Chapter 7

Stability of a Euclidean Cloud

(‘*STABILITY*’)

Introduction

In this chapter, we present stability problems in Geometric Data Analysis. The aim is to study the sensitivity of the methods of determination of the principal axes and variables of a Euclidean cloud to some data modifications, or *perturbations,* so as to bring out their mechanism of variation.

Stability studies have been tackled along several lines.

The line followed by Escofier & Le Roux (1972, 1975, 1976, 1977) consists in comparing the analysis of a *reference Euclidean cloud* to that of a cloud obtained by some modification. The influence of the perturbation is evaluated by an *interval for eigenvalues and inertia rates* and by *upper bounds for the rotation angles of principal subspaces;* intervals and upper bounds are expressed as functions of the results of the analysis of the reference cloud: eigenvalues, contributions and qualities of representation of points. This approach refers to the following linear algebra problems: comparing the eigenvalues and studying the rotation of invariant subspaces of a symmetric endomorphism by the addition of a symmetric endomorphism and/or multiplication by a positive–definite symmetric one.

Another line of study is that of *sensitivity analysis* and influence functions, in the sense of functional analysis[1]. This method has been mainly applied to PCA: see Krzanowski (1984), Critchley (1985), Tanaka (1988);

[1] See e.g. Hampel (1974) or Lecoutre & Tassi (1987, chapter 2).

for its application to CA, see Pack & Joliffe (1992), Bénasséni (1993).

Still another line is that of *bootstrap,* discussed e.g. by Diaconis & Efron (1983), Daudin & al (1988), Lebart & al (1995), Lebart & al (2003).

In this chapter, we present four stability problems, following the first line, which is the most congenial to the geometric approach followed elsewhere in this book. To begin with, we study the effect of the coding according to a partition (§7.1). Then, we deal with the influence of a group of points (§7.2). Then we study the effect of a change of metric (§7.3). We proceed with the study of the influence of a variable or a modality (§7.4). For each problem, we firstly present the results for a Euclidean cloud, then we particularize to the clouds stemming from PCA, CA, and MCA. The chapter closes with the linear algebra theorems on which our stability studies are based, namely, the comparison of eigenvalues and the rotation of the invariant subspaces of a symmetric endomorphism by a perturbation (§7.5).

7.1 Effect of Coding according to a Partition

In GDA, one often proceeds to the grouping of elements. It is a common saying that in CA, since the distance satisfies the principle of distributional equivalence, grouping modalities, whose profiles are similar, entails "little change" in the outcome of analysis. In this section we state mathematical results that formalize this assertion.

Grouping problems may concern either individuals or modalities.

— When grouping individuals according to a partition, the stability study permits to identify the principal variables that are little modified and therefore reflect differences among groups.

— In a questionnaire analysis, one attempts to balance the contributions of questions (at least within a same heading), which motivates the search of groupings of modalities that leave the analysis stable.

To begin with, we will recall some definitions and properties pertaining to the partition of a cloud (§7.1.1). Then we will present the effect of grouping into C classes $(C \geq 2)$, into two classes (§7.1.3), and the grouping of two elements, e.g. quasi–distributional equivalence in CA (§7.1.4).

7.1.1 Partition of a Cloud

Let M^J be a cloud (reference cloud) in a Euclidean space $\mathcal{U}$; we denote G its mean point and $\mathbf{Var}\,\mathrm{M}^J$ its variance. Let $\mathcal{V}$ be the underlying vector space of $\mathcal{U}$. With the cloud M^J there is associated the endomorphism $Som : \mathcal{V} \to \mathcal{V}$

(see CLOUD, Definition 3.13, p.86). Let $\mathcal{M} \subseteq \mathcal{U}$ be the geometric support of the cloud M^J with the underlying vector subspace $\mathcal{L} \subseteq \mathcal{V}$, and L be its dimensionality.

From now on all endomorphisms are restricted to $\mathcal{L}$.

$$\forall \overrightarrow{\alpha} \in \mathcal{L} : Som(\overrightarrow{\alpha}) = \sum_{j \in J} f_j \langle \overrightarrow{\mathrm{GM}}^j | \overrightarrow{\alpha} \rangle \overrightarrow{\mathrm{GM}}^j \in \mathcal{L} \tag{7.1}$$

Consider the grouping $J{<}c{>}$ of elements of J in class c, and the associated subcloud $\mathrm{M}^{J<c>}$ with the three endomorphisms on $\mathcal{L}$ — namely $Som_{J<c>}$, Som_c and $Som_{J(c)}$ — which are defined in chapter CLOUD (p.103), and which verify the between–within relation $Som_{J<c>} = Som_c + f_c Som_{J(c)}$ (cf. Lemma 3.1, p.103). For a partition into C classes, one has the between–within breakdown of the endomorphism Som (cf. Theorem 3.6, p.105):

$$Som = Som_C + Som_{J(C)} \tag{7.2}$$

The endomorphism $Som_C = \sum_{c \in C} Som_c$ is associated with the between–cloud M^C and the endomorphism $Som_{J(C)} = \sum_{c \in C} f_c Som_{J(c)}$ with the within–cloud $\mathrm{M}^{J(C)}$ (see Definitions 3.24 and 3.25, p.103). Let $(\lambda_\ell)_{\ell=1,\ldots L}$ denote the eigenvalues of Som, $(\lambda'_\ell)_{\ell=1,\ldots L}$ the ones of $Som_{J(C)}$ and $(\mu_\ell)_{\ell=1,\ldots L}$ the ones of Som_C. Comparing the between–cloud M^C to the reference cloud M^J amounts to studying the influence of the perturbation $-Som_{J(C)}$ on the eigenvalues and the eigenvectors of Som. In the same way, comparing the within–cloud $\mathrm{M}^{J(C)}$ to the reference cloud M^J— that is, studying the effect of grouping into classes — amounts to comparing the eigenvalues and eigenvectors of Som to those of $Som_{J(C)}$ under the perturbation $-Som_C$.

The study of stability by grouping does not entail any change of metric: the clouds to be compared lie in the same Euclidean space.

7.1.2 Stability and Partition into C Classes ($C \geq 2$)

Property 7.1 (Eigenvalues). *Between the eigenvalue λ_ℓ of the cloud M^J and the eigenvalue λ'_ℓ of the within–cloud $\mathrm{M}^{J(C)}$, one has the inequalities:*

$$\lambda_\ell - \mu_1 \leq \lambda'_\ell \leq \lambda_\ell - \mu_L$$

This property follows from Theorem 7.1 (relation (2b), p.292).

If $\mathrm{Var}\,\mathrm{M}^C$ denotes the between–variance, one has $\mu_1 \leq \mathrm{Var}\,\mathrm{M}^C$ and $\mu_L \geq 0$, hence $\lambda_\ell - \mathrm{Var}\,\mathrm{M}^C \leq \lambda'_\ell \leq \lambda_\ell$.

The inertia rates pertaining to the first ℓ axes are denoted $\tau_{1\to\ell} = \sum\limits_{\ell'=1}^{\ell} \lambda_{\ell'} / \operatorname{Var} \mathrm{M}^J$ for the reference cloud M^J, $\tau'_{1\to\ell} = \sum\limits_{\ell'=1}^{\ell} \lambda'_{\ell'} / \operatorname{Var} \mathrm{M}^{J(C)}$ for the within–cloud $\mathrm{M}^{J(C)}$, and $\upsilon_{1\to\ell} = \sum\limits_{\ell'=1}^{\ell} \mu_{\ell'} / \operatorname{Var} \mathrm{M}^C$ for the between–cloud M^C. Then letting $\eta^2 = \operatorname{Var} \mathrm{M}^C / \operatorname{Var} \mathrm{M}^J$ (correlation ratio), one has the following property.

Property 7.2 (Inertia rates). *Between the inertia rate $\tau_{1\to\ell}$ of the cloud M^J and the inertia rate $\tau'_{1\to\ell}$ of the within–cloud, pertaining to the first ℓ principal axes, one has the following inequalities:*

$$\frac{\tau_{1\to\ell} - \eta^2 \upsilon_{1\to\ell}}{1-\eta^2} \leq \tau'_{1\to\ell} \leq \frac{\tau_{1\to\ell} + \eta^2(1-\upsilon_{1\to\ell})}{1-\eta^2}$$

This property follows from the relation (2c) of Theorem 7.1 (p.292).

Property 7.3 (Principal subspaces). *The greatest canonical angle θ between the principal subspaces associated with the ℓ_0–th, ... , $(\ell_0+\ell-1)$–th eigenvalues is such that, letting $\epsilon = \inf\{(\lambda_{\ell_0-1} - \lambda_{\ell_0})\ ;\ (\lambda_{\ell_0+\ell-1} - \lambda_{\ell_0+\ell})\}$ for $\ell_0 > 1$ and $\epsilon = (\lambda_\ell - \lambda_{\ell+1})$ for $\ell_0 = 1$ (subspace of the first ℓ axes):*

$$\text{If} \quad \mu_1 - \mu_L < \epsilon \quad \text{then} \quad \theta < \pi/4 \quad \text{and} \quad \sin 2\theta \leq (\mu_1 - \mu_L)/\epsilon$$

This property follows from Theorem 7.4, p. 294.

Comments

The *eigenvalues* decrease when points are grouped, but nothing can be said about the increasing or decreasing of inertia rates. The upper bounds of the *angle between principal subspaces* are increasing functions of the deviations between the largest and the smallest eigenvalues of the perturbation cloud (between or within), and they are decreasing functions of the minimal deviation between the eigenvalues that "border" the subspace under study.

The preceding properties are directly applicable to clouds stemming from CA or PCA (see Le Roux, 1977). For an example of application to CA, see Escofier & Le Roux (1972).

7.1.3 Stability and Partition into Two Classes

For a partition into two classes $C = \{c, c'\}$, the rank of the endomorphism Som_C is equal to one and its nonnull eigenvalue is equal to $\operatorname{Var} \mathrm{M}^C$. The previous intervals and upper bounds can be refined by taking into account the position of the line going through the points M^c and $\mathrm{M}^{c'}$ (and consequently through G) with respect to the principal axes of the cloud.

Let $\mathrm{Cta}_\ell(c,c')$ denote the absolute within–contribution of the dipole $(\mathrm{M}^c, \mathrm{M}^{c'})$ to axis ℓ, and $\mathrm{Cta}_{1\to\ell}(c,c') = \sum\limits_{r=1}^{\ell} \mathrm{Cta}_r(c,c')$ the one of the dipole to the first ℓ principal axes.

Let Ψ_ℓ be the angle between the subspace of the first ℓ axes and the line going through M^c and $\mathrm{M}^{c'}$; one has: $\cos^2\Psi_\ell = \mathrm{Cta}_{1\to\ell}(c,c')/\operatorname{Var}\mathrm{M}^C$.

