Descent for algebraic stacks

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ABSTRACT. We prove that algebraic stacks satisfy 2-descent for fppf coverings. We generalize Galois descent for schemes to stacks, by considering the case where the fppf covering is a finite Galois covering and reformulating 2-descent data in terms of group actions on the stack.

1. Introduction

Let $S' \to S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor $X \mapsto X \times_S S'$ induces an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X' over S' such that X' admits an open covering by S'-affine schemes which are stable under ϕ (cf. [Gro60]). In case $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(k')$ and the morphism $S' \to S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil (cf. [Wei56]).

The aim of this paper is to present the most natural analogue of this result in the setting of algebraic stacks. To do so, we assemble existing results on descent theory for stacks, based on work of Giraud [Gir71], Duskin [Dus89], and Breen [Bre94]. It follows from these results that stacks with descent data over a scheme S' descend along fppf morphisms of schemes $p \colon S' \to S$. The main contribution of this paper is to show that algebraic stacks descend not only as stacks, but actually as algebraic stacks. Moreover, if $p \colon S' \to S$ is étale and surjective, then Deligne–Mumford stacks descend along p as well.

Let us explain this in more detail. In the case of stacks, the analogue of the aforementioned descent-theory for schemes is a notion called 2-*descent*, which seems due to Duskin [Dus89]. As it turns out, with respect to a faithfully flat locally finitely presented morphism of schemes $S' \to S$, *every* 2-*descent datum for an algebraic stack is effective.* More precisely, we have the following result. For a scheme S, let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of S (cf. [Stacks, Tag 021S]); a *stack over* S is a stack in groupoids $\mathcal{X} \to (\mathrm{Sch}/S)_{fppf}$, see [Stacks, Tag 0304].

THEOREM 1.1. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and let $\mathcal{X}'$ be a stack over S'. Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S', see Definition 3.1. Then the following holds.*

(1) *The 2-descent datum (ϕ, ψ) is effective. That is, there exists a stack $\mathcal{X}$ over S, an isomorphism of stacks over S',*

$$\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

and a 2-isomorphism $\chi \colon p_2^* \rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^* \rho$ as in the following diagram:

$$
\begin{array}{ccc}
p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\ \mathrm{can}\ } & p_2^*(\mathcal{X} \times_S S') \\
\downarrow{\scriptstyle p_1^* \rho} & \swarrow & \downarrow{\scriptstyle p_2^* \rho} \\
p_1^* \mathcal{X}' & \xrightarrow[\ \phi\ ]{} & p_2^* \mathcal{X}',
\end{array}
$$

such that the natural compatibility between χ and ψ is satisfied.

(2) *The stack $\mathcal{X}'$ over S' is algebraic, with separated and quasi-compact diagonal, if and only if the analogous properties hold for $\mathcal{X}$ over S.*

(3) *If the morphism $S' \to S$ is étale, then $\mathcal{X}'$ is a Deligne–Mumford stack over S' if and only if $\mathcal{X}$ is a Deligne–Mumford stack over S.*

REMARK 1.2. The conditions on the diagonal in item (2) of Theorem 1.1 ensure that we can apply Artin's result [Art74, Theorem (6.1)]. Without these separation conditions, the statement in item (2) still holds if $S' \to S$ is smooth.

Note that even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack $\mathcal{X}$ is obtained by defining $\mathcal{X}(T)$ as the groupoid of objects of $\mathcal{X}'(T \times_S S')$ equipped with a descent datum relative to the 2-descent datum of $\mathcal{X}'$, for any scheme T over S. More precisely, the first assertion in the above theorem follows from:

THEOREM 1.3 (Breen, Giraud). *Consider the 2-fibred category*

$$\underline{Stack}_S \to (\mathrm{Sch}/S)_{fppf},$$

whose fibre over $U \in (\mathrm{Sch}/S)_{fppf}$ is the 2-category $\underline{Stack}(U)$ of stacks over U. Then $\underline{Stack}_S$ is a 2-stack over S.

