On the topology of real algebraic stacks

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Abstract

We investigate the topology of the real locus of a separated Deligne—Mumford stack of finite type over the real numbers. Specifically, we propose a natural generalization of the classical Smith–Thom inequality for real varieties to real Deligne–Mumford stacks, and establish this conjecture in several cases. In the process, we develop methods for studying the real locus of various types of real algebraic stacks. This requires a combination of techniques from group theory, algebraic geometry, and topology.

1 Introduction

1.1 Smith–Thom inequality for real algebraic varieties. Let X be a real algebraic variety, by which we mean a finite type scheme over $\mathbb{R}$. The topological space $X(\mathbb{C})$ is endowed with an involution $\sigma_X\colon X(\mathbb{C}) \to X(\mathbb{C})$ such that $X(\mathbb{R})$ is equal to set of fixed point $X(\mathbb{C})^{\sigma_X}$ of the involution σ_X.

One of the foundational result in real algebraic geometry (see [Flo52; Bor60; Tho65; DIK00b; Man17] for various proofs) is the Smith–Thom inequality

$$h^*(X(\mathbb{R})) = \sum_{i\geq 0} \dim \mathrm{H}^i(X(\mathbb{R}), \mathbb{Z}/2) \leq \sum_{i\geq 0} \dim \mathrm{H}^i(X(\mathbb{C}), \mathbb{Z}/2) = h^*(X(\mathbb{C})). \quad (1)$$

It allows one to bound the cohomology of $X(\mathbb{R})$ in terms of the one of $X(\mathbb{C})$, usually much easier to compute. Here, and in the sequel, $h^*(Y)$ denotes the dimension of the cohomology ring $\mathrm{H}^*(Y, \mathbb{Z}/2)$ of a topological space Y.

1.2 Failure of the naive Smith–Thom inequality for real algebraic stacks. In recent years, there has been increasing interest in moduli problems over $\mathbb{R}$, particularly in determining whether (1) attains equality for the associated coarse moduli space. Notable cases include moduli spaces of stable vector bundles on a curve [BS22], Hilbert schemes of points on a surface [Fu23; KR24], and symmetric powers of varieties [BD17; Fra18].

Note, however, that such a study says something about the *real moduli space* associated to the moduli problem only if this real moduli spaces arises as the real locus of the coarse moduli space, a phenomenon which in fact seems rare. For instance, if A_1 is the coarse moduli space of elliptic curves, then $\mathsf{A}_1(\mathbb{R}) = \mathbb{R}$ parametrizes complex elliptic curves that admit a real structure up to complex isomorphism, whereas the real moduli space of real elliptic curves has two connected components (there are exactly two real models for a complex elliptic curve that can be defined over $\mathbb{R}$).

To bypass this limitation, and start a systematic approach to study the topology of real moduli spaces, one is led to consider real algebraic stacks. If $\mathcal{X}$ is such a stack, then $\mathcal{X}(\mathbb{R})$ is a category rather than a set. To obtain a topological space in a way that generalizes the euclidean topology on $X(\mathbb{R})$ when X is a real variety, one considers the set $|\mathcal{X}(\mathbb{R})|$ of isomorphism classes of $\mathcal{X}(\mathbb{R})$, and defines a natural topology on $|\mathcal{X}(\mathbb{R})|$ as in [GF22b]. A similar procedure defines a topology on the set $|\mathcal{X}(\mathbb{C})|$ of isomorphism classes of $\mathcal{X}(\mathbb{C})$ (if $\mathcal{X}$ is separated Deligne–Mumford, the latter coincides with the topology on $|\mathcal{X}(\mathbb{C})|$ induced by the coarse moduli space).

The advantage of this perspective is that when the algebraic stack $\mathcal{X}$ represents a moduli problem—parametrizing equivalence classes of certain algebraic objects (such as genus g curves or sheaves on a fixed variety)—the set $|\mathcal{X}(\mathbb{R})|$ corresponds to the real isomorphism classes of the real objects. For instance, $|\mathcal{M}_g(\mathbb{R})|$ represents the space of isomorphism classes of real algebraic curves of genus g.

It is then natural to wonder whether the foundational inequality (1) generalizes to this setting. In other words: do we have $h^*(|\mathcal{X}(\mathbb{R})|) \leq h^*(|\mathcal{X}(\mathbb{C})|)$ for each algebraic stack $\mathcal{X}$ over $\mathbb{R}$? This is not the case, as the elliptic curve example shows.

Example 1.1. Let $\mathcal{X} = \mathcal{A}_1$ be the moduli stack of elliptic curves. The j-invariant gives an homeomorphism $|\mathcal{X}(\mathbb{C})| \xrightarrow{\sim} \mathbb{C}$, while $|\mathcal{X}(\mathbb{R})|$ has two connected components both homeomorphic to $\mathbb{R}$, one corresponding to elliptic curves with a connected real locus and the other to those with a disconnected real locus. In particular, $h^*(|\mathcal{X}(\mathbb{R})|) = 2$, which is larger than $h^*(|\mathcal{X}(\mathbb{C})|) = 1$.

The aim of this paper is twofold. First, we propose a conjectural alternative to the

Smith–Thom inequality, expected to hold for all real Deligne–Mumford stacks $\mathcal{X}$ (see Conjecture 1.2 below). Second, we develop several techniques to study the topological space $|\mathcal{X}(\mathbb{R})|$ associated with such a real stack $\mathcal{X}$. These techniques, which allow us to verify the conjecture in numerous examples, appear to be of independent interest.

1.3 Conjectural Smith–Thom inequality for real algebraic stacks. The main challenge in extending the Smith–Thom inequality (1) to algebraic stacks is that, although $|\mathcal{X}(\mathbb{C})|$ is equipped with an involution $\sigma_{\mathcal{X}}\colon |\mathcal{X}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})|$ which generalizes complex conjugation on the complex locus of a real variety, the natural map

$$|\mathcal{X}(\mathbb{R})| \longrightarrow |\mathcal{X}(\mathbb{C})|^{\sigma_{\mathcal{X}}} \tag{2}$$

is, in general, neither injective nor surjective. For instance, points of $|\mathcal{X}(\mathbb{C})|^{\sigma_{\mathcal{X}}}$ which are not in the image of (2) correspond to isomorphism classes of objects $x \in \mathcal{X}(\mathbb{C})$ which are isomorphic to their complex conjugate, but not defined over $\mathbb{R}$.

The failure of injectivity is measured by the following observation: for $x \in \mathcal{X}(\mathbb{R})$, the fibre of (2) above the image of x in $|\mathcal{X}(\mathbb{C})|^{\sigma_{\mathcal{X}}}$ is in canonical bijection with the first Galois cohomology group $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(x_{\mathbb{C}}))$. Therefore, in a sense, the topological space $|\mathcal{X}(\mathbb{C})|$ is too small to fully encode information about $|\mathcal{X}(\mathbb{R})|$, as it does not capture, for instance, the automorphisms of objects in $\mathcal{X}(\mathbb{C})$. To take these into account, we consider the inertia stack $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, whose complex locus consists of pairs (x, ϕ), where $x \in \mathcal{X}(\mathbb{C})$ and ϕ is an automorphism of x. The fiber of π over an object $x \in \mathcal{X}(\mathbb{C})$ is given by the constant group scheme of automorphisms of x.

With these considerations in mind, we propose the following conjectural generalization of the Smith–Thom inequality (1) to real Deligne–Mumford stacks.

Conjecture 1.2. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, with inertia stack $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$. Then the following inequality holds:*

$$\sum_{i\geq 0} \dim \mathrm{H}^i(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2) \leq \sum_{i\geq 0} \dim \mathrm{H}^i(|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|, \mathbb{Z}/2). \tag{3}$$

When $\mathcal{X}$ is a scheme, the map $\mathcal{I}_{\mathcal{X}} \to X$ is an isomorphism, hence (3) reduces to the usual Smith–Thom inequality (1). Moreover, we construct various examples of stacks which are not schemes for which the inequality (3) is an equality. As we explain below, we prove Conjecture 1 in various cases.

We warn the reader that, in general, there is no natural closed embedding of $|\mathcal{X}(\mathbb{R})|$ into $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|$. For example, take an elliptic curve E over $\mathbb{R}$ such that $h^*(E(\mathbb{R})) = 4$, and consider the stacky quotient $\mathcal{X} \coloneqq [E/\langle -1\rangle]$, where $-1\colon E \to E$ is the multiplication by -1. Then one can show (see Section 6.6) that $|\mathcal{X}(\mathbb{R})| \simeq E(\mathbb{R}) \coprod E(\mathbb{R})$, and that $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \simeq E(\mathbb{C}) \coprod \left(\coprod_{x\in E(\mathbb{C})[2]}\{x\})\right)$. Note that inequality (3) holds in this case (and is an equality): we have $h^*(|\mathcal{X}(\mathbb{R})| = 8$ and $h^*(|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|) = 4+4 = 8$.

Since Conjecture 1.2 is purely topological in nature, it is natural to consider a more general formulation within the category of topological groupoids with involution. We provide a precise statement of this generalized conjecture in Section 9.2.

Remark 1.3. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$. If $p\colon |\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})|$ is the map induced by π, then we have $\mathrm{H}^i(|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|, \mathbb{Z}/2) = \mathrm{H}^i(|\mathcal{X}(\mathbb{C})|, p_*\mathbb{Z}/2)$. Therefore, defining $F_{\mathcal{X}} = p_*\mathbb{Z}/2$, the inequality (3) becomes equivalent to the inequality $\dim \mathrm{H}^*(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2) \leq \dim \mathrm{H}^*(|\mathcal{X}(\mathbb{C})|, F_{\mathcal{X}})$. The latter might be closer in analogy to the classical inequality (1).

Remark 1.4. When the Deligne–Mumford stack $\mathcal{X}$ over $\mathbb{R}$ is smooth, the space $|\mathcal{X}(\mathbb{R})|$ carries a natural real analytic orbifold structure; see [GF22a, Section 2.2.3]. This orbifold structure on $|\mathcal{X}(\mathbb{R})|$ is analogous to the natural complex analytic orbifold structure on $|\mathcal{X}(\mathbb{C})|$. It is natural to ask whether the classical Smith–Thom inequality (1) admits an analogue in terms of orbifold cohomology. We explore this question in Section 9.1.

1.4 Topology of real quotient stacks. A distinctive feature of the Smith–Thom inequality is its inherently global nature. Since varieties are locally contractible, the inequality holds trivially at the local level. In contrast, the inequality proposed in Conjecture 1.2 does not seem locally trivial. Indeed, for any separated Deligne–Mumford stack $\mathcal{X}$ over $\mathbb{R}$ and any $x \in \mathcal{X}(\mathbb{R})$, there exists a real algebraic variety U, a finite group scheme Γ over $\mathbb{R}$, a point $y \in U(\mathbb{R})$ and an étale map $[U/\Gamma] \to \mathcal{X}$ such that y maps to x (see [AV02, Lemma 2.2.3] and its proof). Even for $[U/\Gamma]$, Conjecture 1.2 does not appear to be straightforward.

1.4.1 Topology of real quotient stacks. As it turns out, the topology of a real quotient stack can be quite complicated, as the following theorem shows.

Let Γ be a finite group scheme over $\mathbb{R}$, with associated real structure $\sigma_\Gamma\colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$. Define $\mathrm{Z}^1(G,\Gamma)$ as the set of $\gamma \in \Gamma(\mathbb{C})$ with $\sigma_\Gamma(\gamma)\cdot\gamma = e$. Recall that the non-abelian Galois module $\mathrm{H}^1(G,\Gamma)$ can be canonically identified with $\mathrm{Z}^1(G,\Gamma)/\sim$ where $\sim$ is the equivalence relation $\gamma \sim \beta\gamma\sigma(\beta)^{-1}$ for $\beta \in \Gamma(\mathbb{C})$. Choose a set of representative $H \subset \mathrm{Z}^1(G,\Gamma)$ for this equivalence relation, such that $e \in H$. For $\gamma \in H$, define an involution $\sigma_\Gamma^\gamma\colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$ as $\sigma_\Gamma^\gamma(g) \coloneqq \gamma\sigma_\Gamma(g)\gamma^{-1}$.

Let X be a quasi-projective scheme over $\mathbb{R}$ with real structure $\sigma_X\colon X(\mathbb{C}) \to X(\mathbb{C})$, acted upon from the left by the finite group scheme Γ over $\mathbb{R}$. For $\gamma \in H$, define an involution $\sigma_X^\gamma \colon X(\mathbb{C}) \to X(\mathbb{C})$ as $\sigma_X^\gamma(x) = \gamma \cdot \sigma(x)$. By Galois descent, the pair $(X(\mathbb{C}), \sigma_X^\gamma)$ corresponds to a quasi-projective scheme X_γ over $\mathbb{R}$. Similarly, for $\gamma \in H$, the pair $(\Gamma(\mathbb{C}), \sigma_\Gamma^\gamma)$ corresponds to a finite group scheme Γ_γ over $\mathbb{R}$. Note that

$$X_\gamma(\mathbb{R}) = X(\mathbb{C})^{\sigma_X^\gamma} \quad \text{and} \quad \Gamma_\gamma(\mathbb{R}) = \Gamma(\mathbb{C})^{\sigma_\Gamma^\gamma} \quad \text{for each} \quad \gamma \in H.$$

Theorem 1.5. *Consider the above notation. There is a canonical homeomorphism*

$$|[X/\Gamma](\mathbb{R})| \xrightarrow{\sim} \coprod_{\gamma \in H} X_\gamma(\mathbb{R})/\Gamma_\gamma(\mathbb{R}). \tag{4}$$

We use Theorem 1.5 to prove Conjecture 1.2 in a number of examples, such as stacky symmetric products and quotients of abelian varieties by -1 (see Section 1.4.3 below). Theorem 1.5 will also used in the proof of Conjecture 1.2 for stacky quotients of curves by a finite group (which is abelian or acts faithfully, cf. Theorem 1.9 below).

Remark 1.6. In the notation of Theorem 1.5, assume that X is smooth over $\mathbb{R}$. Then the topological space $|[X/\Gamma](\mathbb{R})|$ can naturally be enhanced with the structure of a real analytic orbifold (cf. [GF22a, Section 2.2.3]). For this orbifold structure on $|[X/\Gamma](\mathbb{R})|$, the homeomorphism (4) is an isomorphism of real analytic orbifolds, see Corollary 6.2.

Remark 1.7. Theorem 1.5 suggests a different formulation of Conjecture 1.2. Indeed, in the notation of Theorem 1.5, assume that X is smooth over $\mathbb{R}$. One may try to bound the orbifold cohomology ring of $[X/\Gamma](\mathbb{R})$, which by Theorem 1.5 and Remark 1.6 is the direct sum of the Γ^{σ_γ}-equivariant cohomology ring of $X_\gamma(\mathbb{R})$ for $\gamma \in \mathrm{H}^1(G, \Gamma)$, in terms of the the orbifold cohomology of $[X/\Gamma](\mathbb{C})$, i.e. the Γ-equivariant cohomology of $X(\mathbb{C})$. In Section 9.1 we make the question whether such a bound exists precise.

1.4.2 Positive results for quotient stacks of dimension ≤ 1. Let us first focus on Conjecture 1.2 for finite quotient stacks, and explain our main results in this setting. We start with the zero-dimensional case.

Proposition 1.8. *Let Γ be a finite $\mathbb{R}$-group scheme and set $\mathcal{X} \coloneqq [\mathrm{Spec}(\mathbb{R})/\Gamma]$. Then the inequality (3) holds for $\mathcal{X}$.*

In this case, one show that $|\mathcal{X}(\mathbb{R})|$ and $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|$ are discrete topological spaces, in bijection with $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \Gamma(\mathbb{C}))$ and $\Gamma(\mathbb{C})/\Gamma(\mathbb{C})$ respectively, where $\Gamma(\mathbb{C})$ acts on itself by conjugation. Thus, the inequality (3) reduces to a group theoretic statement (see Lemma 5.1).

We then move to dimension one. A *real curve* is a one-dimensional variety over $\mathbb{R}$ (see Section 2), not necessarily proper.

Theorem 1.9. *Let X be a real curve, and let Γ be a finite group scheme over $\mathbb{R}$ which acts on X over $\mathbb{R}$. Assume one of the following conditions:*

1. *The action of Γ on X is faithful.*
2. *The group scheme Γ is abelian.*

Then Conjecture 1.2 holds for the quotient stack $\mathcal{X} = [X/\Gamma]$.

The proof of Theorem 1.9 is rather indirect, in the sense that we do not compare directly the topology of $|[X/\Gamma](\mathbb{R})|$ with $\left|\mathcal{I}_{[X/\Gamma]}(\mathbb{C})\right|$, but rather we compute separately $h^*(|[X/\Gamma](\mathbb{R})|)$ and $h^*(\left|\mathcal{I}_{[X/\Gamma]}(\mathbb{C})\right|)$ by combining local and global methods. Then we compare the two numbers by using the classical Smith–Thom inequality and the group theoretic inequality of Lemma 5.1.

Remark 1.10. Either one of the conditions in Theorem 1.9 guarantees that $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})|$ is the union of a trivial topological covering with the inclusion of a finite set of points, which allows one to compute the topology of $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|$ in terms of the topology of $|\mathcal{X}(\mathbb{C})|$. Possibly, one could remove these conditions by refining the techniques.

1.4.3 Positive results for higher dimensional quotient stacks. Next, we study Conjecture 3 in higher dimensions. In fact, constructing examples of stacks of arbitrary dimension, which satisfy the conjecture and are not schemes, is relatively straightforward. For instance, if $\mathcal{X}$ and $\mathcal{Y}$ are separated Deligne–Mumford stacks of finite type over $\mathbb{R}$ for which Conjecture 1.2 holds, then it also holds for their product $\mathcal{X} \times_{\mathbb{R}} \mathcal{Y}$ (by the Künneth formula and the fact that inertia commutes with products).

The following theorem provides further evidence for the conjectural Smith–Thom inequality (3) in arbitrary dimension, by verifying it for certain higher-dimensional quotient stacks that do not arise as products of lower-dimensional examples.

Theorem 1.11. *Let $\mathcal{X}$ be a Deligne–Mumford stack over $\mathbb{R}$. Assume that one of the following two conditions holds:*

1. *We have $\mathcal{X} = [(X \times X)/\mathbb{Z}/2]$ for a real variety X, where $1 \in \mathbb{Z}/2$ acts on $X \times X$ by permuting the factors.*
2. *We have $\mathcal{X} = [A/\langle -1 \rangle]$, where A is an abelian variety over $\mathbb{R}$ and $-1\colon A \to A$ the multiplication by -1 homomorphism.*

Then Conjecture 1.2 holds for $\mathcal{X}$.

1.5 Topology of split gerbes over a real variety. A nice example of a real Deligne–Mumford stack which is not the quotient of a real variety by a finite group scheme over $\mathbb{R}$, is any split gerbe over a real variety, i.e., a stack of the form $\mathcal{X} = [U/H]$, where U is a real variety and $H \to U$ a non-constant, finite étale group scheme over U (and where the action of H on U over U is the trivial action). This example seems important in the study of the topology of real Deligne–Mumford stacks in general, and of Conjecture 1.2 in particular, as any Deligne–Mumford stack $\mathcal{X}$ over $\mathbb{R}$ admits a stratification $\{\mathcal{X}_n\}_{n\geq 0}$ by stabilizer order, where the automorphism groups in the stratum $\mathcal{X}_n$ have order exactly n. The stacks $\mathcal{X}_n$ are gerbes over their coarse moduli spaces $\mathcal{X}_n \to M_n$, hence étale locally on M_n of the form $[U_n/H_n]$, where $H_n \to U_n$ is a finite étale group scheme of order n.

We develop a technique for computing $|[U/H](\mathbb{R})|$. As before, let $G \coloneqq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Theorem 1.12. *Let U be a geometrically connected $\mathbb{R}$-variety such that $U(\mathbb{R}) \neq \emptyset$. Let $H \to U$ be a finite étale group scheme and set $\mathcal{X} \coloneqq [U/H]$. The following holds.*

1. *The canonical map $f\colon |\mathcal{X}(\mathbb{R})| \to U(\mathbb{R})$ is a topological covering over each connected component of $U(\mathbb{R})$, with fibre $\mathrm{H}^1(G, H_p(\mathbb{C}))$ above a point $p \in U(\mathbb{R})$.*
2. *Let C be a connected component of $U(\mathbb{R})$, and fix $p \in C$. The image of the natural map $\pi_1(C,p) \to \pi_1(U(\mathbb{C}),p)$ lies in the subgroup of elements $g \in \pi_1(U(\mathbb{C}),p)$ whose action on $H_p(\mathbb{C})$ is G-equivariant. In particular, the group $\pi_1(C,p)$ acts naturally on $\mathrm{H}^1(G, H_p(\mathbb{C}))$.*
3. *The covering space associated to the above action of $\pi_1(C,p)$ on $\mathrm{H}^1(G, H_p(\mathbb{C}))$ is canonically isomorphic to the covering space $f^{-1}(C) \to C$.*

In Theorem 4.12 we prove something more general than the first item in Theorem 1.12. Namely, consider the map $f\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ induced by the coarse moduli space map $p\colon \mathcal{X} \to M$ and assume that the latter is a gerbe. Then f is open and a topological covering over each connected component of its image, see Theorem 4.12.

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2 Notation and conventions

We indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z. For an algebraic stack $\mathcal{X}$, we let $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ denote the inertia stack over $\mathcal{X}$, and $\mathcal{X} \to M_{\mathcal{X}}$ the coarse moduli space. We let $I_{\mathcal{X}} \coloneqq M_{\mathcal{I}_{\mathcal{X}}}$ denote the coarse moduli space of the inertia stack. The morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ induces a morphism $I_{\mathcal{X}} \to M_{\mathcal{X}}$.

