

# ELASTIC CURVES WITH VARIABLE BENDING STIFFNESS

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**ABSTRACT.** We study stationary points of the bending energy of curves  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  subject to constraints on the arc-length and the curve's holonomy while simultaneously allowing for a variable bending stiffness along the arc-length of the curve. Physically, this can be understood as a model for an elastic wire with isotropic cross-section of varying thickness. We derive the corresponding Euler-Lagrange equations for variations that are compactly supported away from the end points thus obtaining characterizations for elastic curves with variable bending stiffness. Moreover, we provide a collection of alternative characterizations, e.g., in terms of the curvature function. Adding to numerous known results relating elastic curves to dynamics, we explore connections between elastic curves with variable bending stiffness, variable length pendulums and the flow of vortex filaments with finite thickness.

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## 1. INTRODUCTION

In mathematics and the natural sciences alike, one-dimensional flexible structures have diverse applications ranging from architectural beams to molecular modeling. Despite a longstanding history of research, they remain a prominent research topic.

Stationary points of the so-called *bending energy* on the space of curves  $\gamma : [a, b] \rightarrow \mathbb{R}^n$ , are used to model, e.g., the bent shapes of an ideal infinitesimally thin elastic rod (without stretching) [33]. This classical problem dates back to the 13th century and the current state of knowledge is still largely based on the work of mathematical pioneers, such as Bernoulli and Euler [33]. Only the rigorous definition of the curvature  $\kappa$  of a planar curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  paved the way for their seminal work, culminating in the characterization of *plane elastic curves* as the stationary points of the energy functional ([13], Fig. 1)

$$\frac{1}{2} \int \kappa^2 ds.$$

It was soon discovered that planar elastic curves relate to a variety of other phenomena in the natural sciences [33]. For regular space curves  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  the bending energy is given by

$$\mathcal{B}(\gamma) = \frac{1}{2} \int_a^b \left\langle \frac{d^2\gamma}{ds^2}, \frac{d^2\gamma}{ds^2} \right\rangle ds.$$

Kirchhoff realized the importance of torsion and related the problem of *elastic curves* in three-dimensions with the dynamics of a spinning top [28]. For more in-depth reviews of the history of elastic curves see, e.g., [33, 22].

Elastic curves have broad utility, praised for their aesthetics and practicality. They are used as models for plant stems [20, 21], DNA strands [2, 16, 38], for preventing kinks and

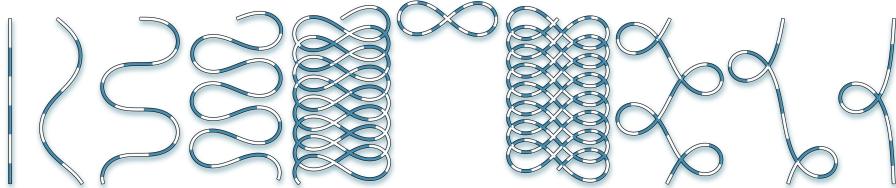


FIGURE 1. Examples of planar elastic curves, i.e., solutions to the Euler-Lagrange equations resulting from considering variations of the bending energy with a length-constraint.

loops in marine cables [23, 51], guiding track layouts, and even in medical applications such as surgical wires [48]. In design and computer graphics they serve as decorative elements in [55] and play a crucial role in simulation software, particularly in complex tasks like hair simulation [10, 49, 5, 29]. The discretization [3, 7] and efforts to approximate elastic curves with computationally more efficient splines [8] form a crucial facet of current research.

Elastic curves' energy-minimizing characteristics and their direct link to material bending make them appealing for architecture and modern fabrication. Fabrication-aware design and cost-effective manufacturing processes, such as active bending [34], leverage certain material properties. Bending and twisting, once challenges, are now utilized as tools, reflecting a shift "from failure to function" [1]. This shift has amplified the interest in inverse problems, which seek to control material parameters to achieve specific shapes, a significant facet of elasticity research [6, 24, 25]. Economic considerations, including ease of manufacturing, transportation, and installation for curved shapes, are pivotal in architecture and fabrication. Techniques like active bending offer numerous advantages [34], and morphing structures leverage bending to attain or modify their desired shapes [35].

Various surface design methods extend elastic curve theory to approximate 2-dimensional curved shapes using networks of single rods. These approaches are instrumental in modern fabrication, including *rod meshes* and *gridshells* [1, 50, 43], and *deployable structures* [41, 45]. Another modeling technique combines elastic curves with minimal surfaces, as seen in Plateau surfaces [18, 44]. In this context, the energy of the whole system depends on the energy of the bounding elastic curve [4, 17]. Also closed curves and "elastic knots" offer a range of interesting results, both in theoretical studies [52, 32] and practical applications [53].

On a theoretical level, elastic curves are known to relate to the motion of a spinning top's symmetry axis [28] and to satisfy the pendulum equation [46]. Langer and Singer made significant contributions [30, 31], elucidating connections between Kirchhoff rods and soliton theory, which lead to integrable Hamiltonian systems—a development built upon Hasimoto's analogy between vortex filament flow and the nonlinear Schrödinger equation [26]. Specifically, the curves' evolution under the *vortex-filament flow* is (up to reparametrization) a rigid motion. Moreover, they can be described as the orbits of charged particles moving in a magnetic field [12, 46]. The tangent vectors of stationary points of the bending energy with no constraints on the holonomy of the curve are known to relate to the motion of the axis of a spinning top [28] and solve the pendulum equation [46].

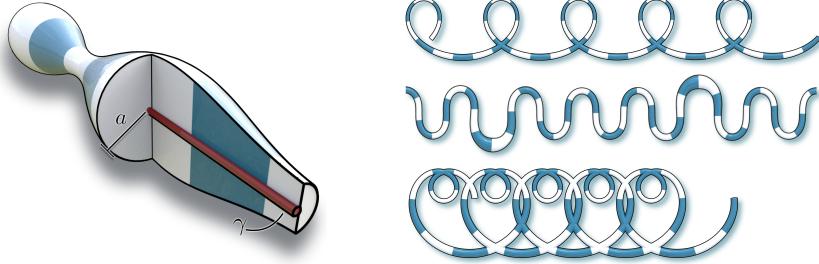


FIGURE 2. Elastic rods with circular cross-section of varying thickness are modeled by weighting the integrand of the bending energy with a positive function  $\varrho$  proportional to the thickness  $a > 0$  (left). The resulting elastica are less bent in regions with greater stiffness (right).

Complementing previous work, our main focus is on space curves  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  that exhibit an isotropic resistance to bending which may vary along the arc-length of the curve. We will refer to such a curve as a curve with *variable bending stiffness*. From a physical viewpoint, these elastic curves model idealized elastic rods with circular cross-section of varying diameter. The study of curves with additional free parameters such as thickness has become an important task in the natural sciences, as they appear as key structural motifs in a variety of contexts (see, e.g., [36, 19, 11, 37] and references therein). The variable bending stiffness is formalized by weighting the integrand of the bending energy by a strictly positive function

$$\varrho : [a, b] \rightarrow \mathbb{R}_{>0},$$

which represents the *bending stiffness* of the curve  $\gamma$  along the arc-length. Consequently, the bending energy takes the form

$$(1.1) \quad \mathcal{B}_\varrho(\gamma) = \frac{1}{2} \int_a^b \varrho \left\langle \frac{d^2\gamma}{ds^2}, \frac{d^2\gamma}{ds^2} \right\rangle ds.$$

Previous investigations on this topic focused on closed planar elastic curves whose stiffness depends on an additional density variable [27, 9], or developed computational methods and applications [40, 24, 25].

Inspired by [25], who considered a significantly larger configuration space allowing for anisotropic cross sections and focused on the inverse problem, on a conceptual level, our approach differs significantly from previous work (see also, e.g., [14]). We consider a somewhat more restrictive variational problem asking for isotropic cross-sections, thus remaining closer to the classical problem. However, similar to the 2-dimensional case, where the step from a “bending beam” to solving the formal Euler-Lagrange equations reveals considerably more interesting curves, we obtain a large number of novel elastic curves with variable bending stiffness. Most importantly, we observe connections to dynamical systems, which are already well established in the classical setup [46]. Further investigation of these aspects seems to be an exciting avenue for future research on their own.

**1.1. Structure of the Article.** The article is structured as follows: in Section 1 we outline related work and place our own work in the context of existing literature. In Section 2 we fix the notation and introduce preliminary definitions and theorems for further

discussion. The Euler-Lagrange equations for different types of elastic curves with variable bending stiffness characterized by constraints on the arc-length and/or holonomy are derived in Section 3. The Euler-Lagrange equations can be reformulated into equivalent characterizations, which we do in Section 4, while in Section 4.1 we express them in terms of the curvature  $\kappa : [a, b] \rightarrow \mathbb{R}^{n-1}$  function. Last, in Section 5, we explore how established connections to dynamical systems, such as physical pendulums or the vortex filament flow, could translate to our novel setup.

## 2. PRELIMINARIES

In this section, we briefly introduce preliminaries and fix the notation used throughout the article. We will denote the set of all smooth functions from an interval  $[a, b]$  into  $\mathbb{R}^n$  by  $C^\infty([a, b]; \mathbb{R}^n)$ . For our purposes we always assume that  $n \geq 2$ . A *curve* is a map  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with  $\gamma' \neq 0$ . Denoting the standard Euclidean norm on  $\mathbb{R}^n$  by  $|\cdot|$ , we refer to  $|\gamma'|$  as the *arc-length* of  $\gamma$  and the *length* of a curve is given by

$$\mathcal{L}(\gamma) := \int_a^b |\gamma'|.$$

A curve is said to be parameterized by *arc-length* if  $|\gamma'| \equiv 1$ . The *derivative* of a function  $g \in C^\infty([a, b]; \mathbb{R}^k)$  with respect to the *arc-length* of  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  is defined as  $\frac{dg}{ds} := \frac{g'}{|\gamma'|}$ . In particular, the *unit tangent* vector field  $T \in C^\infty([a, b]; S^{n-1})$  of a curve is given by  $T := \frac{dy}{ds}$ .

**2.1. The Bending Energy with Bending Stiffness.** The norm of the derivative with respect to the arc-length of the unit tangent vector, measures the failure of the curve  $\gamma$  to be a straight line segment. Integrating the squared norm of this error over the arc-length therefore measures how much the curve bends in space. Consequently the *bending energy* of a curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  is given by

$$(2.1) \quad \mathcal{B}(\gamma) := \frac{1}{2} \int_a^b \left\langle \frac{dT}{ds}, \frac{dT}{ds} \right\rangle ds.$$

Historically, the motivation for studying elastic curves has been to gain a better understanding of the shapes that a thin elastic cable takes when the end points are held fixed. We extend the theory in the sense that we do not assume for a cable of constant thickness, but allow for varying thickness which is accounted for by weighting the bending energy density along the curve.

