defines on $\mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y$. The basis $\{x, y, p_1, \dots, p_{n-2}\}$ for Λ then associates a matrix M_Λ to E, and we have

$$\det(M_P) \cdot \det(M_{x,y}) = \det(M) = \pm 1,$$

where the last equality holds because E is unimodular. Therefore, $\det(M_P) = \pm 1$, hence the restriction of E to $P \otimes P$ is unimodular. The lemma follows by induction on the rank n of Λ. □

Corollary 2.4.3. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be an odd-dimensional smooth cubic hypersurface. Then $H^n(X, \mathbb{Z})$ is free of rank $b_n(X) = 2m$ over $\mathbb{Z}$, and admits a basis $\{\gamma_1, \dots, \gamma_{2m}\}$ with respect to which the intersection matrix of the pairing $H^n(X, \mathbb{Z}) \otimes H^n(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$ has the following form, where $\mathrm{Id} \in \mathrm{GL}_m(\mathbb{Z})$ denotes the identity matrix:*

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Proof. Torsion-freeness follows from Lemma 2.4.1. As the dimension of X is odd, (2.25) is a unimodular, alternating bilinear form, and we can apply Lemma 2.4.2. □

2.4.2 Even-dimensional cubic hypersurfaces

We are going to use the following result, without providing a proof:

Proposition 2.4.4. *If a smooth projective variety X over $\mathbb{C}$ (or, more generally, a compact Kähler manifold) has even dimension $2m$, and if $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$, then the intersection pairing on $H^n(X, \mathbb{R})$ has signature*

$$\mathrm{sgn}(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X).$$

Proof. See [Voi02, Théorème 6.33] or [Huy05, Corollary 3.3.18]. □

Corollary 2.4.5. *Let X be a smooth projective variety of dimension $n = 2m$ over $\mathbb{C}$. Consider the Hirzebruch χ_y-genus $\chi_y(X)$, see (2.21). Then $\chi_{y=1}(X) = \tau(X)$.* □

We shall also need the following result, whose proof we omit:

Theorem 2.4.6 (Hirzebruch)**.** *Let $X_n \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a sequence of smooth hypersurfaces of degree d. For each n, let $\chi_y(X_n)$ be the Hirzebruch χ_y-genus of X_n, cf. (2.21). Then*

$$\sum_{n=0}^{\infty} \chi_y(X_n) z^{n+1} = \frac{1}{(1+yz)(1-z)} \cdot \frac{(1+yz)^d - (1-z)^d}{(1+yz)^d + y(1-z)^d}. \tag{2.28}$$

Proof. See [Hir95, Theorem 22.1.1]. □

Notice that, by Proposition 2.4.4, for $y = 1$ and $d = 3$ we can rewrite (2.28) as

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(X_n) z^{n+1} &= \frac{1}{(1+z)(1-z)} \cdot \frac{(1+z)^3 - (1-z)^3}{(1+z)^3 + (1-z)^3} \\ &= (-1) \cdot \frac{z^3 + 3z}{3z^4 - 2z^2 - 1} \\ &= z \cdot \frac{3 + z^2}{(1+3z^2) \cdot (1-z^2)}.\end{aligned}$$

Lemma 2.4.7. *Consider the power series expansion*

$$z \cdot \frac{3+z^2}{(1+3z^2) \cdot (1-z^2)} = z \cdot \sum_{i=0}^{\infty} a_i \cdot z^i.$$

Then $a_{2m} = (-1)^m \cdot 2 \cdot 3^m + 1$ for each $m \geq 0$.

Proof. Exercise. □

Combining the above, we obtain:

Proposition 2.4.8. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. Let $\tau(X)$ be the signature of the pairing $H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to \mathbb{R}$. Then $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.* □

Let Λ be a *lattice*, i.e. a free $\mathbb{Z}$-module equipped with a symmetric bilinear form $(\cdot\,,\cdot)$. We say that Λ is unimodular when the pairing is perfect, i.e. when the determinant of any intersection matrix is ± 1. We say that a unimodular lattice is *even* if $(\alpha, \alpha) \equiv 0 \bmod 2$ for all $\alpha \in \Lambda$; otherwise, we say that Λ is *odd*. For example, the rank one lattice $\mathbb{Z}(a)$ with $(1, 1) = a$ is odd if and only if a is odd.

If Λ is unimodular, odd, and indefinite, then for some positive integers r, s, we have

$$\Lambda \cong I_{r,s} := \mathbb{Z}(1)^{\oplus r} \oplus \mathbb{Z}(-1)^{\oplus s}. \tag{2.29}$$

For this, see for example [Ser73, Chapter V, Theorem 4].

Theorem 2.4.9. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of even dimension $n = 2m$. The intersection form on $H^n(X, \mathbb{Z})$ turns $H^n(X, \mathbb{Z})$ into a unimodular lattice, and there exists an isomorphism of lattices*

$$H^n(X, \mathbb{Z}) \cong \mathbb{Z}(1)^{\oplus b_n^+} \oplus \mathbb{Z}(-1)^{\oplus b_n^-} = I_{b_n^+, b_n^-}. \tag{2.30}$$

Here, $b_n^+ := b_n^+(X)$ is defined as the number of positive eigenvalues of an intersection matrix of the associated form on $H^n(X, \mathbb{R})$, and $b_n^- = b_n^-(X) = b_n(X) - b_n^+(X)$. The two integers b_n^+ and b_n^- can be calculated from the two equalities $b_n^+ + b_n^- = b_n(X) = (1/6) \cdot (2^{n+3} + (-1)^n \cdot 7 + 3)$ and $b_n^+ - b_n^- = \tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$.

Proof. We prove that $H^n(X, \mathbb{Z})$ is odd. This is easy: the class $h^m = c_1(\mathcal{O}_X(1))$ in $H^n(X, \mathbb{Z})$ satisfies $(h^m, h^m) = \int_X h^n = d$. Moreover, it was shown in Corollary 2.3.17 that we have $b_n(X) = (1/6) \cdot (2^{n+3} + (-1)^n \cdot 7 + 3)$, and the fact that $\tau(X) = (-1)^m \cdot 2 \cdot 3^m + 1$ follows from Proposition 2.4.8. In particular, $b_n(X) \neq \pm\tau(X)$, hence $H^n(X, \mathbb{Z})$ is indefinite. The isomorphism (2.30) follows then by the above-mentioned classification of odd indefinite unimodular lattices. ☐

2.4.3 Cubic surfaces

Proposition 2.4.10. *Let X be a compact complex manifold of dimension two. Let L be a line bundle on X. Then*

$$\chi(X, \mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} \quad \text{and}$$
$$\chi(X, L) = \int_X \frac{c_1(L)^2 + c_1(L) \cdot c_1(X)}{2} + \chi(X, \mathcal{O}_X).$$

Proof. One calculates the value of $\mathrm{td}(X)$, which is $\mathrm{td}(X) = 1 + c_1(X)/2 + c_1(X)^2/12 + c_2(X)/12$. Moreover, $\mathrm{ch}(L) = e^{c_1(L)}$, and the result follows from Theorem 2.3.8. ☐

Lemma 2.4.11. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface, and let L be a line bundle on X. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$. Then*

$$\chi(X, L) = \frac{(L, L) + (L, h)}{2} + 1. \tag{2.31}$$

Proof. Let X be a smooth cubic hypersurface. Then

$$c(X) = \frac{(1+h)^{n+2}}{1+3h} = \left(1 - 3h + (3h)^2 \pm \cdots\right) \cdot \sum_{i=0}^{n} \binom{n+2}{i} h^i.$$

Hence, $c(X) = (1 - 3h + (3h)^2 \pm \cdots) \cdot \left(1 + (n+2) \cdot h + \binom{n+2}{2} \cdot h^2 + \cdots\right)$, which gives

$$\begin{aligned} c_1(X) &= (n+2) \cdot h - 3h = (n-1) \cdot h \\ c_2(X) &= \left(9 - 3 \cdot (n+2) + \binom{n+2}{2}\right) \cdot h^2. \end{aligned}$$

For $n = 2$, this becomes $c_1(X) = h \in H^2(X, \mathbb{Z})$ and $c_2(X) = 3 \cdot h^2$. Together with Proposition 2.4.10, this implies that $\chi(X, L) = (1/2) \cdot ((L, L) + (L, h)) + \chi(X, \mathcal{O}_X)$.

It remains to show that

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1. \tag{2.32}$$

This follows immediately from the fact that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ by Claim 2.1.15. Alternatively, we can use the fact that $c_1(X)^2 + c_2(X) = h^2 + 3h^2 = 4h^2$; applying Proposition 2.4.10 yields

$$\chi(X, \mathcal{O}_X) = \int_X \frac{c_1(X)^2 + c_2(X)}{12} = \int_X \frac{4h^2}{12} = 1.$$

This proves (2.32), and hence the lemma. □

We can now study the intersection form on $H^2(X, \mathbb{Z})$ for a smooth cubic surface $X \subset \mathbb{P}^3_{\mathbb{C}}$.

Lemma 2.4.12. *Let X be a smooth cubic surface. Then $H^2(X, \mathbb{Z}) \cong I_{1,6}$.*

Proof. We have $H^2(X, \mathbb{Z}) \cong I_{r,s}$ for some $r, s \in \mathbb{Z}_{\geq 1}$ by Theorem 2.4.9. We need to prove $r = 1$ and $s = 6$. This follows, as $\tau(X) = -5$, see Theorem 2.4.9. □

Let Λ be an odd unimodular lattice. A primitive vector $\alpha \in \Lambda$ is called *characteristic* if $(\alpha, \beta) \equiv (\beta, \beta) \bmod 2$ for all $\beta \in \Lambda$.

Consider the lattice $I_{1,6}$, see (2.29). Let $\alpha = (3, 1, 1, 1, 1, 1, 1)$ and define $e_1 = (0, 1, -1, 0, 0, 0, 0)$. Similarly, define $e_2 = (0, 0, 1, -1, 0, 0, 0)$, $e_3 = (0, 0, 0, 1, -1, 0, 0)$, $e_4 = (1, 0, 0, 0, 1, 1, 1)$, $e_5 = (0, 0, 0, 0, 1, -1, 0)$ and $(e_7 = (0, 0, 0, 0, 1, -1)$.

Lemma 2.4.13. *The element $\alpha \in I_{1,6}$ is characteristic. Moreover, the elements e_i with $i \in \{1, 2, 3, 4, 5, 7\}$ span the lattice $\alpha^\perp$, and their intersection matrix is an intersection matrix for the lattice $E_6(-1)$. In particular, $\alpha^\perp \cong E_6(-1)$.*

Proof. Exercise. □

Theorem 2.4.14. *Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface. Let $h = c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$, and consider the sublattice $H^2(X, \mathbb{Z})_{prim} := \langle h \rangle^\perp$ of $H^2(X, \mathbb{Z})$. The lattice $H^2(X, \mathbb{Z})_{prim}$ is isomorphic to $E_6(-1)$.*

Proof. We claim that $h \in H^2(X, \mathbb{Z})$ is characteristic. Indeed, as $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ by Corollary 2.1.14, it suffices to show that $(L, L) \equiv (L, h) \bmod 2$ for every $L \in \mathrm{Pic}(X)$, which follows from Lemma 2.4.11. We then apply a general result for unimodular lattices: two primitive vectors $x, y \in \Lambda$ are in the same $O(\Lambda)$ orbit if and only if $(x, x) = (y, y)$ and either both are characteristic or both are not. As $\alpha = (3, 1, 1, 1, 1, 1, 1) \in I_{1,6}$ is characteristic by Lemma 2.4.13, and as $h \in H^2(X, \mathbb{Z})$ is characteristic by the above, it follows that α and the image of h in $I_{1,6}$ are in the same $O(I_{1,6})$-orbit. In particular, $H^2(X, \mathbb{Z})_{prim} = \langle h \rangle^{\perp} \cong \langle \alpha \rangle^{\perp}$, which is isomorphic to $E_6(-1)$, see Lemma 2.4.13. $\square$

Chapter 3

Hodge theory: general theory

3.1 Lecture 5: Hodge decomposition theorem (statement)

3.1.1 Abstract Hodge structures

Definition 3.1.1. Let $k \in \mathbb{Z}_{\geq 0}$. An *integral Hodge structure of weight k* consists of a finitely generated abelian group $V_{\mathbb{Z}}$ and a decomposition of $V_{\mathbb{C}}$ into complex vector subspaces

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}, \tag{3.1}$$

such that $V^{p,q} = \overline{V^{q,p}}$. Here, $x \mapsto \bar{x}$ is the anti-linear $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on $V_{\mathbb{C}}$.

Let $V_{\mathbb{Z}}$ be a Hodge structure of weight k. Define the *Hodge filtration* $F^{\bullet}V_{\mathbb{C}}$ as the filtration

$$F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r}. \tag{3.2}$$

This is a decreasing filtration on $V_{\mathbb{C}}$ and satisfies the property that

$$F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} = V_{\mathbb{C}}.$$

One retrieves the Hodge decomposition (3.1) as follows:

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

Definition 3.1.2. Let $V_{\mathbb{Z}}$ be an integral Hodge structure of weight k. The *Weil operator* is the automorphism $C \colon V_{\mathbb{C}} \xrightarrow{\sim} V_{\mathbb{C}}$ defined by $C \cdot v = i^{p-q} v$ for $v \in V^{p,q}$. A *polarization* of $V_{\mathbb{Z}}$ is a bilinear form $Q \colon V \otimes V \to \mathbb{Z}$ which is $(-1)^k$-symmetric and such that, for the $\mathbb{C}$-bilinear extension $Q_{\mathbb{C}}$ of Q to $V_{\mathbb{C}}$, one has:

(1) The orthogonal complement of F^p is F^{k-p+1};

(2) The hermitian form $(u, v) \mapsto Q_{\mathbb{C}}(C \cdot u, \bar{v})$ is positive definite.

Remark that as the Weil operator is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, it descends to an automorphism $C\colon V_{\mathbb{R}} \xrightarrow{\sim} V_{\mathbb{R}}$, where $V_{\mathbb{R}} = V \otimes \mathbb{R}$. Moreover, the above definitions readily extend to subrings $R \subset \mathbb{R}$ other than $\mathbb{Z}$. In particular, one defines (polarized) *rational* and *real Hodge structures* in a similar way, replacing $\mathbb{Z}$ by $\mathbb{Q}$ or $\mathbb{R}$ respectively, in the definitions above. A Hodge structure is *polarizable* if it admits a polarization. The category of polarizable rational Hodge structures is abelian and semi-simple.

Example 3.1.3. Let C be a compact Riemann surface of genus $g \geq 1$. The exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$$

gives rise to a surjection

$$H^1(C, \mathbb{C}) = H^1(C, \mathbb{Z}) \otimes \mathbb{Z} \to H^1(C, \mathcal{O}_C)$$

whose kernel is the subspace $H^0(C, \Omega_C)$ of holomorphic one-forms on C. Indeed, the exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_C \to \Omega_C \to 0$$

induces a long exact sequence

$$\mathbb{C} = H^0(C, \mathcal{O}_C) \xrightarrow{0} H^0(C, \Omega_C) \hookrightarrow H^1(C, \mathbb{C}) \twoheadrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{0} H^1(C, \Omega_C) \xrightarrow{\sim} H^2(C, \mathbb{C}).$$

Consider the complex conjugate subspace $\overline{H^0(C, \Omega_C)}$ of $H^0(C, \Omega_C)$ in $H^1(C, \mathbb{C})$. As $H^0(C, \Omega_C) \cap \overline{H^0(C, \Omega_C)} = 0$ and $\dim H^1(C, \mathbb{C}) = 2g = 2 \cdot \dim H^0(C, \Omega_C)$, we have

$$H^1(C, \mathbb{C}) = H^0(C, \Omega_C) \oplus \overline{H^0(C, \Omega_C)}.$$

Therefore, the projection $H^1(C, \mathbb{C}) \to H^1(C, \mathcal{O}_C)$ induces a canonical isomorphism

$$\overline{H^0(C, \Omega_C)} = H^1(C, \mathcal{O}_C).$$

Finally, consider the pairing

$$H\colon H^1(C, \mathbb{C}) \times H^1(C, \mathbb{C}) \to \mathbb{C}, \quad H(\alpha, \beta) = i \cdot Q(\alpha, \bar{\beta}) = i \cdot \int_C \alpha \wedge \overline{\beta}.$$

Then $H(\alpha, \alpha) > 0$ for $\alpha \in H^0(C, \Omega_C) \subset H^1(C, \mathbb{C})$ non-zero.

The goal of Lectures 5 and 6 is to generalize the above example by proving the following:

Theorem 3.1.4 (Hodge)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Then for each integer $k \geq 0$, the singular cohomology group $H^k(X, \mathbb{Z})$ admits an integral Hodge structure of weight k in a canonical way, and $H^k(X, \mathbb{Z})_{prim}$ admits a sub-Hodge structure of $H^k(X, \mathbb{Z})$ which has a canonical polarization. Moreover, associating a weight k integral Hodge structure to a smooth projective variety X is contravariantly functorial in X, as well as compatible with cup-products and Gysin homomorphisms.*

3.1.2 Algebraic De Rham complex

We remark that although the above theorem only makes sense for varieties over $\mathbb{C}$, the *Hodge filtration* has a meaning in much larger generality. Namely, for a smooth projective variety X over a field k, one can consider the *algebraic De Rham complex*

$$\Omega_X^\bullet := \left(0 \to \mathcal{O}_X \to \Omega_X \to \Omega_X^2 \to \cdots \to \Omega_X^{\dim X} \to 0\right), \tag{3.3}$$

as well as, for each integer $p \geq 0$, the sub-complex

$$\Omega_X^\bullet \supset F^p\Omega_X^\bullet = \left(0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^{\dim X} \to 0\right).$$

We may then define

$$\begin{aligned} H^k_{dR}(X/k) &= H^k\left(R\Gamma(X, \Omega_X^\bullet)\right) \qquad \text{and} \\ F^p H^k_{dR}(X/k) &= \operatorname{Im}\left(H^k\left(R\Gamma(X, F^p\Omega_X^\bullet)\right) \to H^k_{dR}(X/k)\right). \end{aligned} \tag{3.4}$$

If $k = \mathbb{C}$, then $H^k_{dR}(X/\mathbb{C}) = H^k(X, \mathbb{C})$. Indeed, one has $(\Omega_X^\bullet)^{an} = \Omega_{X^{an}}^\bullet$ by Serre's GAGA theorem, and this complexification $\Omega_{X^{an}}^\bullet$ of (3.3) provides a resolution of the constant sheaf $\mathbb{C}$ on X. The filtration $F^\bullet$ on $H^k(X, \mathbb{C})$ induced by (3.4) is exactly the Hodge filtration (3.2) associated to the Hodge structure on $H^k(X, \mathbb{Z})$ provided by Theorem 3.1.4.

There are two crucial differences between the complex case and the general case. First of all, even though for any smooth projective variety X over a field k, there is a canonical spectral sequence, de *Hodge to De Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}_{dR}(X/k) \tag{3.5}$$

with corresponding filtration on $H^k_{dR}(X/k)$ given by (3.4), this spectral sequence does (in contrast to the case $k = \mathbb{C}$) not always degenerate. Secondly, even if (3.5) degenerates for a certain smooth projective variety X over k, there is on the one hand no natural analogue of complex conjugation on $H^k_{dR}(X/k)$ if $k \neq \mathbb{C}$, and on the other in general no natural inclusion of $H^q(X, \Omega^p)$ into $H^{p+q}_{dR}(X/k)$.

3.1.3 Hodge star operator

Let X be a differentiable manifold, provided with a (smooth) Riemannian metric g. Suppose that X is oriented and compact, and let Vol be the volume form of X relative to g. This means that $\mathrm{Vol} \in A^n(X)$ is a smooth n-form, where $n = \dim(X)$, which is everywhere non-zero and such that for each $x \in X$, $\mathrm{Vol}(x) \in \Omega^n_{X,x}$ is the unique n-form which is positive on each oriented basis of $T_{X,x}$ and of norm one with respect to the induced metric on $\Omega^n_{X,x}$.

Observe that g induces a metric $(\ ,\ )_x$ on each vector space $\Omega^k_{X,x}$. For $\alpha, \beta \in A^k(X)$, one obtains a smooth function $(\alpha, \beta) \colon X \to \mathbb{R}$ sending x to $(\alpha, \beta)(x) = (\alpha_x, \beta_x)_x$. Define the *$L^2$-metric* on the space of real differentiable k-forms as follows:

$$(\ ,\ )_{L^2} \colon A^k(X) \times A^k(X) \to \mathbb{R}, \quad (\alpha, \beta)_{L^2} = \int_X (\alpha, \beta)\mathrm{Vol}.$$

For $x \in X$, consider the canonical isomorphism $\mathrm{Vol}(x)\colon \wedge^n \Omega_{X,x} \to \mathbb{R}$ provided by the volume form. We have a natural composition of isomorphisms

$$\bigwedge^{n-k} \Omega_{X,x} \xrightarrow[\sim]{p} \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \bigwedge^n \Omega_{X,x}\right) \xrightarrow[\sim]{\mathrm{Vol}(x)} \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \mathbb{R}\right).$$

Moreover, the metric $(\ ,\ )_x$ provides an isomorphism

$$m\colon \bigwedge^k \Omega_{X,x} \xrightarrow{\sim} \mathrm{Hom}\left(\bigwedge^k \Omega_{X,x}, \mathbb{R}\right).$$

Definition 3.1.5. Let X be an oriented compact Riemannian manifold. Define

$$\star_x\colon \bigwedge^k \Omega_{X,x} \xrightarrow{\sim} \bigwedge^{n-k} \Omega_{X,x} \qquad \text{as the isomorphism} \qquad p^{-1} \circ m.$$

Similarly, denote by $\star$ the induced isomorphism of vector bundles, respectively spaces of global sections:

$$\star\colon \Omega_X^k \xrightarrow{\sim} \Omega_X^{n-k}, \qquad \text{respectively} \qquad \star\colon A^k(X) \xrightarrow{\sim} A^{n-k}(X).$$

We call $\star\colon A^k(X) \xrightarrow{\sim} A^{n-k}(X)$ the *Hodge star operator*. We extend $\star$ by $\mathbb{C}$-linearity to an isomorphism $\star\colon A_{\mathbb{C}}^k(X) \xrightarrow{\sim} A_{\mathbb{C}}^{n-k}(X)$ of spaces of complex differential forms on X.

Lemma 3.1.6. *Let X be an oriented compact Riemannian manifold. We have*

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star\beta \qquad \forall\, \alpha, \beta \in A^k(X).$$

Proof. It suffices to show that for each $x \in X$, we have $(\alpha_x, \beta_x)_x \mathrm{Vol}_x = \alpha_x \wedge \star\beta_x$. By construction, the following diagram commutes:

$$\begin{tikzcd}
\bigwedge^k \Omega_{X,x} \arrow[r, "m"] \arrow[d, "\star"] & \mathrm{Hom}(\bigwedge^k \Omega_{X,x}, \mathbb{R}) \\
\bigwedge^{n-k} \Omega_{X,x} \arrow[r, "p"] & \mathrm{Hom}(\bigwedge^k \Omega_{X,x}, \bigwedge^n \Omega_{X,x}). \arrow[u, "\mathrm{Vol}(x)"']
\end{tikzcd}$$

The equality $(\alpha_x, \beta_x)_x \mathrm{Vol}_x = \alpha_x \wedge \star\beta_x$ follows from this. $\square$

Lemma 3.1.7. *Let X be an oriented compact Riemannian manifold. Consider the composition $\star^2 = \star \circ \star$. Then $\star^2 = (-1)^{k(n-k)}$ as maps $A^k(X) \to A^k(X)$.*

Proof. Indeed, for every $\alpha, \beta \in A^k(X)$, we have

$$\alpha_x \wedge \star\beta_x = (\alpha_x, \beta_x)\mathrm{Vol}_x = (\star\alpha_x, \star\beta_x)\mathrm{Vol}_x = \star\beta_x \wedge \star\star\,\alpha_x = (-1)^{k(n-k)} \star\star\alpha_x \wedge \star\beta_x.$$

As this holds for every $\beta_x \in A^k(X)$, we have $(-1)^{k(n-k)} \star\star\alpha_x = \alpha_x$ as desired. $\square$

Define an operator d^* as

$$d^*\colon A^k(X) \to A^{k-1}(X), \quad d^* = (-1)^k \star^{-1} d \star .$$

Lemma 3.1.8. *Let X be an oriented compact Riemannian manifold. Let $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in A^{k-1}(X)$ and $\beta \in A^k(X)$. Then*

$$(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}.$$

Proof. On the one hand, we have

$$(d\alpha, \beta)_{L^2} = \int_X d\alpha \wedge \star\beta = \int_X d(\alpha \wedge \star\beta) - \int_X (-1)^{k-1}\alpha \wedge d \star \beta = -\int_X (-1)^{k-1}\alpha \wedge d \star \beta.$$

On the other hand, we have

$$(\alpha, d^*\beta)_{L^2} = (-1)^k \int_X \alpha \wedge d \star \beta.$$

We are done. $\square$

Corollary 3.1.9. *Let X be an oriented compact Riemannian manifold. Let $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in A^k(X)$ and $\beta \in A^{k-1}(X)$. Then*

$$(d^*\alpha, \beta)_{L^2} = (\alpha, d\beta)_{L^2}.$$

Proof. Let $n = \dim(X)$. Using Lemma 3.1.8 and the fact that $\star$ preserves the L^2-metric, we get

$$\begin{aligned}
(d^*\alpha, \beta)_{L^2} &= \left((-1)^k \star^{-1} d \star \alpha, \beta\right)_{L^2} \\
&= (-1)^k \cdot \left(\star \star^{-1} d \star \alpha, \star\beta\right)_{L^2} \\
&= (-1)^k \cdot (d \star \alpha, \star\beta)_{L^2} \\
&= (-1)^k \cdot (\star\alpha, d^* \star \beta)_{L^2} \\
&= (-1)^k \cdot \left(\star\alpha, (-1)^{n-k+1} \star^{-1} d \star \star\beta\right)_{L^2} \\
&= (-1)^k \cdot (-1)^{n-k+1} \cdot (-1)^{(k-1)(n-k+1)} \left(\star\alpha, \star^{-1} d\beta\right)_{L^2} \\
&= (-1)^k \cdot (-1)^{(n-k+1)k} (\star \star \alpha, d\beta)_{L^2} \\
&= (-1)^k \cdot (-1)^{(n-k+1)k} \cdot (-1)^{k(n-k)} \cdot (\alpha, d\beta)_{L^2} \\
&= (-1)^k \cdot (-1)^{k(n-k+n-k+1)} \cdot (\alpha, d\beta)_{L^2} \\
&= (\alpha, d\beta)_{L^2} .
\end{aligned}$$

This proves the corollary. $\square$

Let X be an oriented and compact Riemannian manifold. Let $x \in X$ and consider the metric $(\ ,\ )_x \colon \Omega^k_{X,x} \times \Omega^k_{X,x} \to \mathbb{R}$. We can extend it $\mathbb{C}$-bilinearly to obtain a $\mathbb{C}$-bilinear form

$$(\ ,\ )_x \colon \Omega^k_{X,x,\mathbb{C}} \times \Omega^k_{X,x,\mathbb{C}} \to \mathbb{C}$$

and hence an $\mathbb{R}$-bilinear form

$$\langle\ ,\ \rangle_x\colon \Omega^k_{X,x,\mathbb{C}} \times \Omega^k_{X,x,\mathbb{C}} \to \mathbb{C}, \quad \langle \alpha_x, \beta_x \rangle_x = \left(\alpha_x, \overline{\beta}_x\right)_x. \tag{3.6}$$

Let $\alpha_x = \sum_i \lambda_i u_i \in \Omega^k_{X,x,\mathbb{C}}$ with $\lambda_i \in \mathbb{C}$ and $u_i \in \Omega^k_{X,x}$. Similarly, let $\beta_x = \sum_j \mu_j v_j \in \Omega^k_{X,x,\mathbb{C}}$ with $\mu_j \in \mathbb{C}$ and $v_j \in \Omega^k_{X,x}$. Then

$$\langle \alpha_x, \beta_x \rangle_x = \sum_{i,j} \lambda_i \cdot \overline{\mu}_j \cdot (u_i, v_j)_x = \overline{\sum_{j,i} \mu_j \cdot \overline{\lambda}_i \cdot (v_j, u_i)_x} = \overline{\langle \beta_x, \alpha_x \rangle_x}.$$

Next, let $\{e_1, \ldots, e_r\} \subset \Omega^k_{X,x}$ be an orthonormal basis for $(\ ,\ )_x$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, and define $\alpha = \sum_{i=1}^r \lambda_i e_i$. Then

$$\langle \alpha_x, \alpha_x \rangle_x = \sum_{i,j} \lambda_i \cdot \overline{\lambda}_j \cdot (e_i, e_j)_x = \sum_{i=1}^r |\lambda_i|^2.$$

We conclude that (3.6) is a positive definite hermitian form, i.e. a hermitian metric.

Notice that, for $x \in X$, the hermitian metric $\langle\ ,\rangle_x$ satisfies the property that

$$\langle \alpha_x, \beta_x \rangle_x \mathrm{Vol}_x = \alpha_x \wedge \overline{\star \beta_x}, \qquad \alpha_x, \beta_x \in \Omega^k_{X,x,\mathbb{C}}.$$

For $\alpha, \beta \in A^k_{\mathbb{C}}(X)$, the function $\langle \alpha, \beta \rangle\colon X \to \mathbb{C}$ defined as $\langle \alpha, \beta \rangle(x) = \langle \alpha_x, \beta_x \rangle_x$ is smooth, and we obtain a metric, the *Hermitian L^2-metric*, on the space of complex differentiable forms:

$$\langle\ ,\ \rangle_{L^2}\colon A^k_{\mathbb{C}}(X) \times A^k_{\mathbb{C}}(X) \to \mathbb{C}, \quad \langle \alpha, \beta \rangle_{L^2} = \int_X \langle \alpha, \beta \rangle \mathrm{Vol} = \int_X \alpha \wedge \overline{\star \beta} = \left(\alpha, \overline{\beta}\right)_{L^2}.$$

3.2 Lecture 6: Hodge decomposition theorem (proof)

3.2.1 *Complex differentiable forms*

Let X be an n-dimensional complex manifold. For $k \geq 0$, let $A^k_{\mathbb{C}}(X)$ be the space of complex differentiable forms on X, and consider the differential

$$d\colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X).$$

It decomposes as $d = \partial + \bar{\partial}$. To explain this, define $\Omega^{p,q}_X = \wedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$. Then, by Lemma 2.1.2, we have:

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega_{X,\mathbb{C}} = \bigwedge^k \left(\Omega^{1,0}_X \oplus \Omega^{0,1}_X\right) = \bigoplus_{p+q=k} \bigwedge^p \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X = \bigoplus_{p+q=k} \Omega^{p,q}_X.$$

Let $f\colon X \to \mathbb{C}$ be a complex differentiable function on X. In local coordinates $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, we can write

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i =: \partial f + \bar{\partial} f. \tag{3.7}$$

We conclude that, for our $f \in A^0_{\mathbb{C}}(X)$, we have

$$df = \partial f + \bar{\partial} f \tag{3.8}$$

for unique $\partial f \in A^{1,0}(X)$ and $\bar{\partial} f \in A^{0,1}(X)$, where $A^{p,q}(X)$ is the space of global sections of the bundle $\Omega^{p,q}_X$.

More generally, let $\alpha \in A^{p,q}(X)$ be a global section of $\Omega^{p,q}_X$. Then locally, α is of the form $\sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$ with $\alpha_{I,J}$ of type $(0,0)$. Consequently, $d\alpha$ can locally be written as

$$d\left(\sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J.$$

Now by (3.7), we have $d\alpha_{I,J} = \partial\alpha_{I,J} + \bar{\partial}\alpha_{I,J}$. Remark that $\sum_{I,J} \partial\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$ is a form of type $(p+1,q)$. Similarly, $\sum_{I,J} \bar{\partial}\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$ is a form of type $(p,q+1)$. We conclude:

Lemma 3.2.1. *Let X be a complex manifold of dimension n. There are unique operators ∂ and $\bar{\partial}$ on $A^k_{\mathbb{C}}(X)$ such that $\partial(A^{p,q}(X)) \subset A^{p+1,q}(X)$ and $\bar{\partial} A^{p,q}(X) \subset A^{p,q+1}(X)$ and such that the differential $d\colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X)$ decomposes as $d = \partial + \bar{\partial}$.* □

3.2.2 Hermitian manifolds

Let X be an n-dimensional compact *hermitian manifold.* Thus, X is a complex manifold of dimension n equipped with a Riemannian metric g that preserves the almost complex structure $I\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ on the real tangent bundle $T_{X,\mathbb{R}}$ of X.

Lemma 3.2.2. *The operators $\partial^* := -\star\bar{\partial}\star$ and $\bar{\partial}^* = -\star\partial\star$ are adjoints of ∂ and $\bar{\partial}$ respectively, for the hermitian metric $\langle\ ,\ \rangle_{L^2}$ on the space of complex differential forms.*

Proof. We prove the result only for $\bar{\partial}$; the other case is similar. Let $k \geq 1$. For $u, v \in A^k_{\mathbb{C}}(X)$, we have $\langle u, v\rangle_{L^2} = \int_X u \wedge \overline{\star v}$. In particular, for $\alpha \in A^{k-1}_{\mathbb{C}}(X)$ and $\beta \in A^k_{\mathbb{C}}(X)$, we have

$$\langle \bar{\partial}\alpha, \beta\rangle_{L^2} = \int_X \bar{\partial}\alpha \wedge \overline{\star\beta}.$$

As $\int_X \bar{\partial}\phi = 0$ for every $\phi \in A^{2n-1}_{\mathbb{C}}(X)$, we get (via the Leibniz formula) that

$$\langle \bar{\partial}\alpha, \beta\rangle_{L^2} = \int_X \bar{\partial}\alpha \wedge \overline{\star\beta} = -\int_X (-1)^{k-1}\alpha \wedge \overline{\bar{\partial}\star\beta} = -\int_X (-1)^{k-1}\alpha \wedge \overline{\star\star^{-1}\,\partial\star\beta}.$$

Moreover, $\star^{-1}\partial\star\beta = (-1)^{k-1}\star\partial\star\beta$ because $\deg(\partial\star\beta) = 2n-k+1 = 2n-\deg(\alpha)$. Therefore,

$$-\int_X (-1)^{k-1}\alpha \wedge \overline{\star\star^{-1}\,\partial\star\beta} = -\int_X \alpha \wedge \overline{\star\star\partial\star\beta} = (\alpha, \bar{\partial}^*\beta)_{L^2}$$

and the result follows. □

Definition 3.2.3. Let (X, g) be an oriented compact Riemannian manifold. Define $\Delta_d = dd^* + d^*d$. If X has a complex structure compatible with g, let $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. We say that a form $\alpha \in A^k_{\mathbb{C}}(X)$ is *Δ_d-harmonic* if $\Delta_d(\alpha) = 0$.

Lemma 3.2.4. *Let (X, g) be an oriented compact Riemannian manifold, and consider a complex differentiable k-form $\alpha \in A^k_{\mathbb{C}}(X)$. We have*

$$(\alpha, \Delta_d\alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2}.$$

Proof. Indeed, by Lemma 3.1.8 and Corollary 3.1.9, we have

$$(\alpha, \Delta_d\alpha)_{L^2} = (\alpha, dd^*\alpha + d^*d\alpha)_{L^2} = (\alpha, dd^*\alpha)_{L^2} + (\alpha, d^*d\alpha)_{L^2} = (d^*\alpha, d^*\alpha)_{L^2} + (d\alpha, d\alpha)_{L^2}.$$

This proves the lemma. □

Corollary 3.2.5. *Let X be an oriented compact Riemannian manifold. For each integer $k \in \mathbb{Z}_{\geq 1}$, we have $\mathrm{Ker}(\Delta_d) = \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*) \subset A^k(X)$.*

Proof. The inclusion $\mathrm{Ker}(\Delta_d) \supset \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$ being clear, we claim that any $\alpha \in \mathrm{Ker}(\Delta_d)$ is killed by d and by d^*. By Lemma 3.2.4,

$$0 = (\alpha, \Delta_d\alpha)_{L^2} = (d\alpha, d\alpha)_{L^2} + (d^*\alpha, d^*\alpha)_{L^2}.$$

As $(\ , \ )_{L^2}$ is positive definite, this implies that $d\alpha$ and $d^*\alpha$ must be zero. □

Theorem 3.2.6. *Let (X, g) be an oriented compact Riemannian manifold. For $k \geq 0$, consider the Laplacian $\Delta_d \colon A^k(X) \to A^k(X)$ and its kernel $\mathscr{H}^k = \mathrm{Ker}(\Delta_d)$. We have*

$$A^k(X) = \mathscr{H}^k \oplus \Delta_d\left(A^k(X)\right).$$

Proof. This follows from [Voi02, Corollaire 5.20] and [Voi02, Théorème 5.22]. □

Theorem 3.2.7. *Let (X, g) be an oriented compact Riemannian manifold. Any k-form $\alpha \in \mathrm{Ker}(\Delta_d) \subset A^k(X)$ is closed. Moreover, the linear map*

$$\begin{aligned} \mathrm{Ker}(\Delta_d) = \{\Delta_d\text{-harmonic } k\text{-forms on } X\} = \mathscr{H}^k \to H^k_{dR}(X, \mathbb{R}) = H^k(X, \mathbb{R}), \\ \alpha \mapsto [\alpha], \end{aligned} \tag{3.9}$$

that sends a harmonic form to its De Rham cohomology class, is an isomorphism.

Proof. The injectivity of (3.9) can be seen as follows. Let $\beta \in \mathscr{H}^k$ and suppose that $[\beta] = 0$. Then $\beta = d\alpha$ for some $k-1$-form α on X. Moreover, as $\Delta_d(\beta) = 0$, we have $d^*(\beta) = 0$ by Corollary 3.2.5. Hence $d^*d(\alpha) = 0$. But then, by Lemma 3.1.8, we obtain

$$0 = (\alpha, d^*d(\alpha))_{L^2} = (d\alpha, d\alpha)_{L^2}$$

which implies that $d\alpha = \beta = 0$. Thus, (3.9) is injective.

As for the surjectivity of (3.9), let $\beta \in A^k(X)$ be a closed form. By Theorem 3.2.7, we can write $\beta = \alpha + \Delta_d\gamma$ for a harmonic form α. Thus,

$$\beta = \alpha + dd^*\gamma + d^*d\gamma.$$

As β is closed by assumption, and as α is closed by Corollary 3.2.5, we have $dd^*(d\gamma) = 0$. Hence, by Corollary 3.1.9, we have

$$0 = (d\gamma, dd^*(d\gamma))_{L^2} = (d^*d\gamma, d^*d\gamma)_{L^2}$$

which implies that $d^*d\gamma = 0$. Therefore, we have $\beta = \alpha + dd^*\gamma$, and we deduce that $[\beta] = [\alpha] \in H^k(X, \mathbb{R})$. The k-form α is harmonic, and we are done. □

3.2.3 Kähler manifolds

Lemma 3.2.8. *Let V be a complex vector space of finite dimension. Consider the sets S_1, S_2 and S_3 defined as follows:*

$S_1 =$ *The set of hermitian forms $h\colon V \times V \to \mathbb{C}$.*

$S_2 =$ *The set of symmetric $\mathbb{R}$-bilinear forms $g\colon V \times V \to \mathbb{R}$ such that $g(i \cdot u, i \cdot v) = g(u, v)$ for each $u, v \in V$.*

$S_3 =$ *The set of anti-symmetric $\mathbb{R}$-bilinear forms $\omega\colon V \times V \to \mathbb{R}$ such that the $\mathbb{C}$-bilinear extension $\omega_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is zero on $V^{1,0} \times V^{1,0}$ and on $V^{0,1} \times V^{0,1}$.*

Let $h \in S_1$. The function $-\Im h\colon (u, v) \mapsto -\Im(h(u, v))$ defines an element $\omega \in S_3$. Moreover, for $\omega \in S_3$, the function $g(u, v) = \omega(u, i \cdot v)$ defines an element $g \in S_2$. This construction defines bijections $S_1 \cong S_2 \cong S_3$.

Proof. Exercise. □

Lemma 3.2.9. *Let V be a finite dimensional complex vector space and $h\colon V \times V \to \mathbb{C}$ and $g\colon V \times V \to \mathbb{R}$ be a hermitian (resp. symmetric $\mathbb{R}$-bilinear) form such that g and h correspond to each other via the bijection in Lemma 3.2.8. Then h is positive definite as a hermitian form if and only if g is positive definite as a symmetric bilinear form.*

Proof. Exercise. □

Definition 3.2.10. We call an anti-symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$ *of type* $(1, 1)$ if it satisfies property (3) above. We say that ω is *positive* if the hermitian form $h\colon V \times V \to \mathbb{C}$ is positive definite.

Let X be a hermitian manifold. Let g be the Riemannian metric of X. As g is compatible with the almost complex structure of X, it yields a hermitian metric on the tangent bundle $T_{X,\mathbb{R}}$, see Lemmas 3.2.8 and 3.2.9. In other words, for every $x \in X$, the real tangent bundle $T_{X,x,\mathbb{R}}$ with its natural complex structure $I\colon T_{X,x,\mathbb{R}} \to T_{X,x,\mathbb{R}}$ has a hermitian metric h_x, and these metrics vary differentiably with x.

Definition 3.2.11. We say that the hermitian metric h on $T_{X,\mathbb{R}}$ is *Kähler* if the real differentiable two-form

$$\omega = -\Im(h) \in A^{1,1}(X) \cap A^2_{\mathbb{R}}(X)$$

is closed. If this is the case, we call (X, ω) a *Kähler manifold.*

Theorem 3.2.12. *Let (X, ω) be a Kähler manifold. Let $\Delta_d, \Delta_\partial, \Delta_{\bar\partial}$ the Laplacians associated to the respective operators $d, \partial, \bar\partial$. Then $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.*

Proof. See [Voi02, Théorème 6.7]. □

Corollary 3.2.13. *Let (X, ω) be a Kähler manifold. Then $\Delta_d(A^{p,q}(X)) \subset A^{p,q}(X)$.*

Proof. Let $\alpha \in A^{p,q}(X)$. Then $\Delta_\partial(\alpha) = \partial^*\partial(\alpha) + \partial\partial^*(\alpha) \in A^{p,q}(X)$. The result follows because of Theorem 3.2.12. □

Corollary 3.2.14. *Let (X,ω) be a Kähler manifold. Let $\alpha \in A^k_{\mathbb{C}}(X)$. Define*

$$\mathscr{H}_{\mathbb{C}} = \operatorname{Ker}\left(\Delta_d \colon A^k_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X)\right), \quad \mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} \cap A^{p,q}(X) \subset \mathscr{H}^k_{\mathbb{C}}.$$

Thus, $\mathscr{H}^{p,q} \subset \mathscr{H}^k_{\mathbb{C}}$ is the space of Δ_d-harmonic forms of type (p,q).

(1) If α is harmonic, then each of its components $\alpha^{p,q} \in A^{p,q}(X)$ is harmonic.

(2) There is a canonical decomposition

$$\mathscr{H}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathscr{H}^{p,q}. \tag{3.10}$$

Proof. 1. Indeed, the relation $0 = \Delta_d(\alpha) = \sum_{p+q=k} \Delta_d(\alpha^{p,q})$ implies that $\Delta_d(\alpha^{p,q}) = 0$ by Corollary 3.2.13. 2. This is immediate from item 1. □

Lemma 3.2.15. *Let (X,ω) be a Kähler manifold. Let $\mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} \cap A^{p,q}(X)$ be the space of Δ_d-harmonic forms of type (p,q), $p+q=k$. Let $K^{p,q} \subset H^k(X,\mathbb{C})$ be the space of degree k cohomology classes $[\alpha]$ that admit a closed representative $\alpha' \in [\alpha]$ such that $\alpha' \in A^{p,q}(X)$. The image of the natural map*

$$\mathscr{H}^{p,q} \to H^k(X,\mathbb{C}) \tag{3.11}$$

equals exactly $K^{p,q}$.

Proof. Let $H^{p,q}(X)$ be the image of (3.11). As the elements of $\mathscr{H}^{p,q} \subset A^k_{\mathbb{C}}(X)$ are closed of type (p,q), we have $H^{p,q}(X) \subset K^{p,q}$. Conversely, let $[\omega] \in K^{p,q} \subset H^k(X,\mathbb{C})$ with $\omega \in A^{p,q}(X)$ such that $d\omega = 0$. By Theorem 3.2.7, we can uniquely write

$$\omega = \alpha + \Delta_d\beta,$$

with $\Delta_d\alpha = 0$ and $\beta \in A^k_{\mathbb{C}}(X)$. By looking at the components of type (p,q) with respect to (3.10), it follows from Corollary 3.2.13 that we can write

$$\omega = \omega^{p,q} = \alpha^{p,q} + (\Delta_d\beta)^{p,q} = \alpha^{p,q} + \Delta_d\beta^{p,q}, \quad \alpha^{p,q} \in A^{p,q}(X), \quad \beta^{p,q} \in A^{p,q}(X),$$

where $\alpha^{p,q}$ is harmonic. As ω and $\alpha^{p,q}$ are closed, we have that

$$\Delta_d\beta^{p,q} = dd^*\beta^{p,q} + d^*d\beta^{p,q}$$

is closed, hence $dd^*(d\beta^{p,q}) = 0$, which implies (via Corollary 3.1.9) that $d\beta^{p,q} = 0$. Therefore, $\Delta_d\beta^{p,q} = dd^*\beta^{p,q}$ is exact, hence

$$[\omega] = [\alpha^{p,q}] \in H^k(X,\mathbb{C}).$$

It follows that $[\omega]$ can be represented by a harmonic form of type (p,q), that is, we have $[\omega] \in H^{p,q}(X)$. Thus, $K^{p,q} \subset H^{p,q}(X)$ and we win. □

We proceed to show that any smooth projective variety is naturally a Kähler manifold. To do so, we need to show how to associate a closed real two-form of type $(1,1)$ to any pair (L, h), where L is a hermitian line bundle on a complex manifold X and h a hermitian metric on L. Let $\{U_i\}_{i\in I}$ be an open cover of X that trivializes L. For each i, we get a nowhere vanishing holomorphic section $\sigma_i\colon U_i \to L$. Let $i, j \in J$ with $U_{ij} = U_i \cap U_j \neq \emptyset$. There exists a holomorphic function $g_{ij}\colon U_{ij} \to \mathbb{C}^*$ such that

$$\sigma_i = g_{ij} \cdot \sigma_j.$$

Having fixed the above trivialization of L, for each $x \in X$, the hermitian metric h_x on L_x is determined by a non-zero element in $\mathbb{C}$. Consider the function

$$h_i\colon U_i \to \mathbb{R}, \quad z \mapsto h(\sigma_i(z), \sigma_i(z)).$$

Then $h_i(z) > 0$ for $z \in U_i$, and on $U_{ij} = U_i \cap U_j$, we have

$$h_i(z) = h(\sigma_i(z), \sigma_i(z)) = h\left(g_{ij}(z) \cdot \sigma_j(z), g_{ij}(z) \cdot \sigma_j(z)\right) = |g_{ij}(z)|^2 \cdot h_j(z).$$

We obtain differentiable two-forms

$$\omega_i = \frac{1}{2i\pi} \partial\bar{\partial} \log h_i \in A^2(U_i), \qquad i \in I.$$

Notice that, on U_{ij}, we have

$$\omega_i|_{U_{ij}} = \frac{1}{2i\pi}\partial\bar{\partial}\log h_i = \frac{1}{2i\pi}\partial\bar{\partial}\log\left(|g_{ij}|^2 \cdot h_j\right) = \frac{1}{2i\pi}\partial\bar{\partial}\log|g_{ij}|^2 + \omega_j|_{U_{ij}}.$$

As

$$\frac{1}{2i\pi}\partial\bar{\partial}\log|g_{ij}|^2 = 0$$

we have that ω_i and ω_j coincide on U_{ij}. Therefore, there exists a unique two-form

$$\omega \in A^2(X)$$

such that $\omega|_{U_i} = \omega_i$ for each $i \in I$. Notice that:

(1) The two-form $\omega \in A^2(X)$ is closed. Indeed, $\omega_i \in A^2(U_i)$ is exact.

