RC and RL circuits

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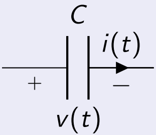
The Capacitor



$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau + v(t_0)$$

From the definition of power

$$p(t) = v(t)i(t) = Cv(t)\frac{dv(t)}{dt}$$

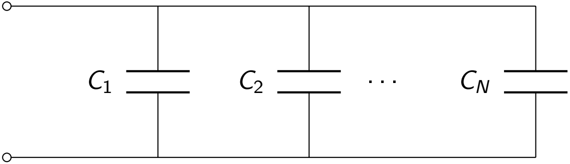
or

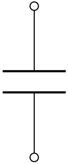
$$p(t) = v(t)i(t) = i(t)\left[\frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau + v(t_0)\right]$$

The energy in the capacitor at any time can be computed as follows:

$$p(t) = \frac{dw(t)}{d(t)} \Rightarrow w(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{-\infty}^{\tau} v(\tau)C\frac{dv(\tau)}{\cancel{d\tau}}\cancel{d\tau} = C\frac{v^2(t)}{2}$$

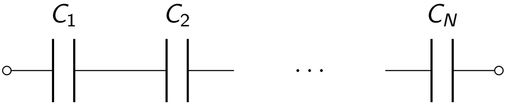
Capacitors in parallel/series

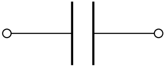




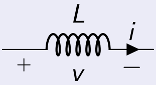
$$C_{eq} = \sum_i C_i$$

$$C_{eq} = \left(\sum_i \frac{1}{C_i} \right)^{-1}$$





The Inductor



$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau + i(t_0)$$

From the definition of power

$$p(t) = v(t)i(t) = Li(t)\frac{di(t)}{dt}$$

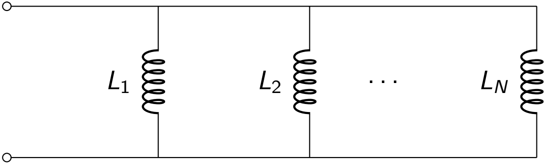
or

$$p(t) = v(t)i(t) = v(t)\left[\frac{1}{L}\int_{t_0}^{t} v(\tau)\mathrm{d}\tau + i(t_0)\right]$$

The energy in the inductor at any time can be computed as follows:

$$p(t) = \frac{dw(t)}{d(t)} \Rightarrow w(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{-\infty}^{t} i(\tau)L\frac{iv(\tau)}{\cancel{d\tau}}\cancel{d\tau} = L\frac{i^2(t)}{2}$$

Inductors in parallel/series



$$L_{eq} = \left(\sum_i \frac{1}{L_i}\right)^{-1}$$

$$L_{eq} = \sum_i L_i$$

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First order linear differential equation with constant coefficients

Compute $y(t)$ such that $y(t_0) = y_0$ if

$$\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \gamma$$

where τ and γ are constants.

Sol.

$$y(t) = \tau\gamma \left(1 - e^{-\frac{(t-t_0)}{\tau}}\right) + y_0 e^{-\frac{(t-t_0)}{\tau}} \qquad t \geq t_0$$

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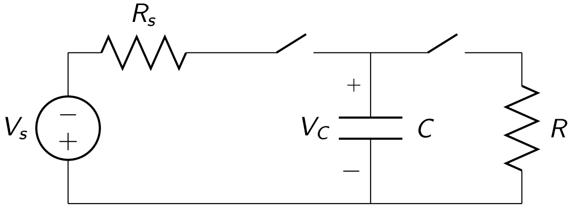
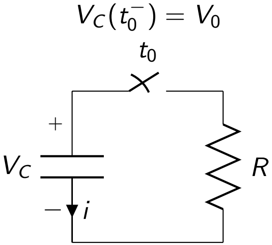
- We now determine the currents and voltages that arise in simple circuits when the energy stored in either an inductor or a capacitor is released or acquired.
- We first focus on the RC circuit: a single capacitor, a resistor and a source.



Natural response of the RC circuit



Kirchhoff's voltage law ($t \geq 0$):

$$V_C(t) + iR = 0 \Rightarrow V_C(t) + RC\frac{dV_C(t)}{dt} = 0$$

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \gamma \qquad y(t_0) = y_0 \qquad y(t) = \tau\gamma\left(1 - e^{-\frac{(t-t_0)}{\tau}}\right) + y_0 e^{-\frac{(t-t_0)}{\tau}}$$

If $V_C(t_0^-) = V_0$ and

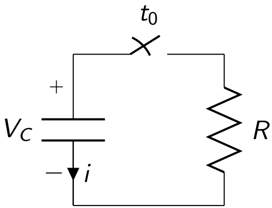
$$V_C(t) + RC\frac{dV_C(t)}{dt} = 0$$

then $\tau = RC$ and $\gamma = 0$.