**Definition 2.1** (Bending energy with bending stiffness). The *bending energy* of a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  with *bending stiffness*  $\varrho : [a, b] \rightarrow \mathbb{R}_{>0}$  is given by

$$(2.2) \quad \mathcal{B}_\varrho(\gamma) := \frac{1}{2} \int_a^b \varrho \left\langle \frac{dT}{ds}, \frac{dT}{ds} \right\rangle ds.$$

**2.2. Holonomy of a Space Curve.** A normal field  $Z : [a, b] \rightarrow \mathbb{R}^n$  satisfies  $\langle Z, T \rangle = 0$  and is said to be *parallel* if  $Z' = \lambda T$  for some  $\lambda \in C^\infty(M)$ . By solving a corresponding initial value problem any normal vector to a curve can be extended to a unique parallel normal field along  $\gamma$  [46, Thm. 4.3]. The vector  $Z_b$  of the solution  $Z$  to this initial value problem for an initial normal vector  $Z_0$  at  $\gamma(a)$  is called the *parallel transport* of  $Z_0$  along the curve  $\gamma$ .

**Definition 2.2.** For a curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  let  $W = (W_a, W_b) \in T(a)^\perp \times T(b)^\perp$  be a pair of unit vectors. Then, the *holonomy* of the curve  $\gamma$  with respect to  $W$  is the unique angle  $\mathcal{T}_W \in \mathbb{R}/_{2\pi\mathbb{Z}}$  such that

$$(2.3) \quad Z_b = \cos(\mathcal{T}_W) W_b + \sin(\mathcal{T}_W) T(b) \times W_b,$$

where  $Z_b$  is the unique vector obtained from a parallel transport of  $W_a$  along  $\gamma$ .

Note that a curve's holonomy does not depend on the bending stiffness  $\varrho$ , but only the geometry of the curve  $\gamma$ .

**2.3. Variations of Curves, Length and Holonomy.** In the forthcoming sections we will define a hierarchy of “elastic curves” as stationary points of bending energy with bending stiffness (2.2) under perturbations with constrained length, arc-length or holonomy. For  $g \in C^\infty([a, b]; \mathbb{R}^n)$  and  $\epsilon > 0$ , a *smooth variation* of  $g$  is a one-parameter family

$$(2.4) \quad t \mapsto g_t \in C^\infty([a, b]; \mathbb{R}^n),$$

where  $t \in [-\epsilon, \epsilon]$  and which satisfies  $g_0 = g$  and such that the map

$$[-\epsilon, \epsilon] \times [a, b] \rightarrow \mathbb{R}^n, (t, x) \mapsto \gamma_t(x)$$

is smooth. Given a smooth variation of a map  $g \in C^\infty([a, b]; \mathbb{R}^n)$ , also  $t \mapsto g'_t$ , as well as  $t \mapsto \dot{g}_t$ , where  $\dot{g}_t \in C^\infty([a, b]; \mathbb{R}^n)$  is defined as

$$\dot{g}_t(x) := \frac{d}{d\tau}|_{\tau=t} g_\tau(x),$$

are smooth. To simplify the notation, we will omit the index when we evaluate at time  $t = 0$  and write  $\dot{g} = \dot{g}_0$ . Moreover, we denote the variation of a smooth functional  $\mathcal{F}$  on  $C^\infty([a, b]; \mathbb{R}^n)$  corresponding to a smooth variation  $t \mapsto g_t$  with variational vector field  $\dot{g}$  by

$$d\mathcal{F}(\dot{g}) := \frac{d}{d\tau}|_{\tau=0} \mathcal{F}(g_\tau).$$

Our main object of interest are smooth variations  $t \mapsto \gamma_t \in C^\infty([a, b]; \mathbb{R}^n)$  of curves, for which we refer to

$$(2.5) \quad \dot{\gamma} \in C^\infty([a, b]; \mathbb{R}^n)$$

as the *variational vector field*. A variation  $t \mapsto \gamma_t$  of a curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  is said to have *compact support in the interior* of  $[a, b]$  if there is  $\delta > 0$  such that for all  $x \in [a, a+\delta] \cup [b-\delta, b]$  it holds that  $\gamma_t(x) = \gamma(x)$ . We denote the vector space of functions with support in the interior of  $[a, b]$  by  $C_0^\infty([a, b]; \mathbb{R}^n)$  and treat variations and corresponding variational vector fields as synonyms. Some useful identities are collected in

**Lemma 2.3** ([46, Ch. 2]). *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  and  $g \in C^\infty([a, b]; \mathbb{R}^n)$ . Then,*

- (1)  $(\dot{g})' = (\dot{g}')$ .
- (2)  $(ds)' = \langle \frac{d\dot{y}}{ds}, T \rangle ds$ .
- (3)  $\left( \frac{dg}{ds} \right)' = \frac{d\dot{g}}{ds} - \langle \frac{d\dot{y}}{ds}, T \rangle \frac{dg}{ds}$ .

**Theorem 2.4.** *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  and  $\dot{\gamma} \in C^\infty([a, b]; \mathbb{R}^n)$ . Then,*

$$d\mathcal{L}(\dot{\gamma}) = \langle \dot{\gamma}, T \rangle|_a^b - \int_a^b \langle \dot{\gamma}, \frac{dT}{ds} \rangle ds.$$

*Proof.* By Lemma 2.3 and the fundamental theorem of calculus we have

$$d\mathcal{L}(\dot{\gamma}) = \int_a^b \langle \frac{d\dot{\gamma}}{ds}, T \rangle ds = \int_a^b \left( \frac{d}{ds} \langle \dot{\gamma}, T \rangle - \langle \dot{\gamma}, \frac{dT}{ds} \rangle \right) ds = \langle \dot{\gamma}, T \rangle|_a^b - \int_a^b \langle \dot{\gamma}, \frac{dT}{ds} \rangle ds.$$

□

We can also compute the variational gradient of the holonomy.

**Theorem 2.5.** *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  and  $\dot{\gamma} \in C_0^\infty([a, b]; \mathbb{R}^n)$ . Then, independent of the choice of  $W = (W_a, W_b)$  it holds that*

$$(2.6) \quad d\mathcal{T}_W(\dot{\gamma}) = \int_a^b \langle \dot{\gamma}, T \times (\frac{dT}{ds})' \rangle.$$

*Proof.* This follows from restricting [46, Thm. 5.3] to variations in  $C_0^\infty([a, b]; \mathbb{R}^n)$ .  $\square$

### 3. EULER-LAGRANGE EQUATIONS

With all necessary preliminaries in place, we start this section with computing the variational formula of the bending energy with bending stiffness (2.2) for a curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  under a general variation  $\dot{\gamma} \in C^\infty([a, b]; \mathbb{R}^n)$ .

**Theorem 3.1** (Variational formula for the bending energy). *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$ . Then, the variation of the bending energy with bending stiffness (2.2) with respect to  $\dot{\gamma} \in C^\infty([a, b]; \mathbb{R}^n)$  is given by*

$$(3.1) \quad dB_\varrho(\dot{\gamma}) = [\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \langle \dot{\gamma}, \varrho \frac{d^2T}{ds^2} + \frac{3}{2} \varrho \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle T + \frac{d\varrho}{ds} \frac{dT}{ds} \rangle]_a^b + \int_a^b \langle \dot{\gamma}, \mathcal{G}^B \rangle ds,$$

where

$$(3.2) \quad \mathcal{G}^B := \varrho \langle \frac{d^3T}{ds^3}, \frac{dT}{ds} \rangle + 3\varrho \langle \frac{d^2T}{ds^2}, \frac{dT}{ds} \rangle T + \frac{3}{2} \varrho \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \frac{dT}{ds} + \frac{d^2\varrho}{ds^2} \frac{dT}{ds} + \frac{d\varrho}{ds} (2 \langle \frac{d^2T}{ds^2}, \frac{dT}{ds} \rangle + \frac{3}{2} \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle T).$$

*Proof.* First note that  $\langle \frac{dT}{ds}, T \rangle = 0$  and Lemma 2.3 3 applied to  $g = \gamma$  gives  $\dot{T} = \frac{d\dot{\gamma}}{ds} - \langle \frac{d\dot{\gamma}}{ds}, T \rangle T$ . Therefore, the variation of Bending Energy with bending stiffness is

$$\begin{aligned} dB_\varrho(\dot{\gamma}) &= \frac{1}{2} \int_a^b (\varrho \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle) ds \\ &= \int_a^b \varrho \langle \langle \frac{dT}{ds} \rangle^\circ, \frac{dT}{ds} \rangle ds + \frac{1}{2} \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle ds \\ &= \int_a^b \varrho \langle \langle \frac{dT}{ds} - \langle \frac{d\dot{\gamma}}{ds}, T \rangle \frac{dT}{ds}, \frac{dT}{ds} \rangle ds + \frac{1}{2} \langle \frac{d\dot{\gamma}}{ds}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle ds \\ &= \int_a^b \varrho \langle \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \frac{1}{2} \langle \frac{d\dot{\gamma}}{ds}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \rangle ds \\ &= \int_a^b \varrho \langle \langle \frac{d}{ds} \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \langle \frac{d\dot{\gamma}}{ds}, T \rangle T, \frac{dT}{ds} \rangle - \frac{1}{2} \langle \frac{d\dot{\gamma}}{ds}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \rangle ds \\ &= \int_a^b \varrho \langle \langle \frac{d^2\dot{\gamma}}{ds^2}, \frac{dT}{ds} \rangle - \frac{3}{2} \langle \frac{d\dot{\gamma}}{ds}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \rangle ds \end{aligned}$$