When $\mathcal{X}$ is an algebraic stack over a scheme S, we let $|\mathcal{X}(S)|$ denote the set of isomorphism classes of the groupoid $\mathcal{X}(S)$. For an object $x \in \mathcal{X}(S)$, we let $[x] \in |\mathcal{X}(S)|$ denote its isomorphism class. For an algebraic stack $\mathcal{X}$ over $\mathbb{R}$, and an object $x \in \mathcal{X}(\mathbb{R})$, let $x_{\mathbb{C}} \in \mathcal{X}(\mathbb{C})$ denote the pull-back of x along $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$.

A *curve* over $\mathbb{R}$ (resp. $\mathbb{C}$) will be a reduced, separated one-dimensional scheme of finite type over $\mathbb{R}$ (resp. $\mathbb{C}$). A curve X over $\mathbb{R}$ will also be called a *real curve*. Note that we do not assume that X is proper. For a smooth curve X over $\mathbb{R}$, any connected component $C \subset X(\mathbb{R})$ is homeomorphic to either the circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ or the open interval $(0, 1)$. By abuse of notation, we call C a *circle* in the first case, and an *open interval* in the second case.

For $n \in \mathbb{Z}_{\geq 1}$, we let μ_n be the $\mathbb{R}$-group scheme with $\mu_n(S) = \{x \in \mathcal{O}_S(S) \mid x^n = 1\}$ for a scheme S over $\mathbb{R}$.

For a topological space X (such that $\dim \mathrm{H}^*(X, \mathbb{Z}/2)$ is finite), we define $h^*(X) = \dim \mathrm{H}^*(X, \mathbb{Z}/2)$. For instance, $h^*(\mathbb{S}^1) = 2$. If Y is any space of endowed with an action of G and $x, y \in Y$, we let $\mathrm{Path}(x, y)$ be the set of topological paths in Y from x to y and observe that G induces a bijection $\sigma_Y \colon \mathrm{Path}(x, y) \to \mathrm{Path}(\sigma(x), \sigma(y))$. In particular, if x, y are fixed by G, the involution $\sigma_Y \colon \mathrm{Path}(x, y) \to \mathrm{Path}(x, y)$ defined an action of G on $\mathrm{Path}(x, y)$. If $\gamma \in \mathrm{Path}(x, y)$, we let $\gamma^{-1} \in \mathrm{Path}(y, x)$ be the inverse path of γ.

3 Topology of the complex inertia

3.1 Topology of the complex locus. For a Deligne–Mumford stack $\mathcal{X}$ locally of finite type over $\mathbb{C}$, we view the set of isomorphism classes $|\mathcal{X}(\mathbb{C})|$ of the groupoid $\mathcal{X}(\mathbb{C})$ as a topological space, by equipping it with the quotient topology induced by the surjective morphism $U(\mathbb{C}) \to |\mathcal{X}(\mathbb{C})|$, where U is a scheme and $U \to \mathcal{X}$ a surjective étale morphism. It is easy to show that this topology on $|\mathcal{X}(\mathbb{C})|$ does not depend on the choice of étale presentation $U \to \mathcal{X}$.

Lemma 3.1. *For a separated Deligne–Mumford stack locally of finite type over $\mathbb{C}$ with coarse moduli space $\mathcal{X} \to M$, the map $|\mathcal{X}(\mathbb{C})| \to M(\mathbb{C})$ is a homeomorphism.*

Proof. As the map $|\mathcal{X}(\mathbb{C})| \to M(\mathbb{C})$ is clearly a bijection, it remains to prove that it is continuous and open. Continuity is clear, hence we need to prove that $|\mathcal{X}(\mathbb{C})| \to M(\mathbb{C})$ is open. For this, we choose étale maps $V_\alpha \to M$ for α in some index set I, such that $\coprod_\alpha V_\alpha \to M$ is surjective, and such that for each α there exists a finite group Γ_α acting on a scheme U over $\mathbb{C}$, such that the fibre product $\mathcal{X} \times_M V_\alpha$ is isomorphic to $[U_\alpha/\Gamma_\alpha]$ (cf. [AV02]). Define $U' = \coprod_\alpha U_\alpha$. Any open set $W \subset |\mathcal{X}(\mathbb{C})|$ is the image of an open set $W' \subset U'(\mathbb{C})$ under the natural map $U'(\mathbb{C}) \to |\mathcal{X}(\mathbb{C})|$. The image of W' in $\coprod_\alpha V_\alpha(\mathbb{C})$ is open as $V_\alpha = U_\alpha/\Gamma_\alpha$ for each α. Since $\coprod_\alpha V_\alpha(\mathbb{C}) \to M(\mathbb{C})$ is open, it follows that the image of W' in $M(\mathbb{C})$ is open, which is exactly the image of W in $M(\mathbb{C})$. □

3.2 Topology of the complex inertia. Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$. The diagonal morphism $\Delta\colon \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is of finite type, see [LMB00, Lemme (4.2)]. Therefore, for each scheme S over $\mathbb{C}$ and each $x \in \mathcal{X}(S)$, the automorphism group algebraic space $\underline{\mathrm{Aut}}_S(x)$ of x over S is of finite type over S.

If the algebraic stack $\mathcal{X}$ is Deligne–Mumford, the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is quasi-finite (see [LMB00, Lemme (4.2)]). In particular, if $\mathcal{X}$ is separated and Deligne–Mumford, then Δ is finite. We conclude the following lemma, which is well-known:

Lemma 3.2. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$. For each scheme S over $\mathbb{C}$ and each $x \in \mathcal{X}(S)$, the automorphism group algebraic space $\underline{\mathrm{Aut}}_S(x)$ is finite over S.* □

Lemma 3.3. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$, with inertia $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$. Let $M_{\mathcal{X}}$ (resp. $I_{\mathcal{X}}$) be the coarse moduli space of $\mathcal{X}$ (resp. $\mathcal{I}_{\mathcal{X}}$), cf. Section 2. The morphism on coarse moduli spaces $I_{\mathcal{X}} \to M_{\mathcal{X}}$ induced by π is finite and surjective.*

Proof. Pick a finite surjective morphism $Z \to \mathcal{X}$ where Z is a scheme; such a morphism exists by [LMB00, Theorem 16.6]. Define $W = Z \times_{\mathcal{X}} \mathcal{I}_X$. The morphisms $W \to Z$ and $Z \to M_{\mathcal{X}}$ are both finite and surjective, hence the composition $W \to M_{\mathcal{X}}$ is finite and surjective. The morphism $W \to \mathcal{I}_{\mathcal{X}}$ is finite and surjective, and the morphism $I_{\mathcal{X}} \to M_{\mathcal{X}}$ is surjective. Hence the latter is also finite. □

Corollary 3.4. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$. The morphism of complex spaces $\mathcal{I}_{\mathcal{X}}(\mathbb{C}) \to M_{\mathcal{X}}(\mathbb{C})$ is closed with finite fibers.*

Proof. This follows from Lemma 3.3 in view of the well-known fact that the morphism of analytic spaces $X(\mathbb{C}) \to Y(\mathbb{C})$ induced by a finite surjective morphism $X \to Y$ of finite type schemes X, Y over $\mathbb{C}$ is closed with finite fibers. $\square$

Lemma 3.5. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack locally of finite type over $\mathbb{C}$, such that $|\mathrm{Aut}(x)|$ is constant for $x \in \mathcal{X}(\mathbb{C})$. The following holds.*

1. *The coarse moduli space map $\mathcal{X} \to M$ is a gerbe.*
2. *The inertia $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite flat over $\mathcal{X}$.*

Proof. To prove that $\mathcal{X} \to M$ is a gerbe, by [Stacks, Tag 06QJ], it suffices to show that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat. Thus, we need to show that for any scheme T and morphism $T \to \mathcal{Y}$, the automorphism group algebraic space $\underline{\mathrm{Aut}}(x)_T \to T$ is flat over T. We know that $\underline{\mathrm{Aut}}(x)_T$ is finite over T, see Lemma 3.2. Moreover, for each $t \in T(\mathbb{C})$, the group scheme $\underline{\mathrm{Aut}}(x)_t = \underline{\mathrm{Aut}}(x_t)$ over $\mathbb{C}$ is reduced. To prove that $\underline{\mathrm{Aut}}(x)_T$ is flat over T, it suffices to show that it has constant fibre cardinality which holds by assumption. This view of Lemma 3.3 this also proves that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite étale. $\square$

Proposition 3.6. *Let X be a scheme of finite type over $\mathbb{C}$. Let Γ be a finite group acting on X over $\mathbb{C}$. Define $\mathcal{X} = [X/\Gamma]$. Let $q\colon X(\mathbb{C}) \to X(\mathbb{C})/\Gamma = M_{\mathcal{X}}(\mathbb{C})$ be the quotient map.*

1. *There is a canonical bijection*

$$|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| = \{(x \in X(\mathbb{C}), \gamma \in \Gamma_x\} \,/\, \left\{(x,\gamma) \sim (gx, g\gamma g^{-1}), g \in \Gamma\right\}. \tag{5}$$

2. *Consider the canonical map $|\pi|: |\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})| = X(\mathbb{C})/\Gamma$. For each $x \in M_{\mathcal{X}}(\mathbb{C}) = X(\mathbb{C})/\Gamma$, there is a canonical bijection*

$$|\pi|^{-1}(x) = \left(\coprod_{y \in q^{-1}(x)} \Gamma_y\right)/\Gamma.$$

   Here, $g \in \Gamma$ acts on $\bigsqcup_{y \in q^{-1}(x)} \Gamma_y$ as follows: for $y \in q^{-1}(x)$, $\gamma \in \Gamma_y$, we define $g \cdot (y, \gamma) = (gy, g\gamma g^{-1})$.

3. *For $x \in M_{\mathcal{X}}(\mathbb{C})$ and fixed $y' \in q^{-1}(x)$, there are bijections*

$$|\pi|^{-1}(x) = \left(\coprod_{y \in q^{-1}(x)} \Gamma_y\right)/\Gamma \cong \Gamma_{y'}/\Gamma_{y'}, \tag{6}$$

of which the second one is in general non-canonical.

Proof. Let $S \to X$ be the stabilizer group scheme attached to the action of Γ on X over $\mathbb{C}$. Then

$$S(\mathbb{C}) = \{(x, \gamma) \in X(\mathbb{C}) \times \Gamma \mid \gamma x = x\}\,.$$

The group Γ acts on the scheme S by

$$g \cdot (x, \gamma) = (gx, g\gamma g^{-1}), \qquad g \in \Gamma, (x, \gamma) \in S,$$

and we have a canonical isomorphism of stacks $\mathcal{I}_{\mathcal{X}} = [S/\Gamma]$. In particular, $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| = S(\mathbb{C})/\Gamma$ from which (5) follows. This proves item 1. Item 2 is clear.

To prove item 3, it remains to provide the second bijection in (6). This holds, since for each $y_1, y_2 \in q^{-1}(x)$, there exists $g \in \Gamma$ such that $gy_1 = y_2$ and $g\Gamma_{y_1}g^{-1} = \Gamma_{y_2}$. $\square$

Remark 3.7.

1. In the notation of Proposition 3.6, assume that Γ acts freely on X. Then $\Gamma_y = \{e\}$ for each $y \in q^{-1}(x)$, and Γ acts freely on $q^{-1}(x)$. Hence $|\pi|^{-1}(x)$ is a singleton.
2. In the notation of Proposition 3.6, assume that Γ is abelian. There is a canonical bijection between $(\bigsqcup_{y \in q^{-1}(x)} \Gamma_y)/\Gamma$ and $\Gamma_y = \Gamma_{y'} \subset \Gamma$ for any $y, y' \in q^{-1}(x)$.

3.3 Examples. The goal of this subsection is to calculate the topology of $I_{\mathcal{X}}(\mathbb{C})$ for certain low-dimensional algebraic quotient stacks $\mathcal{X}$ over $\mathbb{C}$.

Example 3.8. Let Γ be a finite group, and let $B\Gamma = [\mathrm{Spec}(\mathbb{C})/\Gamma]$. Then $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| = \Gamma/\Gamma$, where Γ acts on itself by conjugation. In particular $h^*(|\mathcal{I}_{\mathcal{X}}(\mathbb{C})|) = |\Gamma/\Gamma|$.

Examples 3.9.

1. Let $X := \mathbb{A}^1$ and $\Gamma := \mathbb{Z}/2$ acting on X by sending x to $-x$. Then $I_{[X/\Gamma]}(\mathbb{C}) \simeq \mathbb{C} \coprod \{0\}$. In particular $h^*(I_{[X/\Gamma]}(\mathbb{C})) = 2$.
2. Let $X := \mathbb{A}^1$ and $\Gamma := \mathbb{Z}/2 \times \mathbb{Z}/2$ acting on X by $(a, b) * x$ to $(-1)^{ab}x$. Then $I_{[X/\Gamma]}(\mathbb{C}) \simeq \mathbb{A}^1 \coprod \mathbb{A}^1 \coprod \{0\} \coprod \{0\}$. In particular $h^*(I_{[X/\Gamma]}(\mathbb{C})) = 4$.

Examples 3.10.

1. Let $X := \mathbb{A}^2$ and $\Gamma := \mathbb{Z}/2$, such that $1 \in \mathbb{Z}/2 = \Gamma$ acts on X by $(x, y) \mapsto (y, x)$. Then $I_{[X/\Gamma]}(\mathbb{C}) \simeq \mathbb{A}^2 \coprod \mathbb{A}^1$. In particular $h^*(I_{[X/\Gamma]}(\mathbb{C}) = 2$.
2. Let $X := \mathbb{A}^2$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$. We let $\mathbb{Z}/2$ acts on X by exchanging the coordinates and we let Γ act on X via the addition map $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$. Then, we have $I_{[X/\Gamma]}(\mathbb{C}) \simeq \mathbb{A}^2 \coprod \mathbb{A}^2 \coprod \mathbb{A}^1 \coprod \mathbb{A}^1$. In particular, $h^*(I_{[X/\Gamma]}(\mathbb{C})) = 4$.

Example 3.11. Let U a connected scheme over $\mathbb{C}$ and $H \to U$ a finite étale group scheme over U. Consider the trivial action of H on U. Then $I_{[U/H]}(\mathbb{C}) \simeq H(\mathbb{C})/H(\mathbb{C})$. where $H(\mathbb{C})/H(\mathbb{C}) := \{(p,h) \in U(\mathbb{C}) \times H_p(\mathbb{C})\}/\sim$ with $\sim$ the equivalence relation $(p,h) \sim (p',h')$ if $p = p'$ and h is conjugated to h' in $H_p(\mathbb{C})$. In particular, when H is abelian one has $I_{[U/H]}(\mathbb{C}) \simeq H(\mathbb{C})$. We post-pone the discussion on its Betti number until Section 8, since here the situation is more complicated.

Example 3.12. Let Γ be an abelian group acting faithfully on a variety X and assume that the set $Z := \{x \in X(\mathbb{C})$ such that $\mathrm{Stab}_\Gamma(x) \neq \{0\}\}$ is finite. Then, we have $I_{[X/\Gamma]}(\mathbb{C}) \simeq X(\mathbb{C})/\Gamma \coprod \left(\coprod_{x\in Z} \mathrm{Stab}(x) - \{e\}\right)$.

4 Topology of real DM stacks

The goal of this section is to provide some preliminary definitions and prove some preliminary results on the topology of $|\mathcal{X}(\mathbb{R})|$ when $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over $\mathbb{R}$.

4.1 Generalities on the real locus of a real DM stack. The main object of study in this paper is as follows.

Definition 4.1. A *real DM stack* is a separated Deligne–Mumford stack of finite type over $\mathbb{R}$.

For a real DM stack $\mathcal{X}$, the set of isomorphism classes $|\mathcal{X}(\mathbb{R})|$ of its real locus $\mathcal{X}(\mathbb{R})$ has a natural topology, generalizing the euclidean topology on $X(\mathbb{R})$ when X is a scheme. Indeed, we have the following theorem.

Theorem 4.2. *Let $\mathcal{X}$ be a real DM stack. There exists a scheme U over $\mathbb{R}$ and a surjective étale morphism $U \to \mathcal{X}$ such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective.*

Proof. See [GF22a, Theorem 2.9] or [GF22b, Theorem 7.4]. □

Definition 4.3. (cf. [GF22b, Definition 7.5]) Let $\mathcal{X}$ be a real DM stack. The *real analytic topology* on $|\mathcal{X}(\mathbb{R})|$ is defined as follows. Choose a scheme U over $\mathbb{R}$ and a surjective étale morphism $U \to \mathcal{X}$ such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective. Then consider the real analytic topology on $U(\mathbb{R})$, and give $|\mathcal{X}(\mathbb{R})|$ the quotient topology induced by the surjection $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$.

Proposition 4.4. *The real analytic topology is independent of the choice of an étale presentation that is essentially surjective on real points.*

Proof. See [GF22b, Proposition 7.6]. $\square$

Throughout this paper, whenever we consider the set $|\mathcal{X}(\mathbb{R})|$ of isomorphism classes of real points of a real Deligne–Mumford stack $\mathcal{X}$, we always view it as a topological space via the real analytic topology.

4.2 Fibres of the map to the real locus of the coarse moduli space. We will only need the following proposition in the case of stacky curves, but we state it in arbitrary dimension, since the proof is the same.

Proposition 4.5. *Let $\mathcal{X}$ be a a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, with coarse moduli space $p\colon \mathcal{X} \to M$. Let $f\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ denote the map induced by p, and let $x \in \mathcal{X}(\mathbb{R})$ with isomorphism class $[x] \in |\mathcal{X}(\mathbb{R})|$ (cf. Section 2).*

1. *There is a canonical bijection $f^{-1}(f([x])) = \mathrm{H}^1(G, \mathrm{Aut}(x_{\mathbb{C}}))$.*
2. *We have $\#\mathrm{H}^1(G, \mathrm{Aut}(x_{\mathbb{C}})) = \#\mathrm{H}^1(G, \mathrm{Aut}(x'_{\mathbb{C}}))$ for each pair of objects $x, x' \in \mathcal{X}(\mathbb{R})$ whose induced objects $x_{\mathbb{C}}, x'_{\mathbb{C}} \in \mathcal{X}(\mathbb{C})$ are isomorphic in $\mathcal{X}(\mathbb{C})$.*

Proof. Since two objects in $\mathcal{X}(\mathbb{C})$ are isomorphic if and only if their images in $M(\mathbb{C})$ are the same, the second item is a consequence of the first item. The first item follows from [Gro60, Section 4]. $\square$

This naturally leads us to the following:

Definition 4.6. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, with coarse moduli space $p\colon \mathcal{X} \to M$. For a point $m \in M(\mathbb{R})$ which is in the image of $f\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$, we define $\mathrm{H}^1(G, m) = \#\mathrm{H}^1(G, \mathrm{Aut}(x_{\mathbb{C}}))$, where $x \in \mathcal{X}(\mathbb{R})$ is such that $[x] \in |\mathcal{X}(\mathbb{R})|$ lies in $f^{-1}(m) \subset |\mathcal{X}(\mathbb{R})|$.

By Proposition 4.5, this is well-defined, in the sense that we have $\mathrm{H}^1(G, x) = \#\mathrm{H}^1(G, \mathrm{Aut}(x'_{\mathbb{C}}))$ for any $x' \in \mathcal{X}(\mathbb{R})$ such that $[x'] \in f^{-1}(m)$.

4.3 Covering map between the real locus of the stack and the real locus of the coarse moduli space. The main result of this section is Theorem 4.12 below, which gives a general criterion for the map of topological spaces $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$, induced by the morphism $\mathcal{X} \to M$ of a stack to its coarse moduli space, to be a topological covering. The proof is slightly technical; the reader may wish to skip the proof on a first reading. Before we can start with the proof, we need some preliminary results and definitions.

Lemma 4.7. *Let $f\colon X \to Y$ be a morphism of schemes X, Y which are locally of finite type over $\mathbb{R}$. Assume that f is étale. Then the induced map $f_{\mathbb{R}}\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a local homeomorphism.*

Proof. Consider the map of complex analytic spaces $f_{\mathbb{C}}\colon X(\mathbb{C}) \to Y(\mathbb{C})$. This map is a local homeomorphism by [Gro71, Exposé XII, Proposition 3.1 & Remarque 3.3]. For $x \in X(\mathbb{R})$, let $U \subset X(\mathbb{C})$ be a G-stable open neighbourhood (where $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$) such that $V = f_{\mathbb{C}}(U)$ is open in $Y(\mathbb{C})$ and $f_{\mathbb{C}}|_U\colon U \to V$ is a homeomorphism. Note that $V \subset Y(\mathbb{C})$ is stable under the action of G on $Y(\mathbb{C})$. Indeed, for $v \in V$ and $g \in G$, we have $v = f_{\mathbb{C}}(u)$ for $u \in U$, and since $gu \in U$ we get $gv = gf_{\mathbb{C}}(u) = f_{\mathbb{C}}(gu) \in f_{\mathbb{C}}(U) = V$. The map $f_{\mathbb{C}}|_U\colon U \to V$ is a homeomorphism of topological G-spaces, thus the restriction $f_{\mathbb{R}}|_{U^G} = f_{\mathbb{C}}|_{U^G}\colon U \cap X(\mathbb{R}) = U^G \to V^G = V \cap Y(\mathbb{R})$ is a homeomorphism. $\square$

Lemma 4.8. *Let $f\colon X \to Y$ be a map of topological spaces, let $\pi\colon Y' \to Y$ be a local homeomorphism with $\mathrm{Im}(\pi) = \mathrm{Im}(f)$. Assume that the base change $f'\colon X' \coloneqq X \times_Y Y' \to Y'$ is a topological covering over its image. Then f is a topological covering over its image.*

Proof. Note that $\mathrm{Im}(\pi) \subset Y$ is open in Y because π is a local homeomorphism. Up to pulling everything back along the inclusion $\mathrm{Im}(\pi) \subset Y$, we may assume that f and π are surjective. Let $U' \subset Y'$ and $U \subset Y$ be opens such that the map $\pi\colon Y' \to Y$ induces a homeomorphism $\pi|_{U'}\colon U' \xrightarrow{\sim} U$, and such that $(f')^{-1}(U') \to U'$ is a trivial topological covering (i.e. homeomorphic over U' to a disjoint union of copies of U'). Consider the induced map $\rho\colon X' \to X$, and note that $\rho((f')^{-1}(U')) \subset f^{-1}(U)$, and that the map $\rho\colon (f')^{-1}(U') \to f^{-1}(U)$ is a homeomorphism. Hence $f^{-1}(U)$ is homeomorphic over U to a disjoint union of copies of U. Since π is surjective, this proves the lemma. $\square$

Let $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. This means that π is a finite topological covering, that there are involutions $\sigma\colon H \to H, \sigma\colon U \to U$ commuting with π, and that there is a continuous group law $m\colon H \times_U H \to H$, an inversion $i\colon H \to H$ and identity $e\colon U \to H$ all compatible with the involutions σ; moreover, we require that for each $x \in U$ there exists an open neighbourhood $x \in V \subset U$ such that $H|_V \cong V \times \Gamma$ as families of topological groups, for a finite group Γ.