(2) The two-form ω lies in $A^{1,1}(X) \subset A^2_{\mathbb{C}}(X)$, i.e. ω is of type $(1,1)$.

We have proved:

Lemma 3.2.16. *Let X be a complex manifold. The above construction allows one to associate a closed two-form $\omega \in A^2(X)$ of type $(1,1)$ to any pair (L,h) where L is a line bundle on X and h a hermitian metric on L.* □

Exercise 3.2.17. Show that the construction $(L,h) \mapsto \omega$, where L is a line bundle and h a hermitian metric on L, does not depend on the trivialization $\{U_i\}_{i\in I}$ for L.

Lemma 3.2.18. *Let X be a smooth projective variety. Then X defines a Kähler manifold (X, ω) in a natural way.*

Proof. As X admits a closed embedding into projective space, it suffices to prove this for the projective space $\mathbb{P}^n(\mathbb{C})$. Consider the tautological line bundle

$$S = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-1) \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}.$$

Let h be the standard hermitian metric on $\mathbb{C}^{n+1}$. It induces a hermitian metric on the holomorphic vector bundle

$$\mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1} \to \mathbb{P}^n(\mathbb{C})$$

and hence, by restriction, one on S. Let h^* be the induced hermitian metric on $S^* = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. By Lemma 3.2.16, we obtain a closed two-form

$$\omega \in A^2(\mathbb{P}^n(\mathbb{C})) \cap A^{1,1}(\mathbb{P}^n(\mathbb{C})). \tag{3.12}$$

It remains to prove that ω is positive, in the sense of Definition 3.2.10. We leave this as an exercise for the reader. □

Exercise 3.2.19. Prove that the two-form (3.12) is positive.

Proof of Theorem 3.1.4. Let X be a smooth projective variety. By Lemma 3.2.18, the variety X defines a Kähler manifold (X, ω) in a natural way. Moreover, by Theorem 3.2.7 and Corollary 3.2.14, we have canonical isomorphisms

$$\bigoplus_{p+q=k} \mathscr{H}^{p,q} = \mathscr{H}^k_{\mathbb{C}} = H^k(X, \mathbb{C}).$$

Define $H^{p,q}(X)$ as the image of $\mathscr{H}^{p,q}$ in $H^k(X,\mathbb{C})$ under $\mathscr{H}^k_{\mathbb{C}} \xrightarrow{\sim} H^k(X,\mathbb{C})$. It remains to show that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. This follows from Lemma 3.2.15, which shows that $H^{p,q}(X) = K^{p,q}$, where $K^{p,q} \subset H^k(X,\mathbb{C})$ is the space of De Rham cohomology classes $[\alpha]$ that admit a closed representative $\alpha' \in [\alpha]$ of type (p,q). □

Proposition 3.2.20. *Let X be a smooth projective variety over $\mathbb{C}$. For each $p, q \geq 0$, there is a canonical isomorphism $H^{p,q}(X) = H^q(X, \Omega^p_X)$.*

Proof. Let $n = \dim(X)$. The operator $\bar{\partial}$ induces a complex of sheaves

$$0 \to \Omega^p_X \to \Omega^{p,0}_X \to \Omega^{p,1}_X \to \cdots \to \Omega^{p,q}_X \to \cdots \to \Omega^{p,n}_X \to 0,$$

and this complex is exact, see [Voi02, Proposition 4.19]. In other words, the natural map of complexes

$$\Omega^p_X \to (\Omega^{p,\bullet}_X)$$

defines a resolution of Ω^p_X, and this resolution is in fact acyclic. As $\Gamma(X, \Omega^{p,q}_X) = A^{p,q}(X)$ by definition, we obtain a canonical isomorphism

$$H^q(X, \Omega^p_X) = \frac{\operatorname{Ker}\left(A^{p,q}(X) \xrightarrow{\bar{\partial}} A^{p,q+1}(X)\right)}{\operatorname{Im}\left(A^{p,q-1}(X) \xrightarrow{\bar{\partial}} A^{p,q}(X)\right)}.$$

Moreover, if we put a Kähler metric on X, there is a canonical isomorphism

$$H^{p,q}(X) = \mathrm{Ker}\,(\Delta_d) \cap A^{p,q}(X) = \mathrm{Ker}(\Delta_{\bar\partial}) \cap A^{p,q}(X),$$

and the natural map

$$\mathrm{Ker}(\Delta_{\bar\partial}) \cap A^{p,q}(X) \to \frac{\mathrm{Ker}\left(A^{p,q}(X) \xrightarrow{\bar\partial} A^{p,q+1}(X)\right)}{\mathrm{Im}\left(A^{p,q-1}(X) \xrightarrow{\bar\partial} A^{p,q}(X)\right)} \tag{3.13}$$

is an isomorphism. Indeed, if $\alpha = \partial\beta$ is of type (p,q) and $\bar\partial\alpha = \bar\partial^*\alpha = 0$, then $\bar\partial^*\partial\beta = 0$, which by the adjoint property of $\bar\partial^*$ with respect to $\bar\partial$ (see Lemma 3.2.2) implies that $\partial\beta = \alpha = 0$. This proves the injectivity of (3.13). For the surjectivity of (3.13), see [Voi02, Théorème 5.24].

It remains to verify that the so-constructed isomorphism $H^q(X, \Omega_X^p) \cong H^{p,q}(X)$ is truly canonical, i.e. does not depend on the Kähler metric that we chose to define it. Recall the Hodge to De Rham spectral sequence, see Section 3.1.2:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}(X, \mathbb{C}). \tag{3.14}$$

As we have $\dim H^k(X,\mathbb{C}) = \sum_{p+q=k} H^q(X,\Omega_X^p)$ by Theorem 3.1.4, the spectral sequence (3.14) degenerates. Therefore, there are canonical isomorphisms

$$F^p H^k(X,\mathbb{C})/F^{p+q}(X,\mathbb{C}) = E_\infty^{p,q} = E_1^{p,q} = H^q(X,\Omega_X^p).$$

Finally, the filtration F^p on $H^k(X,\mathbb{C})$ induced by (3.14) is exactly the Hodge filtration (3.2) attached to the Hodge structure on $H^k(X,\mathbb{Z})$ that Theorem 3.1.4 provides, as follows from [Voi02, Proposition 7.5]. In particular, we have

$$F^p H^k(X,\mathbb{C})/F^{p+1}H^k(X,\mathbb{C}) = H^{p,q}(X).$$

This finishes the proof of the proposition. $\square$

3.2.4 Example: complex elliptic curves

Definition 3.2.21. (1) A *complex elliptic curve* is a smooth cubic $E \subset \mathbb{P}^2_{\mathbb{C}}$ equipped with a point $\mathcal{O} \in E(\mathbb{C})$. If $(E_1, \mathcal{O}_1)$ and $(E_2, \mathcal{O}_2)$ are complex elliptic curves, then a *morphism of elliptic curves* $(E_1, \mathcal{O}_1) \to (E_2, \mathcal{O}_2)$ is a morphism of varieties $\phi\colon E_1 \to E_2$ such that $\phi(\mathcal{O}_1) = \mathcal{O}_2$. In this way, elliptic curves form a category.

(2) A *complex torus* is the quotient of a finite dimensional complex vector space $V \cong \mathbb{C}^n$ by a discrete subgroup $\Lambda \subset V$ with $\Lambda \otimes \mathbb{R} = V$. A *morphism of complex tori* is a holomorphic group homomorphism. Thus, complex tori form a category.

Proposition 3.2.22. *There are three compatible functors as in the following diagram:*

$$\begin{tikzcd}[column sep=small] & \big(\textit{Elliptic curves } E_{/\mathbb{C}}\big) \arrow[dl] \arrow[dr] & \\ \big(\textit{One-dimensional complex tori}\big) & & \big(\textit{Weight one Hodge structures on } \mathbb{Z}^2\big) \arrow[ll] \end{tikzcd}$$

These three functors are equivalences of categories.

Remark 3.2.23. It follows from Proposition 3.2.22 that complex elliptic curves are algebraic groups in a natural way, where an *algebraic group* is an algebraic variety X of finite type over a field k which is a group object in the category of schemes over k. The fact that complex elliptic curves E are algebraic groups can be proven directly, by constructing an algebraic group law $E \times E \to E$ explicitly using the defining equation for E in $\mathbb{P}^2_{\mathbb{C}}$. This can be done for smooth cubics $E \subset \mathbb{P}^2_k$ over any field k, as long as $E(k) \neq \emptyset$, leading to the notion of *elliptic curve over* k. For more on this, see [Sil09].

Proof of Proposition 3.2.22. Let E be a complex elliptic curve. By Theorem 3.1.4 (or by Example 3.1.3), there is a natural Hodge structure of weight one on $H^1(E,\mathbb{Z}) \cong \mathbb{Z}^2$, which defines the functor on the right. Next, let $V_{\mathbb{Z}}$ be any weight one Hodge structure on $\mathbb{Z}^2$. The composition

$$V_{\mathbb{R}} \to V_{\mathbb{C}} \to V^{0,1}$$

is an isomorphism, hence the composition $V_{\mathbb{Z}} \to V_{\mathbb{C}} \to V^{0,1}$ is an embedding, and

$$X = V^{0,1}/V_{\mathbb{Z}}$$

is a complex torus of dimension one. These two constructions are functorial, and compatible with the functor that associates the complex torus $X = H^1(E,\mathcal{O}_E)/H^1(E,\mathbb{Z})$ to an elliptic curve E over $\mathbb{C}$. It remains to show that:

($\star$) Any one-dimensional complex torus V/Λ is isomorphic to $H^1(E,\mathcal{O}_E)/H^1(E,\mathbb{Z})$ for a smooth complex elliptic curve $E \subset \mathbb{P}^2_{\mathbb{C}}$.

($\star\star$) If E_1 and E_2 are complex elliptic curves, and X_1 and X_2 the associated complex tori, then any holomorphic group homomorphism $X_1 \to X_2$ is induced by a unique morphism of algebraic groups $E_1 \to E_2$.

In fact, we claim that ($\star\star$) follows from ($\star$). Indeed, if ($\star$) holds, then any one-dimensional complex torus is projective, hence any holomorphic map between two one-dimensional complex tori is uniquely algebraizable by the GAGA principle.

To prove ($\star$), we may assume that $V = \mathbb{C}$, so that Λ is a lattice in $\mathbb{C}$. Then

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z}\cdot\omega \qquad \text{for some } \omega \in \mathbb{C} \text{ with } \Im(\omega) > 0.$$

Consider the meromorphic function

$$\wp\colon \mathbb{C} \to \mathbb{C}, \quad \wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z-n-m\cdot\omega)^2} - \frac{1}{(n+m\cdot\omega)^2} \right).$$

Notice that $\wp$ is periodic with respect to Λ, and that the poles of $\wp$ are given by $z = n + m\cdot\omega$ for $(n,m) \in \mathbb{Z}^2$. The function

$$\mathbb{C} \to \mathbb{P}^2(\mathbb{C}), \quad z \mapsto [\wp(z)\colon \wp'(z)\colon 1]$$

defines a holomorphic and Λ-periodic function, hence induces a morphism

$$X = \mathbb{C}/\Lambda \to \mathbb{P}^2(\mathbb{C}). \tag{3.15}$$

In fact, (3.15) is a closed embedding. To determine its image, define, for $k \in \mathbb{Z}_{\geq 2}$,

$$G_{2k}(\Lambda) = \sum_{x \in \Lambda - \{(0,0)\}} x^{-2k}.$$

Let $g_2(\Lambda) = 60 \cdot G_4(\Lambda)$ and $g_3(\Lambda) = 140 \cdot G_6(\Lambda)$. Then one has:

$$\wp'(z)^2 = 4 \cdot \wp(z)^3 - g_2(\Lambda) \cdot \rho(z) - g_3(\Lambda), \qquad z \in \mathbb{C} \setminus \Lambda. \tag{3.16}$$

Therefore, the closed embedding (3.15) identifies $X = \mathbb{C}/\Lambda$ with the plane cubic curve $E \subset \mathbb{P}^2(\mathbb{C})$ of affine equation $y^2 = 4x^3 - g_2(\Lambda) \cdot x - g_3(\Lambda)$. □

Exercise 3.2.24. Prove that (3.16) holds.

Exercise 3.2.25. Let $V_{\mathbb{R}}$ be a finite dimensional real vector space, and let $k \in \mathbb{Z}_{\geq 0}$. Define $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Prove that to give a Hodge structure of weight k on $V_{\mathbb{R}}$ is to give a continuous homomorphism

$$\rho \colon \mathbb{C}^* \to \mathrm{GL}(V_{\mathbb{C}})$$

such that

$$\rho(t) = t^k \cdot \mathrm{Id} \quad \text{and} \quad \overline{\rho(z)} = \rho(\overline{z}) \qquad \forall t \in \mathbb{R}^*, z \in \mathbb{C}^*.$$

Exercise 3.2.26. Let X be a smooth projective variety of dimension $n \geq 1$. Let k be an integer with $0 \leq k \leq n$, and define

$$\mathrm{Hdg}^{2k}(X, \mathbb{Z}) = \left\{ \alpha \in H^{2k}(X, \mathbb{Z}) \colon \text{the image } \alpha_{\mathbb{C}} \text{ of } \alpha \text{ in } H^{2k}(X, \mathbb{C}) \text{ lies in } H^{k,k}(X) \right\}.$$

Let $Z \subset X$ be a smooth closed subvariety of codimension k. Define φ as the composition

$$\mathbb{Z} = H^0(Z, \mathbb{Z}) = H_{2n-2k}(Z, \mathbb{Z}) \to H_{2n-2k}(X, \mathbb{Z}) = H^{2k}(X, \mathbb{Z}),$$

and put $[Z] = \varphi(1) \in H^{2k}(X, \mathbb{Z})$. Prove that $[Z] \in \mathrm{Hdg}^{2k}(X, \mathbb{Z})$.

3.3 Lecture 7: Period maps and period domains

The goal of this lecture will be to explain the basic concepts and theorems in the theory of variations of Hodge structure. We will restrict ourselves to providing definitions and statements, and refer to [Voi02] for the proofs of the big theorems. Almost all of these fundamental theorems are due to Griffiths.

3.3.1 Family of deformations

Let $\mathscr{X}$ and B be complex manifolds. Let $\phi \colon \mathscr{X} \to B$ be a holomorphic map. We let $X_t = \phi^{-1}(t)$ be the fibre of ϕ over $t \in B$.

Definition 3.3.1. If B is connected with reference point $0 \in B$, and if ϕ is a proper submersion, then each fibre X_t is naturally a compact complex manifold, and we call $\mathscr{X}$ a *family of deformations* of the fibre X_0.

Proposition 3.3.2 (Ehresmann)**.** *Let $\phi \colon \mathscr{X} \to B$ be a proper submersion of differentiable manifolds. Let $0 \in B$ and assume that B is contractible. There is an isomorphism $T \colon \mathscr{X} \cong X_0 \times B$ of differentiable manifolds over B.*

3.3.2 Gauss–Manin connection

Definition 3.3.3. Let B be a topological space and G a group. A *local system on* B is a sheaf of abelian groups F on B such that each $x \in B$ admits an open neighbourhood U such that $F|_U$ is isomorphic to a constant sheaf on U. A *local system with fibres isomorphic to* G on B is a sheaf of abelian groups F on B such that F is locally isomorphic to the constant sheaf attached to G.

Let B be a differentiable manifold, and let H be a local system of $\mathbb{R}$-vector spaces on B. We can then consider the sheaf of free $\mathscr{C}^\infty$-modules

$$\mathscr{H} = H \otimes_{\mathbb{R}} \mathscr{C}^\infty.$$

Definition 3.3.4. Let B be a differentiable manifold and H a local system of $\mathbb{R}$-vector spaces on B. The associated connection $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ is defined as follows. For $\sigma \in \mathscr{H}$, we can locally write $\sigma = \sum_i \sigma_i \otimes \alpha_i$ for $\sigma_i \in H$ and $\alpha_i \in \mathscr{C}^\infty$. Then, we put

$$\nabla\sigma = \sum_i \sigma_i \otimes d\alpha_i \in \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B.$$

Notice that, if $\sigma \in \mathscr{H}$ and $t \in \mathscr{C}^\infty$, then

$$\begin{aligned}\nabla(t\cdot\sigma) &= \nabla\left(\sigma_i \otimes t\alpha_i\right) = \sum_i \sigma_i \otimes d(t\alpha_i) = \sum_i \sigma_i \otimes (td\alpha_i + \alpha_i dt)\\ &= t\nabla(\sigma) + \sigma \otimes dt.\end{aligned}$$

In other words, $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ is a *connection*.

Now let $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ be any connection. Then ∇ induces a map

$$\nabla\colon \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B \to \mathscr{H} \otimes \bigwedge^2 \Omega_B, \quad \nabla(\sigma\otimes\alpha) = \nabla\sigma \wedge \alpha + \sigma \otimes d\alpha.$$

Definition 3.3.5. The *curvature*

$$\Theta\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \bigwedge^2 \Omega_B$$

of ∇ is defined as $\Theta = \nabla \circ \nabla$. The connection $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ is *flat* if $\Theta = 0$.

Lemma 3.3.6. *Let H be a local system of $\mathbb{R}$-vector spaces on a differentiable manifold B. Let $\mathscr{H} = H \otimes \mathscr{C}^\infty$ and let $\nabla\colon \mathscr{H} \to \mathscr{H} \otimes_{\mathscr{C}^\infty} \Omega_B$ be the connection constructed above. Then the curvature of ∇ is zero.*

Proof. Let σ be a local section of H and α a local section of $\mathscr{C}^\infty$. Then

$$\nabla(\sigma\otimes\alpha) = \sigma \otimes d\alpha \qquad \text{and} \qquad \nabla(\sigma) = 0.$$

Hence,

$$\Theta(\sigma\otimes\alpha) = \nabla(\sigma \otimes d\alpha) = \nabla\sigma \wedge d\alpha + \sigma \otimes d^2\alpha = 0.$$

This proves the lemma. □

Proposition 3.3.7. *Let B be a differentiable manifold. Then $H \mapsto (H \otimes_{\mathbb{R}} \mathscr{C}^\infty, \nabla)$ defines a bijection between the set of isomorphism classes of local systems of $\mathbb{R}$-vector spaces and the set of isomorphism classes of differentiable bundles with flat connection.*

Proof. See [Voi02]. □

Let $\pi\colon \mathscr{X} \to B$ be a proper submersion of differentiable manifolds. Let A be a subring of $\mathbb{C}$. By Ehresmann's lemma (see Proposition 3.3.2), the sheaves

$$R^k\pi_*A$$

are local systems of A-modules on B. Defining $\mathscr{H}^k = R^k\pi_*\mathbb{R}$, the flat connection

$$\nabla\colon \mathscr{H}^k \to \mathscr{H}^k \otimes_{\mathscr{C}^\infty} \Omega_B$$

associated to the local system $R^k\pi_*\mathbb{R}$ is called the *Gauss–Manin connection.*

3.3.3 Semi-continuity of Hodge numbers

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds.

Theorem 3.3.8. *Let $\mathcal{F}$ be a holomorphic vector bundle on $\mathscr{X}$. The function $B \to \mathbb{Z}$ defined as $b \mapsto \dim H^q(X_b, \mathcal{F}|_{X_b})$ is upper semi-continuous.*

Proof. See [Voi02, Théorème 9.15]. □

Corollary 3.3.9. *The function $b \mapsto h^{p,q}(X_b)$ is upper semi-continuous.* □

Proposition 3.3.10. *Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds. Assume that the fibre X_0 above $0 \in B$ is a Kähler manifold. Then, in a neighbourhood of 0, we have $h^{p,q}(X_b) = h^{p,q}(X_0)$.*

Proof. Consider the spectral sequence

$$E_1^{p,q} = H^q(X_b, \Omega^p_{X_b}) \implies H^{p+q}(X_b, \mathbb{C}).$$

We have $h^{p,q}(X_b) \leq h^{p,q}(X_0)$ by Corollary 3.3.9. Moreover, $\dim E_\infty^{p,q} \leq \dim E_1^{p,q}$, where

$$E_\infty^{p,q} = F^pH^{p+q}(X_b)/F^{p+1}H^{p+q}(X_b).$$

We get

$$b_k = \sum_{p+q=k} \dim E_\infty^{p,q}(X_b) \leq \sum_{p+q=k} \dim E_1^{p,q}(X_b) = \sum_{p,q} h^{p,q}(X_b) \leq \sum_{p,q} h^{p,q}(X_0) = b_k.$$

The proposition follows. □

3.3.4 Period map and period domain

Let X be a Kähler manifold and $\phi\colon \mathscr{X} \to B$ a proper holomorphic submersion, with B a connected complex manifold, $0 \in B$ a base point, and X_b Kähler for $b \in B$. Suppose that the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$ (which we can always obtain up to shrinking B around $0 \in B$). Define

$$b^{p,k} = \dim F^p H^k(X_0, \mathbb{C}).$$

Then $\dim F^p H^k(X_b, \mathbb{C}) = b^{p,k}$ for each $b \in B$. Suppose also that B is contractible (which we can achieve by shrinking B further around 0). Then, there is a canonical isomorphism $H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C})$ for $b \in B$, given by the composition

$$H^k(X_b, \mathbb{C}) \xleftarrow{\sim} H^k(\mathscr{X}, \mathbb{C}) \xrightarrow{\sim} H^k(X_0, \mathbb{C}).$$

We can then define the *period map* as follows:

$$\mathscr{P}^{p,k}\colon B \to \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C})), \quad b \mapsto F^p H^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C}).$$

Theorem 3.3.11 (Griffiths)**.** *The period map $\mathscr{P}^{p,k}$ is holomorphic for p, k with $p \leq k$.*

Proof. See [Voi02, Théorème 10.9]. □

Define $V = H^k(X_0, \mathbb{C})$ and $G = \mathrm{Grass}(b^{p,k}, V) = \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C}))$ and let $W \subset H^k(X_0, \mathbb{C})$ be a $b^{p,k}$-dimensional subspace. Let $[W] \in G$ be the corresponding point. Recall that there is a canonical isomorphism

$$T_{[W]}G = \mathrm{Hom}(W, V/W).$$

In particular,

$$T_{[F^p H^k(X_b)]}G = \mathrm{Hom}(F^p H^k(X_b), H^k(X_b)/F^p H^k(X_b)).$$

Theorem 3.3.12 (Griffiths)**.** *Let $b \in B$. Consider the differential*

$$d\mathscr{P}^{p,k}\colon T_{B,b} \to T_{G,[F^p H^k(X_b)]} = \mathrm{Hom}(F^p H^k(X_b), H^k(X_b)/F^p H^k(X_b))$$

of the period map $\mathscr{P}^{p,k}\colon B \to G$ at the point $b \in B$. Then the image of $d\mathscr{P}^{p,k}$ is contained in

$$\mathrm{Hom}(F^p H^k(X_b), F^{p-1} H^k(X_b)/F^p H^k(X_b)).$$

Proof. See [Voi02, Proposition 10.12]. □

Let $b \in B$. Then we have the Hodge filtration

$$0 = F^{k+1} H^k(X_b) \subset F^k H^k(X_b) \subset \cdots \subset F^p H^k(X_b) \subset \cdots \subset F^0 H^k(X_b) = H^k(X_0, \mathbb{C}).$$

Hence, if we let

$$F_{b^\bullet,k}(H^k(X_0, \mathbb{C})) = \mathrm{Flag}(b^{k,k}, b^{k-1,k}, \cdots b^{1,k}, H^k(X_0, \mathbb{C}))$$

be the flag variety attached to the numbers $b^{k,k} \leq b^{k-1,k} \leq \cdots \leq b^{1,k}$, then we get a *period map*

$$\begin{aligned} B &\to F_{b^\bullet,k}(H^k(X_0,\mathbb{C})), \\ b &\mapsto \left(F^kH^k(X_b) \subset F^{k-1}H^k(X_b) \subset \cdots \subset F^1H^k(X_b) \subset H^k(X_b,\mathbb{C}) \cong H^k(X_0,\mathbb{C})\right). \end{aligned}$$

By Theorem 3.3.11, this map is holomorphic. Moreover, note that for $b \in B$, we have

$$F^pH^k(X_b) \oplus \overline{F^{k-p+1}H^k(X_b)} = H^k(X_b,\mathbb{C}).$$

This condition defines an open subset

$$\mathscr{D} \subset F_{b^\bullet,k}(H^k(X_0,\mathbb{C})),$$

called a (unpolarized) *period domain*.

Now suppose that there exists a class $\omega \in H^2(\mathscr{X},\mathbb{Z})$ such that $\omega_b := \omega|_{X_b}$ is Kähler for every $b \in B$. Define the primitive cohomology group $H^k(X_b,\mathbb{C})_{prim} \subset H^k(X_b,\mathbb{C})$ as follows. First, define operators

$$L\colon H^i(X_b,\mathbb{Z}) \to H^{i+2}(X_b,\mathbb{Z}), \quad L(\alpha) = \omega_b \cup \alpha,$$

and then put

$$\begin{aligned} H^k(X_b,\mathbb{Z})_{prim} &:= \operatorname{Ker}(L^{n-k+1}) \\ &= \left\{\alpha \in H^k(X_b,\mathbb{Z}) \mid \omega^{n-k+1} \cup \alpha = 0 \in H^{2n-k+2}(X_b,\mathbb{Z})\right\}, \\ H^k(X_b,\mathbb{C})_{prim} &:= H^k(X_b,\mathbb{Z})_{prim} \otimes \mathbb{C} \\ &= \left\{\alpha \in H^k(X_b,\mathbb{C}) \mid \omega^{n-k+1} \cup \alpha = 0 \in H^{2n-k+2}(X_b,\mathbb{C})\right\}. \end{aligned}$$

The subspace $H^k(X_b,\mathbb{C})_{prim}$ is compatible with the Hodge filtration, hence the Hodge decomposition of $H^k(X_b,\mathbb{C})$ restricts to a decomposition on $H^k(X_b,\mathbb{C})_{prim}$. In particular, $H^k(X_b,\mathbb{Z})_{prim}$ carries a natural Hodge structure of weight k. Moreover, $H^k(X_b,\mathbb{Z})_{prim}$ is endowed with a non-degenerate bilinear form

$$Q\colon H^k(X_b,\mathbb{Z})_{prim} \times H^k(X_b,\mathbb{Z}) \to \mathbb{Z}, \quad Q(\alpha,\beta) = \langle L^{n-k}\alpha,\beta\rangle = \int_{X_b} \omega_b^{n-k} \wedge \alpha \wedge \beta,$$

and one has:

(1) $F^pH^k(X_b,\mathbb{C})_{prim} = F^{k-p+1}H^k(X_b,\mathbb{C})^{\perp}_{prim}$.

(2) $F^pH^k(X_b)_{prim} \oplus \overline{F^{k-p+1}H^k(X_b,\mathbb{C})_{prim}} = H^k(X_b,\mathbb{C})_{prim}$.

(3) The hermitian form $(u,v) \mapsto Q(Cu,\bar{v})$ is positive definite on $H^k(X_b,\mathbb{C})_{prim}$.

Condition (3) says that $(H^k(X_b,\mathbb{Z})_{prim},Q)$ is a polarized Hodge structure of weight k, see Definition 3.1.2. This leads us to define

$$\mathscr{D} \subset F_{b^\bullet,k}(H^k(X_0,\mathbb{C})_{prim}) \tag{3.17}$$

as the subset of flags $F^k \subset \cdots \subset F^1 \subset H^k(X_0, \mathbb{C})_{prim}$ satisfying (1), (2) and (3). We call $\mathscr{D}$ the *polarized period domain*. The first condition is closed (in fact already in the Zariski topology). The second and third conditions are open conditions (in the euclidean topology) on the set of filtrations satisfying the first condition. All in all, we obtain a holomorphic period map

$$\mathscr{P}\colon B \to \mathscr{D}. \tag{3.18}$$

Remark 3.3.13. Let us consider Condition (1). It is clear that this defines a closed condition on the flag manifold $F_{b_{\mathrm{pr}}^{\bullet,k}}(H^k(X_0, \mathbb{C})_{prim})$. One would like to know if the closed subvariety defined by this condition is smooth. It turns out that, in fact, it is again a flag variety. To see this, note that if $k = 2m - 1$ is odd, the filtrations

$$F^k \subset F^{k-1} \subset \cdots \subset F^k \subset H^k(X_0, \mathbb{C})_{prim}$$

with $\dim F^p = b_{\mathrm{pr}}^{p,k} = \dim F^p H^k(X_0, \mathbb{C})_{prim}$ satisfying (1) are of the form

$$F^{2m-1} \subset F^{2m-2} \subset \cdots F^m = (F^m)^\perp \subset \cdots \subset (F^{2m-1})^\perp \subset H^k(X_0, \mathbb{C})_{prim}.$$

Similarly, if $k = 2m$ is even, a filtration with $\dim F^p = b_{\mathrm{pr}}^{p,k}$ satisfying (1) is of the form

$$F^{2m} \subset \cdots \subset F^{n+2} \subset (F^{n+2})^\perp \subset \cdots \subset (F^{2m})^\perp \subset H^k(X_0, \mathbb{C})_{prim}.$$

Exercise 3.3.14. Consider Definition 3.3.3. We prove some results on local systems.

(1) Let X be a connected topological space and let G be an abelian group. Let $\mathscr{F}$ be a local system on $X \times [0, 1]$ with stalks isomorphic to G. Let σ be a global section of the restriction $\mathscr{F}|_{X\times 0}$ to $X \times 0 \subset X \times [0, 1]$. Show that σ extends uniquely to a section of $\mathscr{F}$ over X.

(2) Let X be a topological space. Let G be an abelian group and let $\phi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of local systems on X with fibres isomorphic to G. Show that the set of points where ϕ is an isomorphism is open and closed in X.

(3) Let G be an abelian group. Let X be a connected topological space. Let $\mathscr{F}$ be a local system on $X \times [0, 1]$, with fibres isomorphic to G. Let $\pi_1\colon X \times [0, 1] \to X$ be the natural projection. Show that $\mathscr{F} \cong \pi_1^{-1}\left(\mathscr{F}|_{[0,1]}\right)$.

Exercise 3.3.15. Let X be a connected and path connected topological space. Assume X is simply connected. Let $\mathscr{F}$ be a local system on X. Let $x \in X$. Use Exercise 3.3.14 to provide a unique isomorphism $\mathscr{F} \xrightarrow{\sim} \mathscr{F}_x$ that induces the identity on stalks at x. Here, we think of $\mathscr{F}_x$ as the constant sheaf attached to the abelian group $\mathscr{F}_x$.

Exercise 3.3.16. Let X be a connected, path connected topological space, with reference point $x \in X$. Let G be an abelian group.

(1) Let $\mathscr{F}$ be a local system with stalks isomorphic to G on X. Show that $\mathscr{F}$ induces a representation $\rho(\mathscr{F})\colon \pi_1(X, x) \to \mathrm{Aut}(G)$ in a natural way.

(2) Let $\mathscr{F} \to \mathscr{G}$ be a morphism of local systems with stalks isomorphic to G on X. Show that $\rho(\mathscr{F})$ and $\rho(\mathscr{G})$ are equivalent under the conjugation $\mathrm{Aut}(G)$-action on the set of representations $\pi_1(X, x) \to \mathrm{Aut}(G)$ if $\mathscr{F}$ and $\mathscr{G}$ are isomorphic.

(3) Let $\pi\colon \widetilde{X} \to X$ be the universal cover of X. Let $\rho\colon \pi_1(X, x) \to \mathrm{Aut}(G)$ be a representation. Let $\mathscr{G}$ be the constant sheaf on $\widetilde{X}$ attached to G. Show that $\pi_1(X, x)$ acts on the sheaf $\pi_*\mathscr{F}$ via ρ. Let $\mathscr{F}_\rho = (\pi_*\mathscr{G})^\rho$ be the sheaf of fixed points. Show that $\mathscr{F}_\rho$ is a local system. Show that if ρ and ρ' are representations which are conjugate under the $\mathrm{Aut}(G)$-action, then $\mathscr{F}_\rho$ and $\mathscr{F}_{\rho'}$ are isomorphic.

(4) Prove that the above construction establishes a natural bijection

$$\{\text{Local systems with stalk } G \text{ on } X\}_{/\cong} = \{\text{Homomorphisms } \pi_1(X, x) \to \mathrm{Aut}(G)\}_{/\sim}.$$

3.3.5 Kodaira–Spencer map

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds. Let $b \in B$ and consider the following exact sequence of holomorphic vector bundles on X_b:

$$0 \to T_{X_b} \to T_{\mathscr{X}}|_{X_b} \to \phi^*(T_B)|_{X_b} \to 0.$$

Notice that $\phi^*(T_B)|_{X_b} = T_{B,b} \times X_b$ as holomorphic vector bundles on X_b, hence we get an exact sequence

$$0 \to T_{X_b} \to T_{\mathscr{X}}|_{X_b} \to T_{B,b} \times X_b \to 0.$$

In particular, taking cohomology yields a morphism

$$\rho\colon T_{B,b} \to H^1(X_b, T_{X_b})$$

called the *Kodaira–Spencer map.*

3.3.6 Variations of Hodge structure

Let $\phi\colon \mathscr{X} \to B$ be a proper holomorphic submersion of complex manifolds whose fibres are Kähler. For $k \geq 0$, let $\mathscr{H}^k = R^k\phi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$, equipped with its Gauss–Manin connection

$$\nabla\colon \mathscr{H}^k \to \mathscr{H}^k \otimes_{\mathcal{O}_B} \Omega_B.$$

We assume that B is contractible and that the Hodge numbers are constant on B. We know that the period map

$$\mathscr{P}^{p,q}\colon B \to G = \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C}))$$

is holomorphic (see Theorem 3.3.11). In particular, there exists a holomorphic subbundle

$$F^p\mathscr{H}^k \subset \mathscr{H}^k$$

such that $F^p\mathscr{H}^k_b = F^pH^k(X_b, \mathbb{C})$ for $b \in B$. The bundles $F^p\mathscr{H}^k$ are called the *Hodge subbundles* of $\mathscr{H}^k$. We define

$$\mathscr{H}^{p,q} = F^p\mathscr{H}^k/F^{p+1}\mathscr{H}^k,$$

so that $\mathscr{H}^{p,q}_b = H^q(X_b, \Omega^p_{X_b})$ for $p + q = k$.

Theorem 3.3.17 (Griffiths)**.** *We have*

$$\nabla F^p\mathscr{H}^k \subset F^{p-1}\mathscr{H}^k \otimes \Omega_B.$$

Proof. See [Voi02, Proposition 10.18]. □

As a corollary, we get, for $b \in B$, an induced map

$$\overline{\nabla}^p \colon \mathscr{H}^{p,q} \to \mathscr{H}^{p-1,q+1} \otimes \Omega_B.$$

Proposition 3.3.18. *The differential*

$$d\mathscr{P}^{p,k}_b \colon T_{B,b} \to T_{G,F^pH^k(X_b)} = \mathrm{Hom}(F^pH^k(X_b), H^k(X_b)/F^pH^k(XX_b))$$

is the map induced by adjunction from the map

$$\overline{\nabla}^p_b \colon F^pH^k(X_b) \to H^k(X_b)/F^pH^k(X_b) \otimes \Omega_{B,b},$$

which is the map induced by the composition

$$F^p\mathscr{H}^k \xrightarrow{\nabla} \mathscr{H}^k \otimes \Omega_B \to (\mathscr{H}^k/F^p\mathscr{H}^k) \otimes \Omega_B.$$

Proof. See [Voi02, Lemme 10.19]. □

Corollary 3.3.19. *The differential of the period map at b gives a map*

$$d\mathscr{P}^{p,k}_b \colon T_{B,b} \to \mathrm{Hom}(F^pH^k(X_b)/F^{p+1}H^k(X_b), F^{p-1}H^k(X_b)/F^pH^k(X_b)),$$

which is the map induced via adjunction by the map $\overline{\nabla}^p_b$ in the following diagram:

$$\begin{tikzcd}
F^pH^k(X_b)/F^{p+1}H^k(X_b) \arrow[r, "\overline{\nabla}^p_b"] \arrow[d, equal] & F^{p-1}H^k(X_b)/F^pH^k(X_b) \otimes \Omega_{B,b} \arrow[d, equal] \\
H^{p,k-p}(X_b) \arrow[r, "\overline{\nabla}^p_b"] & \mathrm{Hom}(T_{B,b}, H^{p-1,k-p+1}(X_b)).
\end{tikzcd} \tag{3.19}$$

□

This yields an element

$$d\mathscr{P}^{p,k}_b(u) \in \mathrm{Hom}(H^q(X_b, \Omega^p_{X_b}), H^{q+1}(X_b, \Omega^{p-1}_{X_b})), \quad \text{for all} \quad u \in T_{B,b}.$$

Theorem 3.3.20 (Griffiths)**.** *Let $b \in B$ and $u \in T_{B,b}$. Then, the linear map*

$$d\mathscr{P}^{p,k}(u) \colon H^q(X_b, \Omega^p_{X_b}) \to H^{q+1}(X_b, \Omega^{p-1}_{X_b})$$

is the cup-product with the Kodaira–Spencer class $\rho(u) \in H^1(X_b, T_{X_b})$ composed with the contraction induced by the natural map $\Omega^p_{X_b} \otimes T_{X_b} \to \Omega^{p-1}_{X_b}$. In other words, the following diagram commutes for all $u \in T_{B,b}$:

$$\begin{tikzcd} H^q(X_b, \Omega^p_{X_b}) \arrow[d, "\cup\rho(u)"] \arrow[dr, bend left, "d\mathscr{P}^{p,k}(u)"] & \\ H^{q+1}(X_b, \Omega^p_{X_b} \otimes T_{X_b}) \arrow[r] & H^{q+1}(X_b, \Omega^{p-1}_{X_b}). \end{tikzcd}$$

Proof. See [Voi02, Théorème 10.21]. □

Remark 3.3.21. Another way to formulate the theorem is to say that the following diagram commutes for all $b \in B$:

$$\begin{tikzcd} T_{B,b} \arrow[d, "\rho"] \arrow[dr, bend left, "d\mathscr{P}^{p,k}"] & \\ H^1(X_b, T_{X_b}) \arrow[r] & \mathrm{Hom}(H^q(X_b, \Omega^p_{X_b}), H^{q+1}(X_b, \Omega^{p-1}_{X_b})). \end{tikzcd}$$

Finally, consider the flag variety

$$F = F_{b^\bullet,k}(H^k(X_0, \mathbb{C})) = \left\{F^k \subset F^{k-1} \subset \cdots \subset F^1 \subset H^k(X_0, \mathbb{C}) \mid \dim F^p = b^{p,k}\right\},$$

and the period map

$$\begin{aligned} &\mathscr{P}^k \colon B \to F, \\ &\quad b \mapsto [F^\bullet H^k(X_b) \subset H^k(X_0)] = \left[F^k H^k(X_b) \subset \cdots \subset F^1 H^k(X_b) \subset H^k(X_b) = H^k(X_0)\right]. \end{aligned}$$

If we define

$$G_{b^{p,k}} = \mathrm{Grass}(b^{p,k}, H^k(X_0, \mathbb{C})),$$

then there is a natural map

$$\iota \colon F \to G_{b^{k,k}} \times G_{b^{k-1,k}} \times \cdots \times G_{b^{1,k}},$$

and this map is a closed immersion. In particular, for $b \in B$, and $\mathscr{P}$ we have

$$\begin{aligned} T_{F,\mathscr{P}^k(b)} &\subset \left(\text{tangent space of } \prod_p G_{b^{p,k}} \text{ at } \iota(\mathscr{P}^k(b))\right) \\ &= \bigoplus_p \mathrm{Hom}\left(F^p H^k(X_b), H^k(X_b)/F^p H^k(X_b)\right). \end{aligned}$$

Furthermore, by Theorem 3.3.12, the map

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(F^p H^k(X_b), H^k(X_b)/F^p H^k(X_b)\right)$$

factors through a map which we still denote by $d\mathscr{P}_b^k$:

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(F^p H^k(X_b)/F^{p+1} H^k(X_b), F^{p-1} H^k(X_b)/F^p H^k(X_b)\right).$$

As $F^p H^k(X_b)/F^{p+1} H^k(X_b) = H^{p,k-p}(X_b)$, we can, for $b \in B$, view $d\mathscr{P}_b^k$ as a map

$$d\mathscr{P}_b^k \colon T_{B,b} \to \bigoplus_p \operatorname{Hom}\left(H^{p,k-p}(X_b), H^{p-1,k-p+1}(X_b)\right). \tag{3.20}$$

Chapter 4

Formal algebraic geometry

The goal of this chapter is to prove the theorem, due to Grothendieck [SGAII], which says the following. Let k be a field and let $X \subset \mathbb{P}_k^{n+1}$ be a hypersurface of dimension $n \geq 3$. Let $\mathcal{O}_X(1)$ be the pull-back of $\mathcal{O}_{\mathbb{P}_k^n}(1)$. Then $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. See Theorem 4.3.14 at the end of this chapter. To prove this, we need some tools that lie outside the scope of scheme theory. For as is sometimes the case in algebraic geometry, the category of schemes is not big enough to carry out certain constructions. Think of the proof this theorem in the case $k = \mathbb{C}$ (see Corollary 2.2.8): one needed complex analytic functions; algebraic functions were not enough. Similarly, over arbitrary fields, one needs formal functions as we shall see.

4.1 Lecture 8: Formal algebraic geometry

We need to have some tools of formal algebraic geometry to our disposal.

4.1.1 Adic completion of a local ring

Let R be a local ring with maximal ideal $\mathfrak{m}$. The powers of $\mathfrak{m}$ define a topology on R, called the $\mathfrak{m}$-*adic topology*. By definition, the ideals $\mathfrak{m}^n$, $n \geq 1$ define a fundamental system of open neighbourhoods around $0 \in R$. This topology is induced by the pseudometric d on R that is determined by its property that, for each $x \in R$, one has $d(x, 0) = 2^{-n}$ if $n = \max(k \mid x \in \mathfrak{m}^k)$ exists, and $d(x, 0) = 0$ otherwise. One defines $d(x, y) = d(x - y, 0)$.

Let $x, y, z \in R$. Suppose that $x - y \in \mathfrak{m}^{k_1}$ and $y - z \in \mathfrak{m}^{k_2}$, and that $k_1 \geq k_2$. Then

$$x - z = x - y + y - z \in \mathfrak{m}^{k_1} + \mathfrak{m}^{k_2} = \mathfrak{m}^{k_1}\left(1 + \mathfrak{m}^{k_1 - k_2}\right) = \mathfrak{m}^{k_1}.$$

It follows that d indeed defines a pseudometric on R. The *completion* $\widehat{R}$ of R is the completion of R with respect to its $\mathfrak{m}$-adic metric. Alternatively, one can define $\widehat{R}$ as the inverse limit

$$\widehat{R} = \varprojlim R/\mathfrak{m}^n.$$

The basic properties of completion are:

Theorem 4.1.1. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $\widehat{R}$ be its completion.*

(1) The ring $\widehat{R}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$.

(2) If R is a domain, the natural homomorphism $R \to \widehat{R}$ is injective.

(3) If M is a finitely generated R module, then its completion $\widehat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \widehat{A} = \varprojlim_n M/\mathfrak{m}^n M$.

(4) The dimension of R equals the dimension of $\widehat{R}$.

(5) The local ring R is regular if and only if the local ring $\widehat{R}$ is regular.

Proof. See [Atiyah–Macdonald][Ch.10 & 11]. [insert reference] □

For the sake of later use, we also record here the following lemma, known as Nakayama's lemma; see e.g. [Stacks, Tag 07RC] for a reference.

Lemma 4.1.2. *Let R be a ring with Jacobson radical $\mathrm{rad}(R)$. Let M be a finite R-module. Let $I \subset R$ be an ideal.*

(1) If $IM = M$, then there exists $f \in 1 + I$ such that $fM = 0$.

(2) If $IM = M$ and $I \subset \mathrm{rad}(R)$, then $M = 0$.

(3) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.

(4) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and $I \subset \mathrm{Rad}(R)$, then $N \to M$ is surjective.

Proof. To prove part (1), [To insert later]. Part (2) follows from part (1) because for any ideal $J \subset R$, we have $J \subset \mathrm{rad}(R)$ if and only if $(1 + J) \subset R^* = (\text{units of } R)$. To prove part (3), let K be the cokernel of $N \to M$. Then $K = IK$ hence there exists $f \in 1 + I$ such that $fK = 0$ (see part (1). Therefore, $K_f = 0$, so that $N_f \to M_f$ is surjective. To prove part (4), let K be the cokernel of $N \to M$. The hypotheses imply $K = IK$. Hence $K = 0$ by part (2) because K is a finite R-module. □

4.1.2 Adic completion of an arbitrary ring

Let A be a ring and let $I \subset A$ be an ideal. For $n \leq m$, there is a natural surjective homomorphism

$$A/I^m \to A/I^n,$$

and the collection of these maps turns (A/I^n) into an inverse system of topological rings, where each A/I^n is equipped with the discrete topology. The inverse limit

$$\widehat{A} := \varprojlim A/I^n$$

exists in the category of topological rings, and we call it the *completion of A with respect to I* or the *I-adic completion of A*. Similarly, if M is an A-module, we define a topological $\widehat{A}$-module

$$\widehat{M} = \varprojlim M/I^nM$$

and call it the *I-adic completion of M*. The limit is taken in the category of topological abelian groups, where each M/I^nM is equipped with the discrete topology. Notice that $\widehat{M}$ has indeed a natural $\widehat{A}$-module structure. The most important properties of this construction are summarized as follows.

Theorem 4.1.3. *Let A be a noetherian ring and let I be an ideal of A. Then:*

(1) The $\widehat{A}$-module $\widehat{I} = \varprojlim I/I^n$ defines an ideal of $\widehat{A}^n$. Moreover, $\widehat{I}^n = I^n\widehat{A}$ and $\widehat{A}/\widehat{I}^n \cong A/I^n$ for all n.

(2) If M is a finitely generated A module, then $\widehat{M} \cong M \otimes_A \widehat{A}$.

(3) The functor

$$M \mapsto \widehat{M}$$

is an exact functor on the category of finitely generated A-modules.

