Descent for algebraic stacks

Olivier de Gaay Fortman

1. Introduction

Let $S' \to S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor

$$X \mapsto X \times_S S'$$

induces an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X' over S' such that X' admits an open covering by affine schemes which are stable under ϕ (cf. [Gro60]). In case $S = \operatorname{Spec}(k)$, $S' = \operatorname{Spec}(k')$ and the morphism $S' \to S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil (cf. [Wei56]).

The aim of this paper is to present the most natural analogue of this result in the setting of algebraic stacks. Our results follow rather directly from notions and results of Giraud [Gir71], Duskin [Dus89], and Breen [Bre94], who studied descent for stacks over an arbitrary site; as such, the results presented here are possibly well-known to experts. The main contribution of this paper is to assemble existing results on descent theory for stacks, to reformulate them in the specific case of stacks over a scheme, and finally to extend them by considering those stacks over a scheme that are algebraic or Deligne–Mumford.

In the case of stacks, the analogue of the aforementioned descent-theory for schemes is a notion called 2-*descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes $S' \to S$ which is smooth and surjective, *every* 2-*descent datum for an algebraic stack is effective.* More precisely, we have the following result. For a scheme S, let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of S (cf. [Stacks, Tag 021S]); a *stack over* S is a stack in groupoids $\mathcal{X} \to (\mathrm{Sch}/S)_{fppf}$, see [Stacks, Tag 0304].

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THEOREM 1.1. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and let $\mathcal{X}'$ be a stack over S'. Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S', see Definition 3.1.*

(1) The 2-descent datum (ϕ, ψ) is effective. That is, there exists a stack $\mathcal{X}$ over S, an isomorphism of stacks over S'

$$\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

and a 2-isomorphism $\chi\colon p_2^* f \circ \mathrm{can} \Rightarrow \phi \circ p_1^* f$ as in the following diagram:

$$\begin{tikzcd} p_1^*(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\rho"'] \arrow[dr, phantom, "\swarrow"] & p_2^*(\mathcal{X} \times_S S') \arrow[d, "p_2^*\rho"] \\ p_1^*\mathcal{X}' \arrow[r, "\phi"] & p_2^*\mathcal{X}', \end{tikzcd} \tag{1.1}$$

such that the natural compatibility between χ and ψ is satisfied.

(2) If the morphism $S' \to S$ is smooth, then $\mathcal{X}'$ is an algebraic stack over S' if and only if $\mathcal{X}$ is an algebraic stack over S.

(3) If the morphism $S' \to S$ is étale, then $\mathcal{X}'$ is a Deligne–Mumford stack over S' if and only if $\mathcal{X}$ is a Deligne–Mumford stack over S.

Note that even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack $\mathcal{X}$ is obtained by defining $\mathcal{X}(T)$ as the groupoid of objects of $\mathcal{X}'(T \times_S S')$ equipped with a descent datum relative to the 2-descent datum of $\mathcal{X}'$, for any scheme T over S. More precisely, the first assertion in the above theorem follows from:

THEOREM 1.2 (Breen, Giraud). *Consider the 2-fibred category*

$$\underline{Stack}_S \to (\mathrm{Sch}/S)_{fppf},$$

whose fibre over $U \in (\mathrm{Sch}/S)_{fppf}$ is the 2-category $\underline{Stack}(U)$ of stacks over U. Then $\underline{Stack}_S$ is a 2-stack over S.

PROOF. See [Bre94, Example 1.11.(1)] and [Gir71, Chapitre II, §2.1.5]. □

The other two assertions in Theorem 1.1 follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the smooth (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, then for a stack $\mathcal{X}'$ over S', one can reformulate the notion of 2-descent datum for $\mathcal{X}'$ over S' in terms of an action of Γ on $\mathcal{X}'$ over the action of Γ on S' over S, as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define ${}^{\sigma}\mathcal{X}'$ as the pull-back of $\mathcal{X}'$ along $\sigma\colon S' \to S'$.