PROOF. See [Bre94, Example 1.11.(1)] and [Gir71, Chapitre II, §2.1.5]. □

The other two assertions in Theorem 1.1 follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local on the base for the fppf (resp. étale) topology; see Lemma 3.3 for a precise statement. For details, we refer to Section 3.

In case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ, then for a stack $\mathcal{X}'$ over S', one can reformulate the notion of 2-descent datum for $\mathcal{X}'$ over S' in terms of an action of Γ on $\mathcal{X}'$ over the action of Γ on S' over S, as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define ${}^{\sigma}\mathcal{X}'$ as the pull-back of $\mathcal{X}'$ along $\sigma \colon S' \to S'$.

DEFINITION 1.4. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be a stack over S'. A* Galois 2-descent datum *consists of:*

(1) *a family of 1-isomorphisms $f_{\sigma} \colon {}^{\sigma}\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ ($\sigma \in \Gamma$);*
(2) *a family of 2-isomorphisms $\psi_{\sigma,\tau} \colon f_{\sigma} \circ {}^{\sigma}(f_{\tau}) \implies f_{\sigma\tau}$ ($\sigma, \tau \in \Gamma$);*

such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms

$$
\begin{array}{ccc}
f_{\sigma} \circ {}^{\sigma}f_{\tau} \circ {}^{\sigma\tau}f_{\gamma} & \xrightarrow{({}^{\sigma\tau}f_{\gamma})^{*}(\psi_{\sigma,\tau})} & f_{\sigma\tau} \circ {}^{\sigma\tau}f_{\gamma} \\
\big\| {\scriptstyle f_{\sigma *}({}^{\sigma}\psi_{\tau,\gamma})} & & \big\| {\scriptstyle \psi_{\sigma\tau,\gamma}} \\
f_{\sigma} \circ {}^{\sigma}f_{\tau\gamma} & \xrightarrow{\psi_{\sigma,\tau\gamma}} & f_{\sigma\tau\gamma}
\end{array}
$$

is commutative.

One can show that to give a Galois 2-descent datum on $\mathcal{X}'$ over S' is to give a group action (in the sense of [Rom05]) of Γ on $\mathcal{X}'$ as a stack over S, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for $\mathcal{X}'$ over S', see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

THEOREM 1.5. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. Let $\mathcal{X}'$ be an algebraic stack over S', equipped with a Galois 2-descent datum (f_{σ} ($\sigma \in \Gamma$), $\psi_{\sigma,\tau}$ ($\sigma, \tau \in \Gamma$)). Then there exists an algebraic stack $\mathcal{X}$ over S and an isomorphism $\rho \colon \mathcal{X} \times_{S} S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over S'. The stack $\mathcal{X}$ is Deligne–Mumford if and only if $\mathcal{X}'$ is.*

Observe that the statement in Theorem 1.5 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an isomorphism of stacks $\rho \colon \mathcal{X} \times_{S} S' \xrightarrow{\sim} \mathcal{X}'$ over S' as well as a family of 2-isomorphisms $\chi_{\sigma} \colon \rho \circ \mathrm{can} \implies f_{\sigma} \circ {}^{\sigma}\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$
\begin{array}{ccc}
{}^{\sigma}(\mathcal{X} \times_{S} S') & \xrightarrow{\mathrm{can}} & \mathcal{X} \times_{S} S' \\
\downarrow & \swarrow & \downarrow \\
{}^{\sigma}\mathcal{X}' & \longrightarrow & \mathcal{X}',
\end{array}
$$

such that the obvious compatibility conditions are satisfied.