We split the integral just to keep computations more clear. Using integration by parts we obtain

$$\begin{aligned}
\int_a^b \varrho \langle \frac{d^2\dot{\gamma}}{ds^2}, \frac{dT}{ds} \rangle ds &= \int_a^b \frac{d}{ds} (\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle) - \frac{d\varrho}{ds} \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{d^2T}{ds^2} \rangle ds \\
&= [\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle]_a^b + \int_a^b -\frac{d}{ds} (\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle) + \frac{d^2\varrho}{ds^2} \langle \dot{\gamma}, \frac{dT}{ds} \rangle + \frac{d\varrho}{ds} \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle \\
&\quad - \frac{d}{ds} (\varrho \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle) + \frac{d\varrho}{ds} \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle + \varrho \langle \dot{\gamma}, \frac{d^3T}{ds^3} \rangle ds \\
&= [\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \frac{d\varrho}{ds} \langle \dot{\gamma}, \frac{dT}{ds} \rangle - \varrho \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle]_a^b \\
&\quad + \int_a^b \frac{d^2\varrho}{ds^2} \langle \dot{\gamma}, \frac{dT}{ds} \rangle + \frac{d\varrho}{ds} \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle + \frac{d\varrho}{ds} \langle \dot{\gamma}, \frac{d^2T}{ds^2} \rangle + \varrho \langle \dot{\gamma}, \frac{d^3T}{ds^3} \rangle ds \\
&= [\varrho \langle \frac{d\dot{\gamma}}{ds}, \frac{dT}{ds} \rangle - \langle \dot{\gamma}, \varrho \frac{d^2T}{ds^2} + \frac{d\varrho}{ds} \frac{dT}{ds} \rangle]_a^b \\
&\quad + \int_a^b \langle \dot{\gamma}, \varrho \frac{d^3T}{ds^3} + 2 \frac{d\varrho}{ds} \frac{d^2T}{ds^2} + \frac{d^2\varrho}{ds^2} \frac{dT}{ds} \rangle ds.
\end{aligned}$$

Similarly, for the second integral we get

$$\begin{aligned}
\int_a^b \frac{3}{2} \varrho \langle \frac{d\dot{\gamma}}{ds}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle ds &= \int_a^b \frac{d}{ds} \left( \frac{3}{2} \varrho \langle \dot{\gamma}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \right) \\
&\quad - \frac{3}{2} \varrho \langle \dot{\gamma}, \frac{dT}{ds} \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle - 3 \varrho \langle \dot{\gamma}, T \rangle \langle \frac{d^2T}{ds^2}, \frac{dT}{ds} \rangle \\
&\quad - \frac{3}{2} \frac{d\varrho}{ds} \langle \dot{\gamma}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle ds \\
&= \left[ \frac{3}{2} \varrho \langle \dot{\gamma}, T \rangle \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \right]_a^b \\
&\quad - \int_a^b \langle \dot{\gamma}, \frac{3}{2} \varrho \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle \frac{dT}{ds} + 3 \varrho \langle \frac{d^2T}{ds^2}, \frac{dT}{ds} \rangle T + \frac{3}{2} \frac{d\varrho}{ds} \langle \frac{dT}{ds}, \frac{dT}{ds} \rangle T \rangle ds.
\end{aligned}$$

Adding the two results yields the claim.  $\square$

**3.1. Free Elastic Curves with Bending Stiffness.** Theorem 3.1 bears several significant implications. When we imagine an (initially perfectly straight) elastic wire, it naturally wants to minimizes its bending energy. Holding a piece of such wire in our hands, we fix its end-points (and in fact even its tangent directions at the end points). Therefore, from a physical point of view, it is reasonable to restrict our attention to variations of (2.2) with compact support in the interior of  $[a, b]$ .

Already in classical theory, free elastic curves<sup>1</sup> take on a special role because, up to scaling and positioning, there is only one such curve. It turns out that even in our generalized setup, our results in Section 4.1 imply that the constant stiffness ones remain the only solution for curves that are not straight line segments.

**Theorem 3.2.** *Free elastic curves are either straight line segments or have constant bending stiffness.*

**3.2. Elastic Curves with Bending Stiffness.** Our investigations are inspired by deformations of physical rods or cables, possibly with non-uniform thickness distribution. We assume for the thickness to be prescribed along the arc length of the cable. To incorporate this assumption into our model we constrain the set of admissible variations to the arc-length parameterized curves. By standard means of calculus of variations, this can be achieved by introducing a suitable Lagrange multiplier [47, 56]. More specifically, a

<sup>1</sup>i.e., unconstrained stationary points of the bending energy

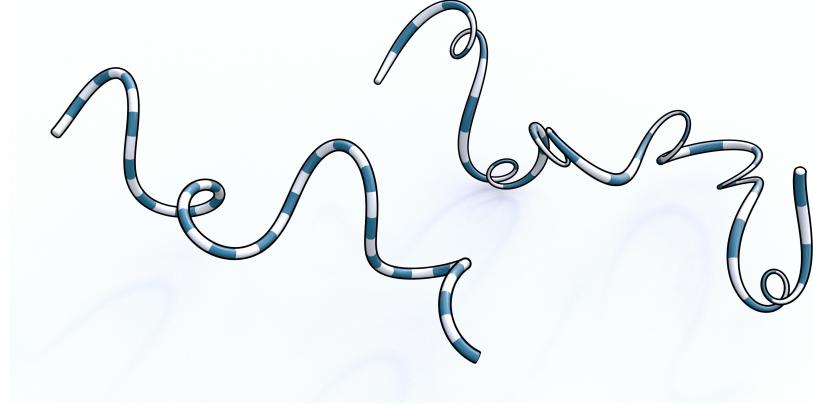


FIGURE 3. A comparison between an elastic curve with constant bending stiffness  $\varrho$  obtained from integrating Eq. (5.5) (left) and a corresponding curve obtained from the same initial conditions, but with a modulated bending stiffness  $\varrho$  (right).

curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is a stationary point of (2.2) subject to the pointwise constraint

$$\langle \gamma', \gamma' \rangle - 1 = 0$$

if and only if there exists a function  $\Lambda \in C^\infty([a, b]; \mathbb{R})$  such that  $\gamma$  is a stationary point of

$$(3.3) \quad \mathcal{B}_\varrho^\Lambda(\gamma) := \int_a^b \frac{1}{2} \varrho \langle T', T' \rangle + \Lambda(\langle \gamma', \gamma' \rangle - 1).$$

*Remark 3.3.* Note that the arc-length constraint is in fact appropriate when dealing with non-uniform bending stiffness. Unlike as for a balloon animal, to mimic the behavior of a physical cable, the thickness shall not be redistributed along the curve. A sole constraint on the length does not rule out such scenarios, though clearly the arc-length constraint also automatically constraints the length of the curve  $\gamma$ .

**Definition 3.4** (Elastic curve with bending stiffness). An arc-length parametrized curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is *elastic with bending stiffness* if it is a critical point of the energy in Eq. (2.2) under all variations  $\dot{\gamma} \in C_0^\infty([a, b]; \mathbb{R}^n)$  constraining the arc-length of the curve.

**Theorem 3.5.** A curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is an elastic curve if and only if there is a  $\Lambda \in C^\infty([a, b]; \mathbb{R})$  such that

$$\mathcal{G}_\Lambda^B := \mathcal{G}^B - 2\Lambda'T - 2\Lambda T' = 0.$$

In terms of the unit tangent field  $T$  this can be expressed by

$$\varrho T''' + 3\varrho \langle T', T'' \rangle T + \frac{3}{2}\varrho \langle T', T' \rangle T' + \varrho'' T' + 2\varrho' T'' + \frac{3}{2}\varrho' \langle T', T' \rangle T - \Lambda'T - \Lambda T' = 0,$$

or equivalently, in terms of  $\gamma$ ,

$$\varrho \gamma'''' + 3\varrho \langle \gamma''', \gamma'' \rangle \gamma' + \frac{3}{2}\varrho \langle \gamma'', \gamma'' \rangle \gamma'' + \varrho'' \gamma'' + 2\varrho' \gamma''' + \frac{3}{2}\varrho' \langle \gamma'', \gamma'' \rangle \gamma' - \Lambda'\gamma' - \Lambda\gamma'' = 0.$$

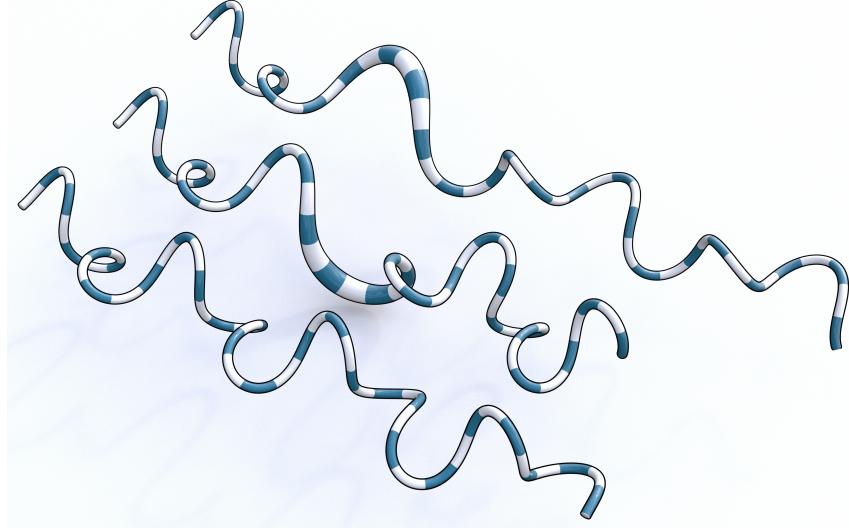


FIGURE 4. A comparison between an elastic curve with constant bending stiffness  $\rho$  obtained from integrating Eq. (5.5) (front) and corresponding curves obtained from the same initial conditions, but with a modulated bending stiffness  $\rho$  (middle, back).

*Proof.* Taking the time derivative of the augmented functional in Eq. (3.3) and using integration by parts we compute

$$\begin{aligned}
 0 &= \int_a^b \langle \ddot{\gamma}, \mathcal{G}^B \rangle + \int_a^b 2\Lambda \langle \dot{\gamma}', T \rangle \\
 &= \int_a^b \langle \ddot{\gamma}, \mathcal{G}^B \rangle + \int_a^b (2\Lambda \langle \dot{\gamma}, T \rangle)' - 2\Lambda' \langle \dot{\gamma}, T \rangle - 2\Lambda \langle \dot{\gamma}, T' \rangle \\
 &= \int_a^b \langle \ddot{\gamma}, \mathcal{G}^B \rangle + \int_a^b -2\Lambda' \langle \dot{\gamma}, T \rangle - 2\Lambda \langle \dot{\gamma}, T' \rangle \\
 &= \int_a^b \langle \ddot{\gamma}, \mathcal{G}^B - 2\Lambda' T - 2\Lambda T' \rangle.
 \end{aligned}$$

Note that the boundary terms vanish since we consider variations  $\dot{\gamma} \in C_0^\infty([a, b]; \mathbb{R}^n)$  and the claim follows.  $\square$

**3.3. Holonomy Constrained Elastic Curves with Bending Stiffness.** In this section we will restrict our attention to curves  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  in  $\mathbb{R}^3$ . As outlined in Section 1, Kirchhoff established torsion as an important parameter when modeling the shapes of elastic wires. One can show that, for the purpose of calculus of variations, constraints on a curve's "total torsion" or holonomy are equivalent [46] and the notion of framed curves is not necessarily required. Therefore, we will refer to stationary points of (2.2) under arc-length preserving variations and with constrained holonomy (Definition 2.2) as *holonomy constrained elastic curves*<sup>2</sup>.

