My background and tutorial on diffraction imaging.

Computational Imaging Group (CIG) 2023

Olivier Leblanc.

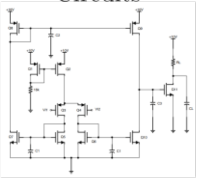
May 23th, 2023.

Outline

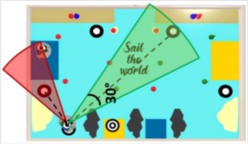
1. Undergrad background
2. PhD - Interferometric Lensless Imaging
3. At WashU: Lippmann-Schwinger for IDT

Undergrad background

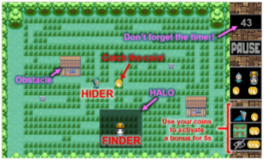
Circuits



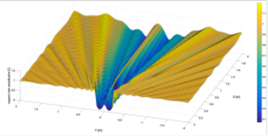
Robotics



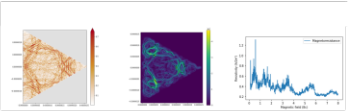
Embedded systems



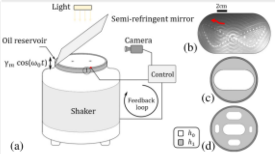
Electromagnetics



Nanoelectronics



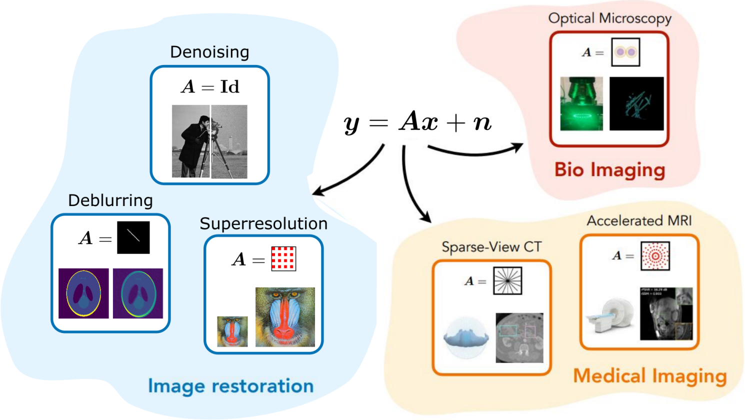
Master thesis



PhD - Interferometric Lensless Imaging

Context - Computational imaging

Many computational imaging problems can be formulated as inverse problems



[1]

Inverse problem: recover x from the observations y.

[1] inspired from Yu Sun

Context - MultiCore Fiber Lensless Imaging (MCF-LI)

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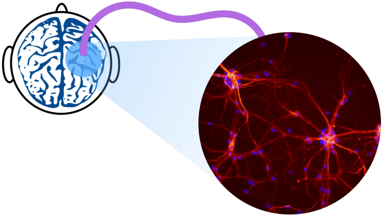
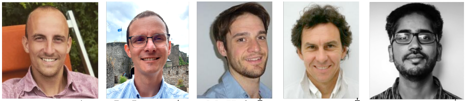
‡ CNRS, Université de Lille



Motivation:

- Deep brain imaging.
- Limited invasiveness.



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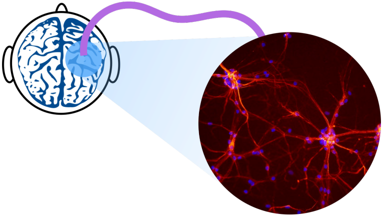
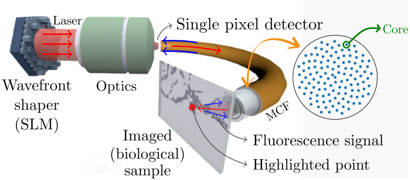


Motivation:

- Deep brain imaging.
- Limited invasiveness.

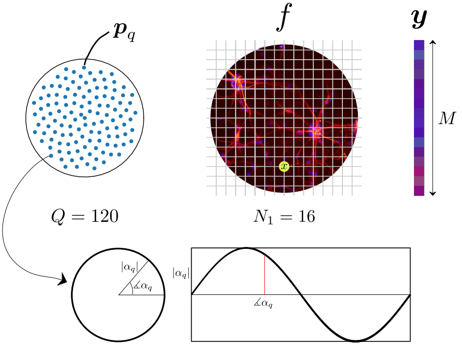
Challenges:

- Fluorescence imaging.
- Single-pixel $\rightarrow$ limited view.
- Small intensity.

