Quantum Resources in Harrow-Hassidim-Lloyd Algorithm

Pradeep Kumar, Tanoy Kanti Konar, Leela Ganesh Chandra Lakkaraju and Aditi Sen(De)

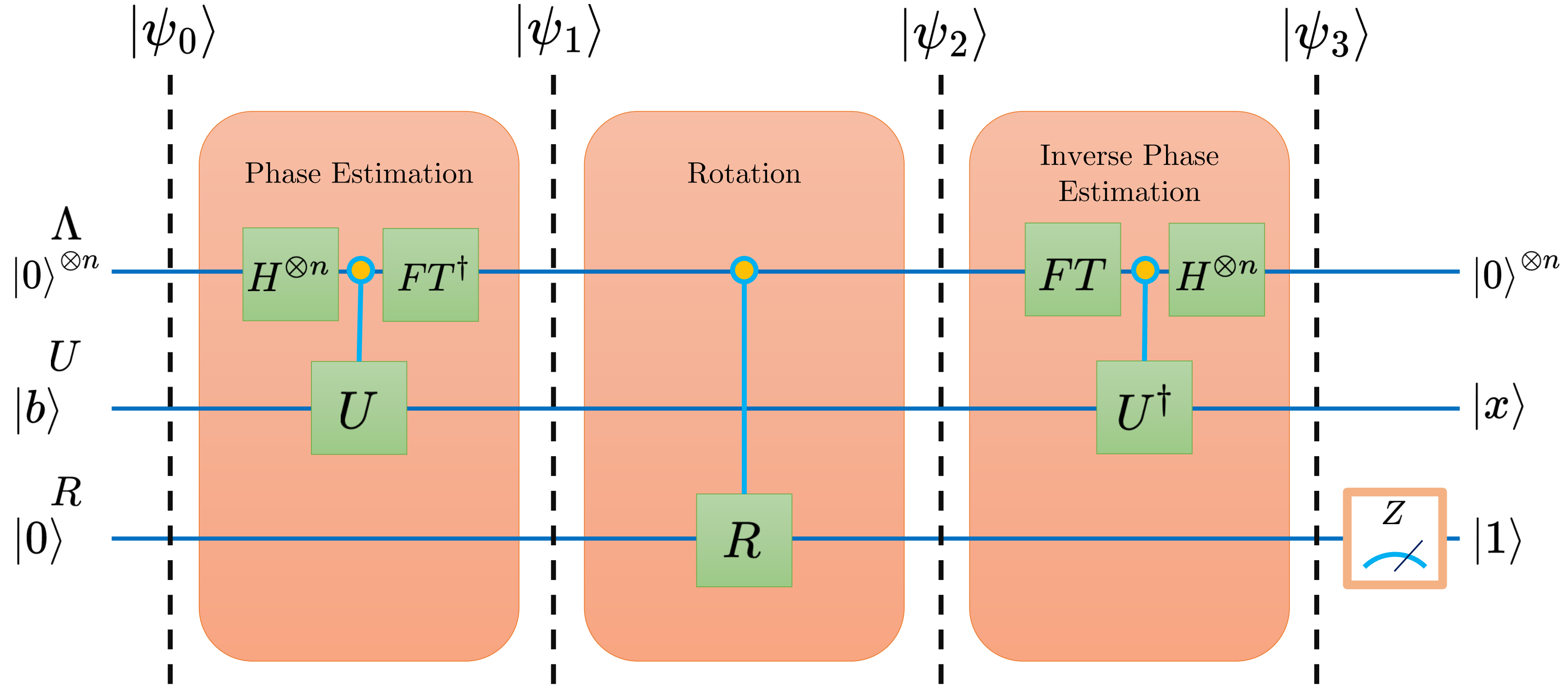
Quantum

$\mathcal{O}(\kappa^2 \log(N))$

Classical

$\mathcal{O}(N\kappa)$

The algorithm



What we need ?