<sup>2</sup>They are also known as Kirchhoff elastica.

**Definition 3.6** (Holonomy constrained elastic curve with bending stiffness). A curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  with *bending stiffness*  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is said to be a *holonomy constrained elastic curve* if it is a critical point of the energy in Eq. (2.2) under all variations  $\dot{\gamma} \in C_0^\infty([a, b]; \mathbb{R}^n)$  constraining the arc-length and holonomy of the curve.

We can again derive the Euler-Lagrange equation characterizing holonomy constrained elastic curves by introducing a suitable Lagrange multiplier constraining the holonomy (Theorem 2.5). For holonomic constraints such as fixed length, or holonomy, the Lagrange multipliers are in fact constants [46, Sec. 2].

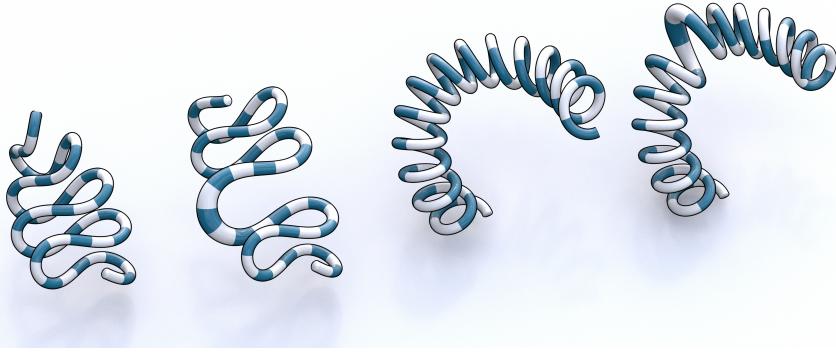


FIGURE 5. Two elastic holonomy constrained curves with constant bending stiffness obtained from integrating condition 5 in Theorem 4.1 with a constant bending stiffness  $\varrho$  (left, middle left) and corresponding curves obtained from the same initial conditions, but with a modulated bending stiffness  $\varrho$  (middle right, right).

**Theorem 3.7.** A curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is a holonomy constrained elastic curve if and only if there is a smooth function  $\Lambda : [a, b] \rightarrow \mathbb{R}$  and a constant  $\mu \in \mathbb{R}$  such that

$$\mathcal{G}_{\Lambda, \mu}^B := \mathcal{G}_\Lambda^B - \mu T \times T'' = \mathcal{G}^B - 2\Lambda'T - 2\Lambda T' - \mu T \times T'' = 0.$$

In terms of the unit tangent field  $T$  this can be expressed by

$$\varrho T''' - \varrho \langle T''', T \rangle T + \frac{3}{2} \varrho \langle T', T' \rangle T' + \varrho'' T' + \varrho'(2T'' + \frac{3}{2} \langle T', T' \rangle T) - \Lambda'T - \Lambda T' - \mu T \times T'' = 0,$$

or equivalently, in terms of  $\gamma$ ,

$$\varrho \gamma'''' - \varrho \langle \gamma'''', \gamma' \rangle \gamma' + \frac{3}{2} \varrho \langle \gamma'', \gamma'' \rangle \gamma'' + \varrho'' \gamma'' + \varrho'(2\gamma'''' + \frac{3}{2} \langle \gamma'', \gamma'' \rangle \gamma') - \Lambda' \gamma' - \Lambda \gamma'' - \mu \gamma' \times \gamma''' = 0.$$

*Proof.* The conditions of [46, Thm. 2.20] are met, so that for constant  $\mu \in \mathbb{R}$  we get by Theorems 2.5 and 3.5 we obtain

$$\varrho T''' + 3\varrho \langle T'', T' \rangle T + \frac{3}{2} \varrho \langle T', T' \rangle T' + \varrho'' T' + \varrho'(2T'' + \frac{3}{2} \langle T', T' \rangle T) - \Lambda'T - \Lambda T' = \mu T \times T''$$

which yields the claim after substituting  $-3\langle T', T'' \rangle = \langle T''', T \rangle$ .  $\square$

#### 4. EQUIVALENT CHARACTERIZATIONS

In this section we derive equivalent characterizations of the elastic curves we derived in the preceding sections. More specifically, we derive analogs of a list of statements which, for the classical case with constant bending-stiffness, relate holonomy constrained elastic curves to dynamical systems such as spinning tops, pendulums, the non-linear Schrödinger equation and the vortex-filament flow [12, 46].

**Theorem 4.1.** *For an arc-length parameterized curve  $\gamma \in C^\infty([a, b], \mathbb{R}^3)$  with unit tangent vector  $T \in C^\infty([a, b]; S^2)$ , the following statements are equivalent:*

(1)  *$\gamma$  is a holonomy constrained elastic curve.*

(2) *There is a smooth function  $\Lambda : [a, b] \rightarrow \mathbb{R}$  and constants  $\mu \in \mathbb{R}$ ,  $\mathbf{a} \in \mathbb{R}^3$  such that*

$$\varrho T'' + \frac{3}{2}\varrho\langle T', T' \rangle T + \varrho' T' - \Lambda T - \mu T \times T' + \mathbf{a} = 0.$$

(3) *There are constants  $\mu \in \mathbb{R}$ ,  $\mathbf{a} \in \mathbb{R}^3$  such that*

$$\varrho T'' - \varrho\langle T'', T \rangle T + \varrho' T' + \mathbf{a} - \langle \mathbf{a}, T \rangle T - \mu T \times T' = 0.$$

(4) *There are constants  $\mu \in \mathbb{R}$ ,  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$  such that*

$$\varrho(\gamma' \times \gamma'') = -\mu T + \mathbf{a} \times \gamma + \mathbf{b}.$$

(5) *There are constants  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$  such that*

$$\varrho\gamma'' = (\mathbf{a} \times \gamma + \mathbf{b}) \times \gamma'.$$

*Proof.* For the first equivalence integrate the differential equation from Theorem 3.7,

$$\varrho T'' + \frac{3}{2}\varrho\langle T', T' \rangle T + \varrho' T' - \Lambda T - \mu T \times T' = \mathbf{a}$$

which is precisely 2.

2  $\Leftrightarrow$  3: Part 3 follows from 2 since its the component orthogonal to  $T$ :

$$\begin{aligned} 0 &= \varrho T'' + \frac{3}{2}\varrho\langle T', T' \rangle T + \varrho' T' - \Lambda T - \mu T \times T' + \mathbf{a} \\ &\quad - \langle \varrho T'' + \frac{3}{2}\varrho\langle T', T' \rangle T + \varrho' T' - \Lambda T - \mu T \times T' + \mathbf{a}, T \rangle T \\ &= \varrho T'' - \varrho\langle T'', T \rangle T + \varrho' T' + \mathbf{a} - \langle \mathbf{a}, T \rangle T - \mu T \times T' \end{aligned}$$

For the converse, note that 3 is the normal component of 2, while defining  $\Lambda$  as

$$\Lambda := \varrho\langle T'', T \rangle + \frac{3}{2}\varrho\langle T', T' \rangle + \langle \mathbf{a}, T \rangle,$$

recovers the component parallel to  $T$ .

3  $\Leftrightarrow$  4: First note that 3 is orhtogonal to  $T$  (as the normal component of 2). Then, taking a cross product with  $T$  amounts to a  $90^\circ$ -rotation in the plane normal to  $T$  and

$$\begin{aligned} 0 &= \varrho T'' \times T + \varrho' T' \times T + \mathbf{a} \times T - \mu(T \times T') \times T \\ &= \varrho T'' \times T + \varrho' T' \times T + \mathbf{a} \times T - \mu T' \\ &= (\varrho T' \times T + \mathbf{a} \times \gamma - \mu T)' \end{aligned}$$

Thus, by integration, there is  $\mathbf{b} \in \mathbb{R}^3$  such that

$$-\mathbf{b} = \varrho\gamma'' \times \gamma' + \mathbf{a} \times \gamma - \mu\gamma'$$

which is equivalent to 4. For the converse direction we take the derivative of 4 and rotate it by  $-90^\circ$  and then take the orthogonal component, which shows equivalence.

4  $\Leftrightarrow$  5: Take the cross product of 4 with  $T$  to obtain

$$\varrho\gamma'' = (\mathbf{a} \times \gamma + \mathbf{b}) \times \gamma'.$$

For the converse direction we must show that the tangential components implied by 5 are constant, as is required by 4. Applying  $\langle \cdot, T \rangle$  to 4, we find

$$\langle \rho T \times T', T \rangle = \mu \langle T, T \rangle - \langle \mathbf{a} \times \gamma + \mathbf{b}, T \rangle$$

which is equivalent to

$$\mu = \langle \mathbf{a} \times \gamma + \mathbf{b}, T \rangle.$$

To verify constancy of this candidate  $\mu$  we check that

$$\mu' = \langle \mathbf{a} \times \gamma', T \rangle + \langle \mathbf{a} \times \gamma + \mathbf{b}, T' \rangle = \frac{1}{\rho} \langle \mathbf{a} \times \gamma + \mathbf{b}, (\mathbf{a} \times \gamma + \mathbf{b}) \times \gamma' \rangle = 0.$$

This concludes the proof.  $\square$

As an immediate consequence we find another necessary condition for a curve to be holonomy constrained elastic.

**Corollary 4.2.** *Let  $\gamma \in C^\infty([a, b], \mathbb{R}^3)$  be an arc-length parameterized holonomy constrained elastic curve with variable bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$ . Then*

$$\varrho \langle \gamma' \times \gamma'', -\mu \gamma' + \gamma \times \mathbf{a} + \mathbf{b} \rangle > 0.$$

Although in our setup the condition of Corollary 4.2 is not sufficient, Hafner and Bickel [25] show that it becomes sufficient when one additionally allows for anisotropic cross-sections.