Property 7.4 (Eigenvalues). *Between the eigenvalue λ_ℓ of the reference cloud and the eigenvalue λ'_ℓ of the within–cloud, one has:*

$$\max\{\lambda_{\ell+1}; \lambda_\ell - \mathrm{Cta}_{1\to\ell}(c,c')\} \le \lambda'_\ell \le \lambda_\ell \quad (1 \le \ell < L)$$

This property follows from Theorem 7.2 (2), p.293.

Property 7.5 (Inertia rates). *Between the inertia rate $\tau_{1\to\ell}$ of the reference cloud and the inertia rate $\tau'_{1\to\ell}$ of the within–cloud pertaining to the first ℓ principal axes, one has the following inequalities:*

$$\frac{\tau_{1\to\ell} - \eta^2\cos^2\Psi_\ell}{1-\eta^2} \le \tau'_{1\to\ell} \le \frac{\tau_{1\to\ell}}{1-\eta^2}$$

Property 7.6 (Principal subspace $1 \to \ell$). *The greatest canonical angle θ between the principal subspace of the cloud M^J and the principal subspace of the within–cloud $\mathrm{M}^{J(C)}$ (generated by the first ℓ principal axes) is such that:*

$$\textit{If } m' = \frac{\lambda_\ell - \lambda_{\ell+1}}{\operatorname{Var}\mathrm{M}^C} > 1 \quad \textit{then } \theta < \pi/4 \textit{ and } \tan 2\theta \le \frac{\sin 2\Psi_\ell}{m' - \cos 2\Psi_\ell}$$

$$\textit{If } \cos^2\Psi_\ell < m' < 1 \qquad \textit{then } \theta < \pi/4 \textit{ and } \tan 2\theta \le \frac{\sin 2\Psi_\ell}{m' - \cos^2\Psi_\ell}$$

This property follows from Theorem 7.5 (2), p.294.

Applications

Culture Example (cf. chapter 5, §5.4, p.221). Consider the partition of the cloud of individuals into two classes $\{c,c'\}$ induced by Gender (cf. p.234). The variance of the between–cloud[2] is found to be 0.0113526.

[2]This variance is equal to the Φ^2 of the table crossing Gender with the 30 active modalities.

We will compare the initial cloud to the within–cloud. For this purpose, we have to compare the eigenvalues and the eigenvectors of the endomorphism *Som* to those of the endomorphism obtained by subtracting the endomorphism Som_C of rank unity.

From Table 5.12 (p.234), one calculates the absolute within–contributions of the dipole (c, c') to the first three axes: $\mathrm{Cta}_1(c, c') = .000007$, $\mathrm{Cta}_2(c, c') = .001295$ and $\mathrm{Cta}_3(c, c') = .000107$.

Hence $\cos^2 \Psi_1 = \frac{.000007}{.0113526} = .000617$; $\cos^2 \Psi_2 = \frac{.000007+.001295}{.0113526} = .114687$; and $\cos^2 \Psi_3 = \frac{.000007+.001295+.000107}{.0113526} = .124113$.

• *Eigenvalues.* Recall (see p.224) the values of the first three eigenvalues of the reference cloud: $\lambda_1 = .29252$, $\lambda_2 = .24149$, $\lambda_3 = .22484$.

From Property 7.4 (p.273), we deduce the *interval of variation of the eigenvalues* of the within–cloud:

$$.29251 \leq \lambda'_1 \leq .29252\ ; \quad .24019 \leq \lambda'_2 \leq .24149\ ; \quad .22213 \leq \lambda'_3 \leq .22484$$

• *Principal subspaces.* From property 7.6 (p.273), we deduce that the angle θ_1 between the *first principal axis* of the reference cloud and the one of the within–cloud is such that $\tan 2\theta_1 \leq .04965/(4.4950 + .99877) = 0.0090$, hence $\theta_1 \leq 0°26$. Similarly, for the *first principal planes,* the greatest canonical angle $\theta_{1\to2}$ is such that $\theta_{1\to2} \leq 7°95$ ($\tan 2\theta_{1\to2} \leq 0.28485$); for the subspaces of the *first three principal axes*[3], one has $\theta_{1\to3} \leq 8°$.

> These results indicate a *quasi–stability of the subspace of the first three principal axes*: in short, for the first three axes, there is practically no difference between genders.

Political space study (cf. chapter 9, §9.2, p.365). Consider the following partition of the 2980 individuals into two classes: class c with the 193 individuals of the NF electorate and class c' with the 2787 others.

Taking again the notation of the specific MCA of a subcloud (*MULTIPLE,* §5.2.2, p.210), let N denote the number of individuals (N = 2980), Q the number of questions (Q = 20) and K' the number of active modalities ($K' = 74$). We denote N_k the total number of individuals (among N = 2980) who have given response k, and n_k the number of those of class c (among $n = 193$) who have given response k.

The between variance is such that:

$$\operatorname{Var} \mathrm{M}^C = \frac{n(N-n)}{N^2}(\mathrm{M}^c\mathrm{M}^{c'})^2 = \frac{1}{Q} \times \frac{1}{n\,N(N-n)} \sum_{k\in K'} \frac{(N\,n_k - n\,N_k)^2}{N_k}$$

We thus find $\operatorname{Var} \mathrm{M}^C = 0.0193456$.

[3]One has $\lambda_4 = .20726$, hence $(\lambda_3 - \lambda_4)/\operatorname{Var} \mathrm{M}^C = 1.54854$ and $\tan 2\theta_{1\to3} \leq 0.28666$.

Let ψ_1 be the angle between the first axis and the line going through M^c and $\mathrm{M}^{c'}$ (hence through G); $\cos^2\psi_1$ is the quality of representation of the mean–point modality NF (cf. Table 9.2.8, p.380). We define similarly ψ_2 (axis 2) and ψ_3 (axis 3).

More precisely, $\cos^2\psi_1 = .88832$, $\cos^2\psi_2 = .04330$, and $\cos^2\psi_3 = .008738$; then for the principal plane 1-2, $\cos^2\Psi_2 = .93162$ and for the principal subspace $1\to 3$, $\cos^2\Psi_3 = .94036$. From properties 7.4 and 7.5 (p. 273), we deduce the intervals of variation for eigenvalues and inertia rates (see Table 7.1).

Axis ℓ	λ_ℓ	Eigenvalues	λ'_ℓ
Axis 1	.21325	$.19606 \leq \cdot \leq .21325$	*.19662*
Axis 2	.16063	$.14261 \leq \cdot \leq .16063$	*.15942*
Axis 3	.11921	$.10102 \leq \cdot \leq .11921$	*.11900*

	$\tau_{1\to\ell}$	Inertia rates	$\tau'_{1\to\ell}$
1	.0785	$.0727 \leq \cdot \leq .0790$	*.0729*
$1\to 2$	.1376	$.1319 \leq \cdot \leq .1386$	*.1319*
$1\to 3$	.1814	$.1759 \leq \cdot \leq .1827$	*.1760*

Table 7.1. Eigenvalues (λ_ℓ) of initial cloud, interval of variation of the eigenvalues (λ'_ℓ) of within–cloud and its eigenvalues (in slanted characters); similarly for inertia rates.

- *First principal axis.* One has $(\lambda_1 - \lambda_2)/\operatorname{Var}\mathrm{M}^C = 2.72$; from Property 7.6, we deduce that the angle θ_1 of the rotation of Axis 1 is such that $\tan 2\theta_1 \leq 0.32415$, that is, $\theta_1 \leq 9°$ (the value θ_1 is found to be 6°18).

- *First principal plane.* $\tan 2\theta_{1\to 2} \leq .050479/(2.141-.86324) = 0.39505$, that is, $\theta_{1\to 2} \leq 10°78$ (the greatest canonical angle between the two planes is found to be 2°33).

- *First three-dimensional principal subspace.* The greatest canonical angle between the two subspaces is such that $\tan 2\theta_{1\to 3} = .047364/(1.995 - .88972) = 0.42515$, that is, $\theta_{1\to 3} \leq 11°52$ (it is found to be 1°91).

> The between–group effect on the rotation of the first three–dimensional principal subspace is very weak .

Remark. In the *Political Space* data, the Ψ_ℓ angles are small, $\cos 2\Psi_\ell$ is positive, therefore the upper bound of the canonical angles between the principal subspaces is larger, and the principal subspaces are less stable than in the *Culture Example,* for which the angle between the line supporting the between–cloud and the principal subspaces is near 90°.

7.1.4 Stability and Grouping of Two Points

If only two points M^j and $\mathrm{M}^{j'}$ are grouped the perturbation of *Som* is of rank unity.

Consider the partition into $C = J - 1$ classes consisting in the class $J\langle c\rangle = \{j, j'\}$ and the $J-2$ singletons $\{j''\}$ ($j'' \notin J\langle c\rangle$). One has (cf. formula (7.2), p.271): $Som = Som_C + Som_{J(C)}$.

For the class $J\langle c'\rangle = \{j''\}$, one has $Som_{c'}(\overrightarrow{\alpha}) = f_{j''}\langle\overrightarrow{\mathrm{GM}}^{j''}|\overrightarrow{\alpha}\rangle\overrightarrow{\mathrm{GM}}^{j''}$ and $Som_{J(c')} = 0$.

For the class $J\langle c\rangle = \{j, j'\}$, one has $Som_c(\overrightarrow{\alpha}) = f_c\langle\overrightarrow{\mathrm{GM}}^c|\overrightarrow{\alpha}\rangle\overrightarrow{\mathrm{GM}}^c$, and $f_c Som_{J(c)} = (f_j\, f_{j'}/f_c)\langle\overrightarrow{\mathrm{M}^j\mathrm{M}^{j''}}\,|\overrightarrow{\alpha}\rangle\overrightarrow{\mathrm{M}^j\mathrm{M}^{j''}}$.

Therefore the perturbation $Som_{J(C)}$ is of rank unity; its nonnull eigenvalue is equal to $\frac{n_j\, n_{j'}}{n\,(n_j+n_{j'})}(\mathrm{M}^j\mathrm{M}^{j'})^2$ (absolute within contribution of dipole) and the associated eigenvector is $\overrightarrow{\mathrm{M}^j\mathrm{M}^{j'}}$. Taking into account the position of the vector $\overrightarrow{\mathrm{M}^j\mathrm{M}^{j'}}$ with respect to the principal subspaces of the cloud M^J, upper bounds can be refined by applying Theorem 7.5 (p.294), as illustrated by the example hereafter.

