Numerical Solutions for Partial Differential Equations
Homework 1

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Show that the FTCS scheme is consistent with equation $u_t + au_x = 0$ where a is a constant and $u(0,x) = f(x)$ for all real numbers x.

The FTCS scheme $P_{k,h}$ is given by:

$$P_{k,h}\Phi = \frac{\Phi_m^{n+1} - \Phi_m^n}{k} + a\frac{\Phi_{m+1}^n - \Phi_{m-1}^n}{2h} \tag{1}$$

To prove consistency we need to show that:

$$P\Phi - P_{k,h}\Phi \to 0 \; as \; k,h \to 0 \tag{2}$$

To do this, we use the Taylor series expansion of the functions Φ_m^{n+1},Φ_{m+1}^n and Φ_{m-1}^n with respect of the point (t_n, x_m).

$$\Phi_m^{n+1} = \Phi_m^n + k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3) \tag{3}$$

$$\Phi_{m+1}^n = \Phi_m^n + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) \tag{4}$$

$$\Phi_{m-1}^n = \Phi_m^n - h\Phi_x + \frac{h^2\Phi_{xx}}{2} - O(h^3) \tag{5}$$

Substituting equations 3, 4 and 5 into 1 we get:

$$\begin{aligned}
P_{k,h}\Phi =& \frac{k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3)}{k} \\
&+ a\frac{\Phi^n_m + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) - (\Phi^n_m - h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3))}{2h} \\
=& \Phi_t + \frac{k}{2}\Phi_{tt} + O(k^2) + a\Phi_x + O(h^2)
\end{aligned} \tag{6}$$

Then we have that:

$$P\Phi - P_{k,h}\Phi = \frac{k}{2}\Phi_{tt} + O(k^2) + O(h^2) \tag{7}$$

This will go to 0 when $k, h \to 0$, therefore the FTCS scheme is consistent for $u_t + au_x = 0$.

Show that the Leapfrog scheme is consistent with equation $u_t + au_x = 0$.
The Leapfrog scheme $P_{k,h}$ is given by:

$$P_{k,h}\Phi = \frac{\Phi^{n+1}_m - \Phi^{n-1}_m}{2k} + a\frac{\Phi^n_{m+1} - \Phi^n_{m-1}}{2h} \tag{8}$$

We use the Taylor series expansion again, using the previous expansion and a new one for Φ^{n-1}_m.

$$\Phi^{n-1}_m = \Phi^n_m - k\Phi_t + \frac{k^2\Phi_{tt}}{2} - O(k^3) \tag{9}$$

Substituting equations 3,4, 5 and 9 into 8 we get:

$$\begin{aligned}
P_{k,h}\Phi =& \frac{\Phi^n_m + k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3) - (\Phi^n_m - k\Phi_t + \frac{k^2\Phi_{tt}}{2} + O(k^3))}{2k} \\
&+ a\frac{\Phi^n_m + h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3) - (\Phi^n_m - h\Phi_x + \frac{h^2\Phi_{xx}}{2} + O(h^3))}{2h} \\
=& \Phi_t + O(k^2) + a\Phi_x + O(h^2)
\end{aligned} \tag{10}$$

Then we have that:

$$P\Phi - P_{k,h}\Phi = O(k^2) + O(h^2) \tag{11}$$

This will go to 0 when $k, h \to 0$, therefore the LeapFrog scheme is consistent for $u_t + au_x = 0$.

Write a computer program that solves $\vec{u}_t + A\vec{u}_x = 0$ with initial condition $u(0,x) = f(x)$, $0 < x < 1$ and assuming periodic boundary conditions.

For this part two schemes that are stables and consistent when $|a\lambda| \leq 1$ where implemented in Matlab ($\lambda = \frac{k}{h}$). The first one uses FTFS when the $a_i > 0$ and FTBS when $a_i < 0$ as shown below:

$$\begin{aligned} w_i^{n+1} =& - a_i\frac{k}{h}(w_i^n - w_{i-1}^n) + w_i^n \quad if \quad a_i > 0 \\ w_i^{n+1} =& - a_i\frac{k}{h}(w_{i+1}^n - w_i^n) + w_i^n \quad if \quad a_i < 0 \end{aligned} \tag{12}$$

The second scheme is Lax-Friedrich's, which is also stable and consistent when $|a\lambda| \leq 1$:

$$w_m^{n+1} = -a\frac{k}{2h}(w_{m+1}^n - w_{m-1}^n) + \frac{1}{2}(w_{m+1}^n + w_{m-1}^n) \tag{13}$$

Both of the previous two schemes obey CFL condition.

The implementation of these schemes is as showed in class. We use $\vec{u}_0$ to indicate the initial boundary conditions and $\vec{v}$ to indicate the numerical approximation. The space and time intervals are given by h, k respectively and the space and time domains are given by $0 \leq m \leq M$, $0 \leq n \leq N$.

1. Set initial condition $\vec{v}_m^0 = \vec{u}_0$.
2. Obtain the eigenvalues Λ and eigenvectors S of the matrix A.
3. Iterate over the next s
   (a) Transform into characteristics equations by $w_m^n = S^{-1}v_m^n$
   (b) Apply any of the two methods to obtain w_m^{n+1}
   (c) Fill ghost cells $w_0^{n+1} = w_{M-1}^{n+1}$ and $w_{M+1}^{n+1} = w_2^{n+1}$
   (d) Transform back to $v_m^{n+1} = Sw_m^{n+1}$

To validate our numerical schemes we first compare the average squared error against to two known solutions while modifying the size of h to show convergence of the schemes.:

1. $A = \begin{pmatrix} .8 & 0 \\ 0 & .6 \end{pmatrix}$ with initial conditions $u_0 = [\sin(2\pi x) \ cos(4\pi x)]$
2. $A = \begin{pmatrix} .5 & 0 \\ 0 & .2 \end{pmatrix}$ with initial conditions $u_0 = [2\sin(2\pi x) \ .1sin(16\pi x)]$.

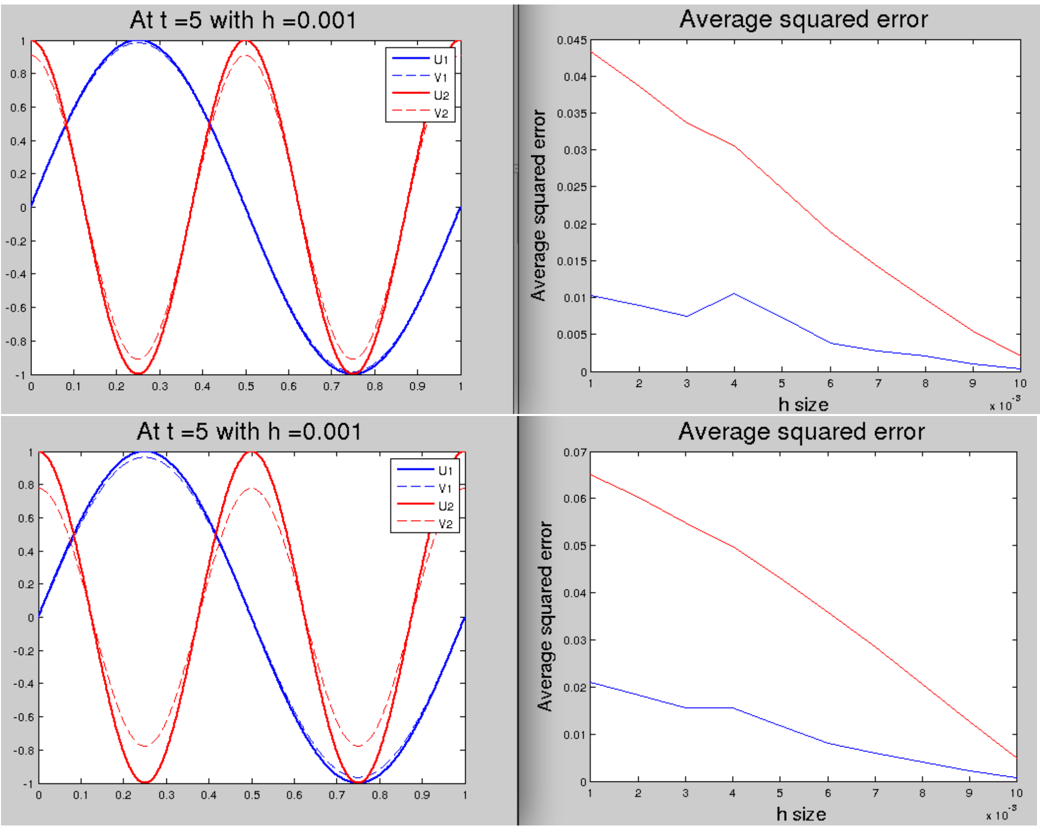

Figure 1: At the top the method using FTFS and FTBS and at the bottom Lax-Friedrich's scheme for the problem 1

The figures 1 and 2 show the average square errors at $t = 5$ for different values of h always using $\lambda = 1 \rightarrow k = .01$.
In all of these cases the two schemes converge to the analytical solution when $h \rightarrow 0$. Finally in figure 3 it is an example where $|a_1\lambda| > 1$ which makes both of this schemes unstable.
The numerical algorithms were programed in Matlab, the documentation on how to run the code is on the file README.txt.

