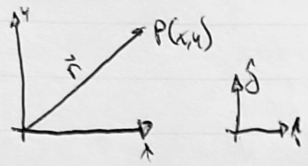
Motion in a plane

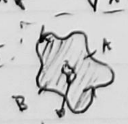
Notation:
$\nearrow V$, scalar
$\nearrow \vec{V}, \underline{V}$ vector
$\underline{V} \cdot \underline{V}$ ← would be dot product.

- 

- $(\hat{i}, \hat{j}, \hat{k})$; $(\underline{i}, \underline{j}, \underline{k})$; $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$; $(\hat{x}, \hat{y}, \hat{z})$
  → All mean the same
- Position $\vec{r} = x\hat{i} + y\hat{j}$
  Velocity $\dot{\vec{r}} = \vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$
  - $\dot{} \equiv \frac{\partial}{\partial t}$
- Acceleration $\ddot{\vec{r}} = \vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{k}$

- Equation of Motion (Newtons 2^{nd} Law)
  - $\vec{F} = m\vec{a}$
    $F_x\hat{i} + F_y\hat{j} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})$.
- Conservative Force
  - $\vec{F} = -\vec{\nabla} V$
    $= -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j}\right]$
  - $V = V(x,y)$ Potential energy function.
    Function of position only.
- Work done by $\vec{F}$ as the body moves (from a to b)
  - $W_{ab} = \int_a^b \vec{F} \cdot d\vec{r} = -\int_a^b \vec{\nabla} V \cdot d\vec{r} = -\int_a^b dV$
    $= V_a - V_b$ ①

$\left[V(x,y) \Rightarrow dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \right.$
$= \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y}\right) \cdot ($
$\left. = \vec{\nabla} \cdot d\vec{V} \right]$

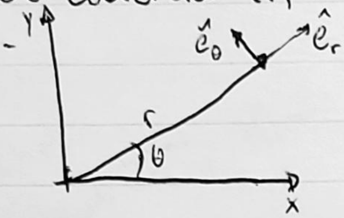
- Alternatively
  - $W_{ab} = \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b m\frac{d\vec{v}}{dt} \cdot d\vec{r} = m\int_a^b d\vec{v} \cdot \frac{d\vec{r}}{dt} = m\int_a^b \vec{v}\, d\vec{v} \Rightarrow$
    $= m\int_a^b \frac{1}{2} d(\vec{v} \cdot \vec{v}) = \frac{1}{2}m\int_a^b d(\vec{v}^2) = \frac{1}{2}m(v_b^2 - v_a^2)$
  - $\frac{1}{2}mv_b^2 + V_b = \frac{1}{2}mv_a^2 + V_A$ (Conservation of mechanical energy)
- Test for a conservative force
  - $\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla} V = 0$.
  - Or.  $\oint \vec{F} \cdot d\vec{r} \xrightarrow{\text{Stokes}} \iint \underbrace{(\vec{\nabla} \times \vec{F})}_{0} \cdot d\vec{r} \Rightarrow 0$, iff $\vec{F}$ is conservative.

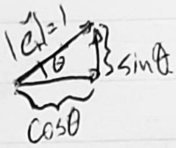
$d\vec{\sigma} = d\sigma\,\hat{n}$
$\hat{n}$ = unit vector coming out.

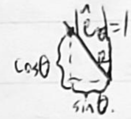
$\oint \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r} + \int_B^A \vec{F} \cdot d\vec{r} = 0$.

$\int_A^B \vec{F} \cdot d\vec{r} = V_A - V_B$ (Path independent).

- Partial Derivative Relation
  - $f = f(x_1, x_2, \ldots x_n, t)$
  - $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 \ldots + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial t}dt.$
  - $\dot{df} = \frac{df}{dt} = \frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial x_2}\dot{x}_2 \ldots \frac{\partial f}{\partial x_n}\dot{x}_n + \frac{\partial f}{\partial t}$
- Polar Coordinates (r, θ)







$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ $\qquad$ $\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}}_{A.} \begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix}$$

$A. \longrightarrow$ Rotation matrix (orthogonal)

Orthogonal matrix. its inverse = transpose $A^{-1} = \tilde{A}$ – – –

$$\begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix} = A^{-1}\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix}, \quad A^{-1} = \tilde{A} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix} \qquad \hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \qquad \hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

Aside:

$\hat{e}_r = \dfrac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$

$\hat{e}_\theta = \dfrac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$

$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$

- (A) Position $\vec{r} = r\hat{e}_r$

  (B) Velocity, $\vec{v} = \dot{\vec{r}} =$ $\dot{r}\hat{e}_r + \dot{\hat{r}}\hat{e}_\theta$

  $\dot{\hat{e}}_r = \frac{\partial \hat{e}_r}{\partial \theta} \cdot \frac{d\theta}{dt} = \dot{\theta}\hat{e}_\theta$

  $\vec{v} = \underbrace{\dot{r}\hat{e}_r}_{\text{radial component}} + \underbrace{r\dot{\theta}\hat{e}_\theta}_{\text{transverse component.}}$

  (C) Acceleration $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta.$

  $$\vec{a} = \ddot{\vec{r}} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{radial component.}}\hat{e}_r + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{\text{transverse Kmt}}\hat{e}_\theta$$

- Vector $\vec{A}$ with fixed magnitude

  $\vec{A}\cdot\vec{A} = A^2$ (constant)

  $\dot{\vec{A}}\cdot\vec{A} = 0$

- Example: Motion with a fixed speed v

  $\vec{v}\cdot\vec{v} = v^2$ (constant)