Returning to the case of an arbitrary faithfully flat locally finitely presented morphism of schemes $S' \to S$, Theorem 1.3 shows that the canonical 2-functor

$$\underline{Stack}(S) \to \underline{Stack}(\{S' \to S\}),$$

that sends a stack over S to the associated stack $\mathcal{X}'$ with canonical 2-descent datum over S', is an equivalence of 2-categories. Here, $\underline{Stack}(\{S' \to S\})$ is the 2-category of stacks over S' equipped with a 2-descent datum (see Definitions 3.1 and 4.1). With regard to morphisms, this has the following consequence.

PROPOSITION 1.6. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and for $i = 1, 2$, let $\mathcal{X}_i$ be a stack over S. Let $\mathcal{X}'_i = \mathcal{X} \times_S S'$ with associated canonical 2-descent datum (ϕ_i, ψ_i). Then the canonical functor*

$$\mathrm{Hom}_S(\mathcal{X}_1, \mathcal{X}_2) \to \mathrm{Hom}_{\mathrm{descent}/S'}\left(\left(\mathcal{X}'_1, \phi_1, \psi_1 \right), \left(\mathcal{X}'_2, \phi_2, \psi_2 \right) \right)$$

is an equivalence of categories.

Here, $\mathrm{Hom}_S(\mathcal{X}_1, \mathcal{X}_2)$ denotes the category of morphisms of stacks $\mathcal{X}_1 \to \mathcal{X}_2$ over S, and $\mathrm{Hom}_{\mathrm{descent}/S'}((\mathcal{X}'_1, \phi_1, \psi_1), (\mathcal{X}'_2, \phi_2, \psi_2))$ the category of morphisms of stacks with 2-descent data over S' (cf. Definition 4.1).

Acknowledgements

2. Descending schemes

Let $p \colon S' \to S$ be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times S' \overset{p_1}{\underset{p_2}{\rightrightarrows}} S' \to S,$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$S''' \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} S'' \rightrightarrows S' \to S$$

where the three arrows $S''' \to S''$ are p_{12}, p_{13} and p_{23}.

Let X' be a scheme over S'. Define

$$p_i^* X' = X' \times_{S', p_i} S'', \quad p_{jk}^* p_i^* X' = \left(p_i^* X' \right) \times_{S'', p_{jk}} S'''$$

and note that

$$p_{jk}^* p_i^* X' = \left(p_i^* X' \right) \times_{S'', p_{jk}} S''' = \left(p_i \circ p_{jk} \right)^* X'.$$

Recall that a *descent datum* for X'/S' is an S''-isomorphism

$$\phi \colon p_1^* X' \overset{\sim}{\to} p_2^* X'$$

such that the following diagram commutes:

$$
\begin{array}{ccccc}
p_{12}^*p_1^*X' & \xrightarrow{\ p_{12}^*\phi\ } & p_{12}^*p_2^*X' & =\!=\!= & p_{23}^*p_1^*X' \\
\Big\| & & & & \Big\downarrow{\scriptstyle p_{23}^*\phi} \\
p_{13}^*p_1^*X' & \xrightarrow{\ p_{13}^*\phi\ } & p_{13}^*p_2^*X & =\!=\!= & p_{23}^*p_2^*X'.
\end{array}
$$

In other words, one requires that

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi \quad \text{as morphisms} \quad p_{12}^*p_1^*X' \to p_{13}^*p_2^*X'.$$

THEOREM 2.1 (Grothendieck). *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. The functor $X \mapsto p^*X$ defines an equivalence of categories between the category of S-schemes X and the category of pairs (X', ϕ) where X' is an S'-scheme and ϕ a descent datum for X'/S' such that X' admits an open covering by S'-affine schemes stable under ϕ.*

PROOF. See [Gro60, Theorem 2] and the discussion below Lemma 1.2 in *loc. cit.* □

Next, recall how to make this explicit in case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group Γ. For instance, S could be the spectrum of a field k, S' the spectrum of a finite field extension $k' \supset k$, and Γ the Galois group of k'/k. Let X' be a scheme over S' and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma \colon {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over S', for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau}X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of Γ on X' as a scheme over S is said to be *compatible with the action of Γ on S' over S* if for each $\sigma \in \Gamma$, the composition $X' \xrightarrow{\sigma} X' \to S'$ agrees with the composition $X' \to S' \xrightarrow{\sigma} S'$.

