— then

$$\text{truncation error} = \tau^h = L_h u - f_h$$

i.e., the truncation error is what is left after one plugs in the exact solution of the PDE into the discretized PDE

— example:

- consider the PDE (actually, it's an ODE)

$$Lu = -\frac{d^2u}{dx^2} = f(x)$$

- consider the discretized PDE [recall that $u_i^h \approx u(x_i)$] using a uniform grid

$$L_h u^h = -\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} = f(x_i)$$

where we choosen $f_h = f(x_i)$

- then, the truncation error is given by

$$\tau^h = L_h u - f_h = -\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - f(x_i)$$

- we do not know the exact solution so what good is this formula?

- suppose we know (from PDE theory) that the exact solution of the PDE has four continuous derivatives

- we then have Taylor series with remainder

$$\begin{aligned} u(x_{i\pm 1}) = u(x_i \pm h) = u(x_i) \;\; &\pm \;\; h\frac{du}{dx}(x_i) \;\; + \;\; \frac{h^2}{2}\frac{d^2u}{dx^2}(x_i) \\ &\pm \;\; \frac{h^3}{6}\frac{d^3u}{dx^3}(x_i) \;\; + \;\; \frac{h^4}{24}\frac{d^4u}{dx^4}(x_i \pm h\theta_\pm) \end{aligned}$$

for some $0 \leq \theta_\pm \leq 1$

- substituting these Taylor series into the the truncation error formula we obtain

$$
\begin{aligned}
\tau^h \; &= \; -\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - f(x_i) \\
&= \; \frac{1}{h^2}\Bigg\{ -\Bigg[u(x_i) \; + \; h\frac{du}{dx}(x_i) \; + \; \frac{h^2}{2}\frac{d^2u}{dx^2}(x_i) \\
&\qquad\qquad + \; \frac{h^3}{6}\frac{d^3u}{dx^3}(x_i) \; + \; \frac{h^4}{24}\frac{d^4u}{dx^4}(x_i + h\theta_+)\Bigg] \\
&\qquad + \; 2u(x_i) \\
&\qquad -\Bigg[u(x_i) \; - \; h\frac{du}{dx}(x_i) \; + \; \frac{h^2}{2}\frac{d^2u}{dx^2}(x_i) \\
&\qquad\qquad - \; \frac{h^3}{6}\frac{d^3u}{dx^3}(x_i) \; + \; \frac{h^4}{24}\frac{d^4u}{dx^4}(x_i - h\theta_-)\Bigg] \Bigg\} \\
&\quad - \; f(x_i)
\end{aligned}
$$

- this simplifies to

$$\tau^h = -\frac{d^2u}{dx^2}(x_i) \;\; + \;\; O(h^2) \;\; - f(x_i)$$

where $O(h^\alpha)$ means that we have a term that goes to zero as h tends to zero as the α power of h, i.e.,

$$\epsilon = O(h^\alpha) \qquad \Rightarrow \qquad \lim_{h\to 0} \frac{\epsilon}{h^\alpha} < \infty$$

or

$$|\epsilon| \approx C_t h^\alpha$$

for some constant C_t whose value is independent of h

- but $u(x)$ is the exact solution of the PDE so it satisfies

$$-\frac{d^2u}{dx^2}(x) \;\; - \;\; f(x) = 0$$

for any x, including $x = x_i$

- we then have obtained an estimate for the truncation error

$$\tau^h = O(h^2) \qquad \text{or} \qquad |\tau^h| \approx C_t h^2$$