Macaulay2 Exercises

Computational Algebraic Geometry Workshop

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These exercises are approximately split into those that explore the M2 language and others about polynomials. They are intended to be solved with liberal use of the M2 help system. To get started with the help system, check out the help help-page with `help "help"`.

1 The M2 Language

0. Find the help page that explains `"continue"`. Copy and modify the first example to produce a loop that prints the numbers 1 to 10 and returns the list of odd numbers between 1 and 10

1. List the first 100 prime numbers

   hint: define variables `numberOfPrimes = 0` and `n = 1`, and use a while loop:

   `while numberOfPrimesFound < 100 list (...)`

2. Compute the 2024th Fibonacci number

3. Write a function `startsWithX` that takes a string `s` and outputs `true` if the first character of `s` is the letter 'x' otherwise it returns false

   Bonus: write `startsWithX` as a method function

   [hint: strings can be indexed using underscores]

   [corner case: does your function work with the empty string?]

4. Use the packages `Graphs` and `Matroids` to count the number of maximal chains of flats of the graphic matroid of the complete graph on 5 vertices

   [hint: use `needsPackage` to load each package. Use `help` or `viewHelp` to find out how to produce the complete graph and how to make a matroid from a graph. Then use the functions `latticeOfFlats` and `chains` to count the number of chains. Note that maximal length chains have length one more than the rank of matroid]

5. Make a function `randomMatrix` that takes an integer `n` and produces an `n` by `n` matrix whose entries are random elements of $\mathbb{Z}/2$. How often is a random $n \times n$ matrix invertible?

6. Modify the `collatz` function so that it can compute `collatz(837799)`.

   [hint: try using a while loop instead of recursion]

7. Compute the total Chern class of the cotangent sheaf of $\mathbb{P}^3$.

   [hint: explore the package `NormalToricVarieties`]

8. Use M2 to create a file called `output.m2` and write code to this file that: defines a ring $R = \mathbb{Q}[x, y]$; defines an ideal $I = \langle x^3 - y, xy - y^3 \rangle \subseteq R$; and prints out the primary decomposition of I. Restart your M2 shell and load your file.

   [hint: remember to use `<< close` to close your file once you have finished writing to it.]

2 Working with Polynomials

1. Make a ring with 5 variables $x_1, \ldots, x_5$ and a term order such that $x_1 > x_2 > \cdots > x_5$ and $x_1^2 > x_2^3$

   [hint: remember the optional argument `MonomialOrder => ...`, perhaps try a lexicographic order]

2. Show, using M2, that the ideal generated by $y - x^2$ and $z - x^3$ contains the polynomial $z - xy$

3. Define the polynomials $f_1 = x + y + z$, $f_2 = xy + xz + yz$, $f_3 = xyz$. Determine how to write $g = x^3 + y^3 + z^3$ as a polynomial combination of f_1, f_2, f_3, i.e., find a polynomial h such that $g = h(f_1, f_2, f_3)$. Check your work using the function `sub`.

   [hint: define a ring $\mathbb{Q}[x, y, z, u_1, u_2, u_3]$ and an ideal I generated by $\{u_i - f_i : i = 1, 2, 3\}$. Write g in 'normal form' with respect to I using `g % I`]

   [follow up: if you used the hint, why does this work?]

4. Let $X = (x_{ij})$ be a 3×3 matrix of variables and I be the ideal generated by the 2×2 minors of X. Show that I is a prime ideal. Find an initial ideal of I.

   Bonus: how many different initial ideals of I are there?

   [hint: M2 can auto-complete double-indexed sequences such as `x_(1,1)..x_(3,3)`. Then use the functions `genericMatrix`, `minors`, `isPrime`, and `leadTerm`]

   [hint for bonus: the package `gfanInterface` can compute the Groebner fan (`gfan`)]

5. Let $X \subseteq \mathbb{C}^n$ be the variety defined by $y^2 = x^3 + x^2$. Find the singular locus of X. Compute the normalisation of X and observe how the singularities are resolved.

   [hint: while there is a function `singularLocus`, try using the differential function `diff`. Then find out about the functions `integralClosure` and `icMap`]

6. Find the tropicalisation of the hypersuface defined by $a_1a_2 + a_3a_4 + a_5a_6 + 2a_7a_8$ as a subfan of the Groebner fan, i.e., with respect to the trivial valuation. Compute its rays and lineality space.

   Bonus: Compute the tropicalisation with respect to 2-adic valuation.

   [hint: see the `Tropical` package.]

   [hint for bonus: if not implemented in `Tropical` yet, then either implement yourself or perform some polyhedral regular subdivisions with `Polyhedra` package]