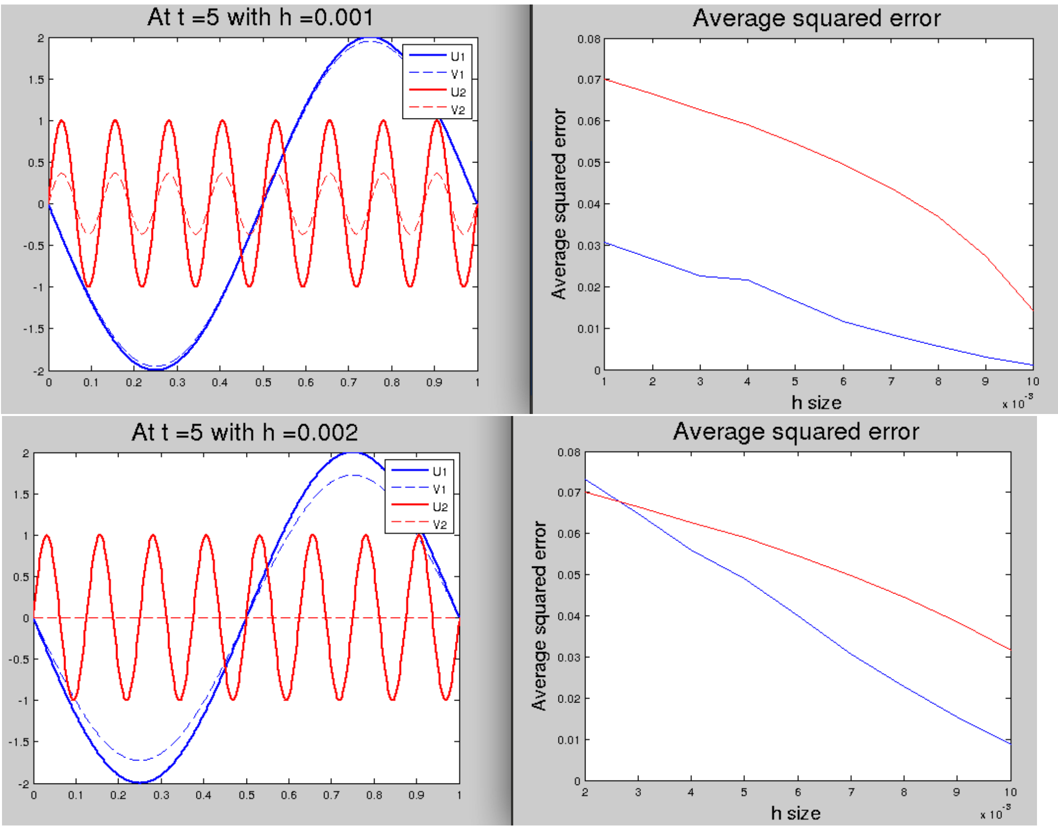

Figure 2: At the top the method using FTFS and FTBS and at the bottom Lax-Friedrich's scheme for the problem 2

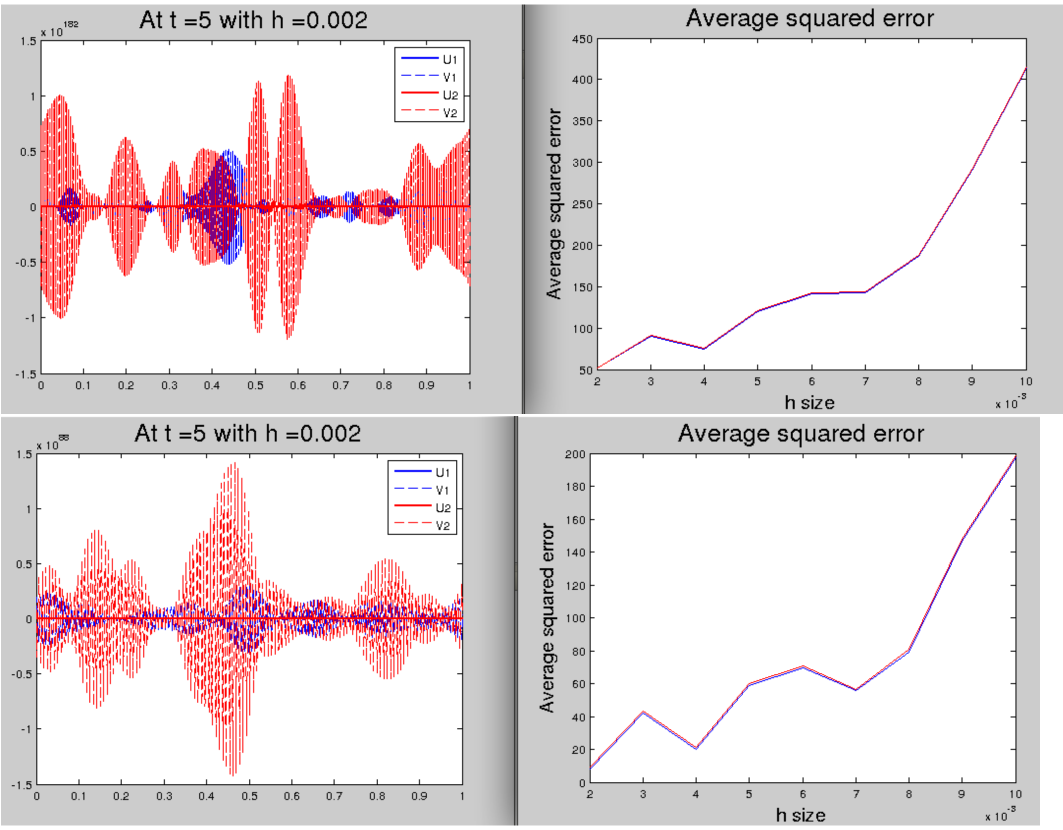

Figure 3: At the top the method using FTFS and FTBS and at the bottom Lax-Friedrich's scheme for $|a_1\lambda| > 1$ for problem