Application

"Hair & Eye Color" (chapter 2, §2.3.5, p.51). The profiles of *Blue* and *Light Eyes* are quite similar (cf. Table 2.4, p.52), i.e. there is a quasi–distributional equivalence. Let us now study the effect of grouping columns $k1$ *(Blue Eyes)* and $k2$ *(Light Eyes)* on the results of CA.

One has $(\mathrm{M}^{k1}\mathrm{M}^{k2})^2 = 0.01676$, hence the absolute contribution of the dipole $(\mathrm{M}^{k1}, \mathrm{M}^{k2})$ is $\mathrm{Cta}_{(k1,k2)} = .00154$. The angle ψ_1 between the first principal line and the line going through points M^{k1} and M^{k2} is such that $\cos^2\psi_1 = (y_1^{k1} - y_1^{k2})^2/(\mathrm{M}^{k1}\mathrm{M}^{k2})^2$ (for y_1^{k1} and y_1^{k2}, coordinates of points M^{k1} and M^{k2} on Axis 1, see Table 2.7, p.53); one has $\cos^2\psi_1 = .0974$, i.e. $\psi_1 = 71°81$. Similarly, the angle ψ_2 between this line and the second principal line is such that $\cos^2\psi_2 = .3506$, hence the angle Ψ_2 between the line and the plane 1-2 is such that $\cos^2\Psi_2 = .0974 + .3506 = .4480$.

- *Interval of variation of eigenvalues.*
$.19909 = .19924 - .00015 \leq \lambda'_1 \leq .19924 \quad ; \quad .02940 \leq \lambda'_2 \leq .03009$

- *First principal line.* The rotation angle θ_1 of the first principal line is such that $\tan 2\theta_1 \leq \sin 2\psi_1/(\frac{.19924-.03009}{.00154} - \cos 2\psi_1) = .0053$ ($\theta_1 < 0°15$).

- *First principal plane.* For the plane of the first two axes, knowing that $\lambda_2 - \lambda_3 = .02923$, one gets: $\tan 2\theta_{1\to 2} \leq .0521$ ($\theta_{1\to 2} < 1°49$).

Remark. After grouping, the two eigenvalues of the contingency table (5×3) are found to be 0.19910 and .02956, and the first two principal axes are virtually unchanged. There is quasi–stability.

7.2 Influence of a Group of Points (Individuals)

In GDA, questions may be raised about the influence of a group of individuals, for example whether they have characteristics that differ from other individuals, or whether the contribution of the group to one or several axes is strong; in short: What would the outcome of analysis become without this group?

Methodologically, the following cases can occur:

— The axis is about the same, i.e. there is stability. The group of individuals *is not characteristic* of the axis.

— The axis vanishes without modification of the other interpretable axes: Then the axis is *specific* to the group of individuals. This case often occurs in opinion questionnaires, when a "nonrespondent" axis exists: Rather than deleting the "nonrespondent" individuals, one will prefer to put the nonresponse modalities as passive ones (cf. specific analysis, *MULTIPLE*, §5.2.2, p.210).

— A rotation of one of the first axes (or first principal plane or subspaces) occurs; then it is advisable to inquire about the *homogeneity* of data; discarding the group may be in order, especially if it appears isolated from the others.

If the aim is to compare the analysis of a table with that of a subtable, one faces the problem of comparing clouds that lie in different Euclidean spaces; deleting data entails a change of metric in CA, in standard PCA as well as in MCA. Most often, it seems methodologically preferable to fix the metric with respect to a reference cloud; this is the viewpoint of *specific analysis,* that we will adopt in this section. The problem of the change of metric will be dealt with in the next section (p.281).

We will firstly present the study of the deletion of a group of several points, with application to the *Political Space* study (§7.2.1), then we will outline the case of a single point (§7.2.2).

7.2.1 Influence of a Group of several Points (Individuals)

Let $J{<}c{>}$ denote the subset of J indexing the deleted points and $J{<}c'{>}$ the complementary subset; hence a partition of J into two classes c and c' $(C = \{c, c'\})$. We take up the notations of §7.1.3 (p.272).

Studying the influence of a group of points consists in comparing the analysis of the L–dimensional cloud M^J to that of the subcloud $\mathrm{M}^{J<c'>}$.

From the between–within breakdown of Som, one has (cf. p.271):

$$Som = Som_C + Som_{J(C)}, \text{ with } Som_{J(C)} = f_c Som_{J(c)} + f_{c'} Som_{J(c')}$$

Then we have to compare the spectral analysis of Som to that of $Som_{J(c')} = \big((Som - Som_C) - f_c Som_{J(c)}\big)/f_{c'}$.

For this purpose, one considers that the initial cloud is submitted to two perturbations: 1) $-Som_C$ on Som (comparison of $Som_{J(C)}$ to Som); 2) $-f_c Som_{J(c)}/f_{c'}$ on $Som_{J(C)}/f_{c'}$ (comparison of $Som_{J(c')}$ to $Som_{J(C)}/f_{c'}$). The second perturbation will be studied hereafter, the first one was studied earlier (see §7.1.3, p.272-273).

We denote λ_ℓ the eigenvalues of Som (initial cloud M^J), λ'_ℓ those of $Som_{J(c')}$, μ_ℓ those of $Som_{J(c)}$ (subcloud of deleted points) and $\widetilde{\mu}_\ell$ those of $Som_{J(C)}$ (within–cloud).

Property 7.7 (Eigenvalues). *Between the eigenvalue λ_ℓ of the cloud M^J and the eigenvalue λ'_ℓ of the subcloud $\mathrm{M}^{J<c'>}$, one has the inequalities:*

$$\frac{1}{1-f_c}\Big(\lambda_\ell - f_c\mu_1 - \mathrm{Cta}_{1\to\ell}(c,c')\Big) \leq \lambda'_\ell \leq \frac{1}{1-f_c}\Big(\lambda_\ell - f_c\mu_L\Big)$$

Proof. From Property 7.4 (p.273), one deduces for the eigenvalues of $Som_{J(C)}$ that $\max\{\lambda_{\ell+1}; \lambda_\ell - \mathrm{Cta}_{1\to\ell}(c,c')\} \leq \widetilde{\mu}_\ell \leq \lambda_\ell$. Applying Weyl's inequalities (Theorem 7.1 *(2b),* p.292) to $Som_{J(C)} - f_c Som_{J(c)}$, one gets the interval $[\widetilde{\mu}_\ell - f_c\mu_1 \,;\, \widetilde{\mu}_\ell - f_c\mu_L]$, with $\widetilde{\mu}_\ell - f_c\mu_1 \geq \lambda_\ell - \mathrm{Cta}_{1\to\ell}(c,c')\} - f_c\mu_1$ and $\widetilde{\mu}_\ell - f_c\mu_L \leq \lambda_\ell - f_c\mu_L$; hence dividing by $f_{c'} = 1 - f_c$, the interval of variation of the eigenvalues λ'_ℓ of $Som_{J(c')}$. ◁

Property 7.8 (Principal subspaces $1\to\ell$). *The greatest canonical angle θ between the principal subspaces generated by the first ℓ principal axes is such that $\theta \leq \theta' + \theta''$, with:*

$$\textit{If } m' = \frac{\lambda_\ell - \lambda_{\ell+1}}{\mathrm{Var}\,\mathrm{M}^C} > 1 \quad \textit{then } \theta' < \pi/4 \textit{ and } \tan 2\theta' \leq \frac{\sin 2\Psi_\ell}{m' - \cos 2\Psi_\ell}$$

$$\textit{If } \cos^2\Psi_\ell < m' < 1 \qquad \textit{then } \theta' < \pi/4 \textit{ and } \tan 2\theta' \leq \frac{\sin 2\Psi_\ell}{m' - \cos^2\Psi_\ell}$$

and letting $\epsilon = \lambda_\ell - \lambda_{\ell+1} - \mathrm{Cta}_{1\to\ell}(c,c')$

$$\textit{If } f_c(\mu_1 - \mu_L) < \epsilon \quad \textit{then } \theta'' < \pi/4 \textit{ and } \sin 2\theta'' \leq \frac{f_c(\mu_1 - \mu_L)}{\epsilon}$$

Proof. The upper bound for θ' follows from Property 7.6 (p.273) applied to the pertubation of rank unity $-Som_C$ on the endomorphism Som.

The upper bound for angle θ'' follows from Theorem 7.4 (p.294) applied to the perturbation $-f_c Som_{J(c)}$ on $Som_{J(C)}$, since the coefficient $1/f_{c'}$ does not modify the eigenvectors. For this purpose a lower bound for $\widetilde{\mu}_\ell - \widetilde{\mu}_{\ell+1}$ is provided by $(\lambda_\ell - \mathrm{Cta}_{1\to\ell}(c,c')) - \lambda_{\ell+1}$.

Lastly, an upper bound of the angle θ is obtained by applying Lemma 7.2 (p.295). ◁

Application

Political space study. We study hereafter the influence of the subgroup of the 193 individuals of the NF electorate. As already seen, the principal axes of the within–cloud are very close to those of the initial cloud, so in the following study we neglect this perturbation and study the second one.

Let us recall that $\mathrm{Cta}_{1\to\ell}(c,c') = \mathrm{Var}\,\mathrm{M}^C \cos^2\Psi_\ell$, hence $\mathrm{Cta}_1(c,c') = .017185$, $\mathrm{Cta}_{1\to 2}(c,c') = .018023$ and $\mathrm{Cta}_{1\to 3}(c,c') = .018192$. The greatest eigenvalue μ_1 of the specific analysis of the subcloud of the NF electorate is equal to .297185, hence $f_c\mu_1 = .019247$ $(\mu_L \simeq 0)$.

Applying Property 7.7 (p.278), one obtains the following intervals of variation for the eigenvalues of $Som_{J(c')}$.

Axis ℓ	λ_ℓ	Eigenvalues	λ'_ℓ
Axis 1	.21325	$.18906 \leq \cdot \leq .22802$	*.19668*
Axis 2	.16063	$.13190 \leq \cdot \leq .17175$	*.16427*
Axis 3	.11921	$.08743 \leq \cdot \leq .12747$	*.11958*

Table 7.2. Eigenvalues (λ_ℓ) of the initial cloud; interval of variation of the eigenvalues (λ'_ℓ) of the within–cloud and its eigenvalues (in slanted characters).

Let us now compare the first principal axes (cf. Property 7.8, p.278). One has $\epsilon = .035435$, hence $\sin 2\theta''_1 \leq .019247/.035435 = .5432$, that is, $\theta''_1 \leq 16°45$ $(\cos\theta''_1 \geq .959)$. One finds $\cos\theta''_1 = .989$ $(\theta''_1 = 8°32)$.

