Review of Substitution (5.5)

- Thought of as the "reverse" of differentiation
- $\int f'(g(x))\,g'(x)\,dx = f(g(x)) + c.$

$$\frac{d(f(g(x)))}{dx} = f'(g(x))\,g'(x).$$

- How to integrate $\int f'(g(x))\,g'(x)\,dx$?
  - Change Variable (u)
  - ① look for function whose derivative also appear (Differs at most by constant).
  - ② "Inside" function of composite function
    - Ex → Expression under "$\sqrt{\quad}$"
      → Exponent of exponential function
  - ③ ~~Try something~~ Try substituting "u" and if it doesnt work try again.

Examples

- $\int \frac{(\ln(x))^2}{x}\,dx$

$= \int \frac{u^2}{u'}\,dx$

$= \frac{u^3}{3}\,dx$

$= \frac{(\ln x)^3}{3}$

$\int \sqrt{3x+2}\,dx$

$= ($

$\int \frac{x}{\sqrt{1+3x}}\,dx$

- Substitute u for $1+3x$
- Solve $u = 1+3x$ for x
- Sub both in.
- Solve integral.

- $\int \tan x\,dx = \ln|\sec x| + c.$

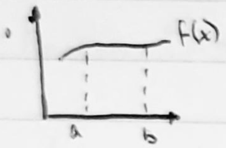
~~$= \int \frac{\sin x}{\cos x}\,dx.$~~

~~$= \int \sin x \sec x\,dx.$~~

# Applications of Integrals (6.2, 6.3).

- Recall: Definite Integrals ( $\int_a^b f(x)\,dx$ )
  - Fundamental Theorem of Calculus
- If $f(x) \geq 0$, $\int_a^b f(x)\,dx$ = area under curve from $\cancel{f(a)}$ $x=a$ to $\cancel{f(b)}$ $x=b$.
- if $f(x)$ takes both positive and negative values
  $\int_a^b f(x)\,dx$ = "net" area = area below curve − area above.

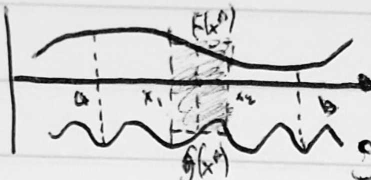
## How did we get here?




- Divide $[a,b]$ into sub intervals of equal width $\Delta x = \frac{b-a}{n}$
- Rectangles can be drawn to approximate the area of the curve. (Sum of the Rectangles)
  - Width = $\Delta x = \frac{b-a}{n}$
  - Height = $f(x)$ ← Value of function at a single point of subsection.
- Approximate area under curve by summing area of approximating rectangles
- $\Sigma (f(x))(\Delta x)$ ← This is a Riemann Sum
- Exact Area = $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$
  $= \int_a^b f(x)\,dx$.

## Area Between 2 Curves (6.2)

- Consider region that lies between $y = f(x)$ and $y = g(x)$ and $x = a$ and $x = b$
- Easiest Case: $f(x) \geq g(x)$ on $[a,b]$.
- Approach: - Divide $[a,b]$ in to $n$ subintervals of equal width
  - Approximate Rectangles and add the areas.
  - This approximates the Area between curves.




$f(x) \} (x, f(x))$

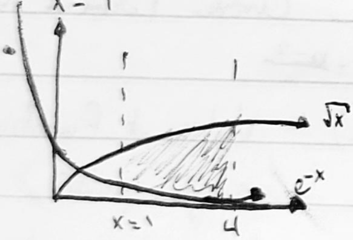
$g(x) \} (x, g(x))$

- Width $= \frac{b-a}{n}$
- Height $= f(x^*) - g(x^*)$

- Approximation to area $= \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \Delta x$
- Exact Area $= \lim_{n \to \infty} \sum_{i=1}^{n} (f(x^*) - g(x^*)) \Delta x$
  $= \int_a^b (f(x) - g(x))\,dx$ ] Provided both $f$ and $g$ are continuous and $f(x) \geq g(x)$ on $[a,b]$.

Example

- Find area between $y = \sqrt{x}$, $y = e^{-x}$ and lines $x = 1$ and $x = 4$



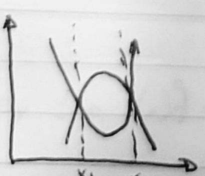

$$\text{Area} = \int_1^4 (\sqrt{x} - e^{-x})\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} + e^{-x} \Big|_1^4$$

$$= \left(\frac{2}{3}(8) + e^{-4}\right) - \left(\frac{2}{3} + e^{-1}\right)$$

$$= \frac{14}{3} + e^{-4} - e^{-1}$$

Want to find area enclosed by 2 curves?




- Need to find x-coordinates of intersection points
- Do this by setting $f(x) = g(x)$ and solve for $x$
- $\int_{x_1}^{x_2} (f(x) - g(x))\,dx$.

Example

- Find area enclosed by $y = 4 - x^2$ and $y = 3x + 4$

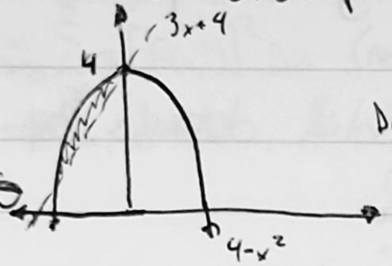
$4 - x^2 = 3x + 4$     intersection points: $x = 0, -3$

$x^2 + 3x = 0$

$x(x+3) = 0$

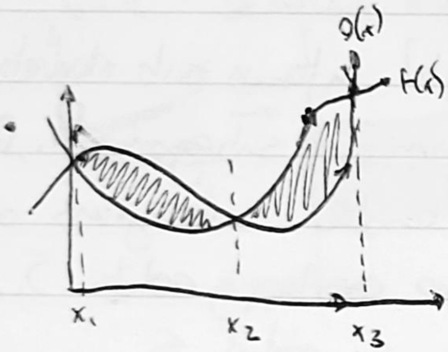
~~$x^2 - 3x = 0$~~

~~$x(x-3)$~~




$$\text{Area} = \int_{-3}^{0} \left[ (4 - x^2) - (3x + 4) \right] dx$$

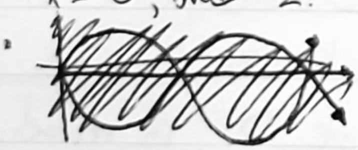
- Another Situation: Suppose $f(x) \geq g(x)$ for some values of $x$ and $g(x) < f(x)$ for other values of $x$

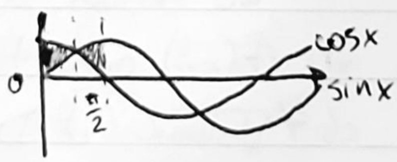



$$\text{Area Enclosed} = \int_{x_1}^{x_2} [f(x) - g(x)]\,dx + \int_{x_2}^{x_3} [g(x) - f(x)]\,dx$$

Example

- Find area enclosed by $y = \sin x$ and $y = \cos x$ between $x = 0$, and $\frac{\pi}{2}$.





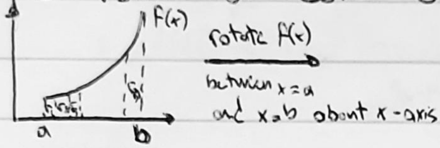

$\cos x = \sin x \rightarrow x = \frac{\pi}{4}$.

- $\text{Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$

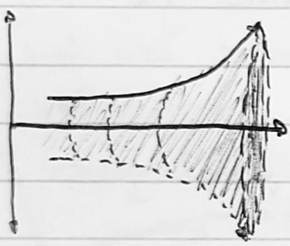
$= 2\sqrt{2} - 2$

# Volumes (6.3).

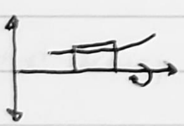
- Solids of Revolution
  - → Obtained by revolving a region about an axis (Axis of Revolution)
  - → Find volume of solid obtained by



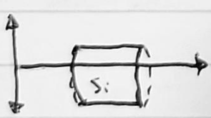

How do we find volume?



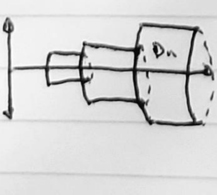
- Divide $[a,b]$ into $n$ subintervals
- Divide regions into subregions $R_1, R_2, \ldots R_n$
- When we revolve subregions about axis we obtain no overlapping solids $S_1, S_2 \ldots S_n$ which make up solid $S$.
- Zoom in on $R_i$ ...
- Choose sample point $x_i$ and approximate area of $R_i$ (width $\Delta x = \frac{b-a}{n}$, height $= f(x_i^*)$)



- If rectangle is rotated about x-axis we obtain a circular disc with radius $f(x_i^*)$ and thickness $\Delta x$.



- Volume of Disc, $D_i$, is $\pi(\text{radius})^2 \cdot \text{thickness}$
  - → i.e. $\pi(f(x_i^*))^2 \cdot \Delta x$ ★
- Volume of Disc approximates volume of solid, $S_i$



- With $n$ approximated Discs we approximate Solid ($S_n$) with the addition of the approximated Discs.

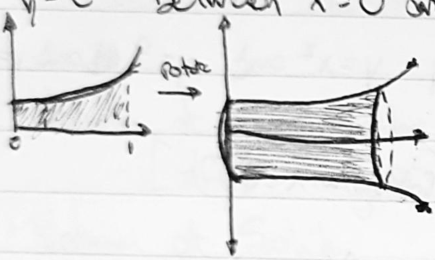
- Volume of Disc $= \pi(f(x_i^*))^2 \Delta x$
- Approx Volume of Solid $= \sum_{i=1}^{n} \pi [f(x_i^*)]^2 \Delta x$
- Exact Volume of Solid $= \lim_{n\to\infty} \sum_{i=1}^{n} \pi [f(x_i^*)]^2 \Delta x = \int_a^b \pi f(x)^2 \, dx$

- Volume of Solid of Revolution (Disc Method)
  - f is continuous on [a,b] and R is a region under f on [a,b]. If a region is rotated about the x-axis, the volume of resulting solid is
  - $V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi [f(x_i^*)]^2 \Delta x$

    $= \int_a^b \pi [f(x_i^*)]^2 dx$ , if $f(x) \geq 0$.

Example(s)

- find volume of solid obtained by rotating region under $y = e^x$ between $x=0$ and $x=1$




Radius of Disc $= f(x_i^*) = e^{x_i^*}$

Volume of Disc $= \pi (e^{x_i^*})^2 \Delta x$

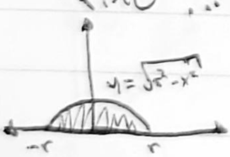
$V = \int_0^1 \pi (e^x)^2 dx.$

$V = \pi \int_0^1 e^{2x} dx.$

$V = \pi \left[\frac{e^{2x}}{2}\right]_0^1$

$V = \frac{\pi e^2}{2} - \frac{\pi}{2}.$

- find ... region under $\sqrt{r^2-x^2}$ from $x = -r$ to $r$




$V = \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx$

$= \pi \int_{-r}^{r} \left(\sqrt{r^2-x^2}\right)^2 dx.$

$= \pi \int_{-r}^{r} (r^2-x^2)\, dx$ by symmetry.

