

A note on descent for algebraic stacks

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1. Introduction

Let $S' \rightarrow S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor $X \mapsto X \times_S S'$ defines an equivalence of categories between the category of S -schemes X and the category of pairs (X', ϕ) where X' is an S' -scheme and ϕ a descent datum for X' over S' such that X' admits an open covering by affine schemes which are stable under ϕ . In case $S = \operatorname{Spec}(k)$, $S' = \operatorname{Spec}(k')$ and the morphism $S' \rightarrow S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil.

The goal of this note is to prove a similar statement for algebraic stacks. In the case of stacks, the analogue of the aforementioned descent-theory is a notion called *2-descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes $S' \rightarrow S$ which is smooth and surjective, *every 2-descent datum for an algebraic stack is effective*. More precisely, we have the following result. For a scheme S , let $(\operatorname{Sch}/S)_{\text{fppf}}$ be the big fppf site of S as in [Stacks, Tag 021S]; a *stack over S* is a stack in groupoids $\mathcal{X} \rightarrow (\operatorname{Sch}/S)_{\text{fppf}}$ over $(\operatorname{Sch}/S)_{\text{fppf}}$, see [Stacks, Tag 0304].

THEOREM 1.1. *Let $S' \rightarrow S$ be a faithfully flat morphism of schemes locally of finite presentation, and let \mathcal{X}' be a stack over S' . Let (ϕ, ψ) be a 2-descent datum for the stack \mathcal{X}' over S' , see Definition 3.1. Then (ϕ, ψ) is effective. That is, there exists a stack \mathcal{X} over S , an isomorphism of stacks over S'*

$$\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

and a 2-isomorphism $\chi: p_2^ f \circ \text{can} \Rightarrow \phi \circ p_1^* f$ as in the following diagram:*

$$(1.1) \quad \begin{array}{ccc} p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\text{can}} & p_2^*(\mathcal{X} \times_S S') \\ \downarrow p_1^* \rho & \swarrow \phi & \downarrow p_2^* \rho \\ p_1^* \mathcal{X}' & \xrightarrow{\quad} & p_2^* \mathcal{X}', \end{array}$$

such that the natural compatibility between χ and ψ is satisfied. Moreover, if $S' \rightarrow S$ is smooth, then \mathcal{X}' is an algebraic stack over S' if and only if \mathcal{X} is an algebraic stack over S , and if $S' \rightarrow S$ is étale, then \mathcal{X}' is a Deligne–Mumford stack over S' if and only if \mathcal{X} is a Deligne–Mumford stack over S .

Note that even the case where \mathcal{X}' is a scheme seems to yield a non-trivial result (cf. Corollary 3.4).

Item (1) in the above theorem follows readily from the fact that the 2-fibred category \underline{Stack}_S over $(\text{Sch}/S)_{fppf}$, whose fibre over $U \in (\text{Sch}/S)_{fppf}$ is the category $\underline{Stack}(U)$ of stacks over U , is a 2-stack over S (see e.g. [Bre94, Example 1.11.(1)]). Items (2) and (3) follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the fppf (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case $S' \rightarrow S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ , and \mathcal{X}' a stack over S' , one can reformulate the notion of 2-descent datum for \mathcal{X}' over S' in terms of an action of Γ on \mathcal{X}' over the action of Γ on S' over S , as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define ${}^\sigma\mathcal{X}'$ as the pull-back of \mathcal{X}' along $\sigma: S' \rightarrow S'$.

DEFINITION 1.2. *Let $S' \rightarrow S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ . Let \mathcal{X}' be a stack over S' . A Galois 2-descent datum consists of:*

- (1) *a set of 1-isomorphisms $f_\sigma: {}^\sigma\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ ($\sigma \in \Gamma$);*
- (2) *a set of 2-isomorphisms $\psi_{\sigma,\tau}: f_\sigma \circ {}^\sigma(f_\tau) \Rightarrow f_{\sigma\tau}$ ($\sigma, \tau \in \Gamma$);*

such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms

$$\begin{array}{ccc} f_\sigma \circ {}^\sigma f_\tau \circ {}^{\sigma\tau} f_\gamma & \Longrightarrow & f_{\sigma\tau} \circ {}^{\sigma\tau} f_\gamma \\ \Downarrow & & \Downarrow \\ f_\sigma \circ {}^\sigma f_{\tau\gamma} & \Longrightarrow & f_{\sigma\tau\gamma} \end{array}$$

is commutative.

