Introduction to Signals

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- Signals: Functions that represent variations in physical magnitudes.
- Information is contained in the variation with respect some independent variables.

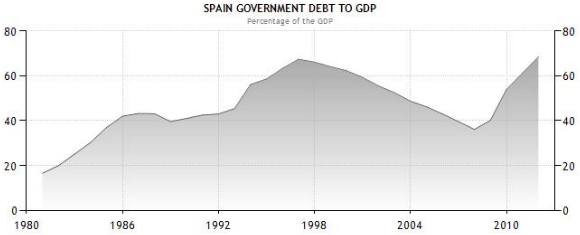

Figure: Spanish debt evolution along the time (GDP: gross domestic product).

Mathematically, we represent signals by functions

- $x(u)$ if a **one-dimensional (unidimensional)** signal with a single independent variable.
- $\mathbf{x}(u)$ is a d-**dimensional (multidimensional)** signal with a single independent variable, where

$$\mathbf{x}(u) = \begin{bmatrix} x_1(u) \\ x_2(u) \\ x_3(u) \\ \cdots \\ x_d(u) \end{bmatrix}$$

- $x(u, t)$ is a unidimensional signal with two independent variables.

Without loss of generality, we restrict to scalar signals with a single independent variable that is referred to as *time*.

E.g.,

- $T(t)$ is the temperature evolution over time.
- $R[n]$ is the average rainfall in 20 consecutive years $n = 1, \ldots, 20$.

Continuous-time signals

The independent variable is continuous in a given interval. Continuous-time signals are defined for a continuum of values of the independent variable.

- $x(t)\ \forall t \in \mathbb{R}$.
- $x(t)\ \forall t \in [a, b]$.
- ...

Figure: Vinyl record

Figure: The shape of the groove encodes the sound that will be played when the stylus goes along the groove

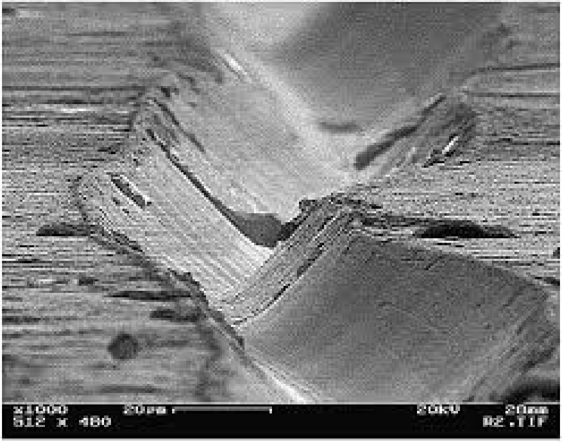

Figure: Source. http://www.synthgear.com/2014/audio-gear/record-grooves-electron-microscope

Discrete-time signals or sequences

They are only defined at discrete times, and consequently for these signals the independent variable takes on only a discrete set of values.

- $x[n]\ \forall n \in \mathbb{Z}$.
- $x[n]\ \forall n \in \{-5, -4, \ldots, 4, 5\}$.
- ...

They are only defined for integer values of the independent variable. Thus, $x[1/2]$ **does not exist**.

- Average temperature per month in the last year $T_{\text{avg}}[n]\ n = 1, 2, \ldots, 12$.
- Benefits in the stock market per day in the last year $P[n]\ n = 1, 2, \ldots, 365$.

Complex and real signals

$x(t)$ and $x[n]$ are **real signals** if $x(t) \in \mathbb{R}\ \forall t$ and $x[n] \in \mathbb{R}\ \forall n$.

E.g. for $\alpha, \beta \in \mathbb{R}$, the following signals are real

$$x(t) = e^{-\alpha t} + \beta t$$
$$x(t) = \cos(2\pi t)t^2 + 3$$
$$x[n] = \alpha^{|n|}$$
$$x[n] = \sum_{u=-\infty}^{n} \alpha^{|n|}$$

Real signals are a part of the *physical world*, physical magnitudes are naturally represented by real signals.

Complex and real signals

$x(t)$ and $x[n]$ are **complex signals** if $x(t) \in \mathbb{C}\ \forall t$ and $x[n] \in \mathbb{C}\ \forall n$.