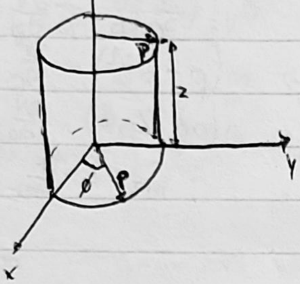
  $\dot{\vec{v}}\cdot\vec{v} = \vec{a}\cdot\vec{v} = 0$, $\vec{a} = 0$ (uniform motion in st. line)

  or $\vec{a} \neq 0$, $\vec{a} \perp \vec{v}$ (uniform circular motion)

- Kinetic Energy, $T = \frac{1}{2} m \vec{v} \cdot \vec{v}$

$$= \frac{1}{2} m [\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta] \cdot [\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta]$$
$$= \frac{1}{2} m [\dot{r}^2 + r^2\dot{\theta}^2]$$

- 3D $\Rightarrow$ start with cylindrical
- Cylindrical Coordinates



$P(\rho, \phi, z)$

$(\rho, \phi) \equiv (r, \theta)$ plane polar.

$(\hat{e}_r, \hat{e}_\theta) \equiv (\hat{e}_\rho, \hat{e}_\theta)$

$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

$\Rightarrow \dot{\hat{e}}_\rho = \dot{\phi}\hat{e}_\phi \quad \Rightarrow \dot{\hat{e}}_\phi = -\dot{\phi}\hat{e}_\rho$

- (A) Position: $\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z$
- (B) Velocity: $\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{e}_\rho + \rho\dot{\hat{e}}_\rho + \dot{z}\hat{e}_z + z\underbrace{\dot{\hat{e}}_z}_{=0}$

$$= \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\theta + \dot{z}\hat{e}_z$$
$$d\vec{v} = d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\theta + dz\hat{e}_z$$

- (C) Acceleration: $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{\rho}\hat{e}_\rho + \dot{\rho}\dot{\hat{e}}_\rho + \dot{\rho}\hat{e}_\theta\dot{\phi} + \rho\dot{\hat{e}}_\theta\dot{\phi} + \rho\ddot{\phi}\hat{e}_\theta + \ddot{z}\hat{e}_z$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\theta + \ddot{z}\hat{e}_z$$

- K.E., $T = \frac{1}{2} m \vec{v} \cdot \vec{v}$

$$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

Equations of Motion

- Plane Polar Coordinates - $\vec{a} = \underbrace{(\ddot{r} - r\dot{\phi}^2)}_{a_r:\text{ radial comp.}}\hat{e}_r + \underbrace{(2\dot{r}\dot{\phi} + r\ddot{\phi})}_{a_\phi:\text{ transverse comp.}}\hat{e}_\phi$

- $\vec{F} = m\vec{a} = F_r\hat{e}_r + F_\phi\hat{e}_\phi$

$F_r = m(\ddot{r} - r\dot{\phi}^2)$

$F_\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$

$\vec{F} = -\vec{\nabla}V$, V = Potential Energy.

$V = V(r, \phi)$, $dV = \frac{\partial V}{\partial r}\Big)_\phi dr + \frac{\partial V}{\partial \phi}\Big)_r d\phi$

$dV = \vec{\nabla}V \cdot d\vec{r}$, $d\vec{r} = dr\hat{e}_r + r\,d\phi\,\hat{e}_\phi$

$\vec{\nabla}V \cdot (dr\hat{e}_r + r\,d\phi\,\hat{e}_\phi) = \frac{\partial V}{\partial r}\Big)_\phi dr + \frac{\partial V}{\partial \phi}\Big)_r d\phi$

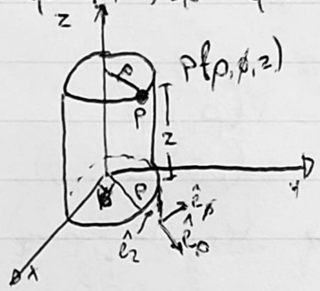
$\vec{\nabla}V\hat{e}_r = \frac{\partial V}{\partial r}$, $\vec{\nabla}V\hat{e}_\phi = \frac{1}{r}\frac{\partial V}{\partial \phi}$

$\therefore \vec{\nabla}V = \frac{\partial V}{\partial r}\Big)\hat{e}_r + \frac{\partial V}{\partial \phi}\Big)\hat{e}_\phi$

Replace ϕ to θ for plane polar.

$$F_r = -\frac{\partial V}{\partial r} = +m(\ddot{r} - \dot{\phi}^2 r), \quad \frac{\partial V}{\partial r} = -m(\ddot{r} - r\dot{\phi}^2)$$
$$F_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = m(2\dot{r}\dot{\phi} + r\ddot{\phi}), \quad \frac{\partial V}{\partial \phi} = -m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

- Cylindrical Coordinates (ρ, ϕ, z).
  - $\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$
  -



$\vec{F} = m\vec{a} = -\nabla V$

$\vec{\nabla} V = \left[\frac{\partial V}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{e}_\phi + \frac{\partial V}{\partial z}\hat{e}_z\right]$

$$\left.\begin{aligned} m(\ddot{\rho} - \rho\dot{\phi}^2) &= -\frac{\partial V}{\partial \rho} \\ m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) &= -\frac{\partial V}{\partial \phi}\frac{1}{\rho} \\ m\ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \text{Newtons } 2^{nd} \text{ Law in cylindrical}$$

Alternative Forms of Newtonian Dynamics.