One can show that to give a Galois 2-descent datum on \mathcal{X}' over S' is to give a group action (in the sense of [Rom05]) of Γ on \mathcal{X}' as a stack over S , such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \rightarrow S'$ agrees with the composition $\mathcal{X}' \rightarrow S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for \mathcal{X}' over S' , see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

THEOREM 1.3. *Let $S' \rightarrow S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ . Let \mathcal{X}' be an algebraic stack over S' , equipped with a Galois 2-descent datum $(f_\sigma$ ($\sigma \in \Gamma$), $\psi_{\sigma,\tau}$ ($\sigma, \tau \in \Gamma$)). There exists an algebraic stack \mathcal{X} over S and an isomorphism $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over S' . The stack \mathcal{X} is Deligne–Mumford if and only if \mathcal{X}' is.*

Observe that the statement in Theorem 1.3 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an isomorphism of stacks $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ over S' as well as a family of 2-isomorphisms $\chi_\sigma: \rho \circ \text{can} \Rightarrow f_\sigma \circ {}^\sigma\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$\begin{array}{ccc} {}^\sigma(\mathcal{X} \times_S S') & \xrightarrow{\text{can}} & \mathcal{X} \times_S S' \\ \downarrow & \Downarrow & \downarrow \\ {}^\sigma\mathcal{X}' & \longrightarrow & \mathcal{X}', \end{array}$$

such that the obvious compatibility conditions are satisfied.

2. Descending schemes

Let

$$p: S' \rightarrow S$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times_{S'} S' \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{matrix} S' \rightarrow S,$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$S''' \rightrightarrows S'' \rightrightarrows S' \rightarrow S$$

where the three arrows $S''' \rightarrow S''$ are p_{12} , p_{13} and p_{23} .

Let X' be a scheme over S' . Define

$$p_i^* X' = X' \times_{S', p_i} S'', \quad p_{jk}^* p_i^* X' = (p_i^* X') \times_{S'', p_{jk}} S'''$$

and note that

$$p_{jk}^* p_i^* X' = (p_i^* X') \times_{S'', p_{jk}} S''' = (p_i \circ p_{jk})^* X'.$$

Recall that a *descent datum* for X'/S' is an S'' -isomorphism

$$\phi: p_1^* X' \xrightarrow{\sim} p_2^* X'$$

such that the following diagram commutes:

$$\begin{array}{ccc} p_{12}^* p_1^* X' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* X' = p_{23}^* p_1^* X' \xrightarrow{p_{23}^* \phi} p_{23}^* p_2^* X' \\ \parallel & & \parallel \\ p_{13}^* p_1^* X' & \xrightarrow{p_{13}^* \phi} & p_{13}^* p_2^* X'. \end{array}$$

In other words, one requires that

$$p_{23}^* \phi \circ p_{12}^* \phi = p_{13}^* \phi \quad \text{as morphisms} \quad p_{12}^* p_1^* X' \rightarrow p_{13}^* p_2^* X'.$$

THEOREM 2.1 (Grothendieck). *Let $p: S' \rightarrow S$ be a faithfully flat locally finitely presented morphism of affine schemes. The functor $X \mapsto p^* X$ defines an equivalence of categories between the category of S -schemes X and the category of pairs (X', ϕ) where X' is an S' -scheme and ϕ a descent datum for X'/S' such that X' admits an open covering by affine schemes stable under ϕ .*

Next, recall how to make this explicit in case $S' \rightarrow S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ . For instance, S could be the spectrum of a field k , S' the spectrum of a finite field extension $k' \supset k$, and Γ the Galois group of k'/k . Let X' be a scheme over S' and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma: {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over S' , for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau} X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of Γ on X' as a scheme over S is said to be *compatible with the action of Γ on S' over S* if for each $\sigma \in \Gamma$, the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma} & S'. \end{array}$$