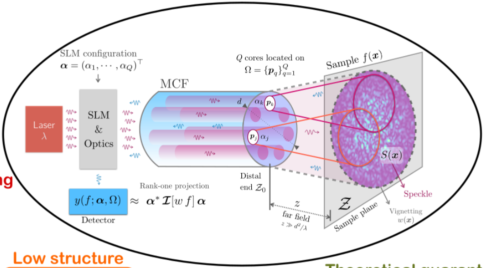


Notations

$\boldsymbol{p}_q$	Position of core q
Q	Number of cores
$\boldsymbol{\alpha} \in \mathbb{C}^Q$	Cores complex amplitudes **(tunable!)**
$\boldsymbol{x} \in \mathbb{R}^2$	2-D object space
$f(\boldsymbol{x})$	Object to be imaged
$N = N_1 \times N_1$	Image resolution
M	Number of measurements
$\boldsymbol{y} \in \mathbb{R}_+^M$	Measurement vector



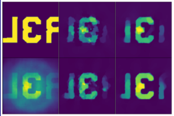
Our contributions



Mathematical modeling

$$\boldsymbol{y} = \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[wf]\boldsymbol{\alpha}_m\}_{m=1}^M$$
$$= \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[wf]$$

Proof of concept

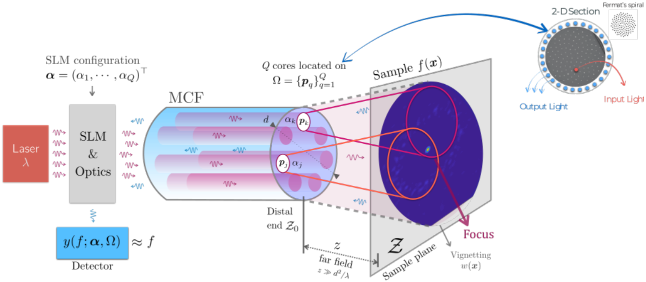


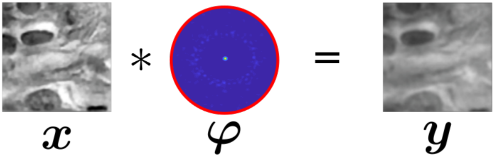
Low structure

Inverse problem & optimisation

Theoretical guarantees

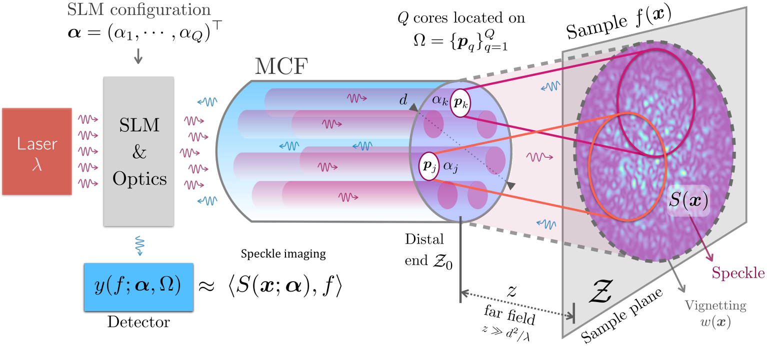
$$\left\|f - \hat{f}\right\|_2 \leqslant C\frac{\|f - f_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}$$


SLM configuration
$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_Q)^\top$
Laser
λ
SLM
&
Optics
$y(f; \boldsymbol{\alpha}, \Omega)$ $\approx f$
Detector
MCF
Q cores located on
$\Omega = \{\boldsymbol{p}_q\}_{q=1}^{Q}$
d
α_k $\boldsymbol{p}_k$
$\boldsymbol{p}_j$ α_j
Distal
end $\mathcal{Z}_0$
z
far field
$z \gg d^2/\lambda$
Sample $f(\boldsymbol{x})$
$\mathcal{Z}$
Sample plane
Focus
Vignetting
$w(\boldsymbol{x})$
2-D Section
Fermat's spiral
Output Light
Input Light

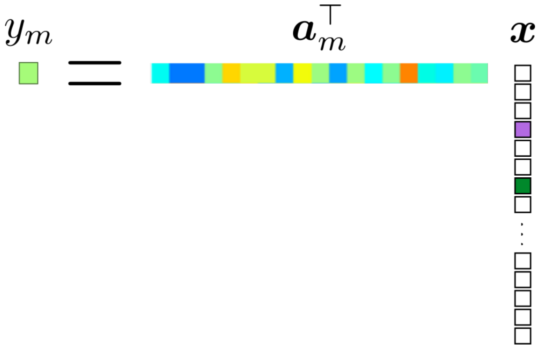

$\boldsymbol{x} * \boldsymbol{\varphi} = \boldsymbol{y}$

Speckle illumination

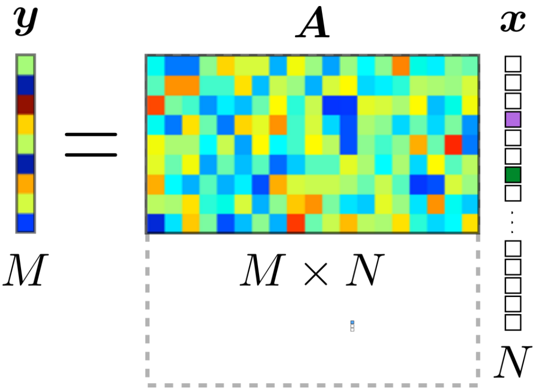
Steph. Guérit



$$y(f; \boldsymbol{\alpha}, \Omega) \approx \langle S(\boldsymbol{x}; \boldsymbol{\alpha}), f \rangle$$



$$y_m = \boldsymbol{a}_m^\top \boldsymbol{x} = \langle \boldsymbol{a}_m, \boldsymbol{x} \rangle$$

