Periodicity, Energy and Power

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Index

Even and odd signals

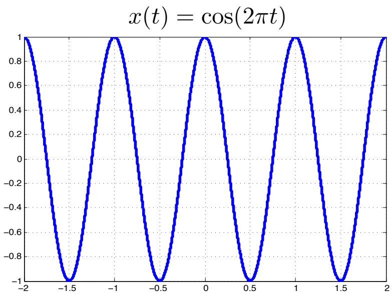
Even signal

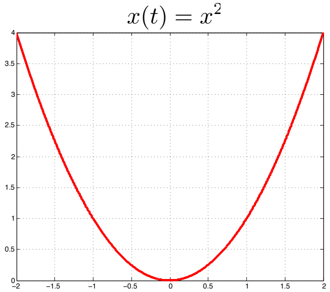
$x(t)$ presents even symmetry (shortly, it is an even signal) if:

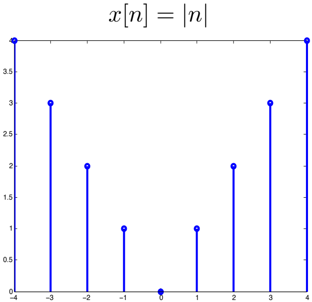
$$x(t) = x(-t) \qquad \forall t$$

The same definition holds for sequences: $x[n]$ is even if $x[n] = x[-n]$ $\forall n$.

By definition, an even signal is symmetric with respect to the coordinate axis.


$x(t) = \cos(2\pi t)$
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1
−2 −1.5 −1 −0.5 0 0.5 1 1.5 2


$x(t) = x^2$
4 3.5 3 2.5 2 1.5 1 0.5 0
−2 −1.5 −1 −0.5 0 0.5 1 1.5 2


$x[n] = |n|$
4 3.5 3 2.5 2 1.5 1 0.5 0
−4 −3 −2 −1 0 1 2 3 4

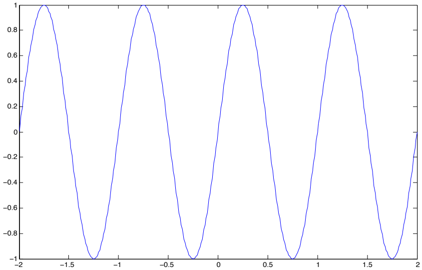
Odd signal

$x(t)$ presents odd symmetry (shortly, it is an odd signal) if:

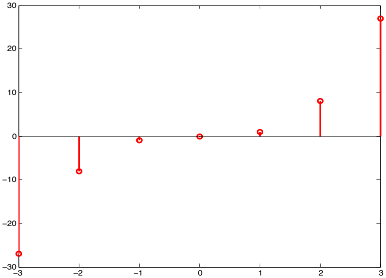
$$x(t) = -x(-t) \qquad \forall t$$

The same definition holds for sequences: $x[n]$ is odd if $x[n] = -x[-n]$ $\forall n$.

Necessarily, if $x(t)$ ($x[n]$) is odd, then $x(0) = 0$ ($x[0] = 0$)!!!



$$x(t) = \sin(2\pi t)$$



$$x[n] = n^3 \qquad n \leq 3$$
$$x[n] = 0 \text{ otherwise}$$

Even and Odd parts of a signal

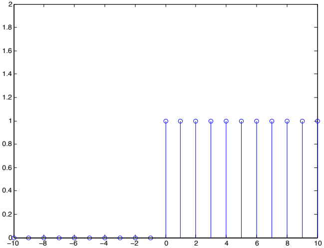
Any signal $x(t)$ can be decomposed as the sum of an even signal $x_{\text{even}}(t)$ plus an odd signal $x_{\text{odd}}(t)$:

$$x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t)$$
$$x[n] = x_{\text{even}}[n] + x_{\text{odd}}[n]$$