**4.1. The Curvature of Elastic Curves.** Equivalent characterizations of the three types of curves can also be stated in terms of their curvature functions. While traditionally, the curvature function is considered a scalar quantity which is only defined for plane curves, we may also define a *curvature function*  $\kappa \in C^\infty([a, b]; \mathbb{R}^{n-1})$  for curves in  $\mathbb{R}^n$  [46, Sec. 4.3]: let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  and  $N := [N_1, \dots, N_{n-1}]$  be made of parallel unit normal fields such that  $\det(T, N_1, \dots, N_{n-1}) = 1$ . From  $1 = \langle T, T \rangle$  we notice that  $\langle T, T' \rangle = 0$ , i.e.,  $T'$  is a normal vector field to  $\gamma$ . Therefore, we define the curvature function  $\kappa \in C^\infty([a, b]; \mathbb{R}^{n-1})$  of  $\gamma$  by

$$(4.1) \quad T' = -N\kappa.$$

Note that, by means of  $\kappa$ , the derivative of any normal vector field  $Y = Ny$  can be expressed as [46]

$$(4.2) \quad Y' = \langle \kappa, y \rangle T + Ny'.$$

**Lemma 4.3.** *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  be an arc-length parameterized curve, then*

$$(4.3) \quad \begin{aligned} T' &= -N\kappa \\ T'' &= -\langle \kappa, \kappa \rangle T - N\kappa' \\ T''' &= -3\langle \kappa, \kappa' \rangle T + N(\langle \kappa, \kappa \rangle \kappa - \kappa''). \end{aligned}$$

*Proof.* This is straightforward computation for which we use that  $\langle N, N \rangle = \text{id}_{T^\perp}$  implies  $\langle N', N \rangle = 0$ , hence by Eq. (4.2)  $N'\kappa = \langle \kappa, \kappa \rangle T$  and  $N'\kappa' = \langle \kappa', \kappa \rangle T$ . Then

$$T'' = -N'\kappa - N\kappa' = -\langle \kappa, \kappa \rangle T - N\kappa'$$

and

$$\begin{aligned} T''' &= -2\langle \kappa', \kappa \rangle T - \langle \kappa, \kappa \rangle T' - N'\kappa' - N\kappa'' \\ &= -3\langle \kappa, \kappa' \rangle T + N(\langle \kappa, \kappa \rangle \kappa - \kappa''). \end{aligned}$$

$\square$

From plugging the expressions in Eq. (4.3) into the Euler-Lagrange equations for the different types of elastic curves we obtain equivalent characterizations in terms of conditions on their curvature functions. For example, Eq. (3.2) for the case of constant  $\varrho$  gives

$$\begin{aligned} 0 &= T''' + 3\langle T', T'' \rangle T + \frac{3}{2}\langle T', T' \rangle T' \\ &= -N(\kappa'' + \frac{1}{2}\langle \kappa, \kappa \rangle \kappa). \end{aligned}$$

We conclude that an arc-length parameterized curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  is free elastic if and only if its curvature function satisfies

$$0 = \kappa'' + \frac{1}{2}\langle \kappa, \kappa \rangle \kappa.$$

For the special case of planar curves we retrieve the well known formula [46, Ch. 2]

$$(4.4) \quad \kappa'' + \frac{\kappa^3}{2} = 0.$$

In more generality, consider an arc-length parameterized curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$ . Then, from Eq. (3.2) we find that

$$\begin{aligned} 0 &= \varrho(T''' + 3\langle T', T'' \rangle T + \frac{3}{2}\langle T', T' \rangle T') + \varrho''T' + 2\varrho'T'' + \frac{3}{2}\varrho'\langle T', T' \rangle T \\ &= N(-\varrho\kappa'' - \frac{1}{2}\varrho\langle \kappa, \kappa \rangle \kappa - \varrho''\kappa - 2\varrho'\kappa') + (-2\varrho\langle \kappa, \kappa \rangle + \frac{3}{2}\varrho'\langle \kappa, \kappa \rangle)T \\ &= N(-(\varrho\kappa)' - \frac{1}{2}\varrho\langle \kappa, \kappa \rangle \kappa) + (-\frac{1}{2}\varrho'\langle \kappa, \kappa \rangle)T. \end{aligned}$$

Hence,  $\gamma$  is a free elastic curve with bending stiffness if and only if

$$\begin{cases} 0 = (\varrho\kappa)'' + \frac{1}{2}\varrho\langle \kappa, \kappa \rangle \kappa \\ 0 = \frac{1}{2}\varrho'\langle \kappa, \kappa \rangle. \end{cases}$$

From the second equation we conclude that whenever  $\gamma$  is not a segment of a straight line, it must have constant bending stiffness  $\varrho$  and the defining equations reduce to Eq. (4.4).

By Theorem 3.5, adding the arc-length constraint leads to an extra term in the Euler-Lagrange equation for elastic curves which can also be expressed in terms of the curvature function  $\kappa$  as

$$0 = -\Lambda'T - \Lambda T' = -\Lambda'T + N(\Lambda\kappa).$$

**Theorem 4.4.** *An arc-length parameterized curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is elastic if and only if its curvature function satisfies*

$$(4.5) \quad \begin{cases} 0 = (\varrho\kappa)'' + \frac{1}{2}\varrho\langle \kappa, \kappa \rangle \kappa - \Lambda\kappa \\ 0 = \Lambda' + \frac{1}{2}\varrho'\langle \kappa, \kappa \rangle. \end{cases}$$

for some  $\Lambda \in C^\infty([a, b])$ .

Notably, by the second condition: whenever  $\varrho$  is constant, so is  $\Lambda$  and vice versa.

Moreover, adding a constraint on the holonomy of a curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  adds the term

$$0 = -\mu T \times T'' = -\mu T \times (-\langle \kappa, \kappa \rangle T - N\kappa') = (T \times N)(\mu\kappa') = N(\mu J\kappa'),$$

where  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is the endomorphism field on  $T^\perp$  which corresponds to a  $90^\circ$  rotation around  $T$  in the normal space.

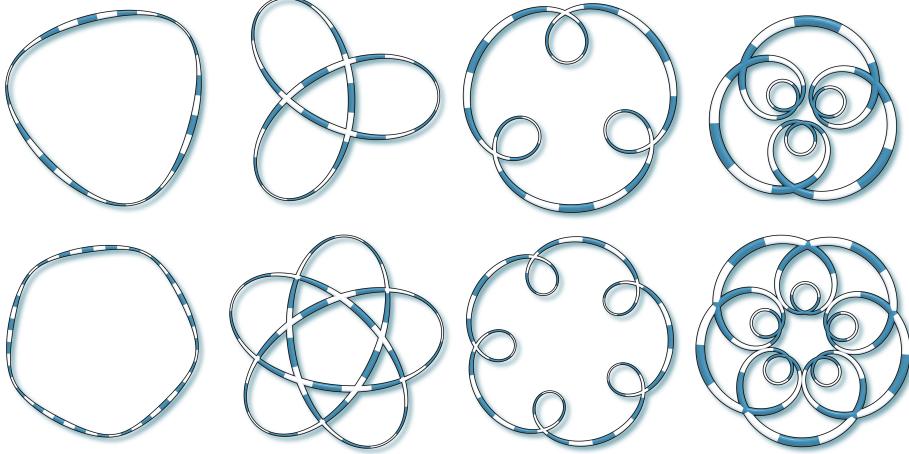


FIGURE 6. Examples of planar elastic curves with variable bending stiffness obtained from integrating Eq. (5.5) which close up and exhibit 3-fold resp. 5-fold symmetries.

**Theorem 4.5.** *An arc-length parameterized curve  $\gamma \in C^\infty([a, b]; \mathbb{R}^3)$  with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  is holonomy constrained elastic if and only if its curvature function satisfies*

$$(4.6) \quad \begin{cases} 0 = (\varrho\kappa)'' + \frac{1}{2}\varrho\langle\kappa, \kappa\rangle\kappa + \mu J\kappa' - \Lambda\kappa \\ 0 = \Lambda' + \frac{1}{2}\varrho'\langle\kappa, \kappa\rangle \end{cases}$$

for some  $\Lambda \in C^\infty([a, b])$  and  $\mu \in \mathbb{R}$ .

**4.2. Closed Examples.** Preliminary numerical experiments suggest that there are a number of interesting closed examples to be discovered. In Fig. 6 we see approximations of planar examples of closed curves which were obtained from integrating Eq. (5.5).

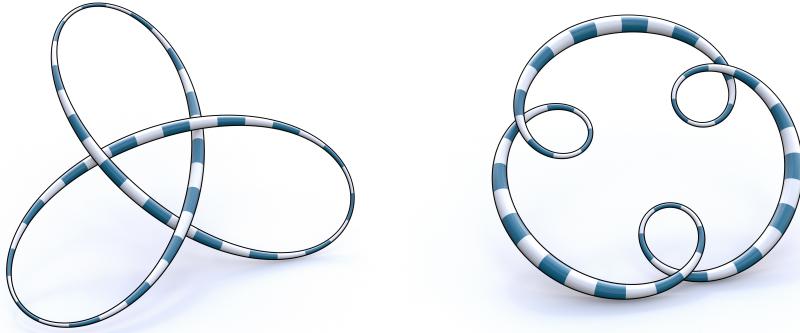


FIGURE 7. Embedded examples of closed holonomy constrained elastic curves with variable bending stiffness obtained from integrating condition 5 in Theorem 4.1.

By adding holonomy, to each of the planar examples, we expect to find a corresponding 1-parameter family of three-dimensional examples [15]. Approximations of such non-planar closed elastic curves corresponding to planar examples in Fig. 6 were obtained from integrating condition 5 in Theorem 4.1 and are depicted in Fig. 7. Note that stationary curves do not generally exhibit any symmetries, and as illustrated by the two examples in Fig. 8, even those that appear to do so can display considerable geometric complexity.

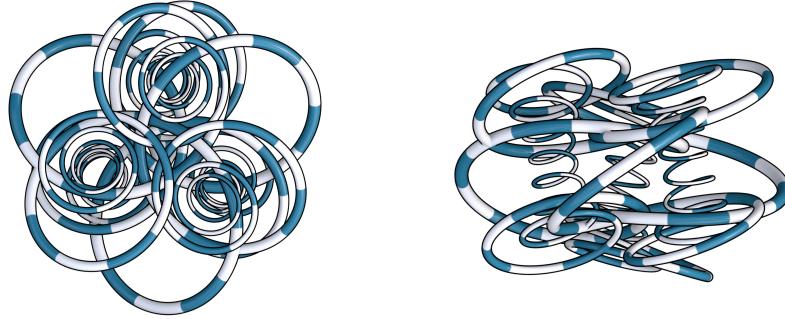


FIGURE 8. Top and side view of a closed holonomy constrained elastic curve with variable bending stiffness comprised of three thin helices connected with a threefold symmetry. This example was obtained from integrating condition 5 in Theorem 4.1.