$= 2\pi \int_0^r (r^2-x^2)\, dx$

$= 2\pi \left[r^2 x - \frac{x^3}{3}\right]_0^r$

$= 2\pi \left[r^3 - \frac{r^3}{3}\right]$

$= \frac{4}{3}\pi r^3$

Suppose this ...

- Suppose we want to find volume of rotating region about x-axis

- 

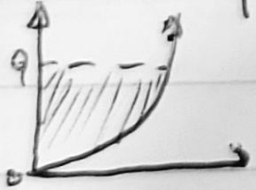
region bounded by $x=g(y)$ at $y=c$, $y=d$

approx. rectangles perpendicular to y with thickness $\Delta y$.

$$V=\int_c^d \pi\left(g(y_i^*)\right)^2 dy.$$

Example.

- Rotate region bounded by $y=x^2$ and $y=9$ and $y=0$ about y-axis

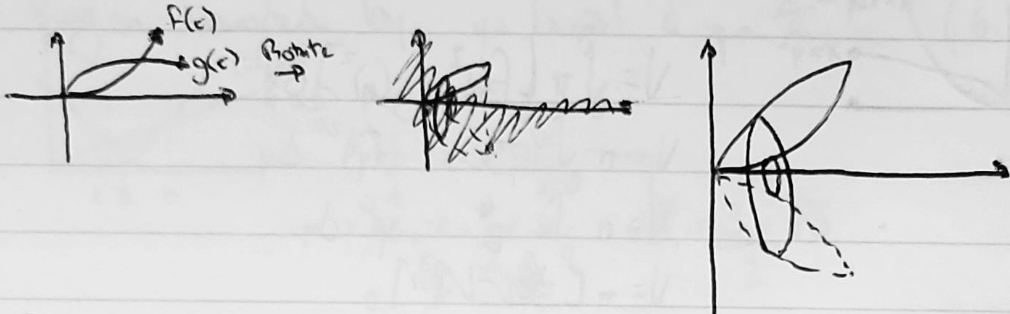
- 

rotate →

radius $= x = \sqrt{y}$.

$$\begin{aligned} V &= \int_0^9 \pi(\sqrt{y})^2\,dy. \\ &= \pi\int_0^9 y\,dy \\ &= \pi\left[\frac{y^2}{2}\right]_0^9 \\ &= \frac{\pi 81}{2} \end{aligned}$$

Region enclosed by 2 curves rotated about an axis.

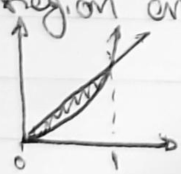



- Solids generated has ~~hole~~ hole in center rotating rectangle gives us washer shape ◎
- Outter radius $= f(x)$
- Inner radius $= g(x)$
- Surface Area of washer = Area of Outter Circle - Area of Inner Circle
  $= \pi(f(x))^2 - \pi(g(x))^2$
- Volume of Approx. Washer is
  - $[\pi(f(x))^2 - \pi(g(x))^2]\Delta x$
- Volume of Solid
  - $\int_a^b \pi[f(x)^2 - g(x)^2]\,dx.$

} Volume by washer method $f \geq g \geq 0$

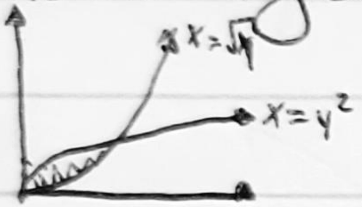
Examples.

- Region enclosed by $y = x$ an $y = x^2$ about x-axis.




$$V = \int_0^1 \pi[f(x)^2 - g(x)^2]\,dx.$$
$$V = \pi\int_0^1 f(x)^2 - g(x)^2\,dx.$$
$$V = \pi\int_0^1 (x)^2 - (x^2)^2\,dx.$$
$$V = \pi\int_0^1 x^2 - x^4\,dx.$$
$$V = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$
$$V = \frac{2\pi}{15}$$

- Rotate region enclosed by $x = y^2$ and $x = \sqrt{y}$ about y-axis




$$V = \int_0^1 \pi \left[ f(y)^2 - g(y)^2 \right] dy$$

$$V = \pi \int_0^1 (\sqrt{y})^2 - (y^2)^2 \, dy$$

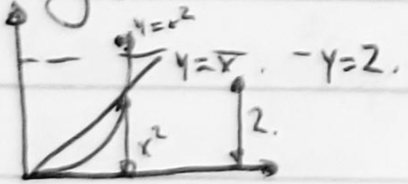
$$V = \pi \int_0^1 y - y^4 \, dy.$$

$$V = \pi \left[ \cancel{\quad} \right]_0^1$$

$$V = \pi \left[ \frac{y^2}{2} - \frac{y^5}{5} \right]_0^1$$

$$V = \frac{3\pi}{10}.$$

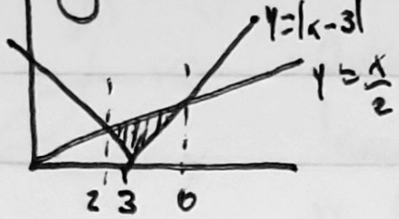
- Region Enclosed by $y = x$ and $y = x^2$ about $y = 2$.




Outter radius $= 2 - x^2$

Inner radius $= 2$

• 1- Region bounded by $y=|x-3|$ & $y=\frac{x}{2}$ (6.2 #21)




$$y = y$$

$$|x-3| = \frac{x}{2}$$

$x \geq 3$, $x - \frac{x}{2} = 3$ $\qquad$ $x<3$, $-\frac{3x}{2} = -3$

$x = 6.$ $\qquad$ $x = 2.$

$$A = \int_2^3 \frac{x}{2} - (3-x)\,dx + \int_3^6 \frac{x}{2} - (x-3)\,dx$$

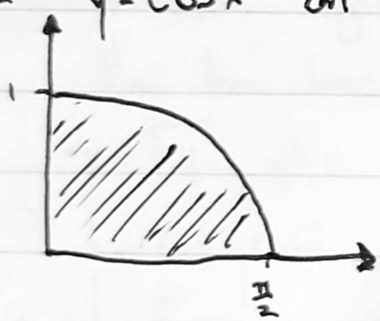
$$= \left(\frac{x^2}{4} - 3x + \frac{x^2}{2}\right)\Big|_2^3 + \left(\frac{x^2}{4} - \left(\frac{x^2}{2} - 3x\right)\right)\Big|_3^6$$

$$= \left[\left(\frac{3^2}{4} - 3(3) + \frac{3^2}{2} - \frac{2^2}{4} + 3(2) - \frac{2^2}{2}\right] + \left[\frac{6^2}{4} - \frac{6^2}{2} + 3(6) - \frac{3^2}{4} + \frac{3^2}{2} - 3(3)\right]$$

$$= \left[\frac{9}{4} - 9 + \frac{9}{2} - 1 + 6 - 2\right] + \left[9 - 18 + 18 - \frac{9}{4} + \frac{9}{2} + 9\right)$$

$$= 3$$

• 2- $y = \cos x$ on $[0, \frac{\pi}{2}]$, $y=0$, $x=0$ (6.3 #20).




$$V = \int_0^{\frac{\pi}{2}} \pi \cos^2 x\,dx.$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\,dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2x\,dx$$

$$= \frac{\pi}{2}\left(x + \frac{\sin 2x}{2}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \cancel{\frac{\sin\pi}{2}}\right) - \left(\cancel{0} + \cancel{\frac{\sin 2(0)}{2}}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{4}$$

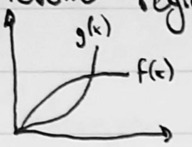
$\cos 2x$ $\qquad$ $u = 2x$

$\int \cos u \frac{du}{2}$ $\qquad$ $du = 2\,dx$

$\frac{\sin u}{2}$ $\qquad$ $\frac{du}{2} = dx$

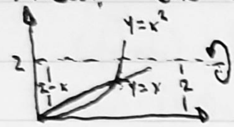
# Review: Volumes of Revolution

- Revolve region bounded by 2 curves about an x-axis




  - Outer Radius ⇒ curve furthest from axis of revolution ($f(x)$)
  - Inner Radius ⇒ curve closest to the axis of revolution. ($g(x)$)
  - $V = \int_a^b \left[\pi (f(x))^2 - \pi (g(x))^2\right] dx$
  - a & b found x. points of intersection

- About another axis



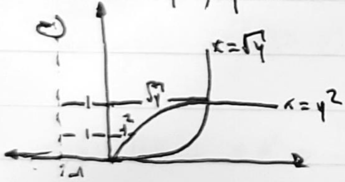

  - Inner radius: 2 - x
  - Outer radius: $2 - x^2$
  - $V = \int_0^1 \pi (2 - x^2)^2 - \pi (2-x)^2 \, dx$

$\vdots$

$V = \frac{8\pi}{15}$

- Ex. Revolve region enclosed by $y = x^2$ and $y^2 = x$ about the line $x = -1$

$x = \sqrt{y}$, $y^2 = x$




  - Outer Radius: $1 + \sqrt{y}$
  - Inner Radius: $1 + y^2$

$V = \int_0^1 \pi (1 + \sqrt{y})^2 - \pi (1 + y^2)^2 \, dy$

$V = \frac{29\pi}{30}$

- When revolving about the x-axis ⇒ integrate with respect to x
  ⇒ Points of intersection described by x coordinates are the upper and low limits of integration
- When revolving about the y-axis ⇒ integrate with respect to y
  ⇒ Points of intersection described by y coordinates are the upper and lower limits of integration.
  ⇒ Fxns in the form must be $x = f(y)$

# Techniques of Integration (7.2-7.5)

## Integration by Parts (7.2)

- Technique of integration associated with product rule of differentiation
- Recall: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $\int (f(x)g(x))' dx = \int [f'(x)g(x) + f(x)g'(x)] dx$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) - \int f'(x)g(x)\,dx = \underbrace{\int f(x)g'(x)\,dx}_{\text{integration by parts.}}$$

- Any integrand that can be interpreted to be a product of a function and a derivative of another.

Example

- $\int \underset{f(x)}{x}\,\underset{g'(x)}{e^x}\,dx \quad f(x) = x \quad f'(x) = 1$

$\quad g'(x) = e^x \quad g(x) = e^x$

$$\int xe^x\,dx = xe^x - \int (1)e^x\,dx$$
$$= xe^x - e^x + C$$
$$= e^x(x-1) + C$$

*When the integral becomes harder to solve than the initial one, correct your $g'(x)$ & $f(x)$ etc.

Alternative Notation

- $\int u\,dv = uv - \int v\,du$, where $u = f(x)$, $v = g(x)$, $dv = g'(x)\,dx$, $du = f'(x)\,dx$
- How to choose $u$ and $dv$?
  - Choose $u$ to be a function that becomes simpler (or not more complicated) when it becomes differentiated
  - $dv \Rightarrow$ easy to integrate.