DEFINITION 1.3. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be a stack over S'. A* Galois 2-descent datum *consists of:*

(1) a family of 1-isomorphisms $f_\sigma \colon {}^\sigma\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ $(\sigma \in \Gamma)$;

(2) a family of 2-isomorphisms $\psi_{\sigma,\tau} \colon f_\sigma \circ {}^\sigma(f_\tau) \implies f_{\sigma\tau}$ $(\sigma, \tau \in \Gamma)$;

such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms

$$\begin{tikzcd} f_\sigma \circ {}^\sigma f_\tau \circ {}^{\sigma\tau} f_\gamma \arrow[r, Rightarrow, "({}^{\sigma\tau} f_\gamma)^*(\psi_{\sigma,\tau})"] \arrow[d, Rightarrow, "f_{\sigma *}({}^\sigma\psi_{\tau,\gamma})"] & f_{\sigma\tau} \circ {}^{\sigma\tau} f_\gamma \arrow[d, Rightarrow, "\psi_{\sigma\tau,\gamma}"] \\ f_\sigma \circ {}^\sigma f_{\tau\gamma} \arrow[r, Rightarrow, "\psi_{\sigma,\tau\gamma}"] & f_{\sigma\tau\gamma} \end{tikzcd}$$

is commutative.

One can show that to give a Galois 2-descent datum on $\mathcal{X}'$ over S' is to give a group action (in the sense of [Rom05]) of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for $\mathcal{X}'$ over S', see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

THEOREM 1.4. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be an algebraic stack over S', equipped with a Galois 2-descent datum $(f_\sigma\ (\sigma \in \Gamma),\ \psi_{\sigma,\tau}\ (\sigma, \tau \in \Gamma))$. There exists an algebraic stack $\mathcal{X}$ over S and an isomorphism $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over S'. The stack $\mathcal{X}$ is Deligne–Mumford if and only if $\mathcal{X}'$ is.*

Observe that the statement in Theorem 1.4 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an isomorphism of stacks $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ over S' as well as a family of 2-isomorphisms $\chi_\sigma \colon \rho \circ \mathrm{can} \implies f_\sigma \circ {}^\sigma\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$\begin{tikzcd} {}^\sigma(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & \mathcal{X} \times_S S' \arrow[d] \\ {}^\sigma\mathcal{X}' \arrow[r] & \mathcal{X}', \end{tikzcd}$$

such that the obvious compatibility conditions are satisfied.

Returning to the case of an arbitrary faithfully flat locally finitely presented morphism of schemes $S' \to S$, Theorem 1.2 shows that the canonical 2-functor

$$\underline{Stack}(S) \to \underline{Stack}(\{S' \to S\}),$$

that sends a stack over S to the associated stack $\mathcal{X}'$ with canonical 2-descent datum over S', is an equivalence of 2-categories. Here, $\underline{Stack}(\{S' \to S\})$ is the 2-category of stacks over S' equipped with a 2-descent datum (see Definitions 3.1 and 4.1). With regard to morphisms, this has the following consequence.

PROPOSITION 1.5. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and for $i = 1, 2$, let $\mathcal{X}_i$ be a stack over S. Let $\mathcal{X}_i' = \mathcal{X} \times_S S'$ with associated canonical 2-descent datum (ϕ_i, ψ_i). Then the canonical functor*

$$\mathrm{Hom}_S(\mathcal{X}_1, \mathcal{X}_2) \to \mathrm{Hom}_{\mathrm{descent}/S'}\left(\left(\mathcal{X}_1', \phi_1, \psi_1\right), \left(\mathcal{X}_2', \phi_2, \psi_2\right)\right)$$

is an equivalence of categories.

Here, $\mathrm{Hom}_S(\mathcal{X}_1, \mathcal{X}_2)$ denotes the category of morphisms of stacks $\mathcal{X}_1 \to \mathcal{X}_2$ over S, and $\mathrm{Hom}_{\mathrm{descent}/S'}((\mathcal{X}_1', \phi_1, \psi_1), (\mathcal{X}_2', \phi_2, \psi_2))$ the category of morphisms of stacks with 2-descent data over S' (cf. Definition 4.1).

2. Descending schemes

Let

$$p \colon S' \to S$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \to S,$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$S''' \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} S'' \rightrightarrows S' \to S$$

where the three arrows $S''' \to S''$ are p_{12}, p_{13} and p_{23}.