LEMMA 2.2. *Let $S' \rightarrow S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ , and let X' be a scheme over S' . To give a descent datum for X' over S' is to give a Galois descent datum for X' over S' . These notions are further equivalent to giving an action of Γ on X' compatible with the action of Γ on S' over S .*

PROOF. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Pool17, Proposition 4.4.4]. \square

3. Descending algebraic stacks

Let $p: S' \rightarrow S$ be a faithfully flat locally finitely presented morphism of schemes. Let \mathcal{X}' be a stack in groupoids on S' , in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$(3.1) \quad r_i: S'''' \rightarrow S'.$$

Similarly, S''' is equipped with three projections $q_i: S''' \rightarrow S'$. Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on S' . One has similar isomorphisms relating the other $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$ with $r_\mu^* \mathcal{X}'$, for $i, j, k \in \{1, 2, 3, 4\}$, $\alpha, \beta \in \{1, 2, 3\}$, $\nu \in \{1, 2\}$ and $\mu \in \{1, 2, 3, 4\}$.

Consider an isomorphism of S'' -stacks (i.e. an equivalence of Sch/S'' -categories):

$$\phi: p_1^* \mathcal{X}' \rightarrow p_2^* \mathcal{X}',$$

and let ψ be a 2-morphism

$$\psi: p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism \Rightarrow in the following diagram:

$$(3.2) \quad \begin{array}{ccccc} p_{12}^* p_1^* \mathcal{X}' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* \mathcal{X}' & \xlongequal{\quad} & p_{23}^* p_1^* \mathcal{X}' & \xrightarrow{p_{23}^* \phi} & p_{23}^* p_2^* \mathcal{X}' \\ \parallel & & & \nearrow \psi & & & \parallel \\ p_{13}^* p_1^* \mathcal{X}' & \xrightarrow{\quad} & & p_{13}^* \phi & \xrightarrow{\quad} & & p_{13}^* p_2^* \mathcal{X}'. \end{array}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234}: S'''' \rightarrow S''',$$

and note that

$$p_{123}^* (p_{23}^* \phi \circ p_{12}^* \phi) = p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and} \\ p_{123}^* p_{13}^* \phi = \pi_{13}^* \phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}: S'''' \rightarrow S''$$

are the canonical morphisms. For $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$, define

$$\psi_{ijk} := p_{ijk}^* \psi.$$

For instance, pulling back ψ along p_{123} gives a 2-morphism

$$\psi_{123} = p_{123}^* \psi: \pi_{23}^* \phi \Rightarrow \pi_{13}^* \phi.$$

Similarly, we obtain 2-morphisms

$$\begin{aligned} \psi_{124}: \pi_{24}^* \phi \circ \pi_{12}^* \phi &\Rightarrow \pi_{14}^* \phi, \\ \psi_{134}: \pi_{34}^* \phi \circ \pi_{13}^* \phi &\Rightarrow \pi_{14}^* \phi, \\ \psi_{234}: \pi_{34}^* \phi \circ \pi_{23}^* \phi &\Rightarrow \pi_{24}^* \phi. \end{aligned}$$

Moreover, observe that under p_{123} , diagram (3.2) pulls back to the diagram

$$(3.3) \quad \begin{array}{ccccc} r_1^* \mathcal{X}' & \xrightarrow{\pi_{12}^* \phi} & r_2^* \mathcal{X}' & \xlongequal{\quad} & r_2^* \mathcal{X}' & \xrightarrow{\pi_{23}^* \phi} & r_3^* \mathcal{X}' \\ \parallel & & & \Downarrow & & & \parallel \\ r_1^* \mathcal{X}' & \xrightarrow{\quad} & & \xrightarrow{\pi_{13}^* \phi} & & & r_3^* \mathcal{X}', \end{array}$$

in which the 2-morphism \Rightarrow is the 2-morphism ψ_{123} defined above (and with r_i is as in (3.1)). Using pull-backs by the other three $p_{ijk}: S'''' \rightarrow S'''$, we thus obtain four triangles, that we may put together to form the following tetrahedron:

$$(3.4) \quad \begin{array}{ccccc} & & r_1^* \mathcal{X}' & & \\ & \swarrow & \downarrow & \searrow & \\ r_2^* \mathcal{X}' & \xleftarrow{\quad} & & \xrightarrow{\quad} & r_4^* \mathcal{X}' \\ & \searrow & \downarrow & \swarrow & \\ & & r_3^* \mathcal{X}' & & \end{array}$$

