Combining the definition of energy with Eq. 6.16 yields

$$dw = Cv\,dv,$$

from which

$$\int_0^w dx = C\int_0^v y\,dy,$$

or

Capacitor energy equation ▶

$$w = \frac{1}{2}Cv^2. \tag{6.18}$$

In the derivation of Eq. 6.18, the reference for zero energy corresponds to zero voltage.

Examples 6.4 and 6.5 illustrate the application of the current, voltage, power, and energy relationships for a capacitor.

Example 6.4 Determining Current, Voltage, Power, and Energy for a Capacitor

The voltage pulse described by the following equations is impressed across the terminals of a 0.5 μF capacitor:

$$v(t) = \begin{cases} 0, & t \leq 0\text{ s}; \\ 4t\text{ V}, & 0\text{ s} \leq t \leq 1\text{ s}; \\ 4e^{-(t-1)}\text{ V}, & t \geq 1\text{ s}. \end{cases}$$

a) Derive the expressions for the capacitor current, power, and energy.
b) Sketch the voltage, current, power, and energy as functions of time. Line up the plots vertically.
c) Specify the interval of time when energy is being stored in the capacitor.
d) Specify the interval of time when energy is being delivered by the capacitor.
e) Evaluate the integrals

$$\int_0^1 p\,dt \quad \text{and} \quad \int_1^\infty p\,dt$$

and comment on their significance.

Solution

a) From Eq. 6.13,

$$i = \begin{cases} (0.5\times 10^{-6})(0) = 0, & t < 0\text{s}; \\ (0.5\times 10^{-6})(4) = 2\,\mu\text{A}, & 0\text{ s} < t < 1\text{ s}; \\ (0.5\times 10^{-6})(-4e^{-(t-1)}) = -2e^{-(t-1)}\,\mu\text{A}, & t > 1\text{ s}. \end{cases}$$

The expression for the power is derived from Eq. 6.16:

$$p = \begin{cases} 0, & t \leq 0\text{ s}; \\ (4t)(2) = 8t\,\mu\text{W}, & 0\text{ s} \leq t < 1\text{ s}; \\ (4e^{-(t-1)})(-2e^{-(t-1)}) = -8e^{-2(t-1)}\,\mu\text{W}, & t > 1\text{ s}. \end{cases}$$

The energy expression follows directly from Eq. 6.18:

$$w = \begin{cases} 0 & t \leq 0\text{ s}; \\ \frac{1}{2}(0.5)16t^2 = 4t^2\mu\text{J}, & 0\text{ s} \leq t \leq 1\text{ s}; \\ \frac{1}{2}(0.5)16e^{-2(t-1)} = 4e^{-2(t-1)}\,\mu\text{J}, & t \geq 1\text{ s}. \end{cases}$$

b) Figure 6.11 shows the voltage, current, power, and energy as functions of time.

c) Energy is being stored in the capacitor whenever the power is positive. Hence energy is being stored in the interval 0–1 s.

d) Energy is being delivered by the capacitor whenever the power is negative. Thus energy is being delivered for all t greater than 1 s.

e) The integral of $p\,dt$ is the energy associated with the time interval corresponding to the limits on the integral. Thus the first integral represents the energy stored in the capacitor between 0 and 1 s, whereas the second integral represents the energy returned, or delivered, by the capacitor in the interval 1 s to ∞:

$$\int_0^1 p\,dt = \int_0^1 8t\,dt = 4t^2\Big|_0^1 = 4\,\mu\text{J},$$

$$\int_1^\infty p\,dt = \int_1^\infty (-8e^{-2(t-1)})dt = (-8)\frac{e^{-2(t-1)}}{-2}\Big|_1^\infty = -4\,\mu\text{J}.$$

The voltage applied to the capacitor returns to zero as time increases without limit, so the energy returned by this ideal capacitor must equal the energy stored.

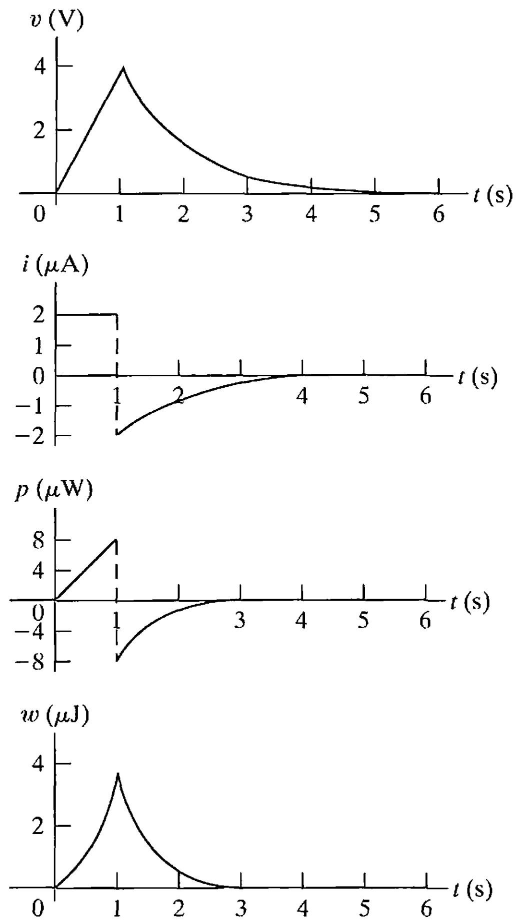


Figure 6.11 ▲ The variables v, i, p, and w versus t for Example 6.4.