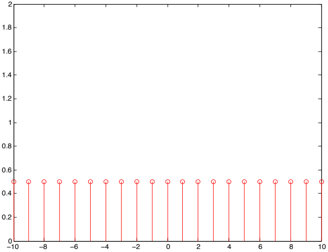
x_{even} and x_{odd} are simply computed as follows:

$$x_{\text{even}}(t) = \frac{x(t) + x(-t)}{2} \qquad x_{\text{even}}[n] = \frac{x[n] + x[-n]}{2}$$
$$x_{\text{odd}}(t) = \frac{x(t) - x(-t)}{2} \qquad x_{\text{odd}}[n] = \frac{x[n] - x[-n]}{2}$$

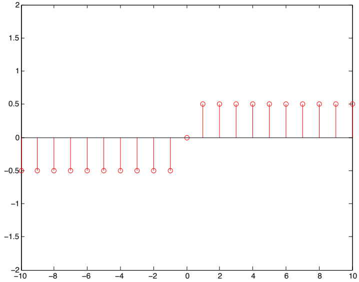
$x[n] = 1 \qquad n \geq 0$



$x_{\text{even}}[n] = 0.5(x[n] + x[-n])$



$x_{\text{odd}}[n] = 0.5(x[n] - x[-n])$



Periodic continuous signals

$x(t)$ is periodic if there exists a real number T such that $x(t) = x(t+T)$ for any t.

$\cos(\frac{2}{3}\pi t)$

$$\cos(\frac{2}{3}\pi(t+T)) = \cos(\frac{2}{3}\pi t) \Leftrightarrow \frac{2}{3}\pi T = 2\pi k \quad k \in \mathbb{Z}$$
$$\Rightarrow T = 3k \qquad k = 1, 2, 3, \ldots$$

$T = 3$ **is the fundamental period.**

Given $x(t) = \cos(2\pi t) + \sin(\frac{\pi}{3}t)$:

- Is $x(t)$ periodic?
- What is the fundamental period?

Example

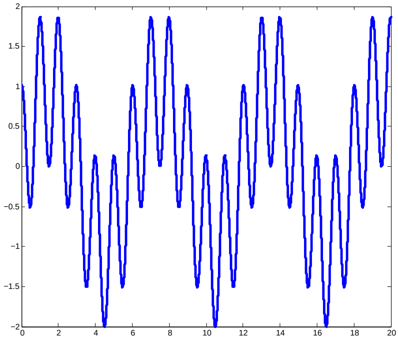
Given $x(t) = \cos(2\pi t) + \sin(\frac{\pi}{3}t)$:

- Is $x(t)$ periodic?
- What is the fundamental period?

Sol.:

1. $\cos(2\pi t)$ is periodic with period $T_1 = k$ for $k = 1, 2, \ldots$
2. $\sin(\frac{\pi}{3}t)$ is periodic with period $T_2 = 6k$ for $k = 1, 2, \ldots$
3. Least common multiple: $6k$.
4. Thus $x(t)$ is periodic with period $T = 6k$ and fundamental period $T = 6$.

$x(t) = \cos(2\pi t) + \sin(\frac{\pi}{3} t)$



Periodic sequences

$x[n]$ is periodic if there exists an integer number N such that $x[n] = x[n + N]$ for any n.

$\cos(\frac{1}{7}\pi n)$

$$\cos(\frac{1}{7}\pi(n+N)) = \cos(\frac{1}{7}\pi n) \Leftrightarrow \frac{1}{7}\pi N = 2\pi k \quad k \in \mathbb{Z}$$

$$\Rightarrow N = 14k \qquad k = 1, 2, 3, \ldots$$

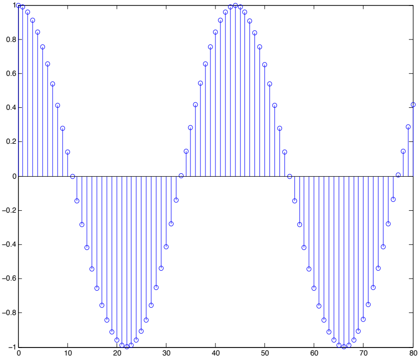
$N = 14$ **is the fundamental period.**

$\cos(\frac{1}{7}n)$

$$\cos(\frac{1}{7}(n+N)) = \cos(\frac{1}{7}n) \Leftrightarrow \frac{1}{7}N = 2\pi k \quad k \in \mathbb{Z}$$

$$\Rightarrow N = 14\pi k \qquad k = 1, 2, 3, \ldots \text{ Never an integer!!}$$