For plane 1-2 and the subspace $1\to 3$, since the deviations $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_4$ are inferior to $\lambda_1 - \lambda_2$, the inequalities cannot lead to smaller bounds for angles; one finds $\cos\theta''_{1\to 2} \geq .886$ $(\theta''_{1\to 2} \leq 27°67)$, and $\cos\theta''_{1\to 3} \geq .816$ $(\theta''_{1\to 3} \leq 35°33)$. If the analyses are performed, one finds $\cos\theta''_{1\to 2} = .9974$ $(\theta''_{1\to 2} = 4°14)$ and $\cos\theta''_{1\to 3} = .9975$ $(\theta''_{1\to 3} = 4°09)$.

The examination of the preceding results shows that the order of axes is not modified even though the variance of the first axis is much smaller. The rotation of the first axis remains weak.

7.2.2 Deleting a single point

Let us delete the point M^{jo} of absolute frequency n_{jo} and relative frequency $f_{jo} = n_{jo}/n$ and denote $J{<}c'{>} = J - \{jo\}$.

One has $Som = (1 - f_{jo})Som_{J(c')} + 0 + Som_{c'} + Som_{jo}$ (cf. equation (7.2), p.271). Now, one has $Som_{c'} + Som_{jo} = Som_{jo}/(1 - f_{jo})$, hence:

$$Som_{J(c')} = \frac{1}{1 - f_{jo}}\left(Som - \frac{1}{1 - f_{jo}} Som_{jo}\right)$$

$\overrightarrow{\mathrm{GM}}^{jo}$ is an eigenvector of Som_{jo}; the associated eigenvalue is equal to the absolute contribution of M^{jo}, denoted Cta_{jo}. Let Ψ_ℓ be the angle between the vector $\overrightarrow{\mathrm{GM}}^{jo}$ and the subspace of the first ℓ principal axes, and $\mathrm{Cta}_{jo(1\to\ell)} = \mathrm{Cta}_{jo} \cos^2 \Psi_\ell$ be the absolute contribution of M^{jo} to this subspace, between the eigenvalue λ_ℓ of Som and the eigenvalue μ_ℓ of $Som_{J(c')}$, one has (Theorems 7.2 and 7.5, p.293-294):

$$\frac{1}{1 - f_{jo}}\left(\lambda_\ell - \frac{1}{1 - f_{jo}} \mathrm{Cta}_{jo(1\to\ell)}\right) \leq \mu_\ell \leq \frac{1}{1 - f_{jo}} \lambda_\ell$$

The greatest canonical angle between the subspaces $1\to\ell$ is such that:

$$\text{If } m' = \frac{\lambda_\ell - \lambda_{\ell+1}}{\frac{1}{1-f_{jo}} \mathrm{Cta}_{jo}} > 1, \text{ then: } \theta < \pi/4 \quad \text{and} \quad \tan 2\theta \leq \frac{\sin 2\Psi_\ell}{m' - \cos 2\Psi_\ell}$$

$$\text{If } \cos^2 \Psi_\ell < m' < 1, \qquad \text{then: } \theta < \pi/4 \quad \text{and} \quad \tan 2\theta \leq \frac{\sin 2\Psi_\ell}{m' - \cos^2 \Psi_\ell}$$

Remarks (1) The problem of adding points is analogous to the one of deleting points.

(2) The above formulas, in terms of eigenvalues of initial cloud, contributions and qualities of representation of deleted points, are general, they apply whether the clouds have been constructed by PCA or MCA.

(3) Modifying the weights of some points of a cloud may be viewed as a particular case of deleting or adding points. Indeed, a point M^j of weight n_j is equivalent to n_j elementary points at the same location, and modifying its weight by putting it to n'_j amounts to deleting $n_j - n'_j$ elementary points if $n_j > n'_j$, or adding $n'_j - n_j$ points if $n'_j > n_j$.

Comment

The influence of a point depends both on its contribution to the subspace under study and on its quality of representation for this subspace. If the point lies in the subspace orthogonal to the subspace under study, the stability is complete; if the point lies in the subspace itself, there is stability if its contribution is not too large.

7.3 Effect of a Change of Metric

The stability of principal subspaces under a change of metric appears in biweighted PCA with the choice of weights of variables and/or individuals. It appears in CA when the influence of modalities is studied by comparing a table to a subtable (cf. Escofier & Le Roux, 1976); when such a study is done in the line of specific GDA, the metric remains that of the reference cloud (cf. below §7.4, p.283).

The problem of the change of metric can be formulated in the following terms. If the space $\mathcal{U}$ is equipped with two Euclidean metrics: one defined by the scalar product denoted $\langle\cdot|\cdot\rangle$ (with the associated norm $\|\cdot\|$), the other by the scalar product denoted $[\cdot|\cdot]$ (with the associated norm $[[\cdot|\cdot]]$).

It is known (e.g. Deheuvels, 1981, p.191-194) that there exists an endomorphism $\langle\cdot|\cdot\rangle$–symmetric and positive $Ech : \mathcal{V} \to \mathcal{V}$, such that:

$$\forall \overrightarrow{\alpha} \in \mathcal{V} \quad \forall \overrightarrow{\alpha}' \in \mathcal{V} : \langle Ech(\overrightarrow{\alpha})|\overrightarrow{\alpha}'\rangle = \langle \overrightarrow{\alpha}|Ech(\overrightarrow{\alpha}')\rangle = [\overrightarrow{\alpha}\,|\,\overrightarrow{\alpha}'] \qquad \textbf{(7.3)}$$

Let $(\overrightarrow{\delta_k})_{k\in K}$ be a $\langle\cdot|\cdot\rangle$–orthogonal basis of $\mathcal{V}$ constituted of eigenvectors of *Ech,* i.e. such that $\forall k \in K \; : \; Ech(\overrightarrow{\delta_k}) = e^k\overrightarrow{\delta_k}$ $(e^k > 0)$.

In the basis $(\overrightarrow{\delta_k})_{k\in K}$, the change of metric is a mere change of scale. If $\|\overrightarrow{\delta_k}\| = \sqrt{\varpi_k}$ and $[[\overrightarrow{\delta_k}]] = \sqrt{\varpi'_k}$, then the eigenvalue e^k of *Ech* is equal to ϖ'_k/ϖ_k. In practice, this is the form under which the change of metric is made explicit.

For the Euclidean structure $\langle\cdot|\cdot\rangle$, the eigenvalue λ_ℓ and the principal axis $\overrightarrow{\alpha}_\ell$ of the cloud (M^J, f_J) verify the equation $Som(\overrightarrow{\alpha_\ell}) = \lambda_\ell\overrightarrow{\alpha_\ell}$ (cf. Theorem 3.3, p.88, where Som is the $\langle\cdot|\cdot\rangle$–symmetric endomorphism defined by $Som(\overrightarrow{\alpha}) = \sum\limits_{j\in J} f_j\langle\overrightarrow{\mathrm{GM}}^j|\overrightarrow{\alpha}\rangle\overrightarrow{\mathrm{GM}}^j$.

For the Euclidean structure $[\cdot|\cdot]$, the eigenvalue λ'_ℓ and the eigenvector $\overrightarrow{\alpha}'_\ell$ verify the equation $Som'(\overrightarrow{\alpha'_\ell}) = \lambda'_\ell\overrightarrow{\alpha}'_\ell$, where Som' is the $[\cdot|\cdot]$–symmetric endomorphism defined by $Som'(\overrightarrow{\alpha}) = \sum\limits_{j\in J} f_j[\overrightarrow{\mathrm{GM}}^j|\overrightarrow{\alpha}]\overrightarrow{\mathrm{GM}}^j$.

From Equation 7.3, one has: $[\overrightarrow{\mathrm{GM}}^j|\overrightarrow{\alpha}] = \langle\overrightarrow{\mathrm{GM}}^j|Ech(\overrightarrow{\alpha})\rangle$; hence one has $Som' = Som \circ Ech$.

Property 7.9 (Eigenvalues). *Let* $e_{max} = \max\limits_{k\in K} e^k$ *and* $e_{min} = \min\limits_{k\in K} e^k$. *One has:* $e_{min}\lambda_\ell \leq \lambda'_\ell \leq e_{max}\lambda_\ell$.

This property follows from Theorem 7.3 (p.293).

Property 7.10 (Principal Variables). *If* $\epsilon = \min\{\frac{\lambda_{\ell-1}}{\lambda_\ell} - 1), (1 - \frac{\lambda_{\ell+1}}{\lambda_\ell)}\}$, *the correlation* r_ℓ *between the* ℓ*–th principal variables, is such that:*

If $(e_{max}/e_{min} - 1)/\epsilon < 1$ *then* $\theta' < \pi/4$ *and* $|r_\ell| \geq \sqrt{\frac{e_{min}}{e_{max}}} \cos\theta'$ *with* θ' *such that* $\sin 2\theta' \leq (e_{max}/e_{min} - 1)/\epsilon$.

Proof. The ℓ–th principal variable y^J_ℓ of the cloud M^J with $\langle\cdot|\cdot\rangle$ metric is defined by $\left(\langle\overrightarrow{\mathrm{GM}}^j|\overrightarrow{\alpha_\ell}\rangle/\|\overrightarrow{\alpha_\ell}\|\right)_{j\in J} = Vac(\overrightarrow{\alpha_\ell}/\|\overrightarrow{\alpha_\ell}\|)$ and its variance is equal to λ_ℓ (cf. *CLOUD*, passage formulas (3.7), p.92). Similarly, the ℓ–th principal variable y'^J_ℓ of the cloud M^J with $[\cdot|\cdot]$ metric is defined by $\left([\overrightarrow{\mathrm{GM}}^j|\overrightarrow{\alpha}'_\ell]/[[\overrightarrow{\alpha}'_\ell]]\right)_{j\in J} = Vac'(\overrightarrow{\alpha}'_\ell/[[\overrightarrow{\alpha}'_\ell]]) = Vac\circ Ech(\overrightarrow{\alpha}'_\ell/[[\overrightarrow{\alpha}'_\ell]])$ and its variance is equal to λ'_ℓ. The correlation r_ℓ between the principal variables y^J_ℓ and y'^J_ℓ is such that:

$$r_\ell = \frac{1}{\sqrt{\lambda_\ell\lambda'_\ell}} \times \frac{\langle Vac(\overrightarrow{\alpha_\ell}) \mid Vac\circ Ech(\overrightarrow{\alpha}'_\ell)\rangle}{\|\overrightarrow{\alpha_\ell}\| \times [[\overrightarrow{\alpha}'_\ell]]} = \frac{1}{\sqrt{\lambda_\ell\lambda'_\ell}} \times \frac{\langle Som(\overrightarrow{\alpha_\ell}) \mid Ech(\overrightarrow{\alpha}'_\ell)\rangle}{\|\overrightarrow{\alpha_\ell}\| \times [[\overrightarrow{\alpha}'_\ell]]}$$

If $\theta^\star$ denotes the angle (for the $\langle\cdot|\cdot\rangle$ metric) between the principal vectors $\overrightarrow{\alpha}_\ell$ and $\overrightarrow{\alpha}'_\ell$, one has $|r_\ell| = \sqrt{\frac{\lambda_\ell}{\lambda'_\ell}} \times \frac{[[\overrightarrow{\alpha}'_\ell]]}{\|Ech(\overrightarrow{\alpha}'_\ell)\|}|\cos\theta^*| \geq \sqrt{\frac{e_{min}}{e_{max}}}|\cos\theta^\star|$.

