MultiCore Fiber Interferometric Imaging

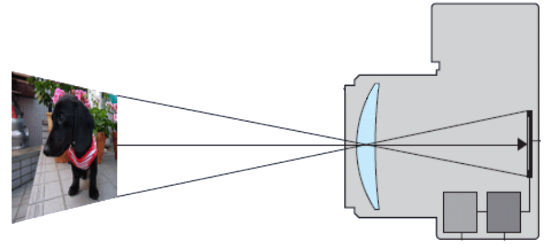
PoMM, Lille.

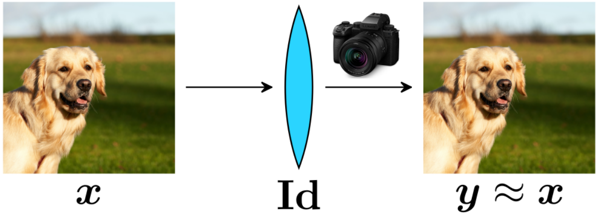
O. Leblanc.

15th May, 2024

Louvain-la-Neuve

Context and notations

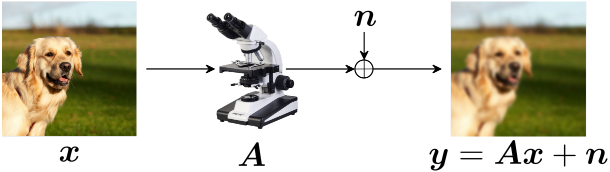



$\boldsymbol{x}$
$\mathbf{Id}$
$\boldsymbol{y} \approx \boldsymbol{x}$

Context - Computational imaging

$\boldsymbol{x} \in \mathbb{R}^N$: object to be imaged.
$\boldsymbol{y} \in \mathbb{R}^M$: measurements.
$\boldsymbol{A} \in \mathbb{R}^{M \times N}$: forward operator containing the physics (generally linear).
$\rightarrow$ $\boldsymbol{A}$ can be partial, ill-conditioned,...

Forward problem: generate $\boldsymbol{y}$ from $\boldsymbol{x}$.



Inverse problem: recover $\boldsymbol{x}$ from the observations $\boldsymbol{y}$.

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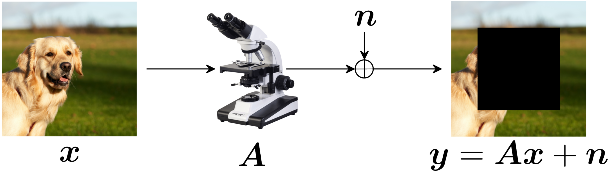
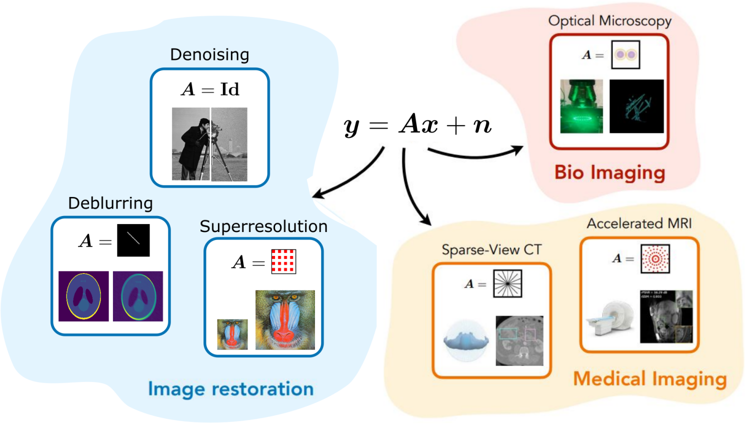


Inverse problem: recover $\boldsymbol{x}$ from the observations $\boldsymbol{y}$.

Many computational imaging problems can be formulated as inverse problems



[1]

[1] inspired from Yu Sun

Context - MultiCore Fiber Lensless Imaging (MCF-LI)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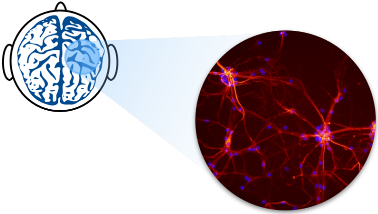
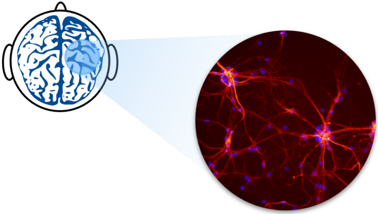


Motivation:

- Deep brain imaging.
- Limited invasiveness.



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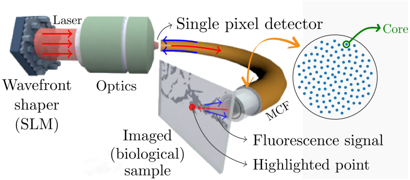
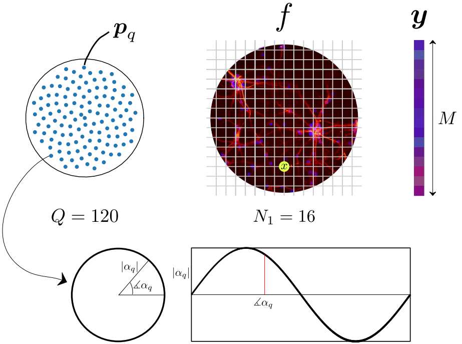


Challenges:

- Fluorescence imaging.
- Single-pixel $\rightarrow$ limited view.
- Small intensity.

