RLC Circuits

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Second order linear differential equation with constant coefficients

Compute $y(t)$ such that $y(t_0) = y_0$, $\frac{dy(t)}{dt}|_{t=t_0} = y_1$ if

$$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = \gamma$$

where α, ω_0^2 are real constants.

Second order linear differential equation with constant coefficients

Compute $y(t)$ such that $y(t_0) = y_0$, $\frac{dy(t)}{dt}|_{t=t_0} = y_1$ if

$$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = \gamma$$

where α, ω_0^2 are real constants.

Characteristic equation:

s_1 and s_2 are the roots of the equation

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Thus

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Case $\alpha > \omega_0$: s_1 and s_2 are negative real constants

$$y(t) = A_1 e^{s_1(t-t_0)} + A_2 e^{s_2(t-t_0)} + \frac{\gamma}{\omega_0^2} \qquad t \geq t_0,$$

where A_1 and A_2 can be found using the initial conditions.

Exercise

Let $x(t) = 3u(t)$ be the input signal to a system defined by the following input/output relationship:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = x(t),$$

where $y(t)$ is the output signal. If we know that $y(0) = 0$ and $\frac{dy(t)}{dt}|_{t=0} = 0$, compute and plot $y(t)$.

Sol.

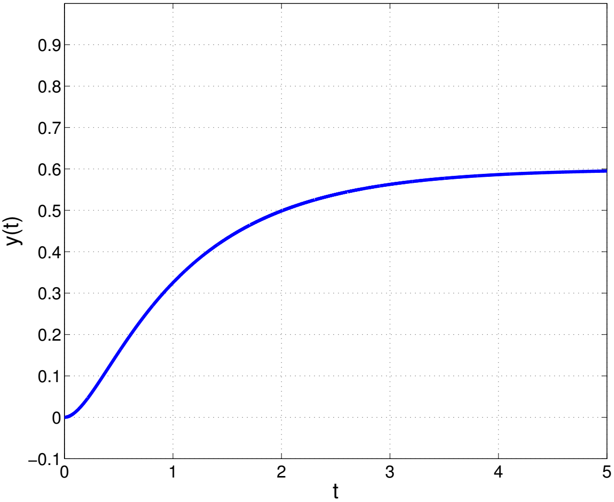
For $t \geq 0$, the equation that gives the output is a second-order differential equation of the form

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = 3 \qquad t \geq 0$$

$$s^2 + 6s + 5 = 0 \Rightarrow s_1 = -3 + \sqrt{9-5} = -1, s_2 = -3 - \sqrt{2} = -5$$

$$y(t) = A_1 e^{-t} + A_2 e^{-5t} + \frac{3}{5} \qquad t \geq 0$$

Using the initial conditions we find $A_1 = \frac{-3}{5}$ and $A_2 = \frac{3}{25}$.



The response of this system to the unit step is said to be *overdamped*.

Case $\alpha < \omega_0$: s_1 and s_2 are complex

$$s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \qquad s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

Define $\omega_d \doteq \sqrt{\omega_0^2 - \alpha^2}$.

$$y(t) = B_1 e^{-\alpha(t-t_0)} \cos(\omega_d(t - t_0)) + B_2 e^{-\alpha(t-t_0)} \sin(\omega_d(t - t_0)) + \frac{\gamma}{\omega_0^2} \qquad t \geq t_0,$$

where B_1 and B_2 can be found using the initial conditions.

$\alpha = 0$

Oscillation around γ/ω_0^2!! The solution **does not** converge to a constant value in the limit $t \to \infty$!!!

Exercise

Let $x(t) = 3u(t)$ be the input signal to a system defined by the following input/output relationship:

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = x(t),$$

where $y(t)$ is the output signal. If we know that $y(0) = 0$ and $\frac{dy(t)}{dt}|_{t=0} = 0$, compute and plot $y(t)$.

Sol.

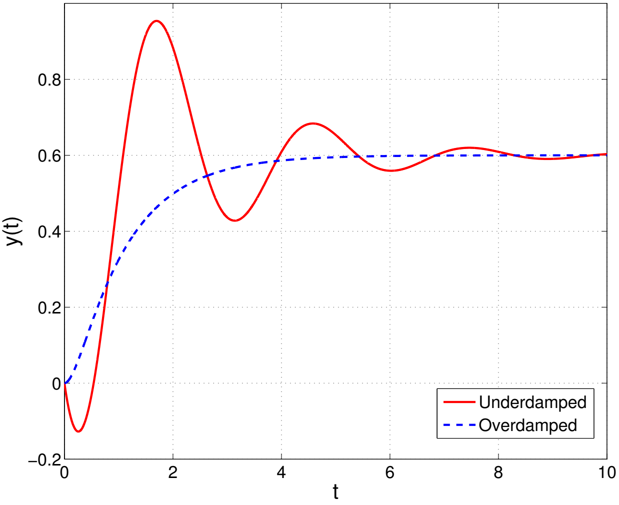
For $t \geq 0$, the equation that gives the output is a second-order differential equation of the form