(4) The topological ring $\widehat{A}$ is noetherian, and flat over A.

(5) Let (M_n) be an inverse system, where each M_n is a finitely generated A/I^n module. Suppose that each transition map

$$\varphi_{m,n} \colon M_m \to M_n, \quad n \leq m,$$

is surjective, with kernel $\mathrm{Ker}(\varphi_{m,n}) = I^nM_m$*. Then $M = \varprojlim M_n$ is a finitely generated $\widehat{A}$-module, and for each n, one has $M_n \cong M/I^nM$.*

Proof. For proofs, see [Atiyah–Macdonald] and [Bourbaki]. □

4.1.3 Adic noetherian rings

Definition 4.1.4. (1) An *adic noetherian ring* is a noetherian ring A equipped with a topology having the following property: there exists a fundamental system of neighbourhoods of zero in A consisting of the powers I^n ($n > 0$) of an ideal I and A is separated and complete for this topology. In other words, A is the projective limit of the discrete rings $A_n = A/I^{n+1}$ ($n \geq 0$).

(2) An *ideal of definition* of A is an ideal I which has the above property. Equivalently, I is an ideal of A which is open and whose powers tend to zero.

(3) If I is an ideal of definition, one says that A is *I-adic*, the topology is called the *I-adic topology*, and the filtration of A by the powers of I is called the *I-adic filtration.*

Remarks 4.1.5. Let $I \subset A$ be an ideal of definition in an adic ring A. Let $J \subset A$ be any ideal.

(1) J is an ideal of definition if and only if there exist integers $p, q > 0$ such that $J^q \subset I^p \subset J$.

(2) If I and J are ideals of a ring A such that $A = \varprojlim A/I^n = \varprojlim A/J^n$, then the topologies that I and J define on A are the same.

(3) The ideal I^n $(n \geq 1)$ is an ideal of definition for each $n \geq 1$.

Let A be an adic noetherian ring, with ideal of definition $I \subset A$. Let

$$u_n \colon A \to A/I^n =: A_n$$

be the canonical morphism, and for $m \geq n$, let $u_{m,n} \colon A_m \to A_n$ be the canonical morphism. Let $S \subset A$ be a multiplicative subset of A, and define $S_n = u_n(S)$. The maps $u_{m,n}$ define natural maps

$$S_m^{-1} A_m \to S_n^{-1} A_n,$$

for which these rings form an inverse system $(S_n^{-1} A_n)$. Define $A\left\{S^{-1}\right\}$ as the projective limit of this system.

Proposition 4.1.6. *The topological ring $A\left\{S^{-1}\right\}$ is topologically isomorphic to the projective limit $\varprojlim_n S^{-1}A/S^{-1}I^n$.*

Proof. Let $v_n \colon S^{-1}A \to S_n^{-1}A_n$ be the canonical morphism. Then the kernel of v_n is $S^{-1}I_n$ and v_n is surjective. In other words, we have an exact sequence

$$0 \to S^{-1}I_n \to S^{-1}A \to S_n^{-1}A_n \to 0.$$

Let $B = \varprojlim_n S^{-1}A/S^{-1}I_n$. Then, we have surjective morphisms

$$B \to B/\left(S^{-1}I_n B\right) = S^{-1}A/S^{-1}I_n = S_n^{-1}A_n,$$

and hence a continuous morphism

$$\varphi \colon B \to \varprojlim_n \left(B/\left(S^{-1}I_n B\right)\right) = \varprojlim_n S_n^{-1}A_n = A\left\{S^{-1}\right\}.$$

As B is separated and complete, φ is an isomorphism. $\square$

Let A be an I-adic noetherian ring, and $S \subset A$ be a multiplicative subset. Consider the natural map

$$A \to S^{-1}A, \tag{4.1}$$

and observe that the inverse image of the ideal $S^{-1}I_n \subset S^{-1}A$ in A contains I_n. Hence (4.1) is continuous. As $S^{-1}A \to A\left\{S^{-1}\right\}$ is continuous as well, we obtain a continuous morphism

$$A \to A\left\{S^{-1}\right\}.$$

Remark 4.1.7. Let A be an I-adic noetherian ring and $S \subset A$ a multiplicative subset. Then up to isomorphism, $A\{S^{-1}\}$ does not depend on the ideal of definition I. Moreover, The pair $(A\{S^{-1}\}, A \to A\{S^{-1}\})$ is characterized by the following universal property: let B be an adic noetherian ring and let $u\colon A \to B$ be a continuous morphism such that $u(S)$ is contained in the set of invertible elements of B. Then u factors uniquely as

$$A \to A\left\{S^{-1}\right\} \xrightarrow{u'} B,$$

where u' is continuous.

Let A be an adic noetherian ring with ideal of definition $I \subset A$, and let $\mathfrak{a}$ be an open ideal of A. Then $I^n \subset \mathfrak{a}$ for almost all $n \geq 1$. In particular, up to replacing I by $J = I^n$ for $n \gg 0$, we may assume that I is an ideal of definition such that $I^n \subset \mathfrak{a}$ for each $n \geq 1$ (see Remarks 4.1.5). In particular, $S^{-1}I^n \subset S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$ for all $n > 0$, hence $S^{-1}\mathfrak{a}$ is an open ideal of $S^{-1}A$. We denote by $\mathfrak{a}\{S^{-1}\}$ its $S^{-1}I$-adic completion. Then $\mathfrak{a}\{S^{-1}\}$ is an open ideal of $A\{S^{-1}\}$. Moreover, there is a canonical isomorphism

$$A\left\{S^{-1}\right\}/\mathfrak{a}\left\{S^{-1}\right\} = S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a}). \tag{4.2}$$

Remark 4.1.8. Let A be an adic noetherian ring. Let $f \in A$ and let S_f be the multiplicative subset of the f^n $(n \geq 0)$ in A. Define $A_{\{f\}} = A\left\{S_f^{-1}\right\}$. For an open ideal $\mathfrak{a} \subset A$, we write $\mathfrak{a}_{\{f\}} = \mathfrak{a}\{S_f^{-1}\}$. If $g \in A$, then we obtain a canonical continuous morphism $A_{\{f\}} \to A_{\{fg\}}$. Let $S \subset A$ be a multiplicative system. We obtain an inductive system $A_{\{f\}}$, and define

$$A_{\{S\}} = \varinjlim_{f \in S} A_{\{f\}}.$$

For every $f \in S$, there is a canonical morphism $A_{\{f\}} \to A\{S^{-1}\}$, and these morphisms form an inductive system. In particular, they define a canonical morphism

$$A_{\{S\}} \to A\left\{S^{-1}\right\}.$$

One can show that, with respect to this morphism, $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.

4.1.4 Sheaves of topological rings and modules

We need a definition.

Definition 4.1.9. Let X be a topological space. Let $\mathcal{C}$ be a category with arbitrary products. A *presheaf* $\mathcal{F}$ on X with values in $\mathcal{C}$ is a contravariant functor from the category of open subsets of X to $\mathcal{C}$. A presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is a *sheaf* if for every open covering $\{U_i\}$ of an open $U \subset X$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

We start with a basic lemma.

Lemma 4.1.10. *Let X be a topological space and let $\mathfrak{C}$ be the category of sheaves of topological abelian groups on X. Then inverse limits exist in $\mathfrak{C}$. Furthermore, for each open $U \subset X$, consider the functor*

$$\mathcal{F} \mapsto \mathcal{F}(U)$$

from topological abelian sheaves on X to topological abelian groups. This functor commutes with projective limits.

Proof. Let $(\mathcal{F}_n)$ be an inverse system of topological abelian sheaves. Let $\mathcal{F}$ be the presheaf defined by $\mathcal{F}(U) = \varprojlim \mathcal{F}_n(U)$ for $U \subset X$ open. Then $\mathcal{F}$ is a sheaf of topological abelian groups. Moreover, $\mathcal{F}$ is the inductive limit of the system $(\mathcal{F}_n)$. $\square$

Assume now that the topological space X has a basis consisting of quasi-compact opens. Given a sheaf $\mathcal{F}$ of sets, groups, rings, modules over a ring, one can endow $\mathcal{F}$ with the structure of a sheaf of topological spaces, topological groups, topological rings, topological modules. Namely, if $U \subset X$ is quasi-compact open, we endow $\mathcal{F}(U)$ with the discrete topology. If $U \subset X$ is arbitrary, then we choose an open covering $U = \cup_i U_i$ by quasi-compact opens U_i. As $\mathcal{F}$ is a sheaf, there is an equalizer diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1}),$$

which is an equalizer diagram in the category of topological spaces, topological groups, topological rings, topological modules. In particular, the first map identifies $\mathcal{F}(U)$ with a subspace of $\prod_{i \in I} \mathcal{F}(U_i)$, which is endowed with the product topology. A sheaf of topological spaces, topological groups, topological rings, topological modules is *pseudo-discrete* if the topology on $\mathcal{F}(U)$ is discrete for every quasi-compact open $U \subset X$. Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological spaces (similarly for groups, rings, modules).

Remark 4.1.11. Naively, one could think that if $\mathcal{F}$ is a sheaf of sets, then we can try to define a sheaf of topological spaces $\mathcal{G}$ by declaring that $\mathcal{G}(U) = \mathcal{F}(U)$ with the discrete topology. However, in general, $\mathcal{G}$ will not be a sheaf of topological spaces.

For an explicit example where $\mathcal{G}$ is not a topological sheaf, let $X = \mathbb{R}$, the real numbers equipped with the euclidean topology. Let S be some set with at least two elements, and let $\mathcal{F}$ be the sheaf of sets on X such that $(V) = S$ for each connected open $V \subset X$. In other words, $\mathcal{F}$ is the constant sheaf attached to the set S. Let $\mathcal{G}$ be the presheaf of topological spaces on X such that, for each open $V \subset X$, we have $\mathcal{G}(V) = \mathcal{F}(V)$ equipped with the discrete topology. Then $\mathcal{G}$ is not a sheaf. Namely, for $n \in \mathbb{Z}$, let U_n be the open interval $(n, n+1)$. Let $U = \cup_n U_n$ be the union of all the $(U_n)'s$. Then the diagram

$$\mathcal{G}(U) \to \prod_{n \in \mathbb{Z}} \mathcal{G}(U_n) \rightrightarrows \prod_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{G}(U_n \cap U_m)$$

is not an equalizer diagram in the category of topological spaces. Indeed, the first map $\mathcal{G}(U) \to \prod_{n\in\mathbb{Z}} \mathcal{G}(U_n)$ is a bijection, and if the diagram were an equalizer diagram, then this map should be a homeomorphism, which is not the case.

4.1.5 Noetherian formal schemes as locally topologically ringed spaces

To an I-adic noetherian ring A, one can define a topologically ringed space

$$\mathrm{Spf}(A) \tag{4.3}$$

as follows. For $n \in \mathbb{Z}_{\geq 0}$, let $X_n = \mathrm{Spec}\ (A_n)$ $(A_n = A/I^{n+1})$, and put $Y = X_0 = \mathrm{Spec}\ (A/I)$. These schemes form an increasing sequence of closed subschemes of Spec (A),

$$Y = \mathrm{Spec}\ (A_0) \to X_1 \to \cdots \to X_n \to \cdots .$$

The schemes X_n all have the same underlying space $|\mathrm{Spf}(A)|$, called the *formal spectrum* of A.

Proposition 4.1.12. *Let I be an ideal of definition of an adic noetherian ring A. Then I is contained the radical* $\mathrm{rad}(A) = \cap_{\mathfrak{m}\text{ maximal}}\, \mathfrak{m} \subset A$ *of A.*

Proof. See [EGAI, §0, 7.1.10]. □

Let $\mathfrak{m} \subset A$ be a maximal ideal, corresponding to a point $x \in \mathrm{Spec}\ (A)$. It follows from Proposition 4.1.12 that $I \subset \mathfrak{m}$, i.e., that $x \in \mathrm{Spec}\ (A/I)$. We conclude that $\mathrm{Spf}(A)$, considered as a closed subset of Spec (A), contains all the closed points of A.

Lemma 4.1.13. *Let A be an I-adic noetherian ring. Every open subset of* Spec (A) *containing* $\mathrm{Spf}(A)$ *is equal to* Spec (A).

Proof. Let $U \subset \mathrm{Spec}\ (A)$ be an open subset containing $\mathrm{Spf}(A)$. Then the complement Z of U in Spec (A) is of the form Spec $(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. In particular, if $\mathfrak{a} \neq A$, then Z contains a closed point. As this is impossible, we conclude that $\mathfrak{a} = A$ hence $U = \mathrm{Spec}\ (A)$. □

Let us now define the sheaf of rings $\mathcal{O}_{\mathrm{Spf}(A)}$ of $\mathrm{Spf}(A)$ as the inverse limit

$$\mathcal{O}_{\mathrm{Spf}(A)} = \varprojlim_n \mathcal{O}'_{X_n}$$

of the pseudo-discrete sheaves $\mathcal{O}'_{X_n}$ on $\mathrm{Spf}(A)$, equipped with the natural topology such that on any open subset U of $\mathrm{Spf}(A)$, we have

$$\mathcal{O}_{\mathrm{Spf}(A)}(U) = \varprojlim_n \mathcal{O}'_{X_n}(U)$$

in the category of topological rings. Here, $\mathcal{O}'_{X_n}$ is the pseudo-discrete sheaf of topological rings associated to $\mathcal{O}_{X_n}$, see Section 4.1.4 (in particular, $\mathcal{O}_{X_n}(U)$ has the discrete topology for each n and each quasi-compact open U). Thus,

$$\mathcal{O}_{\mathrm{Spf}(A)}(\mathrm{Spf}(A)) = \varprojlim_n \mathcal{O}_{X_n}(X_n) = \varprojlim_n A_n = \widehat{A}. \tag{4.4}$$

For $f \in A$, let $D(f) = \text{Spec }(A_f) \subset \text{Spec }(A)$, and define

$$\mathcal{D}(f) = \text{Spec }(A_f) \cap \text{Spf}(A) = \left\{\mathfrak{p} \in \text{Spf}(A) \mid \bar{f}(\mathfrak{p}) \neq 0\right\},$$

where $\bar{f} \in A/I$ is the image of f in $A_0 = A/I$.

Lemma 4.1.14. *Let A be an adic noetherian ring. Let $f \in A$ and write $\mathcal{D}(f) = D(f) \cap \text{Spf}(A)$. Then the topologically ringed space $(\mathcal{D}(f), \mathcal{O}_{\text{Spf}(A)}|_{\mathcal{D}(f)})$ is isomorphic to the formal spectrum*

$$\text{Spf}(\widehat{A_f}) = \text{Spf}(A_{\{f\}}).$$

Proof. Let I be an ideal of definition, and define $A_n = A/I^{n+1}$ and $X_n = \text{Spec }(A_n)$. As topological spaces $\text{Spf}(A) \cap D(f)$ is identified with $\text{Spec }(A/I) \cap D(f) = \text{Spec }(A_f/I_f)$. By (4.2), we have $A_{\{f\}}/I_{\{f\}} = A_f/I_f$. Hence, as topological spaces, $\text{Spf}(A_{\{f\}})$ is identified with $\text{Spec }(A_f/I_f)$, and thus with $\text{Spec }(A) \cap D(f) = \mathcal{D}(f)$.

As for global sections of the structure sheaf, we indeed have

$$\begin{aligned}\mathcal{O}_{\text{Spf}(A)}(\mathcal{D}(f)) &= \varprojlim_n \mathcal{O}_{X_n}(\text{Spec }(A_f) \cap \text{Spf}(A)) = \varprojlim_n \mathcal{O}_{X_n}(\text{Spec }(A_f) \cap \text{Spec }(A/I^n)) \\ &= \varprojlim_n \mathcal{O}_{X_n}\left(\text{Spec }(A_f/I_f^n)\right) = \varprojlim_n A_f/I_f^n = \widehat{A_f}.\end{aligned}$$

More generally, let U' be a quasi-compact open of $\text{Spec }(A)$ contained in $D(f)$ and define $U = U' \cap \text{Spf}(A) \subset \mathcal{D}(f)$. Then $\Gamma(U, \mathcal{O}_{X_n})$ is canonically identified with the module of sections of the structure sheaf of $\text{Spec }(A_f/I_f^n)$, hence, if we define $\mathfrak{Y} = \text{Spf}(A_f)$, then

$$\Gamma(U, \mathcal{O}_{\mathfrak{Y}}) = \varprojlim_n \mathcal{O}_{\text{Spec }(A_f/I_f^n)}(U) = \varprojlim_n \Gamma(U, \mathcal{O}_{X_n}) = \mathcal{O}_{\text{Spf}(A)}(U).$$

Thus, the topologically ringed spaces $\mathcal{D}(f)$ and $\mathfrak{Y} = \text{Spf}(A_f)$ are isomorphic. □

Let A be an adic noetherian ring. Let $x = \{\mathfrak{p}\} \in \text{Spf}(A)$, where $\mathfrak{p} \subset A/I$ is a prime ideal. Then

$$\mathcal{O}_{\text{Spf}(A),x} = \varinjlim_{f \in A|\bar{f}(\mathfrak{p}) \neq 0} \mathcal{O}_{\text{Spf}(A)}(\mathcal{D}(f)) = \varinjlim_{f \in A|\bar{f}(\mathfrak{p}) \neq 0} A_{\{f\}}.$$

This is a noetherian, local, topological ring, but not complete in general. We have:

Lemma 4.1.15. *Let A be an adic noetherian ring.*

(1) The topologically ringed space $\text{Spf}(A)$ *is depends only on A as a topological ring: it does not depend on an ideal of definition.*

(2) The space $\text{Spf}(A)$ *is the subspace of* $\text{Spec }(A)$ *consisting of open prime ideals.*

(3) Moreover, $\mathcal{O}_{\text{Spf}(A)}$ is the inverse limit of the sheaves $(A/J)^\sim$, where J runs through the ideals of definition of A.

Proof. The statements follow directly from the above discussion. Let us be more elaborate about the argument that $\mathrm{Spf}(A)$ is the subspace of Spec (A) consisting of open prime ideals. Indeed, a prime ideal $\mathfrak{p}$ is open if and only if $I^n \subset \mathfrak{p}$ for some integer $n \geq 1$, which, as $\mathfrak{p}$ is prime, is equivalent to the condition that $I \subset \mathfrak{p}$. □

Let X and Y be locally topologically ringed spaces. A *morphism* of locally topologically ringed spaces $f\colon X \to Y$ is a morphism as locally ringed spaces such that for each open $U \subset Y$, the induced morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous.

Definition 4.1.16. An *affine noetherian formal scheme* is a topologically ringed space isomorphic to one of the form (4.3). A *locally noetherian formal scheme* is a topologically ringed space such that any point has an open neighbourhood which is an affine noetherian formal scheme. It is called *noetherian* if the underlying space is noetherian. A *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ between locally noetherian formal schemes is a morphism as locally topologically ringed spaces.

Let A and B be adic rings, and define $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Let $\varphi\colon A \to B$ be a continuous homomorphism of rings. As the inverse image in A of every open prime ideal of B is an open prime ideal of A, the continuous map Spec $(B) \to$ Spec (A) restricts to a continuous map ${}^a\varphi\colon \mathrm{Spf}(B) \to \mathrm{Spf}(A)$. Moreover, for every $f \in A$, we have a canonical morphism

$$\Gamma(\mathcal{D}(f), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} \to B_{\{\varphi(f)\}} = \Gamma(D(\varphi(f)), \mathcal{O}_{\mathrm{Spf}(B)}).$$

As the right compatibility conditions are satisfied, these maps define a continuous morphism of sheaves of topological rings

$$\widetilde{\varphi}\colon \mathcal{O}_{\mathfrak{Y}} \to {}^a\varphi_* \mathcal{O}_{\mathfrak{X}}.$$

This yields a morphism of topologically ringed spaces

$$\Phi = ({}^a\varphi, \widetilde{\varphi})\colon \mathfrak{X} = \mathrm{Spf}(A) \to \mathrm{Spf}(B) = \mathfrak{Y}.$$

Proposition 4.1.17. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes such that $\mathcal{Y} = \mathrm{Spf}(A)$ is a noetherian affine formal scheme. Then, there is a canonical bijection*

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}_{\mathrm{cont}}(A, \mathcal{O}_{\mathcal{X}}(\mathcal{X})).$$

Here, $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ *denotes the set of morphisms of locally noetherian formal schemes* $\mathcal{X} \to \mathcal{Y}$*, see Definition 4.1.16.*

Proof. We first treat the case where $\mathcal{X} = \mathrm{Spf}(B)$ is formally affine. Let $\varphi\colon A \to B$ be a continuous morphism of rings. We need to show that the induced morphism of topologically ringed spaces

$$\Phi = ({}^a\varphi, \widetilde{\varphi})\colon \mathfrak{X} \to \mathfrak{Y} \tag{4.5}$$

is a morphism of locally topologically ringed spaces. Let $x = \mathfrak{p} \in \mathrm{Spf}(B)$ and $y = \mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spf}(A)$. Let $f \in A$ such that $f \notin \mathfrak{q}$. Then $\varphi(f) \notin \mathfrak{p}$, hence we obtain a morphism

$$A_{\{f\}} \to B_{\{\varphi(f)\}},$$

that maps $\mathfrak{q}_{\{f\}}$ into $\mathfrak{p}_{\{\varphi(f)\}}$. Hence, the induced map

$$\mathcal{O}_{\mathfrak{Y},y} = \varinjlim_{f \notin \mathfrak{q}} A_{\{f\}} \to \varinjlim_{g \notin \mathfrak{p}} B_{\{g\}} = \mathcal{O}_{\mathfrak{X},x}$$

is a morphism of local rings. We conclude that (4.5) is a morphism of locally topologically ringed spaces.

Consider a morphism of locally topologically ringed spaces $(\psi, \theta) \colon \mathrm{Spf}(B) \to \mathrm{Spf}(A)$, where ψ is a continuous map of topological spaces and $\theta \colon \mathcal{O}_{\mathrm{Spf}(A)} \to \psi_* \mathcal{O}_{\mathrm{Spf}(B)}$ a continuous map of sheaves of topological rings. By (4.4), we obtain a continuous morphism

$$\varphi \colon A = \Gamma(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) \to \Gamma(\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}) = B.$$

It is readily checked that ${}^a\varphi = \psi$, and that $\widetilde{\varphi} = \theta$. This finished the proof of the proposition in the case where $\mathfrak{X}$ is formally affine.

The proof general case is similar to the proof of the analogous statement for affine schemes, see [EGAI, (2.2.4)]. $\square$

Remark 4.1.18. To conclude the section, we make the trivial but important remark that every noetherian affine scheme $X = \mathrm{Spec}\ (A)$ can be viewed in one and only one way as an affine formal scheme, by considering A as a discrete topological ring. Equivalently, (0) is an ideal of definition for A. As such, the topological rings $\Gamma(U, \mathcal{O}_X)$ are discrete for quasi-compact opens $U \subset X$ (but not in general for arbitrary open subsets of X). In exactly the same way, we can start with any locally noetherian scheme $(X, \mathcal{O}_X)$, and associate to it $(X, \mathcal{O}'_X)$, where $\mathcal{O}'_X$ is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is $\mathcal{O}_X$ (see Section 4.1.4). This construction is compatible with morphisms and defines a functor

$$(\text{Locally noetherian schemes}) \to (\text{Locally noetherian formal schemes})\,.$$

It is straightforward to show that this functor is fully faithful.

4.2 Lecture 9: Algebraizing coherent sheaves on formal schemes

4.2.1 Coherent sheaves of affine noetherian formal schemes

Let $\mathcal{X} = \mathrm{Spf}(A)$ be an affine noetherian formal scheme, and let $I \subset A$ be an ideal of definition for A. Let M be an A-module of finite type. With M is associated a coherent module $\tilde{M}$ on $X = \mathrm{Spec}\ (A)$. In an analogous way, one associates with M a module M^Δ on $\mathrm{Spf}(A)$, defined as follows. For $n \in \mathbb{Z}_{\geq 0}$, let $X_n = \mathrm{Spec}\ (A/I^{n+1})$, and put

$$M^\Delta = \varprojlim_n \tilde{M}_n, \quad M_n = M/I^{n+1}M.$$

Then M^Δ does not depend on the choice of I, and the functor $M \mapsto M^\Delta$ is exact (on the category of finite A-modules). Moreover,

$$\Gamma(\mathcal{X}, M^\Delta) = M,$$

and the formation of M^Δ commutes with tensor products and internal $\mathcal{H}om$. Let $X = \mathrm{Spec}\ (A)$ and let

$$i\colon \mathcal{X} \to X$$

be the natural morphism of locally ringed spaces; it is defined by the inclusion on the underlying topological spaces and the canonical map $\mathcal{O}_X \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings. Then, since M is of finite type, Theorem 4.1.3 implies that

$$M^\Delta = i^* \tilde{M}.$$

Since, for any $f \in A$, $A_{\{f\}}$ is adic noetherian, it follows that $\mathcal{O}_{\mathrm{Spf}(A)}$ is a coherent sheaf of rings, that M^Δ is coherent, and that the coherent modules on $\mathcal{X}$ are exactly those of the form M^Δ for M of finite type over A.

4.2.2 Formal schemes as inductive limits of nilpotent thickenings

A *thickening* is a closed immersion of schemes $X \to X'$ whose ideal I is a nilideal; the schemes X and X' then have the same underlying topological space. If X' is noetherian, the same holds for X and I is nilpotent; conversely, if X is noetherian and I/I^2 is a coherent $\mathcal{O}_X$-module, then X' is noetherian [EGAI, Ch. 0, §7.2.6, §10.6.4]. If X' is noetherian, X' is affine if and only if X is [EGAI, §6.1.7]. We say that a thickening is *of order* n if $I^{n+1} = 0$.

Let $\mathcal{X}$ be a locally noetherian formal scheme. Then $\mathcal{O}_{\mathcal{X}}$ is a coherent sheaf of rings (see Section 4.2.1 above). Moreover, the coherent modules on $\mathcal{X}$ are exactly the modules which are of finite presentation, or equivalently, which on any affine open $U = \mathrm{Spf}(A)$ are of the form M^Δ for an A-module M of finite type.

Definition 4.2.1. Let $\mathcal{X}$ be a locally noetherian scheme. An *ideal of definition* of $\mathcal{X}$ is a coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$ such that, for any $x \in \mathcal{X}$, there exists an affine neighbourhood $U = \mathrm{Spf}(A)$ of x such that $\mathcal{I}|_U$ is of the form I^Δ for an ideal of definition I of A.

Let $\mathcal{X}$ be a locally noetherian formal scheme.

Lemma 4.2.2. *A coherent ideal $\mathcal{I}$ is an ideal of definition if and only if the ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is a scheme having $\mathcal{X}$ as an underlying space.*

Ideals of definition of $\mathcal{X}$ exist, and there is a largest one, $\mathcal{T} = \mathcal{T}_{\mathcal{X}}$, which is the unique ideal of definition $\mathcal{I}$ such that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is reduced. If $U = \mathrm{Spf}(A)$ is an affine open subset, then $\mathcal{T}|_U = T^\Delta$, where T is the ideal of elements $a \in A$ whose image in A/I are nilpotent. If $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$, then so is any power $\mathcal{I}^{n+1}$, $n \geq 0$. If $\mathcal{X}$ is noetherian, and $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$ and $\mathcal{J}$ is any coherent ideal, then $\mathcal{J}$ is an ideal of definition if and only if there exist positive integers p, q such that $J^q \subset I^p \subset J$.

Fix an ideal of definition $\mathcal{I}$ of $\mathcal{X}$. For $n \in \mathbb{Z}_{\geq 1}$, put

$$X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}),$$

which is a locally noetherian scheme. We have an increasing chain of thickenings

$$X_\bullet = (X_0 \to X_1 \to \cdots \to X_n \to \cdots \to \cdots) \to \mathcal{X}. \tag{4.6}$$

Moreover,

$$\mathcal{X} = \varinjlim_n X_n,$$

where the colimit is taken in the category of locally noetherian formal schemes (cf. Remark 4.1.18). Indeed, the underlying topological spaces of the X_n are all equal to the underlying space of $\mathcal{X}$, and

$$\mathcal{O}_{\mathcal{X}} = \varprojlim_n \mathcal{O}'_{X_n}$$

as topological rings, where $\mathcal{O}'_{X_n}$ is the pseudo-discrete sheaf of topological rings attached to the sheaf of rings $\mathcal{O}_{X_n}$ as in Section 4.1.4.

Proposition 4.2.3. *Let $X_\bullet = (X_0 \to X_1 \to \cdots)$ be a sequence of ringed spaces such that*

(1) X_0 is a locally noetherian scheme,

(2) the underlying maps of topological spaces are homeomorphisms, and with respect to them, the maps $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ are surjective,

(3) if $J_n = \mathrm{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0})$, then for $m \leq n$, one has $\mathrm{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}) = J_n^{n+2}$,

(4) J_1 is a coherent $\mathcal{O}_{X_0}$-module.

Then, the topologically ringed space

$$\mathcal{X} = \varinjlim X_i = (X_0, \varprojlim \mathcal{O}_{X_n})$$

is a locally noetherian formal scheme. Moreover, if $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_0}) = \varprojlim J_n$, then $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$ and $\mathcal{I}^{n+1} = \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n})$.

Proof. See [EGAI, §10.6.3 - 10.6.5]. □

4.2.3 Coherent sheaves on formal schemes and adic morphisms

Let $\mathcal{X}$ be a locally noetherian formal scheme. Let $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$, and consider the corresponding chain of thickenings $X_\bullet$ (see (4.6)). For $m \leq n$, denote by

$$u_{m,n} \colon X_m \to X_n, \quad u_n \colon X_n \to \mathcal{X}$$

the canonical morphisms.

Lemma 4.2.4. *Let (F_n) be an inverse system of sheaves of abelian groups on X_0, such that F_n is an $\mathcal{O}_{X_n}$-module for each n and the transition maps $f_{n,m}\colon F_n \to F_m$ are $\mathcal{O}_{X_n}$-linear (with respect to $\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}$). Call (F_n)* coherent *if each F_n is coherent and the transition maps $f_{n,m}$ induce isomorphisms $u_{m,n}^* F_n \cong F_m$. Then the functor*

$$\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(X_\bullet), \quad E \mapsto (u_n^* E), \tag{4.7}$$

from the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules to the category of coherent inverse systems on $X_\bullet$, is an equivalence of categories.

Proof. Exercise. □

Proposition 4.2.5. *Consider the notation in Lemma 4.2.4. Let $r \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathcal{X}}$-module. Then $\mathcal{E}$ is locally free of rank r over $\mathcal{O}_{\mathcal{X}}$ if and only if $u_n^*(\mathcal{E})$ is locally free of rank r over $\mathcal{O}_{X_n}$ for each $n \geq 0$.*

Proof. See the flatness criterion of [Bourbaki61, III, §5, Theorem 1]. □

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes, and let $\mathcal{J}$ be an ideal of definition of $\mathcal{Y}$. Since $\mathcal{J} \subset \mathcal{T}_Y$, the ideal $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is contained in $\mathcal{T}_X$. Fix an ideal of definition $\mathcal{I}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. Let $X_\bullet$ and $Y_\bullet$ be the chains of thickenings induced by $\mathcal{I}$ and $\mathcal{J}$ respectively. Then f induces a morphism of inductive systems

$$f_\bullet\colon X_\bullet \to Y_\bullet.$$

Moreover, one retrieves $f\colon \mathcal{X} \to \mathcal{Y}$ as the colimit $f = \varinjlim f_n$.

Lemma 4.2.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes. The above construction yields a bijection between the set of morphisms from $f\colon \mathcal{X} \to \mathcal{Y}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ and the set of morphisms of inductive systems $f_\bullet\colon X_\bullet \to Y_\bullet$.*

Proof. See [EGAI, §10.6.8]. □

Definition 4.2.7. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes.

(1) If, for some ideal of definition $\mathcal{J}$ of $\mathcal{Y}$, the ideal $\mathcal{I} = f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$, then we say that f is an *adic morphism*.

(2) If f is adic, then f is *flat* if for every $x \in \mathcal{X}$, the stalk $\mathcal{O}_{\mathcal{X},x}$ is flat over $\mathcal{O}_{\mathcal{Y},f(x)}$.

The flatness of an adic morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is equivalent to the condition that $\mathcal{O}_{X_n}$ is flat over $\mathcal{O}_{Y_n}$ for every n (this is a consequence of [Bourbaki61, III, §5, th. 2, prop. 2]. Here, $\mathcal{O}_{X_n} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}$ and $\mathcal{O}_{Y_n} = \mathcal{O}_{\mathcal{Y}}/\mathcal{J}^{n+1}$ for ideals of definition $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{X}$ and $\mathcal{Y}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} = \mathcal{I}$. In other words, $f\colon \mathcal{X} \to \mathcal{Y}$ is flat if and only if it is adic and $f_n\colon X_n \to Y_n$ is flat for each n.

Lemma 4.2.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally noetherian formal schemes, and let $\mathcal{J}$ be an ideal of definition of $\mathcal{Y}$. Let $\mathcal{I}$ be an ideal of definition of $\mathcal{X}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. Consider the inductive systems $X_\bullet$ and $Y_\bullet$ defined by $\mathcal{I}$ and $\mathcal{J}$. Under the correspondence $f \mapsto f_\bullet$ of Lemma 4.2.6, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is adic if and only if the commutative diagram*

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "f_m"] & X_n \arrow[d, "f_n"] \\ Y_m \arrow[r] & Y_n, \end{tikzcd}$$

is cartesian for every n, m.

Proof. Exercise. ☐

4.2.4 Formal completions along closed subschemes

Let X be a locally noetherian scheme, and let X' be a closed subset of the underlying space $|X|$ of X. Choose a coherent ideal I of $\mathcal{O}_X$ such that the closed subscheme of X defined by I has X' as underlying space. Consider the inductive system of locally noetherian schemes, all having X' as underlying space,

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots,$$

where X_n is the closed subscheme of X defined by I^{n+1}. By Proposition 4.2.3, the topologically ringed space

$$\varinjlim_n X_n$$

is a locally noetherian formal scheme, having X' as underlying space.

Definition 4.2.9. Let X be a locally noetherian scheme, and $X' \subset |X|$ a closed subspace. Define

$$X_{/X'} = \varinjlim X_n = (X', \varprojlim \mathcal{O}_X/I^{n+1}).$$

This locally noetherian formal scheme is called the *formal completion of X along X'*. When no confusion can arise, we write $\widehat{X} = X_{/X'}$.

Lemma 4.2.10. *Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. Choose a coherent ideal I of $\mathcal{O}_X$ such that the closed subscheme defined by I has X' as underlying space.*

(1) The locally noetherian formal scheme $X_{/X'}$ does not depend on the choice of I.

(2) If $X = \mathrm{Spec}\,(A)$ is affine and $I = \tilde{J}$, then $\widehat{X} = \mathrm{Spf}(\widehat{A})$, where $\widehat{A}$ is the completion of A with respect to the J-adic topology (see Section 4.1.2).

Proof. Exercise. ☐

Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. The closed immersions $i_n \colon X_n \to X$ define a morphism of ringed spaces

$$i = i_X \colon \widehat{X} \to X. \tag{4.8}$$

Definition 4.2.11. Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset. For a coherent sheaf F on X, define $F_{/X'} = \varprojlim_n i_n^*(F)$. Sometimes, if no confusion can arise, we shall write $\widehat{F} = F_{/X'}$.

Lemma 4.2.12. *Let X be a locally noetherian scheme and $X' \subset |X|$ a closed subset.*

(1) The morphism $i \colon \widehat{X} \to X$ defined in (4.8) is flat.

(2) For any coherent sheaf F on X, the natural map

$$i^*F \to F_{/X'} = \varprojlim_n i_n^*(F)$$

is an isomorphism.

(3) Let $X = \operatorname{Spec}(A)$, and let I be an ideal that defines the closed subset $X' \subset X$. Let $F = \tilde{M}$, with M an A-module of finite type. Then $F_{/X'} = \widehat{M}^{\Delta}$. Here, $\widehat{M} = \varprojlim M_n$ with $M_n = M/I^{n+1}M$, see Section 4.1.2.

Proof. The assertions follow from Theorem 4.1.3. □

Finally, let $f \colon X \to Y$ be a morphism of locally noetherian schemes. Let X' (resp. Y') be a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Choose coherent ideals $J \subset \mathcal{O}_X$ and $K \subset \mathcal{O}_Y$ defining closed subschemes with underlying spaces X' and Y' respectively, and such that $f^*(K)\mathcal{O}_X \subset J$. Then, f induces a morphism of inductive systems

$$f_\bullet \colon X_\bullet \to Y_\bullet,$$

where X_n (resp. Y_n) is the closed subset of X (resp. Y) defined by J^{n+1} (resp. K^{n+1}). By Lemma 4.2.6, we obtain a morphism of locally noetherian formal schemes

$$\widehat{f} = f_{/X'} \colon \widehat{X} = X_{/X'} \to Y_{/Y'} = \widehat{Y}, \tag{4.9}$$

which does not depend on the choices of J and K.

Definition 4.2.13. Let $f \colon X \to Y$ be a morphism of locally noetherian formal schemes. Let X' and Y' be closed subsets of X and Y such that $f(X') \subset Y'$. The morphism $\widehat{f} \colon \widehat{X} \to \widehat{Y}$ defined in (4.9) is called the *extension* of f to the completions.

The reason for the terminology is that the extension $\widehat{f} \colon \widehat{X} \to \widehat{Y}$ makes the following diagram commute:

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "i_X"] & \widehat{Y} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

We conclude the section with the following:

Lemma 4.2.14. *Let $f\colon X \to Y$ be a morphism of locally noetherian formal schemes. Let Y' be a closed subset of Y, and define $X' = f^{-1}(Y')$. The extension $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ of f is an adic morphism of locally noetherian formal schemes.*

Proof. All the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"] & X \arrow[d, "f"] \\ Y_n \arrow[r] & Y \end{tikzcd} \tag{4.10}$$

are cartesian. Let $m \leq n$ and consider the square

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "f_m"] & X_n \arrow[d, "f_n"] \\ Y_m \arrow[r] & Y_n. \end{tikzcd}$$

This square is cartesian because (4.10) is cartesian for each $n \geq 0$. The result follows from this, because of Lemma 4.2.8. ☐

Remark 4.2.15. Let X be a locally noetherian scheme and let $X' \subset X$ be a closed subscheme. Choose an ideal $\mathcal{I} \subset \mathcal{O}_X$ that cuts out the closed subset X' inside X, and let $X_n = (X', \mathcal{O}_X/\mathcal{I}^{n+1})$. Then we have:

(1) closed immersions $X_n \hookrightarrow X$; and

(2) a morphism of ringed spaces $\widehat{X} \to X$.

These fit together in the following commutative diagram:

$$\begin{tikzcd} & & & X & \widehat{X} \arrow[l] \\ \cdots \arrow[r, hook] & X_n \arrow[r, hook] \arrow[urr, hook] \arrow[urrr] & X_{n+1} \arrow[r, hook] \arrow[ur, hook] \arrow[urr] & \cdots . \end{tikzcd}$$

4.2.5 Grothendieck's existence theorem: Algebraizing coherent sheaves

Grothendieck's existence theorem is the following powerful result.

Theorem 4.2.16 (Grothendieck)**.** *Let A be an adic noetherian ring with ideal of definition I, and define $Y = \mathrm{Spec}\,(A)$, $Y_n = \mathrm{Spec}\,(A_n)$, $A_n = A/I^{n+1}$. Let $\widehat{Y} = \mathrm{Spf}(A)$. Let X be a noetherian scheme, separated and of finite type over $Y = \mathrm{Spec}\,(A)$. Then the functor*

$$\mathrm{Coh}_Y(X) \to \mathrm{Coh}_{\widehat{Y}}(\widehat{X})$$

from the category of coherent sheaves with proper support over Y to the category of coherent sheaves on $\widehat{X}$ with proper support over $\widehat{Y}$, is an equivalence.

Two important ingredients in the proof of Theorem 4.2.16 are the following results, that we state without proof.

Theorem 4.2.17. *Let $f\colon X \to Y$ be a finite type morphism of noetherian schemes, Y' a closed subset of Y, $X' = f^{-1}(Y')$. Let $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ be the extension of f to the formal completions. Let F be a coherent sheaf on X whose support is proper over Y. Then, for all q, the canonical maps*

$$(R^q f_* F)^\wedge \to R^q \widehat{f}_* \widehat{F} \qquad \text{and} \qquad R^q \widehat{f}_* \widehat{F} \to \varprojlim R^q (f_n)_* F_n$$

are topological isomorphism.

Proof. See [Fan+05, Theorem 8.2.2]. □

Theorem 4.2.18. *Let A be a noetherian ring and I an ideal. Let $f\colon X \to Y =$ Spec (A) be a morphism of finite type and $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ be its completion along $V(I) \subset Y$ and $f^{-1}(V(I)) \subset X$. Let F and G be coherent sheaves on X whose supports have an intersection which is proper over Y. Then, for all $r \in \mathbb{Z}$, we have that $\operatorname{Ext}^r(F, G)$ is an A-module of finite type, and the natural map $\operatorname{Ext}^r(F, G) \to \operatorname{Ext}^r(\widehat{F}, \widehat{G})$ induces an isomorphism*

$$\operatorname{Ext}^r(F, G)^\wedge \xrightarrow{\sim} \operatorname{Ext}^r(\widehat{F}, \widehat{G}).$$

Proof. See [Fan+05, Corollary 8.2.9] or [EGAIII 1, 4.5.1]. □

Before we start with the proof of Theorem 4.2.16, we introduce:

Definition 4.2.19. Let X be a locally noetherian formal scheme, let $X' \subset |X|$ be a closed subset and let $\widehat{X}$ be the completion of X along X'. A coherent sheaf $\mathcal{F} \in \operatorname{Coh}(\widehat{X})$ is called *algebraizable* if it lies in the essential image of the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$ (see Definition 4.2.11). In other words, a coherent sheaf $\mathcal{F}$ on $\widehat{X}$ is algebraizable if there exists a coherent sheaf F on X such that $\mathcal{F} \cong \widehat{F}$.

Proof of Theorem 4.2.16. We prove the theorem in four steps:

Step 1. Fully faithfulness. Let F and G be coherent sheaves on X with proper supports over Y. By Theorem 4.2.18, $\operatorname{Hom}(F, G)$ is an A-module of finite type. Hence, it is separated and complete for the I-adic topology. Therefore, by Theorem 4.2.18 again, the canonical map

$$\operatorname{Hom}(F, G) \to \operatorname{Hom}(\widehat{F}, \widehat{G})$$

is an isomorphism. This proves that $(-)^\wedge$ is fully faithful.

Step 2. Reduction to the case where $X \to Y$ is quasi-projective. This follows from:

Lemma 4.2.20. *Assume that the theorem holds for quasi-projective morphisms. Assume moreover that for every closed subscheme T of X whose underlying space is strictly contained in that of X, all coherent sheaves on $\widehat{T}$ whose support is proper over $\widehat{Y}$ are algebraizable. Let E be a coherent sheaf on $\widehat{X}$ whose support is proper over $\widehat{Y}$. Then E is algebraizable.*

Proof of Lemma 4.2.20. We proceed in steps. By Chow's lemma, there are morphisms

$$Z \xrightarrow{g} X \xrightarrow{f} Y$$

such that g is projective and surjective, fg is quasi-projective, and there exists an open immersion $j\colon U \to X$ with U nonempty, such that g induces an isomorphism over U.

Let $T = X - U$ with the reduced scheme structure. As we assume that the theorem holds for quasi-projective morphisms, we may assume that $T \neq X$. Let J be the ideal of T in X. Let E be a coherent sheaf on $\widehat{X}$ whose support is proper over $\widehat{Y}$. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow E \longrightarrow \widehat{g}_*\widehat{g}^*E \longrightarrow C \longrightarrow 0. \tag{4.11}$$

We claim that $\widehat{g}_*\widehat{g}^*E$ is algebraizable. Indeed, as we assume that the theorem holds for quasi-projective morphisms, $\widehat{g}^*E$ is algebraizable. Then $\widehat{g}_*\widehat{g}^*E$ is also algebraizable, see Theorem 4.2.17.

Next, observe that K and C are killed by a positive power $\widehat{J}^N$ of $\widehat{J}$. Therefore, these sheaves can be viewed as coherent sheaves on $\widehat{T}'$, where T' is the thickening of T defined by J^N. In particular, by the induction hypothesis, K and C, as coherent sheaves on $\widehat{T}'$, are algebraizable.

Let $\mathscr{C}$ be the category of algebraizable coherent sheaves on $\widehat{X}$ whose support is proper over Y. We claim that $\mathscr{C}$ is closed under kernels, cokernels and extensions. As for kernels and cokernels, this follows from the exactness of $(-)^\wedge$ on coherent sheaves: if $f\colon F \to G$ is a morphism of coherent sheaves on X with proper support over Y, with kernel A and cokernel B, then $\widehat{A} = \mathrm{Ker}(\widehat{f})$ and $\widehat{B} = \mathrm{Coker}(\widehat{f})$.

We are in the situation that K, $\widehat{g}_*\widehat{g}^*E$, and C in (4.11) are algebraizable coherent sheaves on $\widehat{X}$. As $\mathscr{C}$ is closed under kernels, cokernels and extensions, it follows that E is algebraizable. $\square$

Step 2 follows by noetherian induction and Lemma 4.2.20.

Step 3. Reduction to the case where $X \to Y$ is projective. By Step 2, we may assume that $f\colon X \to Y$ is quasi-projective. Thus, we have a factorization of $f\colon X \to Y$ into an open immersion $j\colon X \to Z$ and a projective morphism $g\colon Z \to Y$. Let $\mathcal{E}$ be a coherent sheaf on $\widehat{X}$ whose support T' is proper over $\widehat{Y}$. Suppose that the theorem holds for projective morphisms. We claim that $\mathcal{E}$ is algebraizable. To see this, let $\mathcal{F} = \widehat{j}_!\mathcal{E}$ be the extension by zero of $\mathcal{E}$ on $\widehat{Z}$. Then $\mathcal{F}$ is coherent and has proper support over $\widehat{Y}$. Thus, by the assumption that the theorem holds for projective morphisms, we get that $\mathcal{F} = \widehat{F}$ for a coherent sheaf F on Z.