Notations

$\boldsymbol{p}_q$	Position of core q
Q	Number of cores
$\boldsymbol{\alpha} \in \mathbb{C}^Q$	Cores complex amplitudes **(tunable!)**
$\boldsymbol{x} \in \mathbb{R}^2$	2-D object space
$f(\boldsymbol{x})$	Object to be imaged
$N = N_1 \times N_1$	Image resolution
M	Number of measurements
$\boldsymbol{y} \in \mathbb{R}_+^M$	Measurement vector



Outline

1. Context and notations
2. Existing models: Raster Scanning (RS) and Speckle Illuminations (SI)
3. MCF Interferometric Imaging

Existing models: Raster Scanning (RS) and Speckle Illuminations (SI)

Outline

1. Context and notations

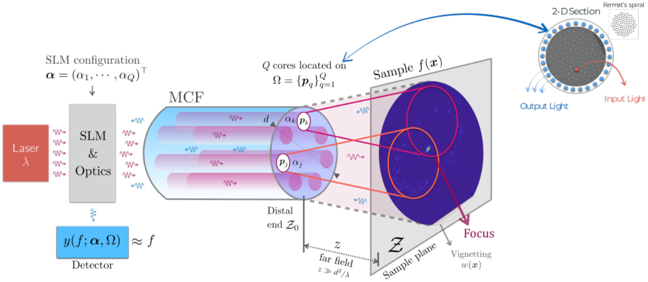
2. Existing models: Raster Scanning (RS) and Speckle Illuminations (SI)

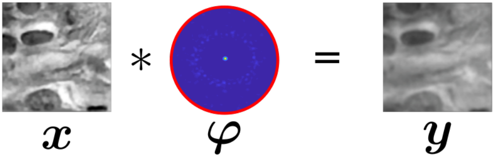
2.1 Raster scanning

2.2 Speckle illumination

3. MCF Interferometric Imaging

Raster scanning (RS) mode

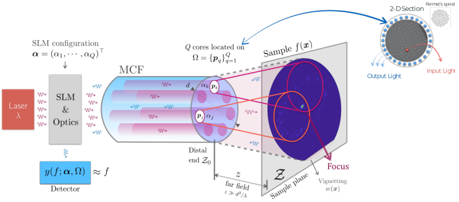


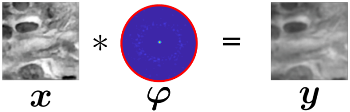


Forward model: $\boldsymbol{y} = \boldsymbol{\varphi} * \boldsymbol{x}$.
→**Deconvolution problem!**

Drawbacks:

- $M = N$.
- φ spatially varying.





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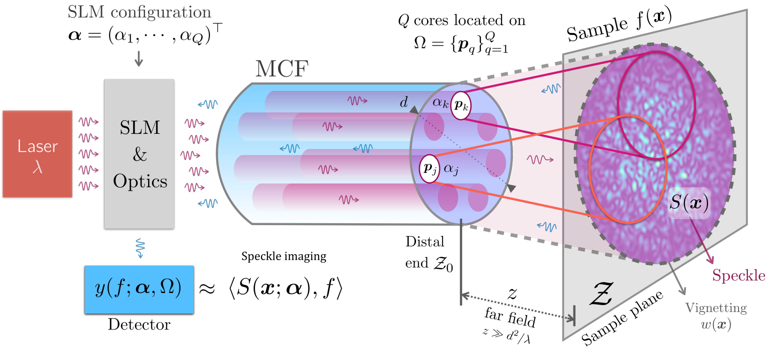
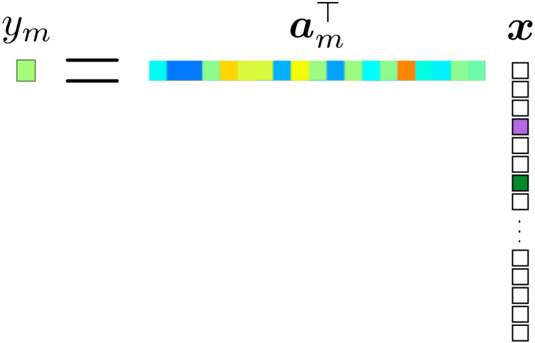
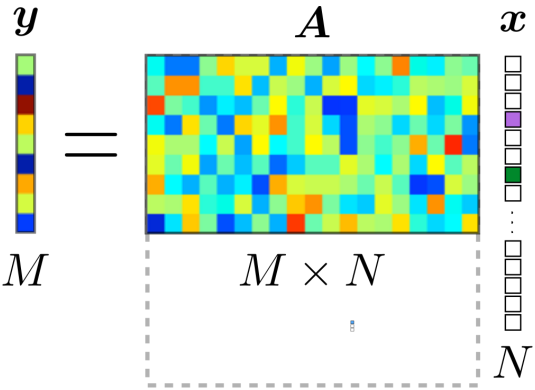
Speckle illumination

Steph. Guérit





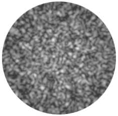
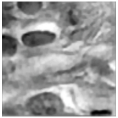
$$y_m = \boldsymbol{a}_m^\top \boldsymbol{x} = \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle$$



Condition: a_{mn} are *iid* random variables (e.g., $a_{mn} \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$).

The m-th single pixel observation is the integration of the m-th speckle pattern $\boldsymbol{s}_m$ illuminating the sample $\boldsymbol{f}$:

$$y_m = \boldsymbol{s}_m^\top \boldsymbol{f} + n_m.$$

$y_m = \Big\langle$  ,  $\Big\rangle$

Classical compressive sensing model (Guerit et al.)

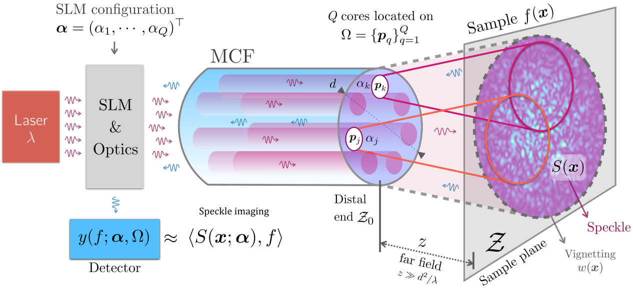
$$\boldsymbol{y} = \boldsymbol{S}\boldsymbol{f} + \boldsymbol{n}$$

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{f}\|_2^2 + \mathcal{R}(\boldsymbol{f})$$

Compressive imaging $\Rightarrow$ number of observations $M \ll N$ image resolution.

Assumption: all coefficients in $\boldsymbol{S}$ are i.i.d. random variables.