From Theorem 7.8 (p.295) (applied to the transposes of the endomorphisms Som and Som'), the angle $\theta' = \min\{\theta^\star, \pi - \theta^\star\}$ is such that if $(e_{max}/e_{min}) - 1 < \epsilon$ then $\theta' < \pi/4$ and $\sin 2\theta' \leq (e_{max}/e_{min} - 1)/\epsilon$. Therefore, one deduces: $|r_\ell| \geq \sqrt{\frac{e_{min}}{e_{max}}} \cos\theta'$. ◁

Particular cases

• *Biweighted* PCA. Suppose that the reference cloud is equipped with the metric defined by ϖ_K, and that the modified metric is defined by ϖ'_K; for a score table (protocol of scores, *PRINCIPAL*, §4.1, p.132), one has $e^k = \varpi'_k/\varpi_k$, and for a table of measures (*PRINCIPAL*, §4.5, p.155), one has $e^k = \varpi_k/\varpi'_k$. These properties also show that if the variances of the initial variables are near one another, then the principal subspaces will also be near one another and the eigenvalues will be almost proportional.

• *Standard* PCA. If, as done in the preceding §, one deletes a group of individuals $J{<}c{>}$, the data can be calibrated by $\varpi'_k = \operatorname{Var} x^{J<c'>k}$ instead of $\operatorname{Var} x^{Jk}$. The study done with this new calibration (contrary to specific analysis) induces a transformation of data that may be viewed as a change of metric, with $e^k = \operatorname{Var} x^{Jk} / \operatorname{Var} x^{J<c>k}$.

• *Correspondence Analysis.* Let n_{JK} be the frequency–measure on $J \times K$ with $n_k = \sum_{j \in J} n_{jk}$ and $n = \sum_{k \in K} n_k$. If one deletes a subset $J{<}c'{>}$ of modalities and compares the CA of the initial table to the CA of the modified table (cf. Escofier & Le Roux, 1976), there is a change of metric.

Letting $n_{c'k} = \sum_{j \in J<c'>} n_{jk}$ and $n_{c'} = \sum_{k \in K} n_{c'k}$, the change of metric in $\mathbb{R}_K$ is defined by

$$e^k = \frac{n_k/n}{(n_k - n_{c'k})/(n-n_{c'})} = \frac{n-n_{c'}}{n} \times \frac{n_k}{n_k - n_{c'k}}$$

Comment

If e_{max}/e_{min} is near 1, the perturbation due to the change of metric is small. More precisely, one can say that there will exist some subspaces of principal variables that are stable under the change of metric if there exists, in the ranking of eigenvalues, a cutoff corresponding to a ratio markedly greater than e_{max}/e_{min}.

7.4 Influence of a Variable and/or a Modality

In PCA, one may discard one or several variables from the analysis; in CA or MCA, one may discard one or several modalities while preserving the same Euclidean structure. If after discarding, one is interested in the stability of an axis (or a principal subspace), one of the following cases may occur.

— The axis of interest remains about the same, i.e. one has stability of axis. The discarded elements do not have a great influence on the determination of axis; which does not mean that they can be neglected in the interpretation of the axis, especially if their quality of representation is good. The axis expresses an *overall trend of data,* such as a general axis as opposed to a specific one.

— The new axis is different but the plane spanned by this axis and the one of next rank is stable (e.g. if the two corresponding eigenvalues are close to each other). If there is only a rotation in the plane, one has *stability of the plane,* to be considered globally.

— The axis under study is found again among interpretable axes but at a more remote rank. In this case the interpretation should take into account the *hierarchy of axes.*

— The axis vanishes without modification of the other interpretable axes: one has a *specific axis* of the discarded variables or modalities.

— There is a perturbation of interpretable axes; this means that the discarded elements play an important role; the question of their *homogeneity* with the other elements should be raised.

All the foregoing cases refer to the stability of a Euclidean cloud of $\mathcal{U}$ by orthogonal projection onto a subspace $\mathcal{A} \subset \mathcal{U}$ ($\mathcal{A}$ strictly included in $\mathcal{U}$) going through the mean point G (cf. Specific analysis, CLOUD, §3.3.4, p.94).

We firstly recall some results and properties relating to the projection of a cloud (§7.4.1). We secondly deal with a perturbation cloud of any dimensionality (§7.4.2), then one–dimensional (§7.4.3); in each case we give formulas for PCA, CA and MCA. Lastly as an application, we study the influence of variables in the standard PCA of the *Parkinson study.*

7.4.1 Projection of a cloud

Let A^j be the orthogonal projection of the point M^j on $\mathcal{A}$, and B^j its orthogonal projection on $\mathcal{A}^\perp$ (supplementary orthogonal subspace of $\mathcal{A}$ going through G), one has: $\overrightarrow{\mathrm{GM}}^j = \overrightarrow{\mathrm{GA}}^j + \overrightarrow{\mathrm{GB}}^j$ with $\overrightarrow{\mathrm{GA}}^j \perp \overrightarrow{\mathrm{GB}}^j$.

Between the variances of clouds one has the following additive relation:

$$\mathrm{Var}\,\mathrm{M}^J = \mathrm{Var}\,\mathrm{A}^J + \mathrm{Var}\,\mathrm{B}^J \tag{7.4}$$

The endomorphism $\mathit{Tom} : \mathbb{R}^J \to \mathbb{R}^J$ associated with the cloud M^J is such that $x^J \mapsto \left(\sum_{j' \in J} f_{j'} x^{j'} \langle \overrightarrow{\mathrm{GM}}^j | \overrightarrow{\mathrm{GM}}^{j'} \rangle\right)_{j \in J}$ (cf. CLOUD, Definition 3.14, p.86); the principal variables of the cloud M^J are eigenvectors of Tom.

Let $\mathcal{X}_0$ be the subspace of $\mathbb{R}^J$ generated by the L nonnull principal variables of the cloud M^J; the endomorphism Tom restricted to $\mathcal{X}_0$ will also be denoted Tom. Similarly, with the cloud A^J there is associated the endomorphism $\mathit{Tom}_A : \mathcal{X}_0 \to \mathcal{X}_0$; and with the perturbation (or residual) cloud B^J there is associated the endomorphism $\mathit{Tom}_B : \mathcal{X}_0 \to \mathcal{X}_0$. Owing to the orthogonality of the supports of the two clouds A^J and B^J, one has:

$$\mathit{Tom} = \mathit{Tom}_A + \mathit{Tom}_B \tag{7.5}$$

To evaluate the influence of the perturbation, one will study the interval of variation of eigenvalues and the rotation of invariant subspaces of the endomorphism Tom (reference cloud) by subtracting endomorphism Tom_B.

7.4.2 Perturbation cloud of any dimensionality

Let L' $(1 \le L' < L)$ denote the dimensionality of the perturbation cloud B^J; and let $(\beta_\ell)_{\ell=1\ldots L}$ denote the eigenvalues of Tom_B (ranked in decreasing order), with $\beta_\ell = 0$ for $\ell > L'$. The variance of the cloud B^J is equal to $\sum_{\ell=1}^{L'} \beta_\ell$; let us denote $\tau_B = \mathrm{Var}\,\mathrm{B}^J / \mathrm{Var}\,\mathrm{M}^J$ the relative contribution of the cloud B^J to the cloud M^J, and $v_{1\to\ell} = \sum_{\ell'=1}^{\ell} \beta_{\ell'} / \operatorname{Var} \mathrm{B}^J$ the inertia rate of the cloud B^J pertaining to its first ℓ axes.

Property 7.11 (Eigenvalues). *Between the eigenvalue λ_ℓ of the cloud M^J and the eigenvalue λ'_ℓ of the cloud A^J, one has the inequalities:*

$$\lambda_\ell - \beta_1 \leq \lambda'_\ell \leq \lambda_\ell \qquad (1 \leq \ell \leq L)$$

This property follows from Weyl's inequalities (cf. Theorem 7.1, p.292).

Property 7.12 (Inertia rates). *Between the inertia rates $\tau_{1\to\ell}$ of the cloud M^J and $\tau'_{1\to\ell}$ of the cloud A^J pertaining to the first ℓ principal axes, one has the following inequalities:*

$$\frac{\tau_{1\to\ell} - \tau_B\, v_{1\to\ell}}{1-\tau_B} \leq \tau'_{1\to\ell} \leq \frac{\tau_{1\to\ell} - \tau_B\, v_{L-\ell'+1\to L}}{1-\tau_B} \left(\leq \frac{\tau_{1\to\ell}}{1-\tau_B} \right)$$

One obtains these inequalities by dividing the relation *(2c)* of Theorem 7.1 (p.292) by $\operatorname{Var} \mathrm{A}^J = \operatorname{Var} \mathrm{M}^J (1-\tau_B)$.

Property 7.13 (Principal subspace $1\to\ell$). *The greatest canonical angle θ between the subspaces generated by the first ℓ principal variables of the two clouds M^J and A^J is such that:*

$$\textit{If } \epsilon = \lambda_\ell - \lambda_{\ell+1} > \beta_1 \textit{ then } \theta < \pi/4 \textit{ and } \sin 2\theta \leq \beta_1/\epsilon$$

This property follows from Theorem 7.4 (p.294).

Particular cases

Biweighted PCA. When in the PCA of a table of scores or of measures one discards a subset K_s of variables, the variance of the cloud M^B is equal to $\sum_{k\in K_s} \mathrm{Cta}_k$, where Cta_k is the absolute contribution of variable k to the variance of the cloud (cf. p.134 and p.155), and β_1 is the greatest eigenvalue of the PCA of the subtable of K_s discarded variables; as a first approximation, one can always majorize β_1 by $\sum_{k\in K_s} \mathrm{Cta}_k$.

Specific MCA. When in specific MCA, one discards a set K_s of modalities of active questions, one has $\operatorname{Var} \mathrm{B}^J = \sum_{k\in K_s} \mathrm{Cta}_k = (K_s - \sum_{k\in K_s} f_k)/Q$, where Q denotes the number of active questions (cf. *MULTIPLE*, property 5.25, p.206). If the K_s discarded modalities are all those of a same question (i.e. the question itself is discarded), one has $\beta_1 = 1/Q$, and the bounds only involve the number Q of questions: This result is uninteresting, it will be improved upon for a question with two modalities.