$\cos(\frac{1}{7}n)$ **is not periodic!! Though it may look as periodic...**



$$\begin{aligned}
x[0] &= 1 \\
x[44] &= 0.990170149822708 \\
x[45] &= 0.999996802135490 \\
x[46] &= 0.989450040493128 \\
x[88] &= 0.990520706338839 \\
x[89] &= 0.999987208562412 \\
x[90] &= 0.989080492285302
\end{aligned}$$

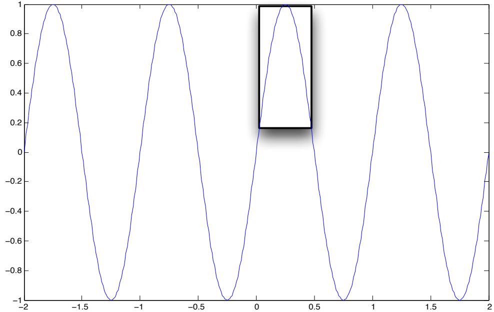
Signal average value

Average value over a given interval (partial average)

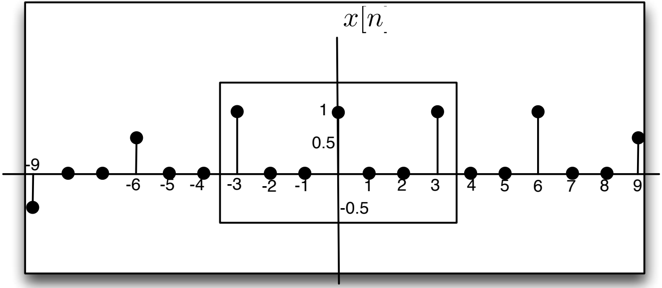
$$< x(t) >_{t_a, t_a+L} = \frac{1}{L} \int_{t_a}^{t_a+L} x(t) dt$$

$$< x[n] >_{n_a, n_a+l} = \frac{1}{l+1} \sum_{n_a}^{n_a+l} x[n]$$

$$x(t) = \sin(2\pi t)$$



$$< x(t) >_{0,\frac{1}{2}} = \frac{1}{1/2}\int_0^{1/2} \sin(2\pi t)dt = -\frac{1}{\pi}[\cos(2\pi t)]_0^{1/2} = \frac{2}{\pi}$$



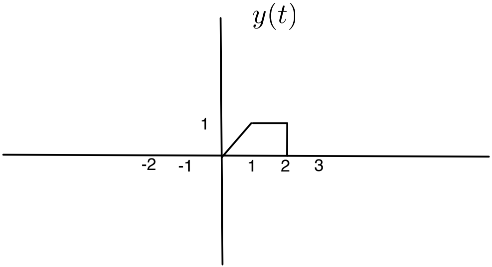
$$< x[n] >_{-3,3} = \frac{3}{7}$$

$$< x[n] >_{-9,9} = \frac{4.5}{19}$$

Average value

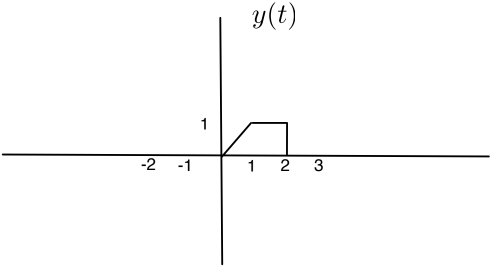
$$< x(t) >= \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} x(t)dt$$