A closer look to sensing model

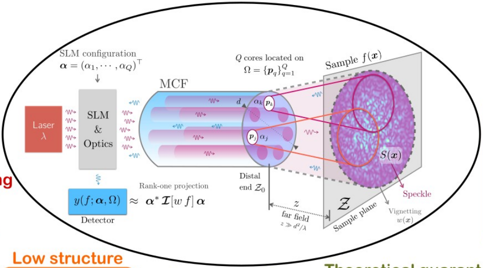


Generating a N pixels speckle from only Q random coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_Q)$...

Why would this behave as a random gaussian waveform?

MCF Interferometric Imaging

Overview



Mathematical modeling

?

Low structure

?

Inverse problem & optimisation

?

?

Theoretical guarantees

?

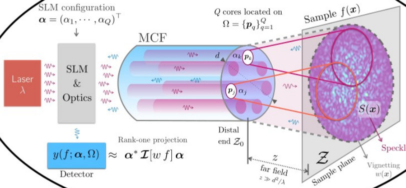
Proof of concept

?

Mathematical modeling

Mathematical modeling

$$y_m = g(f; \boldsymbol{\alpha}_m)?$$



Proof of concept

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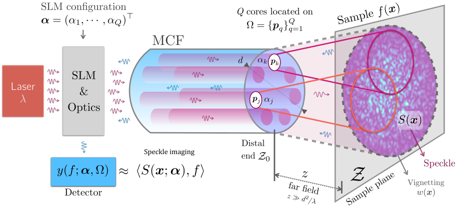
Interferometric imaging

Classical compressive sensing model

$$y_{\boldsymbol{\alpha}} = \int_{\mathbb{R}^2} S_{\boldsymbol{\alpha}}(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \langle S_{\boldsymbol{\alpha}}, f \rangle$$



Speckles are interferences (Under far-field approximation)

$$E_q(\boldsymbol{x}) = \alpha_q \sqrt{w}(\boldsymbol{x}) e^{\frac{2\pi \mathrm{i}}{\lambda z} \boldsymbol{p}_q^\top \boldsymbol{x}}, \quad w(\boldsymbol{x}) = \frac{|\hat{E}_0(\frac{\boldsymbol{x}}{\lambda z})|^2}{(\lambda z)^2} \qquad \text{(Rayleigh-Sommerfeld)}$$

$$S(\boldsymbol{x}; \boldsymbol{\alpha}) \propto w(\boldsymbol{x}) \left| \sum_{q=1}^{Q} \alpha_q e^{\frac{2\pi \mathrm{i}}{\lambda z} \boldsymbol{p}_q^\top \boldsymbol{x}} \right|^2 = \underbrace{w(\boldsymbol{x})}_{\text{Field of view}} \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* e^{\frac{2\pi \mathrm{i}}{\lambda z} (\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}}$$

Hence (noiseless)

$$\begin{aligned} y_{\boldsymbol{\alpha}} &= \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z} (\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right] \\ &= \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[wf] \boldsymbol{\alpha} = \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^*, \boldsymbol{\mathcal{I}}_\Omega[wf] \rangle \end{aligned}$$

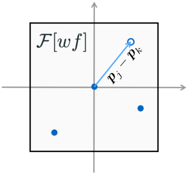
Interferometric imaging

Writing $f^\circ := wf$ we have

$$y_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[f^\circ]\boldsymbol{\alpha}$$

with the interferometric matrix $\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \in \mathbb{C}^{Q\times Q}$ s.t.

$$(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi\mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top} f^\circ(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

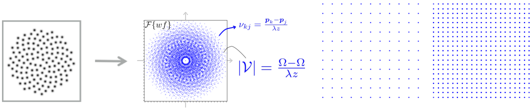


Fourier plane

Observation: denser Fourier sampling if

$$|\{\boldsymbol{p}_j - \boldsymbol{p}_k : \forall 1 \le j,k \le Q\}| \simeq Q^2$$

- Lattices are bad core arrangements
- Fermat's spiral is not bad



MCF-LI Forward Model

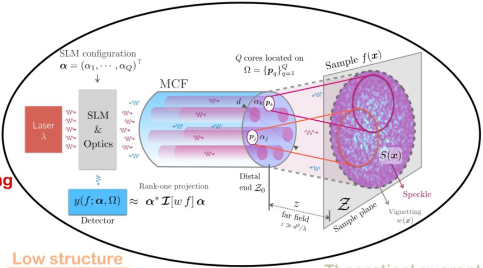
$$
\begin{aligned}
\boldsymbol{y} &= \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[f^\circ]\boldsymbol{\alpha}_m\}_{m=1}^M \\
&= \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[f^\circ]
\end{aligned}
$$

Two-component sensing!

$\boldsymbol{\mathcal{A}}$: Symmetric rank-one projections of a matrix.

$\boldsymbol{\mathcal{I}}_\Omega$: Partial Fourier sensing with replacement.

Mathematical model



Mathematical modeling

$$\boldsymbol{y} = \{\boldsymbol{\alpha}_m^* \mathcal{I}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^{M} = \boldsymbol{\mathcal{A}} \circ \mathcal{I}_\Omega[wf]$$

Proof of concept

?

Low structure

?

Inverse problem & optimisation

?

?

Theoretical guarantees

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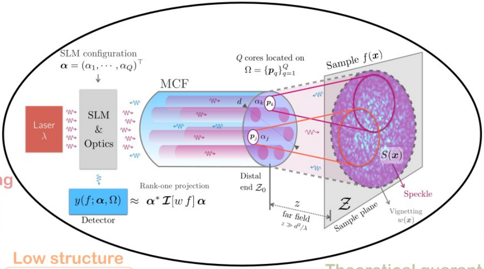

Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^{M} = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[wf]$$

Low structure

$\boldsymbol{\mathcal{I}}_\Omega$

- hermitian
- constant diag
- low-rank?
- low-dim?

