Properties of LTI Systems

Pablo M. Olmos (olmos@tsc.uc3m.es)
Emilio Parrado (emipar@tsc.uc3m.es)

March 3, 2018

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- LTI systems are completely determined by their impulse response $h(t)$ ($h[n]$).
- Consequently, we should be able to deduce further properties of these systems just by inspecting $h(t)$ ($h[n]$):
    1. Memory.
    2. Invertibility.
    3. Causality.
    4. Stability.

Memoryless discrete-time LTI systems

Given the impulse response of a LTI system $h[n]$ and its output for an arbitrary input $x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]h[0] + \sum_{\substack{k=-\infty \\ k\neq n}}^{\infty} x[k]h[n-k],$$

we observe that the system is memoryless only if $h[n] = 0$ for any $n \neq 0$.

Memoryless discrete-time LTI systems

The impulse response is of the form

$$h[n] = a\delta[n]$$

where $a \in \mathbb{C}$ is a constant.

Memoryless continuous-time LTI systems

Given the impulse response of an LTI system $h(t)$ and its output for an arbitrary input $x(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\nu)h(\nu)d\nu,$$

we conclude that the system is memoryless only if $h(t) = a\delta(t)$, where $a \in \mathbb{C}$.

Hint: $h(t)$ must be zero at any point except at $t = 0$ to remove the dependency of $y(t)$ on $x(t')$ at any $t' \neq t$. Besides, because of the differential $d\nu$ inside the integral, $h(0)$ must tend to ∞. Indeed, $h(t)$ must be of the form $a\delta(t)$.

Causality in LTI Systems

A system is causal if the output at any time depends only on the input at the present time and at any time in the past.

Given two LTI systems with impulse response $h[n]$ and $h(t)$ and the output for an arbitrary input $x[n]$ and $x(t)$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

the systems are causal if

$$\begin{aligned} h[n] &= 0 \quad n < 0, \\ h(t) &= 0 \quad t < 0. \end{aligned}$$

Stability in LTI systems

Given a discrete-time LTI system $h[n]$ and an input signal $x[n]$ such that $|x[n]| < B \ \forall n$, the output signal

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

can be bounded as follows

$$\begin{aligned} |y[n]| &= |\sum_{k=-\infty}^{\infty} x[k]h[n-k]| \leq \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \\ &< \sum_{k=-\infty}^{\infty} B|h[n-k]| = B \sum_{k=-\infty}^{\infty} |h[n-k]|. \end{aligned}$$

The LTI system is stable if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

- For continuous-time LTI systems the stability condition can be proven in a similar way (**Exercise**).

BIBO stable LTI systems

A discrete-time LTI system with impulse response $h[n]$ is stable if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

A continuous-time LTI system with impulse response $h(t)$ is stable if

$$\int_{-\infty}^{\infty} |h(\tau)| \mathrm{d}\tau < \infty.$$

Invertibility of LTI Systems

- A system is said to be invertible if distinct inputs lead to distinct outputs.
- We have a one to one input-output correspondence.
- By observing its output, we can determine its input.

A LTI system $h(t)$ is invertible if we are able to find another LTI system with impulse response $h_{\text{inv}}(t)$ such that:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{h_{\text{inv}}(t)} \longrightarrow x(t)$$

By the associative property

$$x(t) \longrightarrow \boxed{h(t) * h_{\mathrm{inv}}(t)} \longrightarrow x(t)$$

Therefore,

$$h(t) * h_{\mathrm{inv}}(t) = \delta(t)$$

Invertible LTI systems

- $h[n]$ is invertible $\Longleftrightarrow$ there exists $h_{\mathrm{inv}}[n]$ such that $h[n] * h_{\mathrm{inv}}[n] = \delta[n]$.
- $h(t)$ is invertible $\Longleftrightarrow$ there exists $h_{\mathrm{inv}}(t)$ such that $h(t) * h_{\mathrm{inv}}(t) = \delta(t)$.

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The unit step response of an LTI systems

- LTI systems are completely characterized by their impulse response.
- Sometimes, it is easier to observe the system's output response to the unit step.
- Let's see how we can obtain the impulse response from the system's output response to the unit step.

Assume an LTI system for which $h[n]$ is unknown:

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n]$$

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=0}^{\infty} h[n-k] = \sum_{v=-\infty}^{n} h[v]$$

Therefore,

$$h[n] = s[n] - s[n-1]$$

Assume a LTI system for which $h(t)$ is unknown:

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t)$$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(t-\tau)d\tau = \int_{-\infty}^{t} h(\psi)d\psi$$

Therefore,

$$h(t) = \frac{\partial s(t)}{\partial t}$$