$$V_C(t) = V_0 e^{-\frac{(t-t_0)}{RC}} \qquad t \geq t_0$$

Natural response of the RC circuit

$$V_C(t) = V_0 e^{-\frac{(t-t_0)}{RC}} \qquad t \geq t_0$$



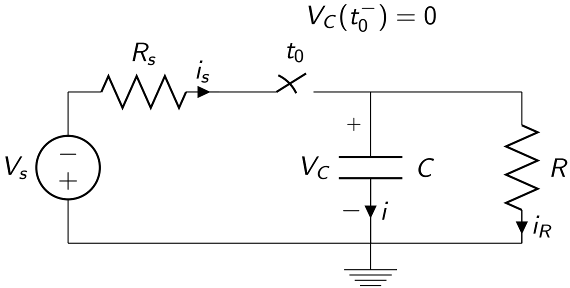
The current $i(t)$ is

$$i(t) = C\frac{dV_C(t)}{dt} = -\frac{V_0}{R} e^{-\frac{(t-t_0)}{RC}} \qquad t \geq t_0$$

The power in the capacitor $p(t) = i(t)V_C(t) < 0$. What does it mean?

How is the natural response of the circuit if $V_0 = 0$?

Step response of the RC circuit



$$i_s = i + i_R \Rightarrow \frac{(V_s - V_C(t))}{R_s} = C\frac{dV_C(t)}{dt} + \frac{V_C(t)}{R}$$

$$V_C(t)(R_s^{-1} + R^{-1}) + C\frac{dV_C(t)}{dt} = \frac{V_s}{R_s}$$

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \gamma \qquad y(t_0) = y_0 \qquad y(t) = \tau\gamma\left(1 - e^{-\frac{(t-t_0)}{\tau}}\right) + y_0 e^{-\frac{(t-t_0)}{\tau}}$$

If $V_C(t_0^-) = 0$ and

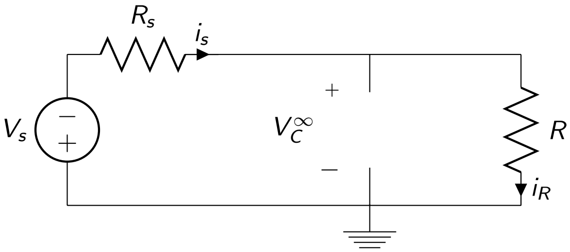
$$V_C(t)(R_s^{-1} + R^{-1}) + C\frac{dV_C(t)}{dt} = \frac{V_s}{R_s}$$

then $\tau = CR_{eq}$, where $R_{eq} = (R_s^{-1} + R^{-1})^{-1}$ and $\gamma = \frac{V_s}{CR_s}$.

$$V_C(t) = V_s\frac{R}{R_s + R}\left(1 - e^{-\frac{(t-t_0)}{R_{eq}C}}\right), \qquad t \geq t_0.$$

Stationary regime

In the limit $t \to \infty$, all currents and voltages are constant and the capacitor behaves like an open circuit. We can easily compute $\lim_{t\to\infty} V_C(t) \doteq V_C^\infty$:

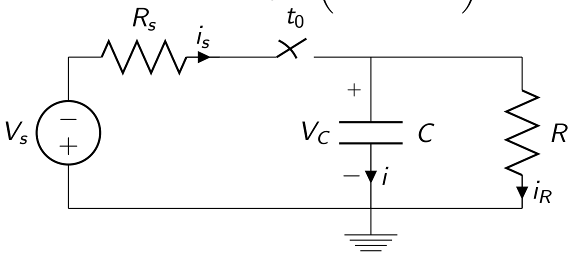


$$V_C^\infty = V_s \frac{R}{R_s + R}$$

Which, obviously, coincides with

$$\lim_{t\to\infty} V_s \frac{R}{R_s + R}\left(1 - e^{-\frac{(t-t_0)}{R_{eq} C}}\right) = V_s \frac{R}{R_s + R}$$

$$V_C(t) = V_s \frac{R}{R_s+R}\left(1 - e^{-\frac{(t-t_0)}{R_{eq}C}}\right), \qquad t \geq t_0$$



Current through the capacitor

$$i(t) = C\frac{\partial V_C(t)}{\partial t} = V_s \frac{R}{R+R_s}\frac{1}{R_{eq}} e^{-\frac{(t-t_0)}{R_{eq}C}} = \frac{V_s}{R_s} e^{-\frac{(t-t_0)}{R_{eq}C}}$$

- Compute the power $p(t)$ in the capacitor over time.
- Homework: Compute the step response of the circuit if $V_C(t_0^-) = V_0$.

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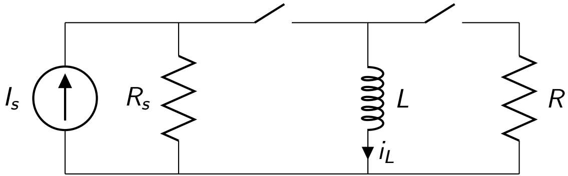
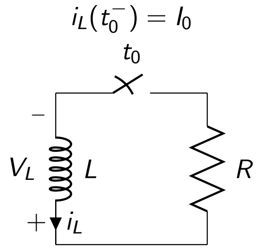
- We now focus on the RL circuit: a single inductor, a resistor network and a source.



