Introduction to Systems.

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Systems

- **System: Any process that results in the transformations of signals**.
- Systems can model the behavior of a chemical process, a hydraulic system, an electric circuit, a communication channel, ...

Microphone (transducer)

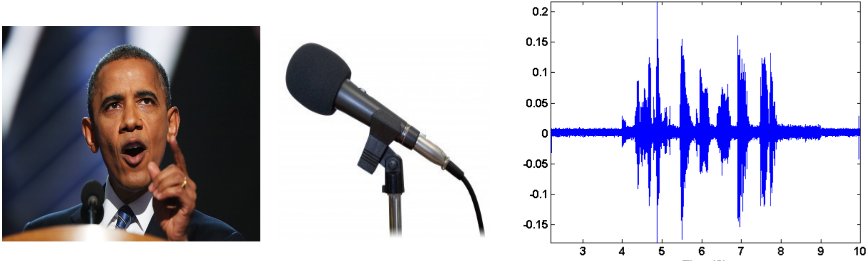
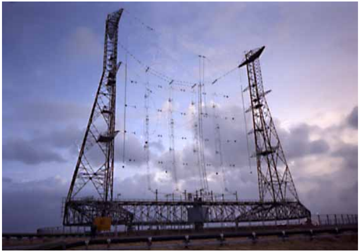


Figure: Voice (pressure) signal $\Rightarrow$ Voltage signal

Communication channel



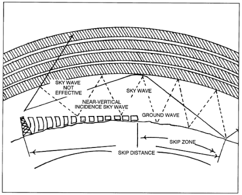


Figure D-4. HF skip zone and use of NVIS.

↓ Generalization

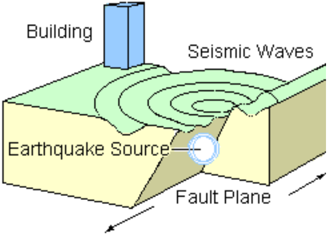
$$\lambda = \frac{c}{f}$$

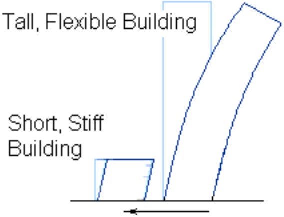
$$10 \text{ m} \leq \lambda_{HF} \leq 100 \text{ m}$$

$x(t)$ → System $T\{x(t)\}$ → $y(t) = \alpha x(t - t_0)$

$$\alpha \leq 1$$

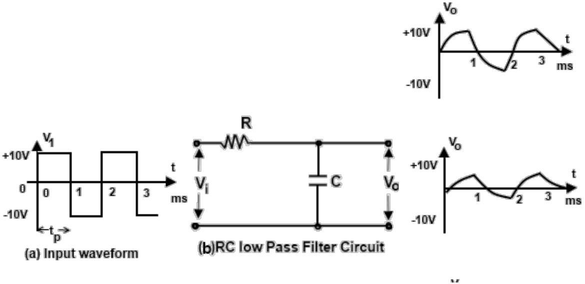
Seismic analysis and earthquake prevention





- The building can be seen as a system.
- Input signal: seismic (sinusoidal) wave with amplitude A and angular frequency ω.
- Output signal: Building curvature and displacement.

Electric Circuits



- Voltages and currents as a function of time in an electrical circuit are examples of signals.
- A circuit is itself an example of a system, which responds to applied input voltage/current signals.

Motivation: Image Filtering

original

filter (3 x 3)

	0	0	0	
	-1	1	0	
	0	0	0	

vertical edge detector

https://docs.gimp.org/en/plug-in-convmatrix.html

Image Filtering

$$
\underset{X_{n\times n}}{\begin{bmatrix}
0_{\times 1} & 1_{\times 0} & 1_{\times 1} & 1 & 0 & 0 & 0 \\
0_{\times 0} & 0_{\times 1} & 1_{\times 0} & 1 & 1 & 0 & 0 \\
0_{\times 1} & 0_{\times 0} & 0_{\times 1} & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
*
\underset{K}{\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}}
=
\underset{Y}{\begin{bmatrix}
1 & 4 & 3 & 4 & 1 \\
1 & 2 & 4 & 3 & 3 \\
1 & 2 & 3 & 4 & 1 \\
1 & 3 & 3 & 1 & 1 \\
3 & 3 & 1 & 1 & 0
\end{bmatrix}}
$$