Example

*3 helpful integrals:

$\int e^{ax} dx = \frac{e^{ax}}{a} + c$

$\int \sin(ax)dx = \frac{-\cos(ax)}{a} + c$

$\int \cos(ax)dx = \frac{\sin(ax)}{a} + c.$

- $\int x \cos 2x \, dx$ $\quad u = x \quad dv = \cos 2x \, dx$

$du = dx \quad v = \frac{\sin 2x}{2}$

$$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x \, dx}{2}$$

$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

- $\int x \ln x \, dx$ $\quad$ ~~$u = 2x$~~ ~~$dv = \ln x \, dx$~~ $\quad u = \ln x \quad dv = x$

~~$du = 2dx$~~ ~~$v =$~~ $\quad du = \frac{1}{x} dx \quad v = \frac{x^2}{2}.$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2x} dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx.$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c.$$

IBP Hint:

$\underbrace{\text{-Single trig fxn} \\ \text{- single inverse trig fxn}}_{\text{use IBP}}$

- $\int \ln x \, dx$ $\quad u = \ln x \quad dv = 1$

$du = \frac{1}{x} dx \quad v = x$

$$\int \ln x \, dx = x \ln x - \int dx$$

$$= x \ln x - x + c.$$

## Checking Integration on Maple

- f:= ____ ;
- F1 := int(f,x); ← if it looks different then
- F2:= your answer;
- simplify(F1 - F2); → if it equals 0, correct. if it is a constant, correct.

Example

- $\int x^3 \sin x \, dx$ $\quad u = x^3 \quad dv = \sin dx$

$du = 3x^2 dx \quad v = -\cos x,$

$\int x^3 \sin dx = -x^3 \cos x - \int 3x^2 \cos dx \quad u = 3x^2 \quad dv = \cos dx$

$du = 6x \quad V = \sin x$

$= x^3 \cos x + [3x^2 \sin x - \int 6x \sin x \, dx] \quad u = 6x \quad dv = \sin x, \; du = 6dx \quad v = -\cos x$

$= -x^3 \cos x + [3x^2 \sin x - [-6x \cos x + 6\int \cos x \, dx]$

$= -x^3 \cos x + [3x^2 \sin x - [-6x \cos x + 6 \sin x]$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

IBP followed by substitution is common.

- $\int \arctan x \, dx$ $\quad u = \arctan x \quad dv = dx$

$du = \frac{1}{1+x^2} dx. \quad v = x$

$\int \arctan x \, dx = (\arctan x)(x) - \int \frac{x}{1+x^2} dx \quad u = 1 + x^2$

$= x \arctan x - \frac{1}{2}\int \frac{1}{u} du \quad du = 2x \, dx.$

$= x \arctan x - \frac{1}{2} \ln u + c.$

$= x \arctan x - \frac{1}{2} \ln(1+x^2) + c$

- $\int e^x \sin x \, dx$ $\quad u = e^x \quad dv = \sin x dx$

$du = e^x dx \quad v = -\cos x$

$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \quad u = e^x \quad dv = \cos x \, dx$

$du = e^x dx \quad v = \sin x$

One like this on assignment # 1.

$\int e^x \sin x \, dx = -e^x \cos x + [e^x \sin x - \int e^x \sin x \, dx]$

$2\int e^x \sin x dx = -e^x \cos x + e^x \sin x.$

$\int e^x \sin dx = -\frac{1}{2} e^x \cos x + e^x \sin x$

$= \frac{1}{2} e^x [\sin x - \cos x.]$

Definite Integrals

- $\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du$

# Trigonometric Integrals (7.3)

- Integrate functions that are powers/products of sine + cosine or tan + secant.
- Ex $\int \sin^3 x \cos^4 x \, dx$
  - How?

Ex → $\int \cos^2 x \, dx$

$= \frac{1}{2} \int (1 + \cos 2x) dx$

$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

$\int \cos^3 x \, dx$

$= \int \cos x \cos^2 x \, dx$

$= \int \cos (1 - \sin^2 x) dx$

$= \int 1 - u^2 \, du$

$= u - \frac{u^3}{3}$

$= \sin x - \frac{\sin^3 x}{3}.$

Trig Properties/Identities

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \sec x \tan x$

★ $\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

★ $\tan^2 x + 1 = \sec^2 x$

★ $\sin 2x = 2 \sin x \cos x$

$\cos 2x = 1 - 2\sin^2 x$

$= \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

★ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

★ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

- $\int \sin^3 x \cos^4 x \, dx.$

$= \int \sin \sin^2 x \cos^4 x \, dx$

$= \int \sin (1 - \cos^2 x) \cos^4 x \, dx.$

$= -\int (1 - u^2) u^4 \, du$

$= \int u^6 - u^4 \, du$

$= \frac{u^7}{7} - \frac{u^5}{5}$

$= \frac{\cos x^7}{7} - \frac{\cos x^5}{5}$

$u = \cos x$

$du = -\sin x \, dx$

- Odd Power
  - Pull out single factor of sin(cos)
  - Rewrite using $\sin^2 x + \cos^2 x = 1$
  - Use Substitution.

Example

- $\int_0^\pi \cos^4 x\,dx$

$= \int_0^\pi (\cos^2 x)^2\,dx$

$= \frac{1}{2}\int_0^\pi (1+\cos 2x)^2\,dx$

$= \frac{1}{2}\int_0^\pi (1+2\cos 2x + \cos^2 2x)\,dx.$

$= \frac{1}{2}\int_0^\pi (1+2\cos 2x + [\frac{1}{2}(1+\cos 4x)]\,dx$

$= \frac{1}{4}\int_0^\pi [\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x]\,dx$

$= \frac{1}{4}(\frac{3x}{2} + \sin 2x + \frac{1}{8}\sin 4x)\Big|_0^\pi$

$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]\Big|_0^\pi$

$= \cancel{\frac{3x}{8}}$ ...

- $\int \sin^4 x\cos^4 x\,dx.$

$= \int (\sin^2 x)^2(\cos^2 x)^2\,dx$

$= \int (\frac{1}{2}(1-\cos 2x))^2(\frac{1}{2}(1+\cos 2x))^2\,dx$

⋮

Very long process fuck that.

$\int \sin^4 x\cos^4 x\,dv$

$= \int (\sin x\cos x)^4\,dx$ $\quad \sin x\cos x = \frac{\sin 2x}{2}$

$= \int (\frac{1}{2}\sin 2x)^4\,dx.$

$= \int (\frac{1}{4}\sin^2 2x)\,dx.$

$= \int (\frac{1}{4}(1-\cos 4x))^2\,dx$

$= \frac{1}{64}\int (1-2\cos 2x - \cos^2 4x)\,dx$

$= \frac{1}{64}\int [1-2\cos 2x - \frac{1}{2}(1\cos 8x)]\,dx$

Summary

- $\int \sin^m x\cos^n x\,dx$

① If m is odd, seperate single factor of $\sin x$, rewrite remainder in terms of $\cos x$ ($\sin^2 x = 1-\cos^2 x$)

② If n is odd, seperate single factor of $\cos x$, rewrit remainder in terms of $\sin x$ ($\cos^2 x = 1-\sin x$)

③ If both are even, use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

" " " " and equal, $\sin 2x = 2\sin x\cos x$

# Integrals of Form $\int \tan^m x \sec^n x \, dx$

- Want to seperate $\sec x \tan x \Rightarrow \frac{d(\sec x)}{dx} = \sec x \tan x$

  $\sec^2 x \Rightarrow \frac{d(\tan x)}{dx} = \sec^2 x$
- If $n$ is even, seperate $\sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to write the remainder in terms of $\tan x$ ($u$ sub $\to \tan x$)
- If $m$ is odd, seperate $\sec x \tan x$ term and use $\tan^2 x = \sec^2 x - 1$ to write remainder in terms of $\sec x$ ($u$ sub $\to \sec x$)

Example

- $\int \tan^2 x \sec^4 x \, dx$

$= \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx$ let $u = \tan x$

$du = \sec^2 x \, dx$

$= \int \cancel{\tan^2 x}\, u^2 (u^2 + 1) \, du$

$= \int u^4 + u^2 \, du$

$= \frac{u^5}{5} + \frac{u^3}{3} \cancel{du}$

$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$

- $\int_0^{\frac{\pi}{3}} \tan^3 x \sec^3 x \, dx$ $u = \sec x$

$= \int_0^{\frac{\pi}{3}} \sec x \tan x \tan^2 x \sec^2 x \, dx$ $du = \sec x \tan x \, dx$

$= \int_0^{\frac{\pi}{3}} (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx$

$= \int_0^{\frac{\pi}{3}} (u^2 - 1)(u^2) \, du$

$= \int_0^{\frac{\pi}{3}} u^4 - u^2 \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} \Big|_0^{\frac{\pi}{3}}$

$= \frac{\sec x^5}{5} - \frac{\sec x^3}{3} \Big|_0^{\frac{\pi}{3}}$

# 7.4 Trigonometric Substitution.

- How do we integrate function like $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$ ?
- The technique of "inverse" substitution might help.
  - take the 'old' variable and express in terms of new variables
  - Instead writing, let u = an expression in x, write x = an expression in $\theta$.
  - i.e. $\int f(x)\,dx = \int f(g(\theta))\, g'(\theta)\, d\theta$.
- If $\sqrt{a^2-x^2}$, let $x = a\sin\theta$  $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- If $\sqrt{a^2+x^2}$, let $x = a\tan\theta$  " " "
- If $\sqrt{x^2-a^2}$, let $x = a\sec\theta$  $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

Example

- $\int \frac{dx}{\sqrt{9-x^2}}$   $x = 3\sin\theta$

$= \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,d\theta$   $dx = 3\cos\theta\,d\theta$

$= \int \frac{3\cos\theta}{3\sqrt{1-\sin^2\theta}}\,d\theta$

$= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta$

$= \int d\theta$

$= \theta + C$.

$x = 3\sin\theta$

$\frac{\sin^{-1}x}{3} = \theta$

- $\int \frac{dx}{\sqrt{x^2+16}}$   $x = 4\tan\theta$

$= \int \frac{4\sec^2\theta\,d\theta}{\sqrt{16\tan^2\theta+16}}$   $dx = 4\sec^2\theta\,d\theta$

$= \int \frac{4\sec^2\theta\,d\theta}{4\sqrt{\tan^2\theta+1}}$

$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta$

$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$.

- $\int \frac{\sqrt{x^2-1}}{x}\,dx$   let $x = \sec\theta$

$= \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$   $dx = \sec\theta\tan\theta\,d\theta$

$= \int \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$

$= \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$.

$= \sqrt{x^2-1} - \operatorname{arcsec}x + C$

$f'g(x)$

- $\int x\sqrt{x^2+4}\,dx$   $u = x^2+4$  $du = 2x$

$= \frac{1}{2}\int x\sqrt{x^2+4}\,dx$.

