Complex Exponential Signals

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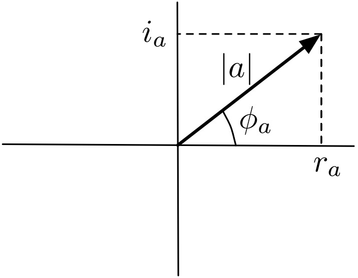
$$a = r_a + j i_a$$



- $|a|^2 = r_a^2 + i_a^2$
- $\phi_a \quad = \arctan\left(\frac{i_a}{r_a}\right)$
- $r_a = |a| \cos(\phi_a)$
- $i_a \quad = |a| \sin(\phi_a)$

Using Euler's formula

$$e^{(r_a + j i_a)} = e^{r_a}\ e^{j i_a} = e^{r_a}\Big(\cos(r_a) + j \sin(i_a)\Big)$$

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1. Continuous-time Complex Exponential Signals: $x(t) = ae^{bt}$
   - a and b are real
   - b is a pure imaginary number
   - a and b are complex
2. Discrete-time Complex Exponential Signals: $x[n] = ae^{bn}$

Complex exponential signals

$$x(t) = a\mathrm{e}^{bt},$$

where a and b are complex numbers:

$$a = |a|\mathrm{e}^{j\phi_a} = r_a + ji_a$$
$$b = |b|\mathrm{e}^{j\phi_b} = r_b + ji_b$$

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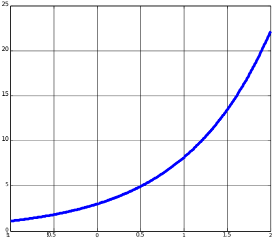
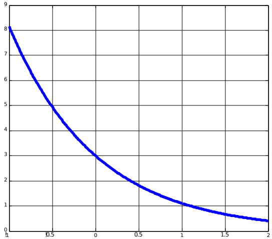
2. Discrete-time Complex Exponential Signals: $x[n] = ae^{bn}$

$x(t) = a\mathbf{e}^{bt}$ **with** $a, b \in \mathbb{R}$

For instance, assume $a = 3$...



b=+1



b=-1

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$x(t) = ae^{bt}$ **with** $b = j\omega$

a is real: $a \in \mathbb{R}$.

$$x(t) = ae^{j\omega t} = a\cos(\omega t) + ja\sin(\omega t)$$

Both the real and imaginary parts of $x(t)$ are periodic with period $T = \frac{2\pi}{\omega}$. ω is the angular frequency (radians/secod).

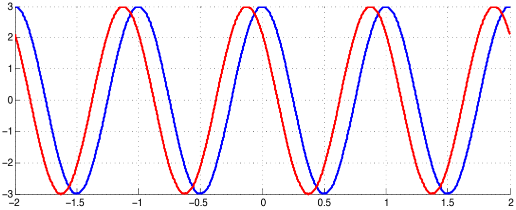
a is complex: $a = r_a + ji_a = |a|e^{j\phi_a}$.

$$x(t) = |a|e^{j(\omega t + \phi_a)} = |a|\cos(\omega t + \phi_a) + j|a|\sin(\omega t + \phi_a)$$

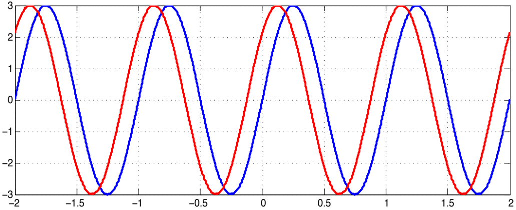
Both the real and imaginary parts of $x(t)$ are periodic with period $T = \frac{2\pi}{\omega}$.

$$x(t) = a\exp(bt), \qquad b = j2\pi$$

Real part of $x(t)$



Imaginary part of $x(t)$



$a = 3$ $\qquad$ $a = 3\cos(\pi/4) + j3\sin(\pi/4)$

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$x(t) = a\mathbf{e}^{bt}$ with $a = |a|\mathbf{e}^{j\phi_a}$ and $b = r + j\omega$

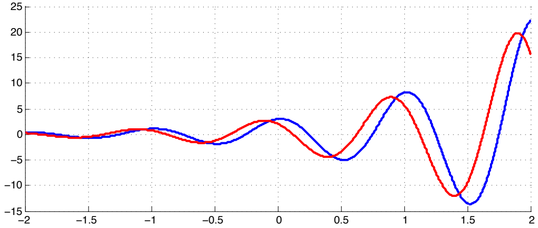
$$
\begin{aligned}
x(t) &= |a|\mathrm{e}^{rt}\mathrm{e}^{j(\omega t+\phi_a)} \\
&= |a|\mathrm{e}^{rt}\cos(\omega t + \phi_a) + j|a|\mathrm{e}^{rt}\sin(\omega t + \phi_a)
\end{aligned}
$$

The real and imaginary parts of $x(t)$ are NOT periodic functions anymore.

