Contents

1 PDE equations

Hyperbolic PDE. The definition is hard, read at wikipedia. The quations of the following form can be transformed to the wave equation:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + (\text{lower order terms}) = 0 \quad \text{with } B^2 - 4AC > 0 \tag{1}$$

2 Schemes

This are numerical schemes for the problem $u_t + au_x = f$.
Leap frog

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h}$$

Upwind

$$FTBS\ if a > 0$$
$$FTFS\ if a < 0$$

Lax-Friedrichs

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

Lax-Wendroff

$$v_m^{n+1} = v_m^n + \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \frac{a^2\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n) + \frac{k}{2}(f_m^{n+1} + f_m^n) - \frac{ak\lambda}{4}(f_{m+1}^n - f_{m-1}^n)$$

Crank-Nicolson

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0$$

Box scheme

$$\begin{aligned} &\frac{1}{2k}\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right] \\ +&\frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] \\ =&\frac{1}{4}(f_{m+1}^{n+1} + f_m^{n+1} + f_{m+1}^n + f_m^n) \end{aligned}$$

3 Summary

3.1 Characteristic method

3.2 Consistency

Given a partial differential equation $Pu = f$, and a finite difference scheme $P_{k,h}v = f$, we say that the finite difference scheme is consistent with the PDE if for any smooth function $\Phi(t, x)$:

$$P\Phi - P_{k,h}\Phi \to 0 \; as \; k, h \to 0 \tag{2}$$

Where P means a first order linear combination of u_t and a linear combination of u_x. For example if $P = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} \to P\Phi = \Phi_t + a\Phi_x$

3.3 Order of accuracy

3.1.1 A scheme $P_{k,h}v = R_{k,h}f$ that is consistent with the differential equation $Pu = f$ is accurete of order p in time and order q in space if for any smooth function $\theta(t, x)$:

$$P_{k,h}\Theta - R_{k,h}P\Theta = O(k^p) + O(h^q)$$

3.1.11 A scheme $P_{k,h}v = R_{k,h}f$ that is consistet with $Pu = f$ is accurate of order (p,q) if and only if for each value of s and ξ:

$$p_{k,h}(s, \xi) - r_{k,h}(s, \xi)p(s, \xi) = O(k^p) + O(h^q) \tag{3}$$

3.4 Stability

Stability for the homogeneous initial value problem
A finite difference scheme $P_{k,h}v_m^n = 0$ is stable in a stability region Λ if there is an integer J such that for any postive time T, there is a constant C_T such that:

$$h\sum_{m=-\infty}^{\infty}|v_m^n|^2 \le C_T h \sum_{j=0}^{j}\sum_{m=-\infty}^{\infty}|v_m^j|^2 \tag{4}$$

for $0 \le nk \le T$, with $(k, h) \in \Lambda$.

3.4.1 CFL (Courant-Friedrichs-Lewy)

For an explicit scheme for the hyperbolic equation of the form $v_m^{n+1} = \alpha v_{m-1}^n + \beta v_m^n + \gamma v_{m+1}^n$ with $k/h = \lambda$ a **necessary condition for stability** is the CFL condition:

$$|a\lambda| \leq 1 \tag{5}$$

3.4.2 Fourier transform

Can't use Fourier transform for variable coefficient $a(x,t)$

We have the following transformations:
Infinite domain

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx \qquad \text{and} \qquad u(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{-i\omega x} \hat{u}(\omega) dx \tag{6}$$

Finite domain The ones in the RIGHT used to test stability

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} v_m h \qquad \text{and} \qquad v(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}_m(\xi)$$

We define he following norms where $||\hat{v}||_h^2 = ||v||_h^2$:

$$||v||_h^2 = \sum_{m=-\infty}^{\infty} |v_m|^2 h \qquad \text{and} \qquad ||\hat{v}||_h^2 = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}(\xi) d\xi \tag{7}$$

3.4.3 Von Neumann Analysis

If we use Fourier transform we normally get:

$$\hat{v}^{n+1} = g(h\xi)\hat{v}^n$$

Where $g(h\xi)$ is the amplification factor of each frequency $\hat{v}^n$.
Theorem A one-step finite difference scheme (with constant coefficients) is stable in a stability region Λ if and only if there is a constant K (independent of θ, k, and h) such that:

$$|g(\theta, k, h)| \leq 1 + Kk$$

If $g(\theta, k, h)$ is independent of h and k, the stabilitiy condition can be replaced by:

$$|g(\theta)| \leq 1$$

Trick, where $\theta = h\xi$ For constant coefficients we can do:

$$v_m^n = g^n e^{imh\xi} = g^n e^{im\theta} \tag{8}$$

3.4.4 Stability for multistep schemes

A finite difference scheme for a scalar equation is stable if and only if all the roots, $g_v(\theta)$ of the amplification polynomial $\Phi(\theta, k, h)$ satisfy the following conditions:

- There is a constant K such that $|g_v| \leq 1 + Kk$
- If $|g_v(\theta)| = 1$ then $g_v(\theta)$ must be a simple root

$$v_m^n = g^n e^{im\theta}$$

3.5 Dissipation

Dissipation: Waves loose energy over time (lost of energy converts into heat, dissipate heat).