Lemma 4.2.21. *Let R be a noetherian ring and $I \subset R$ an ideal. Let M be a finite R-module. Let $\widehat{M} = \varprojlim M/I^{n+1}M$ and $\widehat{R} = \varprojlim R/I^{n+1}$. Let $\mathfrak{p}$ be a prime ideal $\widehat{R}$; abusing notation, let $\mathfrak{p}$ be its inverse image in R. Then $(\widehat{M})_{\mathfrak{p}} = 0$ if and only if $M_{\mathfrak{p}} = 0$.*

Proof. By Nakayama's lemma (see Lemma 4.1.2), we have

$$(\widehat{M})_{\mathfrak{p}} = 0 \qquad \text{if and only if} \qquad (\widehat{M})_{\mathfrak{p}} \otimes_{(\widehat{R})_{\mathfrak{p}}} (\widehat{R})_{\mathfrak{p}}/\mathfrak{p}(\widehat{R})_{\mathfrak{p}} = 0.$$

Moreover,

$$(\widehat{M})_{\mathfrak{p}} \otimes_{(\widehat{R})_{\mathfrak{p}}} (\widehat{R})_{\mathfrak{p}}/\mathfrak{p}(\widehat{R})_{\mathfrak{p}} = (\widehat{M})_{\mathfrak{p}}/\mathfrak{p}(\widehat{M})_{\mathfrak{p}} = \left(\widehat{M}/\mathfrak{p}\widehat{M}\right)_{\mathfrak{p}}.$$

Now note that $I \subset \mathfrak{p}$, hence $\widehat{I} \subset \widehat{\mathfrak{p}}$. This yields

$$\widehat{M}/\mathfrak{p}\widehat{M} = (\widehat{M}/\widehat{IM})/(\widehat{\mathfrak{p}M}/\widehat{IM}) = (M/IM)/(\mathfrak{p}M/IM) = M/\mathfrak{p}M.$$

Hence,

$$(\widehat{M})_{\mathfrak{p}} \otimes_{(\widehat{R})_{\mathfrak{p}}} (\widehat{R})_{\mathfrak{p}}/\mathfrak{p}(\widehat{R})_{\mathfrak{p}} = (M/\mathfrak{p}M)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}.$$

This is zero if and only if $M_{\mathfrak{p}} = 0$. $\square$

Let T be the support of F. We claim that $T \subset X$. To see this, notice that $T' = \mathrm{Supp}(\mathcal{E}) \subset |\widehat{X}| \subset |X|$ and that

$$T' = \mathrm{Supp}(\mathcal{E}) = \mathrm{Supp}(\mathcal{F}) = \widehat{T}, \tag{4.12}$$

where the last equality follows from Lemma 4.2.21. This gives

$$\widehat{T} = T' \subset |\widehat{X}| \cap |\widehat{T}| \subset |X| \cap |T|.$$

Moreover, $X \cap T$ is open in T (because X is open in Z). As it contains $\widehat{T}$, we must have $X \cap T = T$ (see Lemma 4.1.13). This gives $T \subset X$, proving the claim.

Consequently, $F = j_! j^* F$, and hence

$$\widehat{j}_! \mathcal{E} = \mathcal{F} = \widehat{F} = \widehat{j}_! \widehat{j}^* \widehat{F}.$$

This implies that $\mathcal{E} = \widehat{j}^* \widehat{F} = (j^* F)^\wedge$ is algebraizable.

Step 4. Projective case.

Lemma 4.2.22. *Let $\mathcal{X}$ be a $\widehat{Y}$-adic formal scheme such that $X_0 = \mathcal{X} \times_Y Y_0$ is proper, and let L be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that $L_0 = L \otimes \mathcal{O}_{X_0}$ is ample. Let E be a coherent sheaf on $\mathcal{X}$.*

(1) Then, there exist non-negative integers m, r and a surjective morphism

$$(L^{\otimes -m})^{\oplus r} \to E.$$

(2) There exists an integer $n_0 \geq 1$ such that $\Gamma(\mathcal{X}, E(n)) \to \Gamma(X_0, E_0(n))$ is surjective for all $n \geq n_0$, where $E(n) = E \otimes L^{\otimes n}$ for $n \geq 0$.

We will now use the lemma. Assume $f\colon X \to Y$ is projective. Let L be an ample line bundle on X. If M is an $\mathcal{O}_X$-module, and $r \in \mathbb{Z}$, write $M(n) = M \otimes L^{\otimes n}$, and similarly for $\mathcal{O}_{\widehat{X}}$-modules. Let E be a coherent sheaf on $\widehat{X}$. By the lemma, we can find an exact sequence

$$\mathcal{O}(-m_1)^{r_1} \to \mathcal{O}(-m_0)^{r_0} \to E \to 0.$$

By the theorem, there exists a unique morphism $\mathcal{O}(-m_1)^{r_1} \to \mathcal{O}(-m_0)^{r_0}$ that completes to the above one. Define F to be the cokernel. By the fact that $(-)^\wedge$ is exact on coherent modules, we get $E = \widehat{F}$, and we are done.

The proof of Lemma 4.2.22 is left as an exercise for the reader. $\square$

4.3 Lecture 10: Line bundles on hypersurfaces over arbitrary fields

The goal of this section is to study line bundles on hypersurfaces $X \subset \mathbb{P}_k^{n+1}$ of dimension $n \geq 3$ over arbitrary fields k. Before doing so, we provide applications of the existence theorem.

4.3.1 First applications of Grothendieck's existence theorem

Let $\mathcal{X}$ be a locally noetherian formal scheme, and let $\mathcal{A} \subset \mathcal{O}_{\mathcal{X}}$ be a coherent ideal sheaf. Let $\mathcal{Y}$ be the topologically ringed space with underlying space the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{A}$ and sheaf of rings $\mathcal{O}_{\mathcal{Y}} = i^{-1}\mathcal{O}_{\mathcal{X}}/\mathcal{A}$ (with i the inclusion $\mathcal{Y} \to \mathcal{X}$). Then $\mathcal{Y}$ is a locally noetherian formal scheme, adic over $\mathcal{X}$, and is called the *closed formal subscheme* of $\mathcal{X}$ defined by $\mathcal{A}$. If $\mathcal{X} = \mathrm{Spf}(A)$ is affine, for some adic ring A, then there is an ideal $\mathfrak{a} \subset A$ such that $\mathcal{Y} = \mathrm{Spf}(A/\mathfrak{a})$.

Corollary 4.3.1. *Let $Y = \mathrm{Spec}\ (A)$ for some adic ring A with ideal of definition I; define $Y_n = \mathrm{Spec}\ (A/I^{n+1})$. Let X be a noetherian scheme, separated of finite type over Y. Then $Z \mapsto \widehat{Z}$ defines a bijection from the set of closed subschemes of X which are proper over Y to the set of closed formal subschemes of $\widehat{X}$ which are proper over $\widehat{Y}$.*

Proof. Let I and J be coherent ideal sheaves in $\mathcal{O}_X$, and suppose that $\widehat{I} = \widehat{J} \subset \mathcal{O}_{\widehat{X}}$ and that the quotients $\mathcal{O}_X/I$ and $\mathcal{O}_X/J$ have proper support over Y. We get a canonical isomorphism $\mathcal{O}_{\widehat{X}}/\widehat{I} \xrightarrow{\sim} \mathcal{O}_{\widehat{X}}/\widehat{J}$ compatible with the projections. Thus, by Theorem 4.2.16, we get a canonical isomorphism $\mathcal{O}_X/I \xrightarrow{\sim} \mathcal{O}_X/J$ compatible with the projections $\mathcal{O}_X \to \mathcal{O}_X/I$ and $\mathcal{O}_X \to \mathcal{O}_X/J$. This gives $I = J \subset \mathcal{O}_X$.

To prove surjectivity, let $\mathcal{Z}$ be a closed formal subscheme of $\widehat{X}$ which is proper over $\widehat{Y}$. By Theorem 4.2.16, there exists an $\mathcal{O}_X$-module F, unique up to isomorphism, such that $\widehat{F} \cong \mathcal{O}_{\mathcal{Z}}$. We need to algebraize the quotient map $u\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Z}}$. Notice that the support of F is proper over Y. In particular, $\mathcal{O}_X$ and F are two coherent sheaves on X whose supports have intersection which is proper over Y. Thus, we can apply Theorem 4.2.18. We find that $\mathrm{Hom}(\mathcal{O}_X, F) = \mathrm{Hom}(\mathcal{O}_{\widehat{X}}, \widehat{F})$. This gives a unique map $v\colon \mathcal{O}_X \to F$ such that $\widehat{v} = u$. Since $v_0 = u_0$ is surjective, so is v (Nakayama's lemma, see Lemma 4.1.2), hence $F = \mathcal{O}_Z$ for a closed subscheme $Z \subset X$ which is proper over Y, and $\widehat{Z} = \mathcal{Z}$. $\square$

Let $\mathcal{X} = \varinjlim X_n$ be a locally noetherian formal scheme, where X_n is a locally noetherian scheme for each $n \geq 0$. A morphism of locally noetherian formal schemes $f\colon \mathcal{Z} \to \mathcal{X}$ is called *finite* if f is adic and $f_0\colon X_0 \to Z_0$ is finite. If f is finite, then $f_*\mathcal{O}_{\mathcal{Z}}$ is a finite $\mathcal{O}_{\mathcal{X}}$-algebra. Conversely, every finite $\mathcal{O}_{\mathcal{X}}$-algebra arises in this way.

Corollary 4.3.2. *Let $Y = \mathrm{Spec}\ (A)$ for some adic ring A with ideal of definition I; define $Y_n = \mathrm{Spec}\ (A/I^{n+1})$. Let X be a noetherian scheme, separated of finite type over Y. Then $Z \mapsto \widehat{Z}$ is an equivalence of the category of finite X-schemes which are proper over Y to the category of finite $\widehat{X}$-formal schemes which are proper over $\widehat{Y}$.*

Proof. Let A and B be finite $\mathcal{O}_X$-algebras with proper supports over Y, and $u\colon A \to B$ a map of $\mathcal{O}_X$-modules such that $\widehat{u}\colon \widehat{A} \to \widehat{B}$ is a map of $\mathcal{O}_{\widehat{X}}$-algebras, then u preserves the $\mathcal{O}_X$-algebra structures by the fully faithfulness part of Theorem 4.2.16. Hence $Z \mapsto \widehat{Z}$ is fully faithful. Let $\mathcal{A}$ be a finite $\mathcal{O}_{\widehat{X}}$-algebra with proper support over $\widehat{Y}$, and let A be a finite $\mathcal{O}_X$-module with $\widehat{A} = \mathcal{A}$. By Theorem 4.2.16, the maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\mathcal{O}_{\widehat{X}} \to \mathcal{A}$ that give $\mathcal{A}$ the structure of an $\mathcal{O}_{\widehat{X}}$-algebra, uniquely algebraize to give maps that turn A into an $\mathcal{O}_X$-algebra, such that $\widehat{A} = \mathcal{A}$ as $\mathcal{O}_{\widehat{X}}$-algebras. □

Corollary 4.3.3. *Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme which is proper over Y. Let Z be a noetherian scheme, separated and of finite type over Y. Then, the map*

$$\operatorname{Hom}_Y(X, Z) \to \operatorname{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z})$$

is bijective.

Proof. We have a canonical isomorphism

$$(X \times_Y Z)^{\wedge} = \widehat{X} \times_{\widehat{Y}} \widehat{Z},$$

see [EGAI, Proposition 10.9.7]. Under this isomorphism, the completion $(\Gamma_f)^{\wedge}$ of the graph Γ_f of a Y-morphism $f\colon X \to Z$ identifies with the graph $\Gamma_{\widehat{f}}$ of the completion $\widehat{f}\colon \widehat{X} \to \widehat{Z}$ of f, see [EGAI, Corollaire 10.9.8]. Let f and g be morphisms $X \to Z$ over Y such that $\widehat{f} = \widehat{g}$. Then the completions of Γ_f and Γ_g are the same formal subschemes of $(X \times_Y Z)^{\wedge}$, hence, as Γ_f and Γ_g are proper over Y, Γ_f and Γ_g agree as subschemes of $X \times_Y Z$ (see Corollary 4.3.1). Similarly, by algebraizing the graph of a given morphism $f\colon \widehat{X} \to \widehat{Z}$ over $\widehat{Y}$, one algebraizes the morphism f. □

Remark 4.3.4. Let $Y = \operatorname{Spec}(A)$ for some adic ring A with ideal of definition I; define $Y_n = \operatorname{Spec}(A/I^{n+1})$. Let X be a noetherian scheme which is not necessarily proper over Y. Then the conclusion of Corollary 4.3.3 is no longer valid in general. Namely, if $X = Z = \operatorname{Spec} A[t]$, then $\widehat{X} = \widehat{Z} = \operatorname{Spf}(A\{t\})$ and

$$\operatorname{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z}) = \operatorname{Hom}_{A\text{-cont}}(A\{t\}, A\{t\}) = A\{t\},$$

see Proposition 4.1.17, whereas $\operatorname{Hom}_Y(X, Z) = A[t]$. Here, $A\{t\}$ is the ring of restricted formal power series $\sum_n t^n$, which are those power series such that a_n tends to zero for the I-adic topology as n tends to infinity.

4.3.2 Algebraizing projective formal schemes

Before we come to the main theorem of this section, we provide two lemmas.

Lemma 4.3.5. *Let $\mathcal{X}$ be a locally noetherian formal scheme and let $f\colon \mathcal{X} \to \widehat{Y}$ be an adic morphism of locally noetherian formal schemes. Then for each $n \geq 0$, the fibre product $X_n = \mathcal{X} \times_Y Y_n$ is a locally noetherian scheme. Moreover, $\mathcal{X} = \varinjlim X_n$.*

Proof. Consider the ideal of definition $\mathcal{I} = I^{\Delta}$ of $\widehat{Y} = \mathrm{Spf}(A)$. Since f is adic, we know that $\mathcal{J} = f^*(\mathcal{I})\mathcal{O}_{\mathcal{X}}$ is an ideal of definition for $\mathcal{X}$, hence the same holds for $\mathcal{J}^n$ $(n \geq 0)$. In particular, the ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}^n)$ is a scheme, see Lemma 4.2.2. The other assertion is clear. □

Lemma 4.3.6. *Let $f\colon \mathcal{Z} \to \mathcal{X} = \varinjlim X_n$ be a morphism of locally noetherian formal schemes. The following are equivalent.*

(1) f is finite, in other words, f is adic and $f_0\colon \mathcal{Z} \times_{\mathcal{X}} X_0 \to X_0$ is finite.

(2) f is locally of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ for A an adic noetherian ring with ideal of definition A and B a finite IB-adic A-algebra.

(3) f is adic and each $f_n\colon \mathcal{Z} \times_{\mathcal{X}} X_n \to X_n$ is finite.

Proof. See [EGAI, Ch. 0, 7.2.9]. □

We proceed to give a profound application of Theorem 4.2.16. Let A be an adic noetherian ring, I an ideal of definition of A, $Y = \mathrm{Spec}\ (A)$, $Y_n = \mathrm{Spec}\ (A/I^{n+1})$ and

$$\widehat{Y} = \varinjlim Y_n = \mathrm{Spf}(A).$$

Let X be a locally noetherian scheme over Y, and define $X_n = X \times_Y Y_n$. Then the *I-adic completion of X*,

$$\widehat{X} = \varinjlim X_n,$$

is a locally noetherian formal scheme over $\widehat{Y}$. One can ask: which locally noetherian formal schemes $\mathcal{X}$ over $\widehat{Y}$ arise in this way? This leads to the following definition.

Definition 4.3.7. Let $\mathcal{X}$ be a locally noetherian formal scheme and $f\colon \mathcal{X} \to \widehat{Y}$ an adic morphism of locally noetherian formal schemes.

(1) Let $\mathcal{X}$ be a locally noetherian adic formal scheme over $\widehat{Y}$. Define $X_n = \mathcal{X} \times_Y Y_n$ $(n \geq 0)$; this is a scheme by Lemma 4.3.5. The morphism

$$\mathcal{F}\colon \mathcal{X} \to \widehat{Y}$$

is called *proper* if the induced morphism $\mathcal{F}_0\colon X_0 \to Y_0$ is proper.

(2) The $\widehat{Y}$-adic formal scheme $\mathcal{X}$ is *algebraizable* if one of the following equivalent conditions is satisfied.

(a) There exists a locally noetherian scheme X over Y and a $\widehat{Y}$-isomorphism $\mathcal{X} \cong X_{/X'}$, where X' denotes the inverse image of $V(I) = \mathrm{Spec}\ (A/I)$ in X.

(b) There exists a locally noetherian Y-scheme $X \to Y$ such that, if $X_n = X \times_Y Y_n$ and $\widehat{X} = \varinjlim X_n$, then there is an isomorphism $\widehat{X} \cong \mathcal{X}$ over $\widehat{Y}$.

The answer to the above question is then as follows.

Theorem 4.3.8. *Let A be an adic noetherian ring with ideal of definition $I \subset A$ and define $Y = \mathrm{Spec}\,(A)$ and $Y_n = \mathrm{Spec}\,(A/I^{n+1})$. Let $\mathcal{X}$ be a proper, adic formal scheme over $\widehat{Y}$, and define $X_n = \mathcal{X} \times_Y Y_n$, so that X_n is a scheme (cf. Lemma 4.3.5) and $\mathcal{X} = \varinjlim X_n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that*

$$L_0 = \mathcal{L} \otimes \mathcal{O}_{X_0} = \mathcal{L}/I\mathcal{L}$$

is ample on X_0, and so X_0 is projective over Y_0. The following assertions are true.

(1) The noetherian $\widehat{Y}$-adic formal scheme $\mathcal{X}$ is algebraizable.

(2) Let X be a Y-scheme with $\widehat{X} = \mathcal{X}$ over $\widehat{Y}$. There is a line bundle M on X, unique up to isomorphism, with $\mathcal{L} = \widehat{M}$. The line bundle M is ample, so that the morphism $X \to Y$ is projective.

Proof. By Lemma 4.2.22, there exists an integer $n \gg 0$ such that

(1) There exists a closed immersion $i_0 \colon X_0 \to P_0 = \mathbb{P}^r_{Y_0}$ such that $L_0^{\otimes n} = i_0^* \mathcal{O}_{P_0}(1)$.

(2) The map $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes n}) \to \Gamma(X_0, L_0^{\otimes n})$ is surjective.

The map $i_0 \colon X_0 \to P_0$ corresponds to an epimorphism $u_0 \colon \mathcal{O}_{X_0}^{r+1} \to L_0^{\otimes n}$ which we can lift to an $\mathcal{O}_{\mathcal{X}}$-linear map

$$u \colon \mathcal{O}_{\mathcal{X}}^{r+1} \to L^{\otimes n}.$$

By Nakayama's lemma, see Lemma 4.1.2, each $u_p \colon \mathcal{O}_{X_p}^{r+1} \to L_p^{\otimes n}$ is surjective, hence corresponds to a morphism $i_p \colon X_p \to P_p = \mathbb{P}^r_{Y_p}$ of Y_p-schemes such that $L_p^{\otimes n} = i_p^* \mathcal{O}_{P_p}(1)$. These closed immersions form an inductive system $i_\bullet \colon X_\bullet \to P_\bullet$, hence they define a morphism of formal schemes

$$i \colon \mathcal{X} \to \widehat{P}$$

such that $\mathcal{L}^{\otimes n} = i^* \mathcal{O}_{\widehat{P}}(A)$, where $\widehat{P}$ is the completion of $P = \mathbb{P}^r_Y$ over $Y = \mathrm{Spec}\,(A)$. As $i \colon \mathcal{X} \to \mathbb{P}^r_{\widehat{Y}}$ is an adic morphism such that i_0 is finite, it follows that i_p is finite for each $p \geq 0$, see Lemma 4.3.6. By Nakayama's lemma, see Lemma 4.1.2, it follows that i_p is a closed immersion for each p. Consequently, $i \colon \mathcal{X} \to \widehat{P}$ is a closed immersion of formal schemes. By Corollary 4.3.1, there exists a unique closed subscheme

$$j \colon X \to P = \mathbb{P}^r_Y$$

such that $\widehat{X} = \mathcal{X}$ as subschemes of $\mathbb{P}^r_Y$. Moreover, by Theorem 4.2.16, there exists a line bundle M on X, unique up to isomorphism, such that $\mathcal{L} \cong \widehat{M}$. Since $\mathcal{L}^{\otimes n} = i^* \mathcal{O}_{\widehat{P}}(1)$ and $(M^{\otimes n})^\wedge = \widehat{M}^{\otimes n}$, we get $(M^{\otimes n})^\wedge = (j^* \mathcal{O}_P(1))^\wedge$. Hence, by Theorem 4.2.16, we get $M^{\otimes n} = j^* \mathcal{O}_P(1)$, and therefore M is ample. ∎

4.3.3 More on algebraization in formal geometry

We state without proof the following powerful theorems.

Theorem 4.3.9. *Let $f\colon X \to S$ be a projective morphism of schemes, with S noetherian. Let $\mathcal{O}_X(1)$ be an invertible sheaf on X, ample relative to S. Let X_0 be the scheme of zeros of a section t of $\mathcal{O}_X(1)$, and let $\widehat{X}$ be the formal completion of X along X_0. Let $\mathfrak{F}$ be a coherent module on $\widehat{X}$, and let F_0 be the induced coherent module on X_0. Suppose moreover that:*

(1) $\mathfrak{F}$ is flat over S.

(2) For each $s \in S$, the section t_s of $\mathcal{O}_{X_s}(1)$ is $\mathfrak{F}_s$-regular.

(3) For each $s \in S$, F_{0s} is of depth ≥ 2 at the closed points of X_{0s}.

Assume S admits an ample invertible sheaf. There exists a coherent sheaf F on X and an isomorphism between $\widehat{F}$, the formal completion of F, and $\mathfrak{F}$.

Proof. See [SGAII]. □

Theorem 4.3.10. *Let $f\colon X \to S$ be a projective morphism of schemes, with S noetherian. Let $\mathcal{O}_X(1)$ be an invertible sheaf on X, ample relative to S, let Y be the scheme of zeros of a section t of $\mathcal{O}_X(1)$, let J be the ideal that defines $Y \subset X$, let X_n be the subscheme of X defined by J^{n+1}, $\widehat{X}$ the formal completion of X along Y, $\widehat{f}\colon \widehat{X} \to S$ the composition $\widehat{X} \to X \to S$, F a coherent module on X, flat relative to S. We suppose moreover that for each $s \in S$, the coherent $\mathcal{O}_{X_s}$-module F_s is of depth $> n$ at the closed points of X_s, and that t is F-regular. Then the following holds.*

(1) The canonical morphism

$$R^i f_*(F) \to R^i \widehat{f}_*(\widehat{F})$$

is an isomorphism for $i < n$, and a monomorphism for $i = n$.

(2) The canonical morphism

$$R^i \widehat{f}_*(\widehat{F}) \to \varprojlim R^i f_*(F_m)$$

is an isomorphism for $i \leq n$.

Proof. See [SGAII]. □

Corollary 4.3.11. *Assume the conditions in Theorem 4.3.10. Assume S is affine. Then:*

(1) The canonical morphism

$$H^i(X, F) \to H^i(\widehat{X}, \widehat{F})$$

is an isomorphism for $i < n$ and injective for $i = n$.

(2) The canonical morphism

$$H^i(\widehat{X}, \widehat{F}) \to \varprojlim H^i(X_m, F_m)$$

is an isomorphism for $i \leq n$. □

4.3.4 The Picard group of a hypersurface

We begin with:

Lemma 4.3.12. *Let k be a field and $P = \mathbb{P}_k^n$ be the projective space of dimension n over k. Let $X \subset P$ be a closed subscheme of pure dimension $r \geq 1$. Let $U \subset P$ be an open neighbourhood of X in P. Then* $\operatorname{codim}(P - U, P) \geq 2$.

Proof. We may assume that $k = \bar{k}$ and that X is integral. Suppose that $U \neq P$ and let Z be an irreducible component of $P \setminus U$, endowed with its reduced subscheme structure. Consider the subvarieties $X \subset P$ and $Z \subset P$. We have $\dim(X) = r$. Let $s := \dim(Z)$. By [Har77, Chapter I, Theorem 7.2], since $X \cap Z = \emptyset$, we must have $r + s - n < 0$. Therefore, $s < n - r \leq n - 1$ because $r \geq 1$. □

Proposition 4.3.13. *Let X be a smooth variety over a field k. Let $Y \subset X$ be a closed subset of codimension ≥ 2. The restriction* $\operatorname{Pic}(X) \to \operatorname{Pic}(X \setminus Y)$ *is an isomorphism.*

Proof. See [SGAII, Exposé XI, Corollaire 3.8]. □

Theorem 4.3.14. *Let $X_0 \subset \mathbb{P}_k^{n+1}$ be a hypersurface of dimension ≥ 3. Then*

$$\operatorname{Pic}(X_0) = \mathbb{Z} \cdot \mathcal{O}_{X_0}(1).$$

Proof. Define $X := \mathbb{P}_k^{n+1}$. We proceed in steps:

(1) First, we prove that $\operatorname{Pic}(X_0) = \operatorname{Pic}(\widehat{X})$. Let $I \subset \mathcal{O}_X$ be the ideal sheaf of $X_0 \subset X$. For an integer $m \geq 0$, define $X_m = (X_0, \mathcal{O}_X / I^{m+1})$. We have exact sequences

$$0 \to I^{m+1}/I^{m+2} \to \mathcal{O}_{X_0}/I^{m+2} \to \mathcal{O}_{X_0}/I^{m+1} \to 0.$$

Consider the exact sequences

$$0 \to I^{m+1}/I^{m+2} \to \mathcal{O}^*_{X_{m+1}} \to \mathcal{O}^*_{X_m} \to 0, \tag{4.13}$$

where $I^{m+1}/I^{m+2} \to \mathcal{O}^*_{X_{m+1}}$ is the map sending x to $1+x$. Indeed, the sequence (4.13) induces for $m \geq 0$ an exact sequence

$$H^1(X_0, I^{m+1}/I^{m+2}) \to H^1(X_{m+1}, \mathcal{O}^*_{X_{m+1}}) \to H^1(X_m, \mathcal{O}^*_{X_m}) \to H^2(X_0, I^{m+1}/I^{m+2}).$$

Note that $I \cong \mathcal{O}_X(-d) \subset \mathcal{O}_X$. Therefore,

$$I^{m+1}/I^{m+2} \cong I^{m+1} \otimes \mathcal{O}_X/I \cong \mathcal{O}_{X_0}(-(m+1)d),$$

so that by Theorem 2.1.13, we have:

$$H^i(X_0, I^{m+1}/I^{m+2}) = H^i(X_0, \mathcal{O}_{X_0}(-(m+1)d)) = 0, \qquad i = 1, 2, \quad m \geq 0.$$

Consequently, as $H^1(X_n, \mathcal{O}^*_{X_n}) = \operatorname{Pic}(X_n)$, we get $\operatorname{Pic}(X_{n+1}) = \operatorname{Pic}(X_n)$ for $n \geq 0$. Hence $\operatorname{Pic}(X_0) = \operatorname{Pic}(\widehat{X})$ as desired.

(2) Next, we need to show that $\mathrm{Pic}(\widehat{X}) = \mathrm{Pic}(X)$:

(a) Let $\mathcal{L}$ be an invertible sheaf on $\widehat{X}$. By Theorem 4.3.9, there exists a coherent sheaf F on X such that $\widehat{F} \cong \mathcal{L}$. Now $\mathcal{L} = \widehat{F}$ is locally free of rank one on $\widehat{X}$, hence (by Nakayama, see Lemma 4.1.2), there exists an open neighbourhood $U \supset X_0$ such that $F|_U$ is locally free of rank one on U. Thus $F|_U \in \mathrm{Pic}(U)$.

(b) Observe that since U is an open neighbourhood of X_0, we have $\mathrm{codim}(X - U, X) \geq 2$, see Lemma 4.3.12. Hence $\mathrm{Pic}(X) = \mathrm{Pic}(U)$ by Proposition 4.3.13. Thus, we can extend $F|_U$ to a line bundle M on X. We get that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\widehat{X})$ is surjective.

(c) It remains to prove that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\widehat{X})$ is injective. For this, let L_1 and L_2 be line bundles on X and suppose that there is a morphism $f\colon \widehat{L}_1 \to L_2$. This gives a section

$$s(f) \in \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)).$$

Consider the natural map $\mathscr{H}om(L_1, L_2)^\wedge \to \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)$; we claim that

$$\varprojlim_m \Gamma(X_m, \mathscr{H}om(L_1, L_2)|_{X_m}) = \Gamma(\widehat{X}, \varprojlim_m \mathscr{H}om(L_1, L_2)|_{X_m}) = \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2)).$$

Indeed, this follows from item (2) in Corollary 4.3.11. Combining this with item (1) in Corollary 4.3.11, we get that the natural map

$$\Gamma(X, \mathscr{H}om(L_1, L_2)) \to \Gamma(\widehat{X}, \mathscr{H}om(\widehat{L}_1, \widehat{L}_2))$$

is an isomorphism. Hence there exists a unique section $\sigma(g) \in \Gamma(X, \mathscr{H}om(L_1, L_2))$, corresponding to a morphism $L_1 \to L_2$, that induces the section $s(f)$. In other words, $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(\widehat{L}_1, \widehat{L}_2)$. We conclude that, in particular, if $\widehat{L} \cong \mathcal{O}_{\widehat{X}}$ for some line bundle L on X, then $L \cong \mathcal{O}_X$.

This finishes the proof of Theorem 4.3.14. $\square$

Chapter 5

Hodge theory of hypersurfaces

5.1 Lecture 11: Infinitesimal Torelli for hypersurfaces

The goal of Lectures 11 – 14 is to prove that, for a general hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ and open neighbourhood B' of the point 0 corresponding to X in the moduli space of smooth hypersurfaces without additional automorphisms, the differential of the period map

$$d\mathscr{P}^n_0 \colon T_{B',0} \to T_{\mathscr{D},\mathscr{P}^n(0)} \subset \bigoplus_p \operatorname{Hom}\left(H^{p,n-p}(X_0)_{\mathrm{pr}}, H^{p-1,n-p+1}(X_0)_{\mathrm{pr}}\right)$$

(see (3.20)) is an embedding. This is called the *infinitesimal Torelli theorem for hypersurfaces.* It implies that the period map $\mathscr{P}^k$ is an immersion at the point $0 \in B'$ corresponding to X. As B' is smooth around 0, it follows that $\mathscr{P}^n \colon B' \to \mathscr{D}$ is an embedding at the point 0. Thus, in some sense, for $b \in B'$ close to zero, the corresponding hypersurface $X_b \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is determined by the Hodge structure on $H^n(X_b, \mathbb{C})$.

5.1.1 Universal family of hypersurfaces with no automorphisms

Fix positive integers d and n. Let k be a field, let $\mathbb{P}^{n+1} = \mathbb{P}^{n+1}_k$, and define

$$U \subset \mathbb{A} := \operatorname{Spec}\ \left(\operatorname{Sym}\left(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))^\vee\right)\right) \cong \mathbb{A}^N_k \qquad \left(N = \binom{n+2}{d}\right)$$

as the open subset of polynomials F whose attached hypersurface $X = \{F = 0\} \subset \mathbb{P}^{n+1}$ is smooth. We let

$$\mathcal{X} = \left\{(F, x) \in U \times \mathbb{P}^{n+1} \mid F(x) = 0\right\} \subset U \times \mathbb{P}^{n+1},$$

with its natural scheme structure. Moreover, the projection onto U defines a proper flat family of hypersurfaces $\mathcal{X} \to U$, fitting in a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}^{n+1} \times U \arrow[dl] \\ U. & \end{tikzcd}$$

We write $X_t = \pi^{-1} \subset \mathbb{P}^{n+1}$ for $t \in U$. One can show (see [...]), that $U(k) \neq \emptyset$.

As we shall prove in a later chapter (see [insert future reference here]), there exist an open subset

$$B \subset U$$

such that, for $\bar{t} \in U(\bar{k})$, $\bar{t} \in B(\bar{k})$ if and only if $\mathrm{Aut}(X_{\bar{t}}) = \mathrm{Aut}(X_{\bar{t}}, \mathcal{O}_{X_{\bar{t}}}(1)) = \{\mathrm{id}\}$.

Remark 5.1.1. Note that possibly, $B = \emptyset$. It turns out that, if $n \geq 1$, $d \geq 3$, and $(n,d) \neq (1,3)$, then $B \neq \emptyset$ (and hence dense in U), see [Katz and Sarnak]. Note that $B \subset B_{lin}$, where B_{lin} is the variety of smooth hypersurfaces withouth linear automorphisms, i.e. automorphisms induced by automorphisms of $\mathbb{P}^{n+1}$. Poonen has shown that $B_{lin}(k) \neq \emptyset$ in the range $n \geq 1$, $d \geq 3$, and $(n,d) \neq (1,3)$. Moreover, if in addition $(n,d) \neq (2,4)$, then $B(k) \neq \emptyset$. Whether $B(k) \neq \emptyset$ for $(n,d) = (2,4)$ seems an open question (note that these are smooth quartics in $\mathbb{P}^2$, hence K3 surfaces), which has been solved for $\mathrm{char}(k) \in \{0,2,3,5\}$ by Luijk [reference]. We will discuss all this in a future chapter, see [reference].

In any case, without any assumptions on the pair (n,d), the action of the linear algebraic group GL_{n+2} on $\mathbb{A}$ over k (which is induced by the action of $\mathrm{GL}_{n+2}(R)$ on $H^0(\mathbb{P}^{n+1}_R, \mathcal{O}(d))$ for every k-algebra R) preserves the open subsets $B \subset U \subset \mathbb{A}$, and the quotients

$$B' = B/\mathrm{GL}_{n+2} \qquad \text{and} \qquad \mathcal{X}' = \mathcal{X}/\mathrm{GL}_{n+2}$$

exist in the category of schemes over k, see [MFK94]. Moreover, the induced morphism

$$\pi' \colon \mathcal{X}' \to B' \tag{5.1}$$

is a family of smooth projective hypersurfaces in $\mathbb{P}^{n+1}$ over k with no geometric automorphisms, and in fact the *universal* family of such hypersurfaces.

5.1.2 Infinitesimal Torelli theorem for hypersurfaces: statement

We consider now the case $k = \mathbb{C}$. Let $\pi' \colon \mathcal{X}' \to B'$ be the universal family of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of degree d with no non-trivial automorphisms, see (5.1). We fix a point $0 \in B'$. Let $H^n(X_0, \mathbb{C})_{\mathrm{pr}}$ be the primitive n-th cohomology group of X_0, with complex coefficients, and define

$$b^{p,n}_{\mathrm{pr}} = \dim F^p H^n(X_0, \mathbb{C})_{\mathrm{pr}}.$$

This yields a sequence of n numbers $b^{n,n}_{\mathrm{pr}} \leq \cdots \leq b^{1,n}_{\mathrm{pr}}$. Consider the polarized period domain

$$\mathscr{D} \subset \mathrm{Flag}\left(b^{n,n}_{\mathrm{pr}}, \ldots, b^{1,n}_{\mathrm{pr}}, H^n(X_0, \mathbb{C})_{\mathrm{pr}}\right),$$

as well as the period map

$$\mathscr{P}^n \colon B'' \to \mathscr{D}, \quad b \mapsto [F^\bullet H^n(X_b)_{\mathrm{pr}} \subset H^n(X_b)_{\mathrm{pr}} = H^n(X_0)_{\mathrm{pr}}].$$

Here, $B'' \subset B'$ is a sufficiently small open neighbourhood of 0 in B'. See Section 3.3.4, and in particular (3.17), (3.18), and Remark 3.3.13.

Theorem 5.1.2. *Let d and n be positive integers and let $\pi' \colon \mathcal{X}' \to B'$ be the universal family of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of degree d in with trivial automorphism group. Let $b \in B'$. Then the differential of the period map*

$$d\mathscr{P}^n_b \colon T_{B',b} \to T_{\mathscr{D},\mathscr{P}^n(b)} \subset \bigoplus_p \operatorname{Hom}\left(H^{p,n-p}(X_b)_{\mathrm{pr}}, H^{p-1,n-p+1}(X_b)_{\mathrm{pr}}\right) \tag{5.2}$$

is injective.

The strategy of the proof will be as follows:

(1) Prove that the Kodaira–Spencer map $T_{B',b} \to H^1(X_b, T_{X_b})$ is an isomorphism. As a consequence, it suffices to show that the natural map $H^1(X_b, T_{X_b}) \to \operatorname{Hom}(H^{p,n-p}(X_b)_{\mathrm{pr}}, H^{p-1,n-p+1}(X_b)_{\mathrm{pr}})$ is injective for some p with $0 \leq p \leq n$.

(2) Associate a certain graded artinian local $\mathbb{C}$-algebra $R(X_b)$ to the hypersurface X_b and prove that, if $t(p) = (n-p+1)\cdot d - (n+2)$, then $H^{p,n-p}(X_b)_{\mathrm{pr}} \cong R_{t(p)}(X_b)$, compatibly with the maps $H^{p,n-p}(X_b)_{\mathrm{pr}} \times H^{n-p,p}(X_b)_{\mathrm{pr}} \to H^{n,n}(X_b)_{\mathrm{pr}}$ and $R_{t(p)}(X_b) \times R_{t(n-p)}(X_b) \to R_\sigma(X_b)$, for $\sigma = (n+1)\cdot(d-2)$.

(3) Prove that $R_d(X_b) = H^1(X_b, T_{X_b})$.

(4) Prove that under the above isomorphisms, the differential of the period map at b with respect to the p-th piece of the Hodge filtration, which we view as a map $T_{B',b} \to \operatorname{Hom}(H^{p,n-p}(X_b)_{prim}, H^{p-1,n-p+1}(X_b)_{prim})$, becomes identified with the canonical map

$$R_d(X_b) \to \operatorname{Hom}(R_{t(p)}, R_{t(p)+d}). \tag{5.3}$$

(5) Conclude by proving that there exists p such that (5.3) is injective.

5.2 Lecture 12: Logarithmic De Rham complex

Let X be a complex manifold and let $D \subset X$ be a hypersurface, i.e. D is locally defined by the vanishing of a holomorphic equation. We say that D is a *normal crossings divisor* if for each $0 \in X$ there are local holomorphic coordinates $z_1, \ldots, z_n$ for X around 0, in which D is defined by the equation $z_1 \cdots z_r = 0$ for some $r \leq n$ (which depends on the point $0 \in X$).

Definition 5.2.1. Consider a pair (X, D) with X a complex manifold and $D \subset X$ a normal crossings divisor. Fix an integer $k \geq 0$. Let $\Omega^k_X(*D)$ be the sheaf of meromorphic differential k-forms on X which are holomorphic on $X \setminus D$. We define a subsheaf of $\mathcal{O}_X$-modules

$$\Omega^k_X(\log D) \subset \Omega^k_X(*D)$$

in the following way. Let $U \subset X$ be an open subset. For $\alpha \in \Omega^k_X(D)(U)$, we have $\alpha \in \Omega^k_X(\log D)(U)$ if, for each $x \in U$ and each open neighbourhood $x \in V \subset U$ such that $D \cap V = \cup_{i=1}^r D_i$ with the D_i meeting transversally, we have that the orders of the poles of $\alpha|_V$ and $d\alpha|_V$ along D_i are at most one, for each $i \in \{1, \ldots, r\}$.

By construction, there is a natural differential $\Omega^k_X(\log D) \to \Omega^{k+1}_X(\log D)$. This gives a complex which we denote by $\Omega^\bullet_X(\log D)$, and which we call the *logarithmic De Rham complex*. Let $U = X \setminus D$, and let j be the inclusion of U in X. Then, consider the composition

$$\Omega^\bullet_X(\log D) \subset j_*\Omega^\bullet_U \to j_*\mathscr{A}^\bullet_U. \tag{5.4}$$

Lemma 5.2.2. *Consider a pair (X, D) with X a complex manifold and $D \subset X$ a normal crossings divisor. Let $U \subset X$ be an open subset, biholomorphic to $\mathbb{C}^n$ via holomorphic coordinates $z_1, \dots, z_n$, such that $D \cap U = \{z_1 \cdots z_r = 0\} \subset U$ for some $r \leq n$. Then $\Omega^k_X(\log D)|_U$ is a sheaf of free $\mathcal{O}_U$-modules, with basis given by the elements $\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_\ell}}{z_{i_\ell}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$ with $i_s \leq r$, $j_s > r$ and $\ell + m = k$.*

Proof. Exercise. □

Proposition 5.2.3 (Griffiths, Deligne)**.** *Consider a pair (X, D) with X a complex manifold and $D \subset X$ a normal crossings divisor. Let $U = X \setminus D$ with inclusion $j: U \to X$. The morphism $\Omega^\bullet_X(\log D) \to j_*\mathscr{A}^\bullet_U$ defined in (5.4) is a quasi-isomorphism.*

Proof. See [Voi02, Proposition 8.18]. □

Corollary 5.2.4. *In the above notation, let $U = X \setminus D$. There is a canonical isomorphism $H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(\log D))$.*

Proof. On the one hand, by Proposition 5.2.3, we have $\mathbb{H}^k(X, \Omega^\bullet_X(\log D)) = \mathbb{H}^k(X, j_*\mathscr{A}^\bullet_U)$. On the other hand,

$$\mathbb{H}^k(X, j_*\mathscr{A}^\bullet_U) = H^k(\Gamma(X, j_*\mathscr{A}^\bullet_U)) = H^k(\Gamma(U, \mathscr{A}^\bullet_U)) = H^k(U, \mathbb{C})$$

because the sheaves $j_*\mathscr{A}^p_U$ ($p \geq 0$) are acyclic (being sheaves of $\mathscr{C}^\infty_X$-modules), and because $\mathbb{C} \to \mathscr{A}^\bullet_U$ is an acyclic resolution of $\mathbb{C}$. □

5.2.1 Cohomology of smooth hypersurfaces in smooth projective varieties

Let X be a smooth projective complex variety and let

$$Y \hookrightarrow X$$

be a smooth hypersurface. Define

$$U = X \setminus Y \xrightarrow{j} X.$$

Then $\Omega^k_X(\log Y)$ is the locally free $\mathcal{O}_X$-module, locally generated by Ω_X and df/f, where f is a local holomorphic equation for Y. Moreover, under the isomorphism (...), the inclusion $\Omega^\bullet_X \hookrightarrow \Omega^\bullet_X(\log Y)$ induces the restriction $H^k(X, \mathbb{C}) \to H^k(U, \mathbb{C})$. Define a map of sheaves as follows:

$$\mathrm{Res}\colon \Omega^\bullet_X(\log Y) \to \Omega^{\bullet-1}_Y, \quad \alpha \wedge df/f \mapsto \mathrm{Res}(\alpha \wedge df/f) = 2i\pi \cdot \alpha|_Y. \tag{5.5}$$

Example 5.2.5. Let $X = \mathbb{C}$ with coordinate z, and let $D = \{z = 0\} = \{0\} \subset \mathbb{C}$. Let $U \subset X$ be an open subset containing 0. Consider the map

$$\mathrm{Res}\colon \Omega^1_X(\log D)(U) \to \Omega^0_D(D \cap U) = \mathbb{C}, \qquad \alpha \wedge df/f \mapsto 2i\pi \cdot \alpha(0).$$

Let $f\colon U \to \mathbb{C}$ be a meromorphic function which is holomorphic on $U - D$ and which has a pole of order one at 0. Then

$$f dz \in \Omega^1_X(\log D)(U).$$

Moreover, $\mathrm{Res}(f dz)$ equals $2\pi i \cdot a_{-1}$ if $f(z) = a_{-1}z^{-1} + g(z)$ for some $g(z)$ which is holomorphic at z. In other words, $\mathrm{Res}(f dz)$ equals the contour integral of $f(z)$ on a small circle around 0.

As $\mathbb{H}^k(Y, \Omega_Y^{\bullet -1}) = H^{k-1}(Y, \mathbb{C})$, the map (5.5) induces a map in cohomology, called the *residue map*,

$$\mathrm{Res}\colon H^k(U, \mathbb{C}) \to H^{k-1}(Y, \mathbb{C}). \tag{5.6}$$

Let T be a tubular neighbourhood of Y in X; thus T is an open neighbourhood $T \subset Y \subset X$, and if $\pi\colon N \to Y$ is the normal bundle of Y in X, then there is a smooth map $J\colon N \to X$ such that $J \circ 0_N = i\colon Y \to X$, and such that $J(N) = T$ and $J\colon N \to T$ is a diffeomorphism. The Thom isomorphism gives a canonical isomorphism

$$H^k(T, T \setminus Y, \mathbb{Z}) = H^k(N, N \setminus 0_N, \mathbb{Z}) = H^{k-1}(Y, \mathbb{Z}).$$

Moreover, we have inclusions $X \setminus T \subset U \subset X$ and hence, by excision, we have

$$H^k(X, U, \mathbb{Z}) = H^k(T, U \setminus (X \setminus T), \mathbb{Z}) = H^k(T, T - Y, \mathbb{Z}).$$

In particular,

$$H^k(X, U, \mathbb{Z}) = H^k(T, T \setminus Y, \mathbb{Z}) = H^{k-1}(Y, \mathbb{Z}).$$

Lemma 5.2.6. *The map* (5.6) *coincides with the composition*

$$H^k(U, \mathbb{C}) \xrightarrow{\delta^k} H^{k+1}(X, U, \mathbb{C}) \xrightarrow{\sim} H^{k-1}(Y, \mathbb{C}),$$

where δ^k the k-th coboundary map in the long exact sequence of relative cohomology.

Proof. Exercise. □

It follows that (5.6) admits a canonical integral lift $H^k(U, \mathbb{Z}) \to H^{k-1}(Y, \mathbb{Z})$ that fits in a long exact sequence

$$\cdots \to H^k(X, \mathbb{Z}) \to H^k(U, \mathbb{Z}) \xrightarrow{\mathrm{Res}} H^{k-1}(Y, \mathbb{Z}) \to H^{k+1}(X, \mathbb{Z}) \to \cdots. \tag{5.7}$$

The map $H^{k-1}(Y, \mathbb{Z}) \to H^{k+1}(X, \mathbb{Z})$ in (5.7) is given by the Gysin homomorphism; in other words, it is the map induced by Poincaré duality and push-forward on homology.

Lemma 5.2.7. *Let $\ell\colon Y \subset X$ be a smooth ample hypersurface, with $\dim(X) = n+1$, with complement $j\colon U = X \setminus Y \to X$. Define*

$$H^n(Y, \mathbb{Q})_{ev} = \operatorname{Ker}\left(\ell_*\colon H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q})\right).$$

Then (5.7) induces a short exact sequence

$$0 \longrightarrow H^{n+1}(X, \mathbb{Q})_{prim} \xrightarrow{j_*} H^{n+1}(U, \mathbb{Q}) \xrightarrow{\text{Res}} H^n(Y, \mathbb{Q})_{ev} \longrightarrow 0. \tag{5.8}$$

Proof. Exercise. □

Next, we turn to the case where the ambient space is the projective space of dimension $n+1$. In this case, we have:

Lemma 5.2.8. *For a smooth hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, $H^n(X, \mathbb{C})_{ev} = H^n(X, \mathbb{C})_{\mathrm{pr}}$.*

Proof. Exercise. □

Theorem 5.2.9 (Griffiths)**.** *Let $X \subset \mathbb{P} = \mathbb{P}^{n+1} = \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface, with complement $j\colon U = \mathbb{P} \setminus X \to \mathbb{P}$. Let p be an integer with $1 \leq p \leq n$. Then $H^{n+1}(\mathbb{P}, \mathbb{Q})_{prim} = 0$ hence $H^{n+1}(U, \mathbb{Q}) = H^n(X, \mathbb{Q})_{ev}$. Moreover, the image of the composition of maps*

$$H^0(\mathbb{P}, K_{\mathbb{P}}(pX)) \to H^0(U, \Omega^{n+1}_U) \to H^{n+1}(U, \mathbb{C}) \xrightarrow{\sim} H^n(X, \mathbb{C})_{ev} \tag{5.9}$$

equals $F^{n-p+2}H^n(X, \mathbb{C})_{ev} = F^{n-p+2}H^n(X, \mathbb{C}) \cap H^n(X, \mathbb{C})_{ev}$.