However, this approach is highly unintuitive, since the relation between those parameters and the constraints is highly nonlinear. Clearly, it would be favorable to have more intuitive computational methods for generating the curves for which the parameters such as length, holonomy and the bending stiffness can be prescribed as an input. However, these computational aspects as well as an explicit analysis of the closing conditions of the curves (see, e.g., [31]) are beyond the scope of the present work. In Appendix A, we outline how our preliminary results were obtained by numerically optimizing the parameters of the ODEs in Theorems 5.2 and 4.1 to satisfy the closing conditions.

## 5. CONNECTIONS WITH DYNAMICAL SYSTEMS

For the case of planar elastic curves with variable bending stiffness Hafner and Bickel found that planar elastic curves with variable bending stiffness are curves that are non-tangentially intersected in only their inflection points ( $\kappa = 0$ ) by a straight line

$$\{x \in \mathbb{R}^2 \mid \langle \mathbf{a}, x \rangle + c = 0\}$$

for some  $\mathbf{a} \in \mathbb{R}^2, c \in \mathbb{R}$ .

**Theorem 5.1** ([24]). *A planar, unit-speed curve  $\gamma \in C^\infty([a, b], \mathbb{R}^2)$  is elastic with bending stiffness  $\varrho$  if and only if*

$$(5.1) \quad (\frac{1}{2}\varrho\kappa^2 - \Lambda)\mathbf{T} + (\varrho\kappa)'J\mathbf{T} = \mathbf{a}$$

for some  $\mathbf{a} \in \mathbb{R}^2$ . Equivalently, for some  $c \in \mathbb{R}$ , the following equations are satisfied:

$$(5.2) \quad \Lambda = \frac{1}{2}\varrho\kappa^2 - \langle \mathbf{a}, \mathbf{T} \rangle,$$

$$(5.3) \quad \varrho\kappa = -\langle J\mathbf{a}, \gamma \rangle + c.$$

*Proof.* First, integrating the Euler-Lagrange equations in Theorem 3.5 gives

$$(5.4) \quad \varrho' T' + \varrho T'' + \frac{3}{2} \varrho \langle T', T' \rangle T - \Lambda T = \mathbf{a}$$

for some  $\mathbf{a} \in \mathbb{R}^3$ . Now, for planar curves,  $N = -JT$ , hence  $T' = \kappa JT$  and  $T'' = \kappa' JT - \kappa^2 T$ , which plugged into Eq. (5.4) yields Eq. (5.1). The equivalence statement follows from the observation that Eq. (5.2) is the tangential component of (5.1), while, since  $JT$  denotes the  $90^\circ$ -rotation of  $T$ , Eq. (5.3) is its normal component integrated once.  $\square$

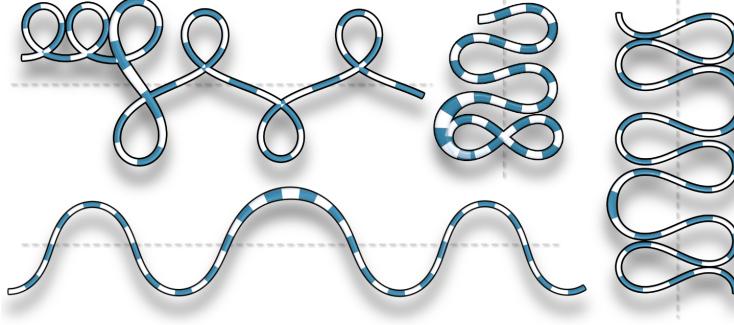


FIGURE 9. A collection of planar elastic curves with variable bending stiffness showcasing the result by [24] that their inflection points are co-linear.

Notably, Theorem 5.1 generalizes a classical result to the case of variable bending stiffness. It is therefore appropriate to look for other results that can be generalized in this context.

**5.1. Pendulum Analogy.** For the case of constant bending stiffness, arc-length parameterized elastic curves and the pendulum equation share an intricate relationship [46]. A generalization of this relationship has been established in [40] who gave an anisotropic version of the pendulum equation. With Theorem 5.2 we give a corresponding generalization of this relationship for the case of variable bending stiffness.

**Theorem 5.2.** *Let  $\gamma \in C^\infty([a, b]; \mathbb{R}^n)$  be an arc-length parameterized curve with bending stiffness  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$ . Then  $\gamma$  is elastic if and only if*

$$(5.5) \quad \varrho T'' + \varrho' T' - \varrho \langle T'', T \rangle T = \mathbf{a} - \langle \mathbf{a}, T \rangle T$$

and the pointwise multiplier  $\Lambda$  is given by:

$$\Lambda = \frac{1}{2} \varrho \langle T', T' \rangle - \langle \mathbf{a}, T \rangle = \frac{1}{2} \varrho \langle \kappa, \kappa \rangle - \langle \mathbf{a}, T \rangle.$$

If  $\varrho$  is constant,  $\Lambda$  is constant.

*Proof.* Similar to the proof of Theorem 4.1 we obtain

$$(5.6) \quad \varrho T'' + \varrho' T' + \frac{3}{2} \varrho \langle T', T' \rangle T - \Lambda T = \mathbf{a}$$

for elastic curves, for some  $\mathbf{a} \in \mathbb{R}^n$ . Eq. (5.5) is the component orthogonal to  $T$ . Taking the inner product of Eq. (5.6) with  $T$  yields

$$\langle \varrho' T' + \varrho T'' + \frac{3}{2} \varrho \langle T', T' \rangle T - \Lambda T, T \rangle = \langle \mathbf{a}, T \rangle,$$

which is equivalent to

$$\varrho \langle T'', T \rangle + \frac{3}{2} \varrho \langle T', T' \rangle - \langle \mathbf{a}, T \rangle = \Lambda.$$

Using  $\langle T'', T \rangle = -\langle T', T' \rangle$  results in the second equation. Conversely, let  $T \in C^\infty([a, b]; \mathbb{S}^{n-1})$  and  $\varrho \in C^\infty([a, b]; \mathbb{R}_{>0})$  satisfy Eq. (5.5) for some  $\mathbf{a} \in \mathbb{R}^n$ . Define

$$\Lambda = \frac{1}{2} \varrho \langle T', T' \rangle - \langle \mathbf{a}, T \rangle$$

and substitute  $-\langle T'', T \rangle$  with  $\langle T', T' \rangle$  and  $-\langle \mathbf{a}, T \rangle$  with  $\Lambda - \frac{1}{2} \varrho \langle T', T' \rangle$  in Eq. (5.5):

$$\varrho T'' + \varrho' T' + \varrho \langle T', T' \rangle T = \mathbf{a} + \Lambda T - \frac{1}{2} \varrho \langle T', T' \rangle T$$

which is equivalent to the integrated differential equation for elastic curves with Lagrange multiplier  $\Lambda$ .  $\square$

**Corollary 5.3.** *The bending energy of an elastic curve  $\gamma \in C^\infty([a, b], \mathbb{R}^n)$  that is parameterized by arc-length is given by*

$$\mathcal{B}_\varrho(\gamma) = \int_a^b \Lambda + \langle \mathbf{a}, \gamma(b) - \gamma(a) \rangle$$

5.1.1. *The Planar Case.* Restricting our attention to the 2-dimensional case and defining  $\mathbf{a} = -qe_2$  as well as  $T = (\begin{smallmatrix} \sin \theta \\ -\cos \theta \end{smallmatrix})$  for  $\theta : [a, b] \rightarrow [0, 2\pi]$  we find that the first entry of the vector-valued (5.5) becomes

$$q \sin \theta \cos \theta = (\varrho \theta'' - \varrho' \theta') \cos \theta.$$

Assuming that  $\theta \notin \frac{\pi}{2} \mathbb{Z}$  this holds if and only if

$$(5.7) \quad \theta'' = -\frac{q}{\varrho} \sin \theta - \frac{\varrho'}{\varrho} \theta'.$$

Notably, Eq. (5.7) describes the motion of a pendulum with time-dependent rod length  $\ell(t)$  [54, Ch. 8], since after a change of variables  $\rho = t^2$  and  $q = g\ell$ , it becomes

$$(5.8) \quad \theta'' = -\frac{g}{\ell} \sin \theta - 2 \frac{\ell'}{\ell} \theta'.$$

Notably, this shows that the relation to the pendulum equation remains intact even when accounting for variable bending stiffness.

Alterations in the length of a pendulum rod can lead to either amplification or damping of its motion. In particular, well-timed adjustments can transform an initially oscillating pendulum into one that loops fully over the top in continuous circles. In the context of elastic curves, this corresponds to modulating the bending stiffness by superimposing a constant term with one or more smooth bump functions. Such modifications can, for example, induce transitions from periodic curve segments without inflection points—corresponding to looping pendulum motion of the tangent vector—to segments with inflection points—corresponding to oscillatory motion of the tangent vector. Fig. 9 illustrates examples of this phenomenon.

5.2. **Vortex Filament Flow.** In this section we will investigate the relations between holonomy constrained elastic curves (with variable bending stiffness) and the dynamics of thin vortex filaments with variable thickness in an incompressible viscous fluid (see, e.g., [39, 12]).

**Definition 5.4.** A *vortex filament* is a map  $\gamma : S^1 \rightarrow \mathbb{R}^3$ . Together with an additional function  $a : S^1 \rightarrow \mathbb{R}_{>0}$  a vortex filament is a *vortex filament with thickness*.

As with rods of variable bending stiffness, we imagine a vortex filament with thickness to be the geometry swept out by a round disc of radius  $a(s)$ , perpendicular to  $T(s)$  and centered at  $\gamma(s)$ . Based on a geometric problem formulation, Padilla *et al.* [39] gave first-order equations of motion of these filaments with variable thickness in an incompressible viscous fluid. In the absence of gravity [39, Eq. (13)] states that, for  $c_1, c_2 \in \mathbb{R}$  the time evolution of a unit strength vortex-filament is (up to a constant) given by

$$(5.9) \quad \dot{\gamma} = u_{\text{BS}}^{(a_0)} - \log\left(\frac{a}{a_0}\right) T \times \frac{dT}{ds} - \frac{c_2}{a} \frac{da}{ds} T,$$

where

$$u_{\text{BS}}^{(a_0)} := \int_{S^1} \frac{T(\tilde{s}) \times (\gamma(s) - \gamma(\tilde{s}))}{(|\gamma(s) - \gamma(\tilde{s})|^2 + c_1 a_0^2)^{3/2}} d\tilde{s},$$

is the cut-off Biot–Savart integral for a vortex filament of constant thickness  $a_0$  (for which  $0 < a \ll a_0 \ll 1$ ),

$$\log\left(\frac{a}{a_0}\right) T \times \frac{dT}{ds}$$

is the *localized induction* term and the last summand is the *tangential velocity*<sup>3</sup> ([39, Eq. (14)]).