$$< x[n] >= \lim_{l \to \infty} \frac{1}{2l+1} \sum_{-l}^{l} x[n]$$



$$< x(t) > = ?$$

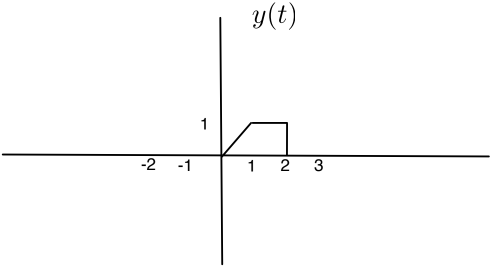
$$< x(t) >_{0,2} = ?$$



$$< x(t) > = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} x(t)dt = \lim_{L\to\infty} \frac{1}{2L} \int_{0}^{2} y(t)dt$$

$$= \lim_{L\to\infty} \frac{1}{2L} \left[\int_{0}^{1} tdt + \int_{1}^{2} 1dt \right] = \lim_{L\to\infty} \frac{1}{2L} \left(\frac{1}{2} + 1 \right) = 0$$

$$< x(t) >_{0,2} = 3/4$$



$$< x(t) > = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} x(t)dt = \lim_{L\to\infty} \frac{1}{2L} \int_{0}^{2} y(t)dt$$

$$= \lim_{L\to\infty} \frac{1}{2L} \left[\int_{0}^{1} tdt + \int_{1}^{2} 1dt \right] = \lim_{L\to\infty} \frac{1}{2L} \left(\frac{1}{2} + 1 \right) = 0$$

$< x(t) >_{0,2} = 3/4$

Average value: Periodic signals

If $x(t)$ is periodic with fundamental period T, then for any $t_0 \in \mathbb{R}$

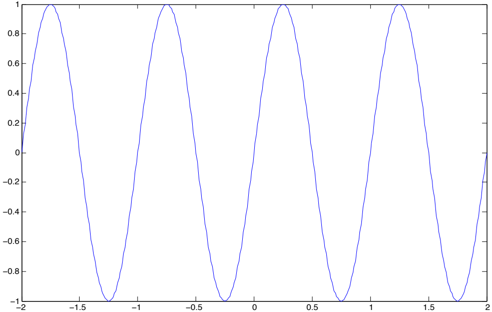
$$< x(t) >= \frac{1}{T} \int_{t_0 - \frac{T}{2}}^{t_0 + \frac{T}{2}} x(t) dt$$

Average value: Periodic signals

If $x[n]$ is periodic with fundamental period N, then for any $n_0 \in \mathbb{Z}$

$$< x[n] >= \frac{1}{N} \sum_{n_0}^{n_0 + N - 1} x[n]$$

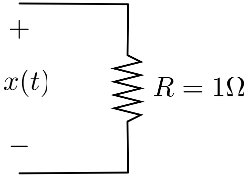
$x(t) = \sin(2\pi t)$



$$< x(t) >= \int_0^1 \sin(2\pi t)dt = -\frac{1}{2\pi}[\cos(2\pi t)]_0^1 = 0$$

Signal Power

Assume $x(t)$ is a voltage signal in an electric circuit.



Dissipated energy per unit time (J/s)

$$p(t) = V(t).I(t) = |x(t)|^2 R = |x(t)|^2 \text{ Watts (W)}$$

$p_x(t) = |x(t)|^2$ is defined as the power signal associated to $x(t)$ (**It is a signal**!!).

The same definition holds for sequences. $p_x[n] = |x[n]|^2$ is the power signal associated to $x[n]$.

Average signal power

Average signal power in a given interval

$$< p_x(t) >_{t_a, t_a+L} = \frac{1}{L}\int_{t_a}^{t_a+L} |x(t)|^2 dt \quad \mathrm{W}$$

$$< p_x[n] >_{n_a, I} = \frac{1}{I+1}\sum_{n_a}^{n_a+I} |x[n]|^2 \quad \mathrm{W}$$

Average power

$$P_x = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} |x(t)|^2 dt \quad \mathrm{W}$$

$$P_x = \lim_{I\to\infty} \frac{1}{2I+1}\sum_{-I}^{I} |x[n]|^2 \quad \mathrm{W}$$

Average power: periodic signals

If $x(t)$ (or $x[n]$) is periodic with fundamental period T, then $p_x(t)$ ($p_x[n]$) is also periodic and T is a valid period. Thus,