Proof. See [Voi02, Théorème 18.5]. □

5.3 Lecture 13: Variation of Hodge structure of hypersurfaces

5.3.1 Jacobian ring: first properties

Let n and d be positive integers. Let k be a field whose characteristic is prime to $d-1$ and d. Define S as the graded polynomial ring

$$S = k[x_0, \ldots, x_{n+1}] = \bigoplus_{i \geq 0} S_i, \qquad S_i = \{\text{homogeneous polynomials of degree } i\}.$$

For a non-zero $F \in S_d$, we write $\partial_i F \in S_{d-1}$ for the partial derivatives $\partial_i F = \partial F / \partial x_i$. The *Hessian* of F is the matrix of homogeneous polynomials of degree $d-2$ defined as

$$H(F) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j}.$$

We put

$$\sigma := (n+2) \cdot (d-2).$$

Then, for $F \in S_d$ non-zero, we have

$$\det H(F) \in S_\sigma.$$

Definition 5.3.1. Let $F \in S_d$ be a non-zero degree d homogeneous polynomial with coefficients in k. The *Jacobian ideal* of F is the homogeneous ideal

$$J(F) = (\partial_i F) = (\partial_0 F, \dots, \partial_{n+1} F) \subset S,$$

generated by the partial derivatives $\partial_i F$ of F. The *Jacobian ring* of F is the quotient

$$S \to R(F) \coloneqq S/J(F),$$

equipped with its natural grading. If $F \in S_d$ defines a smooth hypersurface $X = \{F = 0\} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$, we define $J(X) = J(F)$ and $R(X) = S/J(X) = R(F)$.

Lemma 5.3.2. *Let B be a ring, and let $b_1, \dots, b_m \in B$. Then $b_1, \dots, b_m$ is a regular sequence in B if and only if the image of the sequence $b_1, \dots, b_m$ in $B_{\mathfrak{p}}$ defines a regular sequence in $B_{\mathfrak{p}}$, for each prime ideal $\mathfrak{p}$ of B.*

Proof. Let i be a positive integer with $i \leq m$. Put $A \coloneqq B/(b_1, \dots, b_{i-1})$. Multiplication by b_i defines a map $A \to A$. If $b_1, \dots, b_m$ is a regular sequence, then this map is injective, hence remains injective after localization at a prime ideal (by flatness of the localization map), so that the images form a regular sequence in $B_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of B. Conversely, if, for each prime ideal $\mathfrak{p}$ of B, the images of the b_i in $B_{\mathfrak{p}}$ form a regular sequence and $b_i x = 0$ for some $x \in A = B/(b_1, \dots, b_{i-1})$, then $b_i x = 0$ in the localization of A at each of the prime ideals $\mathfrak{p}$ of B. If $x \neq 0$, then the annihilator of x is contained in a maximal ideal $\mathfrak{m}$ of A; localizing at $\mathfrak{m}$ gives a contradiction. □

Lemma 5.3.3. *Let $F \in S_d \setminus \{0\}$ such that the hypersurface $X = \{F = 0\} \subset \mathbb{P} = \mathbb{P}^{n+1}_k$ is smooth. Then, the Jacobian ring*

$$R(X) = R(F) = S/J(F)$$

is a zero-dimensional local ring, and a finite dimensional k-algebra.

Proof. The Euler equation

$$d \cdot F = \sum_{i=0}^{n+1} x_i \partial_i F \tag{5.10}$$

combined with the Jacobian criterion imply that the affine intersection $\cap V(F_i) \subset \mathbb{A}^{n+2}$ is the point $0 \in \mathbb{A}^{n+2}$. A sequence a_i, $i = 1, \dots, \dim(A)$ in a regular local ring A is a regular sequence if the height of the ideal (a_i) equals the dimension of A. In particular, the elements $\partial_i F$ form a regular sequence in the localization of S at each of its prime ideals, hence they form a regular sequence in S, see Lemma 5.3.2. Thus, $R(X)$ is a zero-dimensional noetherian ring.

In particular, the underlying topological space of the subscheme Spec $(R(X)) \subset \mathbb{A}^{n+2}_k$ consists of a finite set of points. These points must be closed, because if $\mathfrak{p}$ is a prime ideal which is not maximal, and $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{p}$, then $\mathfrak{p} \subsetneq \mathfrak{m}$ contradicts the zero-dimensionality of $R(X)$.

We claim that there is only one maximal ideal of $R(X)$. To see this, note that the subscheme Spec $(R(X)) \subset \mathbb{A}_k^{n+2}$ is $\mathbb{G}_m$-invariant, with respect to the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}_k^{n+2}$, because the polynomials $\partial_i F$ are homogeneous. Hence, indeed, there is only one closed point.

Let $\mathfrak{m} \subset R(X)$ be the corresponding maximal ideal. We claim that $\mathfrak{m}$ is nilpotent. To see this, observe that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n \geq 1$ because $R(X)$ is artinian. Hence $I = \mathfrak{m}^n$ is an ideal such that $\mathfrak{m}I = I$; since $I \subset \mathrm{rad}(R(X)) = \mathfrak{m}$, we get that $I = 0$ by Nakayama's lemma, see Lemma 4.1.2. Consequently, any element $x_{i_1}^{n_1} \cdots x_{i_k}^{n_k} \in R(X)$ satisfies $x_I^n = 0$. We conclude that $R(X)$ is a finite-dimensional k-algebra. □

5.3.2 Hodge decomposition for hypersurfaces

Let d and n be positive integers. Let X be a smooth hypersurface of degree d in $\mathbb{P} = \mathbb{P}_{\mathbb{C}}^{n+1}$. Define

$$S = \mathbb{C}[x_0, \ldots, x_{n+1}] = \bigoplus_{i \geq 0} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(i)),$$
$$S_i = \mathbb{C}[x_0, \ldots, x_{n+1}]_i = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(i)).$$

Recall that there is an isomorphism

$$\mathcal{O}_{\mathbb{P}} \to K_{\mathbb{P}}(n+2), \qquad 1 \mapsto \Omega := \sum (-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n.$$

and

$$\mathcal{O}_{\mathbb{P}}(pd) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}}(pY) = (\text{sheaf of meromorphic functions, poles of order } p \text{ along } Y),$$
$$Q \mapsto Q/F^p.$$

In particular, for $pd \geq n$, we get canonical isomorphisms

$$H^0(\mathbb{P}, \mathcal{O}(pd-n-2)) \to H^0(\mathbb{P}, K_{\mathbb{P}}(pd)), \qquad P \mapsto P\Omega,$$
$$H^0(\mathbb{P}, \mathcal{O}(pd-n-2)) \to H^0(\mathbb{P}, K_{\mathbb{P}}(pX)), \qquad P \mapsto \frac{P\Omega}{F^p}.$$

Recall that $H^n(X, \mathbb{C})_{ev} = H^n(X, \mathbb{C})_{prim}$, see Lemma 5.2.8. We can thus extend the composition of maps (5.9) to get a canonical morphism

$$\begin{aligned}\alpha_p \colon H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(pd-n-2)) \to H^n(U, \mathbb{C}) = H^n(X, \mathbb{C})_{prim},\\ P \mapsto \mathrm{Res}(P\Omega/F^p|_U) \in H^n(X, \mathbb{C})_{prim}.\end{aligned} \tag{5.11}$$

By Theorem 5.2.9, the image of (5.11) is the subspace $F^{n-p+2}H^n(X, \mathbb{C})_{prim}$.

Theorem 5.3.4. *Let d and n be positive integers. Let X be a smooth hypersurface of degree d in $\mathbb{P} = \mathbb{P}_{\mathbb{C}}^{n+1}$, with defining equation $F \in S_d$. Let $J(F) \subset S$ be the Jacobian ideal of X, see Definition 5.3.1. For an integer $p \geq 0$, define $\bar{\alpha}_p$ as the composition*

$$\begin{aligned}\bar{\alpha}_p \colon S_{pd-n-2} = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(pd-n-2)) \to F^{n-p+2}H^n(X, \mathbb{C})_{\mathrm{pr}}\\ \to F^{n-p+2}H^n(X, \mathbb{C})_{\mathrm{pr}}/F^{n-p+3}H^n(X, \mathbb{C})_{\mathrm{pr}} = H^{n-p+2,p-2}(X)_{\mathrm{pr}}.\end{aligned} \tag{5.12}$$

Then the following sequence is exact:

$$0 \to J(F)_{pd-n-2} \to S_{pd-n-2} \to H^{n-p+2,p-2}(X)_{\mathrm{pr}} \to 0. \tag{5.13}$$

Proof. See [Voi02, Théorème 18.10]. □

For convenience, we put

$$t(p) := (n-p+1)\cdot d - (n+2).$$

Corollary 5.3.5. *Let $F \in S_d$ be a non-zero element such that $X = \{F = 0\} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ smooth. Then for $p \geq 0$, the residue map induces a isomorphism*

$$R(F)_{t(p)} = H^{p,n-p}(X)_{\mathrm{pr}}.$$

Here, $R(F)_i = S_i/J(F)_i$ is the i-th component of the Jacobian ring $R(F)$ of X.

Proof. Indeed, we can replace p by $t(p)$ in Theorem 5.3.4 to get the result. □

5.3.3 Infinitesimal variation of Hodge structure for hypersurfaces

We consider the open subset $U \subset \mathbb{A}^N$ defined in Section 5.1.1. Hence, over it, we have the universal hypersurface

$$\pi\colon \mathcal{X} \to U. \tag{5.14}$$

We would like to describe associated the infinitesimal variation of Hodge structure, i.e., the maps given in (3.19):

$$\overline{\nabla}_b^{n-p+2}\colon H^{n-p+2,p-1}(X_b)_{\mathrm{pr}} \to \mathrm{Hom}(T_{U,b}, H^{n-p+1,p}(X_b)_{\mathrm{pr}}).$$

Here, $X_b \subset \mathbb{P}^{n+1}$ is the fibre of (5.14) above a point $b \in B$. To do so, we consider the maps

$$\bar{\alpha}_p\colon H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(pd-n-2)) \to H^{n-p+2,p-2}(X)_{\mathrm{pr}},$$

see (5.12).

Theorem 5.3.6. *Consider the universal family of smooth hypersurfaces $\pi\colon \mathcal{X} \to U$ of degree d in $\mathbb{P}^{n+1}$. Let $b \in U$. Then, for $P \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(pd-n-2))$ and $H \in T_{U,b} = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$, we have*

$$\overline{\nabla}_b^{n-p+2}(\bar{\alpha}_p(P))(H) = -p\cdot \bar{\alpha}_{p+1}(PH).$$

Proof. See [Voi02, Théorème 18.13]. □

Corollary 5.3.7. *Consider the universal family of smooth hypersurfaces $\pi\colon \mathcal{X} \to U$ of degree d in $\mathbb{P}^{n+1}$. Let $b \in U$. Then the following diagram commutes:*

$$\begin{tikzcd}
H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(pd-n-2)) \arrow[r] \arrow[d, "\bar{\alpha}_p"] & \operatorname{Hom}\left(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)), H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((p+1)d-n-2))\right) \arrow[d] \\
H^{n-p+2,p-2}(X_b)_{\mathrm{pr}} \arrow[r, "-\frac{1}{p}\cdot\overline{\nabla}_b^{n-p+2}"] & \operatorname{Hom}(T_{U,b}, H^{n-p+1,p-1}(X_b)_{\mathrm{pr}}).
\end{tikzcd}$$

Here, the upper horizontal arrow is given by multiplication, and the right vertical arrow is induced by the canonical isomorphism $T_{U,b} = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$ and the map $\bar{\alpha}_{p+1}$. □

Lemma 5.3.8. *Let $n \geq \mathbb{Z}_{\geq 1}$ be a positive integer, and let $\mathscr{X} \to U$ be the universal family of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Let $b \in U$. Assume that either $n \geq 3$, or that $n = 2$ and $d \neq 4$. Then the composition*

$$S_d = H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) = T_{U,b} \xrightarrow{\rho} H^1(X_b, T_{X_b}),$$

where ρ is the Kodaira–Spencer map, is surjective with kernel $J(F)_d$. Thus, if the pair (n,d) satisfies $n \geq 3$, or $n = 2$ and $d \neq 4$, then the following sequence is exact:

$$0 \to J(F)_d \to S_d = T_{U,b} \xrightarrow{\rho} H^1(X_b, T_{X_b}) \to 0.$$

Proof. Write $\mathbb{P} = \mathbb{P}^{n+1}$ and $X = X_b \subset \mathbb{P}$. We combine the Euler sequence $0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+2)} \to \mathcal{T}_{\mathbb{P}} \to 0$, see (2.3), with the normal bundle sequence $0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}|_X \to \mathcal{O}_X(d) \to 0$, see (2.8). By tensoring the former with $\mathcal{O}_X$, we obtain an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(n+2)} \to \mathcal{T}_{\mathbb{P}}|_X \to 0.$$

Here, we used that $\mathcal{O}_{\mathbb{P}}$ is flat over $\mathcal{O}_{\mathbb{P}}$. This yields the following diagram with exact rows follows:

$$\begin{tikzcd}
 & H^0(\mathcal{O}_X(1))^{\oplus(n+2)} \arrow[d, two heads] & & & \\
H^0(\mathcal{T}_X) \arrow[r] & H^0(\mathcal{T}_{\mathbb{P}}|_X) \arrow[r] \arrow[d] & H^0(\mathcal{O}_X(d)) \arrow[r] & H^1(\mathcal{T}_X) \arrow[r] & H^1(\mathcal{T}_{\mathbb{P}}|_X) \\
 & H^1(\mathcal{O}_X) \arrow[r, equal] & 0. & &
\end{tikzcd} \tag{5.15}$$

Here, $H^1(\mathcal{O}_X) = H^1(X, \mathcal{O}_X) = 0$ because $n \geq 2$, see Theorems 2.1.13 and 2.1.8. Hence, the kernel of

$$\bar{\rho}\colon H^0(\mathcal{O}_X(d)) \to H^1(\mathcal{T}_X)$$

is given by the image of

$$H^0(\mathcal{O}_X(1))^{\oplus(n+2)} \to H^0(\mathcal{O}_X(d)), \quad (h_0, \ldots, h_{n+1}) \mapsto \sum_i h_i \cdot \partial_i F,$$
$$h_i \in H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_{\mathbb{P}}(1)).$$

Here, we use that the exact sequence $0 \to \mathcal{I}_X(1) \to \mathcal{O}_{\mathbb{P}}(1) \to \mathcal{O}_X(1) \to 0$ and the canonical isomorphism $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}}(-d)$ give rise to an exact sequence

$$0 = H^0(\mathcal{O}_{\mathbb{P}}(1-d)) \to H^0(\mathcal{O}_{\mathbb{P}}(1)) \xrightarrow{\sim} H^0(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_{\mathbb{P}}(1-d)) = 0.$$

Similarly, the exact sequence $0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}}(d) \to \mathcal{O}_X(d) \to 0$ and the canonical isomorphism $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}}(-d)$ give rise to an exact sequence

$$0 \to H^0(\mathcal{O}_{\mathbb{P}}) \xrightarrow{1 \mapsto F} H^0(\mathcal{O}_{\mathbb{P}}(d)) \to H^0(\mathcal{O}_X(d)) \to H^1(\mathcal{O}_{\mathbb{P}}) = 0,$$

so that we can view $H^0(\mathcal{O}_X(d))$ as the quotient of $H^0(\mathcal{O}_{\mathbb{P}}(d))$ by the ideal generated by F.

The sequence (2.10) induces an exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}}(-d) \to \mathcal{T}_{\mathbb{P}} \to \mathcal{T}_{\mathbb{P}}|_X \to 0, \tag{5.16}$$

which gives rise to an exact sequence

$$\begin{aligned} H^0(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) \to H^0(X, \mathcal{T}_{\mathbb{P}}|_X) \to H^1(\mathbb{P}, \mathcal{T}_{\mathbb{P}}(-d)) &= H^n(\mathbb{P}, K_{\mathbb{P}} \otimes \Omega_{\mathbb{P}}(d))^\vee \\ &\cong H^n(\mathbb{P}, \Omega_{\mathbb{P}}(d-n-2))^\vee = 0. \end{aligned} \tag{5.17}$$

Here, the last equality holds by Bott vanishing, because $n \geq 2$ (see Theorem 2.1.8). We conclude that the restriction map $H^0(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) \to H^0(X, \mathcal{T}_{\mathbb{P}}|_X)$ is surjective.

Furthermore, we have a commutative diagram

$$\begin{tikzcd} & H^0(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) \arrow[r] \arrow[d] & H^0(X, \mathcal{T}_{\mathbb{P}}|_X) \arrow[d] \\ H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \arrow[r] & H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \arrow[r] \arrow[dr, bend right, "\rho"'] & H^0(X, \mathcal{O}_X(d)) \arrow[d] \\ & & H^1(X, \mathcal{T}_X). \end{tikzcd}$$

The composition

$$H^0(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) \to H^0(X, \mathcal{T}_{\mathbb{P}}|_X) \to H^0(X, \mathcal{O}_X(d))$$

sends $x_j \partial/\partial x_j$ to $x_i \partial F/\partial x_j|_X = x_i \partial_j F|_X$. The surjectivity of $H^0(\mathcal{T}_{\mathbb{P}}) \to H^0(\mathcal{T}_{\mathbb{P}}|_X)$ implies that the kernel of ρ is generated by the images of $H^0(\mathcal{T}_{\mathbb{P}}) \to H^0(\mathcal{O}_{\mathbb{P}}(d))$ and $H^0(\mathcal{O}_{\mathbb{P}}) \to H^0(\mathcal{O}_{\mathbb{P}}(d))$. Thus, the kernel of ρ is generated by the elements $x_i \partial_j F$ and by F, hence by $J(F)_d$ and by F. The Euler equation (5.10) implies that the ideal generated by the elements F and $x_i \partial_j F$ $(0 \leq i \leq n+1)$ equals the ideal generated by the $x_i \partial_j F$, which proves that the kernel of $\rho\colon H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \to H^1(X, \mathcal{O}_X(d))$ equals $J(F)_d$.

Finally, remark that (5.16) and (5.17) provide us with an exact sequence

$$H^1(\mathbb{P}, \mathcal{T}_{\mathbb{P}}(-d)) = 0 \to H^1(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) \to H^1(X, \mathcal{T}_{\mathbb{P}}|_X) \to H^2(\mathbb{P}, \mathcal{T}_{\mathbb{P}}(-d)). \tag{5.18}$$

By Serre duality, we have

$$H^1(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) = H^n(\mathbb{P}, \Omega_{\mathbb{P}}(-n-2))^\vee,$$
$$H^2(\mathbb{P}, \mathcal{T}_{\mathbb{P}}(-d)) = H^{n-1}(\mathbb{P}, \Omega_{\mathbb{P}}(d-n-2))^\vee.$$

If $n \geq 3$, then we have $1 < n-1 < n < \dim(\mathbb{P}) = n+1$, hence

$$H^n(\mathbb{P}, \Omega_{\mathbb{P}}(-n-2))^\vee = 0 = H^{n-1}(\mathbb{P}, \Omega_{\mathbb{P}}(d-n-2))^\vee$$

by Theorem 2.1.8. If $n = 2$ and $d \neq 4$, then also $H^n(\mathbb{P}, \Omega_{\mathbb{P}}(-n-2))^\vee = 0 = H^{n-1}(\mathbb{P}, \Omega_{\mathbb{P}}(d-n-2))^\vee$, where, in this case, the second equality holds because $d \neq 4 = n+2$, see Theorem 2.1.8.

Consequently, for all (n, d) such that $n \geq 2$ and $d \neq 4$ if $n = 2$, we have

$$H^1(\mathbb{P}, \mathcal{T}_{\mathbb{P}}) = 0 = H^2(\mathbb{P}, \mathcal{T}_{\mathbb{P}}(-d)).$$

Thus, by the exactness of (5.18), we get $H^1(X, \mathcal{T}_{\mathbb{P}}|_X) = 0$ in that range as well. The horizontal exact sequence in (5.15) then allows us to conclude. $\square$

5.4 Lecture 14: Jacobian ring & proof of infinitesimal Torelli

In this section, let again k be any field whose characteristic is coprime to d and $d-1$, and fix a non-zero polynomial $F \in S_d$ such that $X = \{F = 0\} \subset \mathbb{P}_k^{n+1}$ is smooth. The goal will be to prove the following:

Theorem 5.4.1. *Let $F \in S_d$ be a non-zero homogeneous degree d polynomial, with associated hypersurface $X = \{F = 0\} \subset \mathbb{P}^{n+1}$, and assume that X is smooth. Let $R \coloneqq R(X)$ be the Jacobian ring of X. Then R is a naturally graded artinian local ring and a finite-dimensional k-algebra, and has the following properties.*

(1) The Poincaré polynomial of R is given by

$$P(R) \coloneqq \sum_{i=0}^{\infty} \dim(R_i) t^i = \left(\frac{1 - t^{d-1}}{1-t} \right)^{n+2}.$$

(2) We have $R_i = 0$ for $i > \sigma$ and $R_\sigma \cong k$.

(3) The determinant of the Hessian $H(F)$ generates R_σ.

(4) Multiplication defines a perfect pairing

$$R_i \times R_{\sigma - i} \longrightarrow R_\sigma \cong k. \tag{5.19}$$

Before we prove Theorem 5.4.1, we show that it implies Theorem 5.1.2.

Proof of Theorem 5.1.2. First, observe that $J(F)_d \subset S_d$ is the tangent space to F at the orbit $\mathrm{GL}_{n+2}(\mathbb{C}) \cdot [F]$, where $[F] \in S_d = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$ is the point that gives F (see e.g. [Voi02, Remarque 18.16]). Recall that $B' = B/\mathrm{GL}_{n+2}(\mathbb{C})$. From this we onclude that, if $b \in B'$ is the image of $[F] \in B$, then

$$T_b B' = S_d / J_d = R_d, \tag{5.20}$$

which is the degree d part of the Jacobian ring R of F.

In view of Lemma 5.3.8, we can conclude from this that the Kodaira–Spencer map

$$\rho' \colon T_b B' \to H^1(X_b, T_{X_b}) \tag{5.21}$$

is an isomorphism. By Corollary 5.3.7, via the isomorphisms

$$R_{pd-n-2} \cong H^{n+1-p,p-1}_{\mathrm{pr}}, \quad R_{(p+1)d-n-2} \cong H^{n-p,p}_{\mathrm{pr}},$$

and

$$T_b B' \cong H^1(X_b, T_{X_b}) \cong R_d \qquad \text{(given by (5.20) and (5.21))},$$

the map $\overline{\nabla}_b^{n+1-p}$ identifies, up to a non-zero coefficient, with

$$R_{pd-n-2} \to \mathrm{Hom}(R_d, R_{(p+1)d-n-2}).$$

There exists $p \geq 0$ such that either $0 \leq t(p) \leq \sigma - 3$ or $R_d = 0$, unless $(n, d) = (2, 3)$ (cubic surfaces). Hence, by Theorem 5.4.1, multiplication in R gives an injection

$$R_d \hookrightarrow \mathrm{Hom}(R_{t(p)}, R_{t(p)+d}).$$

Via Proposition 3.3.18, the theorem follows. □

Proof of Theorem 5.4.1. Write $f_i \coloneqq \partial_i F$. Then $f_0, \dots, f_{n+1} \in S$ is a regular sequence of homogeneous polynomials of degree $d - 1$. For an exact sequence of graded S-modules

$$0 \to M^m \to \cdots \to M^0 \to 0,$$

the additivity of the Poincaré polynomial implies

$$\sum (-1)^j P(M^j) = 0.$$

Now, define $R^i = S/(f_0, \dots, f_i)$, and consider the sequence

$$0 \to R^{i-1} \xrightarrow{\cdot f_i} R^{i-1} \to R^i \to 0.$$

This is exact, as the sequence $f_0, \dots, f_{n+1}$ is a regular sequence. Consequently,

$$P(R^i) = P(R^{i-1}) - P(f_i \cdot R^{i-1}) = P(R^{i-1}) - t^{d-1} P(R^{i-1}) = (1 - t^{d-1}) P(R^{i-1}).$$

By induction,

$$P(R) = P(R^{n+1}) = (1 - t^{d-1}) P(R^n) = (1 - t^{d-1})^{n+1} P(R^0) = (1 - t^{d-1})^{n+2} P(S).$$

Now $P(S) = P(k[x_0, \ldots, x_{n+1}]) = 1/(1-t)^{n+2}$. The first item follows.

Item (2) follows from Item (1). Indeed, note that

$$\deg(P(R)) = (n+2) \cdot \deg(1 + t + \ldots + t^{d-2}) = (n+2) \cdot (d-2) = \sigma;$$

to prove that $\dim R_\sigma = 1$, we use again that

$$P(R)(t) = P(R) = (1 + t + \ldots + t^{d-2})^{n+2}.$$

It remains to prove that the pairings (5.19) are perfect. Notice that

$$P(R)(t) = P(R)(t^\sigma) \cdot P(R)(1/t).$$

Hence $\dim R_i = \dim R_{\sigma - i}$. Therefore, it suffices to show that for each homogeneous

$$g \notin (f_i)$$

there exists a homogeneous polynomial h with $0 \neq \bar{g} \cdot \bar{h} \in R_\sigma$. Equivalently, one must show that the degree σ part $(\bar{g})_\sigma \subset R$ of the homogeneous ideal $(\bar{g}) \subset R$ is not trivial. Let us prove this. Let i be maximal so that $(\bar{g})_i \neq 0$, and pick $0 \neq \bar{G} \in (\bar{g})_i$. Suppose, for the sake of contradiction, that $i < \sigma$. Then $\bar{G} \cdot (\bar{x}_0, \ldots, \bar{x}_{n+1}) \subset (\bar{g})_{i+1} = 0$. Hence

$$G \cdot (x_0, \ldots, x_{n+1}) \subset (f_0, \ldots, f_{n+1}).$$

Consequently, the map

$$k \to R, \quad 1 \mapsto \bar{G}$$

defines a non-trivial morphism of S-modules, which is not a scalar-multiple of any such a morphism induced by the isomorphism $R_\sigma \cong k$. In particular, we must have $\dim_k \mathrm{Hom}_S(k, R) > 1$. We conclude by using the lemma below. $\square$

Lemma 5.4.2. *Let $S \to R$ be as above. Then* $\dim_k \mathrm{Hom}_S(k, R) = 1$.

Proof. Let V^* be the vector space $V^* = \langle x_0, \ldots, x_{n+1} \rangle$. Then consider the *Koszul complex*

$$K_\bullet(f_i) \colon \left(\bigwedge^{n+2} V^* \to \cdots \to \bigwedge^{k} V^* \to \cdots \to \bigwedge^{2} V^* \to V^* \to k \right) \otimes_k S,$$

concentrated in homological degrees $(n+2, n+1, \ldots, 1, 0)$, with differentials

$$\partial_p(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum (-1)^j f_{i_j} \cdot x_{i_1} \wedge \cdots \widehat{x_{i_j}} \wedge \cdots x_{i_p}.$$

It is a standard fact that for a regular sequence (f_i), the Koszul complex is exact in degree $\neq 0$ with

$$H_0(K_\bullet(f_i)) \cong \mathrm{Coker}(V^* \otimes S \to S) \cong R = S/(f_0, \ldots, f_{n+1}).$$

Then, split the Koszul complex $K_\bullet(f_i)$ into short exact sequences as follows:

$$\begin{aligned}
&0 \to \operatorname{Ker}(\delta_0) \to S \to R \to 0,\\
&0 \to \operatorname{Ker}(\delta_1) \to V^* \otimes_k S \to \operatorname{Ker}(\delta_0) \to 0,\\
&0 \to \operatorname{Ker}(\delta_2) \to \bigwedge^2 V^* \otimes_k S \to \operatorname{Ker}(\delta_1) \to 0,\\
&0 \to \operatorname{Ker}(\delta_3) \to \bigwedge^3 V^* \otimes_k S \to \operatorname{Ker}(\delta_2) \to 0,\\
&\quad \cdots\\
&0 \to \bigwedge^{n+2} V^* \otimes_k S \to \bigwedge^{n+1} V^* \otimes_k S \to \operatorname{Ker}(\delta_n) \to 0.
\end{aligned}$$

Observe that

$$\operatorname{Ext}^i_S(k, \bigwedge^p V^* \otimes S) = 0 \quad \forall i < n+1.$$

Therefore, the above short exact sequences induce a sequence of embeddings

$$\operatorname{Hom}_S(k, R) \hookrightarrow \operatorname{Ext}^1_S(k, \operatorname{Ker}(\partial_0)) \hookrightarrow \cdots \hookrightarrow \operatorname{Ext}^{n+2}_S(k, \bigwedge^{n+2} V^* \otimes S) \cong k.$$

Consequently, $\dim_k \operatorname{Hom}_S(k, R) \leq 1$. The map $k \xrightarrow{\sim} R_\sigma \subset R$ gives a non-zero map of S-modules. Hence $\dim_k \operatorname{Hom}_S(k, R) \geq 1$ and the lemma follows. $\square$

Chapter 6

Linear subspaces and quadric fibrations

6.1 Lecture 15: Linear subspaces of cubic hypersurfaces

Lemma 6.1.1. *Let k be a field of characteristic zero and let $X \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree $d \geq 2$. The dimension of any linear subspace contained in X is not greater than $n/2$.*

Proof. Let $a_1, \ldots, a_N$ be the coefficients of any polynomial F that defines X in $\mathbb{P}^{n+1}_k$. Then X has a model over $K \coloneqq \mathbb{Q}(a_1, \ldots, a_N) \subset k$. Say Y is a smooth hypersurface over K such that $Y_k \cong X$.

Suppose that X contains a linear subspace $P \subset X$ of dimension ℓ. Let $b_1, \ldots, b_M$ be the coefficients of any polynomial that defines P in $\mathbb{P}^{n+1}_k$. Then, over the field

$$L \coloneqq \mathbb{Q}(a_1, \ldots, a_N, b_1, \ldots, b_M) \supset K,$$

the hypersurface $Y_L = Y \times_K L$ contains a linear subspace of dimension ℓ. Choosing an embedding $Y \subset \mathbb{C}$, we get that $Y_{\mathbb{C}} = Y \times_K \mathbb{C}$ contains a linear subspace of dimension ℓ. In particular, to prove the lemma, we may assume that $k = \mathbb{C}$.

In this case, if $h \in H^2(X, \mathbb{Z})$ is the class of a hyperplane, we have

$$H^{2i}(X, \mathbb{Z}) = \mathbb{Z} \cdot h^i \qquad \text{if} \qquad 2i < n. \tag{6.1}$$

See Corollary 2.2.7. Let $\mathbb{P}^\ell \subset X$ be a linear subspace, and suppose that $\ell > n/2$. If

$$c = n - \ell$$

is the codimension of $\mathbb{P}^\ell$ in X, then $c = n - \ell < n - n/2 = n/2$ so that (6.1) gives:

$$[\mathbb{P}^\ell] = m \cdot h^c, \qquad m \in \mathbb{Z}.$$

Let $j \colon X \hookrightarrow \mathbb{P}^{n+1}$ be the inclusion, and let $H \in H^2(\mathbb{P}^{n+1}, \mathbb{Z})$ be the class of a hyperplane. Then $j^*(H) = h$, and

$$[\mathbb{P}^\ell] \cup h^\ell = [\mathbb{P}^\ell] \cup j^*(H^\ell) = j_*[\mathbb{P}^\ell] \cup H^\ell = H^{c+1} \cup H^\ell = 1.$$

However,
$$[\mathbb{P}^\ell] \cup h^\ell = m \cup h^{c+\ell} = m \cup h^n = m \cdot d.$$
We conclude that $m \cdot d = 1$, which is absurd as $d > 1$. Hence $\ell \leq n/2$ as desired. $\square$

Lemma 6.1.2. *Let k be a field, and let $X = V(F) \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Let $P \subset X$ be a linear subspace such that $\dim(P) > n/2$. Then there exists a point $a \in P$ such that $\partial_i F(a) = 0$ for all $i \in \{0, \ldots, n+1\}$.*

In particular, if $\mathrm{char}(k) \nmid d$, then
$$\dim(P) \leq n/2$$
for any linear subspace $P \subset X$.

Proof. Let $\ell = \dim(P)$. We claim that $\ell \leq n/2$. We may assume that $P = V(x_{\ell+1}, \ldots, x_{n+1}) \subset X$. Therefore, we can write
$$F = \sum_{j=\ell+1}^{n+1} x_j G_j, \quad G_j \in k[x_0, \ldots, x_{n+1}].$$
As $d \geq 2$, we have $\deg(G_j) \geq 1$. Notice that if $a \in P = V(x_{\ell+1}, \ldots, x_{n+1})$, then we can write $a = [a_0 \colon \cdots : a_\ell \colon 0 \colon \cdots : 0]$ hence
$$\partial_i F(a) = \left(\sum_{j=\ell+1}^{n+1} x_j \partial_i G_j \right)(a) = 0 \quad \forall i \leq \ell,$$
$$\partial_i F(a) = G_i(a) + (x_i \partial_i G_i)(a) = G_i(a) + a_i \partial_i G_i(a) = G_i(a) \quad \forall i \geq \ell + 1.$$
Assume, for the sake of contradiction, that $\ell > n/2$. Then
$$V(G_{\ell+1}, \ldots, G_{n+1}) \cap \mathbb{P}^\ell \neq \emptyset$$
for codimension reasons (see e.g. [Har77, Chapter I, Theorem 7.2] or [Ful98, §8.2, p. 137]). Therefore, there exists $a \in P$ such that
$$G_i(a) = \partial_i F(a) = 0 \quad \forall i \in \{\ell + 1, \ldots, n + 1\},$$
It follows that $\partial_i(F) = 0$ for all $i \in \{0, \ldots, n+1\}$. As $P \subset X$, we have $a \in X$. Via the Jacobian criterion, this contradicts the smoothness of X at $a \in X$. Hence $\ell < n/2$. $\square$

Exercise 6.1.3. Let $X = V(F) \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d. Assume that for some $i \in \{0, \ldots, n+1\}$, the degree of F as a polynomial in x_i is less than or equal to $d - 2$. Show that X is singular.

Example 6.1.4. Let $F = x_0^2 x_2^2 + x_0 x_2^3 + x_1^4$, and consider the plane quartic curve $X = V(F) \subset \mathbb{P}^2$. Then $d = 4$ and $\deg_{x_0}(F) = 2 = d - 2$. Let $a = [1 : 0 : 0] \in \mathbb{P}^2(k)$. Then $F(a) = 0$. Moreover, $\partial_0 F = 2x_0 x_2^2 + x_2^3$, $\partial_1 F = 4x_1^3$ and $\partial_2 F = 2x_0^2 x_2 + 3x_0 x_2^2$. Thus, we have $F(a) = \partial_i F(a) = 0$ for $i \in \{0, 1, 2\}$, hence X is singular at a.

Remark 6.1.5. Let $k = \mathbb{F}_3$, and consider the cubic

$$X = V(F) \subset \mathbb{P}^1_k, \qquad F = x_0^2 x_1 - x_0 x_2^2 \in k[x_0, x_1, x_2].$$

Then $X = V(F)$ is smooth. However, $\partial_0 F = 2x_0x_1 - x_1^2 = 2x_0x_1 + 2x_1^2$ and $\partial_2 F = x_0^2 - 2x_0x_1 = x_0^2 + x_0x_1$ both vanish at $[1\colon -1]$.

Thus, it is not true, for an arbitrary field k and polynomial $F \in k[x_0, \dots, x_{n+1}]_d$ of degree $d \geq 0$, that the condition $\cap_i V(\partial_i F) \neq \emptyset$ implies that $V(F)$ is singular (even though this does hold when $\mathrm{char}(k) \nmid \deg(F)$ by the Euler equation, see (5.10)).

Exercise 6.1.6. Let $k = \mathbb{F}_3$. Does there exist a smooth cubic threefold $X \subset \mathbb{P}^4_k$ such that X contains a plane $P \subset \mathbb{P}^4_k$? Does there exists a smooth cubic fivefold $X \subset \mathbb{P}^6_k$ such that X contains a linear subspace $P \subset \mathbb{P}^6_k$ of dimension three?

6.1.1 Linear subspaces of generic cubic hypersurfaces

Lemma 6.1.7. *Let (n, d) be positive integers with $(n, d) \notin \{(1, 3), (2, 4)\}$. Let $X \subset \mathbb{P}^{n+1}$ and $X' \subset \mathbb{P}^{n+1}$ be two smooth hypersurfaces of degree d and dimension n. Then any isomorphism $f\colon X \cong X'$ lifts to an automorphism of $\mathbb{P}^{n+1}$.*

Proof. First, assume $n \geq 3$. Then $\mathrm{Pic}(Y) = \mathbb{Z} \cdot \mathcal{O}_Y(1)$ for any smooth degree d hypersurface $Y \subset \mathbb{P}^{n+1}$, see Theorem 4.3.14. In particular, we have $f^*\mathcal{O}_{X'}(1) \cong \mathcal{O}_X(1)$, and hence f lifts to an automorphism of $\mathbb{P}^{n+1}$.

Next, assume $n \in \{1, 2\}$ but $(n, d) \notin \{(1, 3), (2, 4)\}$. We have $\omega_X \cong \mathcal{O}_X(d - n - 2)$, see Proposition 2.1.12. Clearly, $f^*\omega_{X'} \cong \omega_X$. Therefore, there exists $m \in \mathbb{Z} - \{0\}$ such that $m \cdot f^*\mathcal{O}_{X'}(1) \cong m \cdot \mathcal{O}_X(1)$.

If $n = 2$, then the restriction map $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}) \to H^1(X, \mathcal{O}_X)$ is an isomorphism by Theorem [ref], because $p + q = 0 + 1 < n = 2$. In particular, $H^1(X, \mathcal{O}_X) = 0$. Assume $k = \mathbb{C}$. The vanishing of $H^1(X, \mathcal{O}_X)$ implies that the map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Moreover, $H^2(X, \mathbb{Z})$ is torsion-free, see Corollary 2.4.1. Hence, $\mathrm{Pic}(X)$ is torsion-free also. Consequently, $f^*\mathcal{O}_{X'}(1) \cong \mathcal{O}_X(1)$ as desired.

Finally, we treat the case $n = 1$, $d \neq 3$. We may assume that $d \geq 4$, for if $d \leq 2$ then X and X' are rational and thus have Picard group free of rank one. By the above, there exists $m \in \mathbb{Z} - \{0\}$ such that $m \cdot f^*\mathcal{O}_{X'}(1) \cong m \cdot \mathcal{O}_X(1)$. Now X and X' are plane curves of degree $d \geq 4$. In this case, the result follows from Noether's theorem [insert reference]. □

Lemma 6.1.8. *Let $\mathbb{P} = \mathbb{P}^{n+1}$. Let $P \subset \mathbb{P}$ be a linear subspace of dimension $\ell - 1 > 0$. Consider the subspace*

$$H^0(\mathbb{P}, \mathcal{I}_P(3)) \subset H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)) \tag{6.2}$$

classifying cubic homogeneous polynomials $F \in k[x_0, \dots, x_{n+1}]$ such that $X = V(F)$ contains the linear subspace P. Then

$$\dim H^0(\mathbb{P}, \mathcal{I}_P(3)) = \dim H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3) \otimes \mathcal{I}_P) = \binom{n+4}{3} - \binom{\ell+2}{3}.$$

Proof. We have an exact sequence

$$0 \to \mathcal{I}_P(3) \to \mathcal{O}_{\mathbb{P}}(3) \to \mathcal{O}_P(3) \to 0.$$

Therefore, we obtain an exact sequence

$$0 \to H^0(\mathbb{P}, \mathcal{I}_P(3)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)) \to H^0(P, \mathcal{O}_P(3)) \to H^1(\mathbb{P}, \mathcal{I}_P(3)) \to 0.$$

Observe that

$$H^1(\mathbb{P}, \mathcal{I}_P(3)) = 0.$$

Therefore, $h^0(\mathbb{P}, \mathcal{I}_P(3)) = h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)) - h^0(P, \mathcal{O}_P(3))$ and, hence, we have

$$h^0(\mathbb{P}, \mathcal{I}_P(3)) = h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)) - h^0(P, \mathcal{O}_P(3)) = \binom{n+4}{3} - \binom{\ell+2}{3}.$$

The lemma follows. $\square$

Corollary 6.1.9. *Let M_n be the coarse moduli space of smooth cubic hypersurfaces of dimension n. Let $P \subset \mathbb{P}^{n+1}$ be a linear subspace of dimension $\ell - 1 > 0$. Then, we have that* $\dim |\mathcal{O}_{\mathbb{P}}(3) \otimes \mathcal{I}_P| < \dim M_n$ *if and only if* $(n+2)^2 < \binom{\ell+2}{3} + 1$. $\square$

Corollary 6.1.10. *Suppose that n, ℓ and d are positive integers with $\ell > 1$. Assume*

$$(n+2)^2 < \binom{\ell+2}{3} + 1.$$

Then the generic smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}_k$ does not contain a linear subspace of dimension $k - 1$.

Proof. Look at the composition $\left|\mathcal{O}_{\mathbb{P}_{\bar{k}}}(3) \otimes \mathcal{I}_P\right| \to \left|\mathcal{O}_{\mathbb{P}_{\bar{k}}}(3)\right| \to M_n(\bar{k})$. $\square$

Remark 6.1.11. We know that $\mathrm{PGL}_{n+2}(k)$ acts on $H^0(\mathbb{P}^{n+1}_k, \mathcal{O}(1))$. The latter can be identified with the space $(\mathbb{P}^{n+1}(k))^\vee$ of hypersurfaces in $\mathbb{P}^{n+1}_k$. Hence $\mathrm{PGL}_{n+2}(k)$ acts naturally on $(\mathbb{P}^{n+1}(k))^\vee$. One may wonder how this action is defined, explicitly.

Any point $a = [a_0 : \cdots : a_{n+1}]$ defines a hyperplane $H(a) \coloneqq \{\sum_i a_i x_i = 0\} \subset \mathbb{P}^{n+1}$. The association $a \mapsto H(a)$ defines an isomorphism $\mathbb{P}^{n+1} \to (\mathbb{P}^{n+1})^\vee$, where the latter denotes the space of hyperplanes in $\mathbb{P}^{n+1}$. This isomorphism is PGL_{n+2}-equivariant, if we let PGL_{n+2} act on $(\mathbb{P}^{n+1})^\vee$ by sending a linear polynomial $F = F(x_0, \ldots, x_{n+1})$ to $g \cdot F = F(g^{-T}(x_0, \ldots, x_{n+1}))$, for $g \in \mathrm{PGL}_{n+2}(k)$.

Proposition 6.1.12. *Let $\ell \geq 2$ be an integer. Let $n \in \mathbb{Z}_{\geq 0}$. Consider the following assertions $A_1(\ell)$ and $A_2(\ell)$, that depend on ℓ:*

$A_1(\ell)$. *We have the inequality*

$$k(n+2-k) < \binom{k+2}{3}. \tag{6.3}$$

$A_2(\ell)$. *The generic cubic threefold $X \subset \mathbb{P}^{n+1}$ does not contain a linear subspace $P \subset \mathbb{P}^{n+1}$ of dimension $\ell - 1$.*

Then $A_1(\ell) \implies A_2(\ell)$.

Proof. Let $P \subset \mathbb{P}^{n+1}$ be a linear subspace of dimension $\ell - 1$. Assume that $P = \{x_k = \cdots = x_{n+1} = 0\}$. Define

$$G(\bar{k}) \subset \mathrm{GL}_{n+2}(V)$$

as the subgroup of matrices of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \qquad \text{with} \qquad A, B \in \mathrm{GL}_k(\bar{k}), \quad C \in \mathrm{M}_c(\bar{k}), \quad k + c = n + 2.$$

Let $G(\bar{k})$ act on $\mathbb{P}^{n+1}(\bar{k})$ via the embedding $G(\bar{k}) \subset \mathrm{GL}_{n+2}(\bar{k})$ and the action of $\mathrm{GL}_{n+2}(\bar{k})$ on $\mathbb{P}^{n+1}(\bar{k})$. Notice that $G(\bar{k})$ arises as the group of $\bar{k}$-points of a closed subgroup scheme $G \subset \mathrm{GL}_{n+2}$ over k, and that the above action extends to an action of G on $\mathbb{P}^{n+2}$ over k.

Observe that with respect to this action, G stabilizes the subspace $P \subset \mathbb{P}^{n+1}$. In particular, with respect to the induced action on $H^0(\mathbb{P}^{n+1}, \mathcal{O}(3))$, G stabilizes the subspace (6.2) of global sections of $\mathcal{I}_P(3)$. Consider the natural morphism

$$\begin{aligned} \varphi\colon M_n^P(\bar{k}) &:= G(\bar{k}) \setminus \big(H^0(\mathbb{P}^{n+1}_{\bar{k}}, \mathcal{I}_{P_{\bar{k}}}(3)) - \{0\}\big) \\ &\to \mathrm{GL}_{n+2}(\bar{k}) \setminus \big(H^0(\mathbb{P}^{n+1}_{\bar{k}}, \mathcal{O}(3)) - \{0\}\big) = M_n(\bar{k}), \end{aligned}$$

where M_n denotes the coarse moduli space of all cubic hypersurfaces of dimension n. Observe that if φ is not surjective, then assertion $A_2(\ell)$ holds. Thus, it suffices to show that $A_1(\ell)$ implies that φ is not surjective.

We calculate the dimension of G. It is given by the dimension of the spaces of matrices $A \in \mathrm{GL}_{n+2}(\bar{k})$ minus the dimension of the space of matrices $C \in \mathrm{M}_{(n+2-\ell)\times\ell}(\bar{k})$. As such, it equals:

$$\dim(G) = (n+2)^2 - \ell(n+2-\ell).$$

Therefore

$$\begin{aligned} \dim M_n^P(\bar{k}) &< \dim M_n(\bar{k}) &&\iff \\ \binom{n+4}{3} - \binom{\ell+2}{3} - (n+2)^2 + \ell(n+2-\ell) &< \binom{n+4}{3} - (n+2)^2 &&\iff \\ \ell(n+2-\ell) &< \binom{\ell+2}{3}. \end{aligned}$$

The proposition follows. ☐

Corollary 6.1.13. *The generic cubic fourfold does not contain a plane.*

Proof. Then

$$3(n-1) < \binom{5}{3} = 10 \qquad \Longleftrightarrow \qquad n < 5.$$

In view of Proposition 6.1.12 applied to the case $\ell = 3$, the corollary follows. $\square$

Remark 6.1.14. Recall that, if $\mathrm{char}(k) \neq 3$, then a smooth cubic hypersurface of dimension n does not contain a linear subspace of dimension $> n/2$, see Proposition 6.1.2. In particular, if $\mathrm{char}(k) \neq 3$, then no smooth cubic threefold contains a plane.

Proposition 6.1.15. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a cubic hypersurface over $\mathbb{C}$. Assume that $n = \dim(X) \geq 12$. Then X contains a plane.*

Proof. Let $F \in \mathbb{C}[T_0, \dots, T_{n+1}]$ be homogeneous polynomial of degree three. Given three vectors $x, y, z \in \mathbb{C}^{n+2}$, consider the condition

$$(\star) \qquad\qquad F(\lambda x + \mu y + \nu z) = 0 \text{ in } \mathbb{C}[\lambda, \mu, \nu].$$

We prove the proposition via the following steps.

