Master of Science in quantitative and financial modelling

Homework 1

EXAMINER: Prof. L. Laayouni **Due date: Monday vember 7 2022**

FAMILY NAME:
GIVEN NAMES:

Instructions

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in. You may do rough work for this paper anywhere in the booklet.
3. This is a closed book examination.
4. This examination booklet consists of this cover, Pages 1 through 5 containing questions; and Pages 6 and 7, which are blank.
5. You are expected to simplify your answers wherever possible. You are advised to spend the first few minutes scanning the problems.
6. A TOTAL OF 100 MARKS ARE AVAILABLE ON THIS EXAMINATION.

PLEASE DO NOT WRITE INSIDE THIS BOX

1 /20	2 /20	3 /20	4 /20	5 /20				
								Total /100

1. **[20 MARKS]** Find the LU-factorization of the following matrix $\mathbf{A}$ in $\mathbb{R}^{\mathbf{n}\times\mathbf{n}}$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{1}$$

Is the matrix $\mathbf{A}$ positive definite? or semi-definite positive?
Write a python script describing the solution of a linear system associated to the above matrix.

2. **[20 MARKS]** Consider the following boundary value problem modeling the heat flow in a long pipe

$$\begin{cases} y''(x) - p(x)y'(x) - q(x)y(x) = r(x), \quad x \in [a,b] \\ y(a) = \alpha,\ y(b) = \beta \end{cases} \tag{2}$$

   (a) Use a uniform discretization of the interval $[a, b]$ to derive the linear system corresponding to the model problem.

   (b) Solve the linear system using Gaussian elimination method.

   (c) Solve the linear system using QR-factorization
   (**Remark: You are free to decide about the size of the matrix)**

3. **[20 MARKS]** Suppose **A** is an invertible non-singular square matrix of order n and that u, v are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Prove the Sherman-Morrison formula

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}$$

Here, uv^T is a matrix known by the outer product of the two vectors u and v

4. **[20 MARKS]** Let $\mathbf{A}$ be of order $m \times n$ with SVD $\mathbf{A} = \mathbf{U\Sigma V^T}$. Compute the SVDs of the following matrices in terms of $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$

   (a) $(\mathbf{A^T A})^{-1}$

   (b) $(\mathbf{A^T A})^{-1}\mathbf{A^T}$

   (c) $\mathbf{A}(\mathbf{A^T A})^{-1}$

   (d) $\mathbf{A}(\mathbf{A^T A})^{-1}\mathbf{A^T}$

5. **[20 MARKS]** Solve the problem of fitting a polynomial $p(x) = \sum_{i=0}^{d} c_i x^{i-1}$ of degree d to data points $(x_i, yi),\ i = 1, \ldots, m$, in the plane by the method of normal equations and QR decomposition. Choose the degree of the polynomial to be $d = 5$ and then $d = 15$, choose the interval $x \in [-1, 1]$, discretize it using $N = 10$ or $N = 20$ points.

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