Definition 4.9. Let, as above, $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. We define

$$\begin{aligned}
\mathrm{Z}^1(G, H) &\coloneqq \left\{(u, g) \in U \times H \mid u \in U^G, g \in H_u \text{ and } g\sigma(g) = e\right\},\\
\mathrm{H}^1(G, H) &\coloneqq \mathrm{Z}^1(G, H)/\sim
\end{aligned}$$

where $(u, g) \sim (u', g')$ if $u = u'$ and there exists $h \in H_u$ such that $g' = hg\sigma(h)^{-1}$. We equip $\mathrm{Z}^1(G, H)$ with the subspace topology coming from $U \times H$ and we equip $\mathrm{H}^1(G, H)$ with the quotient topology coming from $\mathrm{Z}^1(G, H)$.

Lemma 4.10. *Let $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. There is a natural surjective map $\mathrm{H}^1(G, H) \to U^G$, and for $V \subset U^G$ open such that $H|_V \cong V \times \Gamma$ as families of topological G-groups over V, for a finite G-group Γ, we have $\mathrm{H}^1(G, H)|_V = \mathrm{H}^1(G, H|_V) \cong \mathrm{H}^1(G, \Gamma) \times V$. In particular, the natural map*

$$\mathrm{H}^1(G, H) \longrightarrow U^G$$

is a topological covering, with fibre $\mathrm{H}^1(G, H_u)$ for $u \in U^G$.

Proof. Clear. □

Lemma 4.11. *Let $H \to U$ be a finite étale group scheme over a scheme U of finite type over $\mathbb{R}$. Consider the associated quotient stack $[U/H]$, and also the associated locally trivial family of finite G-groups $H(\mathbb{C}) \to U(\mathbb{C})$. There is a canonical homeomorphism*

$$|[U/H](\mathbb{R})| \xrightarrow{\sim} \mathrm{H}^1(G, H(\mathbb{C}))$$

of spaces over $U(\mathbb{R})$, where the space on the right hand side is defined in Definition 4.9.

Proof. Note that $|[U/H](\mathbb{R})|$ parametrizes pairs (u, P) where $u\colon \mathrm{Spec}(\mathbb{R}) \to U$ is an $\mathbb{R}$-point and P is a H_u-torsor over $\mathbb{R}$. Such a pair (u, P) corresponds to an element $\gamma(u, P) \in \mathrm{H}^1(G, H_u(\mathbb{C})) \subset \mathrm{H}^1(G, H)$ (see e.g. Lemma 6.1). This gives the bijection fibrewise over $U(\mathbb{R})$, and this bijection is a homeomorphism by Lemma 9.14. □

Theorem 4.12. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, such that $|\mathrm{Aut}(y)|$ is constant for $y \in \mathcal{X}(\mathbb{C})$. Let $\mathcal{X} \to M$ be the coarse moduli space of $\mathcal{X}$. Then the induced map $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ is open, and a topological covering over each connected component of its image.*

Proof. By Lemma 3.2, we know that $\mathcal{X} \to M$ is a gerbe, and that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite étale. The proof proceeds in two steps.

Step 1: *If the proposition holds for gerbes $\mathcal{X} \to M$ which have a section, then it holds for all gerbes $\mathcal{X} \to M$.* Indeed, we let $U \to \mathcal{X}$ be a surjective étale morphism where U is a scheme over $\mathbb{R}$, such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective. We then look at the base change $\mathcal{Y} \coloneqq \mathcal{X} \times_M U$, which fits in a 2-cartesian diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{X} \arrow[r] & M. \end{tikzcd}$$

Observe that the map $|\mathcal{Y}(\mathbb{R})| \to |\mathcal{X}(\mathbb{R})| \times_{M(\mathbb{R})} U(\mathbb{R})$ is a homeomorphism. Since $\mathcal{X} \to M$ is étale (as it is étale locally on M of the form $[U/H] \to U$ for a finite flat group scheme $H \to U$, and $H \to U$ is étale since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is étale, so that $[U/H] \to U$ is étale), the composition $U \to \mathcal{X} \to M$ is étale. Therefore, by Lemma 4.7, the map $U(\mathbb{R}) \to M(\mathbb{R})$ is a local homeomorphism, whose image is the image of $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$. Consequently, by Lemma 4.8, if the base change $|Y(\mathbb{R})| \to U(\mathbb{R})$ of $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ by the local homeomorphism $U(\mathbb{R}) \to M(\mathbb{R})$ is a covering map over each connected component of its image, then the same holds for $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$. Step 1 follows.

Step 2: $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ *is a topological covering when* $\mathcal{X} \to M$ *has a section.* Indeed, assuming that $\mathcal{X} \to M$ has a section, we have $\mathcal{X} = [U/H]$ for a scheme U of finite type over $\mathbb{R}$ and a finite flat group scheme $H \to U$, which is étale because $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is étale. We have $|[U/H](\mathbb{R})| \cong \mathrm{H}^1(G, H(\mathbb{C}))$ as spaces over $U(\mathbb{R})$ by Lemma 4.11, and $\mathrm{H}^1(G, H(\mathbb{C})) \to U(\mathbb{R})$ is a topological covering by Lemma 4.10.

5 Smith–Thom for classifying stacks

As a first example of the Smith–Thom inequality, we verify it in the case of a classifying stack over a point. Let Γ be a finite group scheme over $\mathbb{R}$, given by a finite group $\Gamma(\mathbb{C})$ and an involution $\sigma\colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$, and consider the stack $\mathcal{X} = [\mathrm{Spec}(\mathbb{R})/\Gamma]$.

Proof of Proposition 1.8. Recall that, by definition,

$$|\mathcal{X}(\mathbb{R})| = \{\text{isomorphism classes of } \Gamma\text{-torsors over } \mathbb{R}\}.$$

This is a finite discrete set, which is well-know for being in bijection with $\mathrm{H}^1(G, \Gamma)$ (see for example Lemma 6.1). In particular, $h^*(|\mathcal{X}(\mathbb{R})|) = |\mathrm{H}^1(G, \Gamma)|$. On the other hand, by Example 3.3, we have

$$\mathcal{I}_{\mathcal{X}}(\mathbb{C}) = \Gamma(\mathbb{C})/\Gamma(\mathbb{C}) \quad \text{so that} \quad h^*(\mathcal{X}(\mathbb{C})) = |\Gamma(\mathbb{C})/\Gamma(\mathbb{C})|.$$

So the Smith–Thom inequality for $\mathcal{X}$ follows from the following group theoretic lemma, whose proof has been suggested to us by Will Sawin. $\square$

Lemma 5.1. *Let Γ be a finite group with an action of G. Then the following inequality holds:*

$$|\mathrm{H}^1(G,\Gamma)| \leq |\Gamma/\Gamma| \,.$$

Proof. Let $\sigma\colon \Gamma \to \Gamma$ be the involution corresponding to the G-action. Let σ-conj be the equivalence relation on Γ induced by the action of Γ on its self by σ-conjugacy (i.e. h acts by $h(g) = hg\sigma(h^{-1})$. For every $h \in \Gamma$, we let $\mathrm{Stab}_\sigma(h)$ (resp. $[h]_\sigma$) be the stabilizer (resp. the orbit) of h for the σ-conjugacy action and $\mathrm{Stab}(h)$ (resp. $[h]$), the stabilizer (resp. the orbit) for the conjugacy action.

We claim the following chain of inequalities and equalities:

$$|\mathrm{H}^1(G,\Gamma)| \leq |(\Gamma/\sigma\text{-conj})| = |(\Gamma/\Gamma)^G| \leq |(\Gamma/\Gamma)|.$$

Since the first and the last inequalities follow from the inclusions $\mathrm{H}^1(G,\Gamma) \subseteq (\Gamma/\sigma\text{-conj})$ and $(\Gamma/\Gamma)^G \subseteq \Gamma/\Gamma$, we just need to prove the middle equality.

For this, define

$$S \coloneqq \{(g,h) \in \Gamma \times \Gamma \text{ such that } g = hg\sigma(h)^{-1}\} \subseteq \Gamma \times \Gamma$$

and observe that the projections $p_1, p_2 : S \to \Gamma$ into the first and the second factor induce surjective maps $p_1 : S \to (\Gamma/\sigma\text{-conj})$ and $p_2 : S \to (\Gamma/\Gamma)^G$.

We now compute $|S|$ in two different ways, once using p_1 and once p_2.

For any $[g]_\sigma \in (\Gamma/\sigma\text{-conj})$, one has

$$p_1^{-1}([g]_\sigma) = \big\{(g',h) \text{ such that } g' \in [g]_\sigma \text{ and } g' = hg'\sigma(h)^{-1}\big\} =$$

$$= \coprod_{g' \in [g]_\sigma} \{h \in \Gamma \text{ such that } g' = hg'\sigma(h)^{-1}\} = \coprod_{g' \in [g]_\sigma} \mathrm{Stab}_\sigma(g').$$

In particular

$$|S| = \sum_{[g]_\sigma \in (\Gamma/\sigma\text{-conj})} \big(\sum_{g' \in [g]_\sigma} (|\mathrm{Stab}_\sigma(g')|\big).$$

Since for every $g' \in [g]$ one has

$$|\mathrm{Stab}_\sigma(g)| = |\mathrm{Stab}_\sigma(g')| \quad \text{and} \quad |\mathrm{Stab}_\sigma(g')| = |\Gamma|/|[g]_\sigma|$$

we get

$$|S| = \sum_{[g]_\sigma \in (\Gamma/\sigma\text{-conj})} \Big(\sum_{g' \in [g]_\sigma} (|\Gamma|/|[g]_\sigma|\Big) = \sum_{[g]_\sigma \in (\Gamma/\sigma\text{-conj})} |\Gamma| = |\Gamma| \Gamma/\sigma - \mathrm{conj}| \tag{7}$$

On the other hand, For any $[h] \in (\Gamma/\Gamma)^G$, one has

$$p_2^{-1}([h]) = \big\{(g,h') \text{ such that } [h'] = [h] \text{ and } \sigma(h') = g^{-1}h'g\big\} =$$

$$= \coprod_{h' \in [h]} \{g \in \Gamma \text{ such that } \sigma(h') = g^{-1}h'g\}.$$

Observe that

$$|\{g \in \Gamma \text{ such that } \sigma(h') = g^{-1}h'g\}| = |\mathrm{Stab}(h')| = |\mathrm{Stab}(h)$$

so that

$$|p_2^{-1}([h])| = \sum_{h' \in [h]} |\mathrm{Stab}(h)| = \sum_{h' \in [h]} |\Gamma|/|[h]| = |\Gamma|.$$

Hence,

$$|S| = \sum_{[h] \in (\Gamma/\Gamma)^\sigma} |p_2^{-1}([h])| = \sum_{[h] \in (\Gamma/\Gamma)^\sigma} |\Gamma| = |\Gamma| (\Gamma/\Gamma)^\sigma|. \tag{8}$$

Combining Equations (7) and (8), we get the result. □

6 Topology of a real quotient stack

In this section, we fist describe the topology of the real points of the quotient stack $[X/\Gamma]$ of a real variety X on which a finite $\mathbb{R}$-group Γ acts, and prove Theorem 1.5. Then we use this description it to verify the Smith–Thom inequality 1.2 in many examples.

6.1 The real locus of a quotient stack over the real numbers. In this section, we calculate $|\mathcal{X}(\mathbb{R})|$ when $\mathcal{X} = [X/\Gamma]$ is the stacky quotient of a quasi-projective scheme X by a finite group scheme Γ over $\mathbb{R}$.

6.1.1 Group schemes over the reals and torsors. Let Γ be a finite group scheme over $\mathbb{R}$. Let $G = \langle\sigma\rangle \coloneqq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let $\sigma_\Gamma \colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$ be the action of G on $\Gamma(\mathbb{C})$ corresponding to Γ. Define

$$\mathrm{Z}^1(G,\Gamma) \coloneqq \{\gamma \in \Gamma(\mathbb{C}) \text{ such that } \gamma\sigma_\Gamma(\gamma) = e\} \subset \Gamma(\mathbb{C}).$$

Recall (see e.g. [Ser94, Chapitre I, §5]) that there is a canonical identification

$$\mathrm{H}^1(G,\Gamma) = \mathrm{Z}^1(G,\Gamma)/\sim$$

where $\sim$ is the equivalence relation that identifies $\gamma_1, \gamma_2 \in \Gamma(\mathbb{C})$ if there exists a $\beta \in \Gamma(\mathbb{C})$ such that $\gamma_2 = \beta^{-1}\gamma_1\sigma_\Gamma(\beta)$.

Choose a set of representative $H \subset \mathrm{Z}^1(G,\Gamma)$ for the equivalence relation $\sim$ on $\mathrm{Z}^1(G,\Gamma)$, so that the composition $H \subset \mathrm{Z}^1(G,\Gamma) \to \mathrm{H}^1(G,\Gamma)$ is a bijection; we choose H such that $e \in H$. For each $\gamma \in H$, we define an involution

$$\varphi^\gamma \colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C}) \qquad \text{as} \qquad \varphi^\gamma(g) = \sigma(g) \cdot \gamma^{-1}.$$

We consider the resulting G-set $(\Gamma(\mathbb{C}), \varphi^\gamma)$. Note that left multiplication defines an action of the G-group $(\Gamma(\mathbb{C}), \sigma_\Gamma)$ on the G-set $(\Gamma(\mathbb{C}), \varphi^\gamma)$. In particular, if P_γ is the $\mathbb{R}$-scheme associated to $(\Gamma(\mathbb{C}), \varphi^\gamma)$, we get an action of the $\mathbb{R}$-group scheme Γ on the $\mathbb{R}$-scheme P_γ that turns the latter into a Γ-torsor.

Lemma 6.1. *The following map is bijective:*

$$\begin{aligned}\mathrm{H}^1(G,\Gamma) = H &\to \{\textit{isomorphism classes of } \Gamma\textit{-torsors over } \mathrm{Spec}(\mathbb{R})\},\\ \gamma &\mapsto P_\gamma.\end{aligned}$$

Proof. This is well-known. □

6.1.2 The topology of the real locus of a quotient stack. We continue with the above notation. Define an involution

$$\sigma_\Gamma^\gamma \colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C}) \qquad \text{as} \qquad \sigma_\Gamma^\gamma(g) \coloneqq \gamma\sigma_\Gamma(g)\gamma^{-1}.$$

Let X be a quasi-projective scheme over $\mathbb{R}$ with real structure $\sigma_X \colon X(\mathbb{C}) \to X(\mathbb{C})$, acted upon from the left by the finite group scheme Γ over $\mathbb{R}$. For $\gamma \in H$, define an involution $\sigma_X^\gamma \colon X(\mathbb{C}) \to X(\mathbb{C})$ as $\sigma_X^\gamma(x) = \gamma \cdot \sigma(x)$. The pair $(X(\mathbb{C}), \sigma_X^\gamma)$ corresponds to a quasi-projective scheme X_γ over $\mathbb{R}$. Similarly, for $\gamma \in H$, the pair $(\Gamma(\mathbb{C}), \sigma_\Gamma^\gamma)$ corresponds to a finite group scheme Γ_γ over $\mathbb{R}$. Note that

$$X_\gamma(\mathbb{R}) = X(\mathbb{C})^{\sigma_X^\gamma} \quad \text{and} \quad \Gamma_\gamma(\mathbb{R}) = \Gamma(\mathbb{C})^{\sigma_\Gamma^\gamma} \quad \text{for each} \quad \gamma \in H.$$

Proof of Theorem 1.5. Recall that we need to prove that there exists a canonical homeomorphism

$$|[X/\Gamma](\mathbb{R})| \xrightarrow{\sim} \coprod_{\gamma\in H} X_\gamma(\mathbb{R})/\Gamma_\gamma(\mathbb{R}).$$

To prove this, we first observe that the action of $\Gamma(\mathbb{C})$ on $X(\mathbb{C})$ is compatible with the action of σ_X^γ and σ_Γ^γ. Indeed, for $x \in X(\mathbb{C})$ and $g \in \Gamma(\mathbb{C})$, we have:

$$\sigma_X^\gamma(g\cdot x) = \gamma\cdot\sigma_X(g\cdot x) = \gamma\cdot\sigma_\Gamma(g)\cdot\sigma_X(x) = \gamma\cdot\sigma_\Gamma(g)\cdot\gamma^{-1}\cdot\gamma\cdot\sigma_X(x) = \sigma_\Gamma^\gamma(g)\cdot\sigma_X^\gamma(x).$$

Therefore, we obtain an action of the G-group $(\Gamma(\mathbb{C}),\sigma_\Gamma^\gamma)$ on the G-space $(X(\mathbb{C}),\sigma_X^\gamma)$. In particular, the subgroup

$$\Gamma_\gamma(\mathbb{R}) = \Gamma(\mathbb{C})^{\sigma_\Gamma^\gamma} \subset \Gamma(\mathbb{C})$$

of elements fixed under σ_Γ^γ acts on the fixed space $X_\gamma(\mathbb{R}) = X(\mathbb{C})^{\sigma_X^\gamma} \subset X(\mathbb{C})$.

Fix $\gamma \in H$ and take any $x \in X_\gamma(\mathbb{R})/\Gamma_\gamma(\mathbb{R})$. Choose a $y \in X_\gamma(\mathbb{R})$ that lifts x and consider the $\Gamma(\mathbb{C})$-equivariant morphism

$$f_y\colon \Gamma(\mathbb{C}) \to X(\mathbb{C}), \qquad g \mapsto g\cdot y.$$

This morphism is compatible with the G-action φ^γ on $\Gamma(\mathbb{C})$ and with the G-action σ_X on $X(\mathbb{C})$, hence it gives rise to a Γ-equivariant morphism

$$f_y\colon P_\gamma \to X$$

of schemes over $\mathbb{R}$. Define

$$\begin{aligned}\alpha(x) &\coloneqq (P_\gamma, f_y) \in |[X/\Gamma](\mathbb{R})| \\ &= \{\text{pairs } (P,f) \mid P \text{ a } \Gamma\text{-torsor}, f \text{ a } \Gamma\text{-equivariant morphism } P\to X\}/_{\cong}.\end{aligned}$$

We first show that α is well defined, i.e. that it does not depends on the choice of the lift y of X. If $z \in X_\gamma(\mathbb{R})$ is another that lift x, then there exists a $g \in \Gamma_\gamma$ such that $y = g\cdot z$. Since $g \in \Gamma_\gamma(\mathbb{R}) = \Gamma(\mathbb{C})^{\sigma_\Gamma^\gamma}$, the morphism $g\colon P_\gamma \to P_\gamma$ sending h to hg is an isomorphism of torsors over $\mathbb{R}$, fitting into a commutative diagram:

$$\begin{array}{ccc} P_\gamma & \xrightarrow{f_y} & X \\ \downarrow{\scriptstyle g} & & \| \\ P_\gamma & \xrightarrow{f_z} & X. \end{array}$$

In particular, we have an equality of isomorphism classes $[(P_\gamma, f_y)] = [(P_\gamma, f_z)] \in$

$|[X/\Gamma](\mathbb{R})|$. We conclude that we get a canonical map

$$\alpha\colon\ |[X/\Gamma](\mathbb{R})| \longrightarrow \coprod_{\gamma\in H} X_\gamma(\mathbb{R})/\Gamma_\gamma(\mathbb{R}), \tag{9}$$

and it is straightforward to show that α is bijective. It remains to prove that the bijection α is a homeomorphism.

To see this, note that for each $\gamma \in H$, we have a natural morphism

$$X_\gamma \longrightarrow [X/\Gamma]. \tag{10}$$

Namely, to give such a map is to give:

(1) a $\Gamma \times_{\mathbb{R}} X_\gamma$-torsor $P \to X_\gamma$ over X_γ, and

(2) a $\Gamma \times_{\mathbb{R}} X_\gamma$ equivariant morphism $P \to X \times_{\mathbb{R}} X_\gamma$ of schemes over X_γ.

As for (1), we put $P = P_\gamma \times_{\mathbb{R}} X_\gamma$, which is a $\Gamma \times_{\mathbb{R}} X_\gamma$-torsor by base-changing the Γ-torsor structure of $P_\gamma \to \mathrm{Spec}(\mathbb{R})$ along $X_\gamma \to \mathrm{Spec}(\mathbb{R})$. As for (2), we consider the morphism

$$P_\gamma \times_{\mathbb{R}} X_\gamma \longrightarrow X \times_{\mathbb{R}} X_\gamma \tag{11}$$

defined via Galois descent by the map

$$\Gamma(\mathbb{C}) \times X(\mathbb{C}) \longrightarrow X(\mathbb{C}) \times X(\mathbb{C}), \qquad (g, x) \mapsto (gx, x),$$

which is indeed compatible with the anti-holomorphic involution $(g, x) \mapsto (\varphi^\gamma(g), \sigma_X^\gamma(x))$ on the left hand side and the anti-holomorphic involution $(x, y) \mapsto (\sigma_X(x), \sigma_X^\gamma(y))$ on the right hand side. Since the map (11) is $\Gamma \times_{\mathbb{R}} X_\gamma$-equivariant, it yields the desired morphism (10).