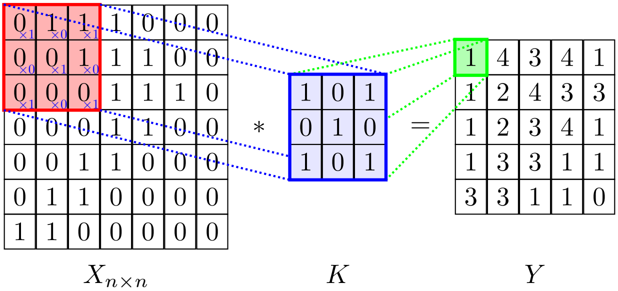
$$
\begin{aligned}
Y[1,1] = {} & X[1,1] * K[1,1] + X[1,2] * K[1,2] + X[1,3] * K[1,3] \\
& + X[2,1] * K[2,1] + X[2,2] * K[2,2] + X[2,3] * K[2,3] \\
& + X[3,1] * K[3,1] + X[3,2] * K[3,2] + X[3,3] * K[3,3]
\end{aligned}
$$

Image Filtering

$$
\underset{X_{n\times n}}{\begin{bmatrix} 0&1&1&1&0&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0\\ 0&1&1&0&0&0&0\\ 1&1&0&0&0&0&0 \end{bmatrix}} * \underset{K}{\begin{bmatrix} 1&0&1\\ 0&1&0\\ 1&0&1 \end{bmatrix}} = \underset{Y}{\begin{bmatrix} 1&\boxed{4}&3&4&1\\ 1&2&4&3&3\\ 1&2&3&4&1\\ 1&3&3&1&1\\ 3&3&1&1&0 \end{bmatrix}}
$$

$$
\begin{aligned}
Y[1,2] = {} & X[1,2] * K[1,1] + X[1,3] * K[1,2] + X[1,4] * K[1,3] \\
& + X[2,2] * K[2,1] + X[2,3] * K[2,2] + X[2,4] * K[2,3] \\
& + X[3,2] * K[3,1] + X[3,3] * K[3,2] + X[3,4] * K[3,3]
\end{aligned}
$$

Image Filtering



$$Y[i,j] = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} X[k_1,k_2]K[i-k_1, j-k_2], \quad K[u,q]=0 \text{ para } u>3, q>3$$

Linear operator, the result does not depend on the position of the image

Image Filtering

original

filter (3 x 3)

	0	0	0	
	-1	1	0	
	0	0	0	

vertical edge detector

https://docs.gimp.org/en/plug-in-convmatrix.html

Image Filtering

original

filter (3 x 3)

	0	1	0	
	1	-4	1	
	0	1	0	

all edge detector

Image Filtering

original

filter (5 x 5)

0	0	0	0	0
0	0	-1	0	0
0	-1	5	-1	0
0	0	-1	0	0
0	0	0	0	0

sharpen

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Symbolically, we represent a system as:

$$x(t) \rightarrow y(t)$$
$$x[n] \rightarrow y[n]$$

Interconnection of systems

Complex system composed of the interconnection of simpler systems:

Cascade interconnection

Parallel interconnection

Series/Parallel interconnection

In general, the order is NOT interchangeable.

$$y[n] = (2x[n] - x^2[n])^2$$

Problem 31

Consider the following interconnection of systems:

where $T_1 : y(t) = 2x(t-2)$, $T_2 : y(t) = dx(t-2)/dt$ and $T_3 : y(t) = x(-t+1)$.

a) Determine the system input-output relationship.

b) Compute the system response when $x(t) = u(t)$.

Sol:

$$e(t) = 2x(t-2) \Rightarrow \; w(t) = \frac{de(t-2)}{dt} = 2\frac{dx(t-4)}{dt}$$
$$r(t) = x(-t+1)$$

For $x(t) = u(t)$

$$y(t) = 2\frac{dx(t-4)}{dt} + x(-t+1) = 2\delta(t-4) + u(-t+1)$$

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Systems with and without memory

A system $x(t) \to y(t)$ is said to be memoryless if the output $y(t_0)$ a one time instant t_0 only depends on the input at the same time instant, i.e., $x(t_0)$.