Inverse problem & optimisation

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Single-step strategy

Basis Pursuit DeNoising in ℓ_1-norm (BPDN$_{\ell_1}$)
How to get a $\hat{f}$ from $\boldsymbol{y} = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[f^\circ]$?

$$\text{Discretise } f^\circ \to \boldsymbol{f} \in \mathbb{R}^{N_1 \times N_1}$$

The interferometric matrix becomes

$$\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \approx \mathcal{T} \boldsymbol{F} \boldsymbol{f}$$

Let us rewrite

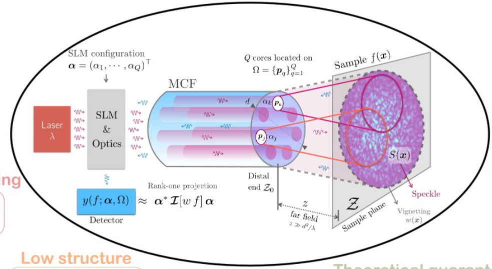
$$\boldsymbol{\mathcal{B}} := \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega \;\text{ s.t. }\; \boldsymbol{y} = \boldsymbol{\mathcal{B}} \boldsymbol{f}$$

Solve

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{u}} \; \|\boldsymbol{u}\|_1 \quad \text{s.t. } \|\boldsymbol{y} - \boldsymbol{\mathcal{B}} \boldsymbol{u}\|_1 \leqslant \epsilon \tag{BPDN$_{\ell_1}$}$$

Here we leverage low structure directly on f°!

Inverse problem & optimisation



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Low structure

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Inverse problem & optimisation

Two-step
1. $\boldsymbol{y} \to \boldsymbol{\mathcal{I}}_\Omega[f^\circ]$
2. $\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \to f^\circ$

Single-step

$\hat{f} \in \arg\min_u \|u\|_1 \;\; \text{s.t.}\; \|\boldsymbol{y} - \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[u]\|_1 \leqslant \epsilon$

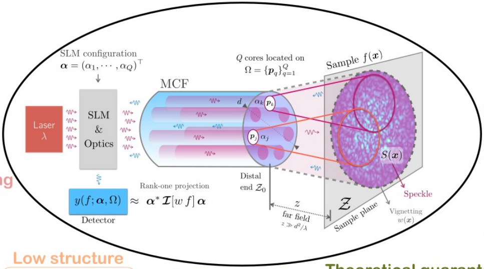
Theoretical guarantees

?

Proof of concept

?

Recovery guarantees



Mathematical modeling

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Single-step

$\hat{f} \in \arg\min_u \|u\|_1$ s.t. $\|y - \mathcal{A} \circ \mathcal{I}_\Omega[u]\|_1 \leqslant \epsilon$

Theoretical guarantees

?

Proof of concept

?

Objective

Show

$$\hat{\boldsymbol{f}} \in \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \epsilon \tag{BPDN_{ℓ_1}}$$

satisfies

$$\left\|\boldsymbol{f} - \hat{\boldsymbol{f}}\right\|_2 \leqslant C \frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}.$$

Prove the RIP-ℓ_2/ℓ_1

Proposition

If

$$M \geqslant CK \log\left(\frac{12eN}{K}\right), \quad Q(Q-1) \geqslant 4K\,plog(N, K, \delta),$$

the operator $\boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega$ *satisfies*

$$\tilde{A}\|\boldsymbol{u}\| \leqslant \tfrac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \tilde{B}\|\boldsymbol{u}\|$$

$\forall \boldsymbol{u} \in \Sigma_K$ *with probability exceeding* $1 - C\exp(-cM)$.

Objective

Show

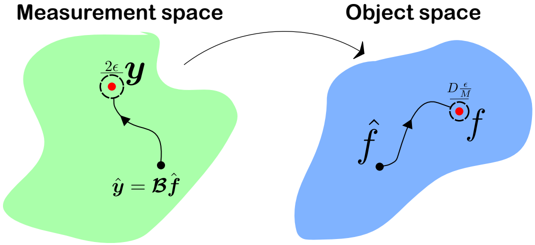
$$\hat{\boldsymbol{f}} \in \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \epsilon \tag{BPDN$_{\ell_1}$}$$

satisfies

$$\left\|\boldsymbol{f} - \hat{\boldsymbol{f}}\right\|_2 \leqslant C \frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D \frac{\epsilon}{m}.$$

Prove the RIP-ℓ_2/ℓ_1

$$\tilde{A}\|\boldsymbol{u}\| \leqslant \tfrac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \tilde{B}\|\boldsymbol{u}\|$$



Recovery Guarantees - Combine RIPs

Objective

Show

$$\hat{\boldsymbol{f}} \in \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \epsilon \qquad (\mathrm{BPDN}_{\ell_1})$$

satisfies

$$\left\|\boldsymbol{f} - \hat{\boldsymbol{f}}\right\|_2 \leqslant C\frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}.$$

- Assume RIP-ℓ_2/ℓ_2 for visibility sampling

$$(1-\delta)\|\boldsymbol{u}\|^2 \leqslant \|\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}]\|_F^2 \leqslant (1+\delta)\|\boldsymbol{u}\|^2, \quad \forall \boldsymbol{u} \in \Sigma_K.$$

- Prove ℓ_2/ℓ_1-concentration for SROPs

$$A\|\boldsymbol{\mathcal{I}}_\Omega\|_F \leqslant \tfrac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega)\|_1 \leqslant B\|\boldsymbol{\mathcal{I}}_\Omega\|_F$$

  with very high probability.
- Combine the two and use union bound to get RIP-ℓ_2/ℓ_1

$$\tilde{A}\|\boldsymbol{u}\| \leqslant \tfrac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \tilde{B}\|\boldsymbol{u}\|$$

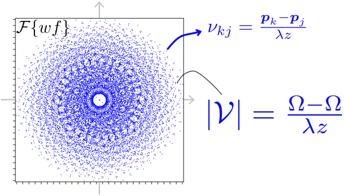
Assumption (inspired by partial Fourier sensing in CS[2])

$$(1-\delta)\|\boldsymbol{u}\|^2 \leqslant \|\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}]\|_F^2 \leqslant (1+\delta)\|\boldsymbol{u}\|^2$$

with the condition

$$|\mathcal{V}_0| = Q(Q-1) \geqslant \delta^{-2} K \operatorname{plog}(N, K, \delta)$$

We need randomness in the sampling
But deterministic visibility set...
Hopefully, we have good reasons think this assumption is not too bad!