Let X' be a scheme over S'. Define

$$p_i^* X' = X' \times_{S', p_i} S'', \quad p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'', p_{jk}} S'''$$

and note that

$$p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'', p_{jk}} S''' = \left(p_i \circ p_{jk}\right)^* X'.$$

Recall that a *descent datum* for X'/S' is an S''-isomorphism

$$\phi \colon p_1^* X' \xrightarrow{\sim} p_2^* X'$$

such that the following diagram commutes:

$$\begin{tikzcd}
p_{12}^* p_1^* X' \arrow[r, "p_{12}^*\phi"] \arrow[d, equal] & p_{12}^* p_2^* X' \arrow[r, equal] & p_{23}^* p_1^* X' \arrow[d, "p_{23}^*\phi"] \\
p_{13}^* p_1^* X' \arrow[r, "p_{13}^*\phi"] & p_{13}^* p_2^* X \arrow[r, equal] & p_{23}^* p_2^* X'.
\end{tikzcd}$$

In other words, one requires that

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi \quad \text{as morphisms} \quad p_{12}^* p_1^* X' \to p_{13}^* p_2^* X'.$$

THEOREM 2.1 (Grothendieck). *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of affine schemes. The functor $X \mapsto p^*X$ defines an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X'/S' such that X' admits an open covering by affine schemes stable under ϕ.*

Next, recall how to make this explicit in case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. For instance, S could be the spectrum of a field k, S' the spectrum of a finite field extension $k' \supset k$, and Γ the Galois group of k'/k. Let X' be a scheme over S' and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma\colon {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over S', for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau}X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of Γ on X' as a scheme over S is said to be *compatible with the action of* Γ *on* S' *over* S if for each $\sigma \in \Gamma$, the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d] & X' \arrow[d] \\ S' \arrow[r, "\sigma"] & S'. \end{tikzcd}$$

LEMMA 2.2. *Let $p\colon S' \to S$ be a finite faithfully flat morphism of schemes. Assume π is a Galois covering with Galois group Γ. Let X' be a scheme over S'.*

(1) To give a descent datum for X' over S' is to give a Galois descent datum for X' over S'.
(2) These notions are further equivalent to giving an action of Γ on X' over S compatible with the action of Γ on S' over S.

PROOF. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4]. □

3. Descending algebraic stacks

Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids on S', in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$r_i\colon S'''' \to S'. \tag{3.1}$$

Similarly, S''' is equipped with three projections $q_i \colon S''' \to S'$. Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on S'. One has similar isomorphisms relating the other $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$ with $r_\mu^* \mathcal{X}'$, for $i, j, k \in \{1, 2, 3, 4\}$, $\alpha, \beta \in \{1, 2, 3\}$, $\nu \in \{1, 2\}$ and $\mu \in \{1, 2, 3, 4\}$.

Consider an isomorphism of S''-stacks (i.e. an equivalence of Sch/S''-categories):

$$\phi \colon p_1^* \mathcal{X}' \to p_2^* \mathcal{X}',$$

and let ψ be a 2-morphism

$$\psi \colon p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism $\Rightarrow$ in the following diagram:

$$\begin{tikzcd}
p_{12}^* p_1^* \mathcal{X}' \arrow[r, "p_{12}^*\phi"] \arrow[d, equal] & p_{12}^* p_2^* \mathcal{X}' \arrow[r, equal] \arrow[dr, phantom, "\Swarrow \psi"] & p_{23}^* p_1^* \mathcal{X}' \arrow[d, "p_{23}^*\phi"] \\
p_{13}^* p_1^* \mathcal{X}' \arrow[r, "p_{13}^*\phi"] & p_{13}^* p_2^* \mathcal{X}' \arrow[r, equal] & p_{23}^* p_2^* \mathcal{X}'.
\end{tikzcd} \tag{3.2}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234} \colon S'''' \to S''',$$

and note that

$$\begin{aligned}
p_{123}^* \left(p_{23}^* \phi \circ p_{12}^* \phi \right) &= p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and} \\
p_{123}^* p_{13}^* \phi &= \pi_{13}^* \phi,
\end{aligned}$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34} \colon S'''' \to S''$$

are the canonical morphisms. For $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$, define

$$\psi_{ijk} \coloneqq p_{ijk}^* \psi.$$

For instance, pulling back ψ along p_{123} gives a 2-morphism

$$\psi_{123} = p_{123}^* \psi \colon \pi_{23}^* \circ \pi_{12}^* \phi \Rightarrow \pi_{13}^* \phi.$$

Similarly, we obtain 2-morphisms

$$\begin{aligned}
\psi_{124} &\colon \pi_{24}^*\phi \circ \pi_{12}^*\phi \Rightarrow \pi_{14}^*\phi,\\
\psi_{134} &\colon \pi_{34}^*\phi \circ \pi_{13}^*\phi \Rightarrow \pi_{14}^*\phi,\\
\psi_{234} &\colon \pi_{34}^*\phi \circ \pi_{23}^*\phi \Rightarrow \pi_{24}^*\phi.
\end{aligned}$$