$= \frac{(x^2+4)^{\frac{3}{2}}}{2}$

• $\int_0^3 \frac{x^3}{\sqrt{9+x^2}}\,dx \qquad \sqrt{9+x^2} = \tan\theta$

$= \int_0^{\frac{\pi}{4}} \frac{27\tan^3\theta}{\sqrt{9+9\tan^2\theta}}\, 3\sec^3\theta\, d\theta \qquad x = \sqrt{\tan^2\theta - 9}$

$= \int_0^{\frac{\pi}{4}} \frac{27\tan^3\theta}{\sqrt{9+9\tan^2\theta}}\; 3\sec^3\theta\, d\theta.$

$= 27\int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\, d\theta.$

$= 27\int_0^{\frac{\pi}{4}} \tan^2\theta\, \tan\theta\sec\theta\, d\theta$

$= 27\int_0^{\frac{\pi}{4}} (1-\sec^2\theta)\tan\theta\sec\theta\, d\theta \qquad u = \sec\theta$

$= 27\int_0^{\frac{\pi}{4}} (u^2-1)\,du. \qquad du = \sec\theta\tan\theta\, d\theta.$

$= 27\int_1^{\sqrt{2}} (u^2-1)\,du$

$= 27\left[\frac{1}{3}u^3 - u\right]_1^{\sqrt{2}}$

$= 18 - 9\sqrt{2}.$

• $\int \sqrt{3-x^2}\,dx. \qquad x = \sqrt{3}\sin\theta$

$= \int \sqrt{3-(\sqrt{3}\sin\theta)^2}\,dx \qquad dx = \sqrt{3}\cos\theta$

$= \int \sqrt{3 - 9\sin^2\theta}\,dx.$

$\vdots\ \frac{3}{2}\arcsin\frac{x}{\sqrt{3}} + \frac{3}{4}\left(\frac{x}{\sqrt{3}}\,\frac{\sqrt{3-x^2}}{\sqrt{3}}\right) + C$

$= \frac{3}{2}\arcsin\frac{x}{\sqrt{3}} + \frac{1}{2}x\sqrt{3-x^2} + C.$

# 7.5 Partial Fractions

- How do we integrate $\frac{P(x)}{Q(x)}$? (Where $P(x)$ and $Q(x)$ are polynomials and substitution doesn't apply.)
- Ex $\int \frac{x-9}{x^2+2x-3}\,dx$. → Express as sum of simpler terms (partial fractions)
  → Partial Fraction Decomposition (Do when $P(x) < Q(x)$)

- Case 1: - Degree of $P(x)$ < Degree of $Q(x)$, and $Q(x)$ can be written as distinct linear factors.
  - $$\frac{P(x)}{(a_1x+b_1)\ldots(a_kx+b_k)} = \underbrace{\frac{A}{a_1x+b_1} + \ldots \frac{k}{a_kx+b_k}}_{\text{Partial Fraction Decomp.}}$$

Example.

- $\int \frac{4x^2-4x+6}{x^3-x^2-6x}\,dx$

$$= \int \frac{4x^2-4x+6}{x(x^2-x-6)}\,dx$$

$$= \int \frac{4x^2-4x+6}{x(x-3)(x+2)}\,dx$$

$$= \int \frac{-1}{x} + \frac{2}{x-3} + \frac{3}{x+2}\,dx$$

$$= -\ln|x| + 2\ln|x-3| + 3\ln|x+2| + c.$$

$$\frac{4x^2-4x+6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$ Multiply both sides

$$4x^2-4x+6 = A(x-3)(x+2) + Bx(x+2) + C(x)(x-3).$$

$$4x^2-4x+6 = (A+B+C)x^2 + (-A+2B-3C)x - 6A.$$ ← match coefficients.

$x^2 \to 4 = A+B+C$ $\quad A=-1$

$x \to -4 = -A+2B-3C$ $\quad B=2.$

$+6 \to 6 = -6A.$ $\quad C=3$

- Grubby Thumb Rule. [By GTR].

$$\frac{4x^2-4x+6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}.$$

$A = \frac{6}{-6}$ $\quad B = \frac{4(3^2)-4(3)+6}{3(5)}$ $\quad C = \frac{4(-2^2)-4(-2)+6}{-2(-5)}$

$A=1$ $\quad B=2$ $\quad C=3$

By GTR

- $\int \frac{x^2-4x+1}{x(x^2-2x-3)}\,dx$. $\quad \frac{x^2-4x+1}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x+1)}$

$\int \frac{x^2-4x+1}{x(x-3)(x+1)}\,dx \qquad A=-\frac{1}{3} \quad B=\frac{-2}{3} \quad C=\frac{3}{2}$

$= \int -\frac{1}{3x} - \frac{1}{6(x-3)} + \frac{3}{2(x+1)}\,dx$

$= -\frac{1}{3}\ln|x| - \frac{1}{6}\ln|x-3| + \frac{3}{2}\ln|x+1| + C$

Case 2:

- What if degree of $P(x) \geq Q(x)$?
- Long Division ~~Rlsts~~ first.
- Ex. $\int \frac{4x^3+x}{2x^2+x-3}\,dx \rightarrow$

$$\begin{array}{r} 2x - 1 \\ 2x^2+x-3 \overline{)\, 4x^3 + x} \\ -(4x^3+2x^2-6x) \\ \hline -2x^2+7x \\ -(-2x^2-x+3) \\ \hline 8x-3 \end{array}$$

$\int \left(2x-1 + \frac{8x-3}{2x^2+x-3}\right)dx \qquad \frac{A}{} + \frac{B}{} + \frac{C}{}$

$= \int \left(2x-1 + \frac{6}{2x+3} + \frac{1}{(x-1)}\right)dx$.

$= x^2 - x + 3\ln|2x+3| + \ln|x-1| + C$.

Summary

- ① Simplify integrand and look for trig expressions
- ② Look for an obvious substitution
- ③ Classify integral.
  - A) Trig Integral ⇒ (product of sinx cosx / secx tanx)
  - B) Rational ⇒ Check degree of numerator and denominator.
  - C.) Integration by parts ⇒ product of powers, trig, exp, and log's.
  - D.) Trig Substitution ⇒ $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$,
- ④ Try again
  - ⇒ look for less obvious substitution
  - ⇒ Single trig or log fxn ⇒ IBP
  - ⇒ May need multiple techniques
    - ⇒ common = subs ⇒ IBP
      IBP ⇒ subs
      IBP ⇒ Partial fractions
      Trig Subs ⇒ Trig integral.
  - ⇒ Use Trig idents to simplify.

Questions

- IDENTIFY TECHNIQUE
  1. $\int \tan^3 x \sec^4 x \, dx$
  2. $\int \ln(x^2)\, dx$
  3. $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx$
  4. $\int \frac{3x^2-2}{x^2-5x+6}$
  5. $\int \frac{du}{u\sqrt{3-u^2}}$
  6. $\int \sin(\ln x)\, dx$
  7. $\int \frac{\cos x + \sin 2x}{\sin x} dx$
  8. $\int x^3 \sqrt{x^2+1}\, dx$

Until this point...

- You have learned all standard techniques of integration
- Does every continuous function have an antiderivative? No
  - Ex $\int e^{x^2}\,dx$ , $\int \sqrt{1+x^4}\,dx$.

Approximate Integration

- This is used for functions we can antiderive or experimental data.
- Recall: Riemann sum - Sum areas of approx. rectangles. ← Bad
- Trapezoid Rule - Approximate region by trapezoids (then sum areas of that) ← Better
- Simpson's Rule - Approximate pieces of a curve by parabolas and sum the areas under parabolas ← Best

Simpson's Rule

- Divide $[a,b]$ into $n$ (even #) subintervals of equal width $\Delta x = \frac{b-a}{n}$
- Approximate curve on consecutive pairs of subintervals and draw parabola through three consecutive points $P_i, P_{i+1}, P_{i+2}$ where $P_i\,(x_i, f(x_i))$
- Start with $P_0(x_0, f(x_0))$ , $P_1(x_1, f(x_1))$ , $P_2(x_2, f(x_2))$
- It can be shown that the area under the parabola joining $P_0, P_1, P_2$ $\dfrac{\Delta x}{3}[f(x_0) + 4f(x_1) + f(x_2)]$
- Next pair $[P_2, P_3]$ and $[P_3, P_4]$
- Area can be shown as $\dfrac{\Delta x}{3}[f(x_2) + 4f(x_3) + f(x_4)]$
- Area under all $n$ parabolas...

$$-\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{\Delta x}{3}[f(x_2) + 4f(x_3) + f(x_4)] \ldots$$

Indicates Simpson's Rule with $n$ subintervals

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \ldots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

- So $\int_a^b f(x)\,dx \approx S_n$ ← The more subintervals used, the better the approximation.

## Illustration

- Use Simpsons Rule with $n=6$ to approximate $\int_1^3 \frac{1}{x^2}\,dx$
  - Exact Value of $\int_1^3 \frac{1}{x^2}dx = \frac{2}{3}$

$\int_1^3 \frac{1}{x^2}\,dx$

$\cancel{\ln x^2 \Big]_1^3}$

$\cancel{2\ln x \Big|_1^3}$

$\cancel{2\ln 3 - 2\ln 1}$

$\cancel{2\ln}$

- Simpsons Rule
  - $\Delta x = \frac{3-1}{6} = \frac{1}{3}$
  - $x_0 = 1,\ x_1 = 4/3,\ x_2 = \frac{5}{3}, \ldots X_6 = 3$
  - $S_6 = \frac{\frac{1}{3}}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots f(x_6)\right]$

$S_6 = \frac{1}{9}\left[1 + 4\left(\frac{9}{16}\right) + 2\left(\frac{9}{25}\right) + 4\left(\frac{1}{4}\right) + 2\left(\frac{9}{49}\right) + 4\left(\frac{9}{64}\right) + \frac{1}{9}\right]$

$S_6 \approx 0.6679$ *(use 4 decimals).

## Error Bound for Simpsons Rule

- Suppose $|f^{4}(x)| \le K$ on $[a,b]$
- If $E_s$ is error incurred when Simpsons rule approximately is used then,

$$|E_s| \le \frac{K(b-a)^5}{180\,n^4}$$

↑
Maximum Error that can occur

Example

- a.) Use Simpson's Rule with $n=4$ to approximate $\int_0^1 e^{x^2}\,dx$.

b) Estimate the error in part a.)

c) How large would $n$ need to be to guarantee an error of no more than $0.0001$?

a.) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

$S_4 = \frac{\frac{1}{4}}{3}\left[f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(\tfrac{3}{4}) + f(1)\right]$

$= \frac{1}{12}\left[1 + 4e^{\frac{1}{16}} + 2e^{\frac{1}{4}} + 4e^{\frac{9}{16}} + e\right]$

$\approx 1.4637$

b.) $f^4(x)$ $\quad f(x) = e^{x^2}$ $\quad f'(x) = 2xe^{x^2}$ $\quad f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ $\quad f'''(x) = 2xe^{x^2}(6+4x^2)$ $\quad f^4(x) = (16x^4+48x^2+12)e^{x^2}$

Need $K$ so that $f^4(x) \le K$ on $[0,1]$

Since $f^4(x) > 0$ and increasing it takes its largest value at 1.