LEMMA 2.2. *Let $p\colon S' \to S$ be a finite faithfully flat morphism of schemes. Assume π is a Galois covering with Galois group Γ. Let X' be a scheme over S'.*

(1) *To give a descent datum for X' over S' is to give a Galois descent datum for X' over S'.*

(2) *These notions are further equivalent to giving an action of Γ on X' over S compatible with the action of Γ on S' over S.*

PROOF. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4]. □

3. Descending algebraic stacks

Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over S', in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$(3.1) \qquad\qquad r_i\colon S'''' \to S'.$$

Similarly, S''' is equipped with three projections $q_i\colon S''' \to S'$. Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on S'. One has similar isomorphisms relating the other $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$ with $r_\mu^* \mathcal{X}'$, for $i, j, k \in \{1, 2, 3, 4\}$, $\alpha, \beta \in \{1, 2, 3\}$, $\nu \in \{1, 2\}$ and $\mu \in \{1, 2, 3, 4\}$.

Consider an isomorphism of S''-stacks (i.e. an equivalence of Sch/S''-categories):

$$\phi\colon p_1^* \mathcal{X}' \to p_2^* \mathcal{X}',$$

and let ψ be a 2-morphism

$$\psi\colon p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism $\Rightarrow$ in the following diagram:

$$(3.2) \qquad
\begin{array}{ccccc}
p_{12}^* p_1^* \mathcal{X}' & \xrightarrow{\;p_{12}^*\phi\;} & p_{12}^* p_2^* \mathcal{X}' & =\!=\!= & p_{23}^* p_1^* \mathcal{X}' \\
\Big\| & & \Swarrow \psi & & \Big\downarrow{\scriptstyle p_{23}^*\phi} \\
p_{13}^* p_1^* \mathcal{X}' & \xrightarrow{\;p_{13}^*\phi\;} & p_{13}^* p_2^* \mathcal{X}' & =\!=\!= & p_{23}^* p_2^* \mathcal{X}'.
\end{array}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234}\colon S'''' \to S''',$$

and note that

$$p_{123}^* \left(p_{23}^* \phi \circ p_{12}^* \phi \right) = p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and}$$
$$p_{123}^* p_{13}^* \phi = \pi_{13}^* \phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}\colon S'''' \to S''$$

are the canonical morphisms. For $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$, define

$$\psi_{ijk} := p^*_{ijk}\psi.$$

For instance, pulling back ψ along p_{123} gives a 2-morphism

$$\psi_{123} = p^*_{123}\psi \colon \pi^*_{23} \circ \pi^*_{12}\phi \Rightarrow \pi^*_{13}\phi.$$

Similarly, we obtain 2-morphisms

$$\psi_{124} \colon \pi^*_{24}\phi \circ \pi^*_{12}\phi \Rightarrow \pi^*_{14}\phi,$$
$$\psi_{134} \colon \pi^*_{34}\phi \circ \pi^*_{13}\phi \Rightarrow \pi^*_{14}\phi,$$
$$\psi_{234} \colon \pi^*_{34}\phi \circ \pi^*_{23}\phi \Rightarrow \pi^*_{24}\phi.$$