We obtain a morphism

$$U := \coprod_{\gamma\in H} X_\gamma \longrightarrow [X/\Gamma],$$

and, by the fact that the map α in (9) is a bijection (which has already been shown), the induced map

$$U(\mathbb{R}) = \coprod_{\gamma\in H} X_\gamma(\mathbb{R}) \longrightarrow |[X/\Gamma](\mathbb{R})| \tag{12}$$

is surjective. By definition of the real analytic topology on $|[X/\Gamma](\mathbb{R})|$, see Definition 4.3, and by independence of the étale surjective cover essentially surjective on real points, see Proposition 4.4, it follows that the topology on $|[X/\Gamma](\mathbb{R})|$ is the quotient topology coming from the surjection (12) and the real analytic topology on $U(\mathbb{R}) = \coprod_\gamma X_\gamma(\mathbb{R})$. As the diagram

$$\begin{tikzcd}
\coprod_{\gamma\in H} X_\gamma(\mathbb{R}) \arrow[r, equal] \arrow[d] & \coprod_{\gamma\in H} X_\gamma(\mathbb{R}) \arrow[d] \\
|[X/\Gamma](\mathbb{R})| \arrow[r, "\alpha"] & \coprod_{\gamma\in H} X_\gamma(\mathbb{R})/\Gamma_\gamma
\end{tikzcd}$$

commutes, and as each quotient $X_\gamma(\mathbb{R})/\Gamma_\gamma$ carries the the quotient topology coming from $X_\gamma(\mathbb{R}) \to X_\gamma(\mathbb{R})/\Gamma_\gamma$, this proves that α is a homeomorphism as wanted. □

In the above notation, assume that X is smooth over $\mathbb{R}$. Then the topological space $|[X/\Gamma](\mathbb{R})|$ can naturally be enhanced with the structure of a real analytic orbifold, see [GF22a, Section 2.2.3]. The proof of Theorem 1.5 shows that the following holds.

Corollary 6.2. *Assume that the quasi-projective scheme X is smooth over $\mathbb{R}$. Then the homeomorphism (4) in Theorem 1.5 is an isomorphism of real analytic orbifolds.*

Proof. As in the proof of Theorem 1.5, we consider the natural surjective morphism

$$U := \coprod_{\gamma\in H} X_\gamma \longrightarrow [X/\Gamma]$$

which is essentially surjective on $\mathbb{R}$-points. Define $\mathcal{X} = [X/\Gamma]$. Then

$$U \times_{\mathcal{X}} U \cong \coprod_{\gamma,\gamma'\in H} X_\gamma \times_{\mathcal{X}} X_{\gamma'} =: R.$$

For $\gamma \in H$, let R_γ for be a scheme such that $R_\gamma \cong X_\gamma \times_{\mathcal{X}} X_\gamma$. Since $(X_\gamma \times_{\mathcal{X}} X_{\gamma'})(\mathbb{R}) = \emptyset$ for $\gamma \neq \gamma' \in H$, we get $R(\mathbb{R}) = \coprod_{\gamma\in H} R_\gamma(\mathbb{R})$. Thus,

$$\coprod_{\gamma\in H} R_\gamma(\mathbb{R}) \rightrightarrows \coprod_{\gamma\in H} X_\gamma(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$$

is a presentation of $|\mathcal{X}(\mathbb{R})|$ by a groupoid object in the category of real analytic manifolds, proving the corollary. □

6.2 Smith–Thom for various quotient stacks. In this section we apply Theorem 1.5 to prove the Smith–Thom inequality (3) in a number of examples.

Example 6.3. Let Γ be any finite $\mathbb{R}$-group scheme. Take $X = \mathrm{Spec}(\mathbb{R})$ with the trivial action of Γ. Then Theorem 1.5 just says that $|[X/\Gamma](\mathbb{R})|$ is the disjoint union of $\#\mathrm{H}^1(G,\Gamma)$ points, which also follows directly from the definitions and Lemma 6.1. We already verified the Smith–Thom inequality (3) in Proposition 1.8.

Example 6.4. Let $X \coloneqq \mathbb{A}^1_{\mathbb{R}}$.

1. Let $\Gamma \coloneqq \mathbb{Z}/2$ endowed with the trivial G-action. We let Γ act on X via the map sending x to $-x$. To compute $\mathcal{X} \coloneqq [\mathbb{A}^1_{\mathbb{R}}/\Gamma]$, we start observing that $\mathrm{H}^1(G,\Gamma)$ has two elements $1, \gamma$. One computes that $X(\mathbb{R})/\Gamma \simeq \mathbb{R}_{\geq 0}$ and also $X_\gamma(\mathbb{R})/\Gamma \simeq \mathbb{R}_{\geq 0}$. Hence, by Theorem 1.5,
$$|\mathcal{X}(\mathbb{R})| = \mathbb{R}_{\geq 0} \coprod \mathbb{R}_{\geq 0}.$$
In conclusion, we find that $h^*(|\mathcal{X}(\mathbb{R})|) = 2$, so that, since $h^*(I_{\mathcal{X}(\mathbb{C})}) = 2$ by Example 3.9.1, we see that the Smith–Thom inequality (3) holds and it is an actual equality.

   For completeness, we also describe the natural map $f\colon |[\mathbb{A}^1_{\mathbb{R}}/\Gamma](\mathbb{R})| \to X/\Gamma(\mathbb{R})$. Identifying $X/\Gamma(\mathbb{C})$ with $\mathbb{C}$ via the map $x \mapsto x^2$, one sees that $X/\Gamma(\mathbb{R}) = \mathbb{R} \subseteq \mathbb{C}$. Under this identification, f induces an homeomorphism of $X(\mathbb{R})/\Gamma$ with $\mathbb{R}_{\geq 0}$ and of $X(\mathbb{R})_\gamma/\Gamma$ with $\mathbb{R}_{\leq 0}$. Hence $\#f^{-1}(x) = 1$ for every $x \neq 0$ and $\#f^{-1}(0) = 2$ as predicted by Proposition 4.5.

2. Let $\Gamma \coloneqq \mathbb{Z}/2 \times \mathbb{Z}/2$ endowed with the G-action exchanging the coordinates. We let Γ act on $\mathbb{A}^1_{\mathbb{R}}$ via $(a,b) * x \coloneqq (-1)^{a+b}x$. To compute $|\mathcal{X}(\mathbb{R})| \coloneqq [\mathbb{A}^1_{\mathbb{R}}/\Gamma]$, we start observing that $\mathrm{H}^1(G,\Gamma) = 0$. Hence
$$|\mathcal{X}(\mathbb{R})| = X(\mathbb{R})/\Gamma(\mathbb{R}) = X(\mathbb{R}) = \mathbb{R}$$
since $\Gamma(\mathbb{R})$ acts trivially on $X(\mathbb{C})$. In conclusion we find that $h^*(|\mathcal{X}(\mathbb{R})|) = 1$, so that, since $h^*(I_{\mathcal{X}(\mathbb{C})}) = 6$ by Example 3.9.2, the Smith–Thom inequality (3) holds and it is a strict inequality.

   For completeness, we also describe the natural map $f\colon |[\mathbb{A}^1_{\mathbb{R}}/\Gamma](\mathbb{R})| \to X/\Gamma(\mathbb{R})$. As in the previous Example 1, one identifies $X/\Gamma(\mathbb{R})$ with $\mathbb{R} \subseteq \mathbb{C}$. Under this identification, the map $f\colon \mathbb{R} \to \mathbb{R}$ becomes the absolute value map, so that it is not surjective, $\#f^{-1}(x) = 2$ for every $x > 0$ and $\#f^{-1}(0) = 1$ as predicted by Proposition 4.5.

Example 6.5. Let $X = \mathbb{A}^2_{\mathbb{R}}$.

1. Let $\Gamma \coloneqq \mathbb{Z}/2$ endowed with the trivial G-action. To compute the real locus of

$\mathcal{X} \coloneqq [\mathbb{A}^2_{\mathbb{R}}/\Gamma]$, we start observing that $\mathrm{H}^1(G,\Gamma)$ has two elements $1, \gamma$. By Theorem 1.5,

$$|\mathcal{X}(\mathbb{R})| = X(\mathbb{R})/\Gamma(\mathbb{R}) \coprod X_\gamma(\mathbb{R})/\Gamma(\mathbb{R})$$

One computes that $X(\mathbb{R})/\Gamma$ and $X_\gamma(\mathbb{R})$ are two half-planes, so that $h^*(|\mathcal{X}(\mathbb{R})| = 2$, $h^*(I_{\mathcal{X}(\mathbb{C})}) = 2$ by Example 3.10.1, we see that the Smith–Thom inequality (3) holds and it is an equality.

2. Let $\Gamma \coloneqq \mathbb{Z}/2 \times \mathbb{Z}/2$ endowed with the G-action exchanging the coordinates. We let Γ act on $\mathbb{A}^2_{\mathbb{R}}$ via its G-equivariant quotient $\mathbb{Z}/2$, acting by exchange of coordinates. To compute $\mathcal{X} \coloneqq [\mathbb{A}^2_{\mathbb{R}}/\Gamma]$, we start observing that $\mathrm{H}^1(G,\Gamma) = 0$, so that, by Theorem 1.5,

$$|\mathcal{X}(\mathbb{R})| = X(\mathbb{R})/\Gamma(\mathbb{R}) = \mathbb{R}^2,$$

since $\Gamma(\mathbb{R})$ acts trivially on $X(\mathbb{C})$. In conclusion, we get that $h^*(|\mathcal{X}(\mathbb{R})| = 1$. Since $h^*(I_{\mathcal{X}(\mathbb{C})}) = 4$ by Example 3.10.2, we see that the Smith–Thom inequality (3) holds and it is a strict inequality.

Example 6.6. Let A be a real abelian variety of dimension g, so that $A(\mathbb{R}) \simeq (S^1)^g \times (\mathbb{Z}/2)^k$ for some $0 \leq k \leq g$ compatibly with the group structure. Consider the inversion $[-1] : A \to A$ and write $\Gamma \coloneqq \mathbb{Z}/2$. Let $\mathcal{X} \coloneqq [A/\mathbb{Z}/2]$ where $\mathbb{Z}/2$ acts via $[-1]$ and let γ be the unique non trivial element of $\mathrm{H}^1(G,\Gamma)$ By Theorem 1.5,

$$|\mathcal{X}(\mathbb{R})| \coloneqq A(\mathbb{R})/[-1] \coprod A_\gamma(\mathbb{R})/[-1].$$

By construction A_γ is the quadratic twist of A, hence $A(\mathbb{R}) \cong A_\gamma(\mathbb{R})$ as topological G-spaces. In particular, we get

$$|\mathcal{X}(\mathbb{R})| \simeq (S^1)^g \times (\mathbb{Z}/2)^k \coprod (S^1)^g \times (\mathbb{Z}/2)^k,$$

hence $h^*(|\mathcal{X}(\mathbb{R})|) = 2^{g+k+1}$. By 3.3(7),

$$\mathcal{I}_{\mathcal{X}}(\mathbb{C}) \simeq A(\mathbb{C})/[-1] \coprod \coprod_{x \in A(\mathbb{C})[2]} \{x\}.$$

Since $A(\mathbb{C})/[-1] \simeq A(\mathbb{C})$ and $\#A(\mathbb{C})[2] = 2^{2g}$, we get $h^*(\mathcal{I}_{\mathcal{X}}(\mathbb{C})) = 2^{2g} + 2^{2g} = 2^{2g+1}$. Since $k \leq g$, the inequality (3) is verified and it is an equality if and only if A is maximal.

Example 6.7. Let Y be a real algebraic variety, let $\Gamma \coloneqq \mathbb{Z}/2$ act on $Y \times Y$ by exchanging the coordinates and let $\mathcal{X} \coloneqq [(Y \times Y)/\Gamma]$. If γ in the non trivial element of $\mathrm{H}^1(G,\Gamma)$, by Theorem 1.5, one has

$$|\mathcal{X}(\mathbb{R})| \simeq (Y(\mathbb{R}) \times Y(\mathbb{R}))/\Gamma \coprod (Y \times Y)_\gamma(\mathbb{R})/\Gamma \simeq (Y(\mathbb{R}) \times Y(\mathbb{R}))/\Gamma \coprod Y(\mathbb{C})/G.$$

Observe that

$$\mathcal{X}(\mathbb{C})^G \simeq (Y(\mathbb{R}) \times Y(\mathbb{R}))/\Gamma \coprod_{Y(\mathbb{R})} Y(\mathbb{C})/G,$$

where $i : Y(\mathbb{R}) \hookrightarrow \mathcal{X}(\mathbb{C})^G$ embeds diagonally in $Y(\mathbb{R}) \times Y(\mathbb{R})$ and naturally in $Y(\mathbb{C})/G$. If $f \colon |\mathcal{X}(\mathbb{R})| \to (Y(\mathbb{R}) \times Y(\mathbb{R})/\Gamma)(\mathbb{R})$ is the natural morphism, the exact sequence of sheaves

$$0 \to \mathbb{Z}/2 \to f_*\mathbb{Z}/2 \to i_*\mathbb{Z}/2 \to 0,$$

shows that

$$h^*(|\mathcal{X}(\mathbb{R})|) \leq h^*(\mathcal{X}(\mathbb{C})^G) + h^*(Y(\mathbb{R})). \tag{13}$$

On the other hand,

$$\mathcal{I}_{\mathcal{X}}(\mathbb{C}) \simeq \mathcal{X}(\mathbb{C}) \coprod Y(\mathbb{C}).$$

while by the classical Smith–Thom inequality for $\mathcal{X}(\mathbb{C}) \coprod Y(\mathbb{C})$ we get

$$h^*(\mathcal{X}(\mathbb{C})^G) + h^*(Y(\mathbb{R})) \leq h^*(\mathcal{X}(\mathbb{C})) + h^*(Y(\mathbb{C})) = h^*(\mathcal{I}_{\mathcal{X}}(\mathbb{C})).$$

Combining this with (13), we get that the Smith–Thom inequality (3) is satisfied.

7 Smith–Thom for real stacky curves

In this section we prove Theorem 1.9. The proof is rather indirect, in the sense that we do not compare directly the topology of $|[X/\Gamma](\mathbb{R})|$ with $I_{[X/\Gamma]}(\mathbb{C})$, but rather we compute separately $h^*(|[X/\Gamma](\mathbb{R})|)$ and $h^*(I_{[X/\Gamma]}(\mathbb{C}))$ and then we compare the two numbers by using the classical Smith–Thom inequality and Lemma 5.1.

In Section 7.1 we compute $h^*(I_{[X/\Gamma]}(C))$, in Section 7.2 $h^*([X/\Gamma](\mathbb{R}))$ and finally in Section 7.3 we combine the two computations to prove Theorem 1.9.

7.1 Inertia of complex stacky curves. Let X be a smooth one-dimensional scheme of finite type over $\mathbb{C}$, and let Γ be a finite abelian group which acts on X over $\mathbb{C}$. We let $K \subset \Gamma$ be the kernel of the homomorphism $\Gamma \to \mathrm{Aut}_{\mathbb{C}}(X)$ associated to the Γ-action, and define $Q \coloneqq \Gamma/K$. This gives a short exact sequence of finite abelian groups

$$0 \to K \to \Gamma \to Q \to 0.$$

The restriction of the action on X of Γ to K yields the trivial action of K on X, and the induced action of Q on X is faithful. Let

$$M = M_{[X/\Gamma]} = M_{[X/Q]} = X/Q$$

be the coarse quotient of X by Q.

Proposition 7.1. *Assume that the subgroup $K \subset \Gamma$ is contained in the center of Γ, so that for every $x \in X(\mathbb{C})$ there is an inclusion $K \subseteq \Gamma_x/\Gamma_x$. Let $\Delta \subset M_{[X/\Gamma]}(\mathbb{C})$ be the branch locus of the quotient map $q\colon X(\mathbb{C}) \to X(\mathbb{C})/Q$, and choose a lift $y_x \in X(\mathbb{C})$ of each $x \in \Delta$. There is a canonical isomorphism isomorphism of complex analytic spaces*

$$I_{[X/\Gamma]}(\mathbb{C}) = \left(K \times M_{[X/\Gamma]}(\mathbb{C})\right) \coprod \coprod_{x\in\Delta} \left(\Gamma_{y_x}/\Gamma_{y_x} - K\right)$$

that commutes with the canonical projections onto $M_{[X/\Gamma]}$.

Proof. We may assume that X is connected. It suffices to show that the map $I_{[X/\Gamma]} \to M_{[X/\Gamma]}$ has $\#K$ disjoint sections. Indeed, $I_{[X/\Gamma]} \to M_{[X/\Gamma]}$ is finite by Lemma 3.3, hence for each irreducible component $Z \subset I_{[X/\Gamma]}$ of dimension one, the restriction $Z \to M_{[X/\Gamma]}$ is a finite morphism of curves, hence an isomorphism if it admits a section; moreover, over the open subset of $M_{[X/\Gamma]}$ where the stabilizer group is exactly K, the fibres of $I_{[X/\Gamma]} \to M_{[X/\Gamma]}$ have cardinality exactly $\#K$ by Proposition 7.1.

Write $\mathcal{X} = [X/\Gamma]$. Let $S \subset X \times_{\mathbb{C}} \Gamma$ be the stabilizer group scheme associated to the action of Γ on X over $\mathbb{C}$, so that S can be described pointwise as

$$S = \{(x, g) \in X \times_{\mathbb{C}} \Gamma \mid g \cdot x = x\}.$$

Then Γ acts on S by $\gamma \cdot (x, g) = (\gamma \cdot x, \gamma g \gamma^{-1})$ for $\gamma \in \Gamma$ and $(x, g) \in S$. Moreover we have a canonical isomorphism $\mathcal{I}_{\mathcal{X}} = S/\Gamma$ (see e.g. [Jar, Exercise 3.2.12]).

Since K is contained in the center of Γ, to any $k \in K$ one can associate the following well defined section s_k of the canonical map $\mathcal{I}_{\mathcal{X}} \to M_{\mathcal{X}}$:

$$s_k\colon X/\Gamma = M_{\mathcal{X}} \longrightarrow \mathcal{I}_{\mathcal{X}} = S/\Gamma, \qquad [x] \mapsto [(x, k)]. \tag{14}$$

By construction, the sections s_k and $s_{k'}$ are disjoint for $k \neq k' \in K$, and so the proposition follows. $\square$

Proposition 7.2. *Assume that $K \subset \Gamma$ is contained in the center of Γ. Then*

$$\begin{aligned}\dim \mathrm{H}^*(M(\mathbb{C}), F_{[X/\Gamma]}) = \#K \cdot \dim \mathrm{H}^*(M(\mathbb{C}), \mathbb{Z}/2) \\ + \left(\sum_{x \in \Delta} \#(\Gamma_{y_x}/\Gamma_{y_x}) \right) - \#\Delta \cdot \#K.\end{aligned} \tag{15}$$

If, in addition, Γ_{y_x} is abelian for each $x \in \Delta$, then

$$\dim \mathrm{H}^*(M(\mathbb{C}), F_{[X/\Gamma]}) = \#K \cdot \dim \mathrm{H}^*(M(\mathbb{C}), F_{[X/Q]}). \tag{16}$$

Proof. By Proposition 7.1, we have

$$\dim \mathrm{H}^*(M(\mathbb{C}), F_{[X/\Gamma]}) = \#K \cdot \dim \mathrm{H}^*(M(\mathbb{C}), \mathbb{Z}/2) + \sum_{x \in \Delta} \left(\#(\Gamma_{y_x}/\Gamma_{y_x}) - \#K \right), \tag{17}$$

and (17) implies (15).

Applying (17) to the quotient stack $[X/Q]$ gives

$$\dim \mathrm{H}^*(M(\mathbb{C}), F_{[X/Q]}) = \dim \mathrm{H}^*(M(\mathbb{C}), \mathbb{Z}/2) + \sum_{x \in \Delta} \left(\#(Q_{y_x}/Q_{y_x}) - 1 \right). \tag{18}$$

If Γ_{y_x} is abelian for each $x \in \Delta$, then one has

$$\Gamma_{y_x}/\Gamma_{y_x} = \Gamma_{y_x}, \quad Q_{y_x}/Q_{y_x} = Q_{y_x}, \quad \#K \cdot \#Q_{y_x} = \#\Gamma_{y_x}.$$

Hence (16) follows from (17) and (18) and we are done. $\square$

Example 7.3. Consider the moduli stack $\mathcal{A}_1$ of elliptic curves over $\mathbb{C}$, with coarse moduli space $\mathcal{A}_1 \to \mathsf{A}_1 = \mathbb{A}^1_{\mathbb{C}}$. Then $\dim \mathrm{H}^*(\mathsf{A}_1(\mathbb{C}), F_{\mathcal{A}_1}) = 8$. Indeed, we let $\ell \geq 3$ be a prime number and let $\mathcal{A}_1[\ell]$ be the moduli space of elliptic curves with level ℓ structure; it is equipped with a $\mathrm{SL}_2(\mathbb{F}_\ell)$-action such that $\mathcal{A}_1 = [\mathrm{SL}_2(\mathbb{F}_\ell) \setminus \mathcal{A}_1[\ell]]$. In this case, we have $K = \langle -1 \cdot \mathrm{Id} \rangle \subset \mathrm{SL}_2(\mathbb{F}_\ell) = \Gamma$, and $\Gamma/K = \mathrm{PSL}_2(\mathbb{F}_\ell)$. The locus $\Delta \subset \mathsf{A}_1(\mathbb{C})$ of isomorphism classes of elliptic curves with automorphism group larger than $\{\pm 1\}$ consists of two points, with respective automorphism groups $\mathbb{Z}/4$ and $\mathbb{Z}/6$. Thus, Proposition 7.2 implies that $\dim \mathrm{H}^*(\mathsf{A}_1(\mathbb{C}), F_{\mathcal{A}_1}) = 2 + 4 + 6 - 2 \cdot 2 = 8$.