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 3 \qquad t \geq 0$$

$$s^2 + s + 5 = 0 \Rightarrow s_1 = -0.5 + j\sqrt{4.75}, s_2 = -0.5 - j\sqrt{4.75}$$

$$y(t) = B_1 e^{-\frac{t}{2}} \cos(\sqrt{4.75}t) + B_2 e^{-\frac{t}{2}} \sin(\sqrt{4.75}t) + \frac{3}{5} \qquad t \geq 0$$

Using the initial conditions we find $B_1 = \frac{-3}{5}$ and $B_2 = \frac{3}{10\sqrt{4.75}}$ (check!).



The response of this system to the unit step is said to be *underdamped*.

Case $\alpha = \omega$: $s_1 = s_2 = -\alpha$

$$y(t) = (D_1 + D_2(t - t_0))\mathrm{e}^{-\alpha(t-t_0)} \qquad t \geq t_0,$$

where D_1 and D_2 can be found using the initial conditions.

Exercise

Let $x(t) = 3u(t)$ be the input signal to a system defined by the following input/output relationship:

$$\frac{d^2y(t)}{dt^2} + 2\sqrt{5}\frac{dy(t)}{dt} + 5y(t) = x(t),$$

where $y(t)$ is the output signal. If we know that $y(0) = 0$ and $\frac{dy(t)}{dt}|_{t=0} = 0$, compute and plot $y(t)$.

Sol.

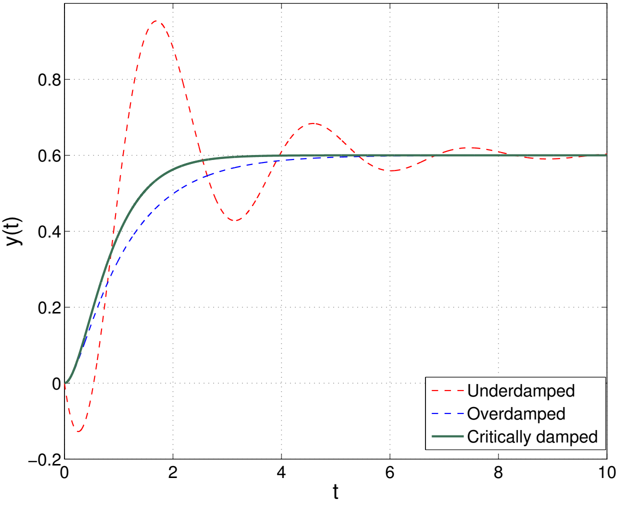
For $t \geq 0$, the equation that gives the output is a second-order differential equation of the form

$$\frac{d^2y(t)}{dt^2} + 2\sqrt{5}\frac{dy(t)}{dt} + 5y(t) = 3 \qquad t \geq 0$$

$$s^2 + 2\sqrt{5}s + 5 = 0 \Rightarrow s_1 = s_2 = -\sqrt{5}$$

$$y(t) = (D_1 + D_2 t)e^{-\sqrt{5}t} + \frac{3}{5} \qquad t \geq t_0$$

Using the initial conditions we find $D_1 = \frac{-3}{5}$ and $D_2 = \frac{-3}{\sqrt{5}}$.



The response of this system to the unit step is said to be *critically damped*.

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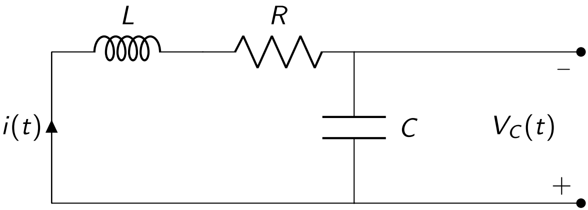
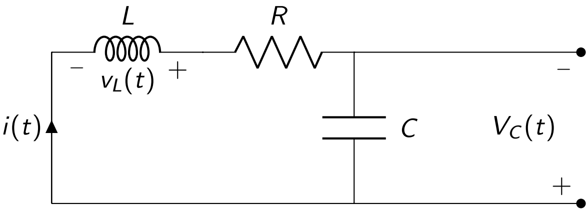
Example



If $i(0) = i_0$ and $\frac{\partial i(t)}{\partial t}|_{t=0} = i_0'$, determine the differential equation for the current $i(t)$ for $t \geq 0$.