$K = e^{1^2}(16(1^4) + 48(1^2) + 12)$

$K = 76e$

$|E_s| \le \dfrac{76e(1-0)^5}{180(4)^4}$

$\le 0.0045$

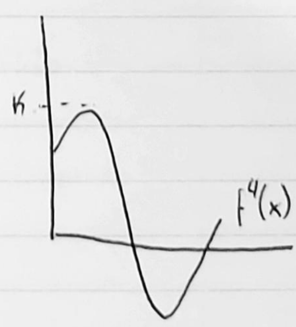
c.) $0.0001 \le \dfrac{K(b-a)^5}{180n^4}$

$n \le \sqrt[4]{\dfrac{K(b-a)^5}{180(0.0001)}}$

$n \ge \sqrt[4]{\dfrac{76e}{180(0.0001)}}$

$n \ge 10.35$

$n$ must be 12 (greatest even number above 10.38).

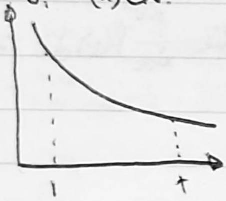


# Improper Integrals (7.8)

- So far, we have integrated continuous functions on finite intervals ($\int_a^b f(x)\,dx$ ~~when~~ $= F(b) - F(a)$ [Fundamental Theorem])
- Now we want to consider
  1. Infinite Intervals ($[a, \infty]$, $[-\infty, b]$, or $[-\infty, \infty]$
  2. Function has infinite discontinuity at some point between $x = a$ and $x = b$.
  - These are improper integrals.

Example

- Consider $f(x) = \frac{1}{x}$ for $x \geq 1$
  - Want to find area under $f(x)$ to the right of $x = 1$
  - $\int_1^\infty f(x)\,dx$.



Pick a finite point much greater than 1

$$\int_1^t \frac{1}{x}\,dx = \ln x \Big|_1^t$$
$$= \ln t - \ln 1$$
$$= \ln t - 0$$
$$= \ln t. \quad \text{As } t \to \infty,\ \ln t \to \infty$$

Infinite region has infinite area.

- Could write

$$\int_1^t \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t \to \infty} \ln t$$
$$= \infty$$

Does this always happen? → NO

- Consider $f(x) = \frac{1}{x^2}dx$ for $x \geq 1$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t \to \infty} \left[\frac{-1}{x}\right]_1^t$$
$$= \lim_{t \to \infty} \left[\frac{-1}{t} + 1\right]$$
$$= 1$$

Infinite region has a finite area in this case.

# Improper Integrals (Type I)

- If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided limit exists.
- Similarly $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then, $\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$ provided limit exists.
- If the limit exists, then we say the integral is convergent otherwise, the integral is divergent.

## ~~Type II~~ Multiple Infinity's Case

- $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$
- If both integrals are convergent then $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent
- If one of the integrals is divergent the $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent.

## Important Result

- $\int_1^\infty \frac{1}{x}\,dx \Rightarrow$ divergent
- $\int_1^\infty \frac{1}{x^2}\,dx \rightarrow$ converges
- $\int_1^\infty \frac{1}{x^p}\,dx$ ; for what values of p does the integral converge
- $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \leq 1. \end{cases}$

Examples

- Determine if the follow integrals are convergent or divergent.

a.) $\int_{-\infty}^{0} e^{2x}\,dx$

$= \lim_{t\to-\infty} \int_{t}^{0} e^{2x}\,dx$

$= \lim_{t\to-\infty} \left[\frac{e^{2x}}{2}\right]_t^0$

$= \frac{1}{2} - \frac{e^{2t}}{2}$ Approaches 0

$= \frac{1}{2}$

$\therefore$ Convergent.

b.) $\int_{1}^{\infty} \frac{\ln x}{x}\,dx$

$= \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x}\,dx.$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \left[\frac{1}{2}\ln x^2\right]_1^t$

$= \lim_{t\to\infty} \left[\frac{\ln t^2}{2} - 0\right]$

$\therefore$ divergent.

c.) $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx.$

$= \int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{x}{(x^2+1)^2}\,dx.$

$= \lim_{t\to-\infty} \int_{t}^{0} \frac{x}{(x^2+1)^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{x}{(x^2+1)^2}\,dx$

$= \lim_{t\to-\infty} \left[\frac{-1}{2(x^2+1)}\right]_t^0 + \lim_{t\to\infty} \left[\frac{-1}{2(x^2+1)}\right]_0^t$

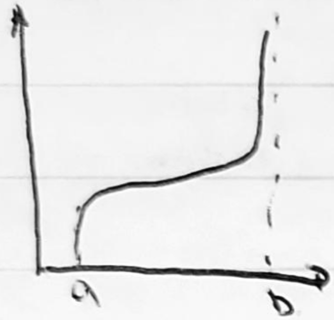
$= \lim_{t\to-\infty} \left[-\frac{1}{2} + \frac{1}{2(t^2+1)}\right]_t^0 + \lim_{t\to\infty} \left[\frac{-1}{2(t^2+1)} + \frac{1}{2}\right]_0^t$

$= -\frac{1}{2} \quad 0 \quad + \frac{1}{2}$

$= 0$ Convergent.

Case II / Type II

- Suppose f is positive and continuous on $[a,b)$ but has a VA at $x=b$




Area of region $= \int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{t \to b} \int_a^t f(x)\,dx.$$

- f is continuous on $[a,b)$, discontinuous at $x=b$, and $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$
- If the limit exists then it is convergent, otherwise divergent.
- Similarly f is continuous $(a,b]$, discontinuous at $x=a$ and $\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$.
- f is discontinuous at $x=c$ and $a<c<b$.
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
- If both are convergent, then $\int_a^b f(x)\,dx$ is convergent
- If one is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Examples

- Determine if the integral is convergent or divergent.

a) $\int_{-3}^{2} \frac{1}{\sqrt{x+3}}\,dx$.

$= \lim_{t \to -3^+} \int_{t}^{2} \frac{1}{\sqrt{x+3}}\,dx$.

$= \lim_{t \to -3^+} \left[ \frac{-2}{3} \frac{1}{(x+3)^{\frac{3}{2}}} \right]_{t}^{2}$

$= \lim_{t \to -3^+} \left[ 2\sqrt{5} - \cancel{\frac{-2}{3(t+3)^{\frac{3}{2}}}} \right]$

$= 2\sqrt{5}$.

Convergent.

b) $\int_{0}^{2} \frac{1}{x^2+x-6}\,dx$.

$= \lim_{t \to 2^-} \int_{0}^{t} \frac{1}{(x+3)(x-2)}\,dx$.

~~$= \lim_{t \to 2^-} \left[ \frac{1}{5}\ln|x-2| - \frac{1}{3}\ln|x+3| \right]_0^t \, dx$~~

$= \lim_{t \to 2^-} \int_{0}^{t} \left( \frac{\frac{1}{5}}{x-2} - \frac{\frac{1}{5}}{x+5} \right) dx$.

$= \lim_{t \to 2^-} \left[ \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| \right]_0^t$

$= \lim_{t \to 2^-} \left[ \left[ \frac{1}{5}\ln|t-2| - \frac{1}{5}\ln|t+3| \right] - \left[ \frac{1}{5}\ln|0-2| - \frac{1}{5}\ln|0+3| \right] \right]$

$= -\infty$ divergent.

# Arc Length (6.5)

- Want to find length of a curve.
- Distance travelled by object when path isnt straight (Application)
- Use what we know
  - length of straight line segements
  - Approx curve by straight line segements and use lengths to approximate length of curve
  - Let number of line segments increase indefinitely
- Length of line segement joining $P_1$ & $P_2$
  - $|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
  - $\sum_{i=1}^{n} \sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}$
- Length of Curve $= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}$
- By the mean value theorem

  Length of Curve $= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+f(x_i^*)^2}\,\Delta x$

  $= \int_a^b \sqrt{1+f'(x)^2}\,dx$

## Arc Length Formula.

- Suppose $f(x)$ is continuous on $[a,b]$ then length of curve, $y=f(x)$, from $x=a$ to $x=b$

  $\rightarrow L = \int_a^b \sqrt{1+f'(x)^2}\,dx$

Example

Often difficult to find length.

- Find the length of $y = 2x^{\frac{3}{2}}+4$ from $x=0$ to $x=2$.
- $\frac{dy}{dx} = 3x^{\frac{1}{2}} = 3\sqrt{x}$ $\quad = 1+\left(\frac{dy}{dx}\right)^2$

  $= 9x$
- $L = \int_0^2 \sqrt{9x+1}\,dx$

  $L = \left[\frac{1}{9}\frac{2}{3}u^{\frac{3}{2}}\right]_1^{19}$

  $L = 6.067$

Differential Equations (8.1, 8.3).

- Definition - Consists of a function (unknown function) and at least one of its derivatives.
  - Eg. $\frac{dy}{dx} + 2y = e^{3x}$
  - Want to find a function, $y = f(x)$ that satisfies the differential equation.
  - Used to model real world problems
    - population growth, motion of object, chemical reactions
- We will be working only with first-order differential equations
- Solution - f is a solution if f and its derivative(s) satisfy the diffential equation.
  - General Solutions - All possible ~~functions~~ solutions
  - Particular Solution - A solution that satifies a given initial condition
  - Initial Value Problem - Differential equation + initial condition

Example

- Show that $y = Ce^{-2x} + e^{x}$ is a solution of $\frac{dy}{dx} + 2y = 3e^{x}$

- $\frac{dy}{dx} = -2Ce^{-2x} + e^{x}$

- $\frac{dy}{dx} + 2y = 3e^{x}$

$-2Ce^{-2x} + e^{x} + 2(Ce^{-2x} + e^{x}) = 3e^{x}$

$\cancel{-2Ce^{-2x}} + e^{x} + \cancel{2Ce^{-2x}} + 2e^{x} = 3e^{x}$

$3e^{x} = 3e^{x}$ $\therefore$, $y = Ce^{-2x} + e^{x}$ is a solution.

Initial Value Problem

- Find solution of $\frac{dy}{dx} + 2y = 3e^{x}$ that satisfies $y(0) = -2$
- General Solution
  - $y = Ce^{-2x} + e^{x}$

$-2 = Ce^{-2(0)} + e^{0}$

$-2 = C + 1$

$-3 = C.$ $\therefore$, the solution is $y = -3e^{-2x} + e^{x}$

Some Models.

- Population growth – ① Simplest Model – Under ideal conditions
  - Population increases at a rate proportional to population size.
  - $\frac{dP}{dt} = kP.$, $(P(t) = Ce^{kt})$
  - ② Logistic Model. – Limited resources etc.
    - Grows to carry capacity $(M)$
    - $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$
  - ③ $P(t) = \frac{M}{1 + Ae^{-kt}}$, $A = \frac{M - P_0}{P_0}$

# Newtons Law of Cooling

- Object cools or warms at a rate that is proportional to the difference in temperature of the object and temperature of the surroundings.
- $y(t)$ = temperature at time, $t$.
- $T$ = temperature of surroundings (constant).
- $\frac{dy}{dt} = K(y-T)$

Note: Solving differential equations is not trivial (need approximation techniques).