- ① Newtons approach: $\vec{F} = m\vec{a}$, initial conditions, identify all forces (may involve newtons 3^{rd} law), constraints possibly, balance forces and accelerations in 3 directions.
- ② Lagrangian (L), Hamiltonian (H) approach:
  - Scalars, no need to work with vector components

Consider 1D Case

- $F_x = m\frac{d^2x}{dt^2} = m\ddot{x}$ (1)
- Introduce Lagrangian function - $L = \underset{(K)}{\overset{K.E.}{T}} - \underset{(U)}{\overset{P.E.}{V}} = L(x, \dot{x}, t)$.
- $x, \dot{x}$ are independent variables: $\frac{\partial \dot{x}}{\partial x} = 0$
- $T = \frac{1}{2}m\dot{x}^2$, $L = \frac{1}{2}m\dot{x}^2 - V(x, t)$.
- $\frac{\partial L}{\partial \dot{x}} = p_x$: momentum conjugate x

  $p_x = m\dot{x}$ (x, p_x): canonically conjugate pair

  $x p_x$: dimension of work/energy.
- $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{\partial}{\partial t}(m\dot{x}) + \frac{\partial V}{\partial x}$

  $= m\ddot{x} + \frac{\partial V}{\partial x}$ ← $\frac{\partial V}{\partial x} = -F_x$.

  $= m\ddot{x} - F_x = 0$.
- Lagrangian for 1D case: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ (2) (equivalent to 1)

- Clearly for 2D case:
  - $L = T(\dot{x},\dot{y}) - V(x,y,t)$, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$
  - $\frac{\partial L}{\partial \dot{x}} = p_x = m\dot{x}$, $\frac{\partial L}{\partial \dot{y}} = p_y = m\dot{y}$
  - (p_x, p_y): momentum conjugate to x, y.
  - 2 L equations
    $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$
    $$\Downarrow \qquad\qquad \Downarrow$$
    $$F_x = m\ddot{x} \qquad\qquad F_y = m\ddot{y}$$
- Number of L Eqns = Number of independent coordinates to describe the system
  = Number of degrees of freedom (d.o.f)
- In D-dimensions ~~with~~ N particles, # of d.o.f = N.D
- If there are 'n' constraints, # of d.o.f = ND − n.
- 2D plane polar coordinates.
  - $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$
  - $V = V(r,\theta)$
  - $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta)$
  - $\frac{\partial V}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r} = p_r$ (momentum conjugate to r), dimensions: MLT^{-1}
    $\frac{\partial V}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta$ (momentum conjugate to θ), dimensions: ML^2T^{-1}
  - (p_r, p_θ) generalized ~~[illegible]~~ momenta.
  - r, θ: generalized position coordinates
  - rp_r, θp_θ: dimensions of work/energy.
  - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dr}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$ ← $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$
    $m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial \theta}$. $\frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}$.
    $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial}{\partial t}(mr\dot{\theta}^2) + \frac{\partial V}{\partial \theta} = 0$.
    $-\frac{\partial V}{\partial \theta} = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$
- Generalized Force:
  - generalized position coordinates: $q_1, q_2, \dots q_n$
    " momentum " : $p_1, p_2, \dots p_n$
  - Position Vector $\vec{r} = \vec{r}(q_1, q_2, \dots q_n)$
    $$q_i \to q_i + \delta q_i$$
    then $\vec{r} \to \vec{r} + \delta\vec{r}$
    $$\delta\vec{r} = \frac{\partial \vec{r}}{\partial q_1}\delta q_1 + \frac{\partial \vec{r}}{\partial q_2}\delta q_2 + \dots \frac{\partial \vec{r}}{\partial q_n}\delta q_n.$$
    $$\delta\vec{r} = \sum_{i=1}^{n}\left(\frac{\partial \vec{r}}{\partial q_i}\right)\delta q_i$$

- Suppose a ~~...~~ force acts on a particle at a displacement of $\delta \vec{r}$

- Work done by $\vec{F}$:

$$\delta W = \vec{F} \cdot \delta \vec{r}$$
$$= \vec{F} \cdot \sum_{i=1}^{n} \frac{\partial \vec{r}}{\partial q_i} \delta q_i$$
$$= \sum_{i=1}^{n} \left(\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} \right) \delta q_i = \sum_{i=1}^{n} Q_i \, \delta q_i$$

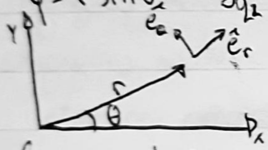
$Q_i = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i}$: generalized force associated with q_i

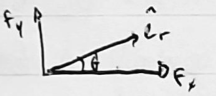
- F is conservative

$$\vec{F} = -\vec{\nabla} V, \quad Q_i = -\vec{\nabla} V \frac{\partial \vec{r}}{\partial q_i} \qquad \vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$
$$Q_i = -\left(\hat{e}_x \frac{\partial V}{\partial x} + \hat{e}_y \frac{\partial V}{\partial y} + \hat{e}_z \frac{\partial V}{\partial z} \right) \cdot \left(\hat{e}_x \frac{\partial x}{\partial q_i} + \hat{e}_y \frac{\partial y}{\partial q_i} + \hat{e}_z \frac{\partial z}{\partial q_i} \right)$$
$$= -\left(\frac{\partial V}{\partial x} \frac{\partial x}{\partial q_i} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial q_i} + \frac{\partial V}{\partial z} \frac{\partial z}{\partial q_i} \right)$$
$$= -\frac{\partial V}{\partial q_i}$$

- Particle in a plane, 2 dof, general coords (r, θ)

$$q_1 = r, \quad q_2 = \theta$$
$$x = r\cos\theta, \quad \frac{\partial x}{\partial q_1} = \frac{\partial x}{\partial r} = \cos\theta$$
$$y = r\sin\theta, \quad \frac{\partial x}{\partial q_2} = \frac{\partial x}{\partial \theta} = -r\sin\theta$$





$F_x\cos\theta + F_y\sin\theta = F_r$

Generalized forces:

$$Q_1 = F_x \frac{\partial x}{\partial q_1} + F_y \frac{\partial y}{\partial q_1}$$
$$= F_x\cos\theta + F_y\sin\theta$$
$$Q_1 = F_r \text{ (radial force)}$$
$$Q_2 = F_x \frac{\partial x}{\partial q_2} + F_y \frac{\partial y}{\partial q_2}$$
$$= -F_x r\sin\theta + F_y r\cos\theta$$
$$= r\underbrace{(-F_x\sin\theta + F_y\cos\theta)}_{F_\theta \text{ } (\hat{e}_\theta \text{ component of } \vec{F})}$$
$$= rF_\theta = (\vec{r} \times \vec{F})_z$$

$Q_2 \rightarrow$ Generalized force associated with an angle is a torque.