Moreover, observe that under p_{123}, diagram (3.2) pulls back to the diagram

$$\begin{tikzcd}
r_1^*\mathcal{X}' \arrow[r, "\pi_{12}^*\phi"] \arrow[d, equal] & r_2^*\mathcal{X}' \arrow[r, equal] \arrow[dr, phantom, "\Swarrow"] & r_2^*\mathcal{X}' \arrow[d, "\pi_{23}^*\phi"] \\
r_1^*\mathcal{X}' \arrow[r, "\pi_{13}^*\phi"] & r_3^*\mathcal{X}' \arrow[r, equal] & r_3^*\mathcal{X}',
\end{tikzcd} \tag{3.3}$$

in which the 2-morphism $\Rightarrow$ is the 2-morphism ψ_{123} defined above (and with r_i is as in (3.1)). Using pull-backs by the other three $p_{ijk}\colon S'''' \to S'''$, we thus obtain four triangles, that we may put together to form the following tetrahedron:

$$\begin{tikzcd}
 & r_1^*\mathcal{X}' \arrow[dl] \arrow[dr] \arrow[dd] & \\
r_2^*\mathcal{X}' \arrow[rr] \arrow[dr] & & r_4^*\mathcal{X}' \\
 & r_3^*\mathcal{X}'. \arrow[ur] &
\end{tikzcd} \tag{3.4}$$

Definition 3.1. *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over S'. A* 2-descent datum *for $\mathcal{X}'$ over S' consists of:*

(1) an isomorphism of stacks (i.e. an equivalence of categories)

$$\phi\colon p_1^*\mathcal{X}' \to p_2^*\mathcal{X}'$$

over S'';

(2) a 2-isomorphism

$$\psi\colon p_{23}^*\phi \circ p_{12}^*\phi \Rightarrow p_{13}^*\phi$$

as in diagram (3.2);

such that the following condition is satisfied: the 2-morphisms ψ_{ijk} between the several compositions in diagram (3.4) are compatible, in the sense that the *following diagram of 2-morphisms commutes:*

$$\begin{tikzcd}
\pi_{34}^*\phi \circ \pi_{23}^*\phi \circ \pi_{12}^*\phi \arrow[r, Rightarrow, "(\pi_{34}^*\phi)_*(\psi_{123})"] \arrow[d, Rightarrow, "(\pi_{12}^*\phi)^*(\psi_{234})"] & p_{34}^*\phi \circ p_{13}^*\phi \arrow[d, Rightarrow, "\psi_{134}"] \\
p_{24}^*\phi \circ p_{12}^*\phi \arrow[r, Rightarrow, "\psi_{124}"] & p_{14}^*\phi.
\end{tikzcd}$$

This gives the following result.

Theorem 3.2 (Breen). *Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S'. Then there exists a stack $\mathcal{X}$ over S, an isomorphism*

$$\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over S', and a 2-isomorphism $\chi\colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ as in diagram

$$\begin{tikzcd}
p_1^*(\mathcal{X} \times_S S') \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\rho"'] \arrow[dr, phantom, "\Swarrow"] & p_2^*(\mathcal{X} \times_S S') \arrow[d, "p_2^*\rho"] \\
p_1^*\mathcal{X}' \arrow[r, "\phi"] & p_2^*\mathcal{X}',
\end{tikzcd} \tag{3.5}$$

such that the natural compatibility condition between χ and ψ is satisfied.

Proof. This follows from [Bre94, Example 1.11.(i)]. $\square$

To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-lemma, which should be well-known but which we include for convenience of the reader.

Lemma 3.3. *Let S be a scheme. The following assertions are true.*

(1) *Let $\pi\colon \mathcal{X}' \to \mathcal{X}$ be a representable, smooth and surjective morphism of stacks in groupoids over S. If $\mathcal{X}'$ is algebraic, then $\mathcal{X}$ is algebraic. If in addition π is étale and $\mathcal{X}'$ is Deligne–Mumford, then $\mathcal{X}$ is Deligne–Mumford.*
(2) *Let $S' \to S$ be a smooth surjective morphism of schemes, let $\mathcal{X}$ be a stack in groupoids over S and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Suppose that $\mathcal{X}'$ is an algebraic stack over S'. Then $\mathcal{X}$ is an algebraic stack over S. If in addition $S' \to S$ is étale and $\mathcal{X}'$ is a Deligne–Mumford stack, then $\mathcal{X}$ is a Deligne–Mumford stack.*