Remark 7.4. Propositions 7.1 and 7.2 have a natural analogue in the complex analytic setting. In fact, these analogues generalize to the case where Γ is a discrete group, not necessarily finite, acting properly discontinuously on a complex manifold. For example, we can consider the complex analytic stack $\mathcal{A}_1^{\mathrm{an}}$ as a quotient stack: $\mathcal{A}_1^{\mathrm{an}} = [\mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}]$, where $\mathbb{H}$ is the upper half plane. In this case, $K \subset \mathrm{Sp}_2(\mathbb{Z})$ is the abelian subgroup of order two generated by -1 times the identity matrix, and $Q = \mathrm{PSL}_2(\mathbb{Z})$. Moreover, the coarse moduli space of $\mathcal{A}_1^{\mathrm{an}}$ is $\mathbb{C}$, and there is one isomorphism class of elliptic curves with automorphism group $\mathbb{Z}/4$, one with automorphism group $\mathbb{Z}/6$, and all other isomorphism classes have automorphism group $\mathbb{Z}/2$. Consequently, the complex analytic analogue of Proposition 7.2 implies as before that $\dim \mathrm{H}^*(\mathbb{C}, F_{\mathcal{A}_1^{\mathrm{an}}}) = 2 + 4 + 6 - 2 \cdot 2 = 8$.

7.2 Topology of real stacky curves. Recall from Definition 4.6, that if $\mathcal{X}$ a Deligne–Mumford stack with coarse moduli space $p : \mathcal{X} \to M$ and if $f\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ is the map induced on the real points, for $x \in M(\mathbb{R})$ we denote by $\mathrm{H}^1(G, x)$ the cardinality of $\mathrm{H}^1(G, \mathrm{Aut}(z))$, where $z \in \mathcal{X}(\mathbb{R})$ is such that $[z] \in f^{-1}(x)$.

7.2.1 Local geometry. We start by study the local topology of a stacky curve around a point with non-trivial stabilizer.

Lemma 7.5. *Let X be a smooth curve over $\mathbb{R}$. Let H be a finite $\mathbb{R}$-group scheme that acts on X over $\mathbb{R}$. Assume that H acts faithfully on X over $\mathbb{R}$.*

1. *For each $x \in X(\mathbb{R})$, there exists an integer $n \geq 1$ such that the stabilizer group scheme H_x is isomorphic to μ_n.*
2. *For $x \in X(\mathbb{R})$, the number $\mathrm{H}^1(G, [x]) = \#\mathrm{H}^1(G, H_x(\mathbb{C}))$ is equal to 1 (resp. 2) if n is odd (resp. even).*

Proof. Let $\Gamma = H(\mathbb{C})$. Since the action of Γ is faithful, there are only finitely many points $x \in X(\mathbb{R})$ with non trivial stabilizer Γ_x. Since the statement is trivial for points with trivial stabilizer, we focus on the points x with $\Gamma_x \neq 0$. Choose a G and Γ stable open neighbourhood U of x not containing any other point with non-trivial stabilizer and G-biholomorphic to an open disk centered in x endowed with the standard G-action. Since the group of biholomorphism of the disk with one fixed point is isomorphic to S^1, we see that Γ_x is cyclic isomorphic to $\mathbb{Z}/n$ for some integer n. Moreover a generator γ acts a $\gamma(z) = e^{i\theta}z$ if z is a local parameter around x. Since the G-action is compatible with the action of Γ, this forces a G-equivariant isomorphism $\Gamma_x \simeq \mu_n$.

The second item follows from the first and the fact that $|\mathrm{H}^1(G, \mu_n)|$ is 1 is n is odd and 2 if n is even. ∎

7.2.2 Global geometry. We now study the possible shapes of the connected components of the real points of a real stacky curve.

Proposition 7.6. *Let X be a smooth curve over $\mathbb{R}$. Let H be a finite étale group scheme over $\mathbb{R}$ which acts on X over $\mathbb{R}$. Let $C \subset |[X/H](\mathbb{R})|$ be a connected component of $|[X/H](\mathbb{R})|$. Then C homeomorphic to either an interval in $\mathbb{R}$ of the form $(0,1), (0,1]$ or $[0,1]$, or to the circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$. If X is proper then only the possibilities $[0,1]$ and $\mathbb{S}^1$ can occur.*

We actually prove something slighlty more general in the following Lemma 7.7. Observe that Proposition 7.6 follows from Theorem 1.5 and Lemma 7.7.

Lemma 7.7. *Let X be a smooth curve over $\mathbb{R}$. Let H be a finite étale group scheme over $\mathbb{R}$ acting on X over $\mathbb{R}$. Then each connected component of $X(\mathbb{R})/H(\mathbb{R})$ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$, to the interval $(0,1]$, to the interval $(0,1)$, or to the circle $\mathbb{S}^1 \subset \mathbb{R}^2$.*

Proof. We may assume that H acts faithfully on X.

First, assume that X is proper, so that $X(\mathbb{R})$ is compact and let C be a connected component of $X(\mathbb{R})$ with stabilizer $\mathrm{Stab}_{H(\mathbb{R})}(C)$ in $H(\mathbb{R})$. We start proving that

$$C/\mathrm{Stab}_{H(\mathbb{R})}(C) \simeq S^1 \quad \text{or} \quad C/\mathrm{Stab}_{H(\mathbb{R})}(C) \simeq [0,1] \tag{19}$$

Recall that every connected Riemann surface S admits a unique complete Riemann metric g with constant curvature being negative (genus ≥ 2), zero (genus zero) or positive (genus one). Moreover, for genus ≥ 2 the group $\mathrm{Bihol}(S)$ coincides with the group $\mathrm{Isom}(S,g)^+$ of orientation preserving isometries of the Riemannian manifold (S,g). In genus zero we have, for the subgroup $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{PGL}_2(\mathbb{C}) = \mathrm{Bihol}(\mathbb{P}^1(\mathbb{C}))$, that $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{SO}_3(\mathbb{R})$ acts as isometries on $\mathbb{P}^1(\mathbb{C}) \cong S^2$. The automorphism group of any complex elliptic curve preserves its Riemannian metric.

In particular, as H acts faithfully on X, there are natural inclusions

$$\mathrm{Stab}_{H(\mathbb{R})}(C) \subset H(\mathbb{R}) \subset \mathrm{Isom}(X(\mathbb{C})) \subset \mathrm{Homeo}(X(\mathbb{C}))$$

where $\mathrm{Isom}(X(\mathbb{C}))$ is the group of isometries with respect to the Riemannian metric of $X(\mathbb{C})$. Consider the connected component

$$C \subset X(\mathbb{R}) \subset X(\mathbb{C}).$$

We endow C with the Riemannian metric induced by the embedding $C \subset X(\mathbb{C})$. Then C is a compact one-dimensional Riemannian manifold, and hence isometric to a circle of some length L: we have $C \cong \mathbb{R}/L\mathbb{Z}$ with the standard Riemannian metric. In particular, $\mathrm{Isom}(C) \cong \mathrm{O}(2)$. By the above, we have $\mathrm{Stab}_{H(\mathbb{R})}(C) \subset \mathrm{Isom}(X(\mathbb{C}))$, and hence $\mathrm{Stab}_{H(\mathbb{R})}(C) \subset \mathrm{Isom}(C) \cong \mathrm{O}(2)$. So, $\mathrm{Stab}_{H(\mathbb{R})}(C)$ is a finite subgroup of $\mathrm{O}(2) = \mathrm{Isom}(S^1)$ with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, hence it is generated by multiplications by some root of unity and, possibly, by the standard complex conjugation on S^1. Hence we get (19).

Let $C_1, \ldots, C_n$ be the connected components of $X(\mathbb{R})$; each C_i is homeomorphic to $\mathbb{S}^1$. Then $I \coloneqq \{1, \ldots, n\}$ admits a partition $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k$ with $k \leq n$ such that the I_j are the orbits for the induced action of $H(\mathbb{R})$ on I. For each $j \in \{1, \ldots, k\}$, choose an element $i_j \in I_j$. Let $H(\mathbb{R})_j = \mathrm{Stab}_{H(\mathbb{R})}(C_{i_j})$ be the stabilizer of C_{i_j} in the group $H(\mathbb{R})$. Then

$$X(\mathbb{R})/H(\mathbb{R}) = \left(\coprod_{i=1}^{n} C_i\right)/H(\mathbb{R}) = \coprod_{j=1}^{k}\left(\left(\coprod_{i \in I_j} C_i\right)/H(\mathbb{R})\right) \cong \coprod_{j=1}^{k} C_{i_j}/H(\mathbb{R})_j.$$

Thus, the lemma in the case where X is proper follows from (19).

In the general case, consider the smooth projective compactification $X \hookrightarrow Y$ of X. The action of H on X extends to an action of H on Y, and the natural map $X(\mathbb{R})/H(\mathbb{R}) \to Y(\mathbb{R})/H(\mathbb{R})$ is an open embedding whose complement is a finite set (possibly empty). By what has already been proved, each connected component of $Y(\mathbb{R})/H(\mathbb{R})$ is homeomorphic to $[0,1]$ or $\mathbb{S}^1$. By removing the points in $\Delta(\mathbb{R}) \subset Y(\mathbb{R})$, where $\Delta = Y - X$, we see that each connected component of $X(\mathbb{R})/H(\mathbb{R})$ is homeomorphic to $[0,1]$, $(0,1]$, $(0,1)$ or $\mathbb{S}^1$, and we are done. $\square$

7.2.3 Map to the coarse moduli space. Finally, we study the map from a real stacky curve to its moduli space.

Proposition 7.8. *Let X be a smooth curve over $\mathbb{R}$. Let H be a finite $\mathbb{R}$-group scheme that acts on X over $\mathbb{R}$, with associated real structure $\sigma\colon H(\mathbb{C}) \to H(\mathbb{C})$. Assume that H acts faithfully on X over $\mathbb{R}$. Let $p\colon [X/H] \to X/H = M$ be the coarse moduli space map, with induced map $f\colon |[X/H](\mathbb{R})| \to M(\mathbb{R})$.*

Let $C \subset M(\mathbb{R})$ be a connected component and let $\mathscr{S} = \{x_1, \ldots, x_n\} \subseteq C$ be the finite set of points such that $H_{x_i} \neq 0$.

1. *Assume that $\mathscr{S} = \emptyset$. Then the map $f^{-1}(C) \to C$ is a homeomorphism.*
2. *Assume that $\mathscr{S} \neq \emptyset$.*

(a) *If C is an interval, then for every homeomorphism $\varphi : C \xrightarrow{\sim} (0,1)$, there exists an homeomorphism*

$$\psi\colon f^{-1}(C) \xrightarrow{\sim} (0,y_1]\coprod[y_1,y_2]\coprod\cdots\coprod[y_{n-1},y_n]\coprod[y_n,1)$$

such that the following diagram commutes:

$$\begin{tikzcd} f^{-1}(C) \arrow[r, "\psi"] \arrow[d] & (0,y_1]\sqcup[y_1,y_2]\coprod\cdots\coprod[y_{n-1},y_n]\coprod[y_n,1) \arrow[d] \\ C \arrow[r, "\varphi"] & (0,1), \end{tikzcd}$$

where the vertical arrows are the canonical ones and $y_i = \varphi(x_i)$.

(b) *If C is a circle, then for every homeomorphism $\varphi\colon C \xrightarrow{\sim} \mathbb{S}^1$, there exists a homeomorphism*

$$\psi\colon f^{-1}(C) \xrightarrow{\sim} [e^{i\theta_1},e^{i\theta_2}]\coprod\cdots\coprod[e^{i\theta_{n-1}},e^{i\theta_n}]\coprod\left([0,e^{i\theta_1}]\coprod_{0\sim 1}[e^{i\theta_n},1]\right)$$

such that the following diagram commutes:

$$\begin{tikzcd} f^{-1}(C) \arrow[r, "\psi"] \arrow[d] & {[e^{i\theta_1},e^{i\theta_2}]\coprod\cdots\coprod[e^{i\theta_{n-1}},e^{i\theta_n}]\coprod\left([0,e^{i\theta_1}]\coprod_{0\sim 1}[e^{i\theta_n},1]\right)} \arrow[d] \\ C \arrow[r, "\varphi"] & \mathbb{S}^1, \end{tikzcd}$$

where the vertical arrows are the canonical ones and $\varphi(x_j) = e^{i\theta_j}$.

Proof. By Lemma 7.5, for each connected component $K \subset M(\mathbb{R})$, the map $f^{-1}(K) \to K$ is an isomorphism outside $\mathscr{S} \subset K$ and has two fibers above each point of $\mathscr{S}$. The proposition follows readily from this and Proposition 7.6. ☐

7.3 Smith–Thom for real stacky curves.

Proof of Theorem 1.9. The action of H on X corresponds to a homomorphism

$$H \longrightarrow \underline{\mathrm{Aut}}_{\mathbb{R}}(X), \tag{20}$$

where the latter denotes the automorphism group scheme of X over $\mathbb{R}$. Let $K \subset H$ be the kernel of (20), and let $Q = H/K$ be the quotient of H by K. The canonical map

$Q \to \underline{\mathrm{Aut}}_{\mathbb{R}}(X)$ is a closed immersion. In particular, the group $Q(\mathbb{C})$ acts faithfully on $X(\mathbb{C})$. We have $M(\mathbb{C}) = X(\mathbb{C})/H(\mathbb{C}) = X(\mathbb{C})/Q(\mathbb{C})$.

Step 1: *If H is abelian, and if the Smith–Thom inequality (3) holds for the quotient stack $[X/Q]$, then it also holds for $[X/H]$.*

Proof. Assume the Smith–Thom inequality (3) for $[X/Q]$, and consider the canonical map

$$g\colon\ |[X/H](\mathbb{R})| \longrightarrow |[X/Q](\mathbb{R})|\,. \tag{21}$$

By Theorem 1.5, we have a commutative diagram of the form

$$\begin{tikzcd} |[X/H](\mathbb{R})| \arrow[r, "g"] \arrow[d, "\wr"] & |[X/Q](\mathbb{R})| \arrow[d, "\wr"] \\ \coprod_{[\gamma]\in \mathrm{H}^1(G,H)} X(\mathbb{C})^{\sigma_\gamma}/H(\mathbb{C})^{\sigma_\gamma} \arrow[r] & \coprod_{[\mu]\in \mathrm{H}^1(G,Q)} X(\mathbb{C})^{\sigma_\mu}/Q(\mathbb{C})^{\sigma_\mu}, \end{tikzcd}$$

where the map on the bottom is induced by the canonical map $\mathrm{H}^1(G,H) \to \mathrm{H}^1(G,Q)$. Since

$$X(\mathbb{C})^{\sigma_\gamma}/H(\mathbb{C})^{\sigma_\gamma} = X(\mathbb{C})^{\sigma_\mu}/Q(\mathbb{C})^{\sigma_\mu}$$

for each $[\gamma] \in \mathrm{H}^1(G,H)$ mapping to $[\mu] \in \mathrm{H}^1(G,K)$, this proves that the map g in (21) is a topological covering over each connected component of its image. Moreover, the exact sequence of pointed sets

$$0 \to K(\mathbb{R}) \to H(\mathbb{R}) \to Q(\mathbb{R}) \to \mathrm{H}^1(G,K) \to \mathrm{H}^1(G,H) \to \mathrm{H}^1(G,Q)$$

shows that the degree of g over a connected component of its image is bounded by $\left|\mathrm{H}^1(G,K(\mathbb{C}))\right|$.

By Proposition 7.6, each connected component C of $|[X/H](\mathbb{R})|$ is homeomorphic to a circle or an interval. If C is a circle then, by the above, $g^{-1}(C)$ consists of at most $\mathrm{H}^1(G,K(\mathbb{C}))$ circles, so that

$$h^*(g^{-1}(C)) \leq 2 \cdot \#\mathrm{H}^1(G,K(\mathbb{C})).$$

Similarly, if C is an interval, then $g^{-1}(C)$ is an union of at most $\mathrm{H}^1(G,K(\mathbb{C}))$ intervals, so that

$$h^*(g^{-1}(I)) \leq \#\mathrm{H}^1(G,K(\mathbb{C}))$$

Therefore, we have:

$$\begin{aligned}
h^*(|[X/H](\mathbb{R})| &= \sum_{C\in\pi_0(|[X/Q](\mathbb{R})|)} h^*\left(g^{-1}(C)\right) \\
&= \sum_{C \text{ circle}} h^*\left(g^{-1}(C)\right) + \sum_{C \text{ interval}} h^*\left(g^{-1}(C)\right) \\
&\overset{\text{(a)}}{\leq} \sum_{C \text{ circle}} 2\cdot \#\mathrm{H}^1(G,K(\mathbb{C})) + \sum_{C \text{ interval}} \#\mathrm{H}^1(G,K(\mathbb{C})) \\
&= \#\mathrm{H}^1(G,K(\mathbb{C}))\cdot h^*(|[X/Q](\mathbb{R})|) \\
&\overset{\text{(b)}}{\leq} \#K(\mathbb{C})\cdot \dim \mathrm{H}^*(M(\mathbb{C}),F_{[X/Q]}) \\
&\overset{\text{(c)}}{=} \dim \mathrm{H}^*(M(\mathbb{C}),F_{[X/H]}),
\end{aligned}$$

where (a) holds by the previous discussion, (b) by the assumption that the Smith–Thom inequality (3) holds for $[X/Q]$ and the fact that $\#\mathrm{H}^1(G,K(\mathbb{C}))\leq \#K(\mathbb{C})$, while (c) holds by Proposition 7.2 which we can apply since H is abelian. This proves what we want. □

Let $\Delta\subset M(\mathbb{C}) = X(\mathbb{C})/Q$ be the branch locus of the quotient map $q\colon X(\mathbb{C})\to X(\mathbb{C})/Q$. For each $x\in\Delta$ choose an element $y_x\in X(\mathbb{C})$ such that $q(y_x)=x$. Define

$$\Delta' \coloneqq \left\{x\in\Delta\cap M(\mathbb{R}) \mid \mathrm{H}^1(G,x)>1\right\},$$

where $\mathrm{H}^1(G,x)=\#\mathrm{H}^1(G,H_y(\mathbb{C}))$ for some $y\in q^{-1}(x)$, see Definition 4.6.

Step 2: *The Smith–Thom inequality* (3) *holds when when the action of H on X over $\mathbb{R}$ is faithful (i.e. $H=Q$).*

Proof. Consider the map $f\colon |\mathcal{X}(\mathbb{R})|\to M(\mathbb{R})$, and note that f is surjective. Let $C\subset M(\mathbb{R})$ be a connected component which is homeomorphic to a circle. By Proposition 7.8, $f^{-1}(C)$ is homeomorphic to a circle if $\mathrm{H}^1(G,x)=1$ for each $x\in C$, and $f^{-1}(C)$ is homeomorphic to the union of $\#(C\cap\Delta')$ intervals if $\Delta'\cap C\neq\emptyset$. In particular, we have:

$$h^*\left(f^{-1}(C)\right) = \begin{cases} 2 & \text{if} \quad C\cap\Delta'=\emptyset, \\ \#(C\cap\Delta') & \text{if} \quad C\cap\Delta'\neq\emptyset. \end{cases}$$

Let $I\subset M(\mathbb{R})$ be a connected component which is homeomorphic to the open interval $(0,1)$. By Proposition 7.8, $f^{-1}(I)$ is homeomorphic to the union of $\#(I\cap\Delta')+1$

intervals. In particular, we have:

$$h^*(f^{-1}(I)) = \#(I \cap \Delta') + 1$$

Therefore, we have:

$$\begin{aligned}
h^*(|\mathcal{X}(\mathbb{R})|) &= \sum_{C\in\pi_0(M(\mathbb{R}))\text{ circle}} h^*(f^{-1}(C)) + \sum_{I\in\pi_0(M(\mathbb{R}))\text{ interval}} h^*(f^{-1}(C)) \\
&\overset{(a)}{=} \sum_{C\cap\Delta'=\emptyset} 2 + \sum_{C\cap\Delta'\neq\emptyset} \#(C\cap\Delta') + \sum_I \big(\#(C\cap\Delta')+1\big) \\
&= \left(\sum_{C\cap\Delta'=\emptyset} 2 + \sum_I 1\right) + \left(\sum_{C\cap\Delta'\neq\emptyset} \#(C\cap\Delta') + \sum_{I\cap\Delta'\neq\emptyset} \#(C\cap\Delta')\right) \\
&\leq \dim \mathrm{H}^*(M(\mathbb{R}),\mathbb{Z}/2) + \sum_{x\in\Delta} 1 \\
&\overset{(b)}{\leq} \dim \mathrm{H}^*(M(\mathbb{R}),\mathbb{Z}/2) + \sum_{x\in\Delta} (\#(H_{y_x}(\mathbb{C})/H_{y_x}(\mathbb{C}))-1) \\
&\overset{(c)}{\leq} \dim \mathrm{H}^*(M(\mathbb{C}),\mathbb{Z}/2) + \sum_{x\in\Delta} (\#(H_{y_x}(\mathbb{C})/H_{y_x}(\mathbb{C}))-1) \\
&\overset{(d)}{=} \dim \mathrm{H}^*(M(\mathbb{C}),F_{\mathcal{X}}),
\end{aligned}$$

where (a) follows from the previous discussion, (b) from $\#(H_{y_x}(\mathbb{C})/H_{y_x}(\mathbb{C})) \geq 2$, (c) from the Smith-Thom inequality 1 for $M(\mathbb{C})$ and finally (d) from Proposition 7.2. This proves Step 2. □

By combining Steps 1 and 2, the theorem follows. □

8 Topology of a split gerbe over a real variety

Let U be a geometrically connected scheme locally of finite type over $\mathbb{R}$. To simplify the discussion, we assume that $U(\mathbb{R}) \neq \emptyset$. Let $H \to U$ be a finite étale group scheme over U. For every $x \in U(\mathbb{R})$, we write $\overline{x} \in U(\mathbb{C})$ for the associated geometric point and H_x (resp. $H_{\overline{x}}$) for the fiber of $H \to U$ over x (resp. $\overline{x}$). The scheme H_x is a group scheme over $\mathrm{Spec}(\mathbb{R})$ so that $H_{\overline{x}}$ is the constant group scheme over $\mathbb{C}$ associated to a finite group which, by abuse of notation, we will also denote by $H_{\overline{x}}$. The finite group

$H_{\overline{x}}$ is endowed with an action of G, hence with an involution

$$\sigma_x \colon H_{\overline{x}} \to H_{\overline{x}}. \tag{22}$$

Let $\mathcal{X} = [U/H]$ be the associated classifying stack, where H acts trivially on U. Recall that the natural quotient map $U \to \mathcal{X}$ is a section of the coarse moduli space map $\mathcal{X} \to U/H = U$, so that, in particular, the map $f \colon |\mathcal{X}(\mathbb{R})| \to U(\mathbb{R})$ is surjective.