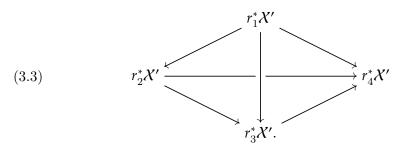Moreover, observe that under p_{123}, diagram (3.2) pulls back to the diagram

$$
\begin{array}{ccc}
r^*_1\mathcal{X}' & \xrightarrow{\pi^*_{12}\phi} & r^*_2\mathcal{X}' === r^*_2\mathcal{X}' \\
\| & \swarrow & \downarrow{\pi^*_{23}\phi} \\
r^*_1\mathcal{X}' & \xrightarrow{\pi^*_{13}\phi} & r^*_3\mathcal{X}' === r^*_3\mathcal{X}',
\end{array}
$$

in which the 2-morphism $\Rightarrow$ is the 2-morphism ψ_{123} defined above (and with r_i is as in (3.1)). Using pull-backs by the other three $p_{ijk} \colon S'''' \to S'''$, we thus obtain four triangles, that we may put together to form the following tetrahedron:

(3.3)

DEFINITION 3.1. *Let $p \colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over S'. A 2-descent datum for $\mathcal{X}'$ over S' consists of:*

(1) an isomorphism of stacks (i.e. an equivalence of categories)

$$\phi \colon p^*_1\mathcal{X}' \to p^*_2\mathcal{X}'$$

over S'';

(2) a 2-isomorphism

$$\psi \colon p^*_{23}\phi \circ p^*_{12}\phi \Rightarrow p^*_{13}\phi$$

as in diagram (3.2);

such that the following condition is satisfied: the 2-morphisms ψ_{ijk} between the several compositions in diagram (3.3) are compatible, in the sense that the following diagram of 2-morphisms commutes:

$$
\begin{array}{ccc}
\pi_{34}^*\phi \circ \pi_{23}^*\phi \circ \pi_{12}^*\phi & \xrightarrow{(\pi_{34}^*\phi)_*(\psi_{123})} & p_{34}^*\phi \circ p_{13}^*\phi \\
\Big\Downarrow{\scriptstyle (\pi_{12}^*\phi)^*(\psi_{234})} & & \Big\Downarrow{\scriptstyle \psi_{134}} \\
p_{24}^*\phi \circ p_{12}^*\phi & \xrightarrow{\psi_{124}} & p_{14}^*\phi.
\end{array}
$$

This gives the following result.

THEOREM 3.2 (Breen). *Let (ϕ, ψ) be a 2-descent datum for the stack $\mathcal{X}'$ over S'. Then there exists a stack $\mathcal{X}$ over S, an isomorphism*

$$\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

of stacks over S', and a 2-isomorphism $\chi \colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ as in diagram
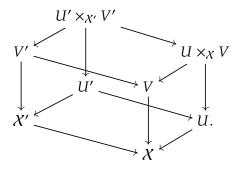
(3.4)
$$
\begin{array}{ccc}
p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & p_2^*(\mathcal{X} \times_S S') \\
\Big\downarrow{\scriptstyle p_1^*\rho} \quad \swarrow\!\!\!\swarrow & & \Big\downarrow{\scriptstyle p_2^*\rho} \\
p_1^*\mathcal{X}' & \xrightarrow{\phi} & p_2^*\mathcal{X}',
\end{array}
$$

such that the natural compatibility condition between χ and ψ is satisfied.

PROOF. This follows from [Bre94, Example 1.11.(i)]. □

To prove Theorem 1.1, we use the fact that any stack over a scheme S which is smooth locally on S an algebraic stack, is actually an algebraic stack. More precisely, we have the following lemma, which is likely well-known but which we include for convenience of the reader.

LEMMA 3.3. *Let $S' \to S$ be an fppf morphism of schemes. Let $\mathcal{X}$ be a stack in groupoids over S and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Then the following holds.*

(1) *The diagonal $\Delta \colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces if and only if the diagonal $\Delta' \colon \mathcal{X}' \to \mathcal{X}' \times_{S'} \mathcal{X}'$ is representable by algebraic spaces. If this is true, then Δ is separated and quasi-compact if and only if Δ' is separated and quasi-compact.*

(2) *Suppose that $\mathcal{X}'$ is an algebraic stack over S' with separated quasi-compact diagonal Δ'. Then $\mathcal{X}$ is an algebraic stack over S with separated quasi-compact diagonal Δ.*