Sol.



KVL in the loop:

$$v_L(t) + i(t)R + V_C(t) = 0$$

We express all terms as a function of $i(t)$:

$$v_L(t) = L\frac{\partial i(t)}{\partial t}$$
$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

Thus

$$L\frac{\partial i(t)}{\partial t} + i(t)R + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0$$

This equation can be transformed into a second order differential equation by computing the time-derivative at both sides of the equation...

$$L\frac{\partial^2 i(t)}{\partial t^2} + \frac{\partial i(t)}{\partial t}R + \frac{1}{C}i(t) = 0$$

To find the solution for $i(t)$ $t \geq 0$ using our "recipe", we divide by L so the coefficient for $\frac{\partial^2 i(t)}{\partial t^2}$ is one...

$$\frac{\partial^2 i(t)}{\partial t^2} + \frac{R}{L}\frac{\partial i(t)}{\partial t} + \frac{1}{LC}i(t) = 0$$

$$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = \gamma, \quad s^2 + 2\alpha s + \omega_0^2 = 0, \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

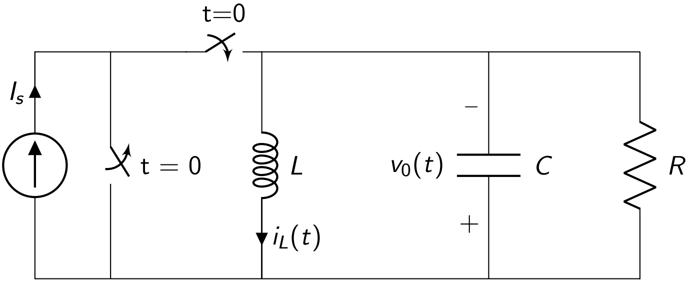
We identify terms

$$\frac{\partial^2 i(t)}{\partial t^2} + \frac{R}{L}\frac{\partial i(t)}{\partial t} + \frac{1}{LC} i(t) = 0 \quad \Rightarrow \omega_0^2 = \frac{1}{LC}, 2\alpha = R/L.$$

$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

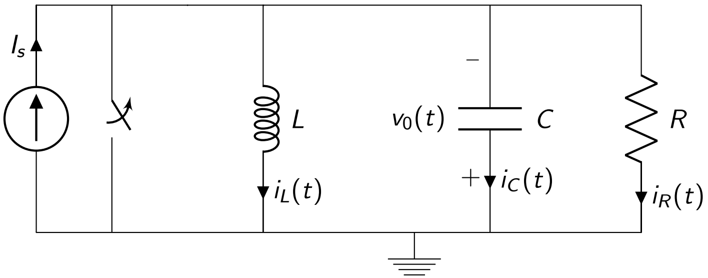
- $R > 2\sqrt{\frac{L}{C}} \rightarrow$ real (negative) roots. Overdamped solution.
- $R < 2\sqrt{\frac{L}{C}} \rightarrow$ complex roots. Underdamped solution.
- $R = 2\sqrt{\frac{L}{C}} \rightarrow$ 2 equal roots. Critically damped solution.
- $R = 0 \rightarrow$ Oscillation.

Example



Determine the differential equation for the current $i_L(t)$ for $t \geq 0$.

Determine the differential equation for the voltage $v_0(t)$ for $t \geq 0$.



$$\text{KCL: } I_s = i_L(t) + i_C(t) + i_R(t)$$

- Inductor: $v_0(t) = L\frac{\partial i_L(t)}{\partial t}$
- Resistor: $i_R(t) = v_0(t)/R = \frac{L}{R}\frac{\partial i_L(t)}{\partial t}$
- Capacitor: $i_C(t) = C\frac{\partial v_0(t)}{\partial t} = CL\frac{\partial^2 i_L(t)}{\partial^2 t}$

$$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_0^2 y(t) = \gamma, \quad s^2 + 2\alpha s + \omega_0^2 = 0, \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Therefore

$$CL\frac{\partial^2 i_L(t)}{\partial^2 t} + \frac{L}{R}\frac{\partial i_L(t)}{\partial t} + i_L(t) = I_s$$