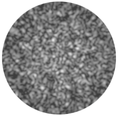
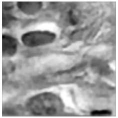


Condition: a_{mn} are *iid* random variables (e.g., $a_{mn} \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$).

Speckle illumination: model

The m-th single pixel observation is the integration of the m-th speckle pattern $\boldsymbol{s}_m$ illuminating the sample $\boldsymbol{f}$:

$$y_m = \boldsymbol{s}_m^\top \boldsymbol{f} + n_m.$$

$y_m = \Big\langle$  ,  $\Big\rangle$

Classical compressive sensing model (Guerit et al.)

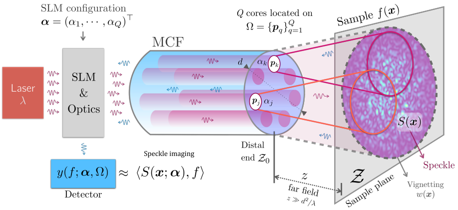
$$\boldsymbol{y} = \boldsymbol{S}\boldsymbol{f} + \boldsymbol{n}$$

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{f}\|_2^2 + \mathcal{R}(\boldsymbol{f})$$

Compressive imaging $\Rightarrow$ number of observations $M \ll N$ image resolution.

Assumption: all coefficients in $\boldsymbol{S}$ are i.i.d. random variables.

Interferometric imaging



Classical compressive sensing model

$$y_{\boldsymbol{\alpha}} = \int_{\mathbb{R}^2} S_{\boldsymbol{\alpha}}(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \langle S_{\boldsymbol{\alpha}}, f \rangle$$

Speckles are interferences (Under far-field approximation)

$$E_q(\boldsymbol{x}) = \alpha_q \sqrt{w}(\boldsymbol{x}) e^{\frac{2\pi \mathrm{i}}{\lambda z} \boldsymbol{p}_q^\top \boldsymbol{x}} \qquad \text{(Rayleigh-Sommerfeld)}$$

$$S(\boldsymbol{x};\boldsymbol{\alpha}) \propto w(\boldsymbol{x}) \Big| \sum_{q=1}^{Q} \alpha_q e^{\frac{2\pi \mathrm{i}}{\lambda z} \boldsymbol{p}_q^\top \boldsymbol{x}} \Big|^2 = \underbrace{w(\boldsymbol{x})}_{\text{Field of view}} \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* e^{\frac{2\pi \mathrm{i}}{\lambda z} (\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}}$$

Hence (noiseless)

$$\begin{aligned} y_{\boldsymbol{\alpha}} &= \sum_{j,k=1}^{Q} \alpha_j \alpha_k^* \left[\int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z} (\boldsymbol{p}_j - \boldsymbol{p}_k)^\top \boldsymbol{x}} w(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right] \\ &= \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[wf] \boldsymbol{\alpha} = \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^*, \boldsymbol{\mathcal{I}}_\Omega[wf] \rangle \end{aligned}$$

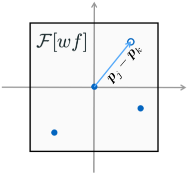
Interferometric imaging

Writing $f^\circ := wf$ we have

$$y_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^* \boldsymbol{\mathcal{I}}_\Omega[f^\circ] \boldsymbol{\alpha}$$

with the interferometric matrix $\boldsymbol{\mathcal{I}}_\Omega[f^\circ] \in \mathbb{C}^{Q \times Q}$ s.t.

$$\left(\boldsymbol{\mathcal{I}}_\Omega[f^\circ]\right)_{j,k} := \int_{\mathbb{R}^2} e^{\frac{2\pi \mathrm{i}}{\lambda z}(\boldsymbol{p}_j - \boldsymbol{p}_k)^\top} f^\circ(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

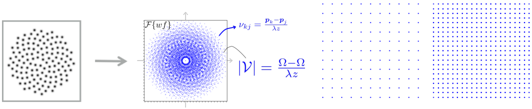


Fourier plane

Observation: denser Fourier sampling if

$$|\{\boldsymbol{p}_j - \boldsymbol{p}_k : \forall 1 \le j, k \le Q\}| \simeq Q^2$$

- Lattices are bad core arrangements
- Fermat's spiral is not bad



MCF-LI Forward Model

$$\begin{aligned} \boldsymbol{y} &= \{\boldsymbol{\alpha}_m^* \boldsymbol{\mathcal{I}}_\Omega[f^\circ]\boldsymbol{\alpha}_m\}_{m=1}^M \\ &= \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{I}}_\Omega[f^\circ] \end{aligned}$$

Two-component sensing!