E.g. for $\alpha, \beta \in \mathbb{C}$, the following signals are complex

$$x(t) = \mathrm{e}^{-\alpha t} + \beta t,$$

$$x(t) = \cos(2\pi t)t^2 + 3tj$$

$$x[n] = \frac{4n + j8n}{n^2 + j4\pi n}$$

$$x[n] = \sum_{u=-\infty}^{n} (-1)^{u/2}\alpha^{|n|}$$

Complex and real signals

- Complex numbers and, by extension, complex signals have no physical existence.
- A huge list of real-life physical effects, though they're described by real numbers, are nevertheless best understood through the mathematics of complex numbers.
- Often, decomposing a real signal in terms of operations between complex signals is very valuable for transforming the problem into a much simpler problem.

- Review of complex numbers!

We need complex signals to understand the electromagnetic spectrum!

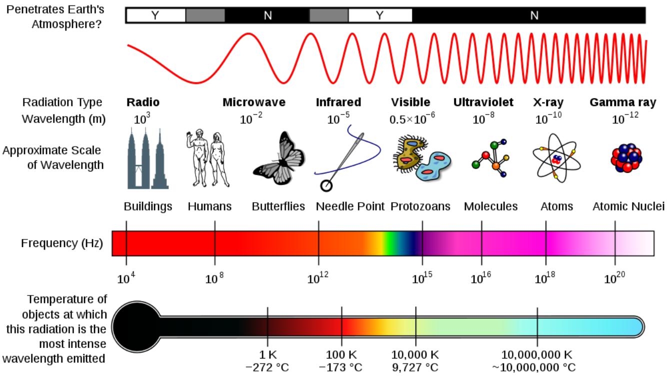

Figure: Source.
http://earthsky.org/space/what-is-the-electromagnetic-spectrum

Complex signals are widely used in signal processing. You can think of a given complex signal $x(t)$ as 2-dimensional signal

$$\begin{bmatrix} x_{\mathcal{R}}(t) \\ x_{\mathcal{I}}(t) \end{bmatrix}, \qquad \begin{bmatrix} |x|(t) \\ \angle(x(t)) \end{bmatrix} = \begin{bmatrix} \sqrt{x_{\mathcal{R}}^2(t) + x_{\mathcal{I}}^2(t)} \\ \arctan(\frac{x_{\mathcal{I}}(t)}{x_{\mathcal{R}}(t)}) \end{bmatrix}$$

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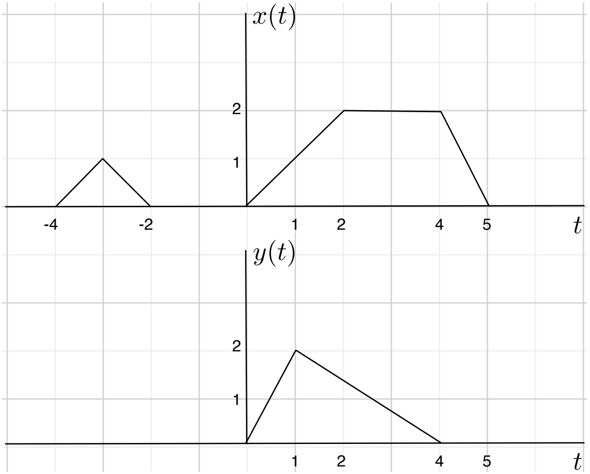
Operations with signals

- Standard operations with mathematical functions.
- They are performed pointwise.

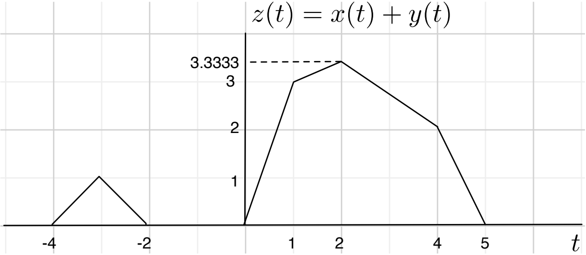
Given $x[n]$ and $y[n]$, $z[n] = x[n] + y[n]$ at $n = 3$ is $x[3] + y[3]$.

Sum of two signals

Compute the sum between the following two signals:



Sol.