**5.2.1. Asymptotic Analysis for Thin Vortex Filaments.** We note that  $\dot{\gamma}$  becomes infinite in the limit  $a \rightarrow 0$ , *i.e.*, the vortex filament moves infinitely fast. Therefore, for small filament thickness  $a$ , the localized induction term becomes the dominating term for the filament evolution. By a suitable re-scaling of the time, we can control this behavior (slowing down the “playback speed” of the filament evolution) and reveal some non-trivial relation to holonomy constrained elastic curves with variable thickness.

The following spells out this procedure in more detail. Define  $\tau := \lambda t$  for some  $\lambda \in \mathbb{R}$ . Then,  $\frac{\partial}{\partial \tau} = \frac{1}{\lambda} \frac{\partial}{\partial t}$  and with  $\dot{\gamma} = \frac{\partial}{\partial t}|_{t=0} \gamma$ , we get

$$(5.10) \quad \frac{\partial}{\partial \tau}|_{\tau=0} \gamma = \frac{1}{\lambda} u_{\text{BS}}^{(a_0)} - \frac{1}{\lambda} \log\left(\frac{a}{a_0}\right) T \times \frac{dT}{ds} - \frac{1}{\lambda} \frac{c_2}{a} \frac{da}{ds} T.$$

Now, choose  $\lambda \sim \log(a_1)$  for some  $a_1$  such that  $0 < a(s) \sim a_1 \ll a_0 \ll 1$ . In other words,  $a(s) = a_1^{\varrho(s)}$  for some  $\varrho : S^1 \rightarrow \mathbb{R}_{>0}$  and

$$\frac{\log(a_0)}{\log(a_1)} =: \varepsilon \ll 1.$$

That is, Eq. (5.10) can be written as

$$(5.11) \quad \frac{\partial}{\partial \tau}|_{\tau=0} \gamma = \frac{1}{\log(a_1)} u_{\text{BS}}^{(a_0)} - \frac{\log(a) - \log(a_0)}{\log(a_1)} T \times \frac{dT}{ds} - \frac{c_2}{\log(a_1)} \frac{d \log(a)}{ds} T$$

$$(5.12) \quad = \frac{1}{\log(a_1)} u_{\text{BS}}^{(a_0)} - (\varrho - \varepsilon) T \times \frac{dT}{ds} - c_2 \frac{d\varrho}{ds} T.$$

Note that  $|u_{\text{BS}}^{(a_0)}|$  is of the order  $\log(a_0)$  the first summand is of order  $\varepsilon$ . Therefore, for thin vortex filaments with  $\varepsilon \rightarrow 0$ , asymptotically (dropping the terms of order  $\varepsilon$ ), Eq. (5.11) is replaced by

$$(5.13) \quad \frac{\partial}{\partial \tau}|_{\tau=0} \gamma = -\varrho T \times \frac{dT}{ds} - c_2 \frac{d\varrho}{ds} T.$$

Since the tangential component of Eq. (5.13) merely amounts to reparametrizations, so that the evolution of the curve’s geometry is determined by the component normal to  $T$  that, after reparametrization by arc-length, agrees with the left-hand side of condition 4 of Theorem 4.1. Therefore, as a corollary of Theorem 4.1 we conclude:

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<sup>3</sup>the Lagrangian form of a viscous Burgers’ equation

**Corollary 5.5.** Let  $\gamma \in C^\infty(S^1; \mathbb{R}^3)$  be a curve with variable bending stiffness  $\varrho \in C^\infty(S^1; \mathbb{R}_{>0})$ . Then  $\gamma$  is holonomy constrained elastic if and only if, when regarded as a thin vortex filament with variable thickness  $a$ , its evolution under equation (5.9) (modulo reparametrization) asymptotically approaches an infinitesimal Euclidean motion as  $a \rightarrow 0$ .

However, the motion of the vortex filament does correspond to a global rigid body motion, since the filament's thickness also evolves over time (see, e.g., [39]). Even if the instantaneous evolution corresponds to an infinitesimal rigid body motion, a time-dependent change in thickness modifies the bending stiffness. As a result, unless the thickness is already constant, the curve will not remain elastic at the next time instance, and its geometry will deform. In the special case of constant thickness, we recover a statement analogous to that known for infinitely thin vortex filaments:

**Corollary 5.6** ([12, Cor. 2 (v)]). A curve  $\gamma \in C^\infty(S^1; \mathbb{R}^3)$  is holonomy constrained elastic if and only if the vortex filament flow  $\dot{\gamma} = \gamma' \times \gamma''$  evolves, modulo reparametrization, by a Euclidean motion with axis along the monodromy of the initial curve  $\gamma$ .

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#### APPENDIX A. NOTES ON VISUALIZATIONS

Throughout the paper we show representative examples encountered during exploration, rather than finely tuned cases. All examples presented in this paper were obtained by integrating ODEs that are equivalent to the Euler–Lagrange equations. Specifically, we use a standard RK4 implementation to solve Eq. (5.5) from Theorem 5.2 to obtain elastic curves, as seen in Figs. 3, 4 and 9, while for holonomy constrained elastic curves we solve condition 5 in Theorem 4.1, as seen in Fig. 5. We use either a sinusoidal type of bending stiffness of the form

$$\varrho(s) = A \sin(s + \xi) + c,$$

or constant bending stiffness modified by Gaussian bump functions, i.e.,

$$\varrho(s) = c + A \exp\left(-\frac{(s-\xi)^2}{2\sigma^2}\right),$$

for  $A, c, \sigma > 0$  and  $\xi \in \mathbb{R}$ .

For the closed examples, which we deem the most interesting examples, we provide specific parameters. All of the closed examples are obtained by integration of condition 5 of Theorem 4.1 and with sinusoidal bending stiffness with  $A = 1$  and  $c = 1.5$ . The initial values are given by  $\gamma(0) = (0, 0, 0)$  and  $\gamma'(0) = (1, 0, 0)$ . The first row of Fig. 6 is obtained with parameters  $a = 0$ , and

	top left	top middle left	top middle right	top right
$\mathbf{b}$	(0, 0.37274, 0)	(0, 0.74556, 0)	(0, 1.4876, 0)	(0, 2.6064, 0)
$\xi$	4.1314	4.6710	4.7618	4.7360

for the top row, while for the bottom row

	bottom left	bottom middle left	bottom middle right	bottom right
<b>b</b>	(0, 0.2234, 0)	(0, 0.67043, 0)	(0, 1.3420, 0)	(0, 2.4584, 0)
$\xi$	1.5859	1.5859	1.5784	1.5862

Moreover, the examples shown in Fig. 7 are obtained with

	left	right
<b>a</b>	$(-1.8502 \times 10^{-6}, 0, 1.2388 \times 10^{-1})$	$(-7.4559 \times 10^{-5}, -6.3287 \times 10^{-13}, 7.5643 \times 10^{-2})$
<b>b</b>	$(-0.1344, 0.1717, -0.7235)$	$(0.0927, 0.0081, 1.4851)$
$\xi$	27.9713	0.1007

while for Fig. 8

<b>a</b>	$(3.8630 \times 10^{-2}, -6.1663 \times 10^{-4}, 23.154)$
<b>b</b>	$(-2.8675, 3.6121, -9.5421)$
$\xi$	36.3588

Since a generic choice of parameters will not yield closed curves, we approximate closed curves that satisfy the Euler–Lagrange equations, by optimizing the parameter set  $\Theta = \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ , the curve’s length  $L$ , and shift  $\xi \in \mathbb{R}$  in the bending-stiffness profile  $\varrho(s) \mapsto \varrho(s + \xi)$ —so that closing conditions hold. Concretely, we define an objective function  $J(\Theta)$  encoding the residual at the endpoints after forward-integrating the ODE with a differentiable implementation of an RK4 solver. Using PyTorch’s reverse-mode automatic differentiation [42], we compute gradients  $\frac{\partial J}{\partial \Phi}$  and solve

$$\operatorname{argmin}_{\Theta} J(\Theta)$$

via standard gradient-based optimizers.

## REFERENCES

- [1] C. Baek and P. M. Reis. “Rigidity of hemispherical elastic gridshells under point load indentation”. In: *J. Mech. Phys. Solids* 124 (2019), pp. 411–426. doi: 10.1016/j.jmps.2018.11.002.
- [2] C. J. Benham and S. P. Mielke. “DNA Mechanics”. In: *Ann. Rev. Biomed. Eng.* 7.1 (2005), pp. 21–53. doi: 10.1146/annurev.bioeng.6.062403.132016.
- [3] M. Bergou, M. Wardetzky, S. Robinson, B. Audoly, and E. Grinpun. “DiscreteElasticRod”. In: *ACM Trans. Graph.* 27.3 (2008). doi: 10.1145/1360612.1360662.
- [4] F. Bernatzki and R. Ye. “Minimal Surfaces with an Elastic Boundary”. In: *Ann. Glob. Anal.* 19 (2001). doi: 10.1023/A:1006734619701.
- [5] F. Bertails, B. Audoly, M.-P. Cani, B. Querleux, F. Leroy, and J.-L. Lévéque. “Superhelices for predicting the dynamics of natural hair”. In: *ACM SIGGRAPH 2006 Papers*. ACM, 2006, pp. 1180–1187. doi: 10.1145/1179352.1142012.
- [6] F. Bertails-Descoubes, A. Derouet-Jourdan, V. Romero, and A. Lazarus. “Inverse design of an isotropic suspended Kirchhoff rod: theoretical and numerical results on the uniqueness of the natural shape”. In: *Proc. R. Soc. Lond. A* 474.2212 (2018). doi: 10.1098/rspa.2017.0837.
- [7] A. I. Bobenko and Y. I. Suris. “A discrete time Lagrange top and discrete elastic curves”. In: 2000. doi: 10.1090/trans2/201.
- [8] D. Brander, J. A. Bærentzen, A.-S. Fisker, and J. Gravesen. “Bézier curves that are close to elastica”. In: *Comp. Aid. Des.* 104 (2018). doi: 10.1016/j.cad.2018.05.003.