DEFINITION 3.1. *Let $p: S' \rightarrow S$ be a faithfully flat locally finitely presented morphism of schemes. Let \mathcal{X}' be a stack in groupoids over S' . A 2-descent datum for \mathcal{X}' over S' consists of:*

- (1) *an isomorphism of stacks (i.e. an equivalence of categories)*

$$\phi: p_1^* \mathcal{X}' \rightarrow p_2^* \mathcal{X}'$$

over S'' ;

(2) a 2-isomorphism

$$\psi: p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi$$

as in diagram (3.2);

such that the following condition is satisfied: the 2-morphisms ψ_{ijk} between the several compositions in diagram (3.4) are compatible, in the sense that the following diagram of 2-morphisms commutes:

$$\begin{array}{ccc} \pi_{34}^* \phi \circ \pi_{23}^* \phi \circ \pi_{12}^* \phi & \xRightarrow{(\pi_{34}^* \phi)_*(\psi_{123})} & p_{34}^* \phi \circ p_{13}^* \phi \\ \Downarrow (\pi_{12}^* \phi)^*(\psi_{234}) & & \Downarrow \psi_{134} \\ p_{24}^* \phi \circ p_{12}^* \phi & \xRightarrow{\psi_{124}} & p_{14}^* \phi. \end{array}$$

This gives the following result.

PROPOSITION 3.2 (Breen). *Let (ϕ, ψ) be a 2-descent datum for the stack \mathcal{X}' over S' . Then there exists a stack \mathcal{X} over S , an isomorphism*

$$\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over S' , and a 2-isomorphism $\chi: p_2^ \rho \circ \text{can} \Rightarrow \phi \circ p_1^* \rho$ as in diagram*

$$(3.5) \quad \begin{array}{ccc} p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\text{can}} & p_2^*(\mathcal{X} \times_S S') \\ \downarrow p_1^* \rho & \not\cong & \downarrow p_2^* \rho \\ p_1^* \mathcal{X}' & \xrightarrow{\phi} & p_2^* \mathcal{X}', \end{array}$$

such that the natural compatibility condition between χ and ψ is satisfied.

PROOF. This follows from [Bre94, Example 1.11.(i)]. \square

To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-lemma, which should be well-known but which we include for convenience of the reader.

LEMMA 3.3. *Let S be a scheme. The following assertions are true.*

- (1) *Let $\pi: \mathcal{X}' \rightarrow \mathcal{X}$ be a representable, smooth and surjective morphism of stacks in groupoids over S . If \mathcal{X}' is algebraic, then \mathcal{X} is algebraic. If in addition π is étale and \mathcal{X}' is Deligne–Mumford, then \mathcal{X} is Deligne–Mumford.*
- (2) *Let $S' \rightarrow S$ be a smooth surjective morphism of schemes, let \mathcal{X} be a stack in groupoids over S and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Suppose that \mathcal{X}' is an algebraic stack over S' . Then \mathcal{X} is an algebraic stack over S . If in addition $S' \rightarrow S$ is étale and \mathcal{X}' is a Deligne–Mumford stack, then \mathcal{X} is a Deligne–Mumford stack.*