Product of two signals

Given

$$x(t) = 2t^2, \qquad y(t) = t$$

then

$$z(t) = x(t) \times y(t) = 2t^3$$

Given

$$x[n] = (3n + 4jn^2) \qquad y[n] = (0.5)^n + 2jn$$

then

$$z[n] = 3n(0.5)^n - 8n^3 + 6jn^2 + 4jn^2(0.5)^n$$

Absolute value or modulus

- Given $x(t)$

$$|x(t)| = \sqrt{x_{\mathcal{R}}^2(t) + x_{\mathcal{I}}^2(t)}$$

- If $x(t)$ is a real signal, then

$$|x(t)| = \begin{cases} x(t) & x(t) \geq 0 \\ -x(t) & x(t) < 0 \end{cases}$$

Other operations that you already know how to compute. Given $x(t) = e^{-4t} + 3j\cos(2\pi t)$, compute

$$y(t) = \frac{\partial x(t)}{\partial t}$$

$$y(t) = \int_{-5}^{t} x(\tau) d\tau$$

$$y(t) = \angle x(t) = \arctan\left(\frac{x_{\mathcal{I}}(t)}{x_{\mathcal{R}}(t)}\right)$$

$$\ldots.$$

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In many situations it is important to consider signals related by a modification of the **independent variable** (time).

Given $x(t)$ or $x[n]$,

- $y[n] = x[-n]$
- $g(t) = x(t+2)$
- $z[n] = x[2n]$
- $w(t) = x(-t/3+4)$
- ....

Time shift

Given $x(t)$,

$$y(t) = x(t)\Big|_{t'=t-t_0} = x(t - t_0)$$

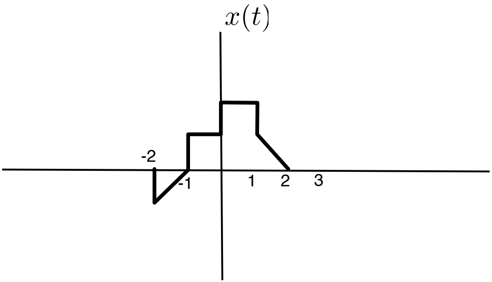
is a shifted version of $x(t)$.

Given $x[n]$,

$$y[n] = x[n]\Big|_{n'=n-n_0} = x[n - n_0]$$

is a shifted version of $x[n]$.

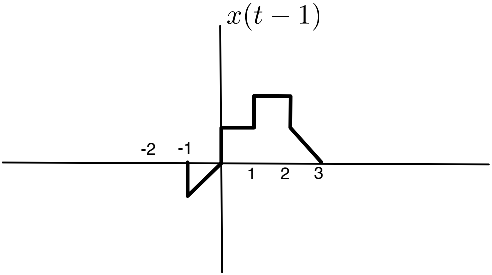
Example (Problem 1.a)

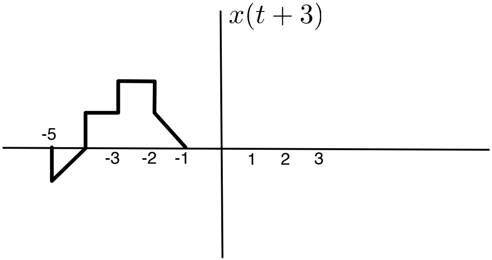


Plot the following signals

- $y(t) = x(t-1)$
- $y(t) = x(t+3)$

Sol.





Temporal Inversion

Given $x(t)$,

$$y(t) = x(-t)$$

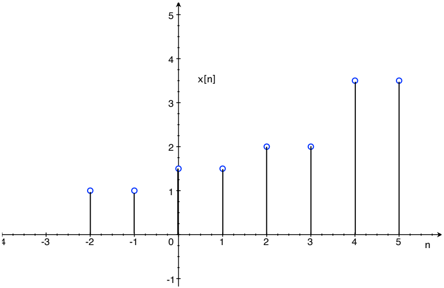
is the temporal inversion of $x(t)$.

Given $x[n]$,

$$y[n] = x[-n]$$

is the temporal inversion of $x[n]$.

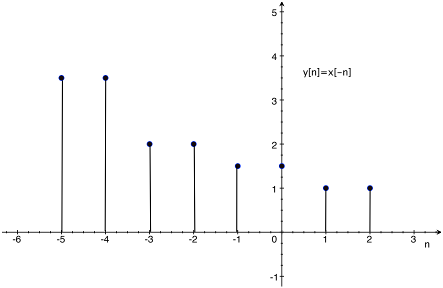
Example



Plot the following signal

- $y[n] = x[-n]$

Sol.



Time scale of a continuous-time signal

Given $x(t)$,

$$y(t) = x(\alpha t)$$

is a scaled version of $x(t)$.

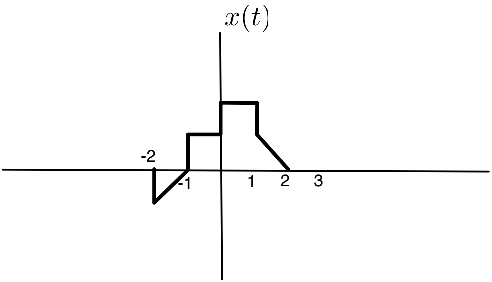
Given $x[n]$,

$$y[n] = x[\alpha n]$$

is a scaled version of $x[n]$.