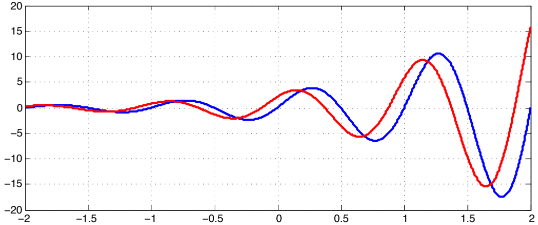
$x(t) = a \exp(bt)$ $\quad$ $a = 3$

$b = 1 + j2\pi$ $\quad$ $a = 3\cos(\pi/4) + j3\sin(\pi/4)$

Real part of $x(t)$



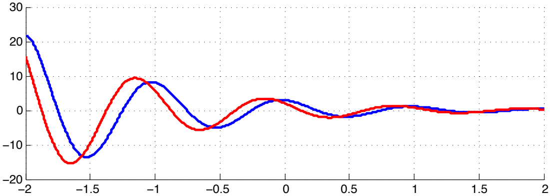
Imaginary part of $x(t)$



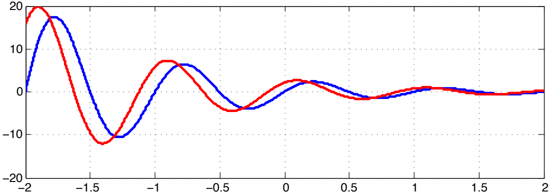
$x(t) = a\exp(bt)$ $\quad$ $a = 3$

$b = -1 + j2\pi$ $\quad$ $a = 3\cos(\pi/4) + j3\sin(\pi/4)$

Real part of $x(t)$



Imaginary part of $x(t)$



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Discrete-time complex exponential signal

$$x[n] = a\mathrm{e}^{bn},$$

where a and b are complex numbers:

$$a = |a|\mathrm{e}^{j\phi_a} = r_a + ji_a$$
$$b = |b|\mathrm{e}^{j\phi_b} = r_b + ji_b$$

$x[n] = a\mathbf{e}^{bn}$ with $a = |a|\mathbf{e}^{j\phi_a}$ and $b = j\Omega$

$$x[n] = |a|e^{j(\Omega n+\phi_a)} = |a|\cos(\Omega n + \phi_a) + j|a|\sin(\Omega n + \phi_a)$$

$x[n]$ is periodic if there exist $N \in \mathbb{Z}$ such that ΩN is a multiple of 2π.

- $x(t) = ae^{j\omega t}$ is always periodic with period $T = 2\pi/\omega$.
- $x[n] = ae^{j\Omega n}$ is only periodic if $\frac{2\pi}{\Omega}$ is a rational number.

Periodicity in the frequency domain **(Discrete case!)**

$$x[n] = ae^{j\Omega n}$$

$$y[n] = ae^{j(\Omega+2\pi k)n} = ae^{\Omega n}e^{j2\pi k} = ae^{\Omega n} = x[n] \qquad \forall n \text{ if } k \in \mathbb{Z}$$

The continuous-time complex exponential $x(t) = ae^{j\omega t}$ is NOT periodic in the frequency domain!.

$x[n] = a\mathbf{e}^{bn}$ with $a = |a|\mathbf{e}^{j\phi_a}$ $b = r + j\Omega$

$$
\begin{aligned}
x[n] &= |a|\mathrm{e}^{rn}\mathrm{e}^{j(\Omega n+\phi_a)} \\
&= |a|\mathrm{e}^{rn}\cos(\Omega n + \phi_a) + j|a|\mathrm{e}^{rn}\sin(\Omega n + \phi_a)
\end{aligned}
$$

$x[n]$ is not periodic.