Examples:

- $y[n] = (2x[n] - x[n]^2)^2$ is memoryless.
- $y(t) = x(t-1)$ is a system with memory.
- $y(t) = x(t-3)x(t+2)$ is a system with memory.

$y(t) = \frac{\partial x(t)}{\partial t}$ is a system **with memory**.

$$y(t) = \frac{\partial x(t)}{\partial t} \doteq \lim_{\Delta \to 0} \frac{x(t + \Delta) - x(t)}{\Delta}$$

- We need $x(t + \Delta)$!
- The system has memory!

Invertibility and inverse systems

- A system is said to be invertible if distinct inputs lead to distinct outputs.
- By observing its output, we can determine its input.

Are invertible the following systems?

- $y[n] = x[2n]$
- $y(t) = x(2t)$
- $y(t) = x^2(t)$

Problem 33

Consider the following discrete-time system

$$y[n] = x[n]x[n-2]$$

a) Is the system memoryless?
b) Determine the system response when the input is $x[n] = A\delta[n]$, where $A \in \mathbb{C}$.
c) Is the system invertible?

Causality

- A system is causal if the output at any time depends only on values of the input at the present time and in the past.
- $y(t) = f(x(t))$ is causal if $y(t_0)$ only depends on $x(t)$ for $t \leq t_0$.
- $y[n] = f(x[n])$ is causal if $y[n_0]$ only depends on $x[n]$ for $n \leq n_0$.
- Memoryless systems are always causal.

Are the following systems causal?

- $y[n] = x[n] - x[n+1]$
- $y(t) = x(t+1)$
- $y[n] = \text{Even}(x[n-1])$?
- $y(t) = x(\sin(t))$

Stability

Bounded signal

Consider an input signal $x(t)$ that verifies

$$|x(t)| \leq B \quad \forall t$$

for some real constant B. We say $x(t)$ is a bounded signal.

Bounded Input, Bounded Output (BIBO) stability

A given system $x(t) \rightarrow y(t)$ is **BIBO stable** if there exists a real constant C for which

$$|y(t)| \leq C \quad \forall t$$

for any input signal $x(t)$ that is bounded.

Same definition holds for discrete-time systems.

Determine whether the following systems are stable:

$$y(t) = x(t/3)$$

$$y[n] = nx[n]$$

$$y(t) = \int_{t-2}^{t-1} x^3(\tau) d\tau$$

$$y[n] = \sum_{k=-\infty}^{n} (1/2)^{n-k} x[k]$$

Time Invariance

A system is time-invariant if a time shift in the input signal causes a time shift in the output signal.

- If $y[n] = f(x[n])$, the system is invariant if $f(x[n - n_0]) = y[n - n_0]$.
- If $y(t) = f(x(t))$, the system is invariant if $f(x(t - t_0)) = y(t - t_0)$.

$y(t) = \sin(x(t))$

Let $x_1(t)$ be the input:

$$y_1(t) = \sin(x_1(t)).$$

Define $x_2(t) = x_1(t - t_0)$:

$$y_2(t) = \sin(x_2(t)) = \sin(x_1(t - t_0)) = y_1(t - t_0).$$

The system is time-invariant.

$y[n] = nx[n]$

Let $x_1[n]$ be the input:

$$y_1[n] = nx_1[n].$$

Define $x_2[n] = x_1[n - n_0]$:

$$\begin{aligned} y_2[n] &= nx_2[n] = nx_1[n - n_0] \\ y_1[n - n_0] &= (n - n_0)x_1[n - n_0]. \end{aligned}$$

Therefore

$$y_1[n - n_0] \neq y_2[n]$$

The system is NOT time-invariant.

Linearity

Linear systems posse the important property of **superposition**.

For any system, consider two arbitrary inputs and their respective outputs:

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t),$$

the system is linear if

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

for any two complex constants $a, b \in \mathbb{C}$.

Linear discrete-time signals

$$ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$

The systems

$$y[n] = (x[n])^2,$$
$$y[n] = \exp(x[n]),$$

are not linear.

The system

$$y[n] = x[n] - x[n-3] + 4x[n-8]$$

is linear.