Sinusoidal Steady-State Analysis

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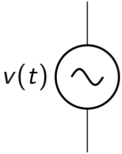
Sinusoidal Steady-State Analysis, why?

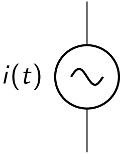
- Thus far, we have focused on circuits with constant sources.
- We are interested now in sources for which the value of the voltage or current varies sinusoidally.
- Important area of study for several reasons:

1. The generation, transmission, distribution and consumption of electric energy occur under essentially sinusoidal steady-sate conditions.
2. An understanding of sinusoidal behavior makes it possible to predict the behavior of circuits with non-sinusoidal sources.
3. Steady-state sinusoidal behavior often simplifies the design of electrical systems.

The sinusoidal source



$$v(t) = V_m \cos(\omega t + \phi).$$



$$i(t) = I_m \cos(\omega t + \phi).$$

- V_m (I_m) the amplitude.
- ω is the angular frequency (rads/s).
- $T = \frac{2\pi}{\omega}$ is the period (seconds).
- ϕ is the phase angle (rads).

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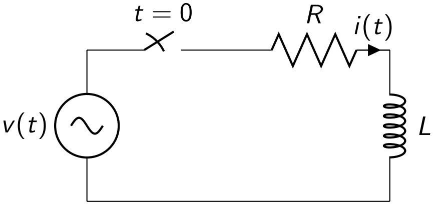
The sinusoidal response: inductor

$$v(t) = V_m \cos(\omega t + \phi).$$



It can be proved that, for $t \to \infty$ (Sinusoidal Steady-State)

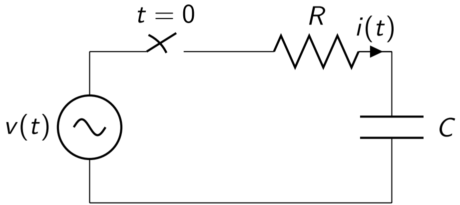
$$\lim_{t\to\infty} i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L}} \cos(\omega t + \phi - \theta) \qquad \theta = \arctan(\frac{\omega L}{R}).$$

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L}} \cos(\omega t + \phi - \theta) \qquad \theta = \arctan(\frac{\omega L}{R}).$$

- $i(t)$ is also a sinusoidal function.
- $i(t)$ has the same frequency that $v(t)$.
- $i(t)$ and $v(t)$ only differ in the amplitude and the phase angle.

The sinusoidal response: capacitor

$$v(t) = V_m \cos(\omega t + \phi).$$



It can be proved that, for $t \to \infty$ (Sinusoidal Steady-State)

$$\lim_{t\to\infty} i(t) = \frac{V_m}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}} \cos(\omega t + \phi + \theta) \qquad \theta = \arctan(\frac{1}{\omega RC}).$$

The capacitor sinusoidal response (II)

$$i(t) = \frac{V_m}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}} \cos(\omega t + \phi + \theta) \qquad \theta = \arctan(\frac{1}{\omega RC}).$$

- $i(t)$ is also a sinusoidal function.
- $i(t)$ has the same frequency that $v(t)$.
- $i(t)$ and $v(t)$ only differ in the amplitude and the phase angle.

Sinusoidal Steady-State Analysis of passive circuits

- Sinusoidal sources (all at the same frequency ω).
- Resistors, capacitors and inductors.
- All magnitudes in the circuit vary according to a sinusoidal function!
- **We only have to compute the amplitude and phase angle of each voltage/current!**
- We use the **phasor method**.

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The phasor

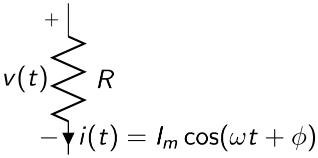
The phasor is a complex number that carries the amplitude and phase angle information about a sinusoidal function.

$$v(t) = V_m \cos(\omega t + \phi) \Rightarrow \mathbf{V} = V_m e^{j\phi}$$

$$i(t) = I_a \cos(\omega t + \phi - \theta) \Rightarrow \mathbf{I} = I_a e^{j(\phi - \theta)}$$

$$i(t) = I_a \sin(\omega t + \phi) = I_a \cos(\omega t + \phi - \frac{\pi}{2}) \Rightarrow \mathbf{I} = I_a e^{j(\phi - \frac{\pi}{2})}$$

The resistor in the Sinusoidal Steady-State regime



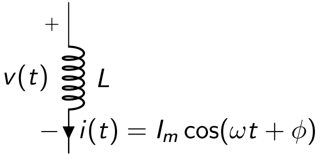
By Ohm's Law

$$v(t) = Ri(t) = RI_m \cos(\omega t + \phi).$$

Therefore, in terms of their respective phasors:

$$\mathbf{V} = R\mathbf{I} = RI_m e^{j\phi}$$

The Inductor in the Sinusoidal Steady-State regime



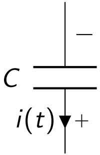
$$v(t) = L\frac{di(t)}{dt} = -\omega L I_m \sin(\omega t + \phi)$$
$$= -\omega L I_m \cos(\omega t + \phi - \frac{\pi}{2})$$