- If $|\alpha| < 1$ the signal is **expanded**
- If $|\alpha| > 1$ the signal is **contracted**

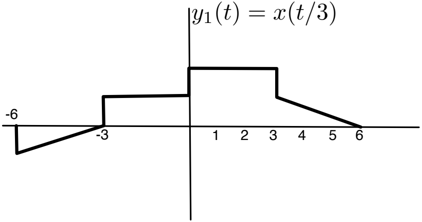
Example (Problem 1.e)

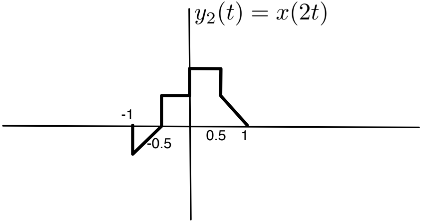


Plot the following signals

- $y_1(t) = x(t/3)$
- $y_2(t) = x(2t)$
- $z_1(t) = y_1(3t)$
- $z_2(t) = y_2(t/2)$
- $y_3(t) = x(-t)$

Sol.

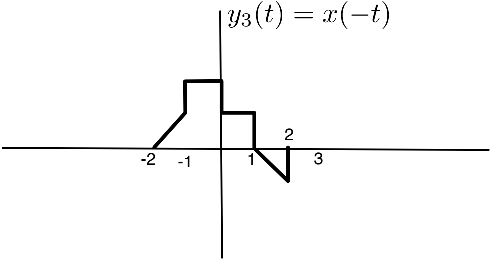




$$z_1(t) = y_1(3t) \quad \stackrel{y_1(t)=x(t/3)}{\Rightarrow} \quad z_1(t) = x(t)$$

$$z_2(t) = y_2(t/2) \quad \stackrel{y_2(t)=x(2t)}{\Rightarrow} \quad z_2(t) = x(t)$$

Time scaling of a continuous-time signal is a **reversible operation**.


$y_3(t) = x(-t)$
-2
-1
1
2
3

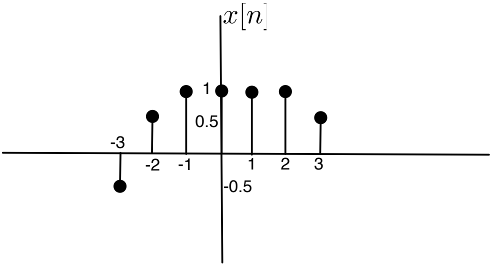
Scaling a discrete-time signal

Given $x[n] \quad n \in \mathbb{Z}$,

- $y[n] = x[2n]$ is well defined for any integer n.
- $y[n]$ is called **a compressed version** of $x[n]$.
- We throw away samples of $x[n]$, **compressing is NOT a reversible operation**.

- $z[n] = x[n/2]$ is NOT defined for n odd.
- We just assume $z[n]$ at n odd is 0.
- We don't throw away samples of $x[n]$, **it is a reversible operation**.

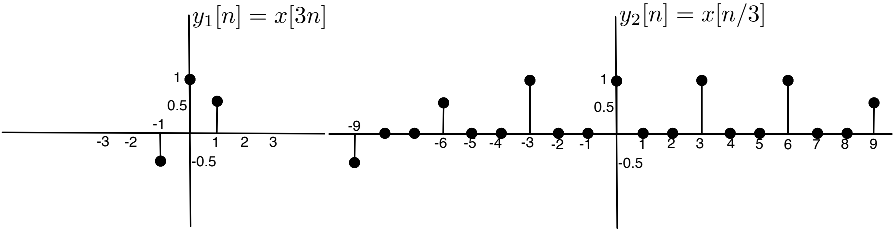
Example



Plot the following signals

- $y_1[n] = x[3n]$
- $y_2[n] = x[n/3]$

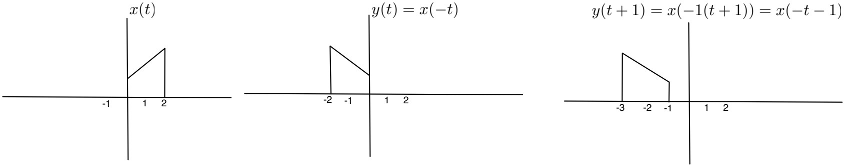
Sol.