Generalized Coordinates, Momentum, Force

- q_i: may not have dimensions of L

  p_i: " " " " " MLT^{-1}

  Q_i: " " " " " MLT^{-2}

  ~~$p_i q_i$: dimensions of work/energy; ML^2T^{-1}~~

  ~~[illegible]~~

  $Q_i q_i$: dimensions of work", ML^2T^{-2}

- $p_i = \frac{\partial L}{\partial \dot{q}_i}$

  $Q_i = \frac{-\partial V}{\partial q_i}$

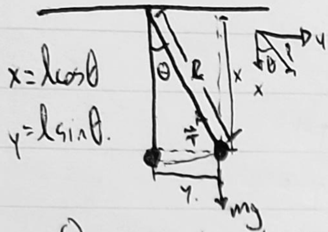
- K.E. in terms of coords.

  $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = T(\dot{x}, \dot{y})$

  $= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = T(\dot{r}, \dot{\theta})$.

- In cartesian coordinates.
  - T is a fxn of $\dot{x}, \dot{y}, \ldots$
- In general
  - T is a fxn of $\{q_i\}, \{\dot{q}_i\}$
- $L = T - V = L(\{q_i, \dot{q}_i\}, i = 1, \ldots n, t)$
- $\frac{\partial L}{\partial \dot{q}_i} = p_i$ (gen. momentum) $p_i q_i$: units of angular momentum.
- L-Equation:
  - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
- General force:
  - $Q_i = \frac{-\partial V}{\partial q_i}$ $Q_i q_i$: units of work/energy.
- K.E., $T = T(\{q_i\}, \{\dot{q}_i\}, t)$.
  - $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = T(r, \dot{r}, \dot{\theta})$
  - In cartesian coordinates, $T = T(\{\dot{q}_i\})$.

① Simple Pendulum



$x = l\cos\theta$

$y = l\sin\theta$

$\vec{T}$ insures that $r = l$ (constraint)

$m\ddot{\vec{r}} = \vec{T} + mg\hat{e}_x$

The force of constraint $\vec{T}$ does not do any work.

L-Equation can be used in its standard form

- One constraint $\Rightarrow$ only one independent coordinate (θ), one L-Eq^n.

$L = T - V$

$T = \frac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2)$

$r = l \Rightarrow \dot{r} = 0 \Rightarrow T = \frac{1}{2}m(l^2\dot{\theta}^2)$

$V = -mgx = -mgl\cos\theta$

$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$, $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

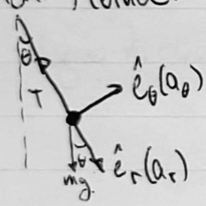
$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$, for small θ, $\sin\theta \simeq \theta$.

$\ddot{\theta} + \omega^2\theta = 0$, $\omega^2 = \frac{g}{l}$.

$\theta = A\sin(\omega t + c)$, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

- Newtons Method:



$a_r = \underbrace{\ddot{r}}_{=0} - \underbrace{r}_{=l}\dot{\theta}^2 = -l\dot{\theta}^2$

$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = l\ddot{\theta}$

Balancing forces in radial direction

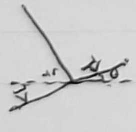
$-T + mg\cos\theta = ma_r = -ml\dot{\theta}^2$

Transverse Direction: $mg\sin\theta = -ml\ddot{\theta}$

$\ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0$. For small θ; $\theta \simeq \sin\theta$.

$\ddot{\theta} + \omega\theta = 0$.

- Suppose we solve the problem using (x, y), say y is the independent coordinates.



Speed of the mass, v, is given by $\dot{y} = \pm v\cos\theta$

$v = \pm\frac{\dot{y}}{\cos\theta} = \pm\dot{y}\frac{1}{\sqrt{l^2 - y^2}/l} = \pm\frac{l\dot{y}}{\sqrt{l^2 - y^2}}$

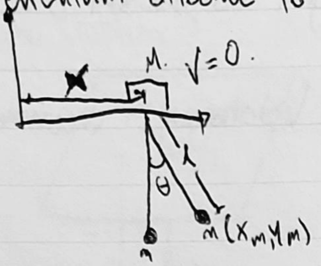
$T = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{l^2\dot{y}^2}{(l^2 - y^2)}$, $V = -mgx = mg\sqrt{l^2 - y^2}$

$L = T - V = \frac{1}{2}\frac{ml^2\dot{y}^2}{(l^2 - y^2)} + mg\sqrt{(l^2 - y^2)}$

$\frac{\partial L}{\partial \dot{y}} = \frac{ml^2\dot{y}}{(l^2 - y^2)}$, $\frac{\partial L}{\partial y} = \frac{ml^2\dot{y}^2 y}{(l^2 - y^2)^2} - \frac{mgy}{(l^2 - y^2)^{\frac{1}{2}}}$

L-Eq^n: $\ddot{y} = \frac{-y\dot{y}^2}{(l^2 - y^2)} - gy\frac{\sqrt{l^2 - y^2}}{l^2}$, correct but unnecessarily complicated, hides the essential physics.