8.1 Topology of the connected components of the real locus. In this section we explain how to compute the topology of $\mathcal{X}(\mathbb{R})$, by comparing it with $U(\mathbb{R})$. To state the main result, recall that, since $H(\mathbb{C}) \to U(\mathbb{C})$ is a topological cover, if $p \in U(\mathbb{C})$ there is a natural action of $\pi_1(U(\mathbb{C}), p)$ on $H_p(\mathbb{C})$.

Theorem 8.1. *Let U be a geometrically connected $\mathbb{R}$-variety such that $U(\mathbb{R}) \neq \emptyset$. Let $H \to U$ be a finite étale group scheme and set $\mathcal{X} := [U/H]$. The following holds.*

1. *The canonical map $f \colon |\mathcal{X}(\mathbb{R})| \to U(\mathbb{R})$ is a topological covering over each connected component of $U(\mathbb{R})$, with fibre $\mathrm{H}^1(G, H_p(\mathbb{C}))$ above a point $p \in U(\mathbb{R})$.*
2. *Let C be a connected component of $U(\mathbb{R})$, and fix $p \in C$. The image of the natural map $\pi_1(C, p) \to \pi_1(U(\mathbb{C}), p)$ lies in the subgroup of elements $g \in \pi_1(U(\mathbb{C}), p)$ whose action on $H_p(\mathbb{C})$ is G-equivariant. In particular, the group $\pi_1(C, p)$ acts naturally on $\mathrm{H}^1(G, H_p(\mathbb{C}))$.*
3. *The covering space associated to the above action of $\pi_1(C, p)$ on $\mathrm{H}^1(G, H_p(\mathbb{C}))$ is canonically isomorphic to the covering space $f^{-1}(C) \to C$.*

The rest of the section is devoted to the proof of Theorem 8.1 and to some of its corollaries. We begin with some preliminaries; the actual proof of Theorem 8.1 is carried out in Section 8.1.2.

8.1.1 Action of fundamental groups. Fix $p \in U(\mathbb{R})$ and write C for the connected component of $U(\mathbb{R})$ containing p. Recall that $H \to U$ corresponds to an action $\pi_1^{\text{ét}}(U, \overline{p})$ on $H_{\overline{p}}$,

$$\rho_p \colon \pi_1^{\text{ét}}(U, \overline{p}) \to \mathrm{Aut}(H_{\overline{p}}),$$

compatible with the group structure of $H_{\overline{p}}$, where $\pi_1^{\text{ét}}(U, \overline{p})$ is the étale fundamental group of U at the geometric point $\overline{p}$.

Since U is geometrically connected, the natural morphisms $U_{\mathbb{C}} \to U \to \mathrm{Spec}(\mathbb{R})$ induce a short exact sequence of groups

$$1 \to \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p}) \to \pi_1^{\text{ét}}(U, \overline{p}) \to G \to 1. \tag{23}$$

Restricting ρ_p to $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$, we get an action of $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ on $H_{\overline{p}}$,

$$\rho_p^{\mathbb{C}} \colon \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p}) \to \operatorname{Aut}(H_{\overline{p}}),$$

which corresponds to the étale $U_{\mathbb{C}}$-group scheme $H_{\mathbb{C}} \to U_{\mathbb{C}}$. Recall that $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ identifies with the profinite completion of usual fundamental group $\pi_1(U(\mathbb{C}), p)$ so that, in particular, there is a map $\pi_1(U(\mathbb{C}), p) \to \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$. We denote again by

$$\rho_p^{\mathbb{C}} \colon \pi_1(U(\mathbb{C}), p) \to \operatorname{Aut}(H_{\overline{p}})$$

the restriction of $\rho_p^{\mathbb{C}} \colon \pi_1^{\text{ét}}(U_{\mathbb{C}}, p) \to \operatorname{Aut}(H_{\overline{p}})$ along the map $\pi_1(U(\mathbb{C}), p) \to \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$; this representation of $\pi_1^{\text{ét}}(U_{\mathbb{C}}, p)$ corresponds to the topological covering $H(\mathbb{C}) \to U(\mathbb{C})$.

Viewing p as a morphism of schemes $p \colon \operatorname{Spec}(\mathbb{R}) \to U$, we get a morphism $\pi_1(p) \colon G = \pi_1(\operatorname{Spec}(\mathbb{R}), \overline{p}) \to \pi_1^{\text{ét}}(U, \overline{p})$ which splits (23), and hence yields an isomorphism

$$\pi_1^{\text{ét}}(U, \overline{p}) \simeq \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p}) \rtimes G. \tag{24}$$

This yields an action of $G = \langle \sigma \rangle$ on $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ by the usual formula $\sigma \cdot \alpha = \sigma \alpha \sigma^{-1}$ for $\alpha \in \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ (where we view G as a subgroup $G \subset \pi_1^{\text{ét}}(U, \overline{p})$), and this action is compatible with the action of G on $\pi_1(U(\mathbb{C}), \overline{p})$ defined as follows: for $\alpha \in \pi_1(U(\mathbb{C}), p)$, we have $\sigma \cdot \alpha = (\sigma_U)_*(\alpha)$, where σ_U is complex conjugation on $U(\mathbb{C})$.

Restricting ρ_p to G via $\pi_1(p)$, we get an action of G on $H_{\overline{p}}$ which identifies with the natural involution σ_p on $H_{\overline{p}}$, see (22). Now consider the morphism $\pi_1(C, p) \to \pi_1(U(\mathbb{C}), p)$ and, by abuse of notation, write

$$\rho_p^{\mathbb{C}} \colon \pi_1(C, p) \to \operatorname{Aut}(H_{\overline{p}}) \tag{25}$$

for the restriction of $\rho_p^{\mathbb{C}}$ to $\pi_1(C, p)$.

Lemma 8.2. *The above action (25) of $\pi_1(C, p)$ on $H_{\overline{p}}$ commutes with σ_p, in the sense that $\sigma_p(\gamma \cdot x) = \gamma \cdot \sigma_p(x)$ for $\gamma \in \pi_1(C, p)$ and $x \in H_{\overline{p}}$. In particular, it preserves $\mathrm{Z}^1(G, H_{\overline{p}}) = \{x \in H_{\overline{p}} \mid x \cdot \sigma_p(x) = e\} \subset H_{\overline{p}}$, and the induced action of $\pi_1(C, p)$ on $\mathrm{Z}^1(G, H_{\overline{p}})$ descends to an action of $\pi_1(C, p)$ on $\mathrm{H}^1(G, H_{\overline{p}})$.*

Proof. We need to show that for every $\alpha \in \pi_1(C, p)$, one has

$$\rho_p^{\mathbb{C}}(\alpha) \circ \sigma_p = \sigma_p \circ \rho_p^{\mathbb{C}}(\alpha) \qquad \text{as maps} \qquad H_{\overline{p}} \to H_{\overline{p}}. \tag{26}$$

Via the isomorphism (24), we write each element $\beta \in \pi_1^{\text{ét}}(U, \overline{p})$ as a pair $\beta = (\beta_1, \beta_2)$

with $\beta \in \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ and $\beta_2 \in G$. Denote by α the image of α in $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$. Then the equation (26) can be rewritten as

$$\rho_p(\alpha, e)^{-1} \circ \rho_p(e, \sigma) \circ \rho_p(\alpha, e) = \rho_p(1, \sigma).$$

Since ρ_p is a group homomorphism we have

$$\rho_p(\alpha, e)^{-1} \circ \rho_p(e, \sigma) \circ \rho_p(\alpha, e) = \rho_p((\alpha^{-1}, e) \cdot (e, \sigma) \cdot (\alpha, e)).$$

By the definition of the semi-direct product group structure, we have

$$(\alpha^{-1}, e) \cdot (e, \sigma) \cdot (\alpha, e) = (\alpha^{-1}\sigma^{-1}\alpha\sigma, \sigma).$$

The image $\alpha \in \pi_1(U(\mathbb{C}), p)$ of $\alpha \in \pi_1(C, p)$ satisfies $(\sigma_U)_*(\alpha) = \alpha \circ \sigma_U = \alpha$. For the image $\alpha \in \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$, one therefore has $\sigma \cdot \alpha = \sigma^{-1}\alpha\sigma = \alpha$. Hence, we get

$$\rho_p(\alpha, e)^{-1} \circ \rho_p(e, \sigma) \circ \rho_p(\alpha, e) = \rho_p((\alpha^{-1}, e)(e, \sigma)(\alpha, e)) = \rho_p(e, \sigma),$$

and the proof is concluded. $\square$

8.1.2 Change of base point. In the previous section we fixed a $p \in U(\mathbb{R})$ to study H_p, but it will be important for us to understand how H_p change with the point. The main result of Section 8.1.2 is the following.

Proposition 8.3. *Let U be a geometrically connected scheme of finite type over $\mathbb{R}$. Let $Y \to U$ be an étale cover. For $p \in U(\mathbb{R})$, consider the natural anti-holomorphic G-action $\sigma_p \colon Y_{\overline{p}} \to Y_{\overline{p}}$. Let $p, q \in U(\mathbb{R})$ and choose a topological path $\gamma_{q,p}$ from q to p in $U(\mathbb{C})$. Consider the element $\omega_{p,q} := (\gamma_{q,p} \circ \sigma_U) * \gamma_{q,p}^{-1} \in \pi_1(U(\mathbb{C}), p)$ (where $*$ denotes the composition of paths). Then the following diagram commutes:*

$$\begin{tikzcd}
Y_{\overline{q}} \arrow[r, "(\gamma_{q,p})_*"] \arrow[d, "\sigma_q"] & Y_{\overline{p}} \arrow[d, "\sigma_p"] \\
Y_{\overline{q}} \arrow[d, equal] & Y_{\overline{p}} \arrow[d, "\omega_{p,q}"] \\
Y_{\overline{q}} \arrow[r, "(\gamma_{q,p})_*"] & Y_{\overline{p}}.
\end{tikzcd}$$

Here, $\omega_{p,q} = (\gamma_{q,p} \circ \sigma_U) * \gamma_{q,p}^{-1} \in \pi_1(U(\mathbb{C}), p)$ acts on $Y_{\overline{p}}$ as an element of $\pi_1(U(\mathbb{C}), p)$, and $(\gamma_{q,p})_* \colon Y_{\overline{q}} \xrightarrow{\sim} Y_{\overline{p}}$ is the canonical isomorphism induced by the path $\gamma_{q,p}$.

Example 8.4. Let $U \subseteq \mathbb{G}_m$ be an open subset whose real part contains $[-1,0)$ and $(0,1]$ and let $\pi\colon E \to U$ be a family of smooth elliptic curves. Let $p = 1 \in U(\mathbb{R})$ and assume that Y_p is a maximal real elliptic curve. Define a local system $\mathcal{F} \coloneqq \pi_*\mathbb{Z}/2$ of finite dimension $\mathbb{Z}/2$-modules on $U_{\text{ét}}$, and let

$$Y \longrightarrow U$$

be the associated finite étale cover. Thus,

$$Y_{\overline{q}} = \mathrm{H}^1(E_q(\mathbb{C}), \mathbb{Z}/2) \qquad \text{for} \qquad q \in U(\mathbb{R}).$$

Since E_p is a maximal real elliptic curve, the action of G on $\mathrm{Y}_{\overline{p}} = \mathrm{H}^1(E_p(\mathbb{C}), \mathbb{Z}/2)$ is trivial.

1. Assume that the action of the standard loop γ around 0 (viewed as an element of $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$) on $\mathrm{H}^1(E_p(\mathbb{C}), \mathbb{Z}/2)$ is not trivial (this happens for example for the family whose affine equation is $y^2 = (x^2 - t)(x+2)$ where t is the coordinate of U). Let $q = -1$ and choose as $\gamma_{q,p}$ the standard "half circle" around 0, so that $\omega_{p,q} = \gamma$, hence it acts non-trivially on $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$. Since the action of G on $\mathrm{H}^1(E_p(\mathbb{C}), \mathbb{Z}/2)$ is trivial, we deduce from Proposition 8.3 that the action of G on $\mathrm{H}^1(E_q(\mathbb{C}), \mathbb{Z}/2)$ is not trivial. In particular, the real elliptic curve E_q is not maximal.
2. Assume now that the action of $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ on $\mathrm{H}^1(E_p(\mathbb{C}), \mathbb{Z}/2)$ is trivial (this happens for example for the family whose affine equation is $y^2 = x(x+2)(x+3)$ where t is the coordinate of U). Let $q = -1$. Since $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ acts trivially on $\mathrm{H}^1(E_{\overline{p}}(\mathbb{C}), \mathbb{Z}/2)$, for every choice of path $\gamma_{q,p}$ from -1 to 1, the loop ω acts trivially on $\mathrm{H}^1(E_p(\mathbb{C}), \mathbb{Z}/2)$, so that from Lemma 8.3, we deduce that E_q is maximal.

Before going to the proof of Proposition 8.3 of let us drawn some consequences.

Corollary 8.5. *Let U be a geometrically connected scheme locally of finite type over $\mathbb{R}$, and let $H \to U$ be a finite étale group scheme. Let $C \subset U(\mathbb{R})$ be a connected component. For $p, q \in C$, there is an isomorphism $H_{\overline{p}} \simeq H_{\overline{q}}$ of finite G-groups. In particular, up to bijection, the set $\mathrm{H}^1(G, H_{\overline{q}})$ does not depend on the choice of $q \in C$.*

Proof. Since p, q are inside the same connected component of $U(\mathbb{R})$, we can choose a path $\gamma_{q,p}\colon [0,1] \to U(\mathbb{C})$ that is fixed by $(\sigma_U)_*$, i.e., $\gamma_{p,q}$ lifts to a path $\gamma_{p,q}\colon [0,1] \to U(\mathbb{R})$. In particular,

$$\omega_{p,q} = (\gamma_{p,q} \circ \sigma_U) * \gamma_{q,p}^{-1} = \gamma_{q,p} * \gamma_{q,p}^{-1} = e \in \pi_1(U(\mathbb{C}), p).$$

Thus, the corollary follows from Proposition 8.3. □

Proof of Theorem 8.1. By Theorem 4.12, Proposition 4.5 and Corollary 8.5 the morphism $f^{-1}(C) \to C$ is finite étale with fibers $\mathrm{H}^1(G, H_{\overline{p}})$. The corresponding action of $\pi_1(C,p)$ on $\mathrm{H}^1(G, H_{\overline{p}})$ identifies with the action $\rho_p^{\mathbb{C}}$ of Lemma 8.2 which is induced, via the morphism $\pi_1(C,p) \to \pi_1(U(\mathbb{C}), \overline{p})$, by the action of $\pi_1(C,p)$ on $H_{\overline{p}}$. □

8.1.3 Proof of Proposition 8.3 We start by recalling how $\gamma_{q,p}$ induces an isomorphism $Y_{\overline{q}} \to Y_{\overline{p}}$. Chose a $y \in Y_{\overline{q}}$. By the theory of the topological fundamental groups, there is a unique path $\widetilde{\gamma}_{q,p}^y$ in $Y(\mathbb{C})$ lifting $\gamma_{q,p}$ starting in y, i.e. such that $\widetilde{\gamma}_{q,p}^y(0) = y$. Then $\varphi_{q,p}^{\mathbb{C}}(y) \coloneqq \widetilde{\gamma}_{q,p}^y(1)$ is the end-point of $\widetilde{\gamma}_{q,p}^y$.

We can now prove Proposition 8.3. By construction,

$$\sigma(\varphi(y)) = \sigma(\widetilde{\gamma}_{q,p}^y(e)) \quad \text{and} \quad \omega\varphi(\sigma(y)) = \omega\widetilde{\gamma}_{q,p}^{\sigma(y)}.$$

Observe that

$$\sigma(\widetilde{\gamma}_{q,p}^y(e)) = \sigma(\widetilde{\gamma}_{p,q}^y)(e)$$

and that $\sigma(\widetilde{\gamma}_{p,q}^y)$ is a path in $Y(\mathbb{C})$ lifting $\sigma(\gamma_{q,p})$ starting at $\sigma(y)$. In other words

$$\sigma(\varphi(y)) = \widetilde{\sigma(\gamma_{q,p})}^{\sigma(y)}(e).$$

On the other hand, by construction of the action of $\pi_1(U(\mathbb{C}), p)$ on $Y_{\overline{p}}$ one has

$$\omega\widetilde{\gamma_{q,p}}_{q,p}^{\sigma(y)}(e) = \widetilde{\omega\gamma_{q,p}}^{\sigma(y)}(e)$$

But $\omega\gamma_{q,p} = \sigma(\gamma_{q,p})$ by definition of ω, hence the proof is concluded. □

8.2 Interpretation in terms of the homotopy exact sequence. In order to do efficiently computations, we interpret Proposition 8.3 in terms of splitting of the homotopy exact sequence (23). Write Fset for the category of finite sets and, for a scheme Z, Fét(Z) for the category of finite étale cover of Z.

8.2.1 Galois formalism Recall that for every $q \in U(\mathbb{R})$, the group $\pi_1^{\text{ét}}(U, \overline{q})$ (resp. $\pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{q})$) is the automorphism of the functor

$$(-)_{\overline{q}} : \text{Fét}(U) \to \text{Fset} \quad (\text{resp. } (-)_{\overline{q}}^{\mathbb{C}} : \text{Fét}(U_{\mathbb{C}}) \to \text{Fset})$$

sending $Y \to U$ (resp. $Y \to U_{\mathbb{C}}$) to the geometric fiber $Y_{\overline{q}}$. By the general formalism of Galois categories, every isomorphism of functors

$$\varphi : (-)_{\overline{q}} \xrightarrow{\simeq} (-)_{\overline{p}},$$

induces an isomorphism

$$\varphi : \pi_1(U, \overline{q}) \xrightarrow{\simeq} \pi_1(U, \overline{p})$$

in a way that the action of $G(= \pi_1(\mathrm{Spec}(\mathbb{R}), \overline{q}))$ on $H_{\overline{q}}$ identifies with restriction of the action of $\pi_1(U, \overline{p})$ on $H_{\overline{q}}$ via the map

$$G = \pi_1(\mathrm{Spec}(\mathbb{R}), \overline{q}) \xrightarrow{\pi_1(q)} \pi_1(U, \overline{q}) \xrightarrow{\varphi} \pi_1(U, \overline{p}).$$

Since both $\pi_1(p)$ and $\varphi \circ \pi_1(q)$ are splitting of the sequence

$$0 \to \pi_1(U_{\mathbb{C}}, \overline{p}) \to \pi_1(U, \overline{p}) \to G \to 0,$$

to understand the action of G on $H_{\overline{q}}$, one has to understand how the different splitting of the sequence are related. This is the main result of the section.

Proposition 8.6. *Let us fix the isomorphism*

$$\pi_1^{\text{ét}}(U, \overline{p}) \simeq \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{x}) \rtimes G$$

induced by the splitting of (23) via $\pi_1(p)$. Chose a topological path $\gamma_{q,p}$ from q to p and write $\omega := \sigma_U(\gamma_{q,p})\gamma_{q,p}^{-1}$. Then the image of the section induced by $\pi_1(q)$ is (conjugated to) (ω, σ).

8.2.2 Splitting of semi-direct products We start recall some properties of splitting of semi-direct products. Let now Γ be any group with an action of G, consider the semi-direct product $\Gamma \rtimes G$ of G so that there is an exact sequence

$$0 \to \Gamma \to \Gamma \rtimes G \xrightarrow{\pi} G \to 0. \tag{27}$$

There is an obvious section s of π, namely the one sending σ to $(0, \sigma)$. Under this section, the action of G on Γ, can be recovered as the conjugation action by $(0, \sigma)$.

On the other hand, there might be many more splitting. Indeed the map

$$\{\epsilon \in \Gamma \text{ such that } \sigma(\epsilon)\epsilon = 1\} \xrightarrow{\simeq} \{\text{ splitting of } \pi\}$$

sending ϵ to the map defined by $s_\epsilon(\sigma) = (\epsilon, \sigma)$ is a bijection. Since we are mainly interested in studying objects up to conjugation, let us remark that the previous bijection, induces a bijection

$$\{\epsilon \in \Gamma \text{ such that } \sigma(\epsilon)\epsilon = 1\}/\sim \ \xrightarrow{\simeq} \{\text{ splitting of } \pi\}/conj,$$

where $\epsilon \sim \epsilon'$ if there exists $\gamma \in \Gamma$ such that $\epsilon = \gamma\epsilon'\sigma(\gamma)^{-1}$ and $conj$ in the conjugation action.

Example 8.7. Let $\Gamma = \mathbb{Z}$ endowed with the action of G by inversion. Then the set of splitting of (27), is in bijection with $\mathbb{Z}$. On the other hand, the set of splitting up to conjugation is only made by two elements, since (n, σ) in conjugated to (m, σ) if and only if n and m have the same parity.

The splitting s_ϵ of 27 induces, by conjugation via (ϵ, σ), the action on Γ given by $\sigma_\epsilon(\gamma) = \epsilon^{-1}\sigma(\gamma)\epsilon$.

