

Introduction to Bayesian statistics with R

1. Motivations

Olivier Gimenez

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This workshop

Objectives

- Try and demystify Bayesian statistics, and what we call MCMC.
- Make the difference between Bayesian and Frequentist analyses.
- Understand the Methods section of ecological papers doing Bayesian stuff.
- Run Bayesian analyses, safely hopefully.

What's on our plate?

- Section 1 - Motivation and Bayes theorem.
- Section 2 - Markov chain Monte Carlo algorithms (MCMC).
- Section 3 - Introduction to NIMBLE and brms.
- Section 4 - Priors.
- Section 5 - Case studies and GLMMs.
- Section 6 - Conclusions.

- Material from previous workshops
(<https://oliviergimenez.github.io/bayesian-stats-with-R/>), a book in progress
(<https://oliviergimenez.github.io/banana-book/>) and a paper on ten quick tips to get you started with Bayesian statistics
(<https://hal.science/CEFE/hal-04731240v2>).
- All material prepared with R, with R Markdown to ensure reproducibility.
- Slides, data and R codes on GitHub at
<https://github.com/oliviergimenez/bayes-workshop>

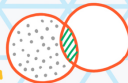
What is Bayesian inference?

THE AMAZING Thomas Bayes

$$P(A/B) = \frac{P(B/A) P(A)}{P(B)}$$



$P(A)$



$P(B)$



A reminder on conditional probabilities

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Screening for vampirism

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- The chance of a negative test given you are mortal is $\Pr(-|\text{mortal}) = 0.95$ (**specificity**).

What is the question?

- From the perspective of the test: Given a person is a vampire, what is the probability that the test is positive? $\Pr(+|\text{vampire}) = 0.90$.

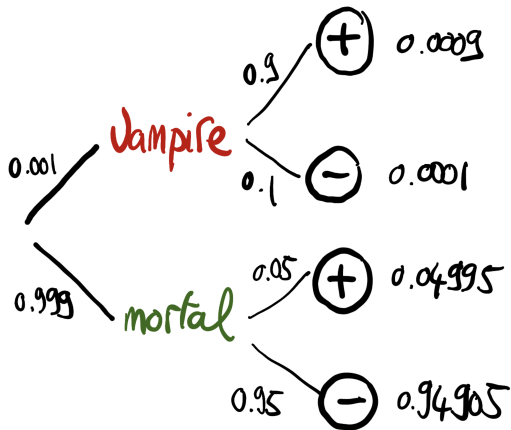
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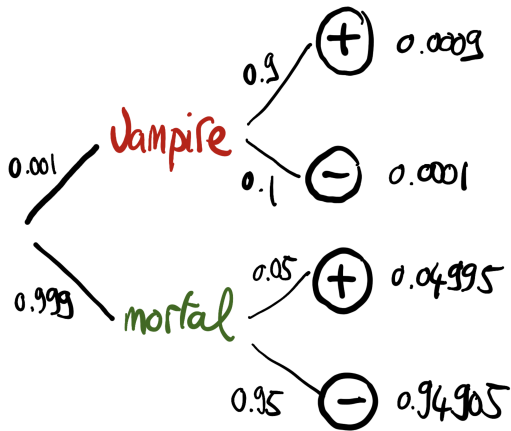
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- From the perspective of a person: Given that the test is positive, what is the probability that this person is a vampire? $\Pr(\text{vampire}|+) = ?$
- Assume that vampires are rare, and represent only 0.1% of the population. This means that $\Pr(\text{vampire}) = 0.001$.

What is the answer? Bayes' theorem to the rescue!