Proof. Let us first prove item (1). If U' is a scheme and $U' \to \mathcal{X}'$ a surjective and smooth morphism, then $U' \to \mathcal{X}' \to \mathcal{X}$ is surjective and smooth, and moreover étale if π and $U' \to \mathcal{X}'$ are étale. Therefore, it suffices to prove that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces if the diagonal $\Delta' \colon \mathcal{X}' \to \mathcal{X}' \times_S \mathcal{X}'$ is representable by algebraic spaces. For this, it suffices to consider to schemes U and V, equipped with morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:

$$\begin{tikzcd}
 & U' \times_{\mathcal{X}'} V' \arrow[dl] \arrow[dd] \arrow[drr] & & \\
V' \arrow[dd] \arrow[drr] & & & U \times_{\mathcal{X}} V \arrow[dl] \arrow[dd] \\
 & U' \arrow[dl] \arrow[drr] & V \arrow[dd] & \\
\mathcal{X}' \arrow[drr] & & & U. \arrow[dl] \\
 & & \mathcal{X} &
\end{tikzcd}$$

The morphism $\mathcal{X}' \to \mathcal{X}$ is representable, hence U' and V' are representable by algebraic spaces. Since $\mathcal{X}'$ is an algebraic stack, the morphism $V' \to \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \to U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \to U$ is étale and surjective, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that the morphism $U \times_{\mathcal{X}} V \to U$ is representable. As U is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, and we are done.

Next, we prove item (2). Via the composition $\mathcal{X}' \to S' \to S$, we may view $\mathcal{X}'$ as an algebraic stack over S, see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over S:

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}$$

As $S' \to S$ is representable, surjective and smooth (resp. étale), the same holds for $\mathcal{X}' \to \mathcal{X}$. Hence, the assertions follow from item (1). □

Proof of Theorem 1.1. Theorem 3.2 yields the stack $\mathcal{X}$ over S together with 1-isomorphism $\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ and the 2-isomorphism $\chi \colon p_2^* \rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^* \rho$ that have the right compatibility properties with respect to ψ, so that we only need to prove that $\mathcal{X}$ is algebraic (resp. Deligne–Mumford if $S' \to S$ is surjective étale). This follows from Lemma 3.3. □

Even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result:

Corollary 3.4. *Let $S' \to S$ be a surjective étale morphism of schemes, and let X' be a scheme over S' equipped with a descent datum ϕ as in Section 2. Then there exists an algebraic space X over S and an S-morphism of algebraic spaces $\pi\colon X' \to X$ such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is cartesian. The pair $(X, \pi\colon X' \to X)$ is compatible with the descent datum ϕ in an appropriate sense, and this makes (X, π) unique up to isomorphism.

Proof. Theorem 1.1 implies the existence of X as a Deligne–Mumford stack, hence we only need to prove that X is an algebraic space. For this, in view of [LMB00, Proposition 2.4.1.1], it suffices to show that the diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is a monomorphism. This is fppf local on S [Stacks, Tag 02YK], thus follows from the fact that $X' \to X' \times_{S'} X'$ is a monomorphism. □

For a scheme S and a stack $\mathcal{X}$, and a finite group Γ, a *group action of* Γ *on* $\mathcal{X}$ *over* S is an action of the functor in groups over S associated to Γ on the stack $\mathcal{X}$ over S, see [Rom05, Definition 1.3].

Lemma 3.5. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, and let $\mathcal{X}'$ be a stack over S'. Then the following sets are in canonical bijection:*

(1) The set of 2-descent data (ϕ, ψ) for $\mathcal{X}'$ over S'.
(2) The set of group actions of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$.
(3) The set of Galois 2-descent data for $\mathcal{X}'$ over S'.

Proof. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] a the proof in the case of schemes. The stacky case is requires some straightforward generalizations; we leave the details to the reader. □

Proof of Theorem 1.4. See Theorem 1.1 and Lemma 3.5. □

4. Morphisms of stacks with descent data

Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and for $i = 1, 2$, let $\mathcal{X}_i$ be a stack over S. Let $\mathcal{X}'_i = \mathcal{X} \times_S S'$ with associated canonical 2-descent datum (ϕ_i, ψ_i).

