Combining the definition of energy with Eq. 6.16 yields

$$dw = Cv\,dv,$$

from which

$$\int_0^w dx = C\int_0^v y\,dy,$$

or

Capacitor energy equation ▶

$$w = \frac{1}{2}Cv^2. \tag{6.18}$$

In the derivation of Eq. 6.18, the reference for zero energy corresponds to zero voltage.

Examples 6.4 and 6.5 illustrate the application of the current, voltage, power, and energy relationships for a capacitor.

Example 6.4 Determining Current, Voltage, Power, and Energy for a Capacitor

The voltage pulse described by the following equations is impressed across the terminals of a 0.5 μF capacitor:

$$v(t) = \begin{cases} 0, & t \leq 0\text{ s}; \\ 4t\text{ V}, & 0\text{ s} \leq t \leq 1\text{ s}; \\ 4e^{-(t-1)}\text{ V}, & t \geq 1\text{ s}. \end{cases}$$

a) Derive the expressions for the capacitor current, power, and energy.
b) Sketch the voltage, current, power, and energy as functions of time. Line up the plots vertically.
c) Specify the interval of time when energy is being stored in the capacitor.
d) Specify the interval of time when energy is being delivered by the capacitor.
e) Evaluate the integrals

$$\int_0^1 p\,dt \quad \text{and} \quad \int_1^\infty p\,dt$$

and comment on their significance.

Solution

a) From Eq. 6.13,

$$i = \begin{cases} (0.5 \times 10^{-6})(0) = 0, & t < 0\text{s}; \\ (0.5 \times 10^{-6})(4) = 2\,\mu\text{A}, & 0\text{ s} < t < 1\text{ s}; \\ (0.5 \times 10^{-6})(-4e^{-(t-1)}) = -2e^{-(t-1)}\,\mu\text{A}, & t > 1\text{ s}. \end{cases}$$

The expression for the power is derived from Eq. 6.16:

$$p = \begin{cases} 0, & t \leq 0\text{ s}; \\ (4t)(2) = 8t\,\mu\text{W}, & 0\text{ s} \leq t < 1\text{ s}; \\ (4e^{-(t-1)})(-2e^{-(t-1)}) = -8e^{-2(t-1)}\,\mu\text{W}, & t > 1\text{ s}. \end{cases}$$

The energy expression follows directly from Eq. 6.18:

$$w = \begin{cases} 0 & t \leq 0\text{ s}; \\ \frac{1}{2}(0.5)16t^2 = 4t^2\mu\text{J}, & 0\text{ s} \leq t \leq 1\text{ s}; \\ \frac{1}{2}(0.5)16e^{-2(t-1)} = 4e^{-2(t-1)}\,\mu\text{J}, & t \geq 1\text{ s}. \end{cases}$$

b) Figure 6.11 shows the voltage, current, power, and energy as functions of time.

c) Energy is being stored in the capacitor whenever the power is positive. Hence energy is being stored in the interval 0–1 s.

d) Energy is being delivered by the capacitor whenever the power is negative. Thus energy is being delivered for all t greater than 1 s.

e) The integral of $p\,dt$ is the energy associated with the time interval corresponding to the limits on the integral. Thus the first integral represents the energy stored in the capacitor between 0 and 1 s, whereas the second integral represents the energy returned, or delivered, by the capacitor in the interval 1 s to ∞:

$$\int_0^1 p\,dt = \int_0^1 8t\,dt = 4t^2\Big|_0^1 = 4\,\mu\text{J},$$

$$\int_1^{\infty} p\,dt = \int_1^{\infty} (-8e^{-2(t-1)})dt = (-8)\frac{e^{-2(t-1)}}{-2}\Bigg|_1^{\infty} = -4\,\mu\text{J}.$$

The voltage applied to the capacitor returns to zero as time increases without limit, so the energy returned by this ideal capacitor must equal the energy stored.