(1) The dimension of $(\mathbb{C}^{n+2})^{\oplus 3}$ equals $3n + 6$.

(2) Condition $(\star)$ yields ten equations on the coordinates of $x, y, z \in \mathbb{C}$.

(3) Let $V(\star) \subset (\mathbb{C}^{n+2})^{\oplus 3}$ be the space of triples x, y, z such that $(\star)$ holds. By (1) and (2), the dimension of $V(\star)$ is at least $3n + 6 - 10 = 3n - 4$.

(4) The dimension of the space of all triples such that x, y, z are linearly dependent is $2(n + 2) + 2 = 2n + 6$.

(5) Hence, if $2n + 6 < 3n - 4$, then $2n + 6 < \dim(V(\star))$ by (3). If that is the case, there must be a linearly independent triple $(x, y, z) \in V(\star)$.

(6) Remark that $2n + 6 < 3n - 4$ if and only if $n \geq 11$. We conclude that if $n \geq 11$, then there is a linear subspace

$$P \subset X = \{F = 0\} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$$

of dimension two.

This proves the proposition. $\square$

6.2 Lecture 16: Quadric fibrations of cubic hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Suppose that X contains a linear subspace $P \subset \mathbb{P}^{n+1}$ with $\dim(P) = \ell - 1$. Let $y \in \mathbb{P}^{n+1} - P$, and let $\overline{yP} \subset \mathbb{P}^{n+1}$ be the linear subspace spanned by y and P. Then $\overline{yP} \cong \mathbb{P}^{\ell}$. Hence $X \cap \overline{yP}$ is an $\ell - 1$-dimensional subspace of X, containing P, so that

$$X \cap \overline{yP} = P \cup Q \qquad \text{for some subscheme} \qquad Q \subset \mathbb{P}^{n+1}.$$

We call Q the *residue* of the subscheme $P \subset X \cap \overline{yP}$. Let us calculate the degree of the closed subscheme $Q \subset \mathbb{P}^{n+1}$. For a hyperplane $H \subset \mathbb{P}^{n+1}$, we have:

$$\deg\left(X \cap \overline{yP}\right) = \deg(P \cup Q) = P \cdot H^{n+1-(\ell-1)} + Q \cdot H^{n+1-(\ell-1)} = 1 + \deg(Q).$$

Notice that $\overline{yP} \equiv H^{n+1-\ell}$ (numerical equivalence), so that

$$\deg\left(X \cap \overline{yP}\right) = \deg\left(3H \cdot H^{n+1-\ell}\right) = 3.$$

We conclude that $\deg(Q) = 2$, i.e., the subscheme $Q \subset \overline{yP} \cong \mathbb{P}^\ell$ is a quadric.

Example 6.2.1. Let $X \subset \mathbb{P}^3$ be a cubic surface and $P = \{p\} \subset X$ for some $p \in X$. Let $y \neq p \in \mathbb{P}^3$. The line $\overline{yp} \subset \mathbb{P}^3$ intersects X in three points (counted with multiplicity). Hence $\overline{yp} = \{p\} \cup Q$ with $Q = \{q_1, q_2\}$ or $Q = 2\{q\}$, as a divisor $Q \subset \overline{yp} \cong \mathbb{P}^1$. In any case, $X \cap \overline{yp}$ is given by a homogeneous cubic polynomial in two variables; splitting off the linear factor that defines $p \in X$ gives a quadratic polynomial that defines Q.

We will consider the following result, without providing a proof.

Theorem 6.2.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Assume that there is a linear subspace $P \subset \mathbb{P}^{n+1}$ with $P \subset X$, such that $\dim(P) = \ell - 1$ and there is no linear subspace $P' \subset \mathbb{P}^{n+1}$ of dimension ℓ such that $P \subset P' \subset X$.*

Let $W \subset V$ be a subspace of dimension ℓ so that $P = \mathbb{P}(W) \subset \mathbb{P}(V) = \mathbb{P}^{n+1}$. Choose a linear subspace $\mathbb{P}(U) \subset \mathbb{P}(V)$ of codimension ℓ that does not intersect $\mathbb{P}(W)$. Then the linear projection from P defines a diagram

$$\begin{tikzcd}
\mathrm{Bl}_P(\mathbb{P}^{n+1}) \arrow[dr, "\phi"] & \\
\mathrm{Bl}_P(X) \arrow[u] \arrow[r, "\psi"] & \mathbb{P}(V/W) \cong \mathbb{P}^{n+1-\ell}
\end{tikzcd}$$

with the following properties:

(1) With respect to the isomorphisms $\mathbb{P}(U) \cong \mathbb{P}(V/W) \cong \mathbb{P}^{n+1-\ell}$, the fibre

$$\psi^{-1}(y) \subset \mathrm{Bl}_P(X), \qquad y \in \mathbb{P}^{n+1-\ell}$$

is the quadric $Q_y \subset \overline{yP} = \phi^{-1}(y)$ which is the residual quadric of $P \subset \overline{yP} \cap X$.

(2) There exists a divisor $D_P \in |\mathcal{O}_{\mathbb{P}^{n+1-\ell}}(\ell+3)|$ such that the fibres of ψ are singular exactly over $D_P \subset \mathbb{P}^{n+1-\ell}$.

(3) The morphism $\psi\colon \mathrm{Bl}_P(X) \to \mathbb{P}^{n+1-\ell}$ is flat.

Proof. See [Huy23, Chapter 1, Proposition 5.3]. □

We discuss some applications, and some questions related to rationality and unirationality of cubic hypersurfaces.

Corollary 6.2.3. *Assume $X \subset \mathbb{P}^{n+1}$ is a smooth cubic hypersurface of even dimension containing a linear subspace $P \cong \mathbb{P}^{n/2} \subset \mathbb{P}^{n+1}$. The linear projection from P defines a quadric fibration $\mathrm{Bl}_P(X) \to \mathbb{P}^{n/2}$ with discriminant divisor $D_P \in |\mathcal{O}((n/2)+3)|$.*

Proof. See Theorem 6.2.2 and Proposition 6.1.2. □

Corollary 6.2.4. *Assume that a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ of even dimension $n = 2m$ contains two linear subspaces $\mathbb{P}(W) \subset X$ and $\mathbb{P}(W') \subset X$, both of dimension m, such that $\mathbb{P}(W) \cap \mathbb{P}(W') = \emptyset$. Then:*

(1) the quadric fibration $\psi\colon \mathrm{Bl}_P(X) \to \mathbb{P}(W')$ of Theorem 6.2.2 admits a section;

(2) X is rational.

Proof. The existence of the section is clear. As any quadric admitting a rational point is rational, the scheme-theoretic generic fibre $\phi^{-1}(\eta)$ is a rational quadric over $K(W') \cong K(\mathbb{P}^{n+1})$. Hence, $\mathrm{Bl}_P(X)$ is rational and, therefore, X itself is.

Alternatively, to prove that X is rational, one may proceed as follows. Consider the rational map

$$f\colon P(W) \times P(W') \dashrightarrow X, \qquad (x, x') \mapsto p \in X\colon \overline{xx'} \cap X = \{x, x', p\}\,. \tag{6.4}$$

The map f is well-defined for all (x, x') for which the line $\overline{xx'}$ is not contained in X, hence on a non-open subset of $P(W) \times P(W')$.

Let $y \in X \setminus (P \cup P')$. Then there exists a unique $x \in P$ and a unique $x' \in P'$ such that $\overline{xx'} \cap X = \{x, x', y\}$. Hence f is generically injective on $\bar{k}$-points, so that, if $\mathrm{char}(k) = 0$, then f is birational. Indeed, this follows from Corollary 6.2.6 below.

With no assumptions on k, (6.4) has a rational inverse $X \dashrightarrow P(W) \times P(W')$. □

Lemma 6.2.5. *Let k be a field and let V and W be integral k-varieties. Suppose that $V \to W$ is a dominant morphism such that $k(W) \to k(V)$ is a separable field extension. Then there is a dense open subscheme $U \subset V$ such that $U \to W$ is smooth.*

Proof. If $k = \bar{k}$ and $\mathrm{char}(k) = 0$, apply [Har77, Chapter III, Section 10, Corollary 10.7] to the composition $\widetilde{X}_1 \to X_1 \to X_2$, where $\widetilde{X}_1 \to X_1$ is a resolution of X_1.

For arbitrary k, see [EGAIV, Quatrième partie, Proposition 17.7.8(ii)]. Alternatively, one may apply [Stacks, Tag 07ND], see also [Stacks, Tag 00TB]. □

Corollary 6.2.6. *Let $f\colon X_1 \to X_2$ be a dominant map of integral varieties over a field k of characteristic zero, generically injective on $\bar{k}$-points. Then f is birational.*

Proof. We have $\dim(X_1) = \dim(X_2)$. There are non-empty affine opens $U \subset X_1$ and $V \subset X_2$ such that $U \to V$ is finite, bijective, and smooth, see [Stacks, Tag 02NW] and Lemma 6.2.5. Thus, f is finite locally free of rank one, see [Stacks, Tag 02KB]. This means that f is an isomorphism. □

Examples 6.2.7. (1) Let

$$X = V(x_0^3 + \cdots + x_5^3) \subset \mathbb{P}^5.$$

Let $P_1 = V(x_0 - \zeta x_1 x_2 - \zeta^2 x_3, x_4 - \zeta^3 x_5)$ and $P_2 = V(x_0 - \zeta^2 x_1 x_2 - \zeta^3 x_3, x_4 - \zeta x_5)$, where ζ is a primitive third root of unity.

(2) Let

$$X = V(F) \subset \mathbb{P}^5, \quad F = x_0^2 x_1 - x_0 x_1^2 + x_2^2 x_3 - x_2 x_3^2 + x_4^2 x_5 - x_4 x_5^2.$$

Then X contains $P = V(x_0, x_2, x_4)$ and $P' = V(x_1, x_3, x_5)$.

Proposition 6.2.8. *Smooth cubic hypersurfaces of dimension $n \geq 2$ are unirational.*

Proof. Let $n \geq 2$ and let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Let $Y_i \subset X$ be generic hyperplane sections for $i = 1, 2$. Consider the map

$$Y_1 \times Y_2 \dashrightarrow X, \quad (y_1, y_2) \mapsto x \in X : \overline{y_1 y_2} \cap X = \{x, y_1, y_2\}.$$

This map is dominant, and has a rational inverse. Thus, by induction, we are reduced to the case $n = \dim(X) = 2$. In this case, it suffices to show that the cubic surface X contains two disjoint lines (see Corollary 6.2.4), which we will prove later [insert future reference]. □

Chapter 7

Deformation theory

7.1 Lecture 17: Deformation theory over the dual numbers

Throughout this section, k is a field, t is a variable, and $D = k[t]/(t^2)$. A good reference for the material in this section is [Har10, Chapter 1].

7.1.1 *Infinitesimal deformations of a subscheme*

Lemma 7.1.1. *Let M be a module over a noetherian ring. Then M is flat over A if and only if for every prime ideal $\mathfrak{p} \subset A$, we have $\operatorname{Tor}_1^A(M, A/\mathfrak{p}) = 0$.*

Proof. By definition, M is flat over A if and only if $\operatorname{Tor}_1^A(M, -) = 0$ as a functor on the category of A-modules. Now Tor commutes with direct limits, and every A-module is a direct limit of finitely generated A-modules. If N is finitely generated, then N has a filtration with quotients of the form $A/\mathfrak{p}_i$ for various prime ideals $\mathfrak{p}_i \subset A$. □

Lemma 7.1.2. *Let A' be a noetherian ring and $J \subset A'$ an ideal of square zero. Define $A = A'/J$. Then an A'-module M' is flat over A' if and only if*

(1) $M = M' \otimes_{A'} A$ is flat over A, and

(2) the natural map $M \otimes_A J \to M'$ is injective.

Proof. Consider the diagram of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] \arrow[d, equal] & \mathfrak{p}' \arrow[r] \arrow[d] & \mathfrak{p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & A'/\mathfrak{p}' \arrow[r, equal] \arrow[d] & A/\mathfrak{p} \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Tensoring it with M', yields:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}) \arrow[r] \arrow[d] & \operatorname{Tor}_1^{A}(M, A/\mathfrak{p}) \arrow[d] & \\
M \otimes_A J \arrow[r] \arrow[d, equal] & M' \otimes_{A'} \mathfrak{p}' \arrow[r] \arrow[d] & M \otimes_A \mathfrak{p} \arrow[r] \arrow[d] & 0 \\
M \otimes_A J \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & M' \otimes_{A'} A'/\mathfrak{p}' \arrow[d] & M \otimes_A A/\mathfrak{p} \arrow[d] & \\
 & 0 & 0. &
\end{tikzcd}$$

By hypothesis (2), the second horizontal sequence is exact, and hence also the first. Thus, the Tors on top are isomorphic. The second Tor is zero by hypothesis (1), and we are done by Lemma 7.1.1. □

Definition 7.1.3. Let $D = k[t]/(t^2)$ for some field k. Let X be a scheme over k and let $Y \subset X$ be a closed subscheme. We define a *deformation of Y over D in X* to be a closed subscheme

$$Y' \subset X' := X \times_k D,$$

flat over D, such that $Y' \times_D k = Y$ as subschemes of X.

Proposition 7.1.4. *Let $X = \operatorname{Spec}(B)$ be an affine scheme over k. Let $Y = \operatorname{Spec}(B/I) \subset X$ for some ideal $I \subset B$. Then the following sets are in bijection:*

(1) The set of deformations of Y over D in X.

(2) The set of ideals $I' \subset B' := B[t]/(t^2)$ with B'/I' flat over D and such that the image of I' in $B = B'/tB'$ is $I \subset B$.

(3) The set $\operatorname{Hom}_B(I, B/I)$.

Proof. The bijection between (1) and (2) is clear. Let $I' \subset B'$ be an ideal with image $I \subset B$. Then the flatness of B'/I' over D is equivalent to the exactness of the sequence

$$0 \to B/I \xrightarrow{t} B'/I' \to B/I \to 0, \tag{7.1}$$

which is in turn equivalent to the exactness of the sequence

$$0 \to I \xrightarrow{t} I' \to I \to 0.$$

Note that, as B-modules, we have $B' = B \oplus tB$. Take an element $x \in I$ and lift it to

$$x + ty \in I' \subset B \otimes tB.$$

Then the image $\bar{y} \in B/I$ of the element $y \in B$ does not depend on the choice of lift of x, and we get a morphism $\varphi\colon I \to B/I$ that sends x to $\bar{y}$.

Conversely, given $\varphi \in \mathrm{Hom}_B(I, B/I)$, consider the map $\pi\colon B \to B/I$ and define

$$I' = \{x + ty \mid x \in I, y \in B, \pi(y) = \varphi(x) \in B/I\}.$$

In other words, I' is the fibre product of I and B over B/I (via the morphisms π and φ). Then I' is an ideal of $B' = B \otimes tB$, and there is an exact sequence

$$0 \to I \xrightarrow{t} I' \to I \to 0.$$

Thus, (7.1) is exact, hence we get an element of the set (2). These constructions are inverse and define the bijection between the sets (2) and (3). $\square$

Corollary 7.1.5. *Let X be a scheme over a field k and let Y be a closed subscheme of X. Then, the deformations of Y over D in X are in natural bijection with the elements of $H^0(Y, \mathcal{N}_{Y/X})$.*

Proof. Let $Y \subset X$ be a closed subscheme of a scheme X over k. By Lemma 7.1.4, we have a natural bijection between the set of deformations $Y' \subset X'$ of $Y \subset X$ and the set $\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Y)$. Now note that

$$\mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_Y) = H^0(X, Hom_X(\mathcal{I}, \mathcal{O}_Y)) = H^0(Y, Hom_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)) = H^0(Y, \mathcal{N}_{Y/X}).$$

The lemma follows. $\square$

7.1.2 Infinitesimal deformations of coherent sheaves

Proposition 7.1.6. *Let X be a scheme over k and let $\mathcal{L}$ be a line bundle on X. Then there is a natural bijection between $H^1(X, \mathcal{O}_X)$ and the set of isomorphism classes of line bundles $\mathcal{L}'$ on $X \times_k D =: X'$ such that $\mathcal{L}' \otimes_{\mathcal{O}'_X} \mathcal{O}_X \cong \mathcal{L}$.*

Proof. The exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

gives rise to an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}^*_{X'} \to \mathcal{O}^*_X \to 0, \qquad \alpha(x) = 1 + tx.$$

The exact sequence $0 \to k \to D \to k \to 0$ splits, hence the above exact sequences do so, too. Thus, taking cohomology yields an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, equal] & H^1(X', \mathcal{O}^*_{X'}) \arrow[r] \arrow[d, equal] & H^1(X, \mathcal{O}^*_X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & \mathrm{Pic}(X') \arrow[r] & \mathrm{Pic}(X) \arrow[r] & 0.
\end{tikzcd}$$

The proposition follows. $\square$

Definition 7.1.7. Let X be a scheme over k and let $\mathcal{F}$ be a coherent sheaf on X. We define a *deformation of $\mathcal{F}$ over D* to be a coherent sheaf $\mathcal{F}'$ on $X' := X \times_k D$, flat over D, together with a homomorphism

$$\mathcal{F}' \to \mathcal{F}$$

such that the induced map $\mathcal{F}' \otimes_D k \to \mathcal{F}$ is an isomorphism. We say that two such deformations $\mathcal{F}'_1 \to \mathcal{F}$ and $\mathcal{F}'_2 \to \mathcal{F}$ are *equivalent* if there is an isomorphism $\mathcal{F}'_1 \cong \mathcal{F}'_2$ compatible with the given maps to $\mathcal{F}$.

Proposition 7.1.8. *Let X be a scheme over k and $\mathcal{F}$ a coherent sheaf on X. Equivalence classes of deformations of $\mathcal{F}$ over D are naturally parametrized by $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$.*

Proof. Let $\mathcal{F}'$ be a coherent sheaf on X' together with a morphism of $\mathcal{O}_X$-modules $\mathcal{F}' \to \mathcal{F}$. By Lemma 7.1.2, the flatness of $\mathcal{F}'$ over D is equivalent to the exactness of the sequence

$$0 \to \mathcal{F} \xrightarrow{t} \mathcal{F}' \to \mathcal{F} \to 0$$

obtained by tensoring $\mathcal{F}'$ with $0 \to k \to D \to k \to 0$. The latter sequence splits, so that we have a splitting $\mathcal{O}_X \to \mathcal{O}_{X'}$, which allows us to view the above sequence as a sequence of $\mathcal{O}_X$-modules. This yields an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$. Conversely, given an extension $\mathcal{F}'$ of $\mathcal{F}$ by $\mathcal{F}$ as $\mathcal{O}_X$-modules, we can give $\mathcal{F}'$ a D-module structure by specifying how $t \in D$ acts; we let t act on $\mathcal{F}'$ via the endomorphism $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'$. □

Corollary 7.1.9. *If $\mathcal{E}$ is a vector bundle on a scheme X over k, then deformations of $\mathcal{E}$ over D are in natural bijection with elements of $H^1(X, \mathscr{E}nd(\mathcal{E}))$.*

Proof. See Exercise 7.2.9. □

Example 7.1.10. Consider the vector bundle

$$\mathcal{E}_0 := \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

on $\mathbb{P}^1$. Then the following assertions hold:

(1) We have $\operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E}_0) = H^1(\mathbb{P}^1, \mathscr{E}nd(\mathcal{E}_0))$.

(2) We have $\dim H^1(\mathbb{P}, \mathscr{E}nd(\mathcal{E}_0)) = 1$.

(3) We have $\operatorname{Ext}^1(\mathcal{E}_0, \mathcal{E}_0) = \operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$.

(4) We have $\operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}(-2))$.

(5) For $t \in \operatorname{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$, let $\mathcal{E}_t$ be the associated extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E}_t \to \mathcal{O}(1) \to 0.$$

Then for $t \neq 0$, we have $\mathcal{E}_t \cong \mathcal{O} \oplus \mathcal{O}$, whereas for $t = 0$ we get $\mathcal{E}_0$.

Proof. Item (1) follows from Corollary 7.1.9. Item (2) follows from the fact that

$$\mathscr{E}nd(\mathcal{E}_0) = \mathscr{E}_0^\vee \otimes \mathscr{E}_0 \cong \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2) \oplus \mathcal{O}(-2).$$

Item (3) holds because

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_0) =& \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(1)) \oplus \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \\ & \oplus \mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}(1)) \oplus \mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-1)) \\ =& \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)).\end{aligned}$$

The latter equality follows from the vanishings

$$\mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}(b)) = H^1(\mathbb{P}^1, \mathcal{O}(b-a)) = 0 \qquad \forall a, b \in \mathbb{Z} \mid b - a \geq -1.$$

Item (4) is clear.

It remains to prove item (5). For $t = 0 \in \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$, the associated extension $\mathcal{E}_t$ is the trivial extension $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Let $t \neq 0 \in \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$. As

$$\dim \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = \dim H^1(\mathbb{P}^1, \mathscr{E}nd(\mathcal{E}_0)) = 1$$

by what has already been proved, it suffices to show that there exists a vector bundle $\mathcal{F}$ that fits into a non-trivial extension of the form

$$0 \to \mathcal{O}(-1) \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{O}(1) \to 0, \quad \text{such that} \quad \mathcal{F} \cong \mathcal{O} \oplus \mathcal{O}.$$

Indeed, one obtains a one-dimensional family of such extensions, parametrized by $t \in k^*$, by considering

$$0 \to \mathcal{O}(-1) \xrightarrow{\lambda \cdot f} \mathcal{F} \xrightarrow{\lambda \cdot g} \mathcal{O}(1) \to 0 \quad \text{for} \quad \lambda \in k^*;$$

moreover, the sheaves $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ are clearly non-isomorphic.

To obtain such $\mathcal{F}$, we consider the twisted Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2} \to \mathcal{T}_{\mathbb{P}^1}(-1) \to 0, \quad \text{where} \quad \mathcal{T}_{\mathbb{P}^1}(-1) \cong \mathcal{O}(2-1) = \mathcal{O}(1).$$

This finishes the proof, and concludes the example. □

7.1.3 Infinitesimal deformations of smooth affine schemes

For the following theorem, we need:

Definition 7.1.11. Let $f \colon X \to S$ be a morphism of schemes. Then f is called *formally smooth* if for any ring R with an ideal $I \subset R$ such that $I^2 = 0$, any morphism Spec $(R) \to S$ and any morphism Spec $(R/I) \to X$ so that the composition

$$\mathrm{Spec}\ (R/I) \to X \to S$$

agrees with the composition

$$\mathrm{Spec}\ (R/I) \to \mathrm{Spec}\ (R) \to S,$$

there exists a morphism Spec $(R) \to X$ compatible with all the other given maps.

In other words, for any pair (R, I) as above, and any solid commutative diagram

$$\begin{tikzcd} & & X \arrow[d] \\ \mathrm{Spec}\ (R/I) \arrow[r] \arrow[urr, bend left] & \mathrm{Spec}\ (R) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{7.2}$$

there exists an arrow Spec $(R) \to X$ that makes diagram (7.2) commute.

Theorem 7.1.12 (Grothendieck)**.** *Let $f\colon X \to S$ be a morphism of schemes. The following are equivalent:*

(1) The morphism f is smooth.

(2) The morphism f is locally of finite presentation and formally smooth.

Proof. See [EGAIV]. □

Definition 7.1.13. (1) Let B be a k-algebra. Then a *deformation of B over D* is a flat D-algebra B' together with a morphism of D-algebras $B' \to B$ (where B is a D-algebra via the map $D \to k$) such that the induced morphism of k-algebras

$$B' \otimes_D k \to B$$

is an isomorphism. Two such $B'_i \to B$ $(i = 1, 2)$ are *equivalent* if there exists an isomorphism $B'_1 \to B'_2$ compatible with the given maps to B.

(2) Let A be an artinian ring over k, and let X be a scheme over k. Define a *deformation of X over A* to be a flat A-scheme X' together with a closed immersion $i\colon X \hookrightarrow X'$ such that the induced map

$$i \times_A k\colon X \to X' \times_A k$$

is an isomorphism. Two such deformations are *equivalent* if there is an isomorphism $f\colon X'_1 \to X'_2$ over A compatible with i_1 and i_2, i.e., such that $i_2 = f \circ i_1$.

Corollary 7.1.14. *Let X be a smooth affine scheme over k. Let A be a local Artin k-algebra. Let X' be a scheme, flat over* Spec (A)*, with $X' \times_A k \cong X$. Then $X' \cong X \times_k A$. In other words, infinitesimal deformations of smooth affine schemes are trivial.*

Proof. Apply Theorem 7.1.12 to the diagram

$$\begin{tikzcd} & & X \arrow[d] \\ X \arrow[r, hook] \arrow[urr, bend left, "\mathrm{id}"] & X' \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\ (k) \end{tikzcd}$$

We obtain a morphism $\rho\colon X' \to X$ such that $\rho \circ \iota = \mathrm{id}$, where ι is the closed embedding $X \hookrightarrow X'$. Then if π is the morphism $X' \to D = \mathrm{Spec}\ k[t]/(t^2)$, we get a morphism $\rho \times_k \pi\colon X' \to X \times_k D$. This morphism is an isomorphism over the closed fibres over $0 \in D$, hence $\rho \times_k \pi$ is an isomorphism, see Lemma 7.1.16 below. □

Lemma 7.1.15. *Let A be an artinian local k-algebra with residue field k. Then the closed immersion ι: : $\mathrm{Spec}\ (k) \to \mathrm{Spec}\ (A)$ is a universal homeomorphism. In other words, for any A-scheme X, the map $X \times_A k \to X$ is a homeomorphism.*

Proof. The map ι is integral, universally injective, and surjective. Therefore, by [Stacks, Tag 04DF], the map ι is a universal homeomorphism. □

Lemma 7.1.16. *Let A be a local artinian k-algebra with residue field k. Let X_i ($i = 1, 2$) be flat, finite type A-schemes. Let $f\colon X_1 \to X_2$ be a morphism of A-schemes inducing an isomorphism*

$$f \otimes_A k\colon X_1 \times_A k \xrightarrow{\sim} X_2 \times_A k.$$

Then f is an isomorphism.

Proof. By Lemma 7.1.15, the map $f\colon X_1 \to X_2$ is a homeomorphism, since it becomes one after passing to the fibres over $\mathrm{Spec}\ (k) \hookrightarrow \mathrm{Spec}\ (A)$. Thus, it suffices to show that the map $f^{-1}\mathcal{O}_{X_2} \to \mathcal{O}_{X_1}$ is an isomorphism of sheaves. We may therefore assume that

$$X_i = \mathrm{Spec}\ (B_i)$$

is an affine scheme, with B_i a flat A-algebra ($i = 1, 2$). In this case, the exact sequence of A-modules

$$0 \to \mathrm{Ker}(f^\#) \to B_2 \xrightarrow{f^\#} B_1 \to \mathrm{Coker}(B_2 \xrightarrow{f^\#} B_1) \to 0 \tag{7.3}$$

yields an exact sequence of A-modules

$$0 \to K := \mathrm{Ker}(\pi) \to B_1 \xrightarrow{\pi} Q := \mathrm{Coker}(B_2 \xrightarrow{f^\#} B_1) \to 0.$$

Let $\mathfrak{m}$ be the maximal ideal of A. Tensoring with $A/\mathfrak{m}$ over A, using the flatness of B_1 over A, yields an exact sequence

$$\mathrm{Tor}_1^A(B_1, A/\mathfrak{m}) = 0 \to \mathrm{Tor}_1^A(Q, A/\mathfrak{m}) \to K \otimes_A A/\mathfrak{m} \to B_1 \otimes_A A/\mathfrak{m} \to Q \otimes_A A/\mathfrak{m} \to 0.$$

As $B_2 \otimes_A A/\mathfrak{m} \xrightarrow{\sim} B_1 \otimes_A A/\mathfrak{m}$, we get that $Q \otimes_A A/\mathfrak{m} = 0$, so that $Q = \mathfrak{m}Q$. By Nakayama's lemma, see Lemma 4.1.2, this gives

$$Q = 0.$$

In particular, (7.3) gives rise to an exact sequence

$$0 \to \mathrm{Ker}(f^\#) \to B_2 \xrightarrow{f^\#} B_1 \to 0,$$

and hence an exact sequence

$$\mathrm{Tor}_1^A(B_1, A/\mathfrak{m}) = 0 \to \mathrm{Ker}(f^\#) \otimes_A A/\mathfrak{m} \to B_2 \otimes_A A/\mathfrak{m} \to B_1 \otimes_A A/\mathfrak{m} \to 0.$$

As the map $B_2 \otimes_A A/\mathfrak{m} \to B_1 \otimes_A A/\mathfrak{m}$ is an isomorphism, we get

$$\mathrm{Ker}(f^\#) \otimes_A A/\mathfrak{m} = 0,$$

and hence $\mathrm{Ker}(f^\#) = 0$ by Nakayama's lemma, see Lemma 4.1.2. □

7.2 Lecture 18: Deformations of cubic hypersurfaces

Lemma 7.2.1. *Let B be a k-algebra. To give a deformation of B over D is to give a k-algebra B' with a homomorphism of k-algebras $B' \to B$ fitting in an exact sequence of k-modules*

$$0 \to B \to B' \to B \to 0. \tag{7.4}$$

Proof. Compare Proposition 7.1.8. Given such an exact sequence (7.4), we let D act on B' via the endomorphism $t\colon B' \to B'$ which is the composition $B' \to B \to B'$. Then $B' \otimes_D k = B$. Moreover, $B = B' \otimes_D k$ is flat over k, and the exact sequence

$$0 \to k \xrightarrow{t} D = k[t]/(t^2) \to k \to 0$$

induces, via tensoring with B' over D, an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B' \otimes_D k \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d, equal] & B' \otimes_D k \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "t"] & B' \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

Hence, by Lemma 7.1.2, we see that B' is flat over D. Conversely, if B' is a flat D-algebra and $B' \to B$ a homomorphism of k-algebras such that $B' \otimes_D k \to B$ is an isomorphism, the map $B' \to B$ is surjective with kernel $t \cdot B \subset B'$. □

Remark 7.2.2. Let B be a finite k-algebra. Then equivalence classes of deformations of B over D *as a module* are parametrized by $\mathrm{Ext}^1_k(B, B) = 0$, see Proposition 7.1.8. In particular, any deformation B' over D of the k-vector space B is of the form $B' = B \otimes_k D = B[t]/(t^2)$. This is also reflected by the isomorphism of D-modules

$$B' = B' \otimes_D D = B' \otimes_D k \otimes_k D = (B' \otimes_D k) \otimes_k D = B \otimes_k D.$$

Thus, it is only the algebra structure of B that can deform non-trivially over D.

Proposition 7.2.3. *Let B be a k-algebra and let B' be a deformation of B over D (cf. Lemma 7.2.1). The automorphism group of the extension $0 \to B \to B' \to B \to 0$ is in canonical bijection with the tangent module $T_{B/k}(B) = \mathrm{Der}_k(B, B)$.*

Proof. Consider the exact sequence

$$0 \to B \xrightarrow{\nu} B' \xrightarrow{\pi} B \to 0. \tag{7.5}$$

Let $\mathrm{Aut}_{B-B-\mathrm{Ext}}(B')$ be the automorphism group of the extension (7.5). Let $\phi \in \mathrm{Aut}_{B-B-\mathrm{Ext}}(B')$,

$$\phi\colon B' \xrightarrow{\sim} B',$$

be an automorphism of the extension (7.5). Let $b \in B$ and let $b' \in B'$ such that $\pi(b') = b$. Then there exists a unique element $b \in B$ such that

$$\phi(b') = b' + \nu\left(d(b)\right) \in B'$$

Let $b'_2 = b' + \nu(a')$ be another lift of B to B'. Then

$$\phi(b'_2) = \phi(b' + \nu(a')) = \phi(b') + \nu(a') = b' + \nu(d(b)) + \nu(a') = b'_2 + \nu(d(b)).$$

Hence the element $d(b) \in B$ depends only on b, and not on the lift $b' \in B'$ of B. This gives a map $d\colon B \to B$.

We claimt that d is a k-derivation. Let $a, b \in B$. Then

$$ab + \nu d(ab) = \phi(ab) = \phi(a)\phi(b) = (a + \nu d(a)) \cdot (b + \nu d(b)) = ab + \nu(ad(b) + bd(a)).$$

Therefore,

$$d(ab) = ad(b) + bd(a) \qquad \forall a, b \in B.$$

Moreover, if $\lambda_i \in k, b_i \in B$ $(i = 1, 2)$, then

$$\begin{aligned}\lambda_1 b_1 + \nu\lambda_1 d(b_1) + \lambda_2 b_2 + \nu\lambda_2 d(b_2) &= \lambda_1\phi(b_1) + \lambda_2\phi(b_2)\\ &= \phi(\lambda_1 b_1 + \lambda_2 b_2) = \lambda_1 b_1 + \lambda_2 b_2 + \nu d(\lambda_1 b_1 + \lambda_2 b_2),\end{aligned}$$

which gives $d(\lambda_1 b_1 + \lambda_2 b_2) = \lambda_1 d(b_1) + \lambda_2 d(b_2)$. We conclude that, indeed, the map $d\colon B \to B$ is a k-derivation.

Conversely, given a k-derivation $d\colon B \to B$, the map $B' \to B'$ defined as

$$b' \mapsto b' + \nu(d(\pi(b')))$$

defines an automorphism $B' \cong B'$ as an extension of B by B. The two constructions are inverse, which proves that $\mathrm{Aut}_{B-B-\mathrm{Ext}}(B') = \mathrm{Der}_k(B, B)$ as desired. □

As a corollary, we obtain the following result.

Theorem 7.2.4. *Let X be a smooth scheme over a field k. Then deformations of X over the dual numbers $D = k[t]/(t^2)$ are in natural bijection with elements of the group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X = \mathcal{H}om_X(\Omega_X, \mathcal{O}_X)$ is the tangent sheaf of X.*

Proof. Let X' be a deformation of X. Let $\mathcal{U} = (U_i)$ be an affine open covering of X. Consider the deformation $U_i' = X' \times_X U_i$ of U_i over D, see Lemma 7.2.5 below. This deformation is trivial because U_i is affine and smooth over k, see Corollary 7.1.14. Choose an isomorphism $\varphi_i \colon U_i \times_k D \cong U_i'$ for each i. Then on U_{ij} we have an automorphism

$$\psi_{ij} = \varphi_j^{-1}\varphi_i \colon U_{ij} \times_k D \xrightarrow{\sim} U_{ij} \times_k D, \qquad \text{with inverse} \qquad \psi_{ij}^{-1} = \varphi_i^{-1}\varphi_j = \psi_{ji}.$$

Notice that on $U_{ijk} \times_k D$, we have:

$$\psi_{jk}\psi_{ij} = \varphi_k^{-1}\varphi_j\varphi_j^{-1}\varphi_i = \varphi_k^{-1}\varphi_i = \psi_{ik} = \psi_{ki}^{-1}.$$

In other words:

$$\psi_{ki} \circ \psi_{jk} \circ \psi_{ij} = \mathrm{id} \qquad \text{on} \qquad U_{ijk} \times_k D. \tag{7.6}$$

By Proposition 7.2.3, we have canonical isomorphisms

$$\mathrm{Aut}_{def}(U_{ij} \times_k D) = \mathrm{Der}_k(\mathcal{O}(U_{ij}), \mathcal{O}(U_{ij})) = H^0(U_{ij}, \mathcal{T}_X).$$

Via these identifications, the automorphism ψ_{ij} corresponds to an element

$$\theta_{ij} \in H^0(U_{ij}, \mathcal{T}_X).$$

Furthermore, in view of (7.6), we have

$$\theta_{ij} + \theta_{jk} + \theta_{ki} = 0 \in H^0(U_{ijk}, \mathcal{T}_X).$$

One verifies that the induced Čech cohomology element

$$\theta \in \check{H}^1(\mathcal{U}, \mathcal{T}_X)$$

does not depend on the chosen isomorphisms φ_i. Since $\mathcal{U}$ is an affine open covering and $\mathcal{T}_X$ is a coherent sheaf, we have $\check{H}^1(\mathcal{U}, \mathcal{T}_X) = H^1(X, \mathcal{T}_X)$. The construction $[X'] \mapsto \theta$ defines the desired bijection between the set of infinitesimal deformations of X and the set $H^1(X, \mathcal{T}_X)$; we leave the details to the reader. $\square$

Lemma 7.2.5. *Let X be a scheme over a field k. Let $i \colon U \hookrightarrow X$ be an open immersion of schemes. Note that the composition $X'|_U = X' \times_X U \to X' \to \mathrm{Spec}\ (D)$ is flat and that $X'|_U \times_D k = (X' \times_X U) \times_D k = (X' \times_X U) \times_{X'} X = U$. Thus $X'|_U$ induces a deformation of U. The morphism $H^1(X, \mathcal{T}_X) \to H^1(X, i_*\mathcal{T}_U) = H^1(U, \mathcal{T}_U)$ induced by the adjunction morphism $\mathcal{T}_X \to i_*i^*\mathcal{T}_X = i_*\mathcal{T}_U$ makes the following diagram commute:*

$$\begin{tikzcd} \mathrm{Def}_D(X) \arrow[r] \arrow[d, equal] & \mathrm{Def}_D(U) \arrow[d, equal] \\ H^1(X, \mathcal{T}_X) \arrow[r] & H^1(U, \mathcal{T}_U). \end{tikzcd}$$

Exercise 7.2.6. Use Theorem 7.1.12 to show that if

$$f\colon X \to Y$$

is a smooth morphism of varieties over k, then for each $p \in X(k)$, the induced morphism on Zariski tangent spaces $T_{X,x} \to T_{Y,f(p)}$ is surjective.

Let P be a smooth scheme over a field k. Let $X \subset P$ be a smooth closed subscheme. Let $\mathrm{Def}_D(X)$ be the set of isomorphism classes of deformations of X over the dual numbers D (see Definition 7.1.13) and let $\mathrm{Def}_D(X \subset P)$ be the set of deformations of the subscheme $X \subset P$ over the dual numbers (see Definition 7.1.3).

On the one hand, consider the map

$$\mathrm{Def}_D(X \subset P) \to \mathrm{Def}_D(X)$$

that sends an infinitesimal deformation of X as a subscheme of P to the underlying infinitesimal deformation of X as a scheme over k. On the other hand, consider the conormal exact sequence (see [Stacks, Tag 06AA]):

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_P|_X \to \Omega_X \to 0.$$

Its dual gives rise to the normal bundle sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_P|_X \to \mathcal{N}_{X/P} \to 0. \tag{7.7}$$

Taking global sections of (7.7) yields a morphism $\rho\colon H^0(X, \mathcal{N}_{X/P}) \to H^1(X, \mathcal{T}_X)$.

Lemma 7.2.7. *Consider the above notation. Then, with respect to the canonical isomorphisms* $\mathrm{Def}_D(X) = H^1(X, \mathcal{T}_X)$ *of Theorem 7.2.4 and* $\mathrm{Def}_D(X \subset P) = H^0(X, \mathcal{N}_{X/P})$ *of Corollary 7.1.5, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Def}_D(X \subset P) \arrow[r] \arrow[d] & \mathrm{Def}_D(X) \arrow[d] \\
H^0(X, \mathcal{N}_{X/P}) \arrow[r, "\rho"] & H^1(X, \mathcal{T}_X).
\end{tikzcd}$$

Proof. Let $s \in H^0(X, \mathcal{N}_{X/P})$ correspond to a deformation

$$X' \subset P \times_k D$$

of X in P over D. Let $\{U_i\}$ be an affine open cover of X so that $s|_{U_i}$ lifts to a section $t_i \in H^0(U_i, \mathcal{T}_P|_{U_i})$. Then on $U_{ij} = U_i \cap U_j$, we have

$$\theta_{ij} := t_i|_{U_{ij}} - t_j|_{U_{ij}} \in H^0(U_{ij}, \mathcal{T}_X).$$

As $\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$, we obtain a Čech cohomology element

$$\theta = \rho(s) \in \check{H}^1(\mathcal{U}, \mathcal{T}_X) = H^1(X, \mathcal{T}_X).$$

[Finish proof.] The strategy of the remaining part of the proof is as follows. First, consider the isomorphism $\mathcal{N}_{X/P} \overset{\text{def}}{=} (\mathcal{I}/\mathcal{I}^2)^\vee = \mathrm{Coker}(\mathcal{T}_X \to \mathcal{T}_P|_X)$. Then, consider the isomorphism $H^0(X, \mathcal{N}_{X/P}) = \mathrm{Hom}_{\mathcal{O}_P}(\mathcal{I}, \mathcal{O}_X)$, see the proof of Corollary 7.1.5. Via the latter isomorphism, the section $s \in \mathcal{N}_{X/P}$ corresponds to a morphism of $\mathcal{O}_P$-modules $\varphi_s \colon \mathcal{I} \to \mathcal{O}_P/\mathcal{I} = \mathcal{O}_X$. Define $P' = P \times_k D$ and consider the ideal sheaf $\mathcal{I}' \subset \mathcal{O}_{P'}$ defined as $\mathcal{I}' = \{x + ty \mid x \in \mathcal{I}, y \in \mathcal{O}_P \mid \bar{y} = \varphi_s(x) \in \mathcal{O}_P/\mathcal{I}\}$. Then $X' = V(\mathcal{I}') \subset P'$, see the proof of Proposition 7.1.4. □

Theorem 7.2.8. *Any infinitesimal deformation of a smooth cubic hypersurface as a variety over k is again a smooth cubic hypersurface.*

Proof. Let $X \subset P := \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. The infinitesimal deformations of X as a scheme are parametrized by $H^1(X, \mathcal{T}_X)$. The infinitesimal deformations of X as a subscheme $X \subset \mathbb{P}^{n+1}$ are parametrized by $H^0(X, \mathcal{N}_{X/\mathbb{P}^{n+1}})$. Consider the normal bundle sequence $0 \to \mathcal{T}_X \to \mathcal{T}_P|_X \to \mathcal{N}_{X/P} \to 0$. Applying cohomology yields a morphism

$$\rho \colon H^0(X, \mathcal{N}_{X/P}) \to H^1(X, \mathcal{T}_X)$$

whose cokernel is contained in $H^1(X, \mathcal{T}_P|_X)$. By Lemma 7.2.7, it suffices to show that ρ is surjective, hence that $H^1(X, \mathcal{T}_P|_X) = 0$.

Consider the exact sequence

$$(0 \to \mathcal{O}_P(-3) \to \mathcal{O}_P \to \mathcal{O}_X \to 0) \otimes_{\mathcal{O}_P} \mathcal{T}_P = (0 \to \mathcal{T}_P(-3) \to \mathcal{T}_P \to \mathcal{T}_P|_X \to 0).$$

It yields an exact sequence

$$H^1(P, \mathcal{T}_P) \to H^1(X, \mathcal{T}_P|_X) \to H^2(P, \mathcal{T}_P(-3)). \tag{7.8}$$

Assume first that $n \geq 2$. Then by Serre duality (see Theorem 2.1.6) and Bott vanishing (see Theorem 2.1.8), we have

$$H^1(\mathbb{P}, \mathcal{T}_\mathbb{P}) = H^n(\mathbb{P}, \Omega_\mathbb{P}(-n-2))^\vee = 0,$$
$$H^2(\mathbb{P}, \mathcal{T}_\mathbb{P}(-3)) = H^{n-1}(\mathbb{P}, \Omega_\mathbb{P}(3-n-2))^\vee = 0.$$

Hence $H^1(X, \mathcal{T}_P|_X) = 0$ so that ρ is surjective.

If $n = 0$ then $H^1(X, \mathcal{T}_X) = 0$ hence the result is trivial.

Let $n = 1$, so that $X \subset \mathbb{P}^2$ is a smooth plane cubic curve. Consider the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to \mathcal{T}_{\mathbb{P}^2} \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \to \mathcal{T}_{\mathbb{P}^2}(-3) \to 0.$$

The first one yields

$$0 = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 3} \to H^1(\mathbb{P}, \mathcal{T}_\mathbb{P}) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0 \quad \Longrightarrow \quad H^1(\mathbb{P}, \mathcal{T}_\mathbb{P}) = 0. \tag{7.9}$$

The second one yields

$$0 = H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2))^{\oplus 3} \to H^2(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-3)) \to H^3(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0 \implies H^2(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-3)) = 0. \tag{7.10}$$

By (7.8), (7.9) and (7.10), we get that $H^1(X, \mathcal{T}_P|_X) = 0$. Hence, ρ is surjective. □

Exercise 7.2.9. Recall (see Proposition 7.1.8) that given a scheme X over k and a coherent sheaf $\mathcal{F}$ on X, equivalence classes of deformations $\mathcal{F}' \to \mathcal{F}$ over D of $\mathcal{F}$ over k are in natural correspondence with elements of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$.

(1) Show that if $\mathcal{F}$ is locally free, then $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F}) = H^1(X, \mathscr{E}nd(\mathcal{F}))$.

(2) Reprove the fact (see Proposition 7.1.6) that infinitesimal deformations of a line bundle on X are parametrized by $H^1(X, \mathcal{O}_X)$.

(3) Show that if X is smooth over k, then there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T}_X)$. Here, an *infinitesimal extension of* X *by* $\mathcal{F}$ is a scheme X' over k together with a sheaf of ideals $\mathcal{I}' \subset \mathcal{O}_{X'}$ with $(\mathcal{I}')^2 = 0$, such that

$$(X', \mathcal{O}_{X'}/\mathcal{I}') \cong (X, \mathcal{O}_X)$$

and such that $\mathcal{I}$, with its resulting $\mathcal{O}_X$-module structure, is isomorphic to the $\mathcal{O}_X$-module $\mathcal{F}$.

(4) Show that a deformation $(X', X \hookrightarrow X')$ of X over D corresponds to an exact extension as $\mathcal{O}_X$-modules of the form

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_{X'} \to \mathcal{O}_X$ is a morphism of sheaves of k-algebras. Thus, X' is an infinitesimal extension of the scheme X by the sheaf $\mathcal{O}_X$ in the above sense, and we see again that infinitesimal deformations of X are classified by $H^1(X, \mathcal{T}_X)$.

Exercise 7.2.10. Let C be an elliptic curve over a field k. Let $\mathcal{E}_P$ be a rank two vector bundle obtained as a non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E}_P \to \mathcal{O}_C(P) \to 0, \quad \text{for some} \quad P \in C(k).$$

(1) Calculate the determinant line bundle $\det(\mathcal{E}_P)$ of $\mathcal{E}_P$. Show that the degree $\deg(\mathcal{E}_P)$ of $\mathcal{E}_P$ equals one, where $\deg(\mathcal{E}_P) := \deg(\det(\mathcal{E}_P))$.

(2) For a vector bundle $\mathcal{E}$ on C, consider the following property:

($\star$) $H^0(C, \mathcal{E}) \neq 0$ and for any line bundle $\mathcal{L}$ on C with $\deg(\mathcal{L}) < 0$, we have $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$.

Show that $\mathcal{E}_P$ satisfies ($\star$).