$$\mathbf{A}\vec{x} = \vec{b}$$

$$\vec{b} = \sum_{i=1}^{N} \beta_i \vec{u}_i$$

$$\vec{x} = \mathbf{A}^{-1}\vec{b}_i = \sum_{i=1}^{N} \mathbf{A}^{-1}\vec{u}_i$$

$$\vec{x} = \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i}\vec{u}_i$$

$$|\psi_0\rangle = \sum_{i=1}^{N} |0\rangle_{\Lambda}^{\otimes n} \otimes \beta_i |u_i\rangle_U \otimes |0\rangle_R$$

$$QPE[\sum_{i=1}^{N} |0\rangle_{\Lambda}^{\otimes n} \beta_i |u_i\rangle_U |0\rangle_R] = \sum_{i=1}^{N} |2^n \phi\rangle_{\Lambda} \beta_i |u_i\rangle_U |0\rangle_R = \sum_{i=1}^{N} |\lambda_i\rangle_{\Lambda} \beta_i |u_i\rangle_U |0\rangle_R$$

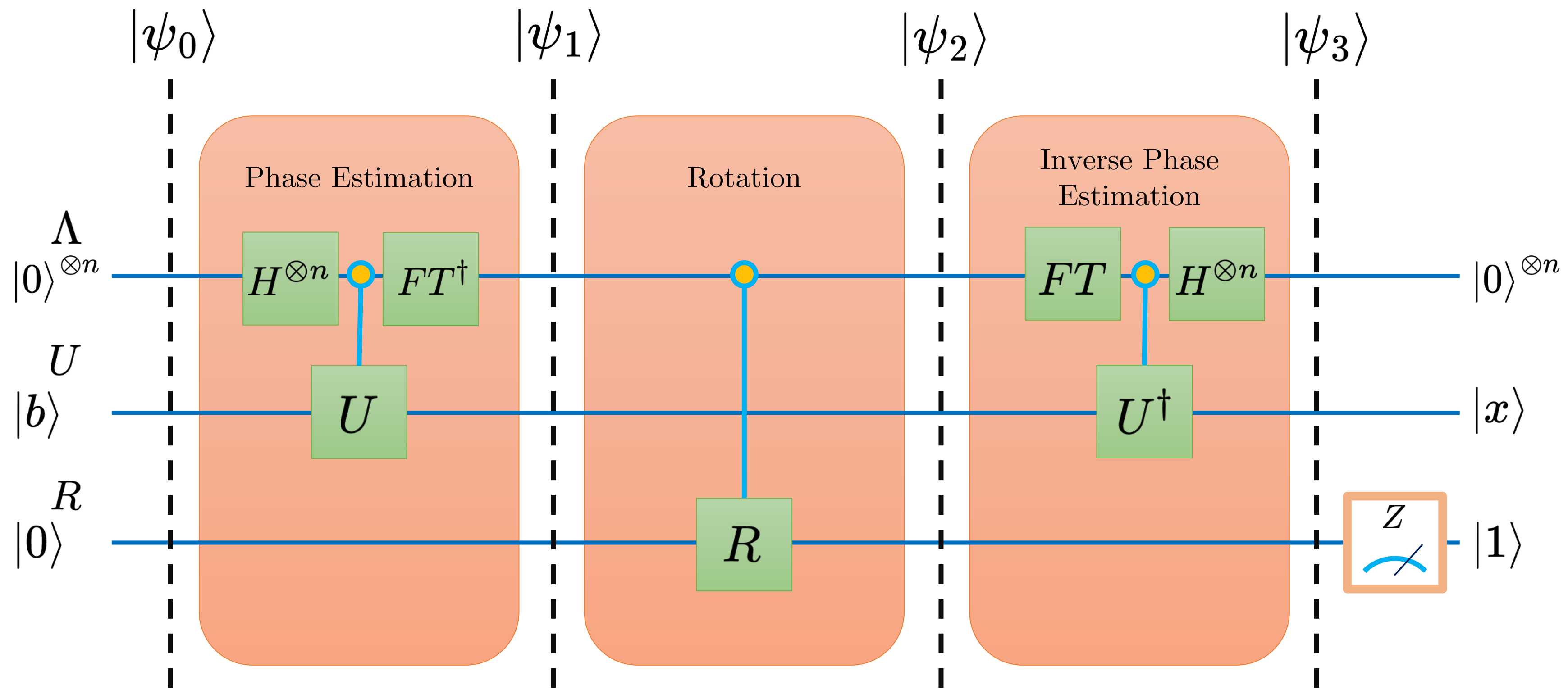
HHL Algorithm

$$|\psi_1\rangle = \sum_{i=1}^{N} |\lambda_i\rangle_{\mathbf{\Lambda}} \otimes \beta_i |u_i\rangle_{\mathbf{U}} \otimes |0\rangle_{\mathbf{R}},$$