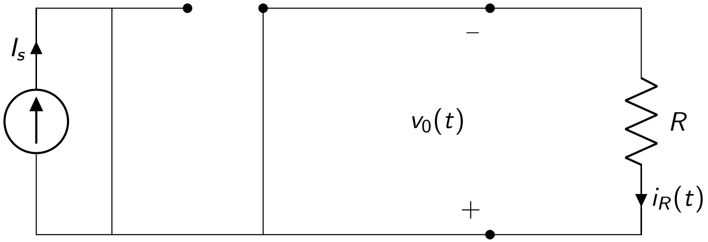
$$\Rightarrow \omega_0^2 = \frac{1}{LC}, 2\alpha = \frac{1}{RC}, \gamma = \frac{I_s}{CL}$$

$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4(RC)^2} - \frac{1}{LC}}$

- $R < \frac{1}{2}\sqrt{\frac{L}{C}} \to$ real (negative) roots. Overdamped solution.
- $R > \frac{1}{2}\sqrt{\frac{L}{C}} \to$ complex roots. Underdamped solution.
- $R = \frac{1}{2}\sqrt{\frac{L}{C}} \to$ 2 equal roots. Critically damped solution.
- $R = \infty \to$ Oscillation.

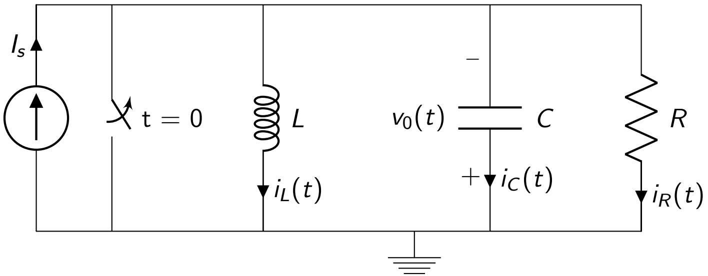
Auxiliary conditions

We need $i_L(0^+)$ and $\frac{\partial i_L(t)}{\partial t}|_{t=0^+}$. If the circuit has been operating for a long time before the switch is open, then the equivalent circuit is

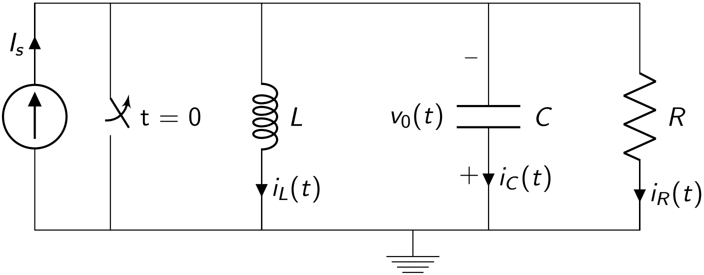


Therefore, $i_L(0^-) = i_L(0^+) = 0$ A and $v_0(0^-) = v_0(0^+) = 0$ V. Since $v_0(t) = I\frac{\partial i_L(t)}{\partial t}$, this means

$$\frac{\partial i_L(t)}{\partial t}|_{t=0^+} = 0 \quad \text{A/s}$$



- If we already have the solution to $i_L(t)$, then $v_0(t) = L\frac{\partial i_L(t)}{\partial t}$.
- We can directly obtain the differential equation for $v_0(t)$.



KCL: $I_s = i_L(t) + i_C(t) + i_R(t)$

- Resistor: $i_R(t) = v_0(t)/R$
- Capacitor: $i_c(t) = C\frac{\partial v_0(t)}{\partial t}$
- Inductor: $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_0(\tau)d\tau$

Therefore

$$I_s = \frac{1}{L}\int_{-\infty}^{t} v_0(\tau)d\tau + C\frac{\partial v_0(t)}{\partial t} + \frac{1}{R}v_0(t)$$

If we compute the time-derivative at both sides of the equation

$$\frac{\partial^2 v_0(t)}{\partial t^2} + \frac{1}{RC}\frac{\partial v_0(t)}{\partial t}\frac{1}{LC}v_0(t) = 0$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4(RC)^2} - \frac{1}{LC}}$$

- $R < \frac{1}{2}\sqrt{\frac{L}{C}} \rightarrow$ real (negative) roots. Overdamped solution.
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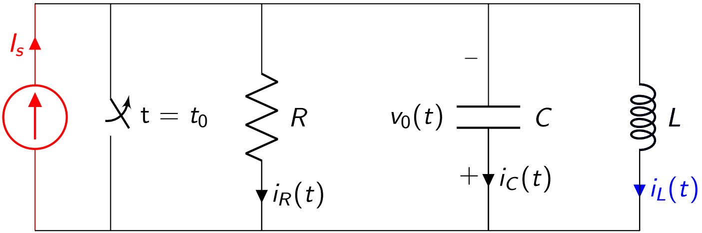
RLC circuits as systems



Differential equation for $i_L(t)$:

$$\frac{\partial^2 i_L(t)}{\partial^2 t} + \frac{1}{RC}\frac{\partial i_L(t)}{\partial t} + \frac{1}{L} i_L(t) = \frac{I_s}{LC} u(t - t_0)$$

with auxiliary conditions $i_L(t_0) = A$, $\frac{\partial i_L(t)}{\partial t}|_{t=t_0} = B$
We can consider $I_s u(t - t_0)$ as the input and $i_L(t)$ as the output of a system that is implemented by the RLC circuit. These systems are called **filters**.