(3) Show that $\mathcal{E}_P$ is uniquely determined by P, up to isomorphism.

(4) Show that $h^0(C, \mathcal{E}_P) = 1$ and $h^1(C, \mathscr{E}nd(\mathcal{E}_P)) = 1$.

(5) Let $\mathcal{E}$ be any rank two vector bundle of degree one on C satisfying $(\star)$. Show that there exists a unique point $P \in C(k)$ such that $\mathcal{E} \cong \mathcal{E}_P$.

(6) Conclude that the family of rank two vector bundles of degree one on C that satisfy $(\star)$ are parametrized by C itself. Remark that this is consistent with the calculation $h^1(C, \mathscr{E}nd(\mathcal{E})) = 1$ of item (4).

7.3 Lecture 19: Pro-representable functors and formal moduli

Before we pass to deformation theory in algebraic geometry, based on formal geometry (see Chapter 4), we give an overview of deformation theory in complex geometry.

7.3.1 Universal deformations of complex manifolds

Definition 7.3.1. Let X_0 be a compact complex manifold. A *deformation* of X_0 consists of:

(1) a germ of complex analytic spaces $(B, 0)$,

(2) a smooth proper morphism

$$\phi\colon X \to B, \tag{7.11}$$

(3) an isomorphism $f\colon \phi^{-1}(0) \cong X_0$.

We say that the deformation (7.11) of X_0 is *complete* if for any pointed complex analytic space $(B', 0')$ and any smooth proper family $\phi'\colon X' \to B'$ and isomorphism $f'\colon (\phi')^{-1}(0) \cong X_0$, there exists a morphism of germs of complex analytic spaces

$$(B', 0') \to (B, 0)$$

and an isomorphism

$$X' \cong X \times_B B' \quad \text{as families over the germ } (B', 0'), \text{ compatibly with } f \text{ and } f'. \tag{7.12}$$

We say that $(\phi\colon X \to B \ni 0, f)$ is a *universal deformation of* X_0 if for any deformation $(\phi'\colon X' \to B' \ni 0', f')$ of X_0, there is a unique morphism of germs $(B', 0') \to (B, 0)$ such that (7.12) holds. If a universal deformation of X_0 exists, it is unique up to unique isomorphism. If the deformation $(X \to B \ni 0, f)$ of X_0 is complete, and for any deformation $(\phi'\colon X' \to B' \ni 0', f')$ of X_0 and any morphism of germs $(B', 0') \to (B, 0)$ such that (7.12) holds, the induced morphism on tangent spaces $T_{B',0'} \to T_{B,0}$ is unique, then the deformation is called *versal*.

Let $(\phi\colon X \to B \ni 0, f)$ be a deformation of a compact complex manifold X_0. Then we have an exact sequence of coherent sheaves on X:

$$0 \to \phi^*\Omega_B \to \Omega_X \to \Omega_{X/B} \to 0,$$

and hence, after restricting to X_0 and dualizing, an exact sequence of vector bundles on X_0:

$$0 \to T_{X_0} \to T_X|_{X_0} \to \phi^*(T_B)|_{X_0} = T_{B,0} \times X_0 \to 0.$$

The Kodaira–Spencer map, see Section 3.3.5, is the induced morphism

$$\rho\colon T_{B,0} \to H^1(X_0, T_{X_0}).$$

Theorem 7.3.2 (Kodaira, Kuranishi, Nirenberg, Spencer)**.** *Let X_0 be a compact complex manifold. Then X_0 admits a versal deformation $(\phi\colon X \to B \ni 0, f)$ for which the Kodaira–Spencer map $T_{B,0} \to H^1(X_0, T_{X_0})$ is an isomorphism. Moreover:*

(1) If $H^2(X_0, \mathcal{T}_{X_0}) = 0$, then there exists a versal deformation $(\phi\colon X \to B \ni 0, f)$ of X_0 such that B is smooth (i.e., a complex manifold).

(2) If $H^0(X_0, \mathcal{T}_{X_0}) = 0$, then a universal deformation of X_0 exists.

(3) Let $(X \to B \ni 0, f)$ be a versal deformation of X_0, and suppose that the function $t \mapsto h^1(X_t, \mathcal{T}_{X_t})$ is constant on B. Then the family $X \to B$ defines a versal deformation of any of its fibres X_t.

Example 7.3.3. Let C be a compact Riemann surface of genus $g \geq 2$. Then $H^2(C, \mathcal{T}_C) = 0$ because $2 > \dim(C) = 1$, and $\deg(K_C) = 2g - 2 > 0$. In particular, $\deg(T_C) < 0$, so that $H^0(C, T_C) = 0$. Thus, by Theorem 7.3.2, the curve C admits a universal deformation $(\phi\colon X \to B, 0 \in B, f\colon \phi^{-1}(0) \cong C)$, such that B is a complex manifold, and $T_0B \cong H^1(C, T_C)$ via the Kodaira–Spencer map. Remark that $\dim(B) = \dim(T_0B) = \dim H^1(C, T_C) = \dim H^0(C, K_C^{\otimes 2})$. Riemann–Roch gives

$$h^0(C, K_C^{\otimes 2}) = h^0(C, K_C^{\otimes 2}) - h^0(C, K_C^{-1}) = 1 - g + \deg(K_C^{\otimes 2}) = 1 - g + 2 \cdot (2g - 2),$$

so that $h^0(C, K_C^{\otimes 2}) = 3g - 3$. Hence the universal deformation space $(B, 0)$ of C is smooth of dimension $\dim(B) = 3g - 3$.

7.3.2 Cotangent complex of a morphism of schemes

For a morphism of rings $A \to B$, one can consider its *cotangent complex* $L_{B/A}$, a naturally defined complex of B-modules, see [Stacks, Tag 08PL]. More generally, for $X \to S$ be a morphism of schemes, consider the complex $L_{X/S}$ of $\mathcal{O}_X$-modules called *cotangent complex of X over S* (see [Stacks, Tag 08UT]). It has following properties:

(1) We have a canonical isomorphism $H^0(L_{X/S}) = \Omega_{X/S}$ of sheaves of $\mathcal{O}_X$-modules.

(2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of schemes, then there is a canonical distinguished triangle

$$Lf^*(L_{Y/Z}) \to L_{X/Z} \to L_{X/Y} \to Lf^*L_{Y/Z}[1]$$

in the derived category $D(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules.

For any B-module M, and integer $i \geq 0$, we define

$$T^i(B/A, M) = h^i(\mathrm{Hom}_B(L_{B/A}, M)).$$

Similarly, for a morphism of schemes $X \to S$, and an $\mathcal{O}_X$-module $\mathcal{M}$, define

$$\mathcal{T}^i(X/S, \mathcal{M}) = h^i(\mathscr{H}om_{\mathcal{O}_X}(L_{X/S}, \mathcal{M})).$$

These are covariant additive functors in M. Moreover, if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules, then there is a long exact sequence

$$0 \to T^0(B/A, M') \to T^0(B/A, M) \to T^0(B/A, M'') \to T^1(B/A, M') \to \cdots .$$

Furthermore, if

$$A \to B \to C$$

is a sequence of ring homomorphisms and if M is a C-module, then there is a canonical long exact sequence of C-modules

$$0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \to T^1(C/B, M) \to \cdots .$$

Notice that $T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$, and hence that

$$T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B) = \mathrm{Der}_A(B, B).$$

Theorem 7.3.4. *Let X be a scheme over a field k. Define $\mathcal{T}^1_X = \mathcal{T}^1(X/k, \mathcal{O}_X)$. Then there exists a canonical exact sequence*

$$0 \to H^1(X, \mathcal{T}_X) \to \mathrm{Def}(X/k, D) \to H^0(X, \mathcal{T}^1_X) \to H^2(X, \mathcal{T}_X).$$

Proof. See [Har10, Exercise 5.7]. □

7.3.3 Pro-representable functors

Fix a field k. Throughout Section 7.3.3, we assume that k is algebraically closed. Let $\mathcal{C}$ be the category of local artinian k-algebras with residue field k. Let $\hat{\mathcal{C}}$ be the category of complete local k-algebras with residue field k. For $R \in \hat{\mathcal{C}}$, let

$$h_R\colon \mathcal{C} \to (\mathrm{Sets})$$

be the restriction to $\mathcal{C}$ of the functor $h_R\colon \mathcal{C} \to (\mathrm{Sets})$ with $h_R(A) = \mathrm{Hom}_{k-alg}(R, A)$.

Definition 7.3.5. A covariant functor

$$F\colon \mathcal{C} \to (\text{Sets})$$

is called *pro-representable* if there exists a complete local k-algebra R with residue field k, such that $h_R \cong F$ as functors $\mathcal{C} \to (\text{Sets})$.

Lemma 7.3.6. *Let $G\colon \mathcal{C} \to (\text{Sets})$ be a functor, and define $\hat{G}\colon \hat{\mathcal{C}} \to (\text{Sets})$ as the functor with*

$$\hat{G}(R) = \varprojlim G(R/\mathfrak{m}^n).$$

Let $H\colon \hat{\mathcal{C}} \to (\text{Sets})$ be a functor, and define $H_{res}\colon \mathcal{C} \to (\text{Sets})$ as the composition of the natural functor $\mathcal{C} \to \hat{\mathcal{C}}$ and the functor H. Then

$$\operatorname{Hom}(G_{res}, F) = \operatorname{Hom}(G, \hat{F}).$$

Proof. Let $\varphi\colon G_{res} \to F$ be a morphism of functors. Let $A = \varprojlim A/\mathfrak{m}^n \in \hat{\mathcal{C}}$. We construct a map

$$G(A) \to \varprojlim F(A/\mathfrak{m}^n)$$

as follows. By the universal property of the projective limit, it suffices to construct a compatible system of morphisms

$$G(A) \to F(A/\mathfrak{m}^n) \tag{7.13}$$

for each $n \in \mathbb{Z}_{\geq 1}$, and we define (7.13) as the composition

$$G(A) \to G(A/\mathfrak{m}^n) = G_{res}(A/\mathfrak{m}^n) \xrightarrow{\varphi} F(A/\mathfrak{m}^n).$$

It is clear that the so-constructed map (7.13) is functorial in A, hence we obtain an element $\psi \in \operatorname{Hom}(G, \hat{F})$.

Conversely, given $\psi \in \operatorname{Hom}(G, \hat{F})$, let us construct a morphism $\varphi\colon G_{res} \to F$. Let $A \in \mathcal{C}$. Then ψ defines a morphism

$$G_{res}(A) = G(A) \xrightarrow{\psi} \hat{F}(A) = F(A),$$

and it is clear that this yields a morphism of functors $\varphi\colon G_{res} \to F$. The constructions are inverse, hence we are done. □

Exercise 7.3.7. Let $F\colon \mathcal{C} \to (\text{Sets})$ be a covariant functor. Let R be a complete local k-algebra with residue field k. Show that to give a homomorphism of functors $\varphi\colon h_R \to F$ is to give an element

$$\xi \in \varprojlim F(R/\mathfrak{m}^n),$$

called a *formal family* of F over the ring R. Thus, $\hat{F}(R)$ is the set of formal families of F over R, and $\hat{F}(R) = \operatorname{Hom}(h_R, F) = \operatorname{Hom}(h_R, \hat{F})$ (see Lemma 7.3.6).

Definition 7.3.8. If $\xi\colon h_R \to F$ is an isomorphism, we get an element $\xi \in \hat{F}(R)$, and we say that the pair (R,ξ) *pro-represents* the functor F.

More generally, a pair (R,ξ) with $R \in \hat{\mathcal{C}}$ and $\xi \in \hat{F}(R)$ is called a *versal family* for F if the associated map $h_R \to F$ is strongly surjective. Here, we say that a morphism of functors $G \to F$ is *strongly surjective* if for every $A \in \mathcal{C}$, the map $G(A) \to F(A)$ is surjective, and furthermore, for every surjection $B \to A$ in $\mathcal{C}$, the map

$$G(B) \to G(A) \times_{F(A)} F(B)$$

is surjective. If, in addition, the map $h_R(D) \to F(D)$ is bijective, where $D = k[t]/(t^2)$, then (R,ξ) is called a *versal family*, and we say that F has a *pro-representable hull* (R,ξ). We say that (R,ξ) is a *universal family* if it pro-represents the functor F.

Theorem 7.3.9. *Let X_0 be a scheme over k. Then the functor* Def_{X_0} *of deformations of X_0 over local artin rings has a versal family if*

(1) X_0 is affine with isolated singularities

(2) X_0 is projective.

Moreover, if X_0 is projective with $H^0(X_0, \mathcal{T}_{X_0}) = 0$, then the functor of deformations of X_0 is pro-representable.

Let X be a projective scheme over a field k. By Theorem [ref], the functor Def_{X_0} of deformations of X_0 has a pro-representable hull (R,ξ), see Theorem [ref]. In particular,

(1) R is a complete local k-algebra with residue field k;

(2) there exists a formal scheme $\mathcal{X}$ and a morphism of formal schemes

$$\mathcal{X} \to \mathrm{Spf}(R)$$

together with a closed embedding $\iota\colon X \hookrightarrow \mathcal{X}$ inducing an isomorphism $X \cong \mathcal{X} \times_{\widehat{R}} k$, such that for each artinian local k-algebra A with residue field k, and each deformation $(X', i'\colon X \hookrightarrow X')$ of X over A, there exists a morphism $\mathrm{Spf}(A) \to \mathrm{Spf}(R)$ that extends to a cartesian diagram of formal schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spf}(A) \arrow[r] & \mathrm{Spf}(R). \end{tikzcd}$$

which is compatible with the closed embeddings ι and i', and such that, for each deformation $(X', i'\colon X \hookrightarrow X')$ of X over $D = k[t]/(t^2)$, there is a unique morphism of formal schemes $\mathrm{Spf}(D) \to \mathrm{Spf}(R)$ with $X' \cong \mathcal{X} \times_R D$ over D, compatibly with ι and i'.

Chapter 8

Intersection theory

8.1 Lecture 20: Intersecting Cartier divisors on a proper scheme

Recall (see [Har77, Chapter I, Theorem 7.5]) that a theorem due to Hilbert and Serre says that if $X \subset \mathbb{P}^N$ is an n-dimensional closed subscheme of the N-dimensional projective space $\mathbb{P}^N = \mathbb{P}^N_k$ over a field k, then the function

$$\mathbb{Z} \mapsto \mathbb{Z}, \quad m \mapsto \chi(X, \mathcal{O}_X(m)) = \sum_i (-1)^i h^i(X, \mathcal{O}_X(m))$$

is *polynomial of degree* n, that is, there exists a unique polynomial $P \in \mathbb{Q}[t]$ such that

$$P(m) = \chi(X, \mathcal{O}_X(m))$$

for each $m \in \mathbb{Z}$. The *degree* of X in $\mathbb{P}^N$ is then defined as $n!$ times the coefficient of m^n. If $H \subset X$ is a hyperplane section, then the degree is also written as H^n. We would like to generalize this definition and associate an intersection number

$$\deg(D_1 \cdots D_n) \in \mathbb{Z}$$

to any set of Cartier divisors $D_1, \ldots, D_n$ on a proper n-dimensional scheme X over k.

8.1.1 *Projection formula*

First, we state without proof the following theorem.

Theorem 8.1.1. *Let $f \colon X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module. There exists an isomorphism of $\mathcal{O}_Y$-modules*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \xrightarrow{\sim} R^q f_* \left(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \right)$$

for each $q \in \mathbb{Z}_{\geq 0}$.

Proof. See [Stacks, Tag 01E6]. $\square$

Remark 8.1.2. More precisely, in the notation of Theorem 8.1.1, there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_*\mathcal{F} \xrightarrow{\sim} Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}),$$

in the derived category of bounded below complexes of $\mathcal{O}_Y$-modules. Moreover, if we assume that $f\colon X \to Y$ is a proper morphism of projective varieties over a field, then there exists an isomorphism

$$\mathcal{E}^\bullet \otimes^L_{\mathcal{O}_Y} Rf_*(\mathcal{F}^\bullet) \xrightarrow{\sim} Rf_*\left(Lf^*(\mathcal{E}^\bullet) \otimes^L_{\mathcal{O}_X} \mathcal{F}^\bullet\right)$$

in the derived category of bounded complexes of coherent sheaves on Y, for any bounded complexes of coherent sheaves $\mathcal{F}^\bullet$ on X, $\mathcal{E}^\bullet$ on Y, cf. [Huy06, p. 83].

Corollary 8.1.3. *Let $f\colon X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{L}$ be a line bundle on Y. Then there exists an isomorphism of $\mathcal{O}_Y$-modules*

$$\mathcal{L} \otimes_{\mathcal{O}_Y} f_*\mathcal{F} \cong f_*(f^*\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

Proof. This is clear from Theorem 8.1.1. □

8.1.2 Intersection numbers of Cartier divisors

Before we prove that n Cartier divisors $D_1, \dots, D_n$ on an n-dimensional proper scheme X give rise to a well-defined intersection number $\deg(D_1 \cdots D_n) \in \mathbb{Z}$, we recall the following result.

Definition 8.1.4. Let $\mathbb{K}$ be an abelian category. Let $\mathbb{K}'$ be a subset of the set of objects of $\mathbb{K}$. We say that $\mathbb{K}'$ is *exact* if $0 \in \mathbb{K}'$ and if, for every exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathbb{K}$ such that two of the three objects A, A', A'' are in $\mathbb{K}'$, the third object is also in $\mathbb{K}'$.

Theorem 8.1.5. *Let X be a noetherian scheme. Let $\mathbb{K}'$ be an exact subset of the abelian category* $\mathrm{Coh}(X)$ *of coherent sheaves on X. Suppose that one of the following two conditions is verified.*

(1) For every integral closed subscheme $Y \subset X$ with generic point $y \in Y$, there exists an $\mathcal{O}_X$-module $\mathcal{G} \in \mathbb{K}'$ such that the $\mathcal{O}_{X,y}$-module structure on $\mathcal{G}_y$ extends to an $\mathcal{O}_{Y,y} = k(y)$-module structure making $\mathcal{G}_y$ a one-dimensional $k(y)$-vector space.

(2) Every coherent direct factor of a coherent $\mathcal{O}_X$-module $\mathcal{M} \in \mathbb{K}'$ belongs to $\mathbb{K}'$, and for every integral closed subscheme $j\colon Y \hookrightarrow X$ with generic point $y \in Y$, there exists an $\mathcal{O}_X$-module $\mathcal{G} \in \mathbb{K}'$ such that $\mathrm{Supp}(\mathcal{G}) = Y$ *(equivalently, such that $\mathcal{G} = j_*j^*\mathcal{G}$ and $j^*\mathcal{G}$ is a torsion-free coherent $\mathcal{O}_Y$-module).*

Then $\mathbb{K}' = \mathrm{Coh}(X)$.

Proof. See [EGAIII, Première partie, §3, Théorème 3.1.2, Corollaire 3.1.3, p. 115]. □

Corollary 8.1.6. *Let X be a noetherian scheme. Let $\mathbb{K}'$ be an exact subset of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X. Suppose that for every integral closed subscheme $j\colon Y \hookrightarrow X$, we have $j_*\mathcal{O}_Y \in \mathbb{K}'$. Then $\mathbb{K}' = \mathrm{Coh}(X)$.*

Proof. Indeed, if $\mathcal{G} = j_*\mathcal{O}_Y$, and if $y \in Y$ is the generic point of Y, then $\mathcal{G}_y = \mathcal{O}_{Y,y} = k(y)$, hence the result follows from Theorem 8.1.5. □

Theorem 8.1.7. *Let $D_1, \dots, D_r$ be Cartier divisors on a proper scheme X over k. For any coherent sheaf $\mathcal{F}$ on X, the function*

$$(m_1, \dots, m_r) \mapsto \chi(X, \mathcal{F}(m_1D_1 + \cdots + m_rD_r)) \tag{8.1}$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathcal{F}$.

Proof. **Step 1: Reduction to X integral and $\mathcal{F} = \mathcal{O}_X$.** Consider the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X. Let $\mathbb{K}'$ be the subset of coherent sheaves $\mathcal{F}$ on X such that the function (8.1) takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathcal{F}$. Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of coherent sheaves on X. Note that, for $(m_1, \dots, m_r) \in \mathbb{Z}^{\oplus r}$, we have

$$\chi(X, \mathcal{F}(\sum m_iD_i)) = \chi(X, \mathcal{F}'(\sum m_iD_i)) + \chi(X, \mathcal{F}''(\sum m_iD_i)).$$

In particular,

if two of the three sheaves $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are in $\mathbb{K}'$, then the third one is also in $\mathbb{K}'$.

Hence $\mathbb{K}' \subset \mathrm{Coh}(X)$ is exact, see Definition 8.1.4. In order to prove Theorem 8.1.7, it suffices to show $\mathbb{K}' = \mathrm{Coh}(X)$, which, By Corollary 8.1.6, is equivalent to the assertion that for every integral closed subscheme $j\colon Y \hookrightarrow X$, the coherent $\mathcal{O}_X$-module $j_*\mathcal{O}_Y$ is in $\mathbb{K}'$. Now note that, for such a closed subscheme Y, we have by Corollary 8.1.3 that

$$j_*\mathcal{O}_Y\left(\sum m_iD_i\right) = j_*j^*\mathcal{O}_X\left(\sum m_iD_i\right) = j_*\left(\mathcal{O}_Y\left(\sum m_iD_i|_Y\right)\right).$$

Therefore,

$$\chi(X, j_*\mathcal{O}_Y(\sum m_iD_i)) = \sum(-1)^j h^j\left(Y, \mathcal{O}_Y\left(\sum m_iD_i\right)\right).$$

In particular, to prove Theorem 8.1.7, we may assume that X is integral and $\mathcal{F} = \mathcal{O}_X$.

Step 2: Reduction to the case $r = 1$. Assume that the theorem holds for $r = 1$. If d is a positive integer and $f\colon \mathbb{Z}^{\oplus r} \to \mathbb{Z}$ be a map such that for each i, $(n_1, \dots, \hat{n}_i, \dots, n_r) \in \mathbb{Z}^{\oplus(r-1)}$, the map $m \mapsto f(n_1, \dots, \hat{n}_i, \dots, n_r)$ is polynomial of degree at most d. Then f takes the same values on $\mathbb{Z}^{\oplus r}$ as a polynomial in r indeterminates with rational coefficients.

It follows that there exists a polynomial

$$P \in \mathbb{Q}[t_1, \dots, t_r]$$

such that $\chi(X, m_1 D_1 + \cdots m_r D_r) = P(m_1, \dots, m_r)$. Let d be its total degree. If $t_1^{n_1} \cdots t_r^{n_r}$ is a monomial of degree $\sum n_i = d$ in $P(t_1, \dots, t_r)$, for some integers $n_1, \dots, n_r$, then the degree of $Q(t) = P(n_1 t, \dots, n_r t)$ is still d. Now note that

$$Q(m) = P(n_1 m, \dots, n_r m) = \chi(X, m(n_1 D_1 + \cdots + n_r D_r)) = \chi(X, mD), \quad D = \sum n_i D_i.$$

From the $r = 1$ case, it follows that $d \leq \dim(X)$.

Step 3: Finish the proof. By Steps 1 and 2, we may assume that X is integral, $\mathcal{F} = \mathcal{O}_X$ and $r = 1$. Put $D \coloneqq D_1$. Choose an embedding of $\mathcal{O}_X(D)$ into $\mathcal{K}_X$. Set

$$\mathcal{I}_1 = \mathcal{O}_X(-D) \cap \mathcal{O}_X \qquad \mathcal{I}_2 = \mathcal{O}_X(D) \cap \mathcal{O}_X.$$

Let $Y_j \subset X$ be the subscheme of X defined by the ideal $\mathcal{I}_j$. Since X is integral, Y_j has dimension smaller than X. Note that $\mathcal{I}_1(D)$ is isomorphic to $\mathcal{I}_2$, so there are exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_1(mD) \arrow[r] \arrow[d, equal] & \mathcal{O}_X(mD) \arrow[r] & \mathcal{O}_{Y_1}(mD) \arrow[r] & 0, \\
0 \arrow[r] & \mathcal{I}_2((m-1)D) \arrow[r] & \mathcal{O}_X((m-1)D) \arrow[r] & \mathcal{O}_{Y_2}((m-1)D) \arrow[r] & 0.
\end{tikzcd}$$

This gives

$$\chi(X, mD) - \chi(X, (m-1)D) = \chi(Y_1, mD) - \chi(Y_2, (m-1)D).$$

By induction, the right hand side is a polynomial function in m of degree $d < \dim(X)$. But if a function $f\colon \mathbb{Z} \to \mathbb{Z}$ is such that $f(m) - f(m-1)$ is polynomial of degree d, then f is polynomial of degree $d+1$. Therefore, $\chi(X, mD)$ is polynomial in m of degree $\leq d + 1 \leq \dim(X)$, with rational coefficients. $\square$

Definition 8.1.8. Let $D_1, \dots, D_r$ be Cartier divisors on a proper scheme X, with $r \geq \dim(X)$. The *intersection number*

$$\deg(D_1 \cdots D_r)$$

is the coefficient of $m_1 \cdots m_r$ in the polynomial $\chi(X, m_1 D_1 + \cdots + m_r D_r)$.

If $Y \subset X$ is a subscheme of dimension at most s, we put

$$\deg(D_1 \cdots D_s \cdot Y) = \deg(D_1|_Y \cdots D_s|_Y).$$

Proposition 8.1.9. *Let $D_1, \dots, D_r$ be Cartier divisors on a proper scheme X of dimension $n \leq r$. Then*

(1) $\deg(D_1 \cdots D_r) = 0$ *for* $r > \dim(X)$*;*

(2) if $D_1, \ldots, D_r$ *are effective and meet properly in a finite number of points, then* $\deg(D_1 \cdots D_r)$ *equals the number of points in the zero-dimensional closed subscheme* $D_1 \cap \cdots \cap D_n \subset X$*, counted with multiplicity.*

(3) The map

$$(D_1, \ldots, D_n) \mapsto \deg(D_1 \cdots D_n)$$

is multilinear, symmetric, and takes integral values.

(4) If D_n *is effective with associated subscheme* Y*, then* $D_1 \cdots D_n = D_1 \cdots D_{n-1} \cdot Y$*.*

Proof. See [Deb01, Proposition 1.8]. □

8.1.3 Curves

Let X be a proper curve over k. If $x \in X$ is a closed point and f a regular element of $\mathcal{O}_{X,x}$, we define the *vanishing order* of f at x as

$$\operatorname{ord}_x(f) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f)) \qquad \text{(the length of } \mathcal{O}_{X,x}/(f) \text{ as an } \mathcal{O}_{X,x}\text{-module).}$$

If $x \in X$ is a regular point, then $\mathcal{O}_{X,x}$ is a discrete valuation ring and this is the usual order of f in $\mathcal{O}_{X,x}$. For f/g in the total ring of fractions of $\mathcal{O}_{X,x}$, we put

$$\operatorname{ord}_x(f/g) = \operatorname{ord}_x(f) - \operatorname{ord}_x(g).$$

We extend this to Cartier divisors D on X as follows: if, in a neighbourhood of $x \in X$, the Cartier divisor D is defined by a rational function f/g, then

$$\operatorname{ord}_x(D) = \operatorname{ord}_x(f_x/g_x) = \operatorname{ord}_x(f_x) - \operatorname{ord}_x(g_x).$$

We can associate to D the Weil divisor

$$\sum_{x \in X} \operatorname{ord}_x(D) \cdot x.$$

Lemma 8.1.10. *Let* X *be a proper curve over a field* k*. Consider the integer* $\deg(D) \in \mathbb{Z}$*, see Definition 8.1.8 and Proposition 8.1.9. Then*

$$\deg(D) = \sum \operatorname{ord}_x(D) \cdot [k(x) \cdot k].$$

If D *is effective, then* $\deg(D) = \dim_k H^0(X, \mathcal{O}_D)$*.*

Lemma 8.1.11. *If* X *is a reduced proper curve over a field* k*, with normalization* $\widetilde{X} \to X$*, then* $\deg(D) = \deg(\widetilde{D})$*.*

A Cartier divisor D on a proper curve X is called principal if there exists a rational function f on X such that $D = div(f)$. For a principal divisor D we have $\deg(D) = 0$. Two Cartier divisors D_1, D_2 are linearly equivalent if $D_1 - D_2$ is principal. Recall that each line bundle $\mathcal{L}$ on X corresponds to a Cartier divisor D on X which is unique up to linear equivalence. Thus, we can put $\deg(\mathcal{L}) = \deg(D)$.

Theorem 8.1.12 (Riemann–Roch)**.** *Let X be a proper curve over a field k, and let $\mathcal{L}$ be a line bundle on X. Then*

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X).$$

Proposition 8.1.13. *Let X be a proper variety. Let D be a Cartier divisor and let $C \subset X$ be a curve contained in X. Then* $\deg(C \cdot D) = \deg \mathcal{O}_C(D)$*.*

Proof. We have

$$C \cdot D = \deg(D|_C) = \text{coefficient of } m \text{ in the polynomial } \chi(C, mD) =$$
$$\deg(mD) + \chi(\mathcal{O}_C) = m\deg(D) + \chi(\mathcal{O}_C) = m \cdot \deg \mathcal{O}_C(D) + \chi(\mathcal{O}_C).$$

Hence $\deg(C \cdot D) = \deg(D|_C) = \deg(\mathcal{O}_C(D))$ as wanted. □

8.1.4 Surfaces

Definition 8.1.14. Let X be a proper surface over k. Let D and E be two effective divisors on X with no common irreducible component. Let $x \in X$ be a closed point. Define

$$i_x(D, E) = \operatorname{length}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} / \left(\mathcal{O}_X(-D)_x + \mathcal{O}_X(-D)_x\right)\right).$$

We call this the *intersection multiplicity of D and E at x.*

Example 8.1.15. Let $X = \operatorname{Spec} k[u, v]$. Let $x \in X$ correspond to (U, v). Let $D = V(u)$ and $E = V(u + v^r)$, for $r \geq 2$. Then

$$i_x(D, E) = length(k[u, v]/(u, u + v^r)) = \dim_k \left(k[v]/(v^r)\right) = r.$$

[Insert picture]

The following theorem is known as the Riemann–Roch theorem for surfaces.

Theorem 8.1.16. *Let X be a smooth proper surface over a field k, with canonical bundle K_X. Let D be a Cartier divisor on X. Then*

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{D \cdot D - D \cdot K_X}{2}.$$

Example 8.1.17. Consider the projective plane $\mathbb{P}^2$. Then $\chi(\mathcal{O}_{\mathbb{P}^2}(1)) = h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$, $(\deg(\mathcal{O}_{\mathbb{P}^2}(1)) = 1$ and $\deg(\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2}))) = -3$. This gives $\chi(\mathcal{O}_{\mathbb{P}^2}(1)) = \chi(\mathcal{O}_{\mathbb{P}^2}) + \frac{1}{2} \cdot (\deg(\mathcal{O}_{\mathbb{P}^2}(1)) - \deg(\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2})))$, which is compatible with Theorem 8.1.16.

Example 8.1.18. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface. We have seen in Proposition 2.4.11 that for a Cartier divisor D on X, we have $\chi(D) = \frac{1}{2} \cdot (D \cdot D + D \cdot \mathcal{O}_X(1)) + 1$, see (2.31). This also follows from Theorem 8.1.16 by observing that $\chi(\mathcal{O}_X) = 1$ in view of Claim 2.1.15, and that $K_X = \mathcal{O}_X(-1)$ in view of Proposition 2.1.12.

Chapter 9

Étale cohomology

9.1 Lecture 21: Étale morphisms of schemes

In this lecture, we introduce smooth and étale morphisms of schemes. We also define the notion of site, and consider fpqc, fppf, étale and Zariski coverings, leading to various sites associated to a single scheme. This yields various cohomology theories.

To begin with, we would like to define what it means for a morphism of schemes $X \to Y$ to be smooth or étale. We will first show how to get the right notion in the locally noetherian case. However, note that the category of locally noetherian schemes does not have all fibre products. For example, the ring $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not noetherian. This implies in particular that even if we manage obtain a good definition of smooth or étale morphism for locally noetherian schemes, this property will not be stable under base change unless we extend it to arbitrary schemes; in particular, one cannot use étale morphisms of locally noetherian schemes to endow Sch/S, the category of schemes over a fixed base scheme S, with the structure of a site (see Definition 9.1.22 below) to obtain the big étale site of S. One alternative way to fix this, would be to assume that S is locally noetherian and consider the full subcategory $\mathsf{LFT}/S \subset \mathsf{Sch}/S$ whose objects are schemes X/S whose structure morphism $X \to S$ is locally of finite type; this is the approach adopted in [Mil80], see §1 in Chapter II of *loc. cit.* We shall not take this approach, but follow [Stacks, Tag 03X7] instead, using Sch/S as underlying category of the big étale site (denoted $(\mathsf{Sch}/S)_{\text{ét}}$) of an arbitrary scheme S. Thus, we define the notion of smooth and étale morphisms of arbitrary schemes.

Before we do so, let us consider the locally noetherian case.

9.1.1 Smooth and étale morphisms of locally noetherian schemes

Lemma 9.1.1. *Let $f\colon X \to Y$ be a morphism between locally noetherian schemes.*

(1) For each $y \in Y$, the fibre $X_y = X \times_Y \operatorname{Spec}(k(y))$ is locally noetherian.

(2) Suppose $Y = \operatorname{Spec} k$ is the spectrum of a field k. Let k'/k be a finitely generated field extension. Then $X \times_k k'$ is locally Noetherian.

Proof. Exercise. □

Recall that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. A Noetherian ring R is *regular* if every local ring $R_{\mathfrak{p}}$ of R is regular. A locally noetherian scheme X is *regular at* $x \in X$ if there exists an affine open neighbourhood U of x such that $\mathcal{O}_X(U)$ is noetherian and regular. Finally, a locally noetherian scheme X is *regular* if X is regular at all of points. Equivalently, X is locally noetherian and all of its local rings are regular.

Definition 9.1.2 (cf. Définition (6.7.6) in [EGAIV])**.** Let k be a field. Let X be a scheme over k which is locally noetherian. We say that X is *geometrically regular at a point* $x \in X$ if, for every finite field extension k' of k, the scheme $X' = X \times_k k'$ is regular at all points $x' \in X'$ that lie over $x \in X$. We say that X is *geometrically regular* if X is geometrically regular at all of its points.

Definition 9.1.3 (cf. [EGAIV], Définition (6.8.1))**.** Let X and Y be locally noetherian schemes. Let $f \colon X \to Y$ be a morphism locally of finite presentation. Let $x \in X$.

(1) Write $y = f(x) \in Y$. We say that f is *smooth at* x if f is flat at x (i.e. $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$), and if the fibre $X_y = f^{-1}(y)$ is geometrically regular at $x \in X_y$.

(2) We say that f is *smooth* if f is locally of finite presentation and if f is smooth at x for each $x \in X$.

Definition 9.1.4. Let $f \colon X \to Y$ be a morphism locally of finite presentation of locally noetherian schemes. Let $x \in X$ and $y = f(x)$.

(1) We say that f is *unramified at* x if the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ verifies $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ and the extension of residue fields $k(y) \to k(x)$ is separable.

(2) We say that f is *étale at* x if it is unramified and flat at x.

9.1.2 Smooth and étale morphisms of arbitrary schemes

Definition 9.1.5 (cf. Définition (17.1.1) & Remarque (17.1.2).(ii) in [EGAIV])**.** A morphism of schemes $f \colon X \to Y$ is *formally smooth (resp. formally unramified, resp. formally étale)* if for every affine scheme Y' and every closed subscheme $Y'_0 \subset Y'$ defined by an ideal $I \subset \mathcal{O}(Y')$ with $I^2 = 0$, and every morphism $Y' \to Y$, the map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X)$$

deduced by the embedding $Y'_0 \to Y'$, is surjective (resp. injective, resp. bijective).

Definition 9.1.6 (cf. Définition (17.3.1) in [EGAIV])**.** Let f be a morphism of schemes $X \to Y$. We say that f is *smooth* (resp. *unramified*, resp. *étale*) if f is locally of finite presentation and formally smooth (resp. unramified, resp. étale).

Recall that if X, Y and S are schemes, and $f \colon X \to Y$ and $g \colon Y \to S$ are morphisms, then there is a canonical exact sequence of quasi-coherent sheaves

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega_{X/Y} \to 0. \tag{9.1}$$

Proposition 9.1.7. *Let $f: X \to Y$ be a morphism of schemes; then f is formally unramified if and only if $\Omega^1_{X/Y} = 0$.*

Proof. See Proposition (17.2.1) in [EGAIV]. □

Corollary 9.1.8. *Let $f: X \to Y$ and $g: Y \to S$ be morphisms of schemes. Then f is formally unramified if and only if $f^*\Omega^1_{Y/S}S \to \Omega^1_{X/S}$ is surjective.*

Proof. This follows from Proposition 9.1.7 and the exact sequence (9.1). □

This aligns with our intuition that unramified morphisms of varieties should induce injections between tangent spaces.

Proposition 9.1.9. *Let S be a scheme. Let X and Y be schemes over S. Let $f: X \to Y$ be a formally smooth morphism of schemes over S.*

(1) The quasi-coherent $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is locally projective. If f is locally of finite type, then $\Omega^1_{X/Y}$ is locally free of finite rank.

(2) The morphism of quasi-coherent $\mathcal{O}_X$-modules $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is injective. In other words, the sequence $0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega_{X/Y} \to 0$ is exact.

Proof. See Proposition (17.2.3) in [EGAIV]. □

Corollary 9.1.10. *Let S be a scheme. Let X and Y be schemes over S. Then if $f: X \to Y$ is a morphism of S-schemes which is formally étale, then the canonical morphism of $\mathcal{O}_X$-modules $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.*

Proof. Note that a formally étale morphism is formally smooth and formally unramified. Hence, the propossition follows from Corollaries 9.1.8 and 9.1.10. □

This corresponds to our intuition that étale morphisms should induce bijections on tangent spaces (see also Proposition 9.1.20 in Section 9.1.3 below).

Definition 9.1.11. Let $f: X \to Y$ be a morphism of schemes. We say that f is *smooth* (resp. *unramified*, resp. *étale*) at a point $x \in X$ if there exists an open neighbourhood U of x in X such that the restriction $f|_U$ is a morphism $U \to Y$ which is smooth (resp. unramified, resp. étale).

Theorem 9.1.12. *Let $f: X \to Y$ be a morphism of schemes, locally of finite presentation. Let $x \in X$ and $y = f(x) \in Y$. The following properties are equivalent:*

(1) f is unramified at x.

(2) $(\Omega^1_{Y/X})_x = 0$.

(3) The $k(y)$-scheme $f^{-1}(y)$ is unramified over $k(y)$ at x.

(4) $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a field, which is a finite separable extension of $k(y)$.

If Y is locally noetherian, then these properties are further equivalent to:

(5) the homomorphism between the completed local rings $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ is surjective.

Proof. See Théorème (17.4.1) in [EGAIV]. □

9.1.3 Properties of étale morphisms

Proposition 9.1.13. *Let $f\colon X \to S$ be a flat morphism of schemes which is locally of finite presentation. Then f is open. In particular, smooth morphisms and étale morphisms are open.*

Proof. See [EGAIV, Proposition 2.4.6]. □

We follow [Stacks, Tag 020C] and will use the following definition of variety.

Definition 9.1.14. Let k be a field. A *variety* over k is a scheme X over k which is integral and such that the morphism $X \to \operatorname{Spec} k$ is separated and of finite type.

Remark 9.1.15. With this definition, the base change of a variety over some finite field extension may no longer be a variety.

For a local ring A with maximal ideal $\mathfrak{m}_A$, define

$$\operatorname{gr}(A) = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}, \quad \mathfrak{m} = \mathfrak{m}_A.$$

For a geometrically integral affine variety

$$V = \operatorname{Spec} k[X_1, \dots, X_n]/\mathfrak{a}$$

over an algebraically closed field k such that $0 \in V(k) \subset k^n$, let $\mathfrak{a}_* = \{f_*\colon f \in \mathfrak{a}\}$, where f_* is the homogeneous part of f of lowest degree. Let $P \in V(k)$ be the point which is the origin in $\mathbb{A}^n(k)$. Define the *tangent cone of V at P* to be the k-algebra

$$C_P(V) = k[X_1, \dots, X_n]/\mathfrak{a}_*.$$

Lemma 9.1.16. *Let $V \subset \mathbb{A}^n$ be a geometrically integral affine variety defined by an ideal $\mathfrak{a}$ such that $p = (0, \dots, 0) \in V$. The map $C_p(V) = k[X_1, \dots, X_n]/\mathfrak{a}_* \to \operatorname{gr}(\mathcal{O}_{V,p})$ that sends the class of X_i in $C_p(V)$ to the class of X_i in $\operatorname{gr}(\mathcal{O}_{V,p})$ is an isomorphism.*

Example 9.1.17. Let $X = V(y^2 = x^3) \subset \mathbb{A}^2$. With $p = (0,0) \in X(k)$. Then

$$C_p(X) = k[x,y]/(y^2) = k[x] \otimes_k k[y]/(y^2)$$

is the coordinate ring of the double line through the origin which is tangent to X at p. We have $\mathcal{O}_{X,p} = k[x,y]/(y^2 - x^3)_{(0,0)}$ with maximal ideal $\mathfrak{m} = (x,y)$. As $y \in \mathfrak{m}$ satisfies $y^2 = x^3 \in \mathfrak{m}^3$, the class $[y] \in \operatorname{gr}(\mathcal{O}_{X,p})$ of y satisfies $[y]^2 = 0$. The natural map $k[x] \otimes_k k[y]/(y^2) \to \operatorname{gr}(\mathcal{O}_{X,p})$ sending x to $[x]$ and y to $[y]$, is an isomorphism.

Proposition 9.1.18. *Let $f\colon A \to B$ be a local morphism of local noetherian rings.*

(1) The map f defines a homomorphism of graded rings $\operatorname{gr}(f)\colon \operatorname{gr}(A) \to \operatorname{gr}(B)$.

(2) The map $\operatorname{gr}(f)$ is an isomorphism if and only if $\hat{f}\colon \widehat{A} \to \widehat{B}$ is an isomorphism.

Proposition 9.1.19. *Let V and W be a morphism of geometrically integral varieties over an algebraically closed field k. Then a morphism of varieties $\varphi\colon W \to V$ is étale if and only if for each $w \in W$, the map $\widehat{\mathcal{O}}_{V,\varphi(w)} \to \widehat{\mathcal{O}}_{W,w}$ induced by φ is an isomorphism.*

Proof. See Theorem ... above. We will give an alternative proof. □

Let S be a scheme. For a scheme X over S, and a point $x \in X$, we let $T_{X/S,x}$ denote the (Zariski) tangent space of X over S at x, see [Stacks, Tag 0B28]. If $S = \operatorname{Spec} k$ for a field k, and if X is a variety over k, and $x \in X$, then we define $T_{X,x} \coloneqq T_{X/\operatorname{Spec} k,x}$.

Proposition 9.1.20. *Let X and Y be smooth varieties of an algebraically closed field. Then a morphism of varieties $f\colon X \to Y$ is* étale *if and only if for each $x \in X$, the induced map on Zariski tangent spaces $df_x\colon T_{X,x} \to T_{Y,f(x)}$ is an isomorphism.*

9.1.4 Sites

We begin with the following definition that we shall frequently use.

Definition 9.1.21. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i\colon U_i \to U\}_{i\in I}$ is the data of

(1) an object $U \in \mathcal{C}$,

(2) a set I, possibly empty,

(3) for all $i \in I$, a morphisms $\varphi_i\colon U_i \to U$ of $\mathcal{C}$ with target U.

Definition 9.1.22. A *site* is a tuple

$$\mathbb{T} = (\mathcal{T}, \operatorname{Cov}(\mathcal{T}))$$

where $\mathcal{C}$ is a category and $\operatorname{Cov}(\mathcal{C})$ is a set of families of morphisms with fixed target called *coverings*, such that

(1) If $\varphi\colon V \to U$ is an isomorphism, then $\{\varphi\colon V \to U\}$ is a covering.

(2) If $\{\varphi_i\colon U_i \to U\}_{i\in I}$ is a covering and for all $i \in I$, $\{\psi_{ij}\colon U_{ij} \to U_i\}_{j\in J_i}$ is a covering, then
$$\{\varphi_i \circ \psi_{ij}\colon U_{ij} \to U\}_{i\in I, j\in J_i}$$
is a covering.

(3) If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then

  (a) for all $i \in I$, the fibre product $U_i \times_U V$ exists in $\mathcal{C}$; and

  (b) $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

If $(\mathcal{T}, \operatorname{Cov}(\mathcal{T}))$ is a site, then we shall often abuse notation, and denote it simply by $\mathcal{T}$ (the set of coverings being implicitly understood). This abuse of notation can be compared with the common way of denoting a topological space $(X, \operatorname{Op}(X))$ simply by its underlying set X, the category $\operatorname{Op}(X)$ of opens in X being implicitly understood.

Remark 9.1.23. Explain the difference between terminology: What we call a site is a called a category endowed with a pretopology in [Exposé II, Définition 1.3, SGA4]. In [ArtinTopologies] it is called a category with a Grothendieck topology.

Example 9.1.24. Let X be a topological space, and let $\mathrm{Op}(X)$ be the category of opens $U \subset X$ (the morphisms between opens are the inclusion mappings, if they exist). Then $\mathrm{Op}(X)$ underlies a site X_{Zar} defined as follows: the coverings are the usual topological coverings (for an open $U \subset X$ and a set of inclusions of opens $\mathcal{U} = \{U_i \subset U\}_{i \in I}$, $\mathcal{U}$ is a covering if $\cup_{i \in I} U_i = U$). Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fibre products. The verifications of the axioms are trivial. This site called the *Zariski site* of X.

Example 9.1.25. Let X be a scheme. Let $\mathcal{C}$ be the category of tuples

$$(U, \pi \colon U \to X)$$

where U is a scheme and π an étale morphism. For $U \in \mathcal{C}$, let $\mathrm{Cov}(U)$ be the set of surjective families of étale morphisms $\mathcal{U} = \{\varphi_i \colon U_i \to U\}_{i \in I}$, where $\mathcal{U}$ is *surjective* if the induced morphism $\sqcup_i U_i \to U$ is surjective. Then $\mathcal{C}$ together with the set $\mathrm{Cov}(\mathcal{C}) = \{\mathrm{Cov}(U)\}_{U \in \mathcal{C}}$ is a site.

Definition 9.1.26. Let X be a scheme.

(1) The *small étale site of* X is the site constructed in Example 9.1.25 and is denoted by $X_{ét}$.

(2) The *small Zariski site of* X is the site constructed in Example 9.1.24 and is denoted by X_{Zar}.

Definition 9.1.27. Let T be a scheme. Let $\{f_i \colon T_i \to T\}_{i \in I}$ be a family of morphisms with fixed target (see Definition 9.1.21).

(1) The family of morphisms with fixed target $\{f_i \colon T_i \to T\}_{i \in I}$ is a *Zariski covering* if each f_i is an open immersion and $T = \cup_{i \in I} f_i(T_i)$.

(2) $\{f_i \colon T_i \to T\}_{i \in I}$ is an *étale covering* if each f_i is étale and $T = \cup_{i \in I} f_i(T_i)$.