• ② Pendulum attached to a movable support.



4 coordinates; X_M, Y_M, x_m, y_m

Constraints; $Y_M = 0$

$$\sqrt{(x_m - X_M)^2 + (y_m - Y_M)^2} = l = \text{constant.}$$

⇒ 2 degrees of freedom.

Choose 2 generalized coordinates. $q_1 = x$, $q_2 = \theta$.

$T = \frac{1}{2}M\dot{x}^2 + T_m$ + K.E of the small mass.

$x_m = X + l\sin\theta$, $\dot{x}_m = \dot{X} + l\cos\theta\,\dot\theta$

$y_m = -l\cos\theta$, $\dot{y}_m = l\sin\theta\,\dot\theta$

$T_m = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\,l\cos\theta\,\dot\theta + l^2\dot\theta^2)$

$V = -mgl\cos\theta$.

$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\,l\cos\theta\,\dot\theta + l^2\dot\theta^2) + mgl\cos\theta$.

$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\,\dot\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = (M+m)\ddot{x} + ml\sin\theta\,\dot\theta^2 + ml\cos\theta\,\ddot\theta$

$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial x} = 0$ (x is 'ignorable' or 'cyclic').

(L is invariant under change of X, implies a symmetry of motion w.r.t X).

$$\frac{\partial L}{\partial q_1} = 0 \Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}_1}\right)}_{}=0$$

$= p_1$ is generalized momentum associated with q_1

p_1 = constant of motion. ①

$q_1 = x$, $p_1 = p = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\,\dot\theta = C$ (constant of motion)

Equation of Motion for X:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$.

$\frac{d}{dt}(p_x) = 0 = (M+m)\ddot{x} + ml\cos\theta\,\ddot\theta - ml\sin\theta\,\dot\theta^2$ ②

① is the first integral of ②.

Equation of motion for θ:

$\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta + ml\dot{x}\cos\theta$.

$\frac{\partial L}{\partial q_2} = \frac{\partial L}{\partial \theta} = -ml\dot{x}\dot\theta\sin\theta - mgl\sin\theta$.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$ml^2\ddot\theta + ml\ddot{x}\cos\theta + mgl\sin\theta = 0$

Linearized form for small θ, $\cos\theta \approx 1$, $\sin\theta \approx \theta$.

$ml^2\ddot\theta + ml\ddot{x} + mgl\theta = 0$. ③

from ① $(M+m)\dot{x} + ml\dot\theta = C$

$(M+m)\ddot{x} + ml\ddot\theta = 0$

$\ddot{x} = -\frac{ml\ddot\theta}{(m+M)}$ → sub this in θ equation ③

$\frac{Mm}{(M+m)}l^2\ddot\theta + mgl\theta = 0$

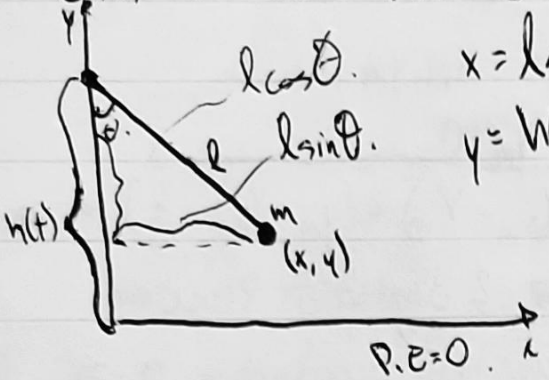
$\ddot\theta + \left(\frac{M+m}{M}\right)\left(\frac{g}{l}\right)\theta = 0$

θ oscillates with angular frequency ω, $\omega^2 = \frac{M+m}{M}\frac{g}{l}$

$\omega = \sqrt{\left(1+\frac{m}{M}\right)\frac{g}{l}}$

If $M \to \infty$ (really heavy support), $\omega = \sqrt{\frac{g}{l}}$ (anticipated).

• Question ② From the sample Lagrangian homework.



$x = l\sin\theta$.

$y = h(t) - l\cos\theta \Rightarrow$ holonomic & rheonomic.

a) - P.E $V = mgy = mg(h(t) - l\cos\theta)$

K.E $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, $\dot{x} = l\cos\theta\,\dot{\theta}$, $\dot{y} = \dot{h}(t) + l\sin\theta\,\dot{\theta}$

$= \frac{1}{2}m(l^2\dot{\theta}^2 + 2\dot{h}l\dot{\theta}\sin\theta + \dot{h}^2)$

$L = T - V = \frac{1}{2}m(l^2\dot{\theta}^2 + 2\dot{h}l\dot{\theta}\sin\theta + \dot{h}^2) - mg(h - l\cos\theta)$

b) - $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + m\dot{h}l\sin\theta$.

$\frac{\partial L}{\partial \theta} = m\dot{h}l\dot{\theta}\cos\theta - mgl\sin\theta$.

L-Eqn - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\Rightarrow ml^2\ddot{\theta} + m(g + \ddot{h})\,l\sin\theta = 0$

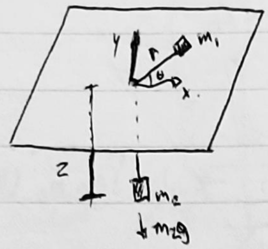
Equation of a simple pendulum in gravitation field $= g + \ddot{h}$

• Sample Problem 1 Answers.

- a) $L = \frac{1}{2}m(1 + 4A^2x^2)\dot{x}^2 - mgAx^2$

b) $m(1 + 4A^2x^2)\ddot{x} + 4mA^2x\dot{x}^2 + 2mgAx = 0$

- Sample Problem 5.