[2] E. J. Candès, J. K. Romberg, and T. Tao, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,” IEEE Transactions on Information Theory, vol. 52, no. 2, pp. 489–509, 2006.

Recovery guarantees - Proving the RIP-ℓ_2/ℓ_1

$$A\|\boldsymbol{\mathcal{I}}_\Omega\|_F \leqslant \frac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega)\|_1 \leqslant B\|\boldsymbol{\mathcal{I}}_\Omega\|_F$$

If $M > CK \log\left(\frac{12eN}{K}\right)$ and $\boldsymbol{\mathcal{A}}$ satisfies

$$\mathbb{P}\left[\left|\frac{1}{\sqrt{M}}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \left\|\boldsymbol{\mathcal{I}}\right\|_F\right| > t\left\|\boldsymbol{\mathcal{I}}\right\|_F\right] \leq Ce^{-cM} \tag{1}$$

$\Rightarrow \boldsymbol{\mathcal{A}}$ satisfies RIP-ℓ_2/ℓ_1 w.h.p.
Prove (1) in two-steps

If $M > CK \log\left(\frac{12eN}{K}\right)$ and $\boldsymbol{\mathcal{A}}$ satisfies

$$\mathbb{P}\left[\left|\frac{1}{\sqrt{M}}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \left\|\boldsymbol{\mathcal{I}}\right\|_F\right| > t\left\|\boldsymbol{\mathcal{I}}\right\|_F\right] \leq Ce^{-cM} \tag{1}$$

$\Rightarrow$ $\boldsymbol{\mathcal{A}}$ satisfies RIP-ℓ_2/ℓ_1 w.h.p.

Prove (1) in two-steps

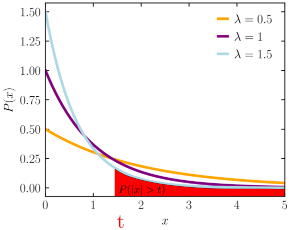
1. $\mathbb{P}\left[\left|\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1\right| > t\right] \leq 2\exp\left(-cM\min\left(\frac{t^2}{4\kappa^2}, \frac{t}{2\kappa}\right)\right)$

$$\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 = \sum_{m=1}^{M}\left|\boldsymbol{\alpha}_m^*\boldsymbol{\mathcal{I}}\boldsymbol{\alpha}_m\right| = \sum_{m=1}^{M}|\xi_m|$$

$$\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 = \sum_{m=1}^{M}|\xi_m| - \mathbb{E}|\xi_m| = \sum_{m=1}^{M}|\tilde{\xi}_m|$$

Recall $\boldsymbol{\alpha}$ is a r.v. $\Rightarrow$ $\tilde{\xi}$ is a r.v. too.
$\tilde{\xi}_m$ **is subexponential!** Property of a subexponential r.v.

$$\mathbb{P}\left[\left|\tilde{\xi}\right| > t\right] \leq 2\exp\left(-cM\min\left(\frac{t^2}{4\kappa^2}, \frac{t}{2\kappa}\right)\right)$$



Recovery guarantees - Proving the RIP-ℓ_2/ℓ_1

If $M > CK \log\left(\frac{12eN}{K}\right)$ and $\boldsymbol{\mathcal{A}}$ satisfies

$$\mathbb{P}\left[\left|\frac{1}{\sqrt{M}}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \left\|\boldsymbol{\mathcal{I}}\right\|_F\right| > t\left\|\boldsymbol{\mathcal{I}}\right\|_F\right] \leq Ce^{-cM} \tag{1}$$

$\Rightarrow \boldsymbol{\mathcal{A}}$ satisfies RIP-ℓ_2/ℓ_1 w.h.p.
Prove (1) in two-steps

1. $\mathbb{P}\left[\left|\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 - \mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1\right| > t\right] \leq 2\exp\left(-cM\min\left(\frac{t^2}{4\kappa^2}, \frac{t}{2\kappa}\right)\right)$

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2. $\boldsymbol{\alpha}_m$ is i.i.d. as the sub-Gaussian r.v.

$$c_\alpha \left\|\boldsymbol{\mathcal{I}}\right\|_F \leqslant \frac{1}{\sqrt{M}}\mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 \leqslant \left\|\boldsymbol{\mathcal{I}}\right\|_F .$$

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$$c_\alpha \left\|\boldsymbol{\mathcal{I}}\right\|_F \leqslant \frac{1}{\sqrt{M}}\mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 \leqslant \left\|\boldsymbol{\mathcal{I}}\right\|_F.$$

$$(1-t)\mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 \leqslant \frac{1}{\sqrt{M}}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 \leqslant (1+t)\mathbb{E}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1$$

$$(1-t)c_\alpha\left\|\boldsymbol{\mathcal{I}}\right\|_F \leqslant \frac{1}{\sqrt{M}}\left\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\right\|_1 \leqslant (1+t)\left\|\boldsymbol{\mathcal{I}}\right\|_F$$

Objective

Show

$$\hat{\boldsymbol{f}} \in \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \epsilon \tag{BPDN$_{\ell_1}$}$$

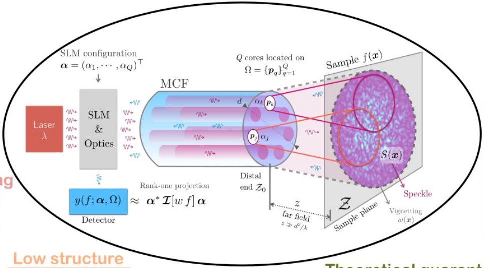
satisfies

$$\left\|\boldsymbol{f} - \hat{\boldsymbol{f}}\right\|_2 \leqslant C\frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}.$$

Recovery guarantees

Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \mathcal{I}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^M = \mathcal{A}\circ\mathcal{I}_\Omega[wf]$$



Proof of concept

?