Shifting a scaled signal

Given $y(t) = x(\alpha t)$, if we shift $y(t)$ a time t_0 we get

$$z(t) = y(t - t_0) \overset{y(t)=x(\alpha t)}{\Rightarrow} z(t) = x(\alpha t - \alpha t_0).$$



Thus given $y(t) = x(\alpha t)$, $y(t - t_0)$ **is NOT equal to** $x(\alpha t - t_0)$.

Scaling a shifted signal

Given $z(t) = x(t - t_0)$, then $z(\alpha t) = x(\alpha t - t_0)$.

Given $x(t)$, plot

$$y(t) = x(\alpha t + b)$$

We have two equivalent alternatives:

Method-1: First shift, then scale

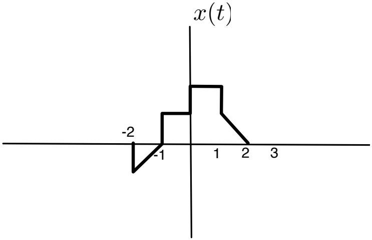
1. $z(t) = x(t + b)$.
2. $y(t) = z(\alpha t) \overset{z(t)=x(t+b)}{\Rightarrow} y(t) = x(\alpha t + b)$.

Method-2: First scale, then shift

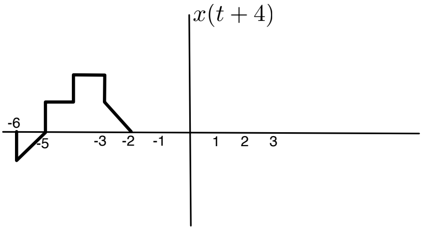
1. $z(t) = x(\alpha t)$
2. $y(t) = z(t + \mathbf{b}/\alpha) \overset{z(t)=x(\alpha t)}{\Rightarrow} y(t) = x(\alpha t + b)$.

Examples from the collection of problems (Problem 1.d)

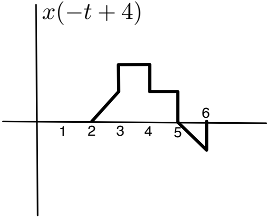
Method-1

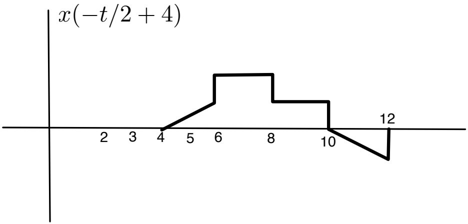


First Step: Shifting



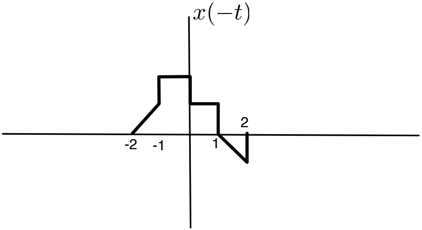
Second Step: Scaling

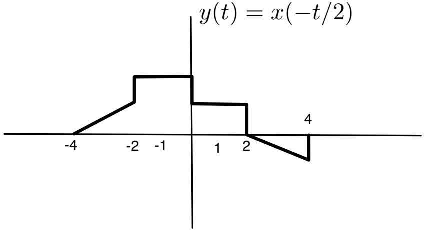




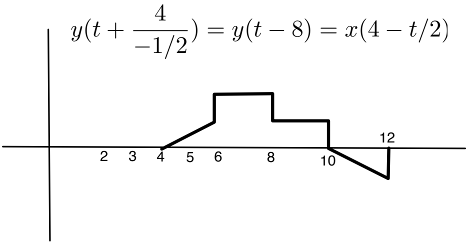
Method-2

First Step: Scaling





Second Step: Shift



Time shift and time scaling. Discrete-time signals

Given $x[n]$, plot

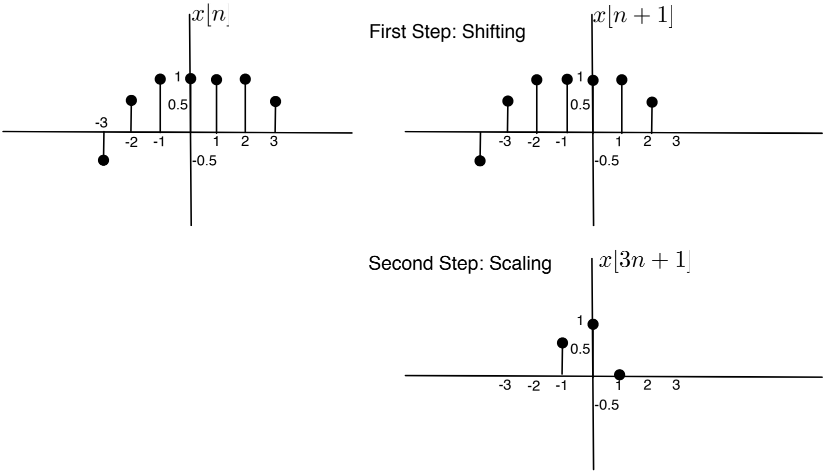
$$y[n] = x[\alpha n + b],$$

where b is an integer.