$\boldsymbol{\mathcal{A}}$: Symmetric rank-one projections of a matrix.
$\boldsymbol{\mathcal{I}}_\Omega$: Partial Fourier sensing with replacement.

Objective

Show

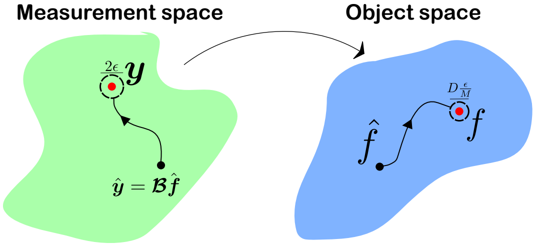
$$\hat{\boldsymbol{f}} \in \arg\min_{\boldsymbol{u}} \|\boldsymbol{u}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \epsilon \tag{BPDN$_{\ell_1}$}$$

satisfies

$$\left\|\boldsymbol{f} - \hat{\boldsymbol{f}}\right\|_2 \leqslant C\frac{\|\boldsymbol{f} - \boldsymbol{f}_K\|_1}{\sqrt{K}} + D\frac{\epsilon}{m}.$$

Prove the RIP-ℓ_2/ℓ_1

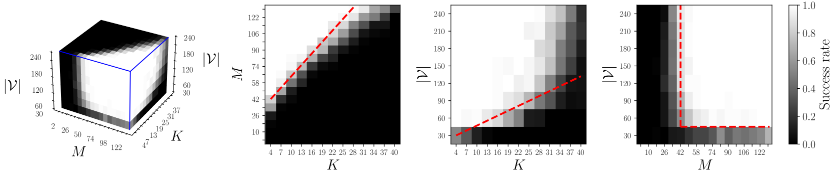
$$\tilde{A}\|\boldsymbol{u}\| \leqslant \frac{1}{M}\|\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{I}}_\Omega[\boldsymbol{u}])\|_1 \leqslant \tilde{B}\|\boldsymbol{u}\|$$



Simulation results

K = sparsity of f, $f \in \Sigma_K$.
$|\mathcal{V}|$ = cardinality of the visibility set in the Fourier space, grows with Q.
M = number of measurements.

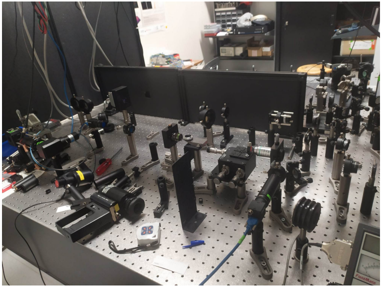
$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{u}} \; \|\boldsymbol{y} - \boldsymbol{\mathcal{B}}\boldsymbol{u}\|_2 \;\; \text{s.t.} \;\; \|\boldsymbol{u}\|_1 \leqslant K \qquad \text{(LASSO)}$$

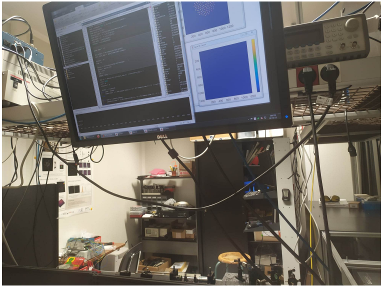


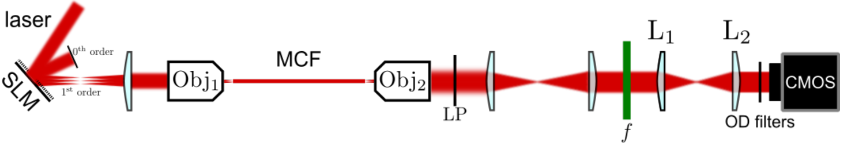
$$M > CK \log\left(\frac{12eN}{K}\right)$$

$$|\mathcal{V}_0| = Q(Q-1) \geqslant \delta^{-2} K \operatorname{plog}(N, K, \delta)$$

Success if SNR> 40dB.