(3) *Assume that $S' \to S$ is étale and surjective. If $\mathcal{X}'$ is a Deligne–Mumford stack over S', then $\mathcal{X}$ is a Deligne–Mumford stack over S.*

PROOF. Let us first prove item (1). Since Δ' is the base change of Δ along $S' \to S$, we may assume that Δ' is representable, separated and quasi-compact, and it suffices under these conditions to prove that Δ is representable (indeed, being separated and quasi-compact is fppf local on the base, see [Stacks, Tag 02YJ]). For this, it suffices to consider to schemes U and V, equipped with morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:

The morphism $\mathcal{X}' \to \mathcal{X}$ of stacks over S is representable as it is the base change of the representable morphism $S' \to S$, hence U' and V' are representable by algebraic spaces. Since $\mathcal{X}'$ is an algebraic stack, the morphism $V' \to \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \to U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \to U$ is an fppf covering, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that $U \times_{\mathcal{X}} V \to U$ is representable. As U is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, proving item (1).

To prove item (2), assume that $\mathcal{X}'$ is algebraic with separated quasi-compact diagonal. By item (1) it suffices to show that $\mathcal{X}$ is algebraic. Let $U' \to \mathcal{X}'$ be a surjective smooth morphism. Then the composition $U' \to \mathcal{X}' \to \mathcal{X}$ is surjective and fppf. In particular, $\mathcal{X}$ is algebraic by [Art74, Theorem (6.1)] (see also [LMB00, Théorème (10.1)]). Hence item (2) follows.

Finally, item (3) follows from the fact that if $S' \to S$ is étale and surjective, and $U' \to \mathcal{X}'$ is an étale surjective presentation by a scheme U', then the composition $U' \to \mathcal{X}' \to \mathcal{X}$ is étale and surjective. We are done. $\square$

PROOF OF THEOREM 1.1. Theorem 3.2 yields the stack $\mathcal{X}$ over S together with the 1-isomorphism $\rho\colon \mathcal{X}\times_S S' \xrightarrow{\sim} \mathcal{X}'$ and the 2-isomorphism $\chi\colon p_2^*\rho\circ\mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ that have the right compatibility properties with respect to ψ. The remaining assertions follow from Lemma 3.3. $\square$

COROLLARY 3.4. *Let $S' \to S$ be a surjective étale morphism of schemes, and let X' be a scheme over S' equipped with a descent datum ϕ as in Section 2. Then there exists an algebraic space X over S and an S-morphism of algebraic*

spaces $\pi\colon X' \to X$ *such that the diagram*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \pi\ } & X \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

is cartesian. The pair $(X, \pi\colon X' \to X)$ *is compatible with the descent datum* ϕ *in an appropriate sense, and this makes* (X, π) *unique up to isomorphism.*

PROOF. Theorem 1.1 implies the existence of X as a Deligne–Mumford stack, hence we only need to prove that X is an algebraic space. For this, in view of [LMB00, Proposition 2.4.1.1], it suffices to show that the diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is a monomorphism. This is fppf local on S [Stacks, Tag 02YK], thus follows from the fact that $X' \to X' \times_{S'} X'$ is a monomorphism. $\quad\square$

For a scheme S and a stack $\mathcal{X}$, and a finite group Γ, a *group action of* Γ *on* $\mathcal{X}$ *over* S is an action of the functor in groups over S associated to Γ on the stack $\mathcal{X}$ over S, see [Rom05, Definition 1.3].

LEMMA 3.5. *Let* $S' \to S$ *be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group* Γ, *and let* $\mathcal{X}'$ *be a stack over* S'. *Then the following sets are in canonical bijection:*

(1) *The set of 2-descent data* (ϕ, ψ) *for* $\mathcal{X}'$ *over* S'.
(2) *The set of group actions of* Γ *on* $\mathcal{X}'$ *as a stack over* S, *such that for each* $\sigma \in \Gamma$, *the composition* $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ *agrees with the composition* $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$.
(3) *The set of Galois 2-descent data for* $\mathcal{X}'$ *over* S'.