Filters

The **system** input/output relationship is given by the differential equation and the auxiliary conditions.

We can characterize the system using the properties defined for systems in the first part of the course. In particular, we will discuss the linearity, time-invariance and causality properties of the filter.

Linearity

If we put an input current I_1 in the source, we will observe a current in the inductor (system output) $i_1(t)$ such that:

$$\frac{\partial^2 i_1(t)}{\partial^2 t} + \frac{1}{RC}\frac{\partial i_1(t)}{\partial t} + \frac{1}{L} i_1(t) = \frac{I_1}{LC} u(t - t_0)$$

with auxiliary contidions $i_1(t_0) = A$, $\frac{\partial i_1(t)}{\partial t}|_{t=t_0} = B$

If we consider a different input I_2, the new output current $i_2(t)$ verifies

$$\frac{\partial^2 i_2(t)}{\partial^2 t} + \frac{1}{RC}\frac{\partial i_2(t)}{\partial t} + \frac{1}{L} i_2(t) = \frac{I_2}{LC} u(t - t_0)$$

with auxiliary conditions $i_2(t_0) = A$, $\frac{\partial i_2(t)}{\partial t}|_{t=t_0} = B$

If we consider an input that is the linear combination of the two former inputs $I_3 = aI_1 + bI_2$ the new output $i_3(t)$ is such that

$$\frac{(aI_1 + bI_2)}{LC} u(t - t_0) = a \left\{ \frac{\partial^2 i_1(t)}{\partial^2 t} + \frac{1}{RC} \frac{\partial i_1(t)}{\partial t} + \frac{1}{L} i_1(t) \right\} +$$

$$b \left\{ \frac{\partial^2 i_2(t)}{\partial^2 t} + \frac{1}{RC} \frac{\partial i_2(t)}{\partial t} + \frac{1}{L} i_2(t) \right\}$$

$$= \frac{\partial^2 (ai_1(t) + bi_2(t))}{\partial^2 t} + \frac{1}{RC} \frac{\partial (ai_1(t) + bi_2(t))}{\partial t} + \frac{1}{L} (ai_1(t) + bi_2(t))$$

Hence, the output is $i_3(t) = ai_1(t) + bi_2(t)$.
But we have to see what happens with the auxiliary conditions.

Filter linearity: auxiliary conditions

The current $i_3(t)$ must verify

$$\begin{array}{rcl} i_3(t_0) & = & A \\ \frac{\partial i_3(t)}{\partial t}\Big|_{t=t_0} & = & B \end{array} \Rightarrow \begin{array}{rcl} ai_1(t_0)+bi_2(t_0) & = & A \\ \frac{\partial(ai_1(t)+bi_2(t))}{\partial t}\Big|_{t=t_0} & = & B \end{array}$$

On the other hand

$$\begin{array}{rcl} ai_1(t_0)+bi_2(t_0) & = & aA+bA \\ \frac{\partial(ai_1(t)+bi_2(t))}{\partial t}\Big|_{t=t_0} & = & aB+bB \end{array}$$

Putting all together

$$\begin{array}{rcl} A & = & (a+b)A \\ B & = & (a+b)B \end{array}$$

which has to be true for any a, b, hence $A=B=0$.

Linearity condition for a filter

Null auxiliary conditions!

Time Invariancy

- $i_L(t)$ is the output when the input is $I_s u(t - t_0)$.
- If we close the switch at $t_1 > t_0$, we are essentially delaying the input: $I_s u(t - t_1)$.
- For this new input, we know the solution is automatically shifted to t_1 (the exponential solution will depend on $(t - t_1)$).
- However, the auxiliary conditions have to be defined at t_1!!! Otherwise, the system is not time-invariant!

Filter time-invariance condition

Auxiliary conditions must be referred to the instant at which the input is activated (auxiliary conditions are initial conditions).

Causality

Filter causality condition

Auxiliary conditions are **null initial conditions**.

- This guarantees that the output signal $i_L(t)$ is zero as long as the input hasn't been activated.