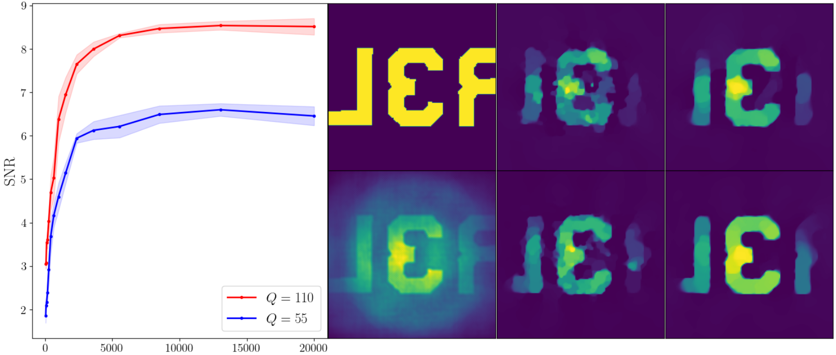





laser
0th order
1st order
SLM
Obj1
MCF
Obj2
LP
f
L1
L2
OD filters
CMOS

Experimental results

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{u}} \ \frac{1}{2} \left\| \boldsymbol{y} - \hat{\boldsymbol{\mathcal{B}}}(\boldsymbol{u}) \right\|_2 + \rho \left\| \boldsymbol{u} \right\|_{\mathrm{TV}} \ \text{ s.t. } \ \boldsymbol{u} \geqslant 0, \tag{1}$$



At WashU: Lippmann-Schwinger for IDT

Outline

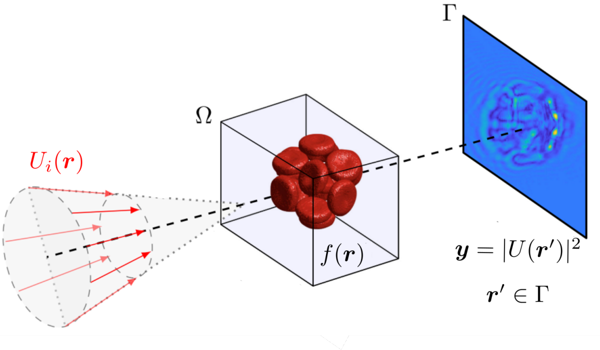


Figure 1: Inspired from T. Pham.

Can we get $\boldsymbol{y} = h(U_i, f)$?

From Maxwell to Wave equation

$$\boxed{\begin{aligned} \nabla\cdot\boldsymbol{D} &= \rho, && \text{(Coulomb)}\\ \nabla\cdot\boldsymbol{B} &= 0, && \text{(Gauss)}\\ \nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t}, && \text{(Ampère)}\\ \nabla\times\boldsymbol{H} &= \frac{\partial\boldsymbol{D}}{\partial t}+\mathcal{J}, && \text{(Faraday)} \end{aligned}}$$

- $\boldsymbol{H}$ is the magnetic excitation.
- $\boldsymbol{B}$ is the magnetic field.
- $\boldsymbol{D}$ is the displacement field.
- $\boldsymbol{E}$ is the electric field.

□: simplifying assumption.
□: rewriting.

From Maxwell to Wave equation

$$\boxed{\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \rho, && \text{(Coulomb)} \\
\nabla \cdot \boldsymbol{B} &= 0, && \text{(Gauss)} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, && \text{(Ampère)} \\
\nabla \times \boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t} + 0, && \text{(Faraday)}
\end{aligned}}$$

No current density (no moving charged particle): $\mathcal{J} = 0$

From Maxwell to Wave equation

$$\boxed{\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \rho \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{E}\right) &= -\nabla \times \left(\frac{1}{\mu}\frac{\partial \boldsymbol{B}}{\partial t}\right) \\
\frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{H}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{D}}{\partial t}\right)
\end{aligned}}$$

From Maxwell to Wave equation

$$\boxed{\begin{aligned}
\nabla \cdot (\epsilon \boldsymbol{E}) &= \rho \\
\nabla \cdot (\mu \boldsymbol{H}) &= 0 \\
\nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) &= -\nabla \times \left(\frac{1}{\mu} \frac{\partial \mu \boldsymbol{H}}{\partial t} \right) \\
\frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}) &= \frac{\partial}{\partial t} \left(\frac{\partial \epsilon \boldsymbol{E}}{\partial t} \right)
\end{aligned}}$$

Linear optics: $\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \epsilon \boldsymbol{E}$

From Maxwell to Wave equation

$$\boxed{\begin{array}{rcl} \nabla \cdot (\epsilon \boldsymbol{E}) & = & \rho \\ \nabla \cdot (\mu \boldsymbol{H}) & = & 0 \\ \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) & = & -\nabla \times \left(1 \frac{\partial \boldsymbol{H}}{\partial t} \right) \\ \nabla \times \left(\frac{\partial \boldsymbol{H}}{\partial t} \right) & = & \epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \end{array}}$$

$$\frac{\partial}{\partial t} (\nabla \times \boldsymbol{B}) = \nabla \times \left(\frac{\partial \boldsymbol{B}}{\partial t} \right), \qquad \mu, \epsilon \perp\!\!\!\perp t$$