(3) We say that $\{f_i \colon T_i \to T\}_{i \in I}$ is an *fppf covering* if each f_i is flat and locally of finite presentation, and $T = \cup_{i \in I} f_i(T_i)$.

(4) We say that $\{f_i \colon T_i \to T\}_{i \in I}$ is an *fpqc covering* of T if the following conditions hold:

   (a) $f_i \colon T_i \to T$ is flat for each $i \in I$;

   (b) for every affine open $U \subset T$, there exists a finite subset $I_U \subset I$ and for each $i \in I_U$, a quasi-compact open $V_i \subset T_i$, such that

$$U = \bigcup_{i \in I_U} f_i(U_i).$$

Definition 9.1.28. Let X be a scheme. Let $\tau \in \{\text{fpqc, fppf, étale, Zariski}\}$. The *big τ-site of X* is the site with underlying category Sch/X, the category of schemes over X, and for which a family of morphisms with fixed target $\mathcal{U} = \{U_i \to T\}_{i \in I}$ is a covering if and only if if it is an τ-covering, see Definition 9.1.27.

Definition 9.1.29. Let S be a scheme. Let F be a contravariant functor on the category of schemes over S with values in sets. Let $\mathcal{U} = \{U_i \to T\}_{i \in I}$ be a family of morphisms of schemes over S with fixed target, see Definition 9.1.21.

(1) We say that F *satisfies the sheaf property for the family* $\mathcal{U}$ if for any collection of elements $s_i \in F(U_i)$ such that $s_i|_{U_i \times_T U_j} = s_j|_{U_i \times_T U_j}$ there exists a unique element $s \in F(T)$ such that $s_i = s|_{U_i} \in F(U_i)$.

(2) We say that F *satisfies is a sheaf for the fpqc (resp. fppf, resp. étale, resp. Zariski) topology* if it satisfies the sheaf property for any fpqc (resp. fppf, resp. étale, resp. Zariski) covering $\mathcal{U}$.

Similarly, suppose $\tau \in \{\text{étale, Zariski}\}$. Let X be a scheme, and suppose F is presheaf on the small τ-site of X, see Definition 9.1.26. Then F is a *sheaf for the τ-topology* if F satisfies the sheaf property for every $\mathcal{U} \in \mathrm{Cov}(X_\tau)$.

Remark 9.1.30. An fppf covering is an fpqc covering. A fortiori, any étale covering is an fpqc covering. In particular, if F is a contravariant functor on the category of schemes over a scheme S with values in sets, and F is a sheaf for the fpqc topology, then F is a sheaf for the fppf topology. Similarly, fppf sheaves are étale sheaves and étale sheaves are Zariski sheaves. For a certain converse, see Proposition 9.3.7.

Remark 9.1.31. Throughout Section 9.1.4, we have been ignoring set-theoretic issues. To resolve these, see [Stacks, Tag 03X7].

9.2 Lecture 22: Sheaves on the étale site

9.2.1 *Morphisms of sites*

Definition 9.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u \colon \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every $\{V_i \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$, we have the following:

(1) $\{u(V_i) \to u(V)\}_{i \in I} \in \mathrm{Cov}(\mathcal{D})$,

(2) for any morphism $T \to V$ and any $i \in I$, the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Let $u \colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. We denote by

$$u^p \colon PSh(\mathcal{D}) \to PSh(\mathcal{C})$$

the functor $F \mapsto F \circ u$.

For $V \in \mathcal{D}$, define $\mathcal{I}_V^u$ as the category with

$$\begin{aligned}\mathrm{Ob}(\mathcal{I}_V^u) &= \{(U,\phi) \mid U \in \mathrm{Ob}(\mathcal{C}), \phi\colon V \to u(U)\} \\ \mathrm{Mor}((U,\phi),(U',\phi')) &= \{f\colon U \to U' \text{ in } \mathcal{C} \mid u(f)\circ\phi = \phi'\}\,.\end{aligned}$$

Given a presheaf $\mathcal{F}$ on $\mathcal{C}$, we obtain a functor

$$\mathcal{F}_V\colon \mathcal{I}_V^{opp} \to \mathsf{Set}, \quad (U,\phi) \mapsto \mathcal{F}(U).$$

We define

$$u_p\mathcal{F}(V) \coloneqq \varinjlim_{\mathcal{I}_V^{opp}} \mathcal{F}_V,$$

where the colimit on the right is an object in Set, see [Stacks, Tag 002F].

Lemma 9.2.2. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories.*

(1) The functor u_p is left adjoint to the functor u^p.

(2) Assume that $\mathcal{C}$ has a final object X and that $u(X)$ is a final object of $\mathcal{D}$. Assume that $\mathcal{C}$ has fibre products and that u commutes with them. Then $(\mathcal{I}_V^u)^{opp}$ is filtered.

Definition 9.2.3. Let $\mathcal{S}$ be a site.

(1) A *presheaf on $\mathcal{S}$* is a contravariant functor $\mathcal{S} \to \mathsf{Set}$.

(2) Let $F\colon \mathcal{S} \to \mathsf{Set}$ be a presheaf. Then F is a *sheaf* if for every covering $\{U_i \to U\}_{i\in I}$, the diagram of sets

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(U_i)\times F(U_j)$$

is exact. In other words, the map $F(U) \to \prod_{i\in I} F(U_i)$ is injective with image the set of $(s_i)_i \in \prod_i F(U_i)$ such that $s_i|_j = s_j|_i$ for each $i,j \in I$.

(3) If F and G are presheaves, then a *morphism of presheaves* $F \to G$ is a morphism of functors, and if F and G are sheaves, then a *morphism of sheaves* is a morphism of presheaves $F \to G$. Hence the category of sheaves on $\mathcal{S}$, denote by $\mathrm{Sh}(\mathcal{S})$, is a full subcategory of the category of presheaves on $\mathcal{S}$.

(4) Let $\mathcal{T}$ be a category. Then $\mathcal{T}$ is called a *topos* if $\mathcal{T}$ is equivalent to the category of sheaves $\mathrm{Sh}(\mathcal{S})$ on a site $\mathcal{S}$.

Let $u\colon \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let

$$u^s\colon Sh(\mathcal{D}) \to Sh(\mathcal{C})$$

be the functor u^p restricted to the subcategory of sheaves of sets (it is readily checked that $u^p\mathcal{F}$ is a sheaf on $\mathcal{C}$ if $\mathcal{F}$ is a sheaf on $\mathcal{D}$). We define

$$u_s\colon Sh(\mathcal{C}) \to Sh(\mathcal{D})$$

as the functor $\mathcal{G} \mapsto (u_p\mathcal{G})^{\#}$, where $(-)^{\#}$ denotes the sheafification functor.

Lemma 9.2.4. *In the above notation, the functor u_s is a left adjoint of u^s.*

Recall that a functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is called *exact* if $\mathcal{C}_1$ has all finite limits and colimits, and if F commutes with them.

Definition 9.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f\colon \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u\colon \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Definition 9.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be sites.

(1) A *morphism of topoi* $f\colon Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is given by a pair of functors $f_*Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1}\colon Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that f^{-1} is left adjoint to f_*, and such that f^{-1} is exact.

For example, let $u\colon \mathcal{C} \to \mathcal{D}$ be a continuous functor inducing a morphism of sites $f\colon \mathcal{D} \to \mathcal{C}$. Then the pair of functors

$$(f^{-1} := u_s, f_* := u^s) \tag{9.2}$$

is a morphism of topoi.

Proposition 9.2.7. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites $\mathcal{C}$ and $\mathcal{D}$. Assume that $\mathcal{C}$ has a final object X, that $u(X)$ is a final object of $\mathcal{D}$, that $\mathcal{C}$ has fibre products, and that u commutes with fibre products. The following assertions hold.*

(1) u defines a morphism of sites $f\colon \mathcal{D} \to \mathcal{C}$. In other words, u_s is exact.

(2) The pullback functor f^{-1} and the pushforward functor f_* (see (9.2)) extend to an adjoint pair

$$(f^{-1}\colon Ab(\mathcal{C}) \to Ab(\mathcal{D}), f_*\colon Ab(\mathcal{D}) \to Ab(\mathcal{C}))$$

of functors of sheaves of abelian groups. Moreover, these functors commute with taking the underlying sheaf of sets.

Examples 9.2.8. (1) Let $f\colon X \to Y$ be a continuous map of topological spaces. The functor

$$\mathrm{Op}(Y) \to \mathrm{Op}(X), \quad U \mapsto f^{-1}(U)$$

defines a morphism of sites $X_{Zar} \to Y_{Zar}$.

(2) Let X be a scheme. There is a natural morphism of sites $X_{\acute{e}t} \to X_{Zar}$.

9.2.2 Galois coverings

Let $\varphi\colon X \to Y$ be a morphism of schemes, and let G be a finite group. A right action of G on X over Y is a morphism $\alpha\colon G \to \mathrm{Aut}_X(Y)$ such that $\alpha(gh) = \alpha(h)\alpha(g)$. In other words, the morphism $\beta\colon G \to \mathrm{Aut}_X(Y)$ defined as $\beta(g) = \alpha(g)^{-1}$ is a group homomorphism (called a *left action of G on X over Y*), because

$$\beta(gh) = \alpha(gh)^{-1} = (\alpha(h)\alpha(g))^{-1} = \alpha(g)^{-1}\alpha(h)^{-1} = \beta(g)\beta(h).$$

Definition 9.2.9. Let $\varphi\colon X \to Y$ be a faithfully flat morphism of schemes, and let G be a finite group actin on X over Y on the right. The morphism $X \to Y$ is called a *Galois covering* if the morphism

$$\coprod_{g\in G} X = X \times G \to X \times_Y X, \quad (x, g) \mapsto (x, xg)$$

is an isomorphism; the group G is called the *Galois group* of the covering. This means that for each scheme T and each $x, y \in X(T)$ with $\varphi(x) = \varphi(y)$, there is a unique $g \in G(T)$ such that $xg = y$.

Lemma 9.2.10. *(1) If $\varphi\colon X \to Y$ is a Galois covering, then φ is surjective, and finite étale of degree equal to the order of G.*

(2) Conversely, if $\varphi\colon X \to Y$ surjective and finite étale of degree $\#G$, where $G = \mathrm{Aut}_Y(X)$, then the canonical right action of G on X over Y turns $X \to Y$ into a Galois covering with Galois group G.

Proof. See [Insert reference.] □

Remark 9.2.11. Galois coverings can be used for questions of *descent*. For instance, if $k \subset L$ is a finite Galois extension of fields, with Galois group G, let $\mathcal{C}_L^G$ be the category of quasi-projective L-schemes $Z \to \mathrm{Spec}\ L$ equipped with an action $G \to \mathrm{Aut}_L(Z)$ compatible with the action of G on L over k, and let $\mathcal{C}_k$ be the category of quasi-projective k-schemes $V \to \mathrm{Spec}\ k$. Then the functor $V \mapsto V \times_k L$ defines an equivalence of categories [reference]. One may think of this as an analogue of the classical fact that the category of k-vector spaces is in natural equivalence with the category of L-vector spaces equipped with a G-action compatible with the action of G on L over k.

Example 9.2.12. Let k be a field and let $f(T) \in k[T]$ be a monic irreducible polynomial of degree $n \geq 1$. Define $L = k[T]/(f(T))$. Then L is a field (because $k[T]$ is one-dimensional) and is called *separable* over k if $f(T)$ has no repeated roots in $\bar{k}$. Moreover, L is called *normal* over k if each embedding $L \to \bar{k}$ identifies L with the same subfield of $\bar{k}$, that is, for any such an embedding, every automorphism of $\bar{k}$ that fixes k maps L into itself, that is, if $f(T)$ splits into linear factors over L, that is, there is a k-algebra isomorphism

$$L \otimes_k L \cong \prod_{i=1}^{n} L.$$

Note for example that $\mathbb{Q}[T]/(T^3 - 2)$ is not normal over $\mathbb{Q}$ because there are three embeddings $\mathbb{Q}[T]/(T^3 - 2) \to \mathbb{C}$, one for which the image of T lies either in $\mathbb{R}$, and two others for which the image of T lies in $\mathbb{C} - \mathbb{R}$. The extension L/k is called *Galois* if L is normal and separable over k. If this is the case, then the morphism of schemes

$$\mathrm{Spec}\ L \to \mathrm{Spec}\ k$$

is a Galois covering, and its Galois group is $G = \mathrm{Aut}_k(L)$.

Lemma 9.2.13. *Let $X \to Y$ be a Galois covering, with Galois group G, and let F be a presheaf on $X_{\text{ét}}$ that takes disjoint unions to products. Then F satisfies the sheaf condition for the covering $X \to Y$ if and only if the restriction map $F(Y) \to F(X)$ is injective with image $F(X)^G$.*

Proof. Indeed, applying F to the diagram $X \times_Y X \rightrightarrows X \to Y$ shows that F satisfies the sheaf condition for the covering $X \to Y$ if and only if the diagram of sets

$$F(Y) \to F(X) \rightrightarrows \prod_{g \in G} F(X)$$

is exact. The lemma follows. □

9.3 Lecture 23: Étale cohomology groups

9.3.1 Definition of étale cohomology with values in an étale sheaf

Definition 9.3.1. Let $f\colon X \to Y$ be a morphism of schemes. We denote by $f_*\colon Sh(X_{ét}) \to Sh(Y_{ét})$ and $f^{-1}\colon Sh(Y_{ét}) \to Sh(X_{ét})$ the functors constructed in Definition 9.2.7.

Tracing through the definitions above shows that

$$f_*\mathcal{F}\colon Y_{ét}^{opp} \to \mathsf{Set} \quad \text{is defined as} \quad (V \to Y) \mapsto \mathcal{F}(X \times_Y V \to X),$$

and that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$(U \to X) \mapsto \varinjlim_{U \to X \times_Y V} \mathcal{G}(V \to Y).$$

Here, the colimit is over the category of pairs

$$(V \to Y, \varphi\colon (U \to X) \to (X \times_Y V \to X)).$$

Example 9.3.2. Let F be a sheaf on $X_{ét}$ for a scheme X. Let $u_{\bar{x}}\colon \bar{x} \to X$ be a geometric point of X. The *stalk* of F at $\bar{x}$, denoted by $F_{\bar{x}}$, is the abelian group $u_{\bar{x}}^{-1}(P)$. More explicitly, $F_{\bar{x}} = \varinjlim P(U)$ where the colimit is over all commutative triangles

$$\begin{tikzcd} U \arrow[d] & \bar{x} \arrow[l] \arrow[dl] \\ X & \end{tikzcd}$$

where $U \to X$ is étale; that is, over all étale neighbourhoods of $\bar{x}$ in X. Note that $F_{\bar{x}}$ is independent of the field $k(\bar{x})$ chosen.

For an étale morphism $U \to X$, a section $s \in F(U)$ and a geometric point $\bar{x} \to U$, we denote by $s_{\bar{x}}$ the image of s in $F_{\bar{x}}$ under the canonical morphism $F(U) \to F_{\bar{x}}$.

Recall that an object I of an abelian category is called *injective* if the functor $M \mapsto \mathrm{Hom}(M, I)$ is exact. For instance, an abelian group I is injective if for every two abelian groups A_i $(i = 1, 2)$ with $A_1 \subset A_2$ an abelian subgroup, the map $\mathrm{Hom}(A_2, I) \to \mathrm{Hom}(A_1, I)$ is surjective. Applying this to $n\mathbb{Z} \subset \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 2}$, we see that for every $g \in I$ there exists $h \in I$ such that $nh = g \in I$. As this holds for every n, we conclude that I is divisible. The converse is also true: divisible abelian groups are injective.

An abelian category $\mathcal{A}$ is said to have *enough injectives* if, for every object M in $\mathcal{A}$ there is an injective object $I \in \mathcal{A}$ and a monomorphism $M \to I$. If $\mathcal{A}$ has enough injectives and if $f\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor from $\mathcal{A}$ into a second abelian category $\mathcal{B}$, then there is an essentially unique sequence of functors $R^i f\colon \mathcal{A} \to \mathcal{B}$ $(i \geq 0)$, called the *right derived functors* of f, such that $R^0 f = f, R^i f(I) = 0$ if I is injective and $i > 0$, and for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{A}$, there exist morphisms $\delta^i\colon R^i f(M_3) \to R^{i+1} f(M_1)$ so that one obtains a long exact sequence

$$\cdots \to R^i f(M_1) \to R^i f(M_2) \to R^i f(M_3) \xrightarrow{\delta^i} R^{i+1} f(M_1) \to \cdots$$

in $\mathcal{B}$, and this association of a long exact sequence to a short exact sequence is functorial. An object M in $\mathcal{A}$ is *f-acyclic* if $R^i f(M) = 0$ for all $i > 0$. If $M \to N^\bullet$ is a resolution of M by f-acyclic objects N^i, then $R^i f(M)$ is canonically isomorphic to the i-th cohomology group of the complex $fN^\bullet$. We have the following:

Lemma 9.3.3. *Let F be a sheaf on $X_{\text{ét}}$. Let $U \to X$ be an étale morphism and let $s \in F(U)$. Suppose that for every geometric point $\bar{x} \to U$, the element $s_{\bar{x}} \in F_{\bar{x}}$ (see Example 9.3.2) is zero. Then $s = 0$.*

Proof. Let $u \in U$ and choose a geometric point $\bar{x} \to U$ with image $u \in U$. As $s \in F(U)$ maps to zero under $F(U) \to F_{\bar{x}}$, there exists an étale neighbourhood $V_{\bar{x}} \to U$ whose image contains u, such that $s|_{V_u} = 0 \in F(V_u)$. The family $\{V_u \to U\}_{u \in U}$ is an étale covering of U. Hence, $s = 0$ by the sheaf condition, and we win. □

Theorem 9.3.4. *Let X be a scheme. Then the category $\mathsf{Ab}(X_{\text{ét}})$ of abelian sheaves on the étale site $X_{\text{ét}}$ of X is an abelian category with enough injectives.*

Proof. We first prove that $\mathsf{Ab}(X_{ét})$ is abelian. Let $\mathsf{PAb}(X_{ét})$ be the category of abelian presheaves on $X_{ét}$. We claim that $\mathsf{PAb}(X_{ét})$ is abelian. Let P and P' be presheaves of abelian groups on $X_{ét}$. Then the presheaf Q with $Q(U) = P(U) \oplus P(U')$ and $Q(f) = P(f) \oplus P(f')$ is the direct sum of P and P' in $\mathsf{PAb}(X_{ét})$. As $\mathrm{Hom}(P, P')$ is naturally an abelian group, $\mathsf{PAb}(X_{ét})$ is an additive category. The kernel (resp. cokernel) of a morphism $\phi\colon P \to P'$ is the presheaf K (resp. Q) with $K(U) = \mathrm{Ker}(\phi(U))$ (resp. $\mathrm{Coker}(\phi(U))$). The direct sum (resp. product) of a family of presheaves is also given U-pointwise, for $U \to X$ étale; a sequence $P' \to P \to P''$ is exact if and only if the corresponding sequences $P'(U) \to P(U) \to P''(U)$ are exact for all $U \to X$ étale. We conclude that, indeed, $\mathsf{PAb}(X_{ét})$ is naturally an abelian category.

Now $\mathsf{Ab}(X_{ét})$, as a full subcategory of $\mathsf{PAb}(X_{ét})$, is an additive category. Let $\phi\colon F \to F'$ be a morphism in $\mathsf{Ab}(X_{ét})$. It remains to verify that $\mathrm{Coim}(\phi) \to \mathrm{Im}(\phi)$ is an isomorphism. The analogous statement is true in $\mathsf{PAb}(X_{ét})$ because the latter is abelian; the sheafification functor $(-)^{\#}\colon \mathsf{PAb}(X_{\acute{e}t}) \to \mathsf{Ab}(X_{\acute{e}t})$ defines an inverse $\mathrm{Hom}_{\mathsf{PAb}(X_{\acute{e}t})}(F, F') \to \mathrm{Hom}_{\mathsf{Ab}(X_{\acute{e}t})}(F, F')$ of the natural bijection $\mathrm{Hom}_{\mathsf{Ab}(X_{\acute{e}t})}(F, F') \to \mathrm{Hom}_{\mathsf{PAb}(X_{\acute{e}t})}(F, F')$. Thus $\mathrm{Coim}(\phi) = \mathrm{Im}(\phi)$ in $\mathsf{Ab}(X_{\acute{e}t})$.

To see that $\mathsf{Ab}(X_{\acute{e}t})$ has enough injectives, let $F \in \mathsf{Ab}(X_{\acute{e}t})$. For any geometric point $u_{\bar{x}}\colon \bar{x} \to X$, choose an embedding $u_{\bar{x}}^*(F) \hookrightarrow G_{\bar{x}}$ into an injective sheaf $G_{\bar{x}}$ on $k(\bar{x})$. Consider the composition

$$F \to \prod_{u_{\bar{x}}\colon \bar{x}\to X} u_{\bar{x},*}(u_{\bar{x}}^{-1}F) \to \prod_{u_{\bar{x}}\colon \bar{x}\to X} u_{\bar{x},*}(G_{\bar{x}}).$$

The first morphism in this composition is a monomorphism because of Lemma 9.3.3. The second morphism in this composition is a monomorphism because $u_{\bar{x},*}$ is left-exact. Hence the composition is a monomorphism. As the object $\prod_{u_{\bar{x}}\colon \bar{x}\to X} u_{\bar{x},*}(G_{\bar{x}})$ is injective because each $G_{\bar{x}}$ is injective by Lemma 9.3.5 below, we are done. □

Lemma 9.3.5. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. Suppose that f has an exact left adjoint $g\colon \mathcal{B} \to \mathcal{A}$. Then f preserves injectives.*

Proof. Let $I \in \mathcal{A}$ be injective. The functor $M \mapsto \mathrm{Hom}_{\mathcal{B}}(M, f(I))$ is isomorphic to $M \mapsto \mathrm{Hom}_{\mathcal{A}}(g(M), I)$. The latter is a composition of exact functors, hence exact. □

Definition 9.3.6. Let X be a scheme. Let $\Gamma(X_{\acute{e}t}, -)$ denote the left exact functor

$$\Gamma(X_{\acute{e}t}, -)\colon \mathsf{Ab}(X_{\acute{e}t}) \to \mathsf{Ab}, \quad F \mapsto \Gamma(X_{\acute{e}t}, F) = F(X).$$

The right derived functors of $\Gamma(X_{\acute{e}t}, -)$ are written

$$H^i_{\acute{e}t}(X, -) := R^i\Gamma(X_{\acute{e}t}, -).$$

For $F \in \mathsf{Ab}(X_{\acute{e}t})$, $H^i_{\acute{e}t}(X, F)$ is the *étale cohomology group* of X with coefficients in F.

9.3.2 Reformulation of the sheaf property

Let X be a scheme. Let

$$\mathcal{F}\colon X_{\acute{e}t} \to \mathsf{Set}$$

be a sheaf for the étale topology. For $U \in X_{Zar}$, the open immersion $U \to X$ is étale, and hence we obtain a well-defined set $\mathcal{F}(U)$. In this way, $\mathcal{F}$ induces a sheaf

$$\mathcal{F}|_{X_{Zar}}\colon X_{Zar} \to \mathsf{Set}.$$

Conversely, we have:

Proposition 9.3.7. *Let X be a scheme. Let Sch/X be the category of schemes over X. Let $\mathcal{F}$ be a contravariant functor $\mathsf{Sch}/X \to \mathsf{Set}$. Then the following are equivalent.*

(1) $\mathcal{F}$ is a sheaf for the étale (resp. fppf) topology;

(2) $\mathcal{F}$ is a sheaf for the Zariski topology and satisfies the sheaf property for every étale (resp. fppf) covering consisting of a single surjective étale (resp. flat, locally finitely presented) morphism of X-schemes $V \to U$, where U and V are affine.

The analogous statement holds for any presheaf $\mathcal{F}$ on the small étale site $X_{\text{ét}}$ of X.

In particular, the proposition says that a presheaf $\mathcal{F}$ on $X_{ét}$ is a sheaf if and only if the restriction $\mathcal{F}|_{X_{Zar}}$ to the small Zariski site X_{Zar} of X is a sheaf for the Zariski topology, and for every surjective étale morphism of X-schemes

$$V \to U,$$

where U and V are affine schemes which are étale over X, the diagram

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is exact. We point out that Proposition 9.3.7 has an analogon in the fpqc topology, see Proposition 9.3.8 below.

Proof of Proposition 9.3.7. Let $\tau \in \{\text{étale, fppf}\}$. Let $U \to X$ be a morphism of schemes. We consider a covering

$$\mathcal{U} = \{U_i \to U\}_{i\in I}$$

of U in the τ-topology. Define $U_{ij} = U_i \times_U U_j$; we need to prove that the sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{ij} \mathcal{F}(U_{ij}) \tag{9.3}$$

is exact.

Step 1. Claim: *Define $U' = \coprod_i U_i$. Then*

$$\mathcal{F}(U) \to \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U') \tag{9.4}$$

is exact if and only if (9.3) is exact.

Indeed, this follows from the fact that $\mathcal{F}$ is a sheaf in the Zariski topology.

Step 2. Claim: *Define $U' = \coprod U_i$. Suppose that U admits an open covering $U = \cup_{j\in J} V_j$ and that U' admits an open covering $U' = \cup_{j\in J, k\in K_j} V_{jk}$ that satisfy the following properties: V_{jk} maps into V_j for each k, and for each j, the family of morphisms with fixed target $\{V_{jk} \to V_j\}_{k\in K_j}$ is surjective (hence defines a τ-covering). Assume that for each $j \in J$, the sequence*

$$\mathcal{F}(V_j) \to \prod_{k\in K_j} F(V_{jk}) \rightrightarrows \prod_{k,k'} F(V_{jk} \times_{V_j} V_{jk'}) \tag{9.5}$$

is exact. Then (9.3) *is exact.*

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(U' \times_U U') \arrow[d] \\
\prod_j \mathcal{F}(V_j) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_j \prod_k \mathcal{F}(V'_{jk}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_j \prod_{k,k'} \mathcal{F}(V'_{jk} \times_{V_j} V'_{jk'}) \\
\prod_{j,j'} \mathcal{F}(V_j \cap V_{j'}) \arrow[r] & \prod_{j,j'} \prod_{k,k'} \mathcal{F}(V'_{jk} \cap V_{j',k'}) &
\end{tikzcd}$$

The two columns are exact because $\mathcal{F}$ is a sheaf in the Zariski topology. The middle horizontal sequence is exact because it is the product of the exact sequences (9.5). Hence, a diagram chase shows that (9.4) is exact. By Step 1, it follows that (9.3) is exact as desired.

Step 3. Finish the proof. Let $U' = \coprod_i U_i$ as above. Let $\pi \colon U' \to U$ be the canonical morphism. Then π is faithfully flat and locally of finite presentation. Consequently, π is open, see Proposition 9.1.13.

Write $U = \cup_{j \in J} V_j$ as a union of open affines. For each $j \in J$, write

$$\pi^{-1}(V_j) = \bigcup_{k \in K'_j} V_{jk}$$

as a union of open affines, for some index set K'_j, possibly infinite. Because $\pi \colon U' \to U$ is open, the sets $\pi(V_{jk}) \subset V_j$ are open in V_j. As V_j is affine, V_j is quasi-compact, hence finitely many of the open subsets $\pi(V_{jk})$ cover V_j. In other words, there exists a finite subset $K_j \subset K'_j$ such that

$$V_j = \bigcup_{k \in K_j} \pi(V_{jk}).$$

By Step 2, to prove that (9.3) is exact, it suffices to show that for each $j \in J$, the sequence (9.5) is exact. Define $V'_j \coloneqq \coprod_{k \in K_j} V_{kj}$. Then V'_j and V_j are affine and $V'_j \to V_j$ is surjective and étale. Moreover, by Step 1, to prove that (9.5) is exact, it suffices to show that

$$\mathcal{F}(V_j) \to \mathcal{F}(V'_j) \rightrightarrows \mathcal{F}(V'_j \times_{V_j} V'_j)$$

is exact for each j. This holds by assumption. The proposition follows. $\square$

Similarly, we have:

Proposition 9.3.8. *Let X be a scheme and let Sch/X be the category of schemes over X. Let $\mathcal{F}$ be a contravariant functor $\mathsf{Sch}/X \to \mathsf{Set}$. Then $\mathcal{F}$ is a sheaf for the fpqc topology if and only if it is a sheaf for the Zariski topology and satisfies the sheaf property for any faithfully flat morphism of X-schemes $V \to U$ with V, U affine.*

Proof. See [Stacks, Tag 022H]. $\square$

9.4 Lecture 24: Representable presheaves are sheaves

Definition 9.4.1. Let $\mathcal{C}$ be a category with fibre products. A morphism $Y \to X$ in $\mathcal{C}$ is called a *strict epimorphism* if the sequence

$$Y \times_X Y \rightrightarrows Y \to X$$

is exact. That is, for every $Z \in \mathcal{C}$, the sequence of sets

$$\mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times_X Y, Z)$$

is exact.

Examples 9.4.2. (1) If the category $\mathcal{C}$ has not only fibre products but also finite products, then the sequence of sets $\mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times_X Y, Z)$ is exact if and only if the sequence $\mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times Y, Z)$ is exact. Namely, the natural morphism $Y \times_X Y \to Y \times Y$ is a monomorphism.

(2) To prove that $Y \times_X Y \to Y \times Y$ is a monomorphism, observe that we have a cartesian diagram

$$\begin{tikzcd} Y \times_X Y \arrow[r] \arrow[d] & Y \times Y \arrow[d] \\ X \arrow[r] & X \times X. \end{tikzcd}$$

The natural diagonal morphism $\Delta \colon X \to X \times X$ is a monomorphism, because if $g_i \colon T \to X$ are morphisms $(i = 1, 2)$ with $\Delta(g_1) = \Delta(g_2)$, then $g_1 = \mathrm{pr}_i \Delta(g_1) = \mathrm{pr}_i \Delta(g_2) = g_2$. Here, $\mathrm{pr}_i \colon X \times X \to X$ is the projection map $(i \in \{1, 2\})$. Furthermore, the base change of a monomorphism is a monomorphism.

(3) Let M_1, M_2, M_3 be objects in an abelian category $\mathcal{A}$. Let f_1, f_2 be morphisms $M_1 \to M_2$, and consider a morphism $g \colon M_2 \to M_3$. Then

$$M_1 \rightrightarrows M_2 \xrightarrow{g} M_3$$

is exact if and only if

$$M_1 \xrightarrow{f_1 - f_2} M_2 \xrightarrow{g} M_3 \to 0$$

is exact, if and only if for any $Z \in \mathcal{A}$, the sequence of abelian groups

$$0 \to \mathrm{Hom}(M_3, Z) \xrightarrow{g_*} \mathrm{Hom}(M_2, Z) \xrightarrow{(f_1 - f_2)_*} \mathrm{Hom}(M_1, Z)$$

is exact, if and only if for any $Z \in \mathcal{A}$, the sequence of sets

$$\mathrm{Hom}(M_3, Z) \xrightarrow{g_*} \mathrm{Hom}(M_2, Z) \rightrightarrows \mathrm{Hom}(M_1, Z)$$

is exact.

(4) Let $f: A \to B$ be a faithfully flat morphism of rings. Then

$$\mathrm{Spec}(f)\colon \mathrm{Spec}\ B \to \mathrm{Spec}\ A$$

is a strict epimorphism in the category of affine schemes. Consider the maps $e_i\colon B \to B \otimes_A B$ defined as $e_0(b) = 1 \otimes b$ and $e_1(b) = b \otimes 1$. To prove that $\mathrm{Spec}(f)$ is a strict epimorphism in the category of affine schemes, it suffices to show that the sequence of A-modules

$$0 \to A \xrightarrow{d^{(-1)}} B \xrightarrow{d^{(0)}} B \otimes_A B, \qquad d^{(-1)} := f, \quad d^{(0)} := e_0 - e_1, \tag{9.6}$$

is exact as a sequence of A-modules. Namely, if such is the case, then for any A-module M the sequence

$$\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, B) \rightrightarrows \mathrm{Hom}_A(M, B \otimes_A B)$$

is exact by item (3).

(5) Let us prove that (9.6) is exact when there exists a ring homomorphism $g\colon B \to A$ such that $gf = \mathrm{id}$. Define $k_0\colon B^{\otimes 2} \to B$ by $k_0(b_0 \otimes b_1) = g(b_0)b_1$. We claim that

$$k_0 \circ d^{(0)} + d^{(-1)} \circ g = \mathrm{id} \qquad \text{as maps} \qquad B \to B. \tag{9.7}$$

Namely, let $b \in B$. Then $\left(k_0 \circ d^{(0)} + d^{(-1)} \circ g\right)(b) =$

$$\left(k_0 \circ d^{(0)} + f \circ g\right)(b) = k_0\left(1 \otimes b - b \otimes 1\right) + f\left(g(b)\right) = b - f(g(b)) + f(g(b)) = b.$$

The existence of a tuple (g, k_0) satisfying (9.7) implies that (9.6) is exact. Indeed, it is clear that $d^{(0)} \circ d^{(-1)} = 0$, which shows that $\mathrm{Im}(d^{(-1)}) \subset \mathrm{Ker}(d^{(0)})$; conversely, equation (9.7) implies that $\mathrm{Ker}(d^{(0)}) \subset \mathrm{Im}(d^{(-1)})$.

(6) To prove that (9.6) is exact without the assumption that a retraction $g\colon B \to A$ of f exists, let $A \to A'$ be a faithfully flat morphism of rings, and define $B' = B \otimes_A A'$. As $A \to A'$ is faithfully flat, we have that

$$0 \to A' \xrightarrow{d^{(-1)}} B' \xrightarrow{d^{(0)}} B' \otimes_{A'} B', \tag{9.8}$$

is exact if and only if (9.6) is exact. Define $A' = B$. Then $A' \to B'$ has a retraction because $B' = B \otimes_A B$: we may define it as $g\colon B \otimes_A B \to B, g(b \otimes b') = bb'$. Thus, by what we have seen in item (5), the sequence (9.8) - and hence also (9.6) - is exact. We conclude that $\mathrm{Spec}\ B \to \mathrm{Spec}\ A$ is a strict epimorphism in the category of affine schemes, whenever $A \to B$ is a faithfully flat ring map.

The goal of Section 9.4 is to prove the following result.

Theorem 9.4.3. *Let X and Y be schemes. Let $f\colon Y \to X$ be a faithfully flat morphism of schemes. Assume that f is quasi-compact, or that f is locally of finite presentation. Then $f\colon Y \to X$ is a strict epimorphism in the category of schemes.*

9.4.1 Preliminary results

To prove Theorem 9.4.3, we need two lemmas and a proposition.

Lemma 9.4.4. *Let $f\colon Y \to X$ be a faithfully flat, quasi-compact morphism of schemes. A subset $T \subset Y$ is open (resp. closed) if and only $f^{-1}(T)$ is open (resp. closed).*

Proof. See [SGA1, Exposé VIII, Corollaire 4.3] and [EGAIV, Corollaire 2.3.12]. $\square$

Lemma 9.4.5. *Let $A \to B$ be a faithfully flat morphism of rings. Then the morphism* Spec $(B) \to$ Spec (A) *is an epimorphism in the category of schemes.*

Proof. Define $X = \text{Spec } A$ and $Y = \text{Spec } B$ and let f be the morphism $Y \to X$. Let Z be a scheme and let $h\colon Y \to Z$ be a morphism of schemes. Assume $g_i\colon X \to Z$ $(i = 1, 2)$ are maps such that $g_1 f = g_2 f$. As $f\colon Y \to X$ is surjective, we must have $g_1 = g_2$ as maps $|X| \to |Z|$ on the underlying topological spaces. Let $x \in X$ and let

$$U \subset Z, \quad U = \text{Spec } (C),$$

be an affine open neighbourhood of $g_1(x) = g_2(x)$ in Z. Let $a \in A$ such that $X_a \subset g_1^{-1}(U) \cap g_2^{-1}(U)$ is an affine open neighbourhood of $x \in g_1^{-1}(U) \cap g_2^{-1}(U) \subset X$. Then $g_1(X_a) = g_2(X_a) \subset U$. Let $b \in B$ be the image of a. Note that we are given two maps

$$g_i|_{X_a}\colon X_a = \text{Spec } (A_a) \to U = \text{Spec } C$$

such that $g_1|_{X_a} \circ f = g_2|_{X_a} \circ f$. By item (4) in Examples 9.4.2, observing that the morphism $A_a \to B_a$ is faithfully flat, it follows that $g_1|_{X_a} = g_2|_{X_a}$ as morphisms $X_a \to Z$. As $x \in X$ was arbitrary, we conclude that $g_1 = g_2$ as morphisms $X \to Z$. $\square$

Proposition 9.4.6. *Let $A \to B$ be a faithfully flat ring map. Then* Spec $B \to$ Spec A *is a strict epimorphism in the category of schemes.*

Proof of Proposition 9.4.6. Define $X = \text{Spec } A, Y = \text{Spec } B$ and let f be the morphism $Y \to X$. Let Z be a scheme and let $h\colon Y \to Z$ be a morphism of schemes such that $hp_1 = hp_2$, where $p_i\colon Y \times_X Y \to Y$ are the two projection maps. We need to prove the following statement:

$$\text{There exists a unique morphism } g\colon X \to Z \text{ such that } gf = h. \tag{9.9}$$

Let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$ so that $X = \text{Spec } A$ and $Y = \text{Spec } B$. We claim that (9.9) holds. To prove this, note that the uniqueness of g follows from Lemma 9.4.5. Consider the induced commutative diagram of topological spaces:

$$\begin{tikzcd}
 & {|Y| \times_{|X|} |Y|} \arrow[dr, shift left] \arrow[dr, shift right] & & \\
{|Y \times_X Y|} \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & {|Y|} \arrow[r, "|f|"] \arrow[d, "|h|"] & {|X|} \\
 & & {|Z|\,.} &
\end{tikzcd}$$

By [Stacks, Tag 01JT], the map $|Y \times_X Y| \to |Y| \times_{|X|} |Y|$ is surjective. Therefore, the two compositions

$$|Y| \times_{|X|} |Y| \rightrightarrows |Y| \to |Z|$$

are the same map $|Y| \times_{|X|} |Y| \to |Z|$. As the map $|Y| \to |X|$ is surjective, there exists a function of sets $|g| : |X| \to |Z|$ such that $|g| \circ |f| = |h|$ as maps $|Y| \to |Z|$.

We claim that $|g|$ is continuous. To prove this, let $T \subset |Z|$ be an open subset of $|Z|$. Then

$$|f|^{-1} |g|^{-1} (T) = |h|^{-1} (T) \subset |Y| ,$$

which is open in $|Y|$ because $|h|$ is continuous. Hence, by Lemma 9.4.4, the set $|g|^{-1} (T)$ is open in $|X|$. This proves that $|g|$ is continuous, as claimed.

We have constructed a continuous map of topological spaces $|g| : |X| \to |Z|$ such that $|g| |f| = |h|$. Our goal is to construct a morphism of schemes $g \colon X \to Z$ such that $gf = h$. By the uniqueness of such a map g, which is something which we have already proved, it suffices to construct g locally. Let $x \in X$ and $y \in f^{-1}(x)$, and let

$$h(y) \in U = \text{Spec } (C) \subset Z$$

be an affine open neighbourhood of $h(y)$ in Z. Let $a \in A$ such that

$$x \in \text{Spec } (A_a) = X_a \subset |g|^{-1} (U) \subset X.$$

As $|g| |f| = |h|$ and $X_a \subset |g|^{-1} (U)$, we have

$$h \left(f^{-1}(X_a)\right) = |g| |f| \left(|f|^{-1} (X_a)\right) \subset |g| (X_a) \subset |g| \left(|g|^{-1} (U)\right) \subset U. \tag{9.10}$$

Let $b \in B$ be the image of $a \in A$, and consider the open $Y_b = \text{Spec } B_b \subset Y$. Note that $Y_b \subset f^{-1}(X_a)$. In view of (9.10), the morphism

$$h|_{Y_b} \colon Y_b \to Z$$

factors through the affine open subscheme $U \subset Z$. Hence, by item (4) in Examples 9.4.2, we conclude that there exists a unique morphism

$$g_a \colon X_a \to U \quad \text{such that} \quad g_a \circ \left(Y_b \xrightarrow{f} X_a\right) = h|_{Y_b} \quad \text{as morphisms} \quad Y_b \to U.$$

By unicity, see Lemma 9.4.5, these morphisms $g_a \colon X_a \to Z$ glue to give a morphism $g \colon X \to Z$. We have proven that (9.9) holds. The proposition follows. □

9.4.2 Proof of Theorem 9.4.3

With the results of Section 9.4.1 in place, we can now proceed with the:

Proof of Theorem 9.4.3. Let $f \colon Y \to X$ be a morphism of schemes which is faithfully flat and quasi-compact. Let Z be a scheme and let $h \colon Y \to Z$ be a morphism of schemes such that $hp_1 = hp_2$, where $p_i\colon Y \times_X Y \to Y$ are the two projection maps. We claim:

$$\text{There exists a unique } g\colon X \to Z \text{ such that } gf = h. \tag{9.11}$$

Consider the morphism $h\colon Y \to Z$. Let $U \subset X$ be an affine open subscheme. Put $V = f^{-1}(U) \subset Y$. Then

$$V \times_U V = (Y \times_X Y) \times_X U \quad \text{with projections} \quad p_i|_U\colon V \times_U V \to V,$$

and $hp_1|_U = hp_2|_U$ as morphisms $V \times_U V \to Z$. If the theorem holds whenever X is affine, then there exists a unique morphism

$$g_U\colon U \to Z \quad \text{such that} \quad g_U \circ f = h|_V \quad \text{as morphisms} \quad V \to Z.$$

By unicity, these g_U glue to a morphism $g\colon X \to Z$ such that $g \circ f = h$. This shows that, in order to prove (9.11), we may and do assume that X is affine. As f is quasi-compact, it follows that Y is quasi-compact, hence $Y = Y_1 \cup \cdots \cup Y_n$ is a finite union of affine opens $Y_i \subset Y$. Define

$$Y^* := Y_1 \coprod \cdots \coprod Y_n.$$

Then Y^* is affine and the obvious morphism $Y^* \to X$ is faithfully flat and of finite type. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(X,Z) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Y,Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Hom}(Y \times_X Y, Z) \arrow[d] \\
\mathrm{Hom}(X,Z) \arrow[r] & \mathrm{Hom}(Y^*,Z) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(Y^* \times_X Y^*, Z).
\end{tikzcd}$$

The lower horizontal row is exact by Step 3. The middle horizontal arrow is injective. A diagram chase shows that the top horizontal row is exact, and (9.11) follows.

It remains to show that, for any scheme Z and any faithfully flat locally finitely presented morphism of schemes $f\colon Y \to X$, the diagram of sets

$$\mathrm{Hom}(X,Z) \to \mathrm{Hom}(Y,Z) \rightrightarrows \mathrm{Hom}(Y \times_X Y, Z)$$

is exact. In other words, if we define h_Z as the contravariant functor

$$h_Z = \mathrm{Hom}(-,Z)\colon \mathsf{Sch} \to \mathsf{Set},$$

then we need to show that h_Z satisfies the sheaf property for any $\{Y \to X\}$ with X, Y schemes and $Y \to X$ faithfully flat and locally of finite presentation. Note that h_Z is a sheaf in the Zariski topology (this is classical, see e.g. [GW20, Proposition 3.5]). Therefore, by Proposition 9.3.7, it suffices to prove that the functor h_Z satisfies the sheaf property for any $\{Y \to X\}$ with X, Y affine schemes and $Y \to X$ faithfully flat and locally of finite presentation. In other words, we may assume that X and Y are affine. But then the result follows from Proposition 9.4.6. $\square$

9.4.3 Group schemes

For the notion of group scheme, see [SGA3, Exposé I]. A nice introduction to group schemes is given in [Tat97]. Here, we briefly recall the basic definitions.

Let X be a scheme. Let Sch/X be the category of schemes over X, and let Grp be the category of groups. Let Set be the category of sets. We let

$$\mathsf{Forget}\colon \mathsf{Grp} \to \mathsf{Set}$$

be the functor that sends a group to its underlying set. For a scheme Y over X, we let h_Y denote the contravariant functor

$$h_Y\colon \mathsf{Sch}/X \to \mathsf{Set}, \quad h_Y(T) = \mathrm{Hom}_X(T, Y),$$

and for any scheme T over X, we define

$$Y(T) \coloneqq h_Y(T) = \mathrm{Hom}_X(T, Y).$$

Definition 9.4.7. A *group scheme* over X is a scheme G over X and a contravariant functor $F\colon \mathsf{Sch}/X \to \mathsf{Grp}$ such that $\mathsf{Forget} \circ F = h_G$.

In concrete terms, to give a group scheme G over X is to define a group structure on the set $G(T) = \mathrm{Hom}_X(T, G)$ for every scheme T over X, in such a way that if $T' \to T$ is a morphism of schemes over X, then the induced map $G(T) \to G(T')$ is a group homomorphism. A third equivalent notion is given by the data of an X-scheme G and a set of morphisms of X-schemes

$$m\colon G \times_X G \to G, \quad \varepsilon\colon S \to G, \quad \mathrm{inv}\colon G \to G,$$

such that the obvious diagrams commute (see [Tat97, (1.1)–(1.4)]).

Theorem 9.4.3 implies:

Corollary 9.4.8. *Let X be a scheme. Let G be a commutative group scheme over X. Then the presheaf $h_G\colon (\mathsf{Sch}/X)^{\mathrm{opp}} \to \mathsf{Set}$ that G defines is a sheaf in the fpqc topology (cf. Definition 9.1.29). A fortiori, h_G is a sheaf in the fppf and étale topologies.*

Proof. As h_G is a sheaf in the Zariski topology (see e.g. [GW20, Proposition 3.5]), in view of Proposition 9.3.8, it suffices to show that h_G satisfies the sheaf property for any covering $\{V \to U\}$ where V and U are X-schemes which are affine, and where $V \to U$ is a faithfully flat morphism of X-schemes. This follows from Theorem 9.4.3. □

Examples 9.4.9. Let X be a scheme. Let t be a variable.

(1) The scheme $\mathbb{G}_{a,X} = \mathrm{Spec}\ \mathcal{O}_X[t]$ underlies a commutative group scheme over X, whose functor of points is defined as $\mathbb{G}_{a,X}(T) = \mathcal{O}_T(T)$.

(2) The scheme $\mathbb{G}_{m,X} = \mathrm{Spec}\ \mathcal{O}_X[t, t^{-1}]$ underlies a commutative group scheme over X, whose functor of points is defined as $\mathbb{G}_{m,X}(T) = \mathcal{O}_T(T)^*$.

(3) Let $n \in \mathbb{Z}_{\geq 2}$. The scheme $\mu_{n,X} = \mathrm{Spec}\ \mathcal{O}_X[t]/(t^n - 1)$ underlies a commutative group scheme over X, with functor of points $\mu_{n,X}(T) = \{x \in \mathcal{O}_T(T) \mid x^n = 1\}$.

9.5 Lecture 25: Fundamental theorems in étale cohomology

Chapter 10

Cubic surfaces

10.1 Lecture 26: Cubic surfaces over finite fields

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