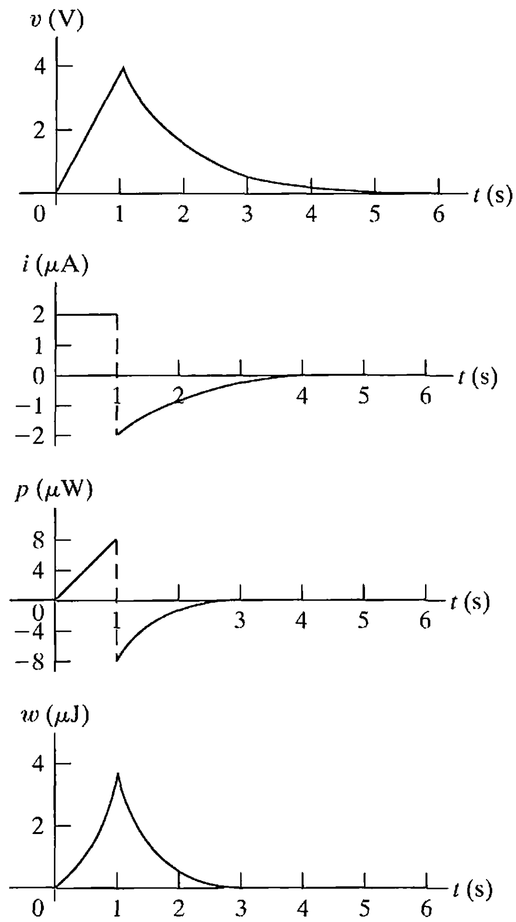


Figure 6.11 ▲ The variables v, i, p, and w versus t for Example 6.4.

Example 6.5 Finding v, p, and w Induced by a Triangular Current Pulse for a Capacitor

An uncharged 0.2 μF capacitor is driven by a triangular current pulse. The current pulse is described by

$$i(t) = \begin{cases} 0, & t \le 0; \\ 5000t \text{ A}, & 0 \le t \le 20\,\mu\text{s}; \\ 0.2 - 5000t \text{ A}, & 20 \le t \le 40\,\mu\text{s}; \\ 0, & t \ge 40\,\mu\text{s}. \end{cases}$$

a) Derive the expressions for the capacitor voltage, power, and energy for each of the four time intervals needed to describe the current.

b) Plot i, v, p, and w versus t. Align the plots as specified in the previous examples.

c) Why does a voltage remain on the capacitor after the current returns to zero?

Solution

a) For $t \le 0$, v, p, and w all are zero.
For $0 \le t \le 20\,\mu s$,

$$v = 5 \times 10^6 \int_0^t (5000\tau)\,d\tau + 0 = 12.5 \times 10^9 t^2 \text{ V},$$

$$p = vi = 62.5 \times 10^{12} t^3 \text{ W},$$

$$w = \frac{1}{2}Cv^2 = 15.625 \times 10^{12} t^4 \text{ J}.$$

For $20\,\mu\text{s} \le t \le 40\,\mu\text{s}$,

$$v = 5 \times 10^6 \int_{20\mu s}^{t} (0.2 - 5000\tau)\,d\tau + 5.$$

(Note that 5 V is the voltage on the capacitor at the end of the preceding interval.) Then,

$$v = (10^6t - 12.5 \times 10^9t^2 - 10)\ \text{V},$$

$$p = vi,$$

$$= (62.5 \times 10^{12}t^3 - 7.5 \times 10^9t^2 + 2.5 \times 10^5t - 2)\ \text{W},$$

$$w = \frac{1}{2}Cv^2,$$

$$= (15.625 \times 10^{12}t^4 - 2.5 \times 10^9t^3 + 0.125 \times 10^6t^2 - 2t + 10^{-5})\ \text{J}.$$

For $t \geq 40\,\mu\text{s}$,

$$v = 10\ \text{V},$$

$$p = vi = 0,$$

$$w = \frac{1}{2}Cv^2 = 10\,\mu\text{J}.$$

b) The excitation current and the resulting voltage, power, and energy are plotted in Fig. 6.12.

c) Note that the power is always positive for the duration of the current pulse, which means that energy is continuously being stored in the capacitor. When the current returns to zero, the stored energy is trapped because the ideal capacitor offers no means for dissipating energy. Thus a voltage remains on the capacitor after i returns to zero.

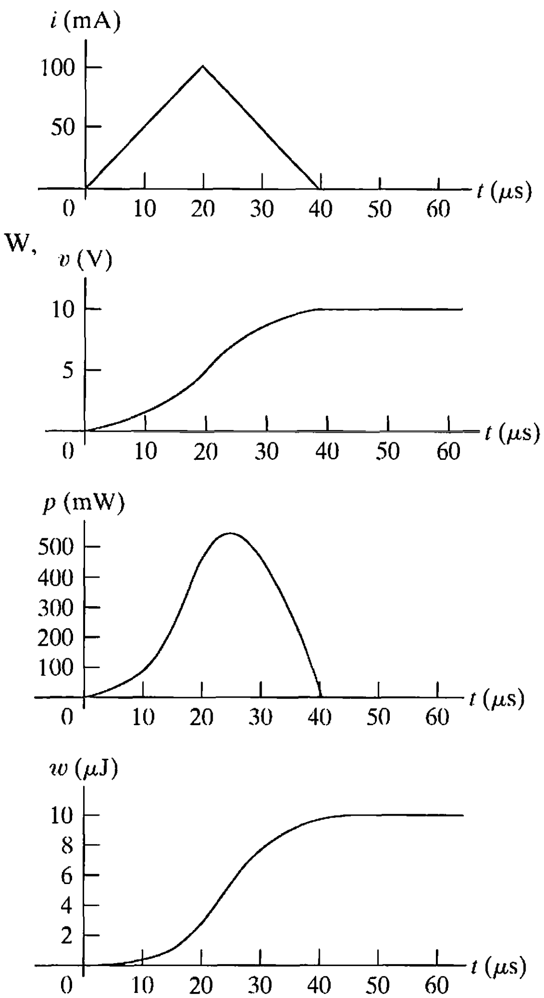
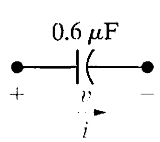