From Maxwell to Wave equation

$$\boxed{\begin{aligned} \nabla \cdot (\epsilon \boldsymbol{E}) &= \rho \\ \cancel{\nabla \cdot (\mu \boldsymbol{H})} &\cancel{= 0} \\ \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) &= -\nabla \times \left(\frac{\partial \boldsymbol{H}}{\partial t} \right) \\ \nabla \times \left(\frac{\partial \boldsymbol{H}}{\partial t} \right) &= \epsilon \ddot{\boldsymbol{E}} \end{aligned}}$$

From Maxwell to Wave equation

$$
\boxed{
\begin{aligned}
\nabla \cdot (\epsilon \boldsymbol{E}) &= \rho \\
\nabla \cdot (\mu \boldsymbol{H}) &= 0 \\
\nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) &= -\epsilon \ddot{\boldsymbol{E}}
\end{aligned}
}
$$

$$\boxed{\begin{aligned} \nabla \cdot (\epsilon \boldsymbol{E}) &= \rho \\ \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) &= -\epsilon \ddot{\boldsymbol{E}} \end{aligned}}$$

From Maxwell to Wave equation

$$\boxed{\nabla \cdot (\epsilon \boldsymbol{E}) \quad = \quad \rho}$$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) \quad = \quad -\epsilon \ddot{\boldsymbol{E}}$$

$$\downarrow$$

stationary conditions $\rightarrow$ get rid of time:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{\sqrt{\epsilon}} \boldsymbol{E}(\boldsymbol{r}) e^{-i\omega t} \implies \ddot{\boldsymbol{E}}(\boldsymbol{r}, t) = -\omega^2 \boldsymbol{E}(\boldsymbol{r}, t)$$

$$\downarrow$$

$$\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} + \nabla \log(\mu) \times (\nabla \times \boldsymbol{E}) + \nabla \left(\frac{\rho}{\epsilon} \right) + \nabla \left(\boldsymbol{E} \cdot \nabla \left(\log(\epsilon) \right) \right) = 0$$

From Maxwell to Wave equation

$$\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} + \nabla \log(\mu) \times (\nabla \times \boldsymbol{E}) + \nabla\left(\frac{\rho}{\epsilon}\right) + \nabla \left(\boldsymbol{E} \cdot \nabla \left(\log(\epsilon)\right)\right) = 0$$

From Maxwell to Wave equation

$$\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} + \nabla \log(\mu) \times (\nabla \times \boldsymbol{E}) + \cancel{\nabla\left(\frac{\rho}{\epsilon}\right)} + \nabla\left(\boldsymbol{E} \cdot \nabla\left(\log(\epsilon)\right)\right) = 0$$

No charge density: $\rho = 0$

From Maxwell to Wave equation

$$\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} + \cancel{\nabla \log(\mu) \times (\nabla \times \boldsymbol{E})} + \cancel{\nabla\left(\frac{\rho}{\epsilon}\right)} + \nabla\left(\boldsymbol{E} \cdot \nabla\left(\log(\epsilon)\right)\right) = 0$$

No charge density: $\rho = 0$

Non magnetic medium: $\mu(\boldsymbol{r}) = \mu_0 \perp\!\!\!\perp \boldsymbol{r}$

From Maxwell to Wave equation

$$\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} + \cancel{\nabla \log(\mu) \times (\nabla \times \boldsymbol{E})} + \cancel{\nabla(\frac{\rho}{\epsilon})} + \cancel{\nabla\left(\boldsymbol{E} \cdot \nabla\left(\log(\epsilon)\right)\right)} = 0$$

No charge density: $\rho = 0$

Non magnetic medium: $\mu(\boldsymbol{r}) = \mu_0 \perp\!\!\!\perp \boldsymbol{r}$

Slow spatial variations of ϵ: $\nabla\left(\boldsymbol{E} \cdot \nabla\left(\log(\epsilon)\right)\right) \ll \nabla^2 \boldsymbol{E}$

From Maxwell to Wave equation

$$\boxed{\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} = 0} \qquad \text{(Wave eq.)}$$

From Maxwell to Wave equation

$$\boxed{\nabla^2 \boldsymbol{E} + k^2 n^2 \boldsymbol{E} = 0} \qquad \text{(Wave eq.)}$$

With $\boldsymbol{E} = (E_x, E_y, E_z)$

$$\begin{cases} \nabla^2 E_x + k^2 n^2 E_x = 0 \\ \nabla^2 E_y + k^2 n^2 E_y = 0 \\ \nabla^2 E_z + k^2 n^2 E_z = 0. \end{cases}$$

Write U for any component E_x, E_y or E_z.

$$\boxed{\nabla^2 U + k^2 n^2 U = 0} \qquad \text{(Scalar Wave eq.)}$$

From Maxwell to Wave equation

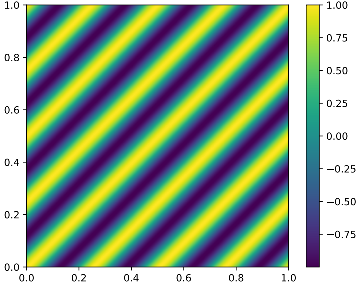
$$\boxed{\nabla^2 U + k^2 n^2 U = 0} \qquad \text{(Scalar Wave eq.)}$$