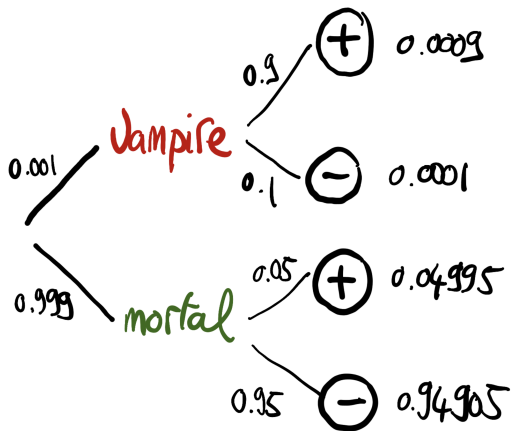
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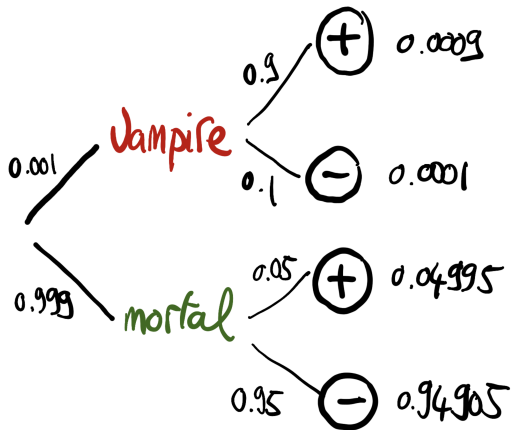
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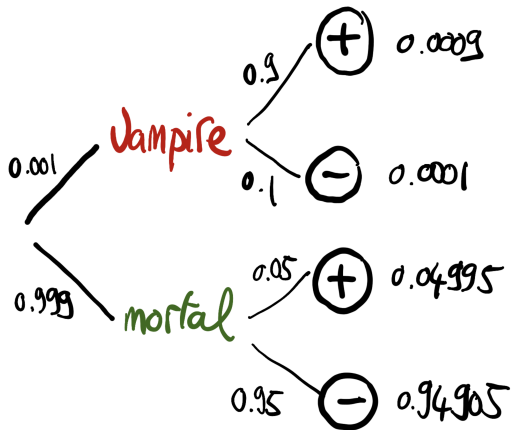
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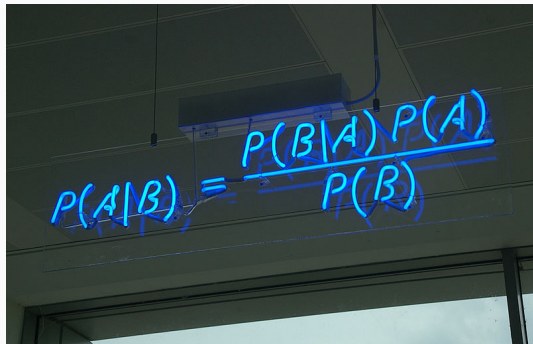


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$$\Pr(\text{vampire}|+) = \frac{\Pr(+|\text{vampire}) \Pr(\text{vampire})}{\Pr(+)}$$

Bayes' theorem

- A theorem about conditional probabilities.
- $\Pr(B | A) = \frac{\Pr(A | B) \Pr(B)}{\Pr(A)}$



A photograph of a blue neon sign mounted on a wall. The sign displays the formula for Bayes' theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. The text is written in a stylized, glowing blue font. The sign is slightly tilted and has some visible wiring and mounting hardware.

Bayes' theorem

- Easy to mess up with letters. Might be easier to remember when written like this:

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- For regression models, the “hypothesis” is a parameter (intercept, slopes or error terms).
- Bayes theorem tells you the probability of the hypothesis given the data.

What is doing science after all?

How plausible is some hypothesis given the data?

$$\Pr(\text{hypothesis} \mid \text{data}) = \frac{\Pr(\text{data} \mid \text{hypothesis}) \Pr(\text{hypothesis})}{\Pr(\text{data})}$$

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- Due to practical problems of implementing the Bayesian approach, and some wars of male statisticians's egos, little advance was made for over two centuries.
- Recent advances in computational power coupled with the development of new methodology have led to a great increase in the application of Bayesian methods within the last two decades.

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- The frequentist approach (**maximum likelihood estimation** – MLE) assumes that the parameters are fixed, but have unknown values to be estimated.
- Classical estimates generally provide a point estimate of the parameter of interest.
- The Bayesian approach assumes that the parameters are not fixed but have some fixed unknown distribution - a distribution for the parameter.

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- And then updates these beliefs on the basis of observed data.
- This updating procedure is based upon the Bayes' Theorem:

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- Translates into:

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- **Prior distribution**: Represents what you know before seeing the data. The source of much discussion about the Bayesian approach.
- $\text{Pr}(\text{data}) = \int \text{Pr}(\text{data} \mid \theta) \text{Pr}(\theta) d\theta$: Possibly high-dimensional integral, difficult if not impossible to calculate. This is one of the reasons why we need simulation (MCMC) methods.

Guiding example

Survival example

- 120 deer were radio-tracked over winter.
- 61 close to a plant, 59 far from any human activity.
- How to estimate winter survival?

	Released	Alive	Dead	Other
treatment	61	19	38	4
control	59	21	38	0

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- How would the classical statistician justify this estimate?

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- K the number of alive individuals at the end of the winter, so that $P(K = k) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$.
- The classical approach is to maximise the corresponding likelihood with respect to θ to obtain the entirely plausible MLE:

$$\hat{\theta} = k/n = 19/57$$

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- What is the Beta distribution?

