Contents

Eqns

Advection

The **Advection or Convection** operator is the following:

$$u \cdot \nabla = \nabla \cdot u = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial u_y} + u_z \frac{\partial}{\partial u_z} \tag{1}$$

For a conserved quantity described by f it is called the **continuity equation** and it models the motion of a conserved scalar field as it is advected by a known velocity field.

$$\frac{df}{dt} + \nabla \cdot f\vec{u} = 0 \tag{2}$$

In 2D $f(x, y, t)$ then $\vec{u} = (u_x, u_y)$ and the **continuity equation** is:

$$\begin{gathered} \frac{df}{dt} + \nabla \cdot f\vec{u} = 0 \\ \text{In 2D:} \\ \frac{df(x,y,t)}{dt} + u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} = 0 \end{gathered} \tag{3}$$

Diffusion/Heat

It describes the Brownian motion. In this case u represents the density of the diffused material.

$$\frac{\partial u}{\partial t} = \nabla \cdot [D \nabla u] \tag{4}$$

With a constant diffusion coefficient D given by λ:

$$\frac{\partial u}{\partial t} = \lambda \nabla^2 u \tag{5}$$

Advection-Diffusion

Combination of diffusion and advection.

$$\frac{\partial f}{\partial t} = \lambda \nabla^2 f - \nabla \cdot f\vec{u} + R \tag{6}$$

R defines the sources or sinks of f.

Wave equation

$u(x_1, x_2, \ldots, x_n)$ can be the pressure in a liquid or the particles of a vibrating solid.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right) \tag{7}$$

c is the speed of propagation.

Burgers' Equation

It occurss in various areas like fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow.
For a given field $u(x,t)$ and diffusion coefficient ν, the general **viscous Burgers' equation**:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{8}$$

Inviscid Burgers' equation. Which is a prototype for conservation equations that can develop discontinuities (shock waves).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{9}$$

Boundary Conditions

Dirichlet. Specifies the values of the function along the boundary of the domain.
For ODEs Dirichlet boundary conditions on the interval $[a,b]$ are:

$$y(a) = \alpha, y(b) = \beta \tag{10}$$

For PDEs Dirichlet boundary conditions on a domain $\Omega \subset R^n$ are

$$y(x) = f(x) \quad \forall x \in \partial\Omega \tag{11}$$

Neumann. Specifies the values in which the derivative of a solution along the boundary of the domain.
For ODEs Neumann boundary conditions on the interval $[a,b]$ are:

$$y'(a) = \alpha, y'(b) = \beta \tag{12}$$

For PDEs Neumann boundary conditions on a domain $\Omega \subset R^n$ are

$$\frac{\partial y}{\partial \mathbf{n}}(x) = f(x) \quad \forall x \in \partial\Omega \tag{13}$$

where $\mathbf{n}$ denots the normal to the boundary

Robin. Weighted combination of Dirichlet and Neumann

Consistency/Stability/Convergent/CFL

Consistency

When $\Delta t, \Delta x \to 0$ then the error of the numerical method will go to 0.

Stability.

• The solution of an ODE/PDE is stable if a small perturbation does not cause divergence from the solution.

• A numerical method is stable if a small perturbation does not cause the numercial solution to diverge without bound.

A FD scheme is stable in a region if there is an integer J such that for any positive time T there is a constant C_T such that:

$$||v^n||_{\Delta x} \leq C_T \sum_{j=0}^{J} ||v^j||_{\Delta x} \tag{14}$$

Convergent scheme ?

CFL It is a necessary condition for convergence, if we reduce the Δx we need to reduce Δt. Important in the following equations u_{x_i} corresponds to the speed of the wave. In the advection equation it is easy, it is v in $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$

$$C = \Delta t \left(\sum_{i=1}^{n} \frac{u_{x_i}}{\Delta x_i} \right) \leq C_{max} \tag{15}$$

For explicit solver $C_{max} = 1$, for implicit C_{max} may be larger. For $C_{max} = 1$

$$\Delta t \leq \frac{1}{\sum_{i=1}^{n} \frac{u_{x_i}}{\Delta x_i}} \tag{16}$$

Methods

FTCS. Replace any time derivative u_t and space derivatives u_x with:

$$u_t \approx \frac{\phi_m^{n+1} - \phi_m^n}{\Delta t}, u_x \approx \frac{\phi_{m+1}^n - \phi_{m-1}^n}{2\Delta x}, \tag{17}$$

Leapfrog (CTCS) . Replace any time derivative u_t and space derivatives u_x with:

$$u_t \approx \frac{\phi_m^{n+1} - \phi_m^{n-1}}{2\Delta t}, u_x \approx \frac{\phi_{m+1}^n - \phi_{m-1}^n}{2\Delta x}, \tag{18}$$

Lax-Friedich. Replace any time derivative u_t and space derivatives u_x with:

$$u_t \approx \frac{\phi_m^{n+1} - \phi_m^{n-1}}{2\Delta t}, u_x \approx \frac{\phi_{m+1}^n - \phi_{m-1}^n}{2\Delta x}, \tag{19}$$