Academic year 2022-2023
Master 1 QFM

Practical work - To be returned before January 27

Finite Differences for the Wave Equation

1 1D Model

We consider the following $1D$ wave equation with initial and boundary conditions:

$$(E_1) \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{for} \quad t > 0\,, \ -L < x < L \\ u(x,0) = f(x) & \text{for} \quad -L \leq x \leq L \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{for} \quad -L \leq x \leq L \\ u(-L,t) = u(L,t) = 0 & \text{for} \quad t > 0 \end{cases}$$

$u(x,t)$ represents the planar wave and c is the speed of the wave (here we take $c > 0$).

1. Show that the solution of equation (E_1) can be written as the sum of two progressive waves F and G propagating respectively with speeds c and $-c$:

$$u(x,t) = F(x-ct) + G(x+ct)$$

We propose to approximate the solution of (E_1) on $[-L,L] \times [0,T]$.

We subdivide the interval $[-L,L]$ into a regular mesh formed of N nodes x_j. We denote by u_j^n the approximate solution at node $x_j = j\Delta x$ and at time $t^n = n\Delta t$, and we set $c\,\dfrac{\Delta t}{\Delta x} = \lambda$.

A finite difference scheme centered in space and time can be written as follows:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - c^2\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0 \qquad (1)$$

2. Study the truncation error, the order and the consistency of the numerical scheme (1).

3. Using Fourier Von-Neumann analysis, show that the scheme (1) is stable under the condition

$$c\frac{\Delta t}{\Delta x} \leq 1$$

4. Implement the numerical algorithm and perform the simulations on the following two test cases, using a mesh with $N = 100$ nodes.

Test case 1:

- $L = 10\,m \quad ; \quad c = 1\,m/s \quad ; \quad f(x) = e^{-x^2}$.
- Represent the results at physical times $t_0 = 0\,s\,,\ t_1 = 1\,s\,,\ t_2 = 4\,s\,,\ t_3 = 7\,s$.

Test case 2:

- $L = 1\,m \quad ; \quad c = 1\,m/s \quad ; \quad f(x) \;=\; \cos\left(\dfrac{(2k+1)\,\pi}{2}\,\dfrac{x}{L}\right)$, with $k = 3$.
- Represent the results at times $t_0 = 0\,s\,,\; t_1 = \dfrac{T}{5}\,,\; t_2 = \dfrac{2T}{3}$, with $T = \dfrac{4L}{(2k+1)\,c}$.

2 2D Model

The $2D$ wave equation can be written as follows:

$$(E_2) \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\,\Delta u \;=\; 0 & \text{for} \quad t > 0\,,\;\; (x,y) \,\in\, \mathring{\Omega} \\ u(x,y,0) \;=\; f(x,y) & \text{for} \quad (x,y) \,\in\, \Omega \\ \dfrac{\partial u}{\partial t}(x,y,0) \;=\; 0 & \text{for} \quad (x,y) \,\in\, \Omega \\ u(x,y,t) \;=\; 1 & \text{for} \quad t > 0\,,\;\; (x,y) \,\in\, \partial\Omega \end{cases}$$

$\Omega = [-L_x, L_x] \times [-L_y, L_y]$ being the computational domain, which is assumed to be rectangular, and $\partial\Omega$ is the boundary of Ω.

The computational domain is discretized by a rectangular mesh with N_x nodes in x direction and N_y nodes in y direction. The space steps are $\Delta x = \dfrac{L_x}{N_x - 1}$ and $\Delta y = \dfrac{L_y}{N_y - 1}$.

We denote by $X_{i,j}$ and $Y_{i,j}$ the coordinates of a node of index (i,j) and by $u^n_{i,j}$ the approximate solution at the node (i,j) and at the time $t^n = n\Delta t$.

5. Based on the scheme (1) for the $1D$ wave equation (E_1), propose a finite difference scheme for the $2D$ wave equation (E_2).

6. Test case:

- $L_x = L_y = 10\,m \quad ; \quad c = 1\,m/s \quad ; \quad f(x,y) \;=\; e^{-(x^2+y^2)} \quad ; \quad N_x = N_y = 30.$
- Represent the contour plots of u at times $t_0 = 0\,s\,,\; t_1 = 1\,s\,,\; t_2 = 4\,s\,,\; t_3 = 7\,s.$