From Maxwell to Wave equation

$$\boxed{\left(\nabla^2 + k^2 n(\boldsymbol{r})^2\right)\, U(\boldsymbol{r}) = 0} \qquad \text{(Scalar Wave eq.)}$$

Example: plane wave
(with $n(\boldsymbol{r}) = 1,\ \forall \boldsymbol{r} \in \mathbb{R}^3$):

$$U(\boldsymbol{r}) = e^{\mathrm{i}\boldsymbol{k}^\top \boldsymbol{r}}$$



Outline

From Wave to Lippmann-Schwinger equation

$$\boxed{\left(\nabla^2 + k^2 n(\boldsymbol{r})^2\right)\, U(\boldsymbol{r}) = 0}$$

k is the wavenumber in vacuum.
$k_\mathrm{m} := k n_\mathrm{m}$ is the wavenumber in medium m with RI n_m.

$$\left(\nabla^2 + k_\mathrm{m}^2 \left(\frac{n(\boldsymbol{r})}{n_\mathrm{m}^2}\right)^2\right) U(\boldsymbol{r}) = 0$$

$$(\nabla^2 + k_\mathrm{m}^2) U(\boldsymbol{r}) = -f(\boldsymbol{r}) U(\boldsymbol{r})$$

with the <u>scattering potential</u>

$$f(\boldsymbol{r}) := k_\mathrm{m}^2 \left[\left(\frac{n(\boldsymbol{r})}{n_\mathrm{m}}\right)^2 - 1\right].$$

$$(\nabla^2 + k_{\mathrm{m}}^2) U(\boldsymbol{r}) = -f(\boldsymbol{r}) U(\boldsymbol{r})$$

Separation into homogeneous and inhomogeneous equations. $U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + U_{\mathrm{s}}(\boldsymbol{r})$ with:

$$\left(\nabla^2 + k_{\mathrm{m}}^2\right) U_i(\boldsymbol{r}) = 0$$
$$\left(\nabla^2 + k_{\mathrm{m}}^2\right) U_{\mathrm{s}}(\boldsymbol{r}) = -f(\boldsymbol{r}) U(\boldsymbol{r})$$

U_i is the incident field.
U_{s} is the scattered field.

From Wave to Lippmann-Schwinger equation

$$\left(\nabla^2 + k_{\mathrm{m}}^2\right) U_{\mathrm{s}}(\boldsymbol{r}) = -f(\boldsymbol{r})U(\boldsymbol{r})$$

The Green's function is defined as:

$$\left(\nabla^2 + k_{\mathrm{m}}^2\right) G(\boldsymbol{r}) = -\delta(\boldsymbol{r})$$

Now

$$\int (\nabla^2 + k_{\mathrm{m}}^2) G(\boldsymbol{r} - \boldsymbol{r}') f(\boldsymbol{r}') U(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}' = \int -\delta(\boldsymbol{r} - \boldsymbol{r}') f(\boldsymbol{r}') U(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}'$$

so that

$$U_{\mathrm{s}}(\boldsymbol{r}) = \int_{\Omega} f(\boldsymbol{r}') U(\boldsymbol{r}') G(\boldsymbol{r} - \boldsymbol{r}') \mathrm{d}\boldsymbol{r}'$$

Finally

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_{\Omega} f(\boldsymbol{r}') U(\boldsymbol{r}') G(\boldsymbol{r} - \boldsymbol{r}') \mathrm{d}\boldsymbol{r}'} \qquad \text{(Lippmann-Schwinger)}$$

Born approximation and interpretation

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'} \qquad \text{(Lippmann-Schwinger)}$$

Born approximation and interpretation

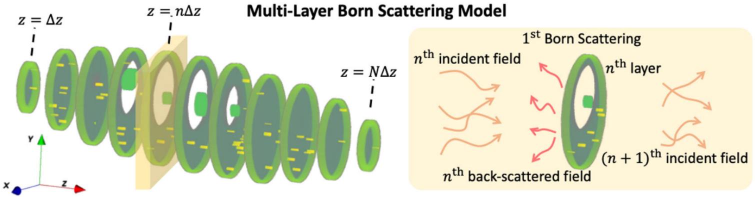
$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}$$ (Lippmann-Schwinger)

$$\boxed{U_1(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U_i(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}$$ (1st Born approx.)

$$\boxed{U_{n+1}(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U_n(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}$$ (n-th Born approx.)

$$\boxed{U_{n+1}(\boldsymbol{r}) = U_n(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U_n(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}$$ (multi-layer Born approx.)

Interpretation:



Outline

Discretizing the 1st Born approx.