Low structure

$\mathcal{I}_\Omega$

- hermitian
- constant diag
- low-rank?
- low-dim?

Inverse problem & optimisation

Two-step

1. $\boldsymbol{y} \to \mathcal{I}_\Omega[f^\circ]$
2. $\mathcal{I}_\Omega[f^\circ] \to f^\circ$

Single-step

$\hat f \in \arg\min_u \|u\|_1 \;\text{ s.t. } \|\boldsymbol{y} - \mathcal{A}\circ\mathcal{I}_\Omega[u]\|_1 \leqslant \epsilon$

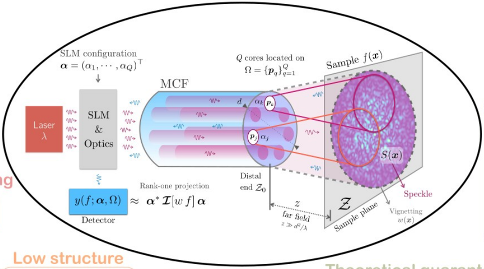
Theoretical guarantees

$$\left\|f - \hat f\right\|_2 \leqslant C\frac{\|f-f_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}$$

Proof of concept

Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \mathcal{I}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^M = \mathcal{A} \circ \mathcal{I}_\Omega[wf]$$



Proof of concept

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Single-step

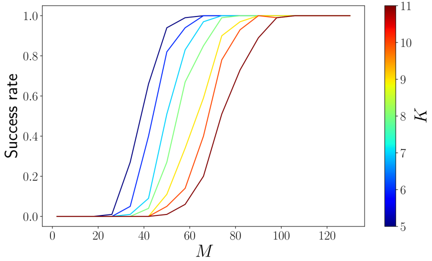
$\hat{f} \in \arg\min_u \|u\|_1 \;\; \text{s.t.}\; \|y - \mathcal{A} \circ \mathcal{I}_\Omega[u]\|_1 \leqslant \epsilon$

Theoretical guarantees

$$\left\| f - \hat{f} \right\|_2 \leqslant C \frac{\|f - f_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}$$

K = sparsity of f, $f \in \Sigma_K$.
$|\mathcal{V}|$ = cardinality of the visibility set in the Fourier space, grows with Q.
M = number of measurements.

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{u}} \ \|\boldsymbol{y} - \boldsymbol{\mathcal{B}}\boldsymbol{u}\|_2 \ \text{ s.t. } \ \|\boldsymbol{u}\|_1 \leqslant K \qquad \text{(LASSO)}$$

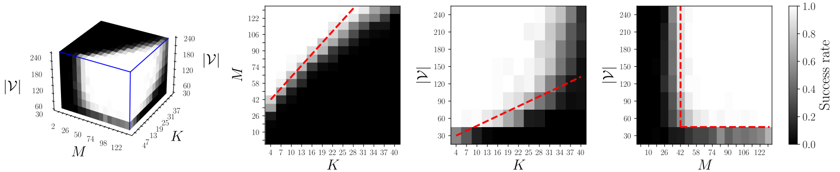


Success if SNR> 40dB.

Simulation results

K = sparsity of f, $f \in \Sigma_K$.
$|\mathcal{V}|$ = cardinality of the visibility set in the Fourier space, grows with Q.
M = number of measurements.

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{u}} \ \|\boldsymbol{y} - \boldsymbol{\mathcal{B}}\boldsymbol{u}\|_2 \ \text{ s.t. } \ \|\boldsymbol{u}\|_1 \leqslant K \qquad \text{(LASSO)}$$

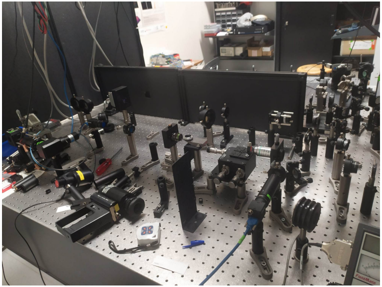


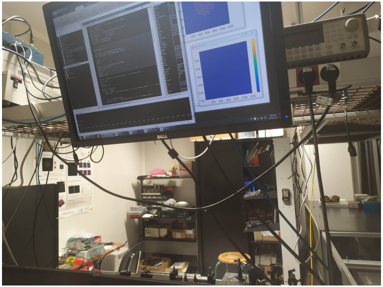
$$M > CK \log\left(\frac{12eN}{K}\right)$$

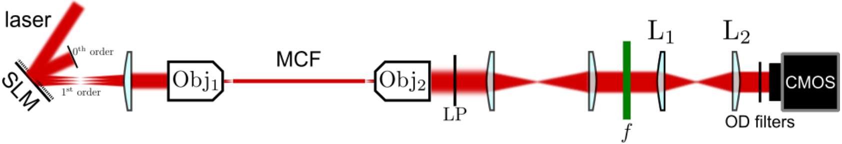
$$|\mathcal{V}_0| = Q(Q-1) \geqslant \delta^{-2} K \operatorname{plog}(N, K, \delta)$$

Success if SNR$>$ 40dB.