## Two Types of Differential Equations.

- ① Separable Differential Equations
- ② Linear Differential Equations (Not in text)
- Swag.

## Separable Differential Equations (8.3).

- Can be written as a production of function of $x$ and of function $y$.
- ex. $\frac{dy}{dx} = f(x)g(y)$

$$\frac{dy}{g(y)} = f(x)\,dx$$ ← Integrate both sides.

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$ ← Then solve for $y$ if possible.

Example

- $\frac{dP}{dt} = kP$

$$\frac{dP}{P} = k\,dt.$$

$$\int \frac{dP}{P} = \int k\,dt.$$

$$= kt + C$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C}$$

$$P(t) = Ae^{kt}, \text{ where } A = \pm e^{C}$$

- Solve $\frac{dy}{dx} = 2xy^2$

$$\frac{dy}{y^2} = 2x\,dx$$

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + C$$

$$-\frac{1}{x^2+C} = y.$$

$y' = \frac{xy - y}{y+1}$, $y(1) = 1$

$$\frac{dy}{dx} = (x-1)\left(\frac{y}{y+1}\right)$$

$$\int \frac{y+1}{y}\,dy = \int x-1\,dx$$

$$y + \ln|y| = \frac{x^2}{2} - x + C$$

$$1 + \ln|1| = \frac{1}{2} - x + C$$

$$-\frac{3}{2} = C$$

Solve the differential equation.

- $\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{ye^y}$

$$\int ye^y = \int x\sqrt{x^2+1}\,dx.$$

$$ye^y - e^y = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

Logistic Population Growth

- $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

$$\int \frac{dP}{P(1-\frac{P}{M})} = \int k\,dt.$$

$$\int \frac{M}{P(M-P)}\,dP = kt.$$

$\vdots$

$P(t) = \frac{M}{1 + Ae^{-kt}}$, $A = \frac{M - P_0}{P_0}$

- As $t \to \infty$, $P(t) \to M$.

# Newtons Law of Cooling

- $\frac{dy}{dt} = k(y-T)$

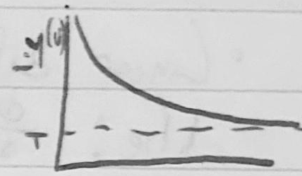
$\int \frac{dy}{y-T} = \int k\,dt$

$\ln|y-T| = kt + C$

$|y-T| = e^{kt+C}$

$y = Ae^{kt+C} + T.$ where $A = \pm e^{C}$

$y = Ae^{kt} + T$

Cooling - 

Heating - 

- A cup of coffee is 95°C when poured into a 22°C room.
  Find the temp when it cools at a rate of $1\frac{°C}{min}$ when temp = 70°C
- a) Find, $y(t)$, at time, $t$.
  b) find temp 15 mins after
  c) How long til room temp?

a) $y(t) = Ae^{kt} + 22.$

$y(0) = 95°$

$\frac{dy}{dt} = -1$ when $y = 70°$

$y(0) = Ae^{kt} + 22$

$95 = A + 22$

$73 = A.$

$\frac{dy}{dt} = k(\cancel{70}-22)$ $(y-22).$

$\frac{-1}{48} = k.$

$y(t) = 73e^{-\frac{1}{48}t} + 22.$

b) $y(15) = 73e^{\frac{-15}{48}} + 22 \approx 75.41$

c) $22 = 73e^{\frac{-15}{48}} + 22$

$0 = 73e^{\frac{-15}{48}}$ ← No solution. * Not perfect model *

# Linear Differential Equations.

- Linear differential equations in standard form look like: $\frac{dy}{dx} + P(x)y = Q(x)$
- Example: $y' + \frac{1}{x}y = x$ $\quad P(x) = \frac{1}{x}$, $Q(x) = x$

$$xy' + y = x^2 \quad \leftarrow \text{Multiply both sides by } x.$$

$$\frac{d}{dx}(xy) = x^2 \quad \leftarrow xy' + x'y = \frac{d(xy)}{dx}$$

$$\int \frac{d(xy)}{dx}dx = \int x^2\,dx$$

$$xy = \frac{1}{3}x^3 + C$$

$$y = \frac{1}{3}x^2 + \frac{C}{x}$$

- Every linear differential equation can be solved by multiplying by integrating factor, $I(x)$
- How to find $I(x)$?
  - $I(x) = e^{\int P(x)dx}$
  - This guarentee's $L.H.S = \frac{d(I(x)y)}{dx}$
  - Must be in standard form.

## To Solve Linear D.E.

1. Write in Standard Form
2. Find $I(x) = e^{\int P(x)dx}$
3. Multiply both sides by $I(x)$
4. Write LHS as $\frac{d}{dx}(I(x)y)$
5. Integrate both sides
6. Solve for $y$

Examples

- (1) $\frac{dy}{dx} + 2xy = x$
- (2) $2xy' + y = 6x \quad , y(4) = 20$
- (3) $xy' = y + x^2 \sin x$
- (4) $(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0$

• (1) $\frac{dy}{dx} + 2xy = x$

$I(x) = e^{\int 2x\,dx}$

$= e^{x^2}$

$e^{x^2}\left(\frac{dy}{dx} + 2xy\right) = xe^{x^2}$

$\int \frac{d(e^{x^2}y)}{dx}dx = \int xe^{x^2}\,dx$

$e^{x^2}y = \frac{e^{x^2}}{2} + C$

$y = \frac{1}{2} + \frac{C}{e^{x^2}}$

(2) $2xy' + y = 6x \quad , y(4) = 20.$

$\frac{dy}{dx} + \frac{y}{2x} = 3$

$I(x) = e^{\int \frac{1}{2x}dx}$

$= \sqrt{x}$

$\sqrt{x}\left(\frac{dy}{dx} + \frac{y}{2x}\right) = 3\sqrt{x}$

$\int \frac{d(\sqrt{x}\,y)\,dx}{dx} = \int 3\sqrt{x}\,dx$

$\sqrt{x}\,y = 2x^{\frac{3}{2}} + C$

$y = 2x + \frac{C}{\sqrt{x}}$

$20 = 2(4) + \frac{C}{\sqrt{4}}$

$12 = \frac{C}{2}$

$24 = C.$

(3) $xy' = y + x^2 \sin x.$

$\frac{dy}{dx} - \frac{y}{x} = x\sin x$

$I(x) = e^{\int -\frac{1}{x}dx}.$

$= \frac{1}{x}$

$\frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right) = \int \sin x$

$\frac{d(\frac{1}{x}y)}{dx} =$

$= -\cos x + C$

$y = -\cos x \, x + xC$

$y = -x\cos x + xC.$

(4)

# 9 Infinite Sequences and Series.

## Introduction

- Sequence - an ordered list of numbers ($a_1, a_2, a_3 \dots a_n$)
- Series - sum of numbers ($a_1 + a_2 + a_3 + \dots a_n$)
- In both, we want to find long term behaviour
- Does the sequence have a limit? Can we find a sum for infinite series?
- Why?
  - Limited ability to evaluate transcendental functions ($\ln x, e^x, \sin x, \dots$).
  - Unable to evaluate some definite integrals $\Rightarrow$ can't find antiderivative.
    - Ex. Writing $\int e^{-x^2} dx$ in simpler terms.

## 9.2 Sequences

- Infinite Sequences - Ordered list of numbers with a property, that for every term, $a_n$, there is subsequent term $a_{n+1}$
  - $\underset{\text{1st term}}{a_1}, a_2, a_3 \dots \underset{\text{General term.}}{a_n}, a_{n+1}, \dots$
- Sequence is a function, $f(n) = a_n$, with a domain equal to the set of positive integers.
- Denoting Sequence $\{a_1, a_2, a_3, \dots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$

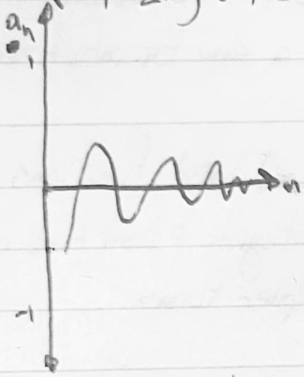
Examples

- $\{\frac{1}{n^2}\} = \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \dots\}$.
- $\{\frac{(-1)^n}{n+1}\} = \{-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5} \dots\}$.
- $\{\cos(n\pi)\} = \{-1, 1, -1, 1 \dots\}$.

# Graphing Sequences.

- Sequences can be plotted by using order pairs
- $(n, a_n)$
- Ex. $\left\{\frac{(-1)^n}{n+1}\right\} = \left\{-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5} \dots\right\}$.

$\{(1, -\frac{1}{2}); (2, \frac{1}{3}); (3, -\frac{1}{4}); (4, \frac{1}{5})$.




- Long Term behaviour
  - Does this sequence have a limit?
  - In the example they seem to approach 0.
- $\lim_{n\to\infty} a_n = L$
  - limit exists $\Rightarrow$ convergent
  - limit does not exist $\Rightarrow$ divergent.
- Note: Only difference between finding $\lim_{n\to\infty} f(x)$ & $\lim_{n\to\infty} a_n$ is that the domain of $f(n) = a_n$ is set of positive integers.

## Strategies

- ① Limit Laws
  - Sum/Difference Law $\Rightarrow \lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
  - Constant Multiple Law
  - Product Law $\Rightarrow \lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
  - Quotient Law
  - Power Law.

② If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$

③ Squeeze Theorem - If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

④ If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

⑤ If $\lim_{n\to\infty} a_n = L$ an $f$ is continuous at $L$, then

$\lim_{n\to\infty}(f(a_n)) = f(\lim_{n\to\infty} a_n) = f(L)$

# Examples

- $\left\{ \dfrac{5n^2-3n+2}{4n^2+1} \right\}$ $\quad \lim\limits_{n\to\infty} \dfrac{5n^2-3n+2}{4n^2+1} = \lim\limits_{n\to\infty} \dfrac{5 - \frac{3}{n} + \cancel{\frac{2}{n^2}}}{4 + \cancel{\frac{1}{n^2}}}$ $\quad \therefore$ convergent.

$$= \frac{5}{4}$$

2 Cases of Limit not existing
- terms do not approach a single value
- terms approach $\pm\infty$

- $\{\cos(n\pi)\} = \{-1, 1, -1, 1\}$ $\quad \lim\limits_{n\to\infty} \cos(n\pi)$ does not exist
- $\left\{\dfrac{(-1)^n}{n+1}\right\}$ $\quad \lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^n}{n+1}\right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0.$ $\quad \therefore$, converges.