Correspondence Analysis. When modalities are discarded in CA, the preceding results apply in the framework of specific analysis, that is, assuming the metric is unchanged; in fact, the metric is usually modified (since the margins change), hence a further perturbation (for a detailed study, see Escofier & Le Roux, 1976).

Comments

The eigenvalues of the specific cloud are less than or equal to those of the reference cloud, but one cannot say anything about the increasing or decreasing of inertia rates.

The rotation of the principal subspaces $1 \to \ell$ is all the weaker as the ratio of the largest eigenvalue to the deviation between the eigenvalues ℓ and $\ell+1$ of the reference cloud is smaller. More precisely, one can state that there will exist subspaces of principal variables that are stable by deletion of variables or modalities only if there exists, in the ordering of eigenvalues, a deviation that is markedly superior to the variance of the discarded variables or modalities.

The above bounds, calculated only as functions of the eigenvalues of the residual cloud, are optimal. Still, they can be improved upon in the case of a one–dimensional residual cloud, when the correlation of its unique principal variable with the principal variables of the reference cloud is known.

7.4.3 One–dimensional Perturbation Cloud

The one–dimensional cloud B^J has a single nonnull principal variable, denoted b^J, which is eigenvector of the endomorphism Tom_B associated with the eigenvalue $\beta = \mathrm{Var}\,\mathrm{B}^J$; one then has $\tau_B = \beta / \mathrm{Var}\,\mathrm{M}^J$. Let r_ℓ denote the correlation coefficient of b^J with the ℓ–th principal variable of the cloud M^J. Let $r_{1\to\ell}$ be the multiple correlation of b^J with the first ℓ principal variables of cloud M^J, that is, $r^2_{1\to\ell} = \sum\limits_{\ell'=1}^{\ell} r^2_{\ell'}$, and $r_{1\to 1} = |r_1|$.

Property 7.14 (Eigenvalues). *Between the eigenvalue λ_ℓ of the cloud M^J and the eigenvalue λ'_ℓ of the cloud A^J, one has the inequalities:*

$$\max\{\lambda_{\ell+1}\,;\,\lambda_\ell - \beta r^2_{1\to\ell}\} \le \lambda'_\ell \le \lambda_\ell$$

This property follows from the first inequality *(2)* of Theorem 7.2 (p.293).

Property 7.15 (Inertia rates). *Between the inertia rates $\tau_{1\to\ell}$ of cloud M^J and $\tau'_{1\to\ell}$ of cloud A^J relating to the subspaces of the first ℓ axes, one* *has the inequalities:*

$$\frac{\tau_{1\to\ell} - \tau_B r^2_{1\to\ell}}{1-\tau_B} \leq \tau'_{1\to\ell} \leq \min\{\frac{\tau_{1\to\ell}}{1-\tau_B}; 1\}$$

Property 7.16 (Principal subspaces $1\to\ell$). *The greatest canonical angle θ between the subspaces of the first ℓ principal variables of cloud M^J and of cloud A^J is such that, letting $\epsilon = \lambda_\ell - \lambda_{\ell+1}$:*

$$\textit{If} \quad \epsilon > \beta, \quad \textit{then } \theta < \pi/4 \textit{ and } \tan 2\theta \leq \beta\, \frac{2r_{1\to\ell}(1-r^2_{1\to\ell})^{1/2}}{\epsilon+\beta-2\beta r^2_{1\to\ell}}$$

$$\textit{If } \beta r^2_{1\to\ell} \leq \epsilon < \beta, \textit{ then } \theta < \pi/4 \textit{ and } \tan 2\theta \leq \beta\, \frac{2r_{1\to\ell}(1-r^2_{1\to\ell})^{1/2}}{\epsilon-\beta r^2_{1\to\ell}}$$

This property follows from Theorem 7.5 (p.294).

Similarly, one studies the rotation of the ℓ–th principal variable, by applying Theorem 7.6.

When the subspace of the ℓ–th and $(\ell+1)$–th principal variables is stable, one can also study the rotation of these two principal variables in this subspace, using Lemma 7.1 (p.294). The formulas for standard PCA are given hereafter and applied later on (§7.4.3).

Particular cases

Biweighted PCA. The foregoing results apply to the study of the influence of a variable x^{Jk} of the biweighted PCA of a table of scores x^{JK} (*PRINCIPAL*, §4.1). In this case, one has $\beta = \varpi_k \operatorname{Var} x^{Jk}$, and r_ℓ is the correlation of the variable x^{Jk} with the ℓ–th principal variable. For the biweighted PCA of a table of measures x^J_K (*PRINCIPAL*, §4.5), one has $\beta = \operatorname{Var} x^J_k / \varpi_k$.

Standard PCA. For the standard PCA of the table of variables $(x^{Jk})_{k\in K}$, one has $\beta = 1$ and the formulas simplify.

- *Eigenvalues.* $\max\{\lambda_{\ell+1}\,;\, \lambda_\ell - r^2_{1\to\ell}\} \leq \lambda'_\ell \leq \lambda_\ell$
- *Variance rates.* $\frac{K}{K-1}(\tau_{1\to\ell} - r^2_{1\to\ell}/K) \leq \tau'_{1\to\ell} \leq \min\{\frac{K}{K-1}\,\tau_{1\to\ell}; 1\}$
- *First principal subspace $1\to\ell$.*

$$\text{If} \quad \epsilon > 1, \quad \text{then } \theta < \pi/4 \text{ and } \tan 2\theta \leq \frac{2r_{1\to\ell}(1-r^2_{1\to\ell})^{1/2}}{\epsilon+1-2r^2_{1\to\ell}}$$

$$\text{If } r^2_{1\to\ell} \leq \epsilon \leq 1, \text{ then } \theta < \pi/4 \text{ and } \tan 2\theta \leq \frac{2r_{1\to\ell}(1-r^2_{1\to\ell})^{1/2}}{\epsilon-r^2_{1\to\ell}}$$

- *Principal variable* ℓ. If $\epsilon \leq \min\{\lambda_{\ell-1}, \lambda_\ell - r^2_{1\to\ell-1}\,;\, \lambda_\ell, \lambda_{\ell+1} - r^2_{1\to\ell}\}$, one has: $\sin\theta \leq |r_\ell|(1-r_\ell^2)^{1/2}/\epsilon$.

  This property follows from Theorem 7.6 (p.295).

- *Principal subspace* $\ell\to\ell+1$. If the subspace of the ℓ–th and $(\ell+1)$–th variables is stable, the rotation in the plane due to the deletion of an initial variable is such that:

$$\tan 2\theta = \frac{2|r'_\ell|(1-r'^2_\ell)^{1/2}}{\lambda_\ell - \lambda_{\ell+1} - 2r'^2_\ell + 1} \quad \text{with } r'_\ell = r_\ell/\sqrt{r_\ell^2 + r_{\ell+1}^2}$$

Specific MCA. To study the *influence of a modality* on the results of a specific MCA[4], one replaces in preceding formulas β by $\mathrm{Cta}_k = (1 - f_k)/Q$, and r_ℓ^2 by the quality of representation of this modality on axis ℓ: $\cos^2\psi_\ell = f_k(y_\ell^k)^2/(1-f_k)$, where y_ℓ^k denotes the coordinate of modality k on axis ℓ; and one lets $\cos^2\Psi_\ell = \sum_{\ell'=1}^{\ell}\cos^2\psi_{\ell'}$.

For a *question with two modalities,* the perturbation cloud is one–dimensional ("lever principle"), and the endomorphism Tom_B is of rank one, with eigenvalue $\beta = 1/Q$, hence:

$$\lambda_\ell - \cos^2\Psi_\ell/Q \leq \lambda'_\ell \leq \lambda_\ell$$

$$\text{If } Q\epsilon > 1 \text{ then } \theta < \pi/4 \text{ and } \tan 2\theta \leq \frac{\sin 2\Psi_\ell}{Q\epsilon - \cos 2\Psi_\ell}$$

Comments

In MCA, the influence of a question with two modalities only depends on the quality of representation $(\cos^2\Psi_\ell)$ of its modalities on the subspace under study, while the influence of a modality depends on the remoteness of this modality from the center of the cloud and on its weight; these facts provide response elements to the problems raised by infrequent modalities. If the modality lies at the periphery of the graph and if its quality of representation is very good (Ψ_ℓ close to 0), discarding the modality changes little the results of the analysis; the same property holds if the modality is poorly represented (Ψ_ℓ near $\pi/2$); if the quality of representation is intermediate, to assess its influence, it may be helpful to study the upper bounds of θ.

[4]For a detailed study, see Le Roux (1999).

Another interesting case in MCA is that of a specific axis, characterized by a set of well–represented modalities (quasi–one–dimensional cloud), which vanishes (among interpretable axes) after deleting these modalities. The study presented here allows to identify such axes, that is, if the elements are very well represented (qualities of representation near 1, or even one–dimensional cloud), one will apply the upper bounds of $\tan 2\theta$ to identify such an axis. Using Theorem 7.7 (p.295) will indicate whether this axis has been shifted to a more remote rank. This case is encountered in opinion questionnaires when among interpretable axes there is a non–response axis; as an example see Bonnet, Le Roux & Lemaine (1996).

Application to Parkinson Data

We now apply the foregoing results to the *Parkinson study* (cf. CASE STUDIES, §9.1, p.336, in particular to the results of §9.1.2) to study the *influence of each initial variable* on the first principal variable and on the subspace of the first two principal variables.

The deviation between the first two eigenvalues (cf. Table 9.1.6-a, p.343) is equal to $\epsilon_1 = 3.99269 - 1.82238 = 2.17031$; the deviation between the second and the third ones is $\epsilon_2 = 1.82238 - 0.17105 = 1.65133.$ Since both deviations are greater than 1, we can assert that the rotation angle is less than $\pi/4,$ which already expresses some conclusion of stability.

For each initial variable, one has $\beta_1 - \beta_L = 1,$ therefore (Property 7.13, p.285), for the first principal variable, $\sin 2\theta_1 \leq 1/\epsilon_1 = 0.4608,$ that is, $\cos\theta_1 \geq 0.971$ $(\theta_1 \leq 13°72)$; and for the subspace of the first two principal variables, $\sin 2\theta_{1\to 2} \leq 0.60557,$ that is, $\cos\theta_{1\to 2} \geq 0.9476$ $(\theta_{1\to 2} \leq 18°64)$[5]. One is thus sure of a good stability of principal axes when some initial variable is deleted.