Experimental results

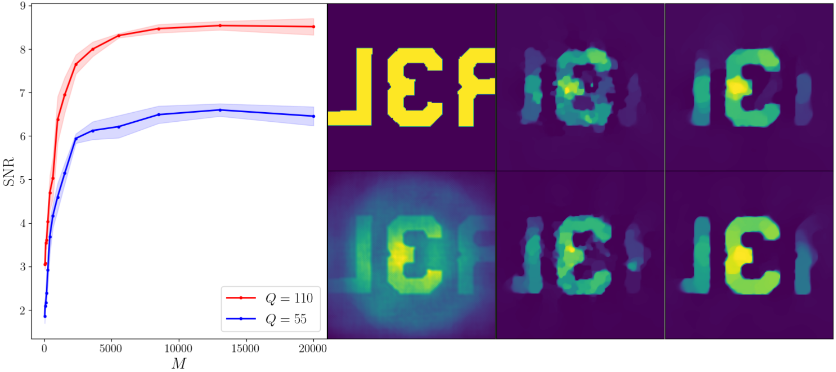


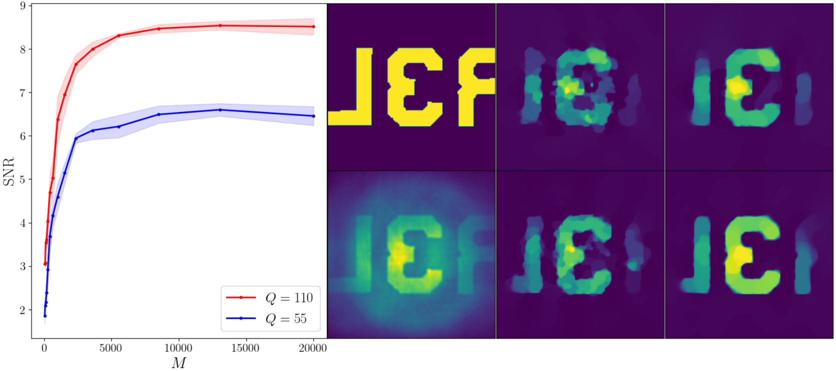




Laser: $\lambda = 1053$nm. L_1 : 75mm, L_2 : 100mm. MCF diameter: 113μm. core diameter: 3.2μm.

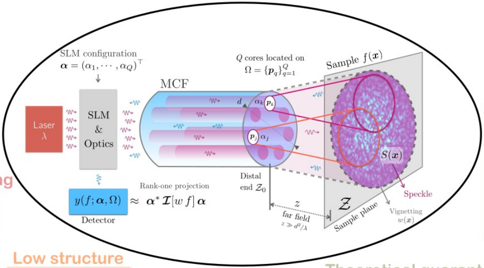
Experimental results



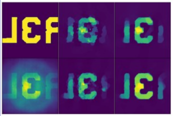


Why such a small SNR?

Proof of concept



Proof of concept



Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^M = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[wf]$$

Low structure

$\mathcal{I}_\Omega$
- hermitian
- constant diag
- low-rank?
- low-dim?

Inverse problem & optimisation

Two-step

1. $y \to \boldsymbol{\mathcal{I}}_\Omega[f^\circ]$
2. $\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \to f^\circ$

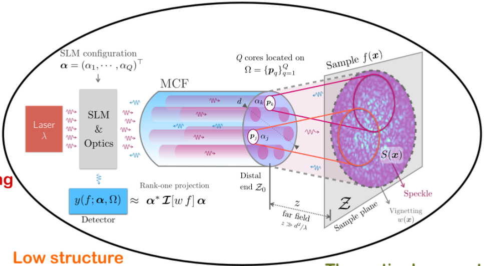
Single-step

$\hat{f} \in \arg\min_u \|u\|_1$ s.t. $\|y - \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[u]\|_1 \leqslant \epsilon$

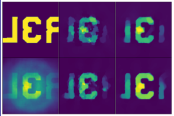
Theoretical guarantees

$$\left\| f - \hat{f} \right\|_2 \leqslant C \frac{\|f - f_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}$$

Proof of concept



Proof of concept



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$$\hat{f} \in \arg\min_u \|u\|_1 \quad \text{s.t.}\ \|\boldsymbol{y} - \mathcal{A} \circ \mathcal{I}_\Omega[u]\|_1 \leqslant \epsilon$$

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$$\left\|f - \hat{f}\right\|_2 \leqslant C\frac{\|f - f_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}$$

Conclusion

Until now

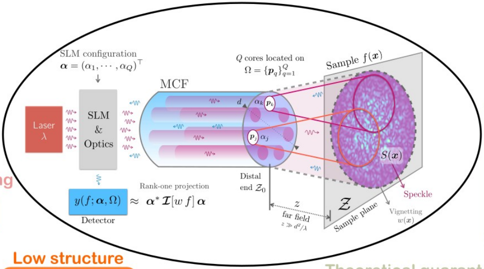
- New interferometric model for MCF-LI.
- Theoretical guarantees.
- Simulated and experimental results.

Perspectives

- Extension to 3D imaging.
- Experiments in endoscopic conditions.
- Acceleration of the sensing using mirrors.
- Manage twisting.

Thank you!

Low structure & complexity



Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[wf] \boldsymbol{\alpha}_m\}_{m=1}^{M} = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[wf]$$

Proof of concept

?

Low structure

?

Inverse problem & optimisation

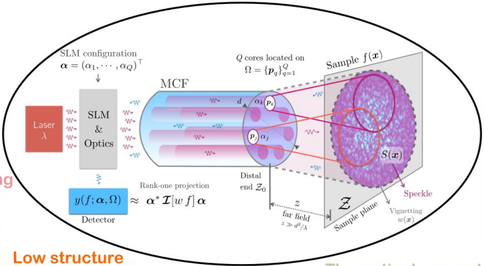
?

?

Theoretical guarantees

?

Low structure & complexity



Mathematical modeling

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Low structure

$\boldsymbol{\mathcal{I}}_\Omega$

Inverse problem & optimisation

?

?

Theoretical guarantees

?

Proof of concept

?

Properties of $\mathcal{I}_\Omega[f^\circ]$

$$(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi\mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} f^\circ(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

- Constant diagonal $(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{q,q} = \int_{\mathbb{R}^2} f^\circ(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \langle f^\circ \rangle, \quad \forall q \in [Q]$
- Hermitian $(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{j,k} = (\boldsymbol{\mathcal{I}}_\Omega[f^\circ])^*_{k,j}$

$\Rightarrow Q \times Q$ matrix with only $Q(Q-1)/2$ degrees of freedom.