Figure 6.12 ▲ The variables i, v, p, and w versus t for Example 6.5.

✓ASSESSMENT PROBLEMS

Objective 2—Know and be able to use the equations for voltage, current, power, and energy in a capacitor

6.2 The voltage at the terminals of the 0.6 μF capacitor shown in the figure is 0 for $t < 0$ and $40e^{-15{,}000t} \sin 30{,}000t$ V for $t \geq 0$. Find (a) $i(0)$; (b) the power delivered to the capacitor at $t = \pi/80$ ms; and (c) the energy stored in the capacitor at $t = \pi/80$ ms.



Answer: (a) 0.72 A;
(b) −649.2 mW;
(c) 126.13 μJ.

6.3 The current in the capacitor of Assessment Problem 6.2 is 0 for $t < 0$ and $3 \cos 50{,}000t$ A for $t \geq 0$. Find (a) $v(t)$; (b) the maximum power delivered to the capacitor at any one instant of time; and (c) the maximum energy stored in the capacitor at any one instant of time.

Answer: (a) $100 \sin 50{,}000t$ V, $t \geq 0$;
(b) 150 W; (c) 3 mJ.

NOTE: Also try Chapter Problems 6.16 and 6.17.

Example 6.1 Determining the Voltage, Given the Current, at the Terminals of an Inductor

The independent current source in the circuit shown in Fig. 6.2 generates zero current for $t < 0$ and a pulse $10te^{-5t}$A, for $t > 0$.

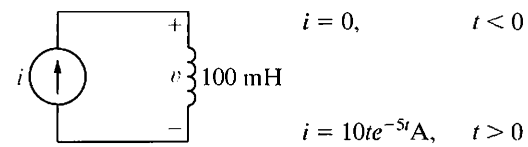


Figure 6.2 ▲ The circuit for Example 6.1.

a) Sketch the current waveform.
b) At what instant of time is the current maximum?
c) Express the voltage across the terminals of the 100 mH inductor as a function of time.
d) Sketch the voltage waveform.
e) Are the voltage and the current at a maximum at the same time?
f) At what instant of time does the voltage change polarity?
g) Is there ever an instantaneous change in voltage across the inductor? If so, at what time?

Solution

a) Figure 6.3 shows the current waveform.
b) $di/dt = 10(-5te^{-5t} + e^{-5t}) = 10e^{-5t}(1 - 5t)$ A/s; $di/dt = 0$ when $t = \frac{1}{5}$ s. (See Fig. 6.3.)
c) $v = Ldi/dt = (0.1)10e^{-5t}(1 - 5t) = e^{-5t}(1 - 5t)$ V, $t > 0$; $v = 0, t < 0$.
d) Figure 6.4 shows the voltage waveform.
e) No; the voltage is proportional to di/dt, not i.
f) At 0.2 s, which corresponds to the moment when di/dt is passing through zero and changing sign.
g) Yes, at $t = 0$. Note that the voltage can change instantaneously across the terminals of an inductor.

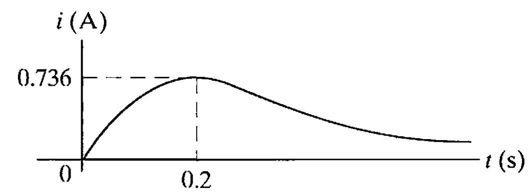


Figure 6.3 ▲ The current waveform for Example 6.1.

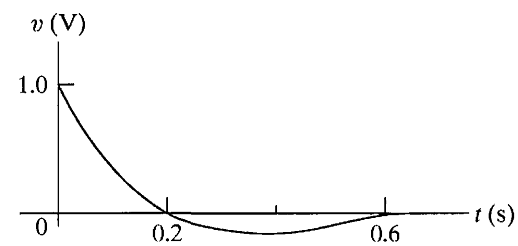


Figure 6.4 ▲ The voltage waveform for Example 6.1.