Constraint: $r + z = \text{constant} = l$.

holonomic & scleronomic system

Without (i) we need 3 coordinates, say r, θ, z

With (i) only 2 coordinates, (r, θ) or (z, θ), are needed.

- Particle 1: $r = l - z$ $\quad \dot{r} = \dot{z}$

$KE \Rightarrow T_1 = \frac{1}{2} m_1 (\dot{r}^2 + r\dot{\theta}^2)$

$= \frac{m_1}{2}(\dot{z}^2 + (l-z)^2\dot{\theta}^2)$

$PE \Rightarrow V_1 = 0$.

- Particle 2:

$KE \Rightarrow T_2 = \frac{1}{2} m_2 \dot{z}^2$

$PE \Rightarrow V_2 = -m_2 g z$

- $L = (T_1 + T_2) - (V_1 + V_2)$

$= \left(\frac{m_1}{2}(\dot{z}^2 + (l-z)^2\dot{\theta}^2) + \frac{m_2}{2}\dot{z}^2\right) - (0 - m_2 g z)$.

- L-Eqn for θ.

$\frac{\partial L}{\partial \dot{\theta}} = m_1 (l-z)^2 \dot{\theta}$.

$\frac{\partial L}{\partial \theta} = 0$, $\quad \theta$ is cyclic or ignorable.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\dot{p}_\theta \qquad 0$

$\dot{p}_\theta = 0$, $\quad p_\theta = \text{constant}$.

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_1 (l-z)^2 \dot{\theta} = L$ (Angular momentum of m_1 about z axis) (ii)

- L-Eqn for z:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow \frac{d}{dt}((m_1 + m_2)\dot{z}) + m_1(z-l)\dot{\theta}^2 - m_2 g = 0$.

$\frac{\partial L}{\partial \dot{z}} = (m_1 + m_2)\dot{z}$

$\frac{\partial L}{\partial z} = m_1(z-l)\dot{\theta}^2 - m_2 g$ $\quad$ using (ii).

$\ddot{z} + \frac{L^2}{m_1(m_1+m_2)}\frac{1}{(l-z)^3} - \frac{m_2}{(m_1+m_2)} g = 0$ $\quad$ (iii)

- LN.R Eq (iii) has the structure

$\ddot{z} + f(z) = 0$

$\ddot{z}\dot{z} + f(z)\dot{z} = 0$

$\frac{d(\dot{z})}{dt}\dot{z} + f(z)\frac{dz}{dt} = 0$

$\frac{1}{2}\int \frac{d}{dt}(\dot{z}^2)\,dt + \int f(z)\frac{dz}{dt}\,dt$

$\frac{1}{2}\int d(\dot{z}^2) + \int f(z)\,dz = \text{constant}$

$\Rightarrow \frac{1}{2}\dot{z}^2 + \int f(z)\,dz = \text{constant}$.

Circular Orbit for any m_1 at $r = r_0$, $z = l - r = l - r_0 = z_0$

$\dot{r} = \ddot{r} = 0$

Eq (iii) $m_1 r_0 \dot{\theta}^2 - m_2 g = 0 \Rightarrow \dot{\theta} = \omega = \sqrt{\frac{m_2}{m_1 r_0} g}$

- Small displacement about the circle orbit: let $z = z_0 + u$. $u \ll z_0$

$$\dot{z} = \dot{u}$$

$$\ddot{z} = \ddot{u}$$

- In general, $\dot{\theta} = \frac{L}{m_1(l-z)^2}$ (ii) $\rightarrow L = \sqrt{g m_1 m_2}\,(l-z_0)^{3/2}$

For the circular orbit, $\dot{\theta} = \sqrt{\frac{m_2 g}{m_1(l-z_0)}}$

- $\therefore$ eqn (iii), $\ddot{u} + \frac{g m_2}{(m_1+m_2)} \frac{(l-z_0)^3}{(l-z)^3} - \frac{m_2}{(m_1+m_2)} g = 0$

$l - z = l - z_0 - u$

$= r_0 - u$

- $\ddot{u} + \frac{g m_2}{(m_1+m_2)} \frac{r_0^3}{(r_0-u)^3} - \frac{g m_2}{(m_1+m_2)} = 0$

$$\frac{1}{(r_0-u)^3} = \frac{1}{r_0^3\left(1-\frac{u_0}{r_0}\right)^3}$$

$$= \frac{1}{r_0^3}\left(1-\frac{u_0}{r_0}\right)^{-3}$$

$$= \frac{1}{r_0^3}\left(1+\frac{3u_0}{r_0}\right)$$

$(1+x)^n$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots$

$+ \frac{n!}{(n-1)!\,r!}$

$C_{n_r} = \frac{n!}{(n-r)!\,r!}$

- $\therefore$, $\ddot{u} + \frac{g m_2}{(m_1+m_2)}\left(1-\frac{u_0}{r_0}\right)^{-3} - \frac{g m_2}{(m_1+m_2)} = 0$

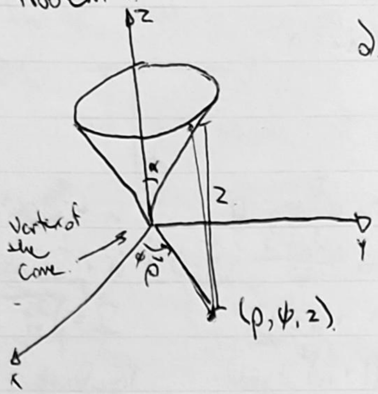
$$\ddot{u} + \frac{3 g m_2}{(m_1+m_2) r_0} u = 0$$

$\ddot{u} + \Omega u = 0$, $\Omega = \sqrt{\frac{3 g m_2}{(m_1+m_2) r_0}}$

Time period, $T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{r_0(m_1+m_2)}{3 g m_2}}$

• Sample Problem 4

a. Constraint: $\frac{\rho}{z} = \tan\alpha$, holonomic, scleronomic.