$$\boxed{|\psi_2\rangle = \sum_{i=1}^{N} |\lambda_i\rangle_{\Lambda} \otimes \beta_i |u_i\rangle_{U} \otimes \left(\sqrt{1 - \frac{C^2}{\lambda_i^2}}\,|0\rangle + \frac{C}{\lambda_i}|1\rangle\right)_R}$$

$$|\psi_3\rangle = |0\rangle_{\Lambda}^{\otimes n} \otimes \sum_{i=1}^{N} \beta_i |u_i\rangle_{U} \otimes \left(\sqrt{1 - \frac{C^2}{\lambda_i^2}}\,|0\rangle + \frac{C}{\lambda_i}|1\rangle\right)_R$$

The algorithm



Theorem -1

Nonvanishing genuine multipartite entanglement

- For a successful run of the HHL algorithm, genuine multipartite entanglement should be nonvanishing in the second step
- We prove that all the single-site local density matrices, $\rho_{\mathbf{R}}$, $\rho_{\mathbf{U}}$ and $\rho_{\mathbf{\Lambda}}$ of the state $|\psi_2\rangle$ are mixed i.e. $Tr(\rho_X^2) < 1$
- Since the total state is pure, mixed local density matrices ensure that no biparition is product, thereby proving nonvanishing GME of $|\psi_2\rangle$

Theorem -2

Bipartite entanglement between **Λ** and **U** is nonvanishing

$$
|\psi_2\rangle \qquad \begin{aligned} \mathcal{LN}(\rho_{\mathbf{\Lambda U}}) &\neq 0 \\ \mathcal{LN}(\rho_{\mathbf{UR}}) &= 0 \\ \mathcal{LN}(\rho_{\mathbf{UR}}) &= 0 \end{aligned}
$$

Unlike GME, bipartite entanglement between subsystems are always present in different stages of the algorithm

Proposition -1

The l1-norm coherences are non-vanishing

- In computational basis C_R, C_U, and $C_{\Lambda UR}$ are non vanishing
- C_Λ is zero as $\boldsymbol{\rho_\Lambda}$ is diagonal in computation basis

Remark : We found that C_R is closely related to Success Probability

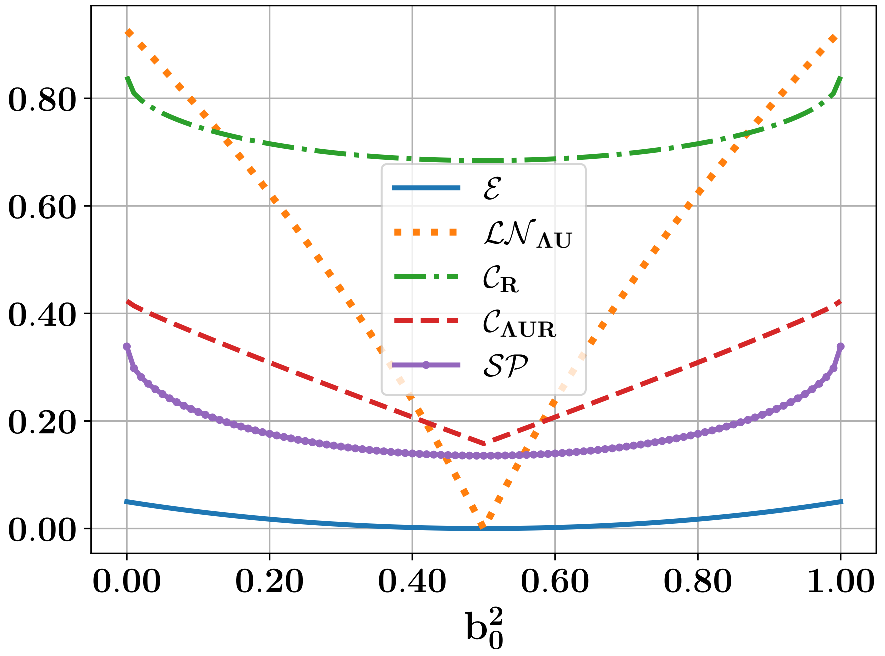
$$\mathcal{SP} = \sum_{i=1}^{N} \beta_i^2 \frac{C^2}{\lambda_i^2}$$