Current in an Inductor in Terms of the Voltage Across the Inductor

Equation 6.1 expresses the voltage across the terminals of an inductor as a function of the current in the inductor. Also desirable is the ability to express the current as a function of the voltage. To find i as a function of v, we start by multiplying both sides of Eq. 6.1 by a differential time dt:

$$v\,dt = L\left(\frac{di}{dt}\right)dt. \tag{6.2}$$

Multiplying the rate at which i varies with t by a differential change in time generates a differential change in i, so we write Eq. 6.2 as

$$v\,dt = L\,di. \tag{6.3}$$

As before, we use different symbols of integration to avoid confusion with the limits placed on the integrals. In Eq. 6.12, the energy is in joules, inductance is in henrys, and current is in amperes. To illustrate the application of Eqs. 6.7 and 6.12, we return to Examples 6.1 and 6.2 by means of Example 6.3.

Example 6.3 Determining the Current, Voltage, Power, and Energy for an Inductor

a) For Example 6.1, plot i, v, p, and w versus time. Line up the plots vertically to allow easy assessment of each variable's behavior.

b) In what time interval is energy being stored in the inductor?

c) In what time interval is energy being extracted from the inductor?

d) What is the maximum energy stored in the inductor?

e) Evaluate the integrals

$$\int_0^{0.2} p\,dt \quad \text{and} \quad \int_{0.2}^{\infty} p\,dt,$$

and comment on their significance.

f) Repeat (a)–(c) for Example 6.2.

g) In Example 6.2, why is there a sustained current in the inductor as the voltage approaches zero?

Solution

a) The plots of i, v, p, and w follow directly from the expressions for i and v obtained in Example 6.1 and are shown in Fig. 6.8. In particular, $p = vi$, and $w = (\frac{1}{2})Li^2$.

b) An increasing energy curve indicates that energy is being stored. Thus energy is being stored in the time interval 0 to 0.2 s. Note that this corresponds to the interval when $p > 0$.

c) A decreasing energy curve indicates that energy is being extracted. Thus energy is being extracted in the time interval 0.2 s to ∞. Note that this corresponds to the interval when $p < 0$.

d) From Eq. 6.12 we see that energy is at a maximum when current is at a maximum; glancing at the graphs confirms this. From Example 6.1, maximum current = 0.736 A. Therefore, $w_{\max} = 27.07$ mJ.

e) From Example 6.1,

$$i = 10te^{-5t}\ \text{A} \quad \text{and} \quad v = e^{-5t}(1 - 5t)\ \text{V}.$$

Therefore,

$$p = vi = 10te^{-10t} - 50t^2e^{-10t}\ \text{W}.$$

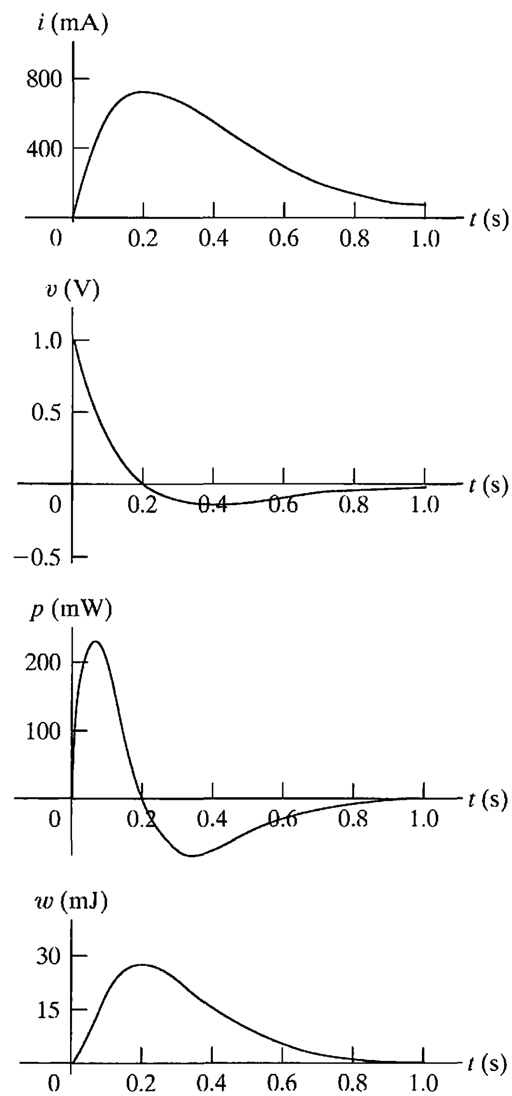


Figure 6.8 ▲ The variables i, v, p, and w versus t for Example 6.1.