- $T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$, $\dot{\rho} = \dot{z}\tan\alpha$.

$= \frac{1}{2}m(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2)$

$= \frac{1}{2}m(\dot{z}^2\sec^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2)$

use (z, ϕ) as independent coordinates.

- $V = mgz$.
- $L = \frac{1}{2}m(\dot{z}^2\sec^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2) - mgz$.

$\frac{\partial L}{\partial z} = mz\tan^2\alpha\,\dot{\phi}^2 - mg$.

$\frac{\partial L}{\partial \dot{z}} = m\dot{z}\sec^2\alpha$

$\frac{\partial L}{\partial \phi} = 0 \Rightarrow p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}mz^2\tan^2\alpha\,2\dot{\phi}$

$= mz^2\tan^2\alpha\,\dot{\phi}$

$p_\phi = \text{constant} = mz^2\tan^2\alpha$.

$\dot{\phi} = \frac{c/m}{z^2\tan^2\alpha} = \frac{A}{z^2\tan^2\alpha}$ (i), $A = \frac{c}{m} = \text{constant}$.

- L-Eqn for z:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$.

$\Rightarrow \frac{d}{dt}(\dot{z}^2\sec^2\alpha) - z\tan^2\alpha\,\dot{\phi}^2 + g = 0$ (ii)

- Circular Orbit

$z = h$; $\rho = \text{constant}$, $\dot{\rho} = \ddot{\rho} = 0$

$\dot{z} = \ddot{z} = 0$.

With $\dot{z} = 0$, $z = h$ in (ii)

$-h\tan^2\alpha\,\dot{\phi}^2 + g = 0$

$\dot{\phi} = \omega = \sqrt{\frac{g}{h}\cot\alpha}$

• b) In general $\dot{\phi} = \frac{A}{z^2\tan\alpha}$

For the special case of circular orbit as:

$z = h$,

$\dot{\phi} = \sqrt{\frac{g}{h}\cot\alpha} = \frac{A}{z^2\tan\alpha}$, $A = \sqrt{g}\,h^{3/2}\tan\alpha$.

In general, $\dot{\phi} = \frac{\sqrt{g}\,h^{3/2}}{z^2}\cot\alpha$

- Consider small displacements

$z = h + u \qquad u << h$

$\dot{z} = \dot{u}, \; \ddot{z} = \ddot{u}$

From (ii)

$\ddot{z}\sec^2\alpha - z\tan^2\alpha\,\dot{\phi}^2 + g = 0.$

$\ddot{z}\sec^2\alpha - z\tan^2\alpha\,\dfrac{gh^3\cot^2\alpha}{z^4} + g = 0.$

$\ddot{u}\sec^2\alpha - \dfrac{gh^3}{(h+u)^3} + g = 0$

$\ddot{u}\sec^2\alpha - \dfrac{gh^3}{h^3}\left(1 + \dfrac{u}{h}\right)^{-3} + g = 0$

$\ddot{u}\sec^2\alpha - g\left(1 - \dfrac{3u}{h}\right) + g = 0.$

$\ddot{u}\sec^2\alpha + \dfrac{3g}{h}u = 0.$

$\ddot{u} + \dfrac{3g}{h}\cos^2\alpha\, u = 0.$

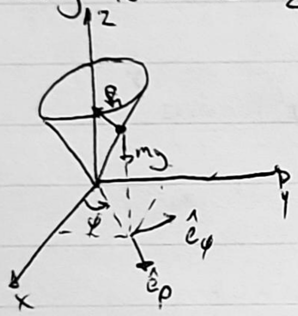
$\ddot{u} + \Omega^2 u = 0$

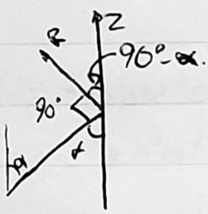
$\Omega = \sqrt{\dfrac{3g}{h}}\cos\alpha$

Time Period

$T = \dfrac{2\pi}{\Omega} = \dfrac{2\pi}{\cos\alpha}\sqrt{\dfrac{h}{3g}}$

- Using Newtons 2^{nd} Law





2^{nd} law along z-direction

$m\ddot{z} = R\sin\alpha - mg$ ①.

2^{nd} law along $\hat{e}_r$-direction

$m(\ddot{\rho} - \rho\dot{\phi}^2) = -R\cos\alpha$ ②.

② $\Rightarrow R = m(\rho\dot{\phi}^2 - \ddot{\rho})/\cos\alpha$.

Sub in ① $\Rightarrow m\ddot{z} = \dfrac{m(\rho\dot{\phi}^2 - \ddot{\rho})\sin\alpha}{\cos\alpha} - mg.$

$= m(\rho\tan\alpha\,\dot{\phi}^2 - \ddot{\rho}\tan\alpha) - mg.$

Acceleration in the $\hat{e}_\phi$ direction

$\rho^2\ddot{\phi} + 2\rho\dot{\rho}\dot{\phi} = \dfrac{d}{dt}(\rho^2\dot{\phi})$

$m\dfrac{d}{dt}(\rho^2\dot{\phi}) = 0. \;,\; \rho^2\dot{\phi} = A$ (constant).