- $a_n = \dfrac{n+1}{e^n}$ $\quad \lim\limits_{n\to\infty} \dfrac{n+1}{e^n}$ ← L'Hopital's Rule.

$$\lim_{n\to\infty} \frac{n+1}{e^n} = \lim_{n\to\infty} \frac{1}{e^n}$$

$$= 0. \qquad \therefore \text{converges.}$$

- $\left\{\dfrac{\cos n}{2n+1}\right\}$ Squeeze Theorem

$$-1 \le \cos n \le 1$$

$$\frac{-1}{2n+1} \le \frac{\cos n}{2n+1} \le \frac{1}{2n+1}$$

$$\lim_{n\to\infty} \frac{-1}{2n+1} = 0 = \lim_{n\to\infty} \frac{1}{2n+1}$$

$$\lim_{n\to\infty} \left(\frac{\cos n}{2n+1}\right) = 0.$$

- $a_n = e^{\frac{1}{n^2}}$ $\quad \lim\limits_{n\to\infty} e^{\frac{1}{n^2}} = e^{\lim\limits_{n\to\infty} \frac{1}{n^2}}$

$$= e^0$$

$$= 1.$$

- $\lim\limits_{n\to\infty} \left(\sqrt{n} - \sqrt{n^2-1}\right) \cdot \dfrac{\sqrt{n} + \sqrt{n^2-1}}{\sqrt{n}\ \sqrt{n^2-1}}$

$= \lim\limits_{n\to\infty}$ $\qquad$ Divergent.

- Tutorial 3.

Quiz. ① - linear D.E.

② - Sequences.

# Infinite Series (9.3).

- Obtained by adding terms of infinite sequences
- Is it possible to find a sum of an infinite number of terms?
  - Negative
  - But $\frac{1}{9} = 0.111111\ldots$

$$= 0.1 + 0.01 + 0.001, + 0.0001 \ldots$$

$$\sum_{n=1}^{\infty} (0.1)^n$$

## Define

- Partial Sum
  - $S_1 = a_1$ , $S_2 = a_1 + a_2$ , $S_3 = a_1 + a_2 + a_3$
  - $S_n = a_1 + a_2 + a_3 \ldots a_n = \sum_{n=1}^{\infty} a_i$
  - $n^{th}$ Partial sum
- As $n \to \infty$ - if the limit exists, then sum of the infinite series can be found.
- If $\lim_{n\to\infty} S_n = S$, the $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = S$.
- If $\lim_{n\to\infty} S_n$ does not exist, the $\sum_{n=1}^{\infty}$ is divergent.

Example: Find is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

- ① Find an expression for the $n^{th}$ partial sum, $S_n$
  ② $\lim_{n\to\infty} S_n$
- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$

$$S_1 = \frac{1}{2} , S_2 = \frac{2}{3} , S_3 = \frac{3}{4} , S_4 = \frac{4}{5} , S_n = \frac{n}{n+1}$$

- $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$$= 1$$

- $\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Geometric Series.

- Series of the form $\Rightarrow a + ar + ar^2 + ar^3 \ldots$

  $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}, \; a \neq 0.$

- $a$ = first term, $r$ = common ratio

- for what values of $r$ does the geometric series converge?
  - If $r = 1$, we'd get $na = S_n$ $\Rightarrow$ Diverge.
  - If $r \neq 1$, $S_n = \frac{a(1-r^n)}{1-r}$

- $\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges if } |r| < 1 \\ \text{diverges if } |r| \geq 1 \end{cases}$

Example

- find the sum of series ...

  $4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} \ldots$

  $4\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} \ldots\right)$

  $4\left(\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 \ldots\right)$

  Since $-1 < r < 1$, it converges

  $\sum_{n=1}^{\infty} 4\left(\frac{2}{3}\right)^{n-1}$

- Converges to $\frac{a}{1-r}$

  $= \frac{4}{1-\frac{2}{3}}$

  $= 12$

- Determine if $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent or divergent.

  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$

  $S_1 = 1, \; S_2 = \frac{3}{2}, \; S_3 = \frac{11}{6}, \; S_4 = \frac{25}{12}, \; S_n = ?$

  $\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots$

  Divergent $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

# Harmonic Series

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series.
- What do we notice?
  - $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
  - In general, if $\lim_{n\to\infty} a_n = 0$ does not imply that $\Sigma$ converges.
- Theorem
  - If $\Sigma a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ but converse is not necessarily true.

# Test for Divergence (9.4)

- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
  - Example $\sum_{n=1}^{\infty} \frac{3n+2}{3n-4}$ $\lim_{n\to\infty} \frac{3+\frac{2}{n}}{3-\frac{4}{n}}$

    $\frac{3}{3} \neq 0$ converges.

What if $S_n$ is difficult to find.

- Used 3 different approaches for all 3 examples so far
- $S_n$ is often not easy to find
- Use test for convergence for infinite series (7 in total).

## Integral Test (9.4)

- Suppose $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$ and $f(n) = a_n$. Then, $\sum a_n$ is convergent if $\int_1^{\infty} f(x)\,dx$ is convergent
- If $\int_1^{\infty} f(x)\,dx$ is convergent, so is $\sum a_n$

  If $\int_1^{\infty} f(x)\,dx$ is divergent, so is $\sum a_n$.
- Note: Value of integral is almost never equal to the sum of the series

Example

- Determine if $\frac{1}{n^2}$ is con. or div.
- $f(x) = \frac{1}{x^2}$
- $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx$

  $= \left[\frac{-1}{x}\right]_1^{t}$

- Convergent.

Important

- $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{Converges if } p>1 \\ \text{Diverges if } p\le 1 \end{cases}$

- P-Series $\frac{1}{x^p}$

Example

- $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ con. or div.
- $f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{1 - \ln x}{x^2}$ when ever $1 - \ln x < 0$

$\ln x > 1$

$x = e$.

$\therefore$, decreasing for $x \geq 3$.

- $\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\,dx$

$= \left[\frac{1}{2}(\ln x)^2\right]_3^\infty$

$= \infty$ Divergent by integral test.

# Comparison Test (9.5)

- Basic Idea: -make use of series with known convergence or divergence
  - Compare given series to one with known convergence and divergence.
  - ↳ geometric, p-series.

Example

- $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is similar to $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ which is a geometric series
- $\sum a_n$ and $\sum b_n$ are series with positive terms
  - i.) If $a_n \leq b_n$ for some $n \geq N$ and $\sum b_n$ is convergent, so is $\sum a_n$.
  - ii.) If $a_n \geq b_n$ for some $n \geq N$ and $\sum b_n$ is divergent, so is $\sum a_n$.
- Need to show $\frac{1}{2^n+1} < \frac{1}{2^n}$

  $2^n+1 > 2^n$ so $\frac{1}{2^n+1} < \frac{1}{2^n}$
- So $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent by the comparison test.

Examples

a.) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n}$ a.) $\sum \frac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n}$ converges by the comparison test

b.) $\sum_{n=1}^{\infty} \frac{3^n+1}{2^n-1}$

c.) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

- Note: To find comparator series, look at highest powers of $n$, in numerator & denominator ("dominate" series)

Example

- $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ similar to $\sum \frac{1}{2^n}$

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

what if terms are greater than terms of convergent series
or what if ~~the~~ terms are less than terms of divergent "
or ~~what if~~ strict > or < relationship is difficult to establish

Limit Comparison Test

- $\sum a_n$ and $\sum b_n$ are series with positive terms.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, finite, then both $\sum a_n$ and $\sum b_n$ converge or both diverge.
- Ex - compare $\sum \frac{1}{2^n-1}$ and $\sum \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}}$$

$$= 1 > 0.$$

Limit converges by the limit comparsion test.

- Keep highest powers in numerator and demominator to find comparator series.

Examples

a.) $\sum \frac{n^2}{n^3+1}$

b.) $\sum \frac{\sqrt{n^2+1}}{n^3+2n^2+5}$

compare $\sum \frac{n^2}{n^3}$ + $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}}$$

$$= 1 > 0.$$ ← Divergent by limit Comparison test.

# Ratio Test

- Given $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$
  - If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges
  - If $L > 1$, then $\sum a_n$ diverges
  - If $L = 1$, the test is inconclusive.

# Example

- Test for convergence $\sum_{n=1}^{\infty} \frac{n}{3^n}$
- $\sum \frac{n}{3^n} \rightarrow$

$$\rightarrow \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}}$$
$$= \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n}$$
$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{3}\left(1 + \cancel{\frac{1}{n}}\right)$$
$$= \frac{1}{3}.$$

Convergent by the ratio test.

- $\sum_{n=1}^{\infty} \frac{(n+1)4^{2n+1}}{10^n}$

$$= \lim_{n\to\infty} \frac{\frac{(n+2)4^{2(n+1)+1}}{10^{n+1}}}{\frac{(n+1)4^{2n+1}}{10^n}}$$
$$= \lim_{n\to\infty} \frac{(n+2)4^{2n+3}}{10^{n+1}} \cdot \frac{10^n}{(n+1)4^{2n+1}}$$
$$= \lim_{n\to\infty} \frac{16}{10}\left(\frac{n+2}{n+1}\right)$$
$$= \lim_{n\to\infty} \frac{8}{5}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)$$
$$= \frac{8}{5}$$

Divergent by ratio test

- $\sum \frac{n^2 2^n}{n!}$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^2 2^{n+1}}{(n+1)!}}{\frac{n^2 2^n}{n!}}$$
$$= \lim_{n\to\infty} \frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n^2 2^n}{n!}$$
$$= \lim_{n\to\infty} 2\left(\frac{n+1}{n^2}\right)$$
$$= \lim_{n\to\infty} 2\left(\cancel{\frac{1}{n}} + \cancel{\frac{1}{n^2}}\right)$$
$$= 0$$

Converges by ratio test.

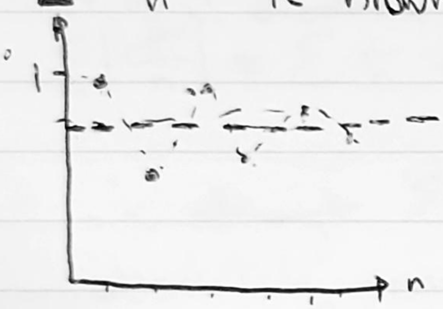
- Strategy is useful for products / Quotients of powers / factorials.

# Alternating Series (9.6).

- Series whose terms have signs that alternate. $(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\ldots)$.
- Commonly written as $(-1)^{n-1}b_n$ or $(-1)^n b_n$ where $b_n > 0$
- If terms of alternating series decrease in absolute value towards 0.

## Illustration

- $\sum \frac{(-1)^{n-1}}{n}$ is known as the alternating harmonic series.



Odd sums decrease $S_{1,3,5,7}$
Even Sum increase $S_{2,4,6,8}$.
Both approach a common value (converges).

- $b_n = \frac{1}{n}$ is decreasing as a series and $\lim_{n\to\infty} b_n = 0$

## Alternating Series Test

- If alternating Series $\sum (-1)^{n-1} b_n$ $(\sum (-1)^n b_n)$ satisfies
  a.) $b_{n+1} \leq b_n$
  b.) $\lim_{n\to\infty} b_n = 0$
- then the alternating series is convergent.
- i) show $b_{n+1} \leq b_n$ or $\frac{b_{n+1}}{b_n} \leq 1$.
  ii) Look at $F(x)$, $f'(x) < 0$ for what values of $x$ (i.e. for some $x > N$).