Example 6.5 Finding v, p, and w Induced by a Triangular Current Pulse for a Capacitor

An uncharged 0.2 μF capacitor is driven by a triangular current pulse. The current pulse is described by

$$i(t) = \begin{cases} 0, & t \le 0; \\ 5000t \text{ A}, & 0 \le t \le 20\,\mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \le t \le 40\,\mu\text{s}; \\ 0, & t \ge 40\,\mu\text{s}. \end{cases}$$

a) Derive the expressions for the capacitor voltage, power, and energy for each of the four time intervals needed to describe the current.

b) Plot i, v, p, and w versus t. Align the plots as specified in the previous examples.

c) Why does a voltage remain on the capacitor after the current returns to zero?

Solution

a) For $t \le 0$, v, p, and w all are zero.
For $0 \le t \le 20\,\mu s$,

$$v = 5 \times 10^6 \int_0^t (5000\tau)\,d\tau + 0 = 12.5 \times 10^9 t^2 \text{ V},$$

$$p = vi = 62.5 \times 10^{12} t^3 \text{ W},$$

$$w = \frac{1}{2}Cv^2 = 15.625 \times 10^{12} t^4 \text{ J}.$$

For $20\,\mu s \le t \le 40\,\mu s$,

$$v = 5 \times 10^6 \int_{20\mu s}^t (0.2 - 5000\tau)\,d\tau + 5.$$

(Note that 5 V is the voltage on the capacitor at the end of the preceding interval.) Then,

$$v = (10^6 t - 12.5 \times 10^9 t^2 - 10)\ \text{V},$$

$$p = vi,$$

$$= (62.5 \times 10^{12} t^3 - 7.5 \times 10^9 t^2 + 2.5 \times 10^5 t - 2)\ \text{W},$$

$$w = \frac{1}{2}Cv^2,$$

$$= (15.625 \times 10^{12} t^4 - 2.5 \times 10^9 t^3 + 0.125 \times 10^6 t^2$$

$$- 2t + 10^{-5})\ \text{J}.$$

For $t \geq 40\,\mu\text{s}$,

$$v = 10\ \text{V},$$

$$p = vi = 0,$$

$$w = \frac{1}{2}Cv^2 = 10\ \mu\text{J}.$$

b) The excitation current and the resulting voltage, power, and energy are plotted in Fig. 6.12.

c) Note that the power is always positive for the duration of the current pulse, which means that energy is continuously being stored in the capacitor. When the current returns to zero, the stored energy is trapped because the ideal capacitor offers no means for dissipating energy. Thus a voltage remains on the capacitor after i returns to zero.

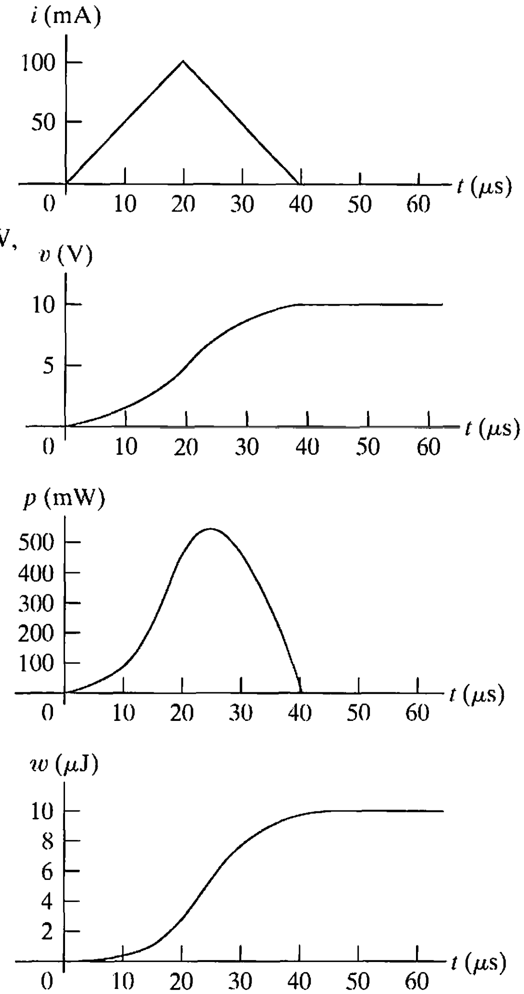
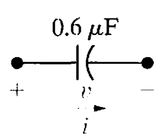


Figure 6.12 ▲ The variables i, v, p, and w versus t for Example 6.5.

✓ASSESSMENT PROBLEMS

Objective 2—Know and be able to use the equations for voltage, current, power, and energy in a capacitor

6.2 The voltage at the terminals of the 0.6 μF capacitor shown in the figure is 0 for $t < 0$ and $40e^{-15{,}000t} \sin 30{,}000t$ V for $t \geq 0$. Find (a) $i(0)$; (b) the power delivered to the capacitor at $t = \pi/80$ ms; and (c) the energy stored in the capacitor at $t = \pi/80$ ms.



Answer: (a) 0.72 A;
(b) −649.2 mW;
(c) 126.13 μJ.

6.3 The current in the capacitor of Assessment Problem 6.2 is 0 for $t < 0$ and $3 \cos 50{,}000t$ A for $t \geq 0$. Find (a) $v(t)$; (b) the maximum power delivered to the capacitor at any one instant of time; and (c) the maximum energy stored in the capacitor at any one instant of time.

Answer: (a) $100 \sin 50{,}000t$ V, $t \geq 0$;
(b) 150 W; (c) 3 mJ.

NOTE: Also try Chapter Problems 6.16 and 6.17.