Therefore, in terms of their respective phasors:

$$\mathbf{V} = j\omega L\mathbf{I}$$

If we treat an inductor as a generalized resistor (called **impedance**) with value $\mathbf{Z} = j\omega L$, Ohm's law applies in the usual form for the phasors: $\mathbf{V} = \mathbf{Z}\mathbf{I}$.

The Capacitor in the Sinusoidal Steady-State regime



$v(t) = V_m \cos(\omega t + \phi)$

$$i(t) = C\frac{dv(t)}{dt} = -\omega C V_m \sin(\omega t + \phi)$$
$$= -\omega C V_m \cos(\omega t + \phi - \frac{\pi}{2}).$$

Therefore, in terms of their respective phasors:

$$\mathbf{I} = j\omega C\mathbf{V} \Rightarrow \mathbf{V} = \frac{1}{j\omega C}\mathbf{I}.$$

If we treat the capacitor as a generalized resistor (called **impedance**) with value $\mathbf{Z} = \frac{1}{j\omega C}$, Ohm's law applies in the usual form for the phasors: $\mathbf{V} = \mathbf{Z}\mathbf{I}$.

Analysis of passive circuits in the Sinusoidal Steady-State regime

- All the magnitudes of interest are sinusoidal of the same frequency ω.
- They are represented by phasors.
- Resistors, inductors and capacitors are regarded as generalized resistors, called impedances, with respective values: $\mathbf{Z}_R = R$, $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = \frac{1}{j\omega C}$. We measure impedance also in Ohms Ω.
- Ohm's law: $\mathbf{V} = \mathbf{Z}\mathbf{I}$.
- Although we don't prove it, **Kirchhoff's laws are also valid for phasors**.

Passive circuits in the Sinusoidal Steady-State regime are analyzed using standard techniques: loop current method, voltage node method, Thévenin/Norton equivalent, ...

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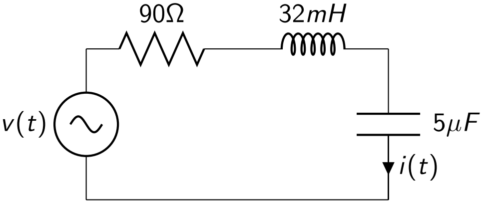
Compute $i(t)$ using the phasor method if $v(t) = 750\cos(5000t + \frac{\pi}{6})$.



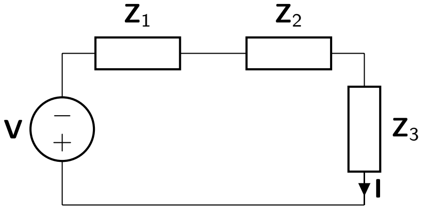
Step 1

Compute the phasor associated to the source and the impedances:

- $v(t) = 750\cos(5000t + \frac{\pi}{6}) \Rightarrow \mathbf{V} = 750e^{j\frac{\pi}{6}}$.
- Resistor of 90 $\Omega \Rightarrow \mathbf{Z}_1 = 90\Omega$.
- Inductor of $32mH \Rightarrow \mathbf{Z}_2 = j\omega L = j160\Omega$.
- Capacitor of $5\mu F \Rightarrow \mathbf{Z}_3 = \frac{1}{j\omega C} = -j40\Omega$.

Step 2

We compute the phasor for the current $i(t)$ working with the phasor circuit.



Kirchhoff's voltage law:

$$\mathbf{I}\mathbf{Z}_1 + \mathbf{I}\mathbf{Z}_2 + \mathbf{I}\mathbf{Z}_3 - \mathbf{V} = 90\mathbf{I} + j160\mathbf{I} - j40\mathbf{I} - \mathbf{V} = 0$$

$$\Rightarrow \mathbf{I} = \frac{\mathbf{V}}{90 + j120} = \frac{750e^{j\frac{\pi}{6}}}{90 + j120}$$

If we apply that $90 + j120 = 150e^{j0.927}$, then

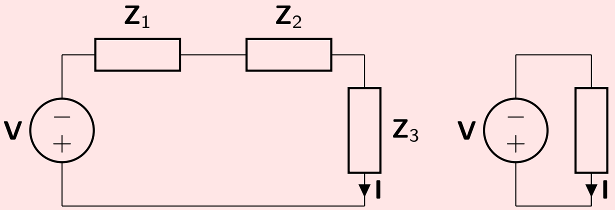
$$\mathbf{I} = \frac{\mathbf{V}}{90 + j120} = \frac{750e^{j\frac{\pi}{6}}}{150e^{j0.927}} = 5e^{-j0.4}A$$

Step 3

Compute $i(t)$ from its phasor $\mathbf{I}$.

$$\mathbf{I} = 5e^{-j0.4} \Rightarrow i(t) = 5\cos(5000t - 0.4)$$

Remark! Impedance in series



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$

As with resistors, the **equivalent impedance** of a set of impedances in series is equal to its sum.