PROOF. Let us first prove item (1). If U' is a scheme and $U' \rightarrow \mathcal{X}'$ a surjective and smooth morphism, then $U' \rightarrow \mathcal{X}' \rightarrow \mathcal{X}$ is surjective and smooth, and moreover étale if π and $U' \rightarrow \mathcal{X}'$ are étale. Therefore, it suffices to prove

that the diagonal $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. For this, it suffices to consider to schemes U and V , equipped with morphisms $U \rightarrow \mathcal{X}$ and $V \rightarrow \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:

$$\begin{array}{ccccc}
 & & U' \times_{\mathcal{X}'} V' & & \\
 & \swarrow & \downarrow & \searrow & \\
 V' & & & & U \times_{\mathcal{X}} V \\
 & \swarrow & \downarrow & \searrow & \\
 & & U' & & V \\
 & \swarrow & \downarrow & \searrow & \\
 \mathcal{X}' & & & & U \\
 & \swarrow & \downarrow & \searrow & \\
 & & \mathcal{X} & &
 \end{array}$$

The morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is representable, hence U' and V' are representable by algebraic spaces. Since \mathcal{X}' is an algebraic stack, the morphism $V' \rightarrow \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \rightarrow U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \rightarrow U$ is étale and surjective, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that the morphism $U \times_{\mathcal{X}} V \rightarrow U$ is representable. As U is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, and we are done.

Next, we prove item (2). Via the composition $\mathcal{X}' \rightarrow S' \rightarrow S$, we may view \mathcal{X}' as an algebraic stack over S , see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over S :

$$\begin{array}{ccc}
 \mathcal{X}' & \longrightarrow & \mathcal{X} \\
 \downarrow & & \downarrow \\
 S' & \longrightarrow & S.
 \end{array}$$

As $S' \rightarrow S$ is representable, surjective and étale, the same holds for $\mathcal{X}' \rightarrow \mathcal{X}$. The stack \mathcal{X}' is algebraic, hence \mathcal{X} is algebraic as well, see item (1). \square

PROOF OF THEOREM 1.1. Proposition 3.2 yields the stack \mathcal{X} over S together with 1-isomorphism $\rho: \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ and the 2-isomorphism $\chi: p_2^* \rho \circ \text{can} \Rightarrow \phi \circ p_1^* \rho$ that have the right compatibility properties with respect to ψ , so that we only need to prove that \mathcal{X} is algebraic (resp. Deligne–Mumford if $S' \rightarrow S$ is surjective étale). This follows from Lemma 3.3. \square

Even the case where \mathcal{X}' is a scheme seems to yield a non-trivial result:

COROLLARY 3.4. *Let $S' \rightarrow S$ be a smooth surjective morphism of schemes, and let X' be a scheme over S' equipped with a descent datum ϕ as in Section 2. Then there exists an algebraic stack \mathcal{X} over S and an S -morphism $\pi: X' \rightarrow \mathcal{X}$ such that the diagram*

$$\begin{array}{ccc}
 X' & \xrightarrow{\pi} & \mathcal{X} \\
 \downarrow & & \downarrow \\
 S' & \longrightarrow & S
 \end{array}$$

is cartesian. The tuple $(X, \pi: X' \rightarrow X)$ is compatible with the descent datum ϕ in an appropriate sense, and this makes (X, π) unique up to isomorphism.

PROOF. This is a straightforward consequence of Theorem 1.1. \square

For a scheme S and a stack X , and a finite group Γ , a *group action of Γ on X over S* is an action of the functor in groups over S associated to Γ on the stack X over S , see [Rom05, Definition 1.3].

LEMMA 3.5. *Let $S' \rightarrow S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ , and let X' be a stack over S' . Then the following sets are in canonical bijection:*

- (1) *The set of 2-descent data (ϕ, ψ) for X' over S' .*
- (2) *The set of group actions of Γ on X' as a stack over S , such that for each $\sigma \in \Gamma$, the composition $X' \xrightarrow{\sigma} X' \rightarrow S'$ agrees with the composition $X' \rightarrow S' \xrightarrow{\sigma} S'$.*
- (3) *The set of Galois 2-descent data for X' over S' .*

PROOF. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] for the proof in the case of schemes. The stacky case requires some straightforward generalizations; we leave the details to the reader. \square

PROOF OF THEOREM 1.3. See Theorem 1.1 and Lemma 3.5. \square

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