$$\mathcal{C}_{\mathbf{R}} = 2\sum_{i=1}^{N} \beta_i^2 \sqrt{1 - \frac{C^2}{\lambda_i^2}} \cdot \frac{C}{\lambda_i}$$

Quantum Advantage

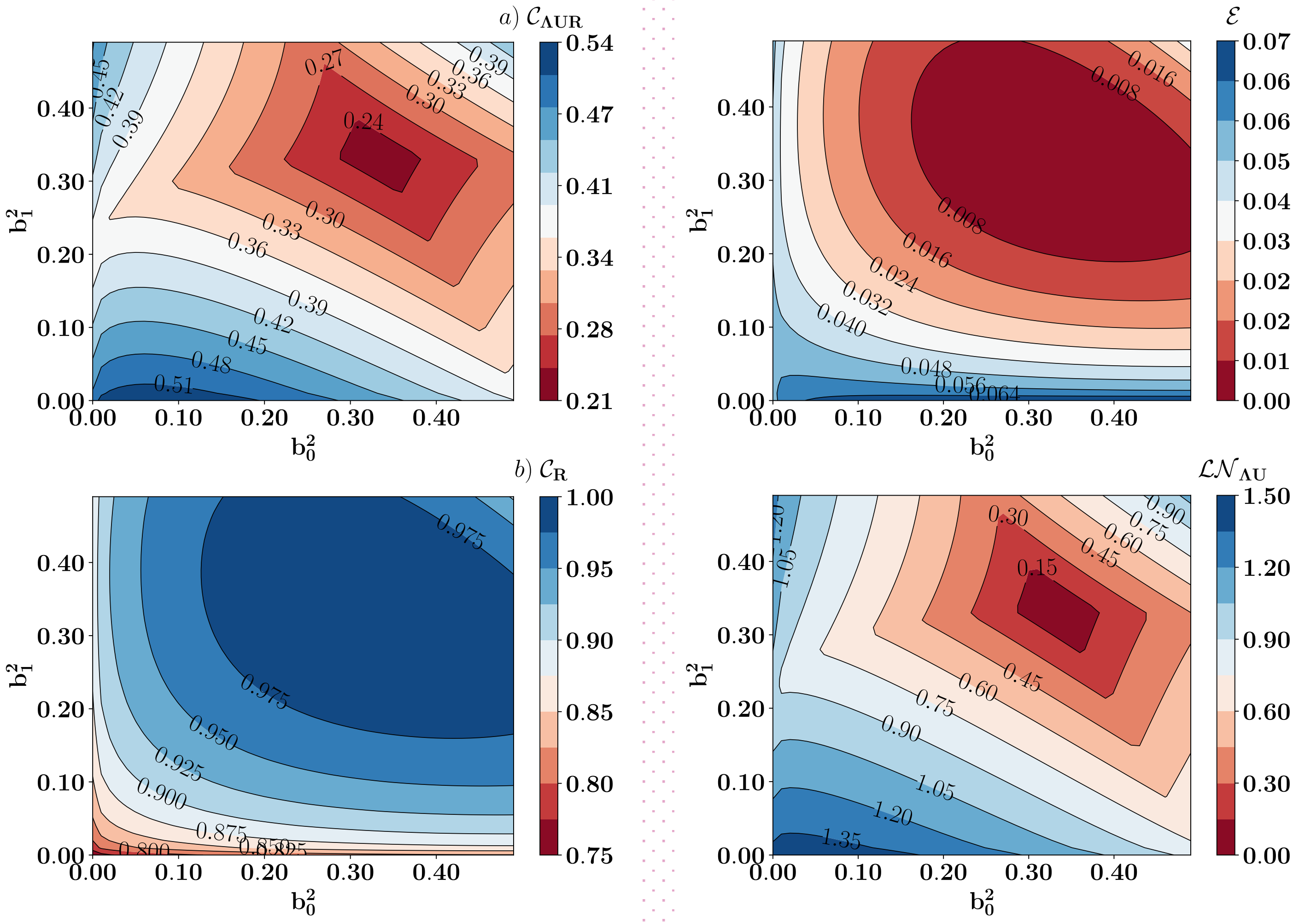
$$\frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}x_0\\ x_1\end{pmatrix}=\begin{pmatrix}b_0\\ b_1\end{pmatrix}$$

$$\mathbf{A}\quad\vec{x}=\vec{b}$$



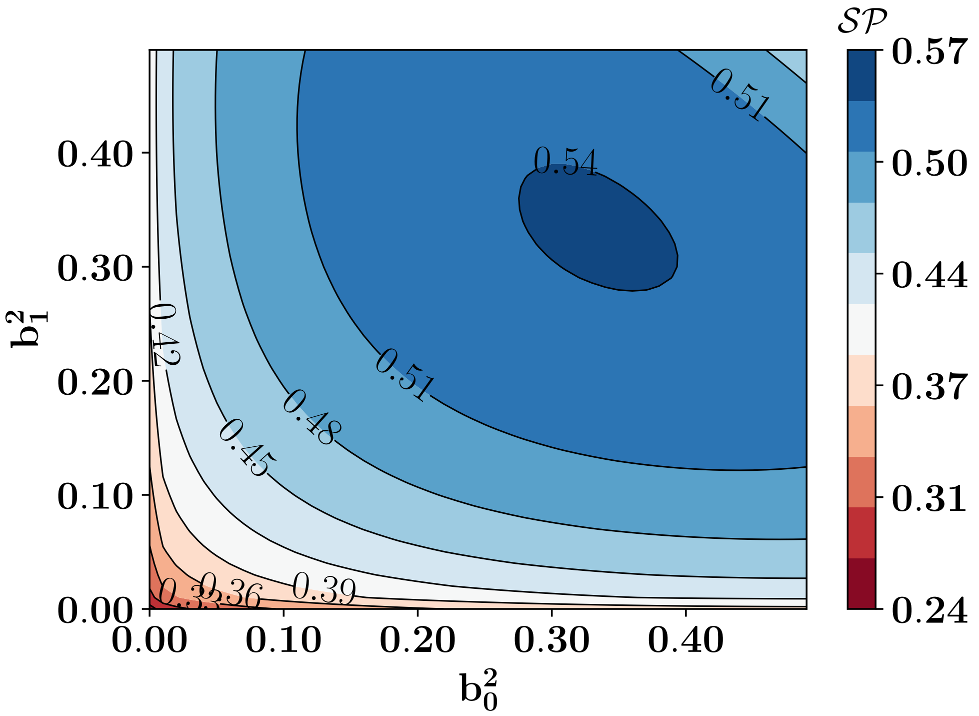
Quantum Advantage in 3D

$$\frac{1}{6}\begin{bmatrix}14 & -4 & -4\\ -4 & 11 & -1\\ -4 & -1 & 11\end{bmatrix}\begin{bmatrix}x_0\\ x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_0\\ b_1\\ b_2\end{bmatrix}$$

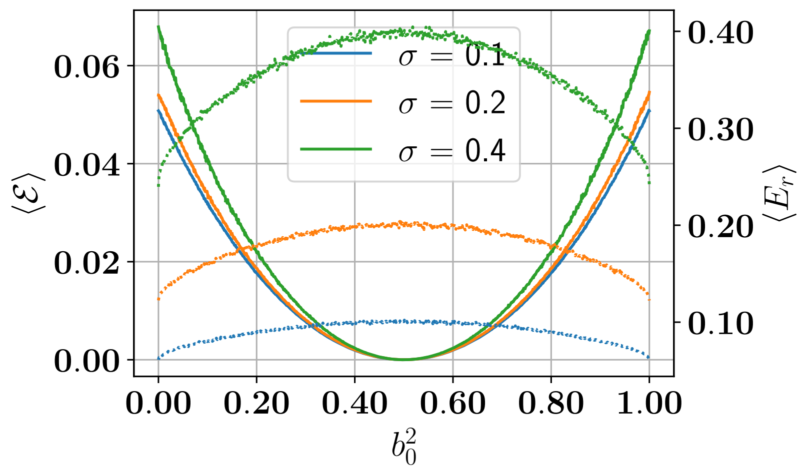


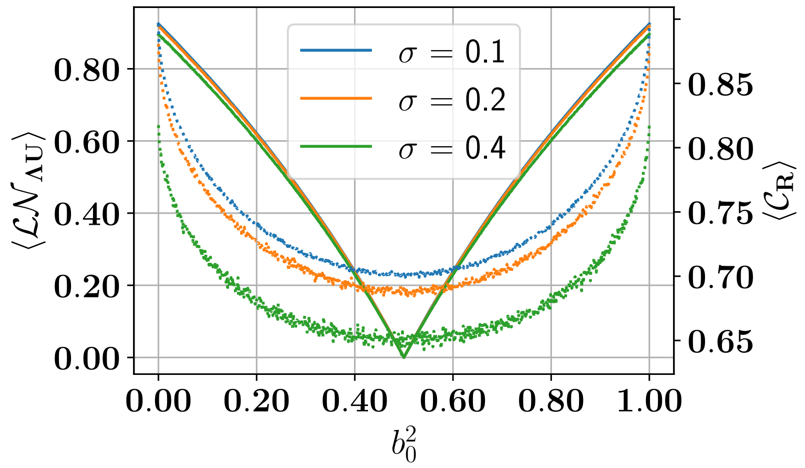
Success Probability

$$\mathcal{SP} = \sum_{i=1}^{N} \beta_i^2 \frac{C^2}{\lambda_i^2}$$



$$\mathcal{C}_{\mathbf{R}} = 2 \sum_{i=1}^{N} \beta_i^2 \sqrt{1 - \frac{C^2}{\lambda_i^2} \cdot \frac{C}{\lambda_i}}$$





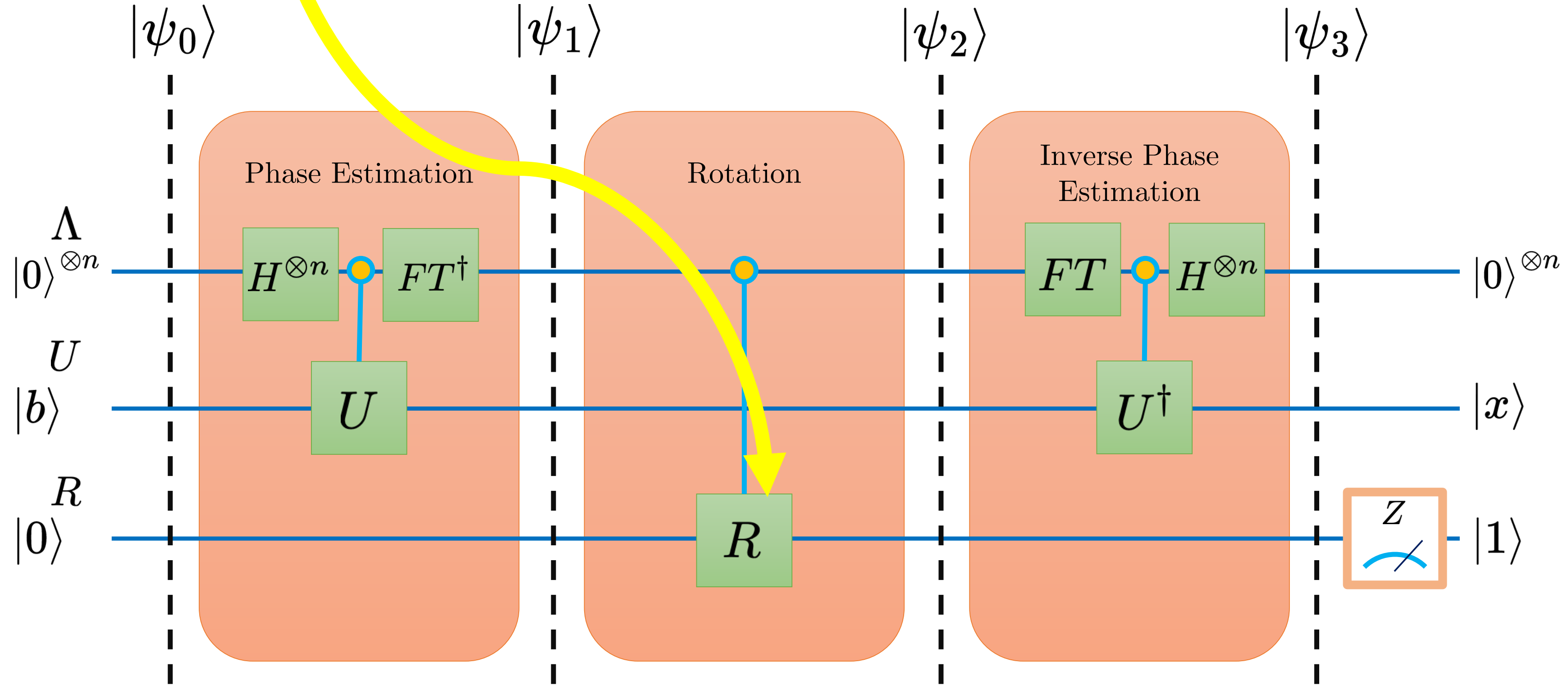
Imperfections

$$R(\lambda^{-1}) = R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$\frac{\tilde{\theta}_i}{2} = \sin^{-1}\left(\frac{C}{\lambda_i}\right) + \epsilon_i$$

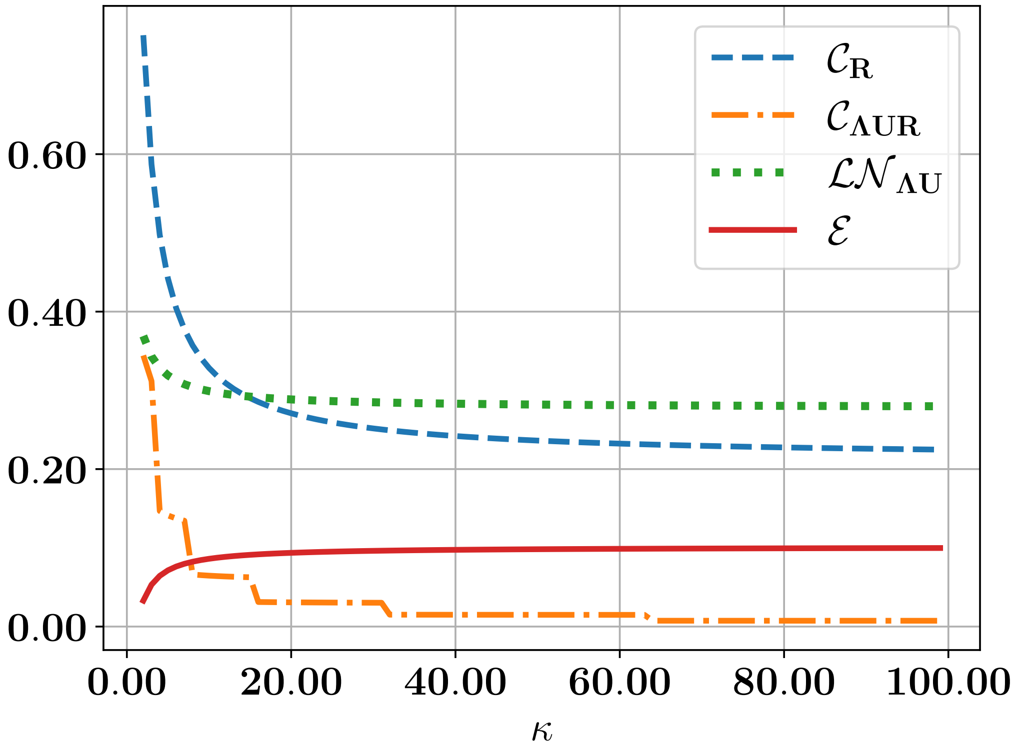
The algorithm

$$R(\lambda^{-1}) = R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$



Kappa – κ
The condition number

$$\kappa = \frac{\lambda_{max}}{\lambda_{min}}$$



$$\mathcal{O}(\kappa^2 \log(N))$$

Thank you