First shift, then scale

1. $z[n] = x[n + b]$.
2. $y[n] = z[\alpha n] \quad \overset{z[n]=x[n+b]}{\Rightarrow} \quad y[n] = x[\alpha n + b]$.

Example


$x[n]$
First Step: Shifting
$x[n+1]$
Second Step: Scaling
$x[3n+1]$
1
0.5
-0.5
-3
-2
-1
1
2
3

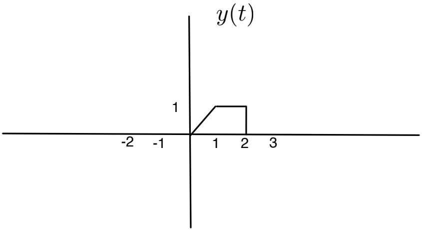
Caution!! In general, doing first the scaling and then shifting the sequence is not possible.

E.g., $y[n] = x[3n + 1]$

- Scaling: $z[n] = x[3n]$.
- Shifting: $w[n] = z[n + 1/3]$!! Not possible! The minimum shift is 1.

Problem 5

Given $y(t)$,

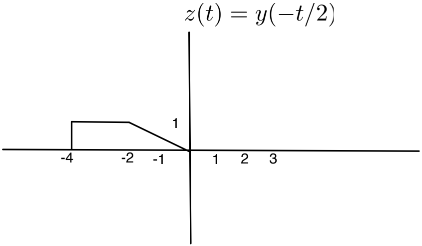


We know that $y(t) = 0.2x(-2t - 3)$. Plot $x(t)$.

Lets solve the problem following two ways.

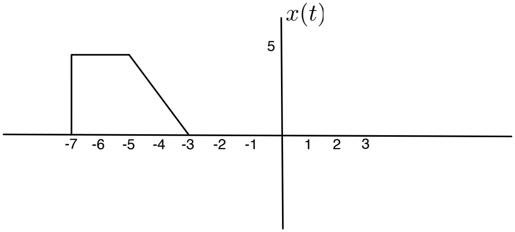
Example 2

Method-1
Step-1: Lets compute $z(t) = y(-t/2) = 0.2x(-2(-t/2) - 3) = 0.2x(t - 3)$.



Example 2

Method-1
Step-2: Lets compute $5z(t+3) = x((t+3)-3) = x(t)$.



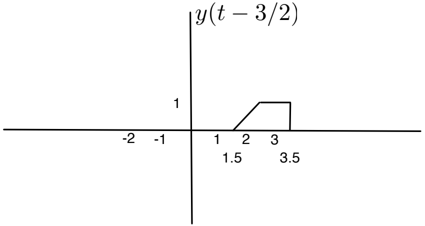
To check the solution, compute $w(t) = 0.2x(t-3)$ and then check that $y(t) = w(-2t)$.

Example 2

Method-2

Step-1: Lets compute

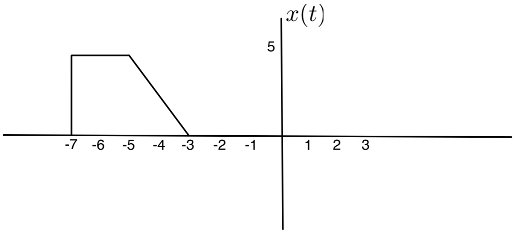
$z(t) = y(t - \frac{3}{2}) = 0.2x(-2(t - \frac{3}{2}) - 3) = 0.2x(-2t + 3 - 3) = 0.2x(-2t)$.



Example 2

Method-2

Step-2: Lets compute $5y(-1/2t) = x(t)$.



To check the solution, compute $w(t) = x(-2t)$ and then check that $y(t) = w(t + \frac{-3}{-2}) = w(t + \frac{3}{2}) = x(t)$.

Example 3 (You should do it at home)

The four signals are just combinations of (scaled) versions of the following ones: