LTI systems and the convolution operation

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May 7, 2018

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Motivation: Image Filtering

original

filter (3 x 3)

	0	0	0	
	-1	1	0	
	0	0	0	

vertical edge detector

https://docs.gimp.org/en/plug-in-convmatrix.html

Motivation: Image Filtering

$$\underset{X_{n\times n}}{\begin{array}{|c|c|c|c|c|c|c|}\hline 0_{\times 1} & 1_{\times 0} & 1_{\times 1} & 1 & 0 & 0 & 0 \\ \hline 0_{\times 0} & 0_{\times 1} & 1_{\times 0} & 1 & 1 & 0 & 0 \\ \hline 0_{\times 1} & 0_{\times 0} & 0_{\times 1} & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}} * \underset{K}{\begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}} = \underset{Y}{\begin{array}{|c|c|c|c|c|}\hline \boxed{1} & 4 & 3 & 4 & 1 \\ \hline 1 & 2 & 4 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 1 \\ \hline 1 & 3 & 3 & 1 & 1 \\ \hline 3 & 3 & 1 & 1 & 0 \\ \hline \end{array}}$$

$$\begin{aligned} Y[1,1] = {} & X[1,1] * K[1,1] + X[1,2] * K[1,2] + X[1,3] * K[1,3] \\ & + X[2,1] * K[2,1] + X[2,2] * K[2,2] + X[2,3] * K[2,3] \\ & + X[3,1] * K[3,1] + X[3,2] * K[3,2] + X[3,3] * K[3,3] \end{aligned}$$

Motivation: Image Filtering

$$
\underbrace{\begin{bmatrix}
0 & 1_{\times 1} & 1_{\times 0} & 1_{\times 1} & 0 & 0 & 0 \\
0 & 0_{\times 0} & 1_{\times 1} & 1_{\times 0} & 1 & 0 & 0 \\
0 & 0_{\times 1} & 0_{\times 0} & 1_{\times 1} & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{X_{n\times n}}
*
\underbrace{\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}}_{K}
=
\underbrace{\begin{bmatrix}
1 & \boxed{4} & 3 & 4 & 1 \\
1 & 2 & 4 & 3 & 3 \\
1 & 2 & 3 & 4 & 1 \\
1 & 3 & 3 & 1 & 1 \\
3 & 3 & 1 & 1 & 0
\end{bmatrix}}_{Y}
$$

$$
\begin{aligned}
Y[1,2] = {} & X[1,2] * K[1,1] + X[1,3] * K[1,2] + X[1,4] * K[1,3] \\
& + X[2,2] * K[2,1] + X[2,3] * K[2,2] + X[2,4] * K[2,3] \\
& + X[3,2] * K[3,1] + X[3,3] * K[3,2] + X[3,4] * K[3,3]
\end{aligned}
$$

Motivation: Image Filtering

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 0_{\times 1} & 1_{\times 0} & 1_{\times 1} & 1 & 0 & 0 & 0 \\ \hline 0_{\times 0} & 0_{\times 1} & 1_{\times 0} & 1 & 1 & 0 & 0 \\ \hline 0_{\times 1} & 0_{\times 0} & 0_{\times 1} & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}}_{X_{n\times n}} * \underbrace{\begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}}_{K} = \underbrace{\begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 3 & 4 & 1 \\ \hline 1 & 2 & 4 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 1 \\ \hline 1 & 3 & 3 & 1 & 1 \\ \hline 3 & 3 & 1 & 1 & 0 \\ \hline \end{array}}_{Y}$$

$$Y[i,j] = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} X[k_1,k_2] K[i-k_1, j-k_2], \quad K[u,q] = 0 \text{ para } u > 3, q > 3$$

Linear operator, the result does not depend on the position of the image

Motivation: Image Filtering

original

filter (3 x 3)

	0	0	0	
	-1	1	0	
	0	0	0	

vertical edge detector

https://docs.gimp.org/en/plug-in-convmatrix.html

Motivation: Image Filtering

original

filter (3 x 3)

	0	1	0	
	1	-4	1	
	0	1	0	

all edge detector

Motivation: Image Filtering

original

filter (5 x 5)

0	0	0	0	0
0	0	-1	0	0
0	-1	5	-1	0
0	0	-1	0	0
0	0	0	0	0

sharpen

- The theory of LTI systems has direct applications in a wide set of technical areas:
  - Nuclear magnetic resonance spectroscopy
  - Seismology
  - Electric circuit design
  - Control Theory
  - Any application that involves Signal Processing
- Our goal in Systems and Circuits: predict the output of a given LTI system for a given input.
- In future courses you will face the design of LTI systems according to certain specifications.

Time Invariance

A system is time-invariant if a time shift in the input signal causes a time shift in the output signal.

- Given $y[n] = f(x[n])$, the system is time-invariant if $f(x[n - n_0]) = y[n - n_0]$ $\forall\ n_0$.
- Given $y(t) = f(x(t))$, the system is time-invariant if $f(x(t - t_0)) = y(t - t_0)$ $\forall\ t_0$.

Linearity

Linear system posses the important property of superposition.

For any system, consider two arbitrary inputs and their respective outputs:

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t),$$

the system is linear if

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

for any two complex constant $a, b \in \mathbb{C}$.

Linear discrete-time signals

$$ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$

Analysis of LTI systems

$$x(t) \longrightarrow \boxed{\text{LIT}} \longrightarrow y(t)?$$

- Decompose input signal by means of a linear combination of simpler signals

$$x(t) = \ldots a_{-1}\phi_{-1}(t) + a_{-2}\phi_0(t) + a_0\phi_0(t) + a_1\phi_1(t) + a_2\phi_2(t) + \ldots$$

- These “basic” signals are chosen to provide a certain degree of analytical convenience, so we can analyze the system’s properties and its response to arbitrary input signals:
  - Delayed Impulses $\Rightarrow$ **Convolution**
  - Complex exponential signals $\Rightarrow$ **Fourier analysis**

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$$x[n] \longrightarrow \boxed{\text{LIT}} \longrightarrow y[n]?$$

Remember that any discrete-time sequence $x[n]$ can be decomposed as a linear combination of unit impulses:

$$\begin{aligned}
x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\
&= \ldots + x[-20]\delta[n+20] + x[-19]\delta[n+19] + \ldots \\
&+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots
\end{aligned}$$

If the system is linear and we are able to compute

$$\delta[n] \longrightarrow \boxed{\text{Lineal}} \longrightarrow h_0[n]$$

$$\delta[n+1] \longrightarrow \boxed{\text{Lineal}} \longrightarrow h_{-1}[n]$$

$$\delta[n-1] \longrightarrow \boxed{\text{Lineal}} \longrightarrow h_1[n]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\delta[n-k] \longrightarrow \boxed{\text{Lineal}} \longrightarrow h_k[n]$$

then we can make use of the superposition property!

$$x[n] \longrightarrow \boxed{\text{Lineal}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

- This is just a consequence of the system linearity.
- The problem is reduced to evaluate the system's output for any $\delta[n-k]$.
- The problem can be further reduced by exploiting that the system is **time-invariant**.

$$\delta[n] \longrightarrow \boxed{\text{LIT}} \longrightarrow h_0[n] = h[n]$$

$$\delta[n+1] \longrightarrow \boxed{\text{LIT}} \longrightarrow h_{-1}[n] = h[n+1]$$

$$\delta[n-1] \longrightarrow \boxed{\text{LIT}} \longrightarrow h_1[n] = h[n-1]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\delta[n-k] \longrightarrow \boxed{\text{LIT}} \longrightarrow h_k[n] = h[n-k]$$

Therefore, for any input $x[n]$, if $h[n]$ is known then the system output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- $h[n]$ is the system's **impulse response**.
- Any LTI system is **completely defined** by $h[n]$!

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Convolution

Given two discrete-time signals $x[n]$ and $h[n]$:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

is the convolution operation between them.

Problem

- $x[n] = \alpha^n u[n]$ with $\alpha \in (0, 1)$
- $h[n] = u[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Sol.

$$y[n] = \begin{cases} 0 & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha} & n \geq 0 \end{cases}$$

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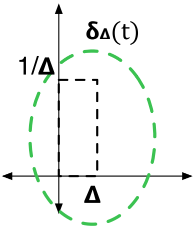
The discussion for continuous-time LTI systems is just a generalization of the discrete-time case.

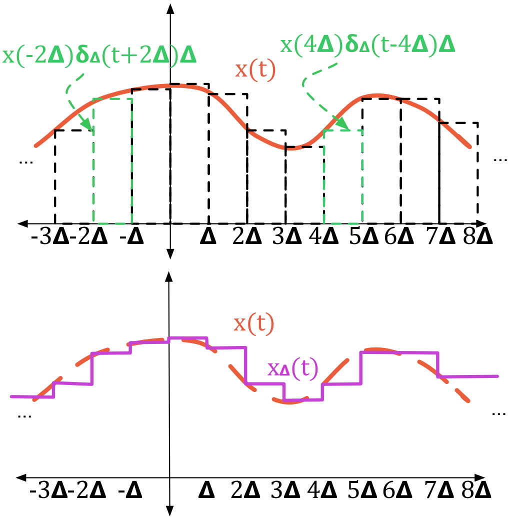
$$x(t) \longrightarrow \boxed{\text{LIT}} \longrightarrow y(t)?$$