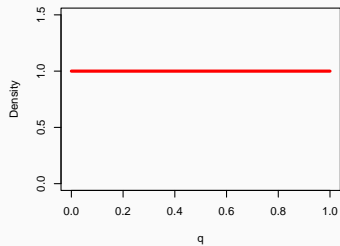
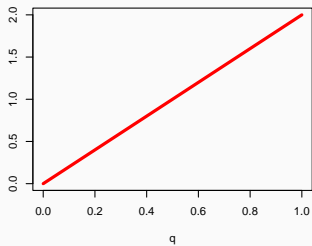
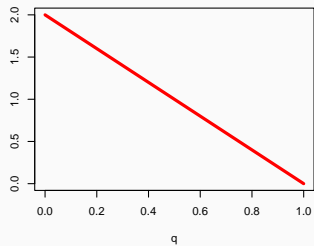
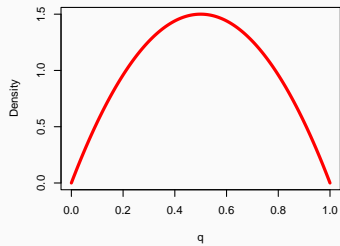
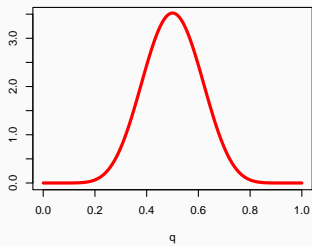
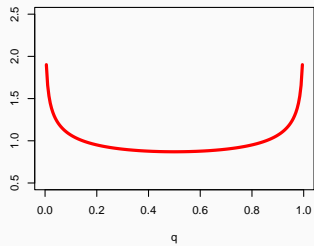
What is the Beta distribution?

$$q(\theta \mid a, b) = \frac{1}{\text{Beta}(a, b)} \theta^{a-1} (1 - \theta)^{b-1}$$

with $\text{Beta}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(n) = (n-1)!$

What is the Beta distribution?

- The beta distribution is continuous between 0 and 1, and extends the uniform distribution to situations where not all outcomes are equally likely.
- It has two parameters a and b that control its shape.
- Note that for $a = b = 1$, we get the uniform distribution between 0 and 1.
- The expectation (or mean) of a $\text{beta}(a,b)$ is $\frac{a}{a+b}$.

beta(1,1)**beta(2,1)****beta(1,2)****beta(2,2)****beta(10,10)****beta(0.8,0.8)**

The Bayesian approach

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- Then we have:

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- Take a Beta prior with a Binomial likelihood, you get a Beta posterior (conjugacy)

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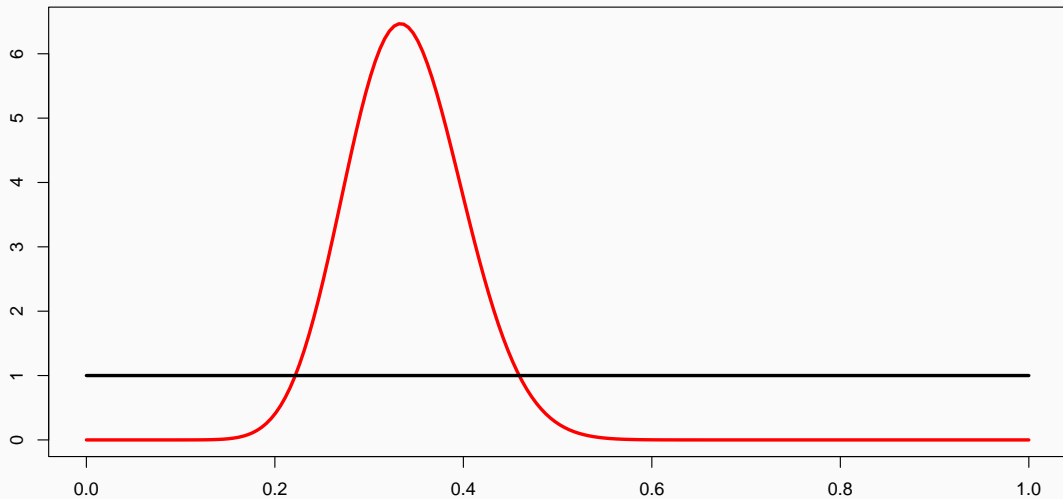
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- Note that in this specific situation, the posterior has an explicit expression, easy to manipulate.
- In particular, $E(\text{Beta}(a, b)) = \frac{a}{a + b} = 20/59$ to be compared with the MLE $19/57$.