PROOF. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] for the proof in the case of schemes. The stacky case requires some straightforward generalizations; we leave the details to the reader. $\quad\square$

PROOF OF THEOREM 1.5. See Theorem 1.1 and Lemma 3.5. $\quad\square$

4. Morphisms of stacks with descent data

Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and for $i = 1, 2$, let $\mathcal{X}_i$ be a stack over S. Let $\mathcal{X}_i' = \mathcal{X} \times_S S'$ with associated canonical 2-descent datum (ϕ_i, ψ_i).

DEFINITION 4.1. *A morphism*

$$
\left(\mathcal{X}_1', \phi_1, \psi_1\right) \to \left(\mathcal{X}_2', \phi_2, \psi_2\right)
$$

of stacks with 2-descent data over S' *consists of a pair* (f, α), *where* $f\colon \mathcal{X}_1' \to \mathcal{X}_2'$ *is a morphism of stacks over* S' *and* $\alpha\colon \phi_2 \circ p_1^* f \implies p_1^* f \circ \phi_1$ *a 2-morphism*

as in the following diagram:

$$
\begin{array}{ccc}
p_1^*(\mathcal{X}_1') & \xrightarrow{\ \phi_1\ } & p_2^*(\mathcal{X}_1') \\
{\scriptstyle p_1^*f}\Big\downarrow & \diagdown\!\diagdown & \Big\downarrow{\scriptstyle p_2^*f} \\
p_1^*\mathcal{X}_2' & \xrightarrow{\ \phi_2\ } & p_2^*\mathcal{X}_{2'}'
\end{array}
$$

such that each square in the following diagram of 2-morphisms commutes:

Here, the 2-morphisms in each square are the canonical ones (induced by α).

PROOF OF PROPOSITION 1.6. This follows from [Gir71, Chapitre II, §2.1.5] (and is also a special case of Theorem 1.3). □

5. Example

Let k be a field and let $k \subset k'$ be a degree two field extension; one may think of $\mathbb{R} \subset \mathbb{C}$ or $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ for a prime power q. Let $\sigma \in \mathrm{Gal}(k'/k)$ be the generator of $\mathrm{Gal}(k'/k)$. Let $\mathcal{X}'$ be a stack over k' equipped with a 1-isomorphism

$$\sigma \colon \mathcal{X}' \to \mathcal{X}'$$

of stacks over k, and a 2-isomorphism $F \colon \sigma^2 \implies \mathrm{id}_{\mathcal{X}'}$ between σ^2 and the identity functor, such that σ commutes with the functor $(\mathsf{Sch}/k') \to (\mathsf{Sch}/k')$ defined as $T \mapsto {}^\sigma T = T \times_{k',\sigma} k'$, and such that for each $x \in \mathcal{X}'(T)$, $T \in (\mathsf{Sch}/k')$, the isomorphism $F(x) \colon \sigma^2(x) \to x$ lies over the canonical isomorphism of k-schemes ${}^\sigma({}^\sigma T) \to T$. One obtains the descended stack $\mathcal{X}$ over k by defining, for $T \in (\mathsf{Sch}/k)$, $\mathcal{X}(T)$ as the groupoid of pairs (x, φ) with $x \in \mathcal{X}'(T_{k'})$ and $\varphi \colon x \to \sigma(x)$ an isomorphism such that the composition

$$x \xrightarrow{\ \varphi\ } \sigma(x) \xrightarrow{\ {}^\sigma\varphi\ } \sigma^2(x) \xrightarrow{\ F\ } x$$

is the identity. There is a natural isomorphism $\mathcal{X} \times_k k' \cong \mathcal{X}'$ of stacks over k'.

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