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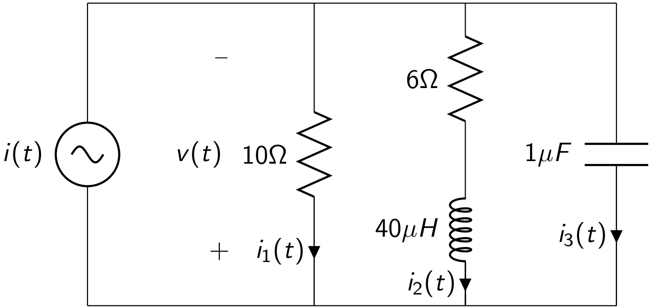
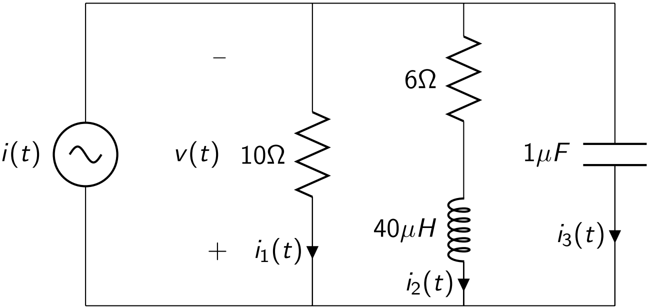
Compute $v(t), i_1(t), i_2(t)$ and $i_3(t)$ using the phasor method if $i_s(t) = 8\cos(\omega t)$, where $\omega = 2 \times 10^5$ rads/s.





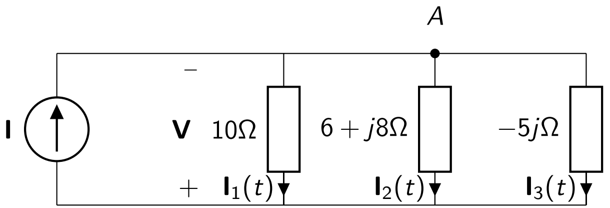
Step 1

Compute the phasor associated to the source and the corresponding impedances:

- $i_s(t) = 8\cos(\omega t) \Rightarrow \mathbf{V} = 8e^{j0} = 8.$
- Resistor of 10 $\Omega \Rightarrow \mathbf{Z}_1 = 10\Omega.$
- Resistor in series with Inductor of $4\mu H \Rightarrow \mathbf{Z}_2 = 6 + j\omega L = 6 + j8\Omega.$
- Capacitor of $1\mu F \Rightarrow \mathbf{Z}_3 = \frac{1}{j\omega C} = -j5\Omega.$

Step 2

We work with the phasor circuit.



Kirchhoff's current law in node A:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} + \frac{\mathbf{V}}{\mathbf{Z}_3} = \frac{\mathbf{V}}{\mathbf{Z}_{eq}}.$$

The equivalent impedance of the set of three impedances in parallel is

$$\mathbf{Z}_{eq} = \left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right)^{-1}, \tag{1}$$

i.e., the same as with simple resistors!!

$$\mathbf{Z}_{eq} = \left(\frac{1}{10} + \frac{1}{6+8j} + \frac{1}{-5j}\right)^{-1} = \left(0.1 + \frac{6-8j}{100} + 0.2j\right)^{-1}$$
$$= \left(0.2e^{j0.64}\right)^{-1} = 5e^{-j0.64}.$$

Therefore

$$\mathbf{V} = \mathbf{Z}_{eq}\mathbf{I} = 40e^{-j0.64} \to v(t) = 40\cos(\omega t - 0.64)$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = 4e^{-j0.64} \to i_1(t) = 4\cos(\omega t - 0.64)$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{40e^{-j0.64}}{6+j8} = \frac{40e^{-j0.64}}{10e^{j0.927}} \approx 4e^{-j0.5\pi} \to i_2(t) = 4\cos(\omega t - 0.5\pi)$$

$$\mathbf{I}_3 = \frac{\mathbf{V}}{\mathbf{Z}_3} = \frac{40e^{-j0.64}}{-5j} = \frac{40e^{-j0.64}}{5e^{-j0.5\pi}} = 8e^{j0.932} \to i_3(t) = 8\cos(\omega t + 0.93).$$