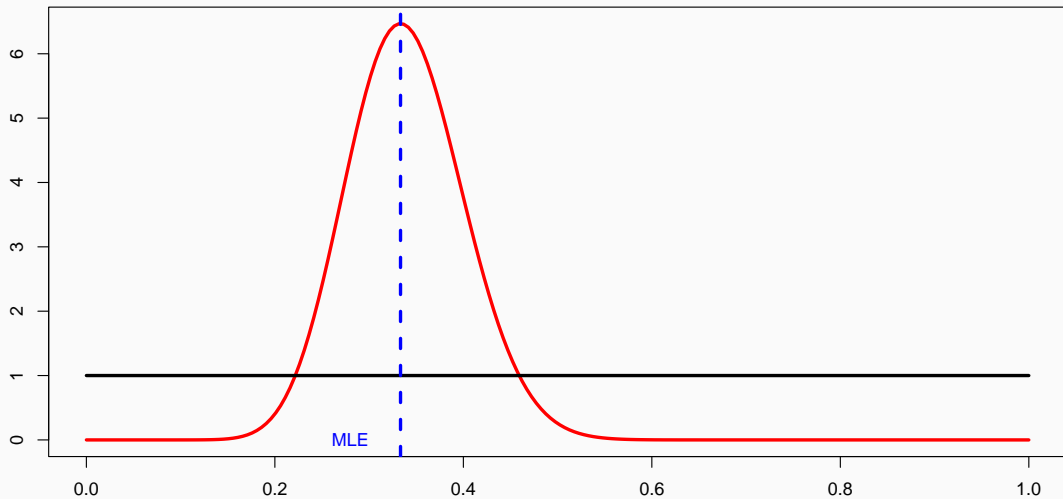
A general result

This is a general result, the Bayesian and frequentist estimates will always agree if there is sufficient data, so long as the likelihood is not explicitly ruled out by the prior.

Prior $Beta(1, 1)$ and posterior survival $Beta(20, 39)$



Prior $Beta(1, 1)$ and posterior survival $Beta(20, 39)$



Summary

- With the Bayes' theorem, you update your beliefs (prior) with new data (likelihood) to get posterior beliefs (posterior): $\text{posterior} \propto \text{likelihood} \times \text{prior}$.
- In some simple situations, it is possible to explicitly calculate the posterior distribution of model parameters.
- In general though, when applying the Bayes' theorem, you get high-dimensional integral, difficult if not impossible to calculate. This is one of the reasons why we need simulation (MCMC) methods.

Further reading

Marc Kéry
Kenneth F. Kellner



APPLIED STATISTICAL MODELLING FOR ECOLOGISTS

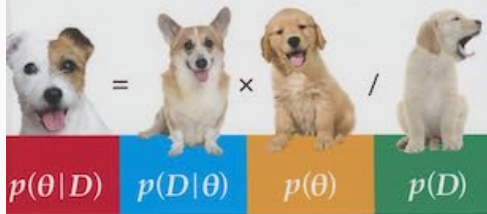
A practical guide to Bayesian and likelihood
inference using R, JAGS, NIMBLE, Stan and TMB



Second Edition

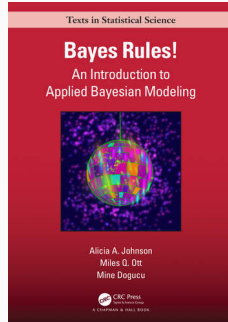
Doing Bayesian Data Analysis

A Tutorial with R, JAGS, and Stan

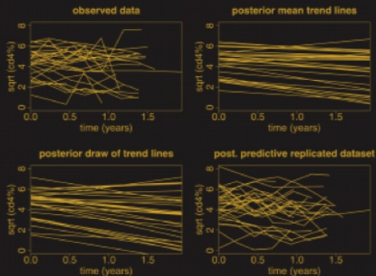


John K. Kruschke





<https://www.bayesrulesbook.com/>



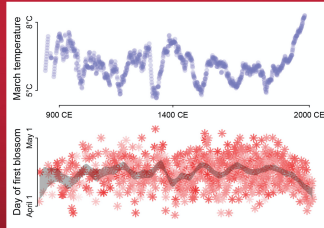
Data Analysis Using Regression and Multilevel/Hierarchical Models

ANDREW GELMAN
JENNIFER HILL

Texts in Statistical Science

Statistical Rethinking

A Bayesian Course
with Examples in R and Stan
SECOND EDITION



Richard McElreath

 **CRC Press**
Taylor & Francis Group
A CHAPMAN & HALL BOOK