In order to study more precisely the *influence* of each initial variable, we take into account their correlation coefficients with the first two principal variables (cf. Table 9.1.6-b, p.343), the results are summarized in Table 7.3 (p.290). Figure 7.1 (p.290) gives the bound of the correlation $(\cos\theta_1)$ between the first two variables (solid line) expressed as a function of the absolute value of correlation coefficient r_1 and the bound of the smallest canonical correlation $(\cos\theta_{1\to 2})$ between the first principal planes expressed as a function of the multiple correlation $r_{1\to 2}$ (dotted line).

Comments. There is a great stability of the first principal variable and mostly of the first principal subspace; the multiple correlations of the initial

[5]These values correspond to the minima of the curves of Figure 7.1, p.290.

	First princ. var. ($\tau_1 = .665$)			Plane 1-2 ($\tau_{1\to2} = .969$)		
	r_1	τ_1'	$\cos\theta_1$	$r_{1\to2}$	$\tau_{1\to2}'$	$\cos\theta_{1\to2}$
Velocity	.902	[.636; .800] *.653*	.973 *.976*	.983	[.970; 1] *.973*	.973 *.996*
Length	.665	[.710; .800] *.730*	.979 *.980*	.984	[.969; 1] *.974*	.974 *.990*
Swing	−.403	[.776; .800] *.775*	.992 *.992*	.976	[.972; 1] *.981*	.967 *.976*
Cycle	−.868	[.625; .800] *.669*	.972 *.972*	.999	[.963; 1] *.964*	.998 *1.*
Stance	−.958	[.615; .800] *.625*	.981 *.984*	.991	[.966; 1] *.968*	.984 *.999*
Db_Support	−.950	[.618; .800] *.627*	.979 *.987*	.973	[.974; 1] *.977*	.965 *.996*

First principal variable: correlation with each initial variable (r_1), interval of variation of the rates of inertia (τ_1'), lower bound of the correlation $(\cos\theta_1)$ with the true value (in slanted characters).
Principal subspace 1-2: multiple correlation of each initial variable $(r_{1\to2})$, interval of variation of inertia rates $(\tau_{1\to2}')$, and lower bound of the minimal canonical correlation $(\cos\theta_{1\to2})$ with the true value (in slanted characters).

Table 7.3. ***Parkinson data*****: Influence of initial variables.**

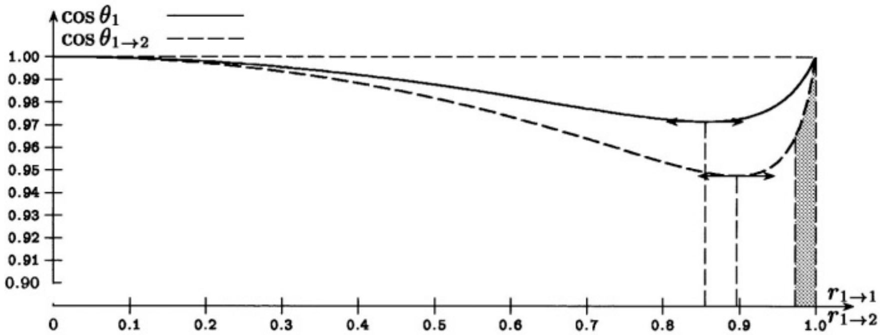

Figure 7.1. ***Parkinson data.*** **Curves giving $\cos\theta$ as a function of $r_{1\to1} = |r_{\ell1}|$ for the first principal variable (solid line) and as a function of $r_{1\to2}$ for the first principal plane (dotted line).**

variables with the first two principal variables are all greater than .973; see gray zone in Figure 7.1.

For the *first principal variable,* deleting one variable among the four variables that are most correlated (in absolute value) with the first principal one is liable to entail a decrease of variance rates, whereas deleting the *Length* or the *Swing* variables may only increase the variance rate. The variables *Cycle* and *Velocity* entail the strongest (though moderate) rotation; they are the most influential on the first principal variable.

As for the *principal subspace 1-2,* deleting *Cycle* or *Stance* variables may entail a decrease of inertia rates, while leaving plane 1-2 stable. Yet, *Swing* remains the most influential variable; its deleting entails the weakest canonical correlation (.967), then comes variable *Length* (.974).

To complement the stability studies, the six corresponding PCA have been performed; their results (in slanted characters in Table 7.3, p.290) are in good agreement with the results of the stability studies.

Further complements. To show how any principal variable can be studied, we hereafter study the second principal variable. From the properties of p.288, we deduce the results of Table 7.4. One notices the influence of *Length* on axis 2.

	ϵ	$\sin\theta$	$\cos\theta$
Velocity	0.686	.523	.852 (*.975*)
Length	0.683	.731	.682 (*.972*)
Swing	0.698	.583	.813 (*.969*)
Cycle	0.653	.658	.753 (*.972*)
Stance	0.669	.370	.929 (*.984*)
Db_Support	0.704	.292	.957 (*.989*)

Table 7.4. ***Parkinson data*****: stability of the second principal variable**

Then, assuming the stability of the subspace of the first two principal variables — in other words, studying the data reconstitution of order 2 — one gets the results of Table 7.5 (cf. formula p.288). One notices that in the subspace of the first two principal variables, *Cycle,* then *Velocity,* have the greatest influences.

	$\lambda'1$	$\frac{\lambda_1-\lambda_2}{r^2_{1\to 2}}$	$r'_1 = \frac{r_1}{r_{1\to 2}}$	$\cos\theta$
Velocity	3.256	2.247	+.918	.976
Length	3.649	2.241	+.676	.980
Swing	3.876	2.277	-.413	.992
Cycle	3.343	2.174	-.869	.972
Stance	3.120	2.209	-.966	.984
Db_Support	3.119	0.704	-.977	.989

Table 7.5. ***Parkinson data*****: rotation in the subspace of the first two principal variables.**

7.5 Rotation of Invariant Subspaces: Basic Theorems

The study of the *eigenvalues* of the sum and product of symmetric endomorphisms has given rise to many publications such as Wielandt (1955), Anderson & Dasgupta (1963), Markus (1964), Wilkinson (1965), Thomson & Freede (1971). The classical results are recalled hereafter (Theorems 7.1 and 7.3).

For the addition of a perturbation B of rank unity, we will give more precise bounds expressed as functions of the position of the invariant subspace of B associated with the nonnull eigenvalue with respect to the invariant subspaces of C or A (Theorem 7.2, p.293).

On the other hand, there are few publications dealing with the study of the *rotation of invariant subspaces,* with the exception of the fundamental paper by Davis & Kahan (1970). These authors deal with the rotation of invariant subspaces of the sum of bounded operators ($C = A + B$) of a Hilbert space, and prove four theorems that give upper bounds for the rotation expressed as functions of the eigenvalues of A and/or C and of the norm of perturbation; the study being made for any norm of operators that is invariant by unitary transformations. We make use of two of these theorems, named by the authors the "$\sin\theta$ theorem" and "$\sin 2\theta$ theorem" (op. cit. p.10-11), to prove Theorems 7.4 (p.294) and 7.6 (p.295).

7.5.1 Eigenvalues

Let $\mathcal{V}$ be an n–dimensional Euclidean vector space. Let a_ℓ, b_ℓ and c_ℓ $(1 \leq \ell \leq n)$ be the eigenvalues (ranked in decreasing order, with their multiplicity order) of endomorphisms A, B and C. Let $(\alpha_\ell)_{\ell=1,\dots n}$ (resp. $(\gamma_\ell)_{\ell=1,\dots n}$) be an orthogonal basis of $\mathcal{V}$ constituted by eigenvectors of A (resp. C). If B is of *rank unity,* let b be its nonnull eigenvalue (assuming $b > 0$) associated with eigenvector β. Let us denote φ_ℓ (resp. ψ_ℓ) the angle between the eigenspace of B associated with b and the eigenspace of A (resp. C) generated by α_ℓ (resp. γ_ℓ); and Φ_ℓ (resp. Ψ_ℓ) the angle between the eigenspace of B associated with b and the invariant subspace of A (resp. C) generated by its ℓ first eigenvectors:

$$\cos^2 \Phi_\ell = \sum_{j=1}^{\ell} \cos^2 \varphi_j \qquad \cos^2 \Psi_\ell = \sum_{j=1}^{\ell} \cos^2 \psi_j$$

Theorem 7.1. *C=A+B, for any B* (e.g. Wilkinson (1965), p.100-101).
(1) Between the eigenvalues of A and C=A+B, one has the following relations:

(1a) For all integers $j, k, \ell,$ *between 1 and* $n,$ *such that* $j + k \leq \ell + 1,$

$$c_\ell \leq a_j + b_k \quad \text{and} \quad c_{n-\ell+1} \geq a_{n-j+1} + b_{n-k+1}$$

(1b) For $1 \leq \ell \leq n\ :\ a_\ell + b_n \leq c_\ell \leq a_\ell + b_1$ *(Weyl's inequalities)*

(1c) For $1 \leq \ell \leq n\ :\ \sum_{j=1}^{\ell} a_j + \sum_{j=n-\ell+1}^{n} b_j \leq \sum_{j=1}^{\ell} c_j \leq \sum_{j=1}^{\ell} a_j + \sum_{j=1}^{\ell} b_j$

(2) Between the eigenvalues of C and of A = C – B, one has the following relations:

(2a) For all integers $j, k, \ell,$ *between 1 and* $n,$ *such that* $j + k \leq \ell + 1,$

$$a_\ell \leq c_j - b_{n-k+1} \quad \textit{and} \quad a_{n-\ell+1} \geq c_{n-j+1} + b_k$$

(2b) For $1 \leq \ell \leq n, \quad c_\ell - b_1 \leq a_\ell \leq c_\ell - b_n$ *(Weyl's inequalities)*

(2c) for $1 \leq \ell \leq n \ : \ \sum_{j=1}^{\ell} c_j - \sum_{j=1}^{\ell} b_j \leq \sum_{j=1}^{\ell} a_j \leq \sum_{j=1}^{\ell} c_j - \sum_{j=n-\ell+1}^{n} b_j$

Theorem 7.2. *$C = A + B$ with B positive and of rank 1* (see Escofier, Le Roux, 1977).

(1) For $1 < \ell \leq n \ : \ a_\ell \leq c_\ell \leq \min\{a_\ell + b\sin^2\Phi_{\ell-1}\,;\, a_{\ell-1}\}$

For $1 \leq \ell \leq n \ : \ \sum_{j=1}^{\ell} a_j + b\cos^2\Phi_\ell \leq \sum_{j=1}^{\ell} c_j \leq \sum_{j=1}^{\ell} a_j + b$

(2) For $1 \leq \ell < n \ : \ \max\{c_{\ell+1}\,;\, c_\ell - b\cos^2\Psi_\ell\} \leq a_\ell \leq c_\ell$

For $1 \leq \ell \leq n \ : \ \sum_{j=1}^{\ell} c_j - b\cos^2\Psi_\ell \leq \sum_{j=1}^{\ell} a_j \leq \sum_{j=1}^{\ell} c_j$

Theorem 7.3. $C{=}B{\circ}A$ *(A positive and B positive–definite)* (see e.g. Anderson & Dasgupta, 1963).