Remember that any continuous-time signal $x(t)$ can be decomposed as a linear combination of an infinite number of impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

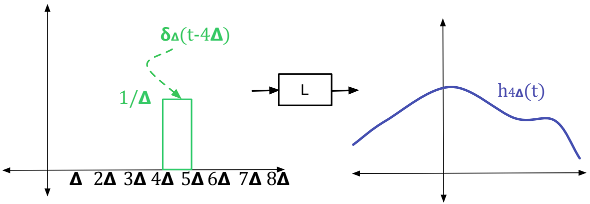
Approximate $x(t)$ **by a combination of scaled and equally spaced versions** $\delta_\Delta(t)$





$$x(t) \approx x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

$$\delta_\Delta(t - k\Delta) \longrightarrow \boxed{\text{Lineal}} \longrightarrow h_{k\Delta}(t)$$



Then applying the superposition property

$$x_\Delta(t) \longrightarrow \boxed{\text{Lineal}} \longrightarrow y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_{k\Delta}(t)\Delta$$

If the system is also **time-invariant**:

$$\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{LIT}} \longrightarrow h_{k\Delta}(t) = h_0(t-k\Delta)$$

$$x_\Delta(t) \longrightarrow \boxed{\text{LIT}} \longrightarrow y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_0(t-k\Delta)\Delta$$

Taking the limit $\Delta \to 0$:

- $k\Delta \to \tau$
- $\sum \to \int$
- $\Delta \to d\tau$
- $h_0(t) = h(t)$

$$y(t) = \lim_{\Delta\to 0} y_\Delta(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- $h(t)$ is the **system's response to the impulse** $\delta(t)$.
- Any LTI system is **completely defined** by $h(t)$!

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Continuous-time Convolution

Given two signals $x(t)$ and $h(t)$:

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

is the continuous-time convolution operation between them.

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Consider the two signals $x(t)$ and $h(t)$ given by:

$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}.$$

If $h(t)$ is the impulse response of an LTI system, compute the system's output when $x(t)$ is the input signal.

Sol.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < T \\ tT - \frac{T^2}{2} & T \leq t < 2T \\ Tt + \frac{3}{2}T^2 - \frac{t^2}{2} & 2T \leq t < 3T \\ 0 & t \geq 3T \end{cases} ,$$

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Commutative property

$$x[n] * h[n] = h[n] * x[n]$$

Proof.:
Given

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

define $v = n - k$, thus

$$x[n] * h[n] = \sum_{v=+\infty}^{-\infty} x[n-v]h[v] = \sum_{v=-\infty}^{+\infty} h[v]x[n-v] = h[n] * x[n]$$

Commutative property

Therefore,

is equivalent to

Commutative property for the continuous-time convolution

$$x(t) * h(t) = h(t) * x(t) \Rightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Associative property

$$x(t) * \Big(h(t) * z(t)\Big) = \Big(x(t) * h(t)\Big) * z(t)$$

$$x[n] * \Big(h[n] * z[n]\Big) = \Big(x[n] * h[n]\Big) * z[n]$$

Therefore, the following configurations are equivalent:

Distributive property with respect to the sum

$$x[n] * \Big(y[n] + z[n]\Big) = x[n] * y[n] + x[n] * z[n]$$

$$x(t) * \Big(y(t) + z(t)\Big) = x(t) * y(t) + x(t) * z(t)$$

The following configurations are equivalent:

Convolution with an impulse signal

Remember that any signal can be decomposed as a linear combination of an infinite number of unit impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Therefore

$$x[n] * \delta[n] = x[n],$$

$$x(t) * \delta(t) = x(t).$$

Convolution with a delayed impulse (discrete-time)

$$x[n] * \delta[n - n_0] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k - n_0]$$

Define $v = k + n_0$, thus

$$\begin{aligned} x[n] * \delta[n - n_0] &= \sum_{v=-\infty}^{\infty} x[v - n_0]\delta[n - v] \\ &= x[n - n_0] * \delta[n] = x[n - n_0]. \end{aligned}$$

Therefore,

$$x[n] * \delta[n - n_0] = x[n - n_0],$$

Convolution with a delayed impulse (continuous-time)

$$x(t) * \delta(t - t_0) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\mathrm{d}\tau$$

Define $v = \tau + t_0$ and $x'(t) = x(t - t_0)$, thus

$$\begin{aligned} x(t) * \delta(t - t_0) &= \int_{v=-\infty}^{\infty} x(v - t_0)\delta(t - v)\mathrm{d}v \\ &= \int_{v=-\infty}^{\infty} x'(v)\delta(t - v)\mathrm{d}v = x'(t) * \delta(t) = x'(t) \end{aligned}$$

Therefore,

$$x(t) * \delta(t - t_0) = x(t - t_0),$$