Unit Step, Impulse

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May 17, 2018

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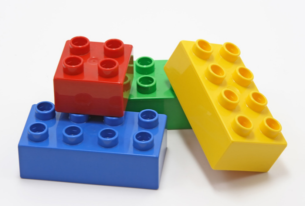
Today

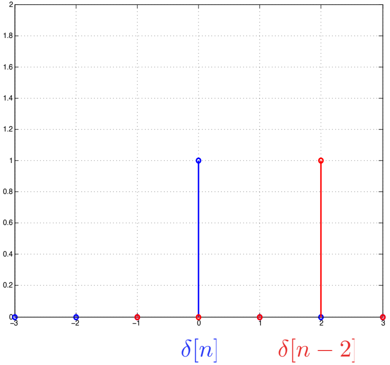
We describe a set of basic signal models whose properties are extremely important to analyze and design complex signal processing systems.



Unit Impulse Sequence (Kronecker delta)

Probably the simplest sequence we can imagine:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad \delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$



Unit Impulse Sequence. Properties

It is an even signal: $\delta[n] = \delta[-n]$.

Is $\delta[n - n_0]$ an even signal?

$$x[n]\delta[n - n_0] = \begin{cases} x[n_0] & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

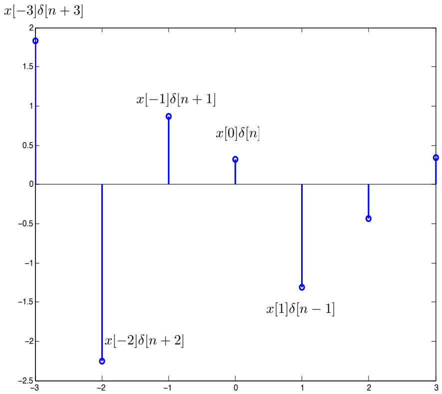
$$\sum_{n=-\infty}^{\infty} \delta[n] = 1$$

$\delta[n]$ is a signal of finite energy.

Unit Impulse Sequence. Properties

Any signal can be decomposed as a sum of unit impulse sequences:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Unit Step Sequence

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

From $\delta[n]$ to $u[n]$

As we have seen, we can decompose it as a sum of unit impulse sequences:

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] = \sum_{0}^{\infty} \delta[n-k]$$

$$u[n-n_0] = \sum_{k=-\infty}^{\infty} u[k-n_0]\delta[n-k] = \sum_{n_0}^{\infty} \delta[n-k]$$

Unit Step Sequence

From $u[n]$ to $\delta[n]$

We can also obtain $\delta[n]$ from $u[n]$:

$$\begin{aligned}
\delta[n] &= u[n] - u[n-1] \\
\delta[n-n_0] &= u[n-n_0] - u[n-n_0-1]
\end{aligned}$$

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Unit Step signal

Similar to the discrete case:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Continuous approximation

Let

$$u_L(t) = \begin{cases} 0 & t < 0 \\ t/L & 0 < t < L \\ 1 & t \geq L \end{cases}$$

then

$$\lim_{L \to 0} u_L(t) = u(t)$$

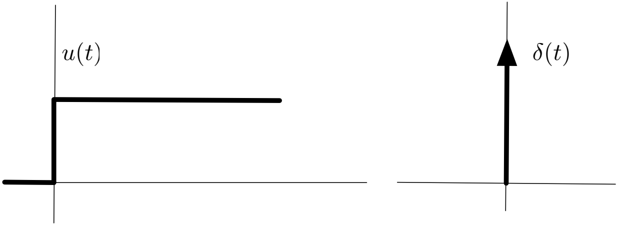
Continuous-time impulse (Dirac Delta function)

The continuous-time impulse function $\delta(t)$ is related to the unit step by the equation

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

and this suggests that

$$\delta(t) = \frac{\partial u(t)}{\partial t} = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$



Continuous-time impulse (Dirac Delta function)

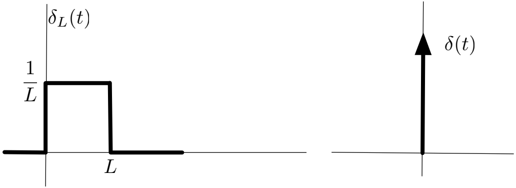
Continuous approximation

Let

$$\delta_L(t) = \frac{\partial u_L(t)}{\partial t} \begin{cases} L^{-1} & 0 < t < L \\ 0 & \text{otherwise} \end{cases}$$

then

$$\delta(t) = \lim_{L \to 0} \delta_L(t)$$



For any L value,

$$\int_{-\infty}^{\infty} \delta_L(t)dt = L\frac{1}{L} = 1 \Rightarrow \int_{-\infty}^{\infty} \delta(t) = 1.$$

The signal $\delta(t)$ has unit area.

$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1$$

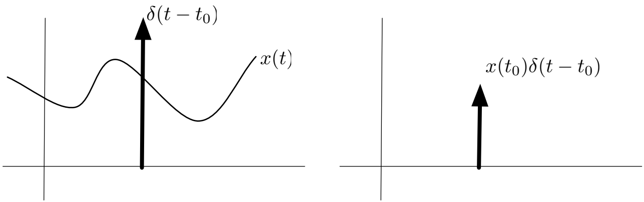
Continuous-time Impulse. Properties.

Multiplication by a constant → Area multiplication

$$\int_{-\infty}^{\infty} k\delta(t)dt = k\int_{-\infty}^{\infty}\delta(t)dt = k$$

$x(t)\delta(t-t_0)$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



Continuous-time Impulse. Properties (II).

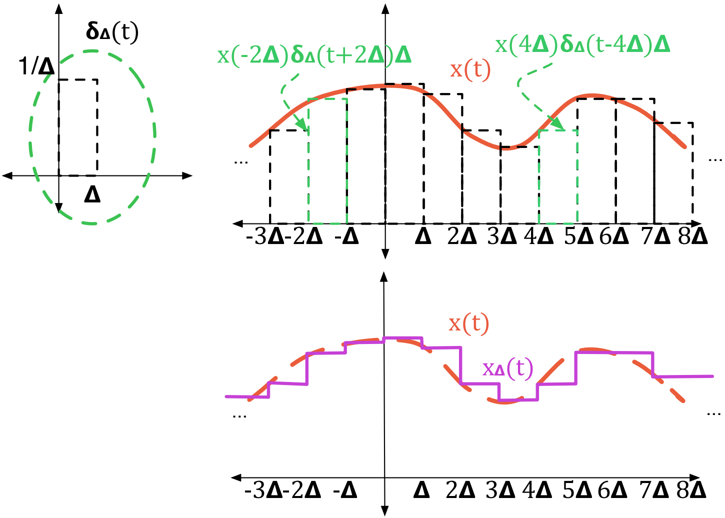
Any signal $x(t)$ can be decomposed as a linear combination of an infinite number of impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Unit step signal

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau$$

We approximate $x(t)$ by a lineal combination of delayed and scaled versions of $\delta_L(t)$



Continuous-time Impulse. Properties (II).

$$x(t) \approx x_L(t) = \sum_{k=-\infty}^{\infty} x(kL)\delta_L(t-kL)L$$

If we take the limit $L \to 0$:

- $kL \to \tau$
- $\sum \to \int$
- $L \to d\tau$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Relationship with the rectangular pulse