8.2.3 Proof of Proposition 8.6 The isomorphisms $Y_{\overline{q}} \simeq Y_{\overline{p}}$ induced by $\gamma_{q,p}$, fits together to give an isomorphism $\varphi^{\mathbb{C}}_{q,p} : (-)^{\mathbb{C}}_{\overline{q}} \xrightarrow{\simeq} (-)^{\mathbb{C}}_{\overline{p}}$ of fiber functors. This in turn, induces an isomorphism $\varphi^{\mathbb{C}}_{q,p} : \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{q}) \xrightarrow{\simeq} \pi_1^{\text{ét}}(U_{\mathbb{C}}, \overline{p})$ (well defined up to conjugation), extending the usual isomorphism $\pi_1(U(\mathbb{C}), q) \to \pi_1(U(\mathbb{C}), p)$ defined by $\alpha \mapsto \gamma_{q,p}\alpha\gamma_{q,p}^{-1}$.

However, in general, the isomorphism of fiber functors $\varphi^{\mathbb{C}}_{q,p} : (-)^{\mathbb{C}}_{\overline{q}} \xrightarrow{\simeq} (-)^{\mathbb{C}}_{\overline{p}}$ is not equivariant for the Galois action, so that it does not always extends to an isomorphism of fiber functors $\varphi^{\mathbb{C}}_{q,p} : (-)_{\overline{q}} \xrightarrow{\simeq} (-)_{\overline{p}}$.

Chose now any isomorphism of fiber functors

$$\varphi_{q,p} : (-)_{\overline{q}} \xrightarrow{\simeq} (-)_{\overline{p}},$$

which induces a isomorphism $\varphi_{q,p} : \pi_1^{\text{ét}}(U, \overline{q}) \to \pi_1^{\text{ét}}(U, \overline{p})$, well defined up to conjugation.

Let $Y \to U$ be a finite connected étale cover. By Lemma 8.3, the action of $G = \pi_1(\mathrm{Spec}(\mathbb{R}, \overline{q}))$ on $Y_{\overline{p}}$ induced the morphism

$$G \xrightarrow{\pi_1(q)} \pi_1^{\text{ét}}(U, \overline{q}) \xrightarrow{\varphi_{q,p}} \pi_1^{\text{ét}}(U, \overline{p})$$

identifies with the natural action of G on $Y_{\overline{p}}$ after multiplication by ω. In other worlds, after identifying $Y_{\overline{p}}$ and $Y_{\overline{q}}$ using $\gamma_{q,p}$ one has $\sigma_q(y) = \omega\sigma_p(x)$.

Hence, if we fix the isomorphism

$$\pi_1(U, \overline{p}) \simeq \pi_1(U^{\mathbb{C}}, \overline{p}) \rtimes G$$

induced by $\pi_1(p)$, the image of section corresponding to $\pi_1(q)$ is (ω, σ). To conclude observe that $\sigma(\omega)\omega = 1$, so that it is actually a well defined section.

Remark 8.8. At the level on geometric fundamental group, one can do a similar procedure. Define $\sigma_U^{\gamma_{q,p}} : \pi_1(U(\mathbb{C}), q) \to \pi_1(U(\mathbb{C}), q)$ via $\omega^{-1}\sigma_U(\alpha)\omega$, i.e. by twisting σ_U on $\pi_1(U(\mathbb{C}), p)$ with the conjugation action of ω. This is still an involution of $\pi_1(U(\mathbb{C}), q)$. Since the action of σ_U on $\pi_1(U(\mathbb{C}), q)$ is well defined up to conjugation, the actions of $\sigma_U^{\gamma_{q,p}}$ and of σ_U on $Fin^{\text{ét}}(U_{\mathbb{C}})$ are isomorphic. With this new involution the isomorphism

$$\varphi_{p,q} : \pi_1(U(\mathbb{C})), p) \xrightarrow{\simeq} \pi_1^{\text{ét}}(U(\mathbb{C}), q),$$

becomes equivariant.

Example 8.9. Let U be $\mathbb{G}_m$ and take $p = 1 \in \mathbb{G}_m(\mathbb{R})$. In this case, one has $\pi_1^{\text{ét}}(U, \overline{p}) \simeq \hat{\mathbb{Z}} \rtimes G$ where G acts on $\hat{\mathbb{Z}}$ by inversion.

1. Take $q = 2$ and as $\gamma_{q,p}$ the natural path contained in the real part from q to p. Then ω is the trivial loop so that the image of section corresponding to $\pi_1(q)$ identifies is $(0, 1_G)$.
2. Take again $q = 2$ but as $\gamma_{q,p}$ a loop not contained in the real part and such that ω is not the trivial loop. By construction, the class of ω in $\pi_1(U(\mathbb{C}), p) \simeq \mathbb{Z}$ is the class of 2, hence via this γ, the image of the section $\pi_1(q)$ is $(2, 1_G)$. Even if it is different from the previous point, we remark that $(2, 1_G)$ is conjugated to $(0, 1_G)$ in $\mathbb{Z} \rtimes G$ (see Example 8.7), so that the conjugacy class of the section does not change.
3. Take $q = -1$ and choose as $\gamma_{q,p}$ the "half circle path" between -1 and 1. One has that γ is the class of 1 in $\mathbb{Z}$, so that the image of the section $\pi_1(q)$ is $(1, 1_G)$. Since this is not conjugated to $(0, 1_G)$, the two sections are different even up to conjugation.

8.3 Examples We now use Theorem 8.1 and the previous discussions, to verify the Smith–Thom inequality in a few examples.

8.3.1 Cover of the multiplicative group Let U be $\mathbb{G}_m$ and take $p = 1 \in \mathbb{G}_m(\mathbb{R})$. In this case, one has $\pi_1^{\text{ét}}(U, \overline{p}) \simeq \hat{\mathbb{Z}} \rtimes G$ where G acts on $\hat{\mathbb{Z}}$ by inversion. Let $\pi_1 : \hat{\mathbb{Z}} \rtimes G \to \mathbb{Z}/2$

be the natural projection on the first factor, which correspond to the cover $(-)^2 : \mathbb{G}_m \to \mathbb{G}_m$.

Since G acts trivially on the fiber in 1 of $(-)^2 : \mathbb{G}_m \to \mathbb{G}_m$ and not trivially on the fiber along -1, the element corresponding to the section associated to -1 is of the form $\epsilon_{-1} \coloneqq (n, \sigma)$ with n odd.

In the rest of the section we let $\mathbb{Z}/2$ acting on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ by exchanging the coordinates.

Example 8.10. Let $\pi_1 : \hat{\mathbb{Z}} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ be the morphism defined by the natural projection $\hat{\mathbb{Z}} \rtimes G \to \mathbb{Z}/2$ and we let $\hat{\mathbb{Z}} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ act on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ via the action of $\mathbb{Z}/2$. Let $H \to U$ be the corresponding group scheme. Since the action of G on the fiber of $\mathbb{G}_m \xrightarrow{2} \mathbb{G}_m$ over 1 is trivial, the action of G on $H_{\overline{1}}$ is trivial.

Hence, the pre-image of $(0{,}+\infty)$ along the map $[U/H](\mathbb{R}) \to U(\mathbb{R})$ is the disjoint union of $4 = |\mathrm{H}^1(G, H_{\overline{1}})|$ copies of $(0{,}+\infty)$.

On the other hand, since the action of G on the fiber of $\pi_1 : \hat{\mathbb{Z}} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ over -1 is not, trivial, the action of G on $H_{\overline{1}}$ is by exchanging the coordinates. Hence $\mathrm{H}^1(G, H_{\overline{1}}) = 1$, so that the pre-image of $(-\infty,\, 0)$ along the map $[U/H](\mathbb{R}) \to U(\mathbb{R})$ is the disjoint union of just one copy of $(-\infty,\, 0)$. In summary $[U/H](\mathbb{R})$ is the union of 5 copies of $\mathbb{R}$, one lying over $(-\infty,\, 0)$ and 4 lying over $(0{,}+\infty)$.

In conclusion, one has $h^*([U/H](\mathbb{R}) = 5$. To compute $I_{[U/H]}(\mathbb{C})$, recall from Example 3.3(6), that $I_{[U/H]}(\mathbb{C}) = H(\mathbb{C})$. In this case $H(\mathbb{C})$ has 3 connected components (corresponding to the 3 orbits of $\pi_1^{\text{ét}}(U_{\mathbb{C}})$ acting on $\mathbb{Z}/2 \times \mathbb{Z}/2$) which are finite étale over $\mathbb{C}^*$ so that $H(\mathbb{C}) \simeq \mathbb{C}^* \coprod \mathbb{C}^* \coprod \mathbb{C}^*$, hence $h^*(I_{[U/H]}(\mathbb{C})) = 6$ and the Smith–Thom inequality (3) holds.

Example 8.11. Let $\rho : \hat{\mathbb{Z}} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ be the morphism defined by the sum of the natural map $\hat{\mathbb{Z}} \to G \to \mathbb{Z}/2$ and $G \to \mathbb{Z}/2$ and we let $\hat{\mathbb{Z}} \rtimes \mathbb{Z}/2$ act on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ via the action of $\mathbb{Z}/2$. Let $H \to U$ be the corresponding group scheme. Since the action of G on the fiber of $\pi_1 : \hat{\mathbb{Z}} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$ over 1 is trivial, the action of G on $H_{\overline{1}}$ is non trivial while the one on $H_{\overline{-1}}$ is non-not.

Hence, as in the previous example, $[U/H](\mathbb{R})$ is the union of 5 copies of $\mathbb{R}$, but now 4 are lying over $(-\infty,\, 0)$ and 1 is lying over $(0{,}+\infty)$. The Smith–Thom inequality (3) is verified exactly has in the previous example.

8.3.2 Enriques surfaces Let U be an Enriques surfaces such that $U(\mathbb{R}) \neq \emptyset$, so that its K3 cover $h : V \to U$ is defined over $\mathbb{R}$. To simplify the discussion, we assume that also $V(\mathbb{R}) \neq \emptyset$.

Fix $p \in U(\mathbb{R})$ be in the image of $h : V(\mathbb{R}) \to U(\mathbb{R})$. Then the section $\pi_1(p)$ induces an isomorphism

$$\pi_1(U, \overline{p}) \simeq \mathbb{Z}/2 \times G$$

such that the K3 cover $h : V \to U$ correspond to the projection on the first factor $\pi_1 : \mathbb{Z}/2 \times G \to \mathbb{Z}/2$. For every $q \in U(\mathbb{R})$, the group G acts trivially on $V_{\overline{x}}$ if and only q is in the image of $h : V(\mathbb{R}) \to U(\mathbb{R})$. Hence the element ϵ_q corresponding to the section associated to q is $(0, 0)$ if q is in the image of $h : V(\mathbb{R}) \to U(\mathbb{R})$ and $(1, 0)$ otherwise. In the rest of the section we let $\mathbb{Z}/2$ acting on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ by exchanging the coordinates.

Example 8.12. Assume that $U(\mathbb{R})$ is the union of 4 $\mathbb{P}^2(\mathbb{R})$ and 2 spheres S^2 and that the map $h : V(\mathbb{R}) \to U(\mathbb{R})$ is surjective (observe that such Enriques surfaces exists by [DIK00a, Table 8, Pag. 180]). Let $\pi_1 : \mathbb{Z}/2 \times G \to \mathbb{Z}/2$ be projection on the first coordinates and we let $\mathbb{Z}/2 \times G$ act on $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ via this map. Let $H \to U$ be the corresponding group scheme. Since $h : V(\mathbb{R}) \to U(\mathbb{R})$ is surjective, for every $q \in U(\mathbb{R})$, the image of the splitting $\pi_1(q)$ is $(0, 1_G)$, hence the action of G on $H_{\overline{q}}$ is trivial, hence $\mathrm{H}^1(G, \overline{q}) = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Let C be a connected component homeomorphic to S^2. Since S^2 is simply connected, the cover $f^{-1}(C) \to C$ is the trivial cover, hence preimage of each C along the map $f \colon [U/H](\mathbb{R}) \to U(\mathbb{R})$ consists of 4 spheres.

On the other hand let C be a connected component homeomorphic to $\mathbb{P}^2(\mathbb{R})$. Then, the natural map $\pi_1(C) \to \pi_1(U(\mathbb{C}))$ is an isomorphism, since over C there is at least one of the spherical connected component of one the real part of the K3-cover (see the discussion in [DK96, Section 3.5]). Hence, that the cover $f^{-1}(C) \to C$ has 3 connected components, corresponding to the orbits $\{(0, 0)\}$,$\{(1, 1_G)\}$, $\{(0, 1_G), (1, 0)\}$ of the action of $\pi_1(U(\mathbb{C}))$ on $\mathrm{H}^1(G, \overline{q}) = \mathbb{Z}/2 \times \mathbb{Z}/2$. Since the $\pi_1(C)$-action on $\{(0, 1_G), (1, 0)\}$ is not trivial, the corresponding cover is homeomorphic to the universal cover $S^2 \to \mathbb{P}^2(\mathbb{R})$, hence $f^{-1}(C)$ is homeomorphic to the disjoint union of 2 copies $\mathbb{P}^2(\mathbb{R})$ and an S^2.

In conclusion, $[U/H](\mathbb{R})$ is homeomorphic to the disjoint union of 4 copies of the space $S^2 \coprod \mathbb{P}^2(\mathbb{R}) \coprod \mathbb{P}^2(\mathbb{R})$ (each one is lying over an $\mathbb{P}^2(\mathbb{R})$) and 2 copies of $\coprod_{1 \le i \le 4} S^2$ (each lying over an S^2). In particular $h^*([U/H](\mathbb{R})) = 4 * (2 + 3 + 3) + 2 * 8 = 48$.

On the other hand, $I_{[U/H](\mathbb{C})} \to U(\mathbb{C})$ is the cover corresponding to the action of $\pi_1(U(\mathbb{C}))$ on H_p, hence it has 3 connected components, corresponding to the orbits $\{(0, 0)\}$,$\{(1, 1_G)\}$, $\{(0, 1_G), (1, 0)\}$. Hence $I_{[U/H](\mathbb{C})}$ is the disjoint union of 2 copies of $U(\mathbb{C})$ and one copy of its K3-cover. In particular $h^*(I_{[U/H](\mathbb{C})}) = 16 * 2 + 24 = 56$, so that the Smith–Thom inequality 3 is verified.

Example 8.13. Retain the notation of the previous Example 8.12, but now assume that the image of the map $V(\mathbb{R}) \to U(\mathbb{R})$ consists of only 3 copies of $\mathbb{P}^2(\mathbb{R})$ and just one S^2 (observe that such Enriques surfaces exists by [DIK00a, Table 8, Pag. 180]). The description of the cover $f^{-1}(C) \to C$) is the same for the connected components in the image of $V(\mathbb{R}) \to U(\mathbb{R})$, but it is different for the connected components C_1, C_2, which are respectively homeomorphic to S^2 and $\mathbb{P}^2(\mathbb{R})$, which are not in the image. Chose a $q_i \in C_i$. Since q_i is not in the image of $V(\mathbb{R}) \to U(\mathbb{R})$, the action of G on $V_{\overline{q}_i}$ is not trivial, so the image of the section $\pi_1(q_i)$ is $(1, 1_G)$, so that G acts on $H_{\overline{q}_i}$ by exchanging the coordinates. In particular $\mathrm{H}^1(G, H_{\overline{q}_i}) = 0$ so that $f^{-1}(C_i) \to C_i$ is an isomorphism. One verifies as in the previous example that the Smith–Thom inequality is verified also in this case.

9 Variants of the Smith–Thom inequality for stacks

Let $\mathcal{X}$ be a real Deligne–Mumford stack. In the previous sections, we studied the topological space $|\mathcal{X}(\mathbb{R})|$ and proposed a conjectural bound on the sum of its Betti numbers. While we believe this to be the most compelling problem related to $\mathcal{X}$ (as understanding the geometry of real moduli spaces of objects could facilitate the classification of their topological types), other interesting directions remain to be explored. In this section, we investigate two such directions.

First, within the algebraic framework, it is known (see [GF22a]) that if $\mathcal{X}$ is smooth, the space $|\mathcal{X}(\mathbb{R})|$ is not merely a topological space but also carries additional structure as a real analytic orbifold. This suggests the natural question of whether a Smith–Thom-type inequality can be formulated for the orbifold cohomology of $|\mathcal{X}(\mathbb{R})|$. The main difficulty in doing so is that, in general, the orbifold cohomology $\mathrm{H}^i_{\mathrm{orb}}(|\mathcal{X}(\mathbb{R})|)$ is nonzero in arbitrarily high degrees, making it unclear how to extend the conjectural inequality (3) to this setting. In Section 9.1, we propose a way to address this issue for quotient stacks by exploiting results of Quillen on equivariant cohomology rings, see [Qui71].

Second, in a more topological direction, it is well known that the classical Smith–Thom inequality (1) is not specific to real algebraic varieties but applies to any topological space equipped with an involution σ, where $X(\mathbb{R})$ is replaced by the fixed locus of σ. This naturally leads to the question of whether (3) admits a generalization to all topological groupoids $\mathscr{X}$ equipped with an involution. The main challenge in this approach is identifying a suitable analogue of the fixed locus of the involution, that coincides with $\mathcal{X}(\mathbb{R})$ when $\mathscr{X}$ is the topological groupoid with involution associated to a real DM stack with étale presentation $U \to \mathcal{X}$. We explore this question and formulate a precise conjecture in Section 9.2.

9.1 Orbifold cohomology version. Let $\mathcal{X}$ be a real smooth Deligne–Mumford stack. In this case the space $|\mathcal{X}(\mathbb{R})|$ can be naturally enriched with the structure of a real analytic orbifold, see [GF22a, Section 2.2.3].

It seems natural to wonder whether the classical Smith–Thom inequality (1) has an analogue for the orbifold cohomology groups $\mathrm{H}^i_{\mathrm{orb}}(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2)$ and $\mathrm{H}^i_{\mathrm{orb}}(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2)$ of $|\mathcal{X}(\mathbb{R})|$ and $|\mathcal{X}(\mathbb{C})|$. We refer the reader to [MP99] for the generalities on orbifold cohomology. For the sequel, it will be usefull to recall what is the orbifold cohomology of a quotient.

Remark 9.1. If a topological orbifold $\mathcal{X}$ is obtained as a quotient of a topological space X by the action of a finite group G, by [MP99, Section 1.3], one has

$$\mathrm{H}^i_{\mathrm{orb}}(\mathcal{X}, \mathbb{Z}/2) \simeq \mathrm{H}^i_{\Gamma}(X, \mathbb{Z}/2),$$

where $\mathrm{H}^i_{\Gamma}(X, \mathbb{Z}/2)$ is the Γ-equivariant cohomology of X.

As already mentioned, in general, the groups $\mathrm{H}^i_{\mathrm{orb}}(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2)$ and $\mathrm{H}^i_{\mathrm{orb}}(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2)$ can be non-zero for infinitely many i.

Example 9.2. Let $X \coloneqq \mathrm{Spec}(\mathbb{R})$ and $\Gamma \coloneqq \mathbb{Z}/2$ viewed as a constant group scheme over $\mathbb{R}$. By Remark 9.1, one has

$$\mathrm{H}^i_{\mathrm{orb}}(|[X/\mathbb{Z}/2](\mathbb{C})|, \mathbb{Z}/2) \simeq \mathrm{H}^i(\Gamma, \mathbb{Z}/2)$$

while

$$\mathrm{H}^i_{\mathrm{orb}}(|[X/\mathbb{Z}/2](\mathbb{R})|, \mathbb{Z}/2) \simeq \mathrm{H}^i(\Gamma, \mathbb{Z}/2) \oplus \mathrm{H}^i(\Gamma, \mathbb{Z}/2)$$

as follows from Remark 9.1 and Theorem 1.5 and [MP99, Section 1.3]. In particular, they are both non zero every integer $i \geq 0$.

Even if it does not make sense to compare the sum of all the dimensions of all the cohomology groups, one can ask the following vague question.

Question 9.3. *Let $\mathcal{X}$ be a smooth separated Deligne–Mumford stack over $\mathbb{R}$. Is there a uniform natural bound on the growth rate of $\mathrm{H}^{\leq i}_{\mathrm{orb}}(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$ when $i \to \infty$ in terms of the growth rate of $\mathrm{H}^{\leq i}_{\mathrm{orb}}(\mathcal{X}(\mathbb{C}), \mathbb{Z}/2)$, that does not depend on the real model $\mathcal{X}$ of $\mathcal{X}_{\mathbb{C}}$?*

Even if we don't know how to make precise Question 9.3 in general, in this section we do it for quotient stacks, where the orbifold cohomology can be identified with equivariant cohomology by Remark 9.1.

9.1.1 Quillen's theorem on Poincaré series. In order to do so, we need to recall a result of Quillen on the structure of the Poincaré series of equivariant cohomology. Let Γ be a finite group and let X be a topological Γ-space such that $\mathrm{H}^\bullet(X, \mathbb{Z}/2)$ is a finite dimensional $\mathbb{F}_2$-vector space. By [Qui71, Corollary 2.2], the equivariant cohomology ring $\mathrm{H}^*_\Gamma(X, \mathbb{Z})$ is a finitely generated graded $\mathbb{F}_2$-algebra. Let

$$P_\Gamma(X)(t) \coloneqq \sum_{i=0}^{\infty} \dim_{\mathbb{F}_2}\left(\mathrm{H}^i_\Gamma(X, \mathbb{Z}/2)\right) \cdot t^i \in \mathbb{Z}[[t]],$$

the associated Poincaré series. If X is just a point, we write $P_\Gamma(t) \coloneqq P_\Gamma(X)(t)$.

Recall from [Qui71, Proposition 2.5, Theorem 7.7] the following theorem.

Theorem 9.4 (Quillen)**.** *Let* $e_\Gamma(X) \coloneqq \max_{n\in\mathbb{N}}\left(\exists A \subseteq \Gamma \mid A \cong (\mathbb{Z}/2)^n \text{ and } X^A \neq \emptyset\right)$. *Then there exists a polynomial* $Q_\Gamma(X)(t) \in \mathbb{Z}[t]$ *with* $Q_\Gamma(X)(1) \neq 0$ *such that*

$$P_\Gamma(X)(t) = \frac{Q_\Gamma(X)(t)}{\prod_{i=1}^{e_\Gamma(X)}(1 - t^{2i})} \in \mathbb{Q}(t).$$

In particular, if a subspace $Y \subseteq X$ is stable under the action of a subgroup $H \subseteq \Gamma$, then the rational function

$$f(t) = \frac{P_H(Y)(t)}{P_\Gamma(X)(t)} \in \mathbb{Q}(t)$$

has no pole at 1. Consequently, one obtains a rational number $f(1) \in \mathbb{Q}$, and $f(1) = 0$ is zero if and only if $e_H(Y) < e_\Gamma(X)$. Morally, the rational number $f(1)$ can be thought as the ratio between the total H-equivariant Betti number of Y and the total Γ-equivariant Betti number of X.