- [9] K. Brazda, G. Jankowiak, C. Schmeiser, and U. Stefanelli. *Bifurcation of elastic curves with modulated stiffness*. 2021. doi: 10.1017/S0956792521000371.
- [10] M. Campen and L. Kobbelt. “Dual Strip Weaving: Interactive Design of Quad Layouts using Elastica Strips”. In: *ACM Trans. Graph.* 33.6 (2014). doi: 10.1145/2661229.2661236.
- [11] J. Cantarella, R. B. Kusner, and J. M. Sullivan. “On the minimum ropelength of knots and links”. In: *Invent. Math.* 150 (2002), pp. 257–286. doi: 10.1007/s00222-002-0234-y.
- [12] A. Chern, F. Knöppel, F. Pedit, and U. Pinkall. “Commuting Hamiltonian Flows of Curves in Real Space Forms”. In: *Integrable Systems and Algebraic Geometry*. Ed. by R. Donagi and T. Shaska. Vol. 1. London Mathematical Society Lecture Note Series. Cambridge University Press, 2020, pp. 291–328. doi: 10.1017/9781108773287.013.
- [13] J. L. Coolidge. “The Unsatisfactory Story of Curvature”. In: *Amer. Math. Month.* 59.6 (1952), pp. 375–379. doi: 10.1080/00029890.1952.11988145.
- [14] A. Dall’Acqua, G. Jankowiak, L. Langer, and F. Rupp. “Conservation, Convergence, and Computation for Evolving Heterogeneous Elastic Wires”. In: *SIAM J. Math. Anal.* 56.4 (2024), pp. 4494–4529. doi: 10.1137/23M159086X.
- [15] F. B. Fuller. “The writhing number of a space curve”. In: *Proc. Nat. Acad. Sci.* 68.4 (1971), pp. 815–819. doi: 10.1073/pnas.68.4.81.
- [16] P. B. Furrer, R. S. Manning, and J. H. Maddocks. “DNA Rings with Multiple Energy Minima”. In: *Biophys. J.* 79 (1 2000). doi: 10.1016/S0006-3495(00)76277-1.
- [17] L. Giomi and L. Mahadevan. “Minimal surfaces bounded by elastic lines”. In: *Proc. R. Soc. Lond. A* 468 (2143 2012). doi: 10.1098/rspa.2011.0627.
- [18] G. G. Giusteri, L. Lussardi, and E. Fried. “Solution of the Kirchhoff Plateau Problem”. In: *J. Nonlin. Sc.* 27.3 (2017), pp. 1043–1063. doi: 10.1007/s00332-017-9359-4.
- [19] O. Gonzalez and J. H. Maddocks. “Global Curvature, Thickness, and the Ideal Shapes of Knots”. In: *Proc. Nat. Acad. Sci.* 96.9 (1999), pp. 4769–4773. doi: 10.1073/pnas.96.9.4769.
- [20] A. Goriely and S. Neukirch. “Mechanics of Climbing and Attachment in Twining Plants”. In: *Phys. R. Lett.* 97 (18 2006), p. 184302. doi: 10.1103/PhysRevLett.97.184302.
- [21] A. Goriely and M. Tabor. “Spontaneous Helix Hand Reversal and Tendril Perversion in Climbing Plants”. In: *Phys. R. Lett.* 80 (7 1998), pp. 1564–1567. doi: 10.1103/PhysRevLett.80.1564.
- [22] V. Goss. “Snap buckling, writhing and loop formation in twisted rods”. PhD thesis. University College London, 2003.
- [23] S. Goyal, N. Perkins, and C. Lee. “Nonlinear dynamics and loop formation in Kirchhoff rods with implications to the mechanics of DNA and cables”. In: *J. Comput. Phys.* 209.1 (2005). doi: 10.1016/j.jcp.2005.03.027.
- [24] C. Hafner and B. Bickel. “The design space of plane elastic curves”. In: *ACM Trans. Graph.* 40 (2021), pp. 1–20. doi: 10.1145/3450626.3459800.
- [25] C. Hafner and B. Bickel. “The Design Space of Kirchhoff Rods”. In: *ACM Trans. Graph.* 40.4 (2023). doi: 10.1145/3606033.
- [26] H. Hasimoto. “A soliton on a vortex filament”. In: *J. Fl. Mech.* 51.3 (1972), pp. 477–485. doi: https://doi.org/10.1017/S0022112072002307.
- [27] M. Helmers. “Snapping elastic curves as a one-dimensional analogue of two-component lipid bilayers”. In: *Math. Models Methods Appl. Sci.* 21.05 (2011), pp. 1027–1042. doi: 10.1142/S0218202511005234.

- [28] G. Kirchhoff. “Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes.” In: *J. Reine Angew. Math.* 1859.56 (1859), pp. 285–313. doi: 10.1515/crll.1859.56.285.
- [29] P. Kmoch, U. Bonanni, and N. Magnenat-Thalmann. “Hair simulation model for real-time environments”. In: *Computer Graphics International Conference*. 2009. doi: 10.1145/1629739.1629740.
- [30] J. Langer and D. A. Singer. “The total squared curvature of closed curves”. In: *J. Diff. Geom.* 20.1 (1984), pp. 1–22. doi: 10.4310/jdg/1214438990.
- [31] J. C. Langer and D. A. Singer. “Lagrangian Aspects of the Kirchhoff Elastic Rod”. In: *SIAM Rev.* 38 (1996). doi: 10.1137/S00361445932532.
- [32] J. C. Langer and D. A. Singer. “Knotted Elastic Curves in  $\mathbb{R}^3$ ”. In: *J. Lond. Math. Soc.* s2-30.3 (1984), pp. 512–520. doi: 10.1112/jlms/s2-30.3.512.
- [33] R. L. Levien. “From Spiral to Spline: Optimal Techniques in Interactive Curve Design”. PhD thesis. University of California, Berkeley, 2009.
- [34] J. Lienhard, H. Alpermann, C. Gengnagel, and J. Knippers. “Active Bending, a Review on Structures where Bending is Used as a Self-Formation Process”. In: *Int. J. Space Struc.* 28.3–4 (2013). doi: 10.1260/0266-3511.28.3-4.187.
- [35] M. Liu, L. Domino, and D. Vella. “Tapered elasticae as a route for axisymmetric morphing structures”. In: *Soft Mat.* 16 (33 2020), pp. 7739–7750. doi: 10.1039/D0SM00714E.
- [36] H. K. Moffatt. “The energy spectrum of knots and links”. In: *Nature* 347.6291 (1990), pp. 367–369. doi: 10.1038/347367a0.
- [37] D. Moulton, T. Lessinnes, and A. Goriely. “Morphoelastic rods. Part I: A single growing elastic rod”. In: *J. Mech. Phys. Solids* 61.2 (2013), pp. 398–427. doi: 10.1016/j.jmps.2012.09.017.
- [38] S. Neukirch. “Extracting DNA Twist Rigidity from Experimental Supercoiling Data”. In: *Phys. R. Lett.* 93 (19 2004). doi: 10.1103/PhysRevLett.93.198107.
- [39] M. Padilla, A. Chern, F. Knöppel, U. Pinkall, and P. Schröder. “On Bubble Rings and Ink Chandeliers”. In: *ACM Trans. Graph.* 38.4 (0730-0301), 129:1–129:14. doi: 10.1145/3306346.3322962.
- [40] B. Palmer and Á. Pámpano. “Anisotropic bending energies of curves”. In: *Ann. Glob. Anal. Geom.* 57 (2020), pp. 257–287. doi: 10.1007/s10455-019-09698-1.
- [41] J. Panetta, M. Konaković-Luković, F. Isvoranu, E. Bouleau, and M. Pauly. “X-Shells: A new class of deployable beam structures”. In: *ACM Trans. Graph.* 38.4 (2019). doi: 10.1145/3306346.3323040.
- [42] A. Paszke et al. “PyTorch: an imperative style, high-performance deep learning library”. In: *Proceedings of the 33rd International Conference on Neural Information Processing Systems*. Curran Associates Inc., 2019. doi: 10.5555/3454287.3455008.
- [43] J. Pérez et al. “Fabrication of Flexible Rod Meshes”. In: *ACM Trans. Graph.* 34.4 (2015). doi: 10.1145/2766998.
- [44] J. Pérez, M. A. Otaduy, and B. Thomaszewski. “Computational Design and Automated Fabrication of Kirchhoff-Plateau Surfaces”. In: *ACM Trans. Graph.* 36.4 (2017). doi: 10.1145/3072959.3073695.
- [45] S. Pillwein and P. Musialski. “Generalized Deployable Elastic Geodesic Grids”. In: *ACM Trans. Graph.* 40.6 (2021). doi: 10.1145/3478513.3480516.
- [46] U. Pinkall and O. Gross. *Differential Geometry: From Elastic Curves to Willmore Surfaces*. Compact Textbooks in Mathematics. Birkhäuser, 2024. doi: 10.1007/978-3-031-39838-4.

- [47] D. A. Singer. “Lectures on Elastic Curves and Rods”. In: *AIP Conf. Proc.* 1002.1 (Apr. 2008), pp. 3–32. doi: 10.1063/1.2918095.
- [48] J. Spillmann and M. Harders. “Inextensible elastic rods with torsional friction based on Lagrange multipliers”. In: *Comp. An. & Virt. Worlds* 21 (2010). doi: 10.1002/cav. 362.
- [49] J. Spillmann and M. Teschner. “CoRdE: Cosserat rod elements for the dynamic simulation of one-dimensional elastic objects”. In: *Proc. Symp. Comp. Anim.* 2007. doi: 10.2312/SCA/SCA07/063-072.
- [50] J. Spillmann and M. Teschner. “Cosserat nets”. In: *IEEE Trans. Vis. Comp. Graph.* 15 (2008). doi: 10.1109/TVCG.2008.102.
- [51] D. Stump. “The hocking of cables: a problem in shearable and extensible rods”. In: *Int. J. Solid Struc.* 37.3 (2000). doi: 10.1016/S0020-7683(99)00019-0.
- [52] E.-H. Tjaden. “Einfache elastische Kurven”. PhD thesis. Technische Universität Berlin, 1991.
- [53] M. Vidulis, Y. Ren, J. Panetta, E. Grinspun, and M. Pauly. “Computational Exploration of Multistable Elastic Knots”. In: *ACM Trans. Graph.* 42.4 (2023), pp. 1–15. doi: 10.1145/3592399.
- [54] X. Xin and Y. Liu. *Control design and analysis for underactuated robotic systems*. Springer, 2014. doi: 10.1007/978-1-4471-6251-3.
- [55] J. Zehnder, S. Coros, and B. Thomaszewski. “Designing Structurally-Sound Ornamental Curve Networks”. In: *ACM Trans. Graph.* 35.4 (2016). doi: 10.1145/2897824. 2925888.
- [56] E. Zeidler. *Applied functional analysis: main principles and their applications*. Vol. 109. Springer, 2012. doi: 10.1007/978-1-4612-0821-1.

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