A scheme is dissipative of order $2r$ if there exist a positive constant c, independent of h an k, such that each amplification factor $g_v(\theta)$ satisfies:

$$|g_v(\theta)| \leq 1 - c(\sin \frac{1}{2}\theta)^{2r}$$

3.6 Dispersion

Dispersion: Different frequencies at different speeds.
Option one:

$$v_m^n = r(h\xi)e^{i\xi(x_m - \alpha(h\xi)t^n)} \tag{9}$$

Option two:

$$\hat{v}^n = g^n e^{i\theta m} \tag{10}$$

4 Math

4.1 Eigenvalues

Eigenvalues and eigenvectors of A_{2x2}.
Eigenvalues is with $det(A - \lambda I) = ab - cd$ and then solve the quadratic function.
Eigenvectors, solve the two equations given by $(A - \lambda I)\vec{x} = 0$ (with the computed values of λ.
Inverse of matrix A:

$$A^{-1} = \frac{1}{ab - cd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{11}$$

4.2 Integration by parts

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x) \tag{12}$$

4.3 Trigonometric identities

$$\begin{aligned}
\cos(-\theta) &= \cos(\theta) \\
\sin(-\theta) &= -\sin(\theta) \\
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin^2 x + \cos^2 x &= 1 \\
\cos(\theta) &= 1 - 2\sin^2(\frac{\theta}{2}) \\
\sin\theta &= 2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta) \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y
\end{aligned}$$

4.4 Complex numbers

Basics

$$\begin{aligned}
&z = a + ib \\
&\bar{z} = a - ib \text{ Complex conjugate} \\
&(z \bar{+} w) = \bar{z} + \bar{w} \\
&\bar{z}^n = \bar{z}^n
\end{aligned} \tag{13}$$

For e:

$$\begin{aligned}
\cos(x) &= Re(e^{ix}) \\
\sin(x) &= Img(e^{ix}) \\
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{i\theta} - e^{-i\theta} &= 2i\sin(\theta) \\
e^{i\pi} &= -1 \\
(a+bi) + (c+di) &= (a+c) + (b+d)i \\
(a-bi)(c-di) &= (ac-bd) + (bc+ad)i \\
|a+bi|^2 &= a^2 + b^2 \\
\cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y \\
\sin(x+iy) &= \sin x \cosh y + i \cos x \sinh y
\end{aligned} \tag{14}$$

4.5 Series

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\
\tan(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \text{ for } |x| < \frac{\pi}{2} \\
\arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
\end{aligned}
\tag{15}
$$

5 Examples

Test consistency example.
Show that the FTCS scheme is consistent with equation $u_t + au_x = 0$ where a is a constant and $u(0, x) = f(x)$ for all real numbers x.

The FTCS scheme $P_{k,h}$ is given by:

$$P_{k,h}\Phi = \frac{\Phi_m^{n+1} - \Phi_m^n}{k} + a\frac{\Phi_{m+1}^n - \Phi_{m-1}^n}{2h} \tag{16}$$

Where $k = \Delta t$ and $h = \Delta x$.
To prove consistency we need to show that:

$$P\Phi - P_{k,h}\Phi \to 0 \ as \ k, h \to 0 \tag{17}$$

To do this, we use the Taylor series expansion of the function Φ in t and x with respect of the point (t_n, x_m).

$$\Phi_m^{n+1} = \Phi_m^n + k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3) \tag{18}$$

$$\Phi_{m+1}^n = \Phi_m^n + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) \tag{19}$$

$$\Phi_{m-1}^n = \Phi_m^n - h\Phi_x + \frac{h^2\Phi_{xx}}{2} - O(h^3) \tag{20}$$

Substituting equations 18, 19 and 20 into 16 we get:

$$
\begin{aligned}
P_{k,h}\Phi =& \frac{k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3)}{k} \\
&+ a\frac{\Phi_m^n + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) - (\Phi_m^n - h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3))}{2h} \\
=& \Phi_t + \frac{k}{2}\Phi_{tt} + O(k^2) + a\Phi_x + O(h^2)
\end{aligned}
\tag{21}
$$

Then we have that:

$$P\Phi - P_{k,h}\Phi = \frac{k}{2}\Phi_{tt} + O(k^2) + O(h^2) \tag{22}$$

This will go to 0 when $k, h \to 0$, therefore the FTCS scheme is consisten for $u_t + au_x = 0$.

Show that the Leapfrom scheme is consistent with equation $u_t + au_x = 0$.

The Leapfrom scheme $P_{k,h}$ is given by:

$$P_{k,h}\Phi = \frac{\Phi_m^{n+1} - \Phi_m^{n-1}}{2k} + a\frac{\Phi_{m+1}^n - \Phi_{m-1}^n}{2h} \tag{23}$$

Where $k = \Delta t$ and $h = \Delta x$.
To prove consistency we need to show that:

$$P\Phi - P_{k,h}\Phi \to 0 \ as \ k, h \to 0 \tag{24}$$

To do this we use the Taylor series expansion again, using the previous expansion and a new one for Φ_m^{n-1}

$$\Phi_m^{n-1} = \Phi_m^n - k\Phi_t + \frac{k^2\Phi_{tt}}{2} - O(k^3) \tag{25}$$

Substituting equations 18, 19 and 20 into 16 we get:

$$\begin{aligned} P_{k,h}\Phi =& \frac{\Phi_m^n + k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3) - (\Phi_m^n - k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3))}{2k} \\ &+ a\frac{\Phi_m^n + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) - (\Phi_m^n - h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3))}{2h} \\ =& \Phi_t + O(k^2) + a\Phi_x + O(h^2) \end{aligned} \tag{26}$$

Then we have that:

$$P\Phi - P_{k,h}\Phi = O(k^2) + O(h^2) \tag{27}$$

This will go to 0 when $k, h \to 0$, therefore the LeapFrog scheme is consisten for $u_t + au_x = 0$.