9.1.2 An orbifold Smith–Thom conjecture for quotient stacks Let Γ be a finite group and $\sigma \colon \Gamma \to \Gamma$ an involution; we call such a pair a *finite G-group*. As usual, define $\mathrm{Z}^1(G, \Gamma)$ as the set of $\gamma \in \Gamma(\mathbb{C})$ such that $\sigma_\Gamma(\gamma) \cdot \gamma = e$, so that $\mathrm{H}^1(G, \Gamma) = \mathrm{Z}^1(G, \Gamma)/\sim$ where $\sim$ is the equivalence relation that identifies $\gamma_1, \gamma_2 \in \Gamma$ if there exists a $\beta \in \Gamma$ such that $\gamma_2 = \beta^{-1}\gamma_1\sigma(\beta)$. Choose a set of representative $H \subset \mathrm{Z}^1(G, \Gamma)$ for this equivalence relation; we choose H such that $e \in H$. For $\gamma \in H$, define an involution

$$\sigma_\gamma \colon \Gamma \to \Gamma, \qquad \sigma_\gamma(g) = \gamma\sigma(g)\gamma^{-1}.$$

Let X a quasi-projective variety over $\mathbb{R}$, and $\Gamma \times X(\mathbb{C}) \to X(\mathbb{C})$ a G-equivariant action of Γ on $X(\mathbb{C})$. Recall from Corollary 1.5 that

$$|[X/\Gamma](\mathbb{R})| = \coprod_{\gamma \in H} X_\gamma(\mathbb{R})/\Gamma^{\sigma_\gamma}.$$

By Theorem 9.4, for $\gamma \in H$, the rational function

$$P_{\gamma,\Gamma}(X,t) \coloneqq \frac{P_{\Gamma^{\sigma_\gamma}}(X_\gamma(\mathbb{R}))(t)}{P_\Gamma(X(\mathbb{C}))(t)} \in \mathbb{Q}(t)$$

has no pole at 1, and we obtain a rational number $P_{\gamma,\Gamma}(X,1) \in \mathbb{Q}$.

Since the orbifold cohomology of a quotient space, identifies with equivariant cohomology (see Remark 9.1), one can ask the following, which make precise Question 9.3 in this setting.

Question 9.5. *Let $\Gamma = H(\mathbb{C})$ for a finite group scheme H over $\mathbb{R}$ acting on a quasi-projective variety X over $\mathbb{R}$. Does there exist a natural number $C > 0$, independent of the real model $H \times_\mathbb{R} X \to X$ of the action $\Gamma \times_\mathbb{C} X_\mathbb{C} \to X_\mathbb{C}$, such that*

$$\sum_{[\gamma] \in \mathrm{H}^1(G,\Gamma)} P_{\gamma,\Gamma}(X,1) = \sum_{[\gamma] \in \mathrm{H}^1(G,\Gamma)} \left. \frac{P_{\Gamma^\gamma}(X_\gamma(\mathbb{R}))(t)}{P_\Gamma(X(\mathbb{C}))(t)} \right|_{t=1} \leq C \ ?$$

Example 9.6. Assume that Γ acts freely on $X(\mathbb{C})$. Then $(X/\Gamma)(\mathbb{R}) = \coprod_{[\gamma]} X_\gamma(\mathbb{R})/\Gamma^{\sigma_\gamma}$ by Theorem 1.5, and Γ^{σ_γ} acts freely on $X_\gamma(\mathbb{R})$ for each γ. Therefore, in this case,

$$\sum_{\gamma \in \mathrm{H}^1(G,\Gamma)} P_{\gamma,\Gamma}(X,1) = \sum_{\gamma \in \mathrm{H}^1(G,\Gamma)} \frac{h^*(X_\gamma(\mathbb{R})/\Gamma^\gamma)}{h^*(X(\mathbb{C})/\Gamma)} = \frac{h^*((X/\Gamma)(\mathbb{R}))}{h^*((X/\Gamma)(\mathbb{C}))} \leq 1,$$

where the first and the second equality follows from the freeness of the action (see [Bor60, 3.4, Pag. 54] for the first and Lemma 4.5 for the second), while the third inequality holds by the classical Smith–Thom inequality (1).

While Example 9.6 seems to suggest that one could take $C = 1$ in Question 9.5, this is not the case, as for example one easily see in Example 9.2, where the ratio is 2.

9.1.3 The zero dimensional case. Assume now that $X = \operatorname{Spec}(\mathbb{R})$. In this case, we can make even more precise Question 9.5, since it is implied by the following.

Question 9.7. *Let Γ be a group and let $\sigma\colon \Gamma \to \Gamma$ be an involution. Do we have*

$$\left. \frac{P_{\Gamma^\sigma}(t)}{P_\Gamma(t)} \right|_{t=1} \leq |\Gamma| \ ?$$

To give a non-trivial example of a finite group for which Question 9.7 has a positive answer, we prove:

Proposition 9.8. *Let $\Gamma \coloneqq \mathfrak{S}_4$ be symmetric group on four letters. Consider Γ as a G-module via the trivial action, where $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$. Then*

$$\sum_{\gamma \in \mathrm{H}^1(G,\Gamma)} P_{\gamma,\Gamma}(\{\mathrm{pt}\}, 1) = \sum_{\gamma \in \mathrm{H}^1(G,\Gamma)} \left. \frac{P_{\Gamma^{\sigma_\gamma}}(t)}{P_\Gamma(t)} \right|_{t=1} = 3.$$

Proof of Proposition 9.8. First observe that the elements $\gamma_1 \coloneqq e, \gamma_2 \coloneqq (12), \gamma_3 \coloneqq (12)(34)$ form a complete set of representatives of the equivalence classes in $\mathrm{H}^1(G, \mathfrak{S}_4)$. In particular

$$|\mathrm{H}^1(G, \mathfrak{S}_4)| = 3. \tag{28}$$

Next observe that

$$\mathfrak{S}_4^{\gamma_1} = \mathfrak{S}_4; \quad \mathfrak{S}_4^{\gamma_2} = \{e, (1,2), (3,4), (12)(34)\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \quad \text{and}$$

$$\mathfrak{S}_4^{\gamma_3} = \{e, (12), (34), (12)(34), (13)(24), (14)(23), (1423), (1324)\} \simeq D_8,$$

where D_8 is the dihedral group with 8 elements.

Next, we compute the Poincaré series of $\mathfrak{S}_4^{\gamma_i}$ for each $i = 1, 2, 3$.

Lemma 9.9. *One has the following equalities of rational functions:*

1. $P_{\mathbb{Z}/2\times\mathbb{Z}/2} = \frac{1}{(1-t)^2}$;
2. $P_{D_8} = \frac{1}{(1-t)^2}$;
3. $P_{\mathfrak{S}_4} = \frac{1+t^2}{(1-t)^2(1+t+t^2)}$.

Before proving Lemma 9.9, let us show that it implies the Proposition 9.8. From Lemma 9.9 one deduce that

$$\frac{P_{\Gamma^{\gamma_1}}(t)}{P_\Gamma(t)} = 1 \quad \text{and} \quad \frac{P_{\Gamma^{\gamma_2}}(t)}{P_\Gamma(t)} = \frac{1+t+t^2}{1+t^2} = \frac{P_{\Gamma^{\gamma_3}}(t)}{P_\Gamma(t)},$$

so that, combined with (28), one has

$$\frac{h^*([X/\Gamma])}{h^*([X/\Gamma]_\mathbb{C})} = \frac{(1+3/2+3/2)}{3} = \frac{4}{3},$$

which what we wanted. We are left to show Lemma 9.9. $\square$

Proof of Lemma 9.9. Let us recall that if $A = \sum_{i\in\mathbb{N}} A_i$ is a graded $\mathbb{F}_2$-algebra, we can consider its Poincaré series

$$P_A(t) \coloneqq \sum_{i\geq 0} \dim(A_i),$$

so that $P_N(t) = P_{H^*(N,\mathbb{F}_2)}(t)$ for every group N. If $f \in A$ is an homogeneous element of degree d, then

$$P_A(t) = \frac{P_{A/(f)}(t)}{(1-t^d)}, \tag{29}$$

In particular if A is freely degenerated by $x_1 \dots x_n$ with x_i of degree j_i, one has

$$P_A(t) = \prod_{1\leq i\leq n} \frac{1}{(1-t^{j_i})} \tag{30}$$

To prove item 1, note that, by the Kunneth-Formula

$$H^*(\mathbb{Z}/2\times\mathbb{Z}/2,\mathbb{Z}/2) \simeq H^*(\mathbb{Z}/2,\mathbb{Z}/2)\otimes H^*(\mathbb{Z}/2,\mathbb{Z}/2) \simeq \mathbb{F}_2[x,y]$$

with x, y of degree 1. Hence item 1 of Lemma 9.9 follows from (30).

By [Han93, Theorem 5.5],

$$H^*(D_8,\mathbb{Z}/2) \simeq \frac{\mathbb{F}_2[x,y,z]}{(x(x+y))} \quad \text{with } deg(x) = 1, deg(y) = 1, deg(z) = 2.$$

Since $x(x+y)$ is an homogenous element of degree 2, we get

$$P_{H^*(D_8,\mathbb{Z}/2)}(t) = P_{\mathbb{F}_2[x,y,z]}(t)(1-t^2) = \frac{1}{(1-t)^2(1-t^2)}(1-t^2) = \frac{1}{(1-t)^2},$$

where the first equality follows from (29) and the second from (30). This proves item 3 of Lemma 9.9.

By [Nak62, Theorem 4.1]

$$H^*(\mathfrak{S}_4,\mathbb{Z}/2) \simeq \frac{\mathbb{F}_2[x,y,z]}{(xz)} \quad \text{with } deg(x) = 1, deg(y) = 2, deg(z) = 3.$$

Since xz is an homogeonous element of degree 4, we get

$$H^*(\mathfrak{S}_4,\mathbb{Z}/2) = P_{\mathbb{F}_2[x,y,z]}(t)(1-t^4) = \frac{1}{(1-t)(1-t^2)(1-t^3)}(1-t^4) = \frac{1+t^2}{(1-t)^2(1+t+t^2)}$$

where the first equality follows from (29) and the second from (30). This finishes the proof of Lemma 9.9. $\square$

9.2 Smith–Thom inequality for topological groupoids with involution. As already mentioned, the statement (and the proof) of the Smith-Thom inequality (1.2) is purely topological, in the sense that it holds for every topological space endowed with an involution σ, replacing $X(\mathbb{R})$ with the fixed locus of σ.

In this section we generalize Conjecture 1.2 from real algebraic stack to more general topological stack endowed with an involution, which we see as an analogous to move from algebraic varieties to topological spaces.

9.2.1 Topological groupoids. Recall that a *topological groupoid* is a groupoid object in the category of topological spaces. In particular, a topological groupoid $\mathscr{X} = [X_1 \rightrightarrows X_0]$ consist of two topological spaces, X_0 (the space of objects) and X_1 (the space of arrows) and a collection of continuous maps $s\colon R \to U$ (source), $t\colon R \to U$ (target), $c\colon R \times_U R \to R$ (composition), $e\colon U \to R$ (unit) and $i\colon R \to R$ (inversion). These maps satisy a number of conditions to ensure that one obtains a groupoid by letting U be the set of objects, R the set of arrows, $s(f)$ and $t(f)$ the source and target of an arrow $f \in R$, $c(f, g) = f \circ g$ the composition of arrows $f, g \in R$, $e(x)$ the identity $x \to x$ of an object $x \in U$ and $i(f) = f^{-1}$ the inverse of an arrow f. For a topological groupoid $\mathscr{X} = [X_1 \rightrightarrows X_0]$, we let $\mathrm{Gr}(\mathscr{X})$ be the associated groupoid (i.e., the associated category in which every arrow is an isomorphism). Thus, the objects of $\mathrm{Gr}(\mathscr{X})$ are given by X_0 and the arrows of $\mathrm{Gr}(\mathscr{X})$ are given by X_1.

Example 9.10. Every complex Deligne–Mumford stack gives rise to a topological groupoid in the following way. Let $\mathcal{X}$ be a complex Deligne–Mumford stack, so that there exists an étale surjective presentation $\pi\colon U \to \mathcal{X}$ by a scheme. Let R be a scheme with $R \cong U \times_{\mathcal{X}} U$, so that the two projection maps $U \times_{\mathcal{X}} U \to U$ yield two maps $R \to U$ that turn $[R \rightrightarrows U]$ into a groupoid scheme. Then $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ is a topological groupoid. Moreover, any $r \in R(\mathbb{C})$ corresponds to an element $(x, y, \alpha) \in (U \times_{\mathcal{X}} U)(\mathbb{C})$ consisting of $x, y \in U(\mathbb{C})$ and an isomorphism $\alpha\colon \pi(x) \xrightarrow{\sim} \pi(y)$. Consider the functor $F\colon \mathrm{Gr}(\mathscr{X}) \to \mathcal{X}(\mathbb{C})$ that sends $x \in U(\mathbb{C})$ to $\pi(x) \in \mathcal{X}(\mathbb{C})$ and $r = (x, y, \alpha)$ to the isomorphism $\alpha\colon \pi(x) \xrightarrow{\sim} \pi(y)$. Then $F\colon \mathrm{Gr}(\mathscr{X}) \to \mathcal{X}(\mathbb{C})$ is an equivalence of categories.

9.2.2 Topological groupoids with involution. Let $\mathscr{X} = [X_1 \rightrightarrows X_0]$ be a topological groupoid. An *involution* $\sigma\colon \mathscr{X} \to \mathscr{X}$ consists of involutions $\sigma_1\colon X_1 \to X_1$ and $\sigma_0\colon X_0 \to X_0$ that are compatible with s, t and all the other structure maps of the topological groupoid. Every real Deligne–Mumford stack give rise to a topological groupoid involution $(\mathscr{X}, \sigma)$ in the following way (see [GF22a, Section 2.3]).

Example 9.11. Let $\mathcal{X}$ be a real Deligne-Mumford stack, and choose a scheme U over

$\mathbb{R}$ and an étale surjective morphism $U \to \mathcal{X}$. Let R be a scheme with $R \cong U \times_{\mathcal{X}} U$, so that we get a groupoid scheme $[R \rightrightarrows U]$, see Example 9.10. Since U and R are schemes locally of finite type over $\mathbb{R}$, $U(\mathbb{C})$ and $R(\mathbb{C})$ admit natural anti-holomorphic involutions $\sigma_0 \colon U(\mathbb{C}) \to U(\mathbb{C})$ and $\sigma_1 \colon R(\mathbb{C}) \to R(\mathbb{C})$, compatible with the structure maps of the groupoid. Hence $[R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ is a topological groupoid with involution.

9.2.3 Fixed locus of an involution. Let $\mathscr{X} = [X_1 \rightrightarrows X_0]$ be a topological groupoid and assume that $\mathscr{X}$ is equipped with an involution $\sigma \colon \mathscr{X} \to \mathscr{X}$. Thus, σ corresponds to involutions $\sigma_1 \colon X_1 \to X_1$ and $\sigma_0 \colon X_0 \to X_0$ that are compatible with the structure maps of the topological groupoid.

We now proceed defining the correct analogue of the fixed point of σ, in a way that, in the setting of Example 9.11 recovers the topological space $|\mathscr{X}(\mathbb{R})|$. Define $\mathrm{Ob}(\mathscr{X}^\sigma) \subset X_0 \times X_1$ as the subspace of pairs $(x, \varphi) \in X_0 \times X_1$ such that φ is an isomorphism $x \xrightarrow{\sim} \sigma_0(x)$ with $\sigma_1(\varphi) \circ \varphi = \mathrm{id}$. Then $\mathrm{Ob}(\mathscr{X}^\sigma)$ is the set of objects of a topological groupoid $\mathscr{X}^\sigma$, whose arrows between $(x, \varphi) \in \mathrm{Ob}(\mathscr{X}^\sigma)$ and $(y, \psi) \in \mathrm{Ob}(\mathscr{X}^\sigma)$ are given by isomorphisms $f \colon x \to y$ in X_1 such that $\psi \circ f = \sigma_1(f) \circ \varphi$.

Definition 9.12. Define $|\mathscr{X}^\sigma| = \mathrm{Ob}(\mathscr{X}^\sigma)/_{\cong}$ and equip it with the quotient topology.

Example 9.13. Let $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ be the topological groupoid with involution associated to a real Deligne-Mumford stack $\mathcal{X} = [U/R]$ as in Examples 9.10 and 9.11. Recall that any $r \in R(\mathbb{C}) = (U \times_{\mathcal{X}} U)(\mathbb{C})$ corresponds to a triple $r = (x, y, \alpha)$ with $x, y \in U(\mathbb{C})$ and $\alpha \colon \pi(x) \xrightarrow{\sim} \pi(y)$ an isomorphism, where π is the map $U \to \mathcal{X}$. We have $\mathrm{Ob}(\mathscr{X}^\sigma) = \{\omega = (x, (x, \sigma(x), \varphi)) \in U(\mathbb{C}) \times R(\mathbb{C}) \mid \sigma(\varphi) \circ \varphi = \mathrm{id}\}$. For such $\omega = (x, (x, \sigma(x), \varphi))$, we get an element $\pi(x) \in \mathcal{X}(\mathbb{C})$ and an isomorphism $\varphi \colon \pi(x) \xrightarrow{\sim} \pi(\sigma(x))$ such that $\sigma(\varphi) \circ \varphi = \mathrm{id}$. By Galois descent (cf. [Gro60]), this yields an object $F(\omega) \in \mathcal{X}(\mathbb{R})$. Similarly, any arrow $f \colon \omega \to \omega'$ in $\mathrm{Gr}(\mathscr{X}^\sigma)$ is given by an arrow $f = (x, x', \alpha) \in R(\mathbb{C})$ such that $\varphi' \circ \alpha = \sigma(\alpha) \circ \varphi$ as maps $\pi(x) \to \pi(\sigma(x'))$, and this yields an arrow $F(f) \colon F(\omega) \to F(\omega')$ in $\mathcal{X}(\mathbb{R})$, again by Galois descent. The resulting functor $F \colon \mathrm{Gr}(\mathscr{X}^\sigma) \to \mathcal{X}(\mathbb{R})$ is an equivalence of categories. In particular, we get a canonical bijection $|F| : |\mathscr{X}^\sigma| \xrightarrow{\sim} |\mathcal{X}(\mathbb{R})|$.

Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$. Choose a surjective étale morphism $U \to \mathcal{X}$ where U is a scheme, and let R be a scheme with $R \cong U \times_{\mathcal{X}} U$. Let $\mathscr{X}$ be the topological groupoid $[R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ equipped with its natural involution $\sigma \colon \mathscr{X} \to \mathscr{X}$.

Lemma 9.14. *Consider the natural bijection $|F| : |\mathscr{X}^\sigma| \to |\mathcal{X}(\mathbb{R})|$, see Example 9.13. Consider $|\mathscr{X}^\sigma|$ as a topological space via Definition 9.12, and consider $|\mathcal{X}(\mathbb{R})|$ as a* *topological space via Definition 4.3. Then the bijection $|F|$ is a homeomorphism.*

Proof. By taking fibre products, one reduces to the case where $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective (cf. Theorem 4.2). We consider the canonical continuous map $U(\mathbb{R}) \to \mathrm{Ob}(\mathscr{X}^\sigma) \subset U(\mathbb{C}) \times R(\mathbb{C})$ defined by sending $x \in U(\mathbb{R})$ to (x, id); indeed, since $\sigma(x) = x$, the identity defines an isomorphism $\varphi\colon \pi(x) \to \pi(\sigma(x))$ with $\sigma(\varphi) \circ \varphi = \mathrm{id}$. The composition $U(\mathbb{R}) \to \mathrm{Ob}(\mathscr{X}^\sigma) \to \mathrm{Ob}(\mathscr{X}^\sigma)_{/\cong} = |\mathscr{X}^\sigma|$ is surjective and closed, and factors through a homeomorphism $U(\mathbb{R})/R(\mathbb{R}) \xrightarrow{\sim} |\mathscr{X}^\sigma|$, proving the lemma. □

9.2.4 A Smith–Thom conjecture for topological groupoids. Let $I_{\mathscr{X}} = [Y_1 \rightrightarrows Y_0]$ be the inertia groupoid of $\mathscr{X}$, so that Y_0 is the space of $(x, \varphi) \in X_0 \times X_1$ with $\varphi \in \mathrm{Aut}(x)$, and Y_1 is the space of isomorphisms $(x, \varphi) \xrightarrow{\sim} (y, \psi)$ for $(x, \varphi), (y, \psi) \in Y_0$. Let $|I_{\mathscr{X}}|$ be the set of isomorphism classes of objects in Y_0.

We can now state a Smith–Thom conjecture for topological groupoids, which, by Examples 9.11 and 9.13 and Lemma 9.14, generalizes Conjecture 1.2.

Conjecture 9.15. *Let $\mathscr{X} = [X_1 \rightrightarrows X_0]$ be a topological groupoid with finite stabilizer groups, equipped with an involution $\sigma\colon \mathscr{X} \to \mathscr{X}$. Assume that $|\mathscr{X}^\sigma|$ and $|I_{\mathscr{X}}|$ have finite dimensional $\mathbb{Z}/2$-cohomology. Then, we have:*

$$\dim \mathrm{H}^*(|\mathscr{X}^\sigma|, \mathbb{Z}/2) \leq \dim \mathrm{H}^*(|I_{\mathscr{X}}|, \mathbb{Z}/2).$$

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