Definition 4.1. *A* morphism

$$(\mathcal{X}'_1, \phi_1, \psi_1) \to (\mathcal{X}'_2, \phi_2, \psi_2)$$

of stacks with 2*-descent data over* S' *consists of a pair* (f, α)*, where* $f\colon \mathcal{X}_1' \to \mathcal{X}_2'$ *is a morphism of stacks over* S' *and* $\alpha\colon \phi_2 \circ p_1^* f \implies p_1^* f \circ \phi_1$ *a* 2*-morphism as in the following diagram:*

$$\begin{tikzcd}
p_1^*(\mathcal{X}_1') \arrow[r, "\phi_1"] \arrow[d, "p_1^*f"] \arrow[dr, phantom, "\swarrow"] & p_2^*(\mathcal{X}_1') \arrow[d, "p_2^*f"] \\
p_1^*\mathcal{X}_2' \arrow[r, "\phi_2"] & p_2^*\mathcal{X}_2',
\end{tikzcd}$$

such that each square in the following diagram of 2*-morphisms commutes:*

$$\begin{tikzcd}
& p_{12}^*p_1^*\mathcal{X}_1' \arrow[rr, "p_{12}^*\phi_1"] \arrow[dl, "p_{12}^*p_1^*f"'] \arrow[dd, equal] & & p_{12}^*p_2^*\mathcal{X}_1' \arrow[rr, equal] \arrow[dl, "p_{12}^*p_2^*f"'] & & p_{23}^*p_1^*\mathcal{X}_1' \arrow[dl, "p_{23}^*p_1^*f"'] \arrow[dd, "p_{23}^*\phi_1"] \\
p_{12}^*p_1^*\mathcal{X}_2' \arrow[rr, "p_{12}^*\phi_2"] \arrow[dd, equal] & & p_{12}^*p_2^*\mathcal{X}_2' \arrow[rr, equal] & & p_{23}^*p_1^*\mathcal{X}_2' \arrow[dd, "p_{23}^*\phi_2"] & \\
& p_{13}^*p_1^*\mathcal{X}_1' \arrow[rr, "p_{13}^*\phi_1"] \arrow[dl, "p_{13}^*p_1^*f"] & & p_{13}^*p_2^*\mathcal{X}_1' \arrow[rr, equal] \arrow[dl, "p_{13}^*p_2^*f"] & & p_{23}^*p_2^*\mathcal{X}_1' \arrow[dl, "p_{23}^*p_2^*f"] \\
p_{13}^*p_1^*\mathcal{X}_2' \arrow[rr, "p_{13}^*\phi_2"] & & p_{13}^*p_2^*\mathcal{X}_2'. \arrow[rr, equal] & & p_{23}^*p_2^*\mathcal{X}_2'. &
\end{tikzcd}$$

Here, the 2*-morphisms in each square are the canonical ones (induced by* α*).*

Proof of Proposition 1.5. This follows from [Gir71, Chapitre II, §2.1.5] (and is also a special case of Theorem 1.2). ◻

5. Example

Let k be a field and let $k \subset k'$ be a degree two field extension; one may think of $\mathbb{R} \subset \mathbb{C}$ or $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ for a prime power q. Let $\sigma \in \mathrm{Gal}(k'/k)$ be the generator of $\mathrm{Gal}(k'/k)$. Let $\mathcal{X}'$ be a stack over k' equipped with a 1-isomorphism

$$\sigma\colon \mathcal{X}' \to \mathcal{X}'$$

of stacks over k, and a 2-isomorphism $F\colon \sigma^2 \implies \mathrm{id}_{\mathcal{X}'}$ between σ^2 and the identity functor, such that σ commutes with the functor $(\mathsf{Sch}/k') \to (\mathsf{Sch}/k')$ defined as

$$T \mapsto {}^{\sigma}T = T \times_{k',\sigma} k',$$

and such that for each $x \in \mathcal{X}'(T)$, $T \in (\mathsf{Sch}/k')$, the isomorphism $F(x)\colon \sigma^2(x) \to x$ lies over the canonical isomorphism of k-schemes ${}^{\sigma}({}^{\sigma}T) \to T$. One obtains the descended stack $\mathcal{X}$ over k by defining, for $T \in (\mathsf{Sch}/k)$, $\mathcal{X}(T)$ as the groupoid of pairs (x, φ) with $x \in \mathcal{X}'(T_{k'})$ and $\varphi\colon x \to \sigma(x)$ an isomorphism such that the composition

$$x \xrightarrow{\varphi} \sigma(x) \xrightarrow{{}^{\sigma}\varphi} \sigma^2(x) \xrightarrow{F} x$$

is the identity. There is a natural isomorphism $\mathcal{X} \times_k k' \cong \mathcal{X}'$ of stacks over k'.

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Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands

Email address: a.o.d.degaayfortman@uu.nl