Example

- $\sum (-1)^n \frac{\sqrt{n}}{n+4}$ $b_n = \frac{\sqrt{n}}{n+4}$

i.) $f(x) = \frac{\sqrt{x}}{x+4}$

$f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$ $x>4$ $\{b_n\}$ decreasing for $n \geq 5$.

ii.) $\lim_{n\to\infty} \frac{\sqrt{n}}{n+4}$

$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1+\frac{4}{n}}$

$= 0.$ Convergent by Alt series test.

- $\sum \frac{(-1)^{n-1} n^2}{2n^2-1}$ $b_n = \frac{n^2}{2n^2-1}$

i.) $f(x) = \frac{n^2}{2n^2-1}$

$f'(x)$

ii.) $\lim_{n\to\infty} \frac{n^2}{2n^2-1} = \frac{1}{2} \neq 0$ Divergent by test for divergence.

- $\sum \frac{\cos(n\pi)}{n^{3/4}}$ $b_n = \frac{1}{n^{3/4}}$

$\sum \frac{(-1)^n}{n^{3/4}}$

Absolute Convergence

- Now we want to look at general series, $\sum a_n$, and the terms may be positive, negative, alternating, positive + negative but not alternating
- look at $\sum |a_n| = |a_1| + |a_2| + |a_3| + |a_4| \ldots |a_n| \ldots$ whose terms are absolute value of original terms
- Definition: If series of absolute value of terms is convergent, then we say $\sum a_n$ is absolutely convergent (stronger type of convergence)
- If $\sum a_n$ is a series of positive terms, then absolute convergence and convergence means the same thing

Example

- Test for absolute convergence

$$\sum \left|\frac{(-1)^n}{3^n}\right| = \sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$$

→ Convergent Geometric Series $r = \frac{1}{3} < 1$

$\therefore$, $\sum \left|\frac{(-1)^n}{3^n}\right|$ converges.

- Alternating Harmonic Series
  - ↳ Is it absolutely convergent?

$$\sum \left|\frac{(-1)^{n-1}}{n}\right| \to \sum \frac{1}{n} \to \text{Harmonic series diverges.}$$

$\therefore$, ~~although~~ $\sum \left|\frac{(-1)^{n-1}}{n}\right|$ ~~is convergent it~~ is not absolutely convergent.

↳ This series is conditionally convergent series that is convergent but not absolutely convergent

- Consider

$$\sum \frac{\cos n}{n^2} = 0.5403 - 0.1040 - 0.11 - 0.0409 + 0.1135.$$

$$\sum \frac{|\cos n|}{n^2} \qquad \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

↖ Convergent P-Series.

$\therefore$, series is absolutely convergent by comparison test.

## Testing for Absolute/Conditional Convergence

- Consider $\sum |a_n|$ and test for convergence.
  - if $\sum |a_n|$ is convergent then $\sum a_n$ is convergent
  - if $\sum |a_n|$ is divergent, need a second series test (Alt series test) to determine if $\sum a_n$ is convergent. → conditionall convergent.

Example

① $\sum \frac{(-1)^{n-1}}{n} = \sum \frac{1}{n} \rightarrow$ Divergent.

② Alt series Test ...

$\therefore$ Conditionally convergent.

- $\sum \left| \frac{(-2)^n n}{(n+1)3^{n+1}} \right| \rightarrow \sum \frac{2^n n}{(n+1)3^{n+1}}$

$$\sum \frac{2^n n}{(n+1)3^{n+1}} \overset{\text{ratio test}}{=} \lim_{n\to\infty} \frac{2^{n+1}(n+1)}{(n+2)3^{n+2}} \cdot \frac{(n+1)3^{n+1}}{2^n n}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{(n+1)^2}{n(n+2)}\right)$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{n^2+2n+1}{n^2+2n}$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{1+\cancel{\frac{2}{n}}+\cancel{\frac{1}{n^2}}}{1+\cancel{\frac{2}{n}}}$$

$$= \frac{2}{3} \leq 1$$

$\therefore$ absolute convergent

- $\sum \left| \frac{(-1)^n}{\sqrt{n-1}} \right| \rightarrow \sum \frac{1}{\sqrt{n-1}}$ Not absolutely convergent.

# Summary / Strategy for Testing Series

① Is it geometric or p-series?

② Test for divergence ($\lim_{n\to\infty} a_n = 0$).

③ Easy integration → f(x) is positive, continuous, and decreasing use integral test.

④ "Similar" to p-series or geometric series → comparison test / limit comparison

- Rational / Algebraic Expressions
- Keep highest powers in numerator and denominator to find comparative series.

⑤ Factorials / Powers → Ratio test

⑥ $\Sigma(-1)^n b_n$ - Alt series Test

⑦ Absolute Convergence.

Test $\Sigma|a_n|$

- if convergent, done
- if divergent, need alt series test to test for conditional convergence.

# 10.3 Taylor Series

- Essential means of evaluation and analyzing transcendental functions Ex. $e^x$, $\ln x$, $\sin x$
- Not all functions have power series representations → assume $f(x)$ one and proceed
- How do we find ~~given~~ power series given $f(x)$?

## Power Series

- $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots$ for $|x-a| < R$.
- Put $x=a$ in $f(x)$ $\quad f(a) = c_0$.
- Now, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 \ldots$
- Put $x=a$, in $f'(x)$ $\quad f'(a) = c_1$
- Now, $f''(x) = 2c_2 + 6c_3(x-a) \ldots$
- Put $x=a$, in $f''(x)$ $\quad f''(a) = 2c_2$.
- In General $\quad c_n = \dfrac{f^n(a)}{n!}$

## Theorem

- If $f$ has a power series representation at $x=a$, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, $R>0$ with coefficients $c_n = \frac{f^n(a)}{n!}$ then $f(x) = \sum \frac{f^n(a)}{n!}(x-a)^n$ is called a taylor series of $f(x)$ at $x=a$.
- Special Case when $a=0$ $\quad f(x) = \sum_{n=1}^{\infty} \dfrac{f^n(0)}{n!} x^n$, $f(0), f'(0)x, \dfrac{f''(0)x^2}{2!} \ldots$

  is the Maclaurin Series.
- Taylor Series $= \sum \dfrac{f^n(a)}{n!}(x-a)^n$
- Maclaurin Series $= \sum \dfrac{f^n(0)}{n!} x^n$

Example

- Maclaurin Series for $e^x$ & find radius of convergence.

$f(x)=e^x \quad f(0)=1$ Maclaurin is $\rightarrow$

$f'(x)=e^x \quad f'(0)=1$

$$\sum \frac{1}{n!}x^n = \sum \frac{x^n}{n!}$$

$f''(x)=e^x \quad f''(0)=1$

$f^n(0)=1$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}|x|\frac{1}{n+1}$$
$$=0<1$$

convergent for all values of x.

- Estimate value of $e^{\frac{1}{2}}$ by using 6 terms in the series.

$$\sqrt{e} \approx \sum_{n=0}^{5}\frac{(0.5)^n}{n!} = 1+\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)^3}{3!}+\frac{\left(\frac{1}{2}\right)^4}{4!}+\frac{\left(\frac{1}{2}\right)^5}{5!}$$
$$=1+\frac{1}{2}+\frac{1}{8}+\frac{1}{48}+\frac{1}{384}+\frac{1}{3840}$$
$$\approx 1.648697917$$

- find $e^{2x}$

$$e^{2x} \approx \sum\frac{(2x)^n}{n!} = \sum\frac{2^n x^n}{n!}$$

- Taylor Series of $\ln x$ about $a=1$ and radius of convergence.

$f(x)=\ln x \quad f(1)=0$

$f'(x)=\frac{1}{x} \quad f'(1)=1$

$$\sum \frac{f^n(1)}{n!}(x-1)^n = \sum\frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

$f''(x)=\frac{-1}{x^2} \quad f''(1)=-1$

$$\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}$$

$f'''(x)=\frac{2}{x^3} \quad f'''(1)=2.$

$f^4(x)=\frac{-6}{x^4} \quad f^4(1)=-6$

$f^5(x)=\frac{24}{x^5} \quad f^5(1)=24.$

$$\lim_{n\to\infty}\left|\frac{-1^n(x-1)^{n+1}}{n+1}\cdot\frac{n}{-1^{n-1}(x-1)^n}\right|$$

$$\lim_{n\to\infty}\frac{n}{n+1}|x-1|$$

$|x-1|$ converges whenever $|x-1|<1$

$R=1.$

Example.

- find Taylor for $\frac{1}{x}$ when $a=2$ and radius of convergence.

- $f(x) = \frac{1}{x}$ $\quad$ $f(2) = \frac{1}{2}$

$f'(x) = \frac{-1}{x^2}$ $\quad$ $f'(2) = -\frac{1}{4}$

$f''(x) = \frac{2}{x^3}$ $\quad$ $f''(2) = \frac{1}{4}$

$f'''(x) = \frac{-6}{x^4}$ $\quad$ $f'''(2) = \frac{-3}{8}$

$f^4(x) = \frac{24}{x^5}$ $\quad$ $f^4(2) = \frac{3}{4}$.

$$\frac{1}{x} = \sum \frac{\frac{(-1)^n n!}{2^{n+1}}}{n!} (x-2)^n$$

$$= \sum \frac{(-1)^n (x-2)^n}{2^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(-1)^n (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{2}|x-2|$$

$$= \frac{1}{2}|x-2| \qquad \frac{1}{2}|x-2| < 1$$

~~convergent when~~ $R = 2$.

• Maclaurin for sin and radius of convergence.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^4(x) = \sin x \qquad f^4(0) = 0$

$f^5(x) = \cos x \qquad f^5(0) = 1$

$$\sum \frac{f^n(0)}{n!} x^n = x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7$$

$$\sin x = \sum \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n-1} x^{2n-1}} \right|$$

$$= \frac{1}{(2n+1)(2n)} |x^2|$$

radius $= \infty$.

• Taylor for cos x when $\pi = a$ and find ROC.

$f(x) = \cos x \qquad f(\pi) = -1$

$f'(x) = -\sin x \qquad f'(\pi) = 0$

$f''(x) = -\cos x \qquad f''(\pi) = 1$

$f'''(x) = \sin x \qquad f'''(\pi) = 0$

$f^4(x) = \cos x \qquad f^4(\pi) = -1$

$$\cos x = \sum \frac{(-1)^{n+1}(x-\pi)^{2n}}{2n!}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-\pi)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^{n+1}(x-\pi)^{2n}} \right|$$

$$= 0 < 1$$

$R = \infty$

• $\int_0^1 e^{-x^2} dx$

$$e^x = \sum \frac{x^n}{n!}$$

$$e^{-x^2} = \sum \frac{(-x^2)^n}{n!} \to \frac{(-1)^n x^{2n}}{n!}$$

$$\int_0^1 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}\, dx \quad = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$= x - \frac{1}{3}x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} - \frac{1}{924}$$

$$\int_0^1 e^{-x^2} dx = 0.747486772$$