Possible structural models on $\mathcal{I}_\Omega[f^\circ]$

Given $\mathcal{I}_\Omega[f^\circ]$

$$(\mathcal{I}_\Omega[f^\circ])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} f^\circ(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

- If $f^\circ(\boldsymbol{x}) = \sum_i^K \rho_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) \Rightarrow f^\circ \in \Sigma_K$, $\mathcal{I}_\Omega[f^\circ]$ is low-rank

$$(\mathcal{I}_\Omega[f^\circ])_{j,k} = \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} \sum_i^K \rho_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) \mathrm{d}\boldsymbol{x}$$

$$\mathcal{I}_\Omega[f^\circ] = \sum_i^K \rho_i \boldsymbol{u}(\boldsymbol{x}_i) \boldsymbol{u}^*(\boldsymbol{x}_i), \qquad \boldsymbol{u}(\boldsymbol{x})_j := e^{\mathrm{i}2\pi \boldsymbol{p}_j^\top \boldsymbol{x}}.$$

Possible structural models on $\boldsymbol{\mathcal{I}}_\Omega[f^\circ]$

Given $\boldsymbol{\mathcal{I}}_\Omega[f^\circ]$

$$(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} f^\circ(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

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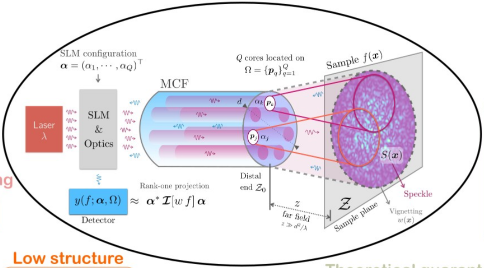
$$(\boldsymbol{\mathcal{I}}_\Omega[f^\circ])_{j,k} = \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top} \boldsymbol{x} \sum_i^K \rho_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) \mathrm{d}\boldsymbol{x}$$

$$\boldsymbol{\mathcal{I}}_\Omega[f^\circ] = \sum_i^K \rho_i \boldsymbol{u}(\boldsymbol{x}_i)\boldsymbol{u}^*(\boldsymbol{x}_i), \qquad \boldsymbol{u}(\boldsymbol{x})_j := e^{\mathrm{i}2\pi \boldsymbol{p}_j^\top \boldsymbol{x}}.$$

- f° sparse in $\boldsymbol{\Psi}$ (dictionary), $\quad f^\circ(\boldsymbol{x}) = \sum_i \rho_i \psi_i(\boldsymbol{x})$
  $\Rightarrow \boldsymbol{\mathcal{I}}_\Omega[f^\circ]$ belongs to a subspace.

$$\boldsymbol{\mathcal{I}}_\Omega[f^\circ] = \frac{1}{Q} \sum_i \rho_i \boldsymbol{\mathcal{I}}_\Omega[\psi_i]$$

Low structure & complexity



Mathematical modeling

$$y = \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^{M} = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[wf]$$

Low structure

$\mathcal{I}_\Omega$
- hermitian
- constant diag
- low-rank?
- low-dim?

Inverse problem & optimisation

?

?

Theoretical guarantees

?

Proof of concept

?

Two-step strategies

Strategy

1. SROPs : $\boldsymbol{y} \to \boldsymbol{\mathcal{I}}_\Omega[f^\circ]$
2. Partial Fourier sampling : $\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \to f^\circ$

1.

- Deterministic schemes

Two-step strategies

- Deterministic schemes

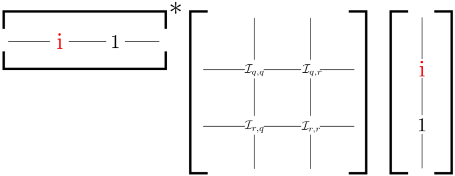
- **Deterministic schemes**

$$\underbrace{\begin{bmatrix} \cdots & 1 & \cdots & 1 & \cdots \end{bmatrix}^*}_{\boldsymbol{\alpha}^*} \underbrace{\begin{bmatrix} \ddots & \vdots & & \vdots & \\ \cdots & \mathcal{I}_{q,q} & \cdots & \mathcal{I}_{q,r} & \cdots \\ & \vdots & & \vdots & \\ \cdots & \mathcal{I}_{r,q} & \cdots & \mathcal{I}_{r,r} & \cdots \\ & \vdots & & \vdots & \ddots \end{bmatrix}}_{\mathcal{I}_\Omega[f^\circ]} \underbrace{\begin{bmatrix} \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}}_{\boldsymbol{\alpha}}$$

$$y_1 = \mathcal{I}_{q,q} + \mathcal{I}_{r,r} + 2\Re\{\mathcal{I}_{q,r}\}$$

Two-step strategies

- **Deterministic schemes**



$$\boldsymbol{\alpha}^* \quad \mathcal{I}_\Omega[f^\circ] \quad \boldsymbol{\alpha}$$

$$y_2 = \mathcal{I}_{q,q} + \mathcal{I}_{r,r} + 2\Im\{\mathcal{I}_{q,r}\}$$

Two-step strategies

- **Deterministic schemes**

$$y_1 = \mathcal{I}_{q,q} + \mathcal{I}_{r,r} + 2\Re\{\mathcal{I}_{q,r}\}$$

$$y_2 = \mathcal{I}_{q,q} + \mathcal{I}_{r,r} + 2\Im\{\mathcal{I}_{q,r}\}$$

$$\begin{aligned} y_1 + \mathrm{i} y_2 &= (1+\mathrm{i})(\mathcal{I}_{q,q} + \mathcal{I}_{r,r}) + 2\mathcal{I}_{q,r} \\ &= 2(1+\mathrm{i})\hat{f}(0) + 2\mathcal{I}_{q,r} \end{aligned}$$

$\Rightarrow M = Q(Q-1)+1$

Two-step strategies

- Deterministic schemes
- **Inverse problem**
  $f^\circ \in \Sigma_K \Rightarrow \boldsymbol{\mathcal{I}}_\Omega$ is rank-K.

Solve

$$\arg\min_{\boldsymbol{\mathcal{I}}} \|\boldsymbol{\mathcal{I}}\|_* \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}})\|_1 \leqslant \varepsilon$$

Upper bound on the reconstruction error

$$\|\hat{\boldsymbol{\mathcal{I}}} - \boldsymbol{\mathcal{I}}_0\|_F \leqslant C\,\frac{\|\boldsymbol{\mathcal{I}}_0 - (\boldsymbol{\mathcal{I}}_0)_K\|_*}{\sqrt{K}} + D\,\frac{\varepsilon}{M}\,.$$