Between the eigenvalues of A and $C = B{\circ}A,$ *one has the relations:*

(1) For all integers j, k, ℓ *between 1 and* n *such that* $j + k \leq \ell + 1$*:*

$$c_\ell \leq a_j\, b_k \quad \textit{and} \quad c_{n-\ell+1} \geq a_{n-j+1}\, b_{n-k+1}$$

(2) For all ℓ *with* $1 \leq \ell \leq n \ : \ a_\ell\, b_n \leq c_\ell \leq a_\ell\, b_1$

7.5.2 Invariant Subspaces

Let us recall that the *relative position of two subspaces* $\mathcal{A}$ and $\mathcal{C}$ of respective dimensionalities r and r' (with $r' \geq r$) is defined by r angles $\pi/2 \geq \theta_1 \geq \ldots \geq \theta_r \geq 0$ (Dixmier, 1948; Benzécri & al. 1973, p. 179) called *canonical angles.* In what follows, we will take as a measure of the deviation between two subspaces the *greatest canonical angle,* denoted $\theta,$ with, if $\theta(a|c)$ (with $0 \leq \theta(a|c) \leq \pi$) denotes the angle between the (nonnull) vectors a and c: $\theta = \max\limits_{\substack{a \in \mathcal{A} \\ a \neq 0}}\{\min\limits_{\substack{c \in \mathcal{C} \\ c \neq 0}} \theta(a|c)\}.$

Theorem 7.4 (p.294) deals with the case of the invariant subspaces associated with the eigenvalues of ranks $\ell_0, \ldots \ell_0 + \ell$ (we will say "intermediate" if $\ell_0 > 1$) for a symmetric perturbation of any rank.

Theorems 7.5 (p.294) and 7.6 (p.295) deal with the case of the adjunction of a symmetrical perturbation B of rank 1; the upper bounds of rotation are expressed not only as functions of the eigenvalues of A or of

C but also taking into account the position of the eigenspace of B with respect to the subspaces under study.

Theorem 7.7 (p.295) allows the comparison of the subspaces associated with eigenvalues of different ranks; a case often encountered in GDA when the rank of a principal axis is shifted without modifying the others.

Theorem 7.8 (p.295) studies the case of the multiplication by a positive–definite endomorphism.

Lastly, to study complex perturbations obtained by summing and/or composing two perturbations, one will use the Lemma 7.2 (p.295).

Theorem 7.4. *$C = A + B$ with any B* (Escofier & Le Roux, 1975).

The angle θ between the invariant subspaces of A and $C = A + B$ associated with the ℓ eigenvalues of ranks $\ell_0, \ldots \ell_0 + \ell - 1$ is such that:

- *for $\ell_0 = 1$ and $1 \leq \ell \leq n$, letting $\epsilon_1 = a_\ell - a_{\ell+1}$, $\epsilon_2 = c_\ell - c_{\ell+1}$;*
- *for $1 < \ell_0 \leq n - 1$ and $1 \leq \ell \leq n - \ell_0$, letting*
  $\epsilon_1 = \inf\{(a_{\ell_0-1} - a_{\ell_0}), (a_{\ell_0+\ell-1} - a_{\ell_0+\ell})\}$,
  $\epsilon_2 = \inf\{(c_{\ell_0-1} - c_{\ell_0}), (c_{\ell_0+\ell-1} - c_{\ell_0+\ell})\}$

$$\text{If } b_1 - b_n < \epsilon_1 \quad \text{then:} \quad \theta < \pi/4 \quad \text{and} \quad \sin 2\theta \leq (b_1 - b_n)/\epsilon_1$$
$$\text{If } b_1 - b_n < \epsilon_2 \quad \text{then:} \quad \theta < \pi/4 \quad \text{and} \quad \sin 2\theta \leq (b_1 - b_n)/\epsilon_2$$

Remark. The bound is optimal: It is reached if B is of rank 1, as well as for endomorphisms of a two–dimensional Euclidean space.

Lemma 7.1. *If A, B and C are endomorphisms of a two–dimensional Euclidean space, with $C = A + B$ and B positive, one has:*

$$\tan 2\theta = \frac{(b_1 - b_2)sin2\phi_1}{(a_1 - a_2) + (b_1 - b_2)cos2\phi_1} = \frac{(b_1 - b_2)sin2\psi_1}{(c_1 - c_2) - (b_1 - b_2)cos2\psi_1}$$

Theorem 7.5. *$C = A + B$ with B positive and of rank 1* (Escofier & Le Roux, 1977). *The angle θ between the invariant subspaces of A and of $C = A + B$ associated with the first ℓ eigenvalues is such that:*

(1) If $m = \dfrac{a_\ell - a_{\ell+1}}{b} > 1$ (or for $\ell = 1$ if $m + \cos 2\varphi_1 > 0$) :

$$\theta < \pi/4 \quad \text{and} \quad \tan 2\theta \leq \frac{\sin 2\Phi_\ell}{m + \cos 2\Phi_\ell}$$

$$\textit{If } \sin^2 \Phi_\ell < m < 1 \;:\; \theta < \pi/4 \quad \text{and} \quad \tan 2\theta \leq \frac{\sin 2\Phi_\ell}{m - \sin^2 \Phi_\ell}$$

(2) If $m' = \dfrac{c_\ell - c_{\ell+1}}{b} > 1$ $: \theta < \pi/4$ and $\tan 2\theta \leq \dfrac{\sin 2\Psi_\ell}{m' - \cos 2\Psi_\ell}$

$$\textit{If } \cos^2 \Psi_\ell < m' < 1 \;:\; \theta < \pi/4 \quad \text{and} \quad \tan 2\theta \leq \frac{\sin 2\Psi_\ell}{m' - \cos^2 \Psi_\ell}$$

Remark. For $m > 1$ and $m' > 1$, *the bounds are optimal*: They are reached if the eigenvector of B associated with b lies in the plane of the eigenvectors of ranks ℓ and $\ell+1$. For $m > 1$, the bound is maximum for $\cos 2\Phi_\ell = -1/m$, and the corresponding angle θ_{max} is such that $\tan 2\theta_{max} = 1/\sqrt{m^2-1}$, or else $\sin 2\theta_{max} = 1/m$ (upper bound of Theorem 7.4). Similarly, for $m' > 1$, the angle θ_{max} is such that $\sin 2\theta_{max} = 1/m'$ for $\cos 2\Psi_\ell = 1/m'$.

Theorem 7.6. *C = A + B with B positive and of rank 1* (Escofier & Le Roux, 1977).
The angle θ between the invariant subspaces of A and of C associated with the eigenvalues of ranks $\ell_0, \dots \ell_0 + \ell$ ($\ell_0 \geq 1$, $0 \leq \ell \leq n - \ell_0$) is such that:
If $\epsilon = \inf\{(a_{\ell_0-1} - a_{\ell_0}) - b\sin^2\Phi_{\ell_0-1}\ ;\ (a_{\ell_0+\ell} - a_{\ell_0+\ell+1}) - b\sin^2\Phi_{\ell_0+\ell}\} > 0$,

$$\textit{then:}\quad \sin\theta \leq \frac{b}{2}\,\frac{\sin 2\Phi}{\epsilon}\quad \textit{with } \Phi \textit{ such that } \cos^2\Phi = \sum_{j=\ell_0}^{\ell_0+\ell}\cos^2\phi_j.$$

If $\epsilon' = \inf\{(c_{\ell_0-1} - c_{\ell_0}) - b\cos^2\Psi_{\ell_0-1}\ ;\ (c_{\ell_0+\ell} - c_{\ell_0+\ell+1}) - b\cos^2\Psi_{\ell_0+\ell}\} > 0$,

$$\textit{then:}\quad \sin\theta \leq \frac{b}{2}\,\frac{\sin 2\Psi}{\epsilon'}\quad \textit{with } \Psi \textit{ such that } \cos^2\Psi = \sum_{j=\ell_0}^{\ell_0+\ell}\cos^2\psi_j.$$

Remarks. This theorem allows to study the eigenspace associated with the eigenvalue ℓ_0. For $\ell_0 = 1$ (and $\ell = 0$), one will use Theorem 7.5 that gives a better upper bound.

Theorem 7.7. *Shifted subspaces* (Le Roux, 1999, p.83).
Let P_ℓ be the orthogonal projection on the eigenspace of C associated with γ_ℓ, and let $C' = C - (b\cos 2\psi_\ell)P_\ell$. The angle θ between the invariant subspaces of C' and of A associated with the eigenvalues of ranks $s, s+1, \dots s+r$ is such that, letting $\epsilon = \inf\{(c'_{s-1} - c'_s), (c'_{s+r} - c'_{s+r+1})\}$:

$$\textit{If}\quad b\sin 2\psi_\ell < \epsilon \quad \textit{then}\quad \theta < \pi/4 \textit{ and } \sin 2\theta \leq \frac{b\sin 2\psi_\ell}{\epsilon}$$

Theorem 7.8. *C = BA, with A positive and B definite–positive.*
Letting $\epsilon = \inf\{(a_{\ell-1}/a_\ell) - 1\,;\, 1 - (a_{\ell+1}/a_\ell)\}$, the angle θ between the one–dimensional eigenspaces of A and of C = BA associated with the (simple) ℓ–th eigenvalue is such that:

$$\textit{If}\ \left(\frac{b_1}{b_n} - 1\right) < \epsilon \quad \textit{then}\quad \theta < \pi/4 \textit{ and } \sin 2\theta \leq \left(\frac{b_1}{b_n} - 1\right)/\epsilon$$

Lemma 7.2. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three subspaces of the same dimensionality. If ϕ_1 is the angle between $\mathcal{A}$ and $\mathcal{B}$, ϕ_2 the angle between $\mathcal{B}$ and $\mathcal{C}$, then the angle θ between $\mathcal{A}$ and $\mathcal{C}$ is such that: $\theta \leq \phi_1 + \phi_2$.*

For example, to get an upper bound of the angle θ between invariant subspaces of the endomorphisms A and $C = B_2(A + B_1)$, one takes, as an upper bound between the invariant subspaces of A and $A+B_1$, the angle ϕ_1 (theorem 7.4 or theorem 7.5), then one takes, as an upper bound between the invariant subspaces of $A+B_1$ and of $B_2(A+B_1)$, the angle ϕ_2 (theorem 7.8). Hence for θ the upper bound $\phi_1 + \phi_2$.