Natural response of the RL circuit



Natural response.

Kirchhoff's voltage law ($t \geq t_0$):

$$V_L(t) + i_L(t)R = 0 \Rightarrow i_L(t)R + L\frac{di_L(t)}{dt} = 0$$

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \gamma \qquad y(t_0) = y_0 \qquad y(t) = \tau\gamma\left(1 - e^{-\frac{(t-t_0)}{\tau}}\right) + y_0 e^{-\frac{(t-t_0)}{\tau}}$$

If $i_L(t_0^-) = I_0$ and

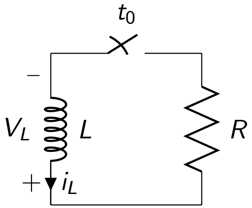
$$i_L(t)R + L\frac{di_L(t)}{dt} = 0$$

then $\tau = L/R$ and $\gamma = 0$.

$$i_L(t) = I_0 e^{-\frac{R}{L}(t-t_0)} \qquad t \geq t_0$$

Natural response of the RL circuit

$$i_L(t) = I_0 e^{-\frac{R}{L}(t-t_0)} \qquad t \geq t_0$$

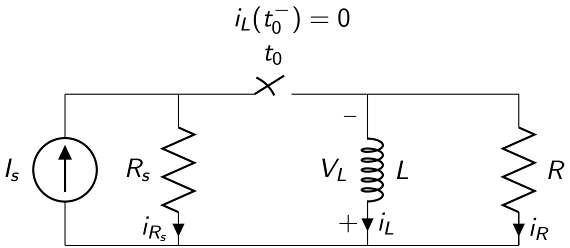


Voltage in the Inductor

$$V_L(t) = L\frac{di_L(t)}{dt} = -I_0 R e^{-\frac{R}{L}(t-t_0)}$$

Compute the power in the inductor.

Step response of the RC circuit



Step response.

$$I_s = i_{R_s} + i_L + i_R \Rightarrow I_s = \frac{V_L}{R_s} + i_L + \frac{V_L}{R}$$

$$I_s = \frac{1}{R_s} L \frac{di_L(t)}{dt} + i_L + \frac{1}{R} L \frac{di_L(t)}{dt} = L(R_s^{-1} + R^{-1}) \frac{di_L(t)}{dt} + i_L$$

$$\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \gamma \qquad y(t_0) = y_0 \qquad y(t) = \tau\gamma\left(1 - e^{-\frac{(t-t_0)}{\tau}}\right) + y_0 e^{-\frac{(t-t_0)}{\tau}}$$

If $i_L(t_0^-) = 0$ and

$$L(R_s^{-1} + R^{-1})\frac{di_L(t)}{dt} + i_L = I_s$$

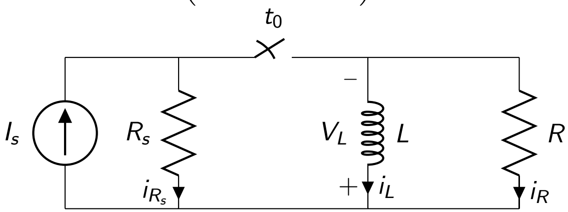
then $\tau = L/R_{eq}$, where $R_{eq} = (R_s^{-1} + R^{-1})^{-1}$, and $\gamma = I_s R_{eq}/L$.

$$i_L(t) = I_s\left(1 - e^{-\frac{R_{eq}}{L}(t-t_0)}\right), \qquad t \geq t_0$$

where $R_{eq} = (R_s^{-1} + R^{-1})^{-1}$.

Step response of the RC circuit

$$i_L(t) = I_s \left(1 - \mathrm{e}^{-\frac{R_{eq}}{L}(t-t_0)}\right), \qquad t \geq t_0$$

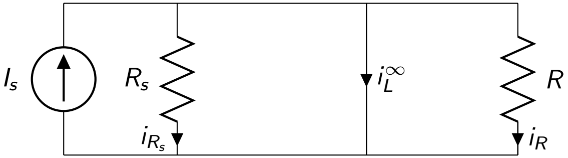


Voltage in the inductor

$$V_L(t) = L\frac{di_L(t)}{dt} = I_s R_{eq} \mathrm{e}^{-\frac{R_{eq}}{L}t}$$

Stationary regime

In the limit $t \to \infty$, all currents and voltages are constant and the inductor behaves like short circuit. We can easily compute $\lim_{t\to\infty} i_L(t) \doteq i_L^\infty$:



$$i_L^\infty = I_s$$

Which, obviously, coincides with

$$\lim_{t\to\infty} I_s \left(1 - e^{-\frac{R_{eq}}{L}(t-t_0)}\right) = I_s$$