Average Power: Periodic signals

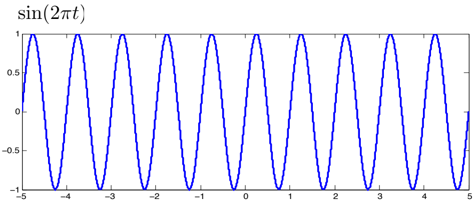
If $x(t)$ is periodic with fundamental period T, then for any $t_0 \in \mathbb{R}$

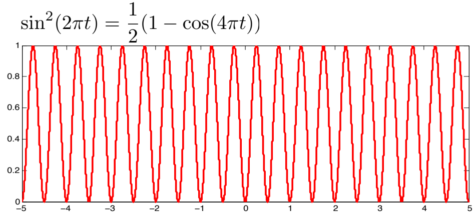
$$P_x = \frac{1}{T} \int_{t_0 - \frac{T}{2}}^{t_0 + \frac{T}{2}} |x(t)|^2 dt \quad \text{W}$$

Average value: Periodic signals

If $x[n]$ is periodic with fundamental period N, then for any $n_0 \in \mathbb{Z}$

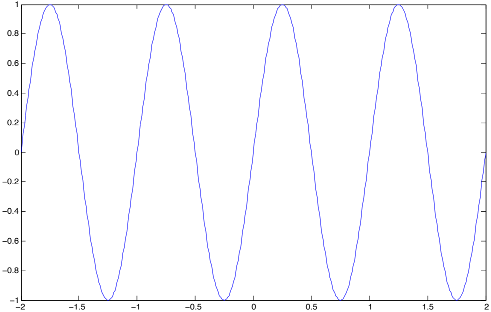
$$P_x = \frac{1}{N} \sum_{no}^{n_0 + N - 1} |x[n]|^2 \quad \text{W}$$





$\sin(2\pi t)$ has fundamental period $T = 1$ s. $\sin^2(2\pi t)$ has fundamental period $T = 0.5$ s but $T = 1$ is a valid period.

$x(t) = \sin(2\pi t)$



$$P_x = \int_0^1 \sin^2(2\pi t)dt = \frac{1}{2}\int_0^1 (1 - \cos(4\pi t))dt = \frac{1}{2} \quad \mathrm{W}$$

http://www.mathwords.com/t/trig_identities.htm

Signal energy

Given $x(t)$, its energy measured over an interval $[t_a, t_b]$ is the integral of the power signal $p_x(t)$:

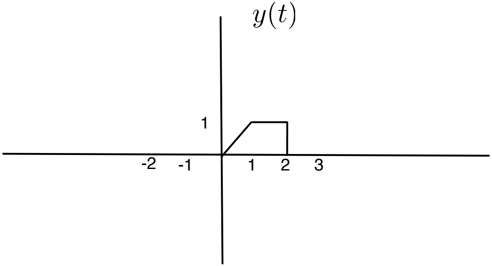
$$E_x(t_a, t_b) = \int_{t_a}^{t_b} |x(t)|^2 dt \quad \text{J (joules)}$$

For sequences,

$$E_x[n_a, n_b] = \sum_{n_a}^{n_b} |x[n]|^2 \quad \text{J}$$

The total energy of a signal is defined as

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E_x = \sum_{-\infty}^{\infty} |x[n]|^2 \quad \text{J}$$



$$E_y = \int_0^1 t^2 dt + \int_1^2 1 dt = \frac{1}{3}[t^3]_0^1 + 1 = \frac{4}{3} \quad \mathrm{J}$$

Energy signals and Power signals

Given $x(t)$,

- If $E_x < \infty$, then $x(t)$ is an energy signal (or a signal defined in terms of energy)
- If $E_x \to \infty$, then $x(t)$ is NOT an energy signal. Besides, if $P_x < \infty$, then $x(t)$ is a power signal (or a signal defined in terms of power).
- If $P_x \to \infty$, then is neither an energy signal nor a power signal.