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U_i(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'} \qquad \text{(1st Born approx.)}$$

$$\boldsymbol{U} = \boldsymbol{U}_i + \boldsymbol{G}\,\mathrm{diag}(\boldsymbol{U}_i)\boldsymbol{f}$$

$$\boldsymbol{y} = |\boldsymbol{U}|^2 = |\boldsymbol{U}_i|^2 + 2\Re\{\boldsymbol{U}_i^*\boldsymbol{G}\,\mathrm{diag}(\boldsymbol{U}_i)\boldsymbol{f}\} + \text{negl.}$$

Discretizing the 1st Born approx.

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U_i(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'} \qquad \text{(1st Born approx.)}$$

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$$\boldsymbol{y}' = |\boldsymbol{U}|^2 - |\boldsymbol{U}_i|^2 = 2\Re\{\boldsymbol{U}_i^*\boldsymbol{G}\,\mathrm{diag}(\boldsymbol{U}_i)\boldsymbol{f}\} + \text{negl.}$$

Remove background intensity

Discretizing the 1st Born approx.

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Remove background intensity

Remind

$$f(\boldsymbol{r}) = k_{\mathrm{m}}^2\left[n^2(\boldsymbol{r}) - n_{\mathrm{m}}^2\right] = k_{\mathrm{m}}^2\left[\epsilon(\boldsymbol{r}) - \epsilon_{\mathrm{m}}\right]\mu_0 = k_{\mathrm{m}}^2\mu_0\Delta\epsilon(\boldsymbol{r})$$

Hence,

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{\Delta\epsilon}$$

$\boldsymbol{A}$ is nice and fast $\rightarrow$ FFT implementation.

Inverse problem

Try to estime the volume of RI as

$$\widetilde{\boldsymbol{\Delta\epsilon}} = \arg\min_{\boldsymbol{x}} \frac{1}{2} \left\| \boldsymbol{y}' - \boldsymbol{A}\boldsymbol{x} \right\|_2^2 + \mathcal{R}(\boldsymbol{x})$$

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Eventually use Neural Fields to get a continuous representation $\Delta\epsilon = \mathcal{M}_{\boldsymbol{\phi}}(\boldsymbol{r})$ and solve

$$\widetilde{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \frac{1}{2} \left\| \boldsymbol{y}' - \boldsymbol{A}\boldsymbol{x} \right\|_2^2 + \mathcal{R}(\boldsymbol{x}) \text{ s.t. } \boldsymbol{x} = \mathcal{M}_{\boldsymbol{\phi}}(\boldsymbol{r}).$$

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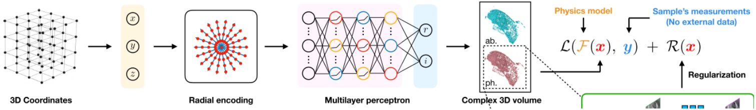
This is the approach used in DeCaf[1].



[1] R. Liu, Y. Sun, J. Zhu, L. Tian, U. Kamilov, "Recovery of Continuous 3D Refractive Index Maps from Discrete Intensity-Only Measurements using Neural Fields," Nature Mach Intell 4, 781-791 (2022)

Discretizing Lippmann-Schwinger

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'} \qquad \text{(Lippmann-Schwinger)}$$

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$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{\Delta\epsilon})$$

$\mathcal{A}$ is less nice. Accurate but slow...

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IDEA 1 Keep Neural Fields approach and solve

$$\widetilde{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \frac{1}{2}\left\|\boldsymbol{y}' - \mathcal{A}(\boldsymbol{x})\right\|_2^2 + \mathcal{R}(\boldsymbol{x}) \text{ s.t. } \boldsymbol{x} = \mathcal{M}_{\boldsymbol{\phi}}(\boldsymbol{r}).$$

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IDEA 2 Can we get a loss diffentiable in $\boldsymbol{\phi}$ with the continuous model?

$$\boldsymbol{y} = \mathcal{B}(\Delta\epsilon)$$

$\mathcal{B}$ is a continuous-to-discrete operator.

$$\widetilde{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \frac{1}{2}\left\|\boldsymbol{y}' - \mathcal{B}(x)\right\|_2^2 + \mathcal{R}(x) \text{ s.t. } x = \mathcal{M}_{\boldsymbol{\phi}}(\boldsymbol{r}).$$

Discretizing Lippmann-Schwinger

$$\boxed{U(\boldsymbol{r}) = U_i(\boldsymbol{r}) + \int_\Omega f(\boldsymbol{r}')U(\boldsymbol{r}')G(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'} \qquad \text{(Lippmann-Schwinger)}$$

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IDEA 3 Could we train a neural field $\mathcal{M}_{\boldsymbol{\phi}}$ for multiple RI volumes?

Conclusion

Today, you learned:

- a little more about me.
- some diffraction theory.
- my contribution perspectives.

From now on

- Your tips? Ideas?
- Please help me with $\mathcal{M}_{\phi}$:).

Thank you!