$\dot{\rho} = \dfrac{A}{\rho^2} = \dfrac{A}{z^2\tan^2\alpha}$ (eqn (i))

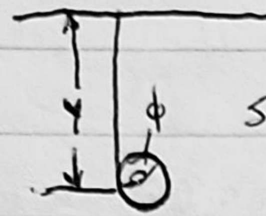
$m\ddot{z} = m(z\tan^2\alpha\,\dot{\phi}^2 - \ddot{z}\tan^2\alpha) - mg$

$m\ddot{z}\sec^2\alpha = mz\tan^2\alpha\,\dot{\phi} - mg.$

$\ddot{z}\sec^2\alpha = z\tan^2\alpha\,\dot{\phi} - g.$ (same as eqn (iii)).

• Hints for the other sample problems

- ③



suppose at $t=0$, $\phi=0$

$y = a\phi$

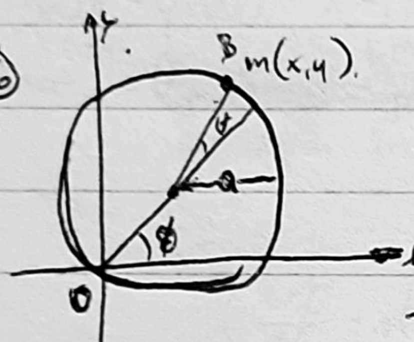
$\omega \Rightarrow \dot{\phi} = \frac{\dot{y}}{a}$, $a \Rightarrow$ radius

$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2$

L-eqn $\left(m + \frac{I}{a^2}\right)\ddot{y} - mg = 0$

$I = \frac{1}{2}ma^2$, $\ddot{y} = \frac{2}{3}g$

- ⑥



$x = a\cos\phi + a\cos(\theta+\phi)$

$y = a\sin\phi + a\sin(\theta+\phi)$

$\dot{\phi} = \omega$, $\phi = \omega t$

P.E. $= V = 0$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Soln of pendulum
$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

L-eqn $\Rightarrow$ $\ddot{\theta} + \omega^2\sin\theta = 0$.

↳ means oscillates about OA like a pendulum of length $l = \frac{g}{\omega^2}$.

Theorem involving K.E.

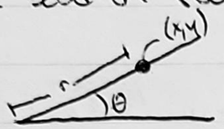
- N-particles in 3D

$$T = \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha i}^2 \rightarrow x_{\alpha i};\ i\text{th cartesian coordinates of particle } \alpha.$$

- $x_{\alpha i} = x_{\alpha i}(q_j, t)$

$j = 1, 2 \ldots s$ → no. of degrees of freedom (i.e. independent gen. position co-ords.

- $\dot{x}_{\alpha i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j + \frac{\partial x_\alpha}{\partial t}$
- Eg. Bead on rod



$\dot\theta = \omega$, $x_1 = x = r\cos\theta = r\cos\omega t$

$\theta = \omega t$ $\quad x_2 = y = r\sin\theta = r\sin\omega t$

$$\dot{x}_1 = \dot{r}\cos\omega t - r\sin\omega t\,\omega$$
$$= \dot{r}\left.\frac{\partial x}{\partial r}\right)_t + \left.\frac{\partial x}{\partial t}\right)_r$$

- $\dot{x}_{\alpha i}^2 = \left(\sum_{j=1}^{s}\frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j + \frac{\partial x_{\alpha i}}{\partial t}\right)\left(\sum_{i=1}^{s}\frac{\partial x_{\alpha i}}{\partial q_k}\dot{q}_k + \frac{\partial x_{\alpha i}}{\partial t}\right)$

$$= \sum_{j=1}^{s}\sum_{k=1}^{s}\left(\frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}\dot{q}_j\dot{q}_k\right) + 2\frac{\partial x_{\alpha i}}{\partial t}\left(\sum_{j=1}^{s}\frac{\partial x_{\alpha i}}{\partial q_i}\dot{q}_j\right) + \left(\frac{\partial x_{\alpha i}}{\partial t}\right)^2$$

- $T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j b_j\dot{q}_j + c$

independent of $\dot{q}_j$'s

$$a_{jk} = \sum_\alpha\sum_i \frac{1}{2}m_\alpha \frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}$$
$$b_j = \sum_\alpha\sum_i m_\alpha \frac{\partial x_{\alpha i}}{\partial t}\frac{\partial x_{\alpha i}}{\partial q_j}$$
$$c = \sum_\alpha\sum_i \frac{1}{2}m_\alpha\left(\frac{\partial x_{\alpha i}}{\partial t}\right)^2$$

- $T = \sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha\sum_{i=1}^{3}\dot{x}_{\alpha i}^2$

$T\{\dot{q}_j, j = 1\ldots s, t\}$

- If the constraints are t-independent (scleronomic)

$\left(\frac{\partial x_{\alpha i}}{\partial t}\right) = 0 \Rightarrow b_j = 0$ and $c = 0$.

- $T = \sum_{j,k=1}^{s} a_{jk}\dot{q}_j\dot{q}_k$ → a homogeneous quadratic function of (i;) generalized velocities, $\dot{q}_j$'s.

- Simple Examples

 - (r,θ): $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot\theta^2)$

$$= \underbrace{\tfrac{1}{2}m}_{a_{11}}\dot{q}_1^2 + \underbrace{\tfrac{1}{2}mq_1^2}_{a_{22}}\dot{q}_2^2 \qquad q_1 = r,\ q_2 = \theta.$$

 - $(\overset{q_1}{r}, \overset{q_2}{\theta}, \overset{q_3}{\phi})$: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2)$.

$$= \underbrace{\tfrac{1}{2}m}_{a_{11}}\dot{q}_1^2 + \underbrace{\tfrac{1}{2}mq_1^2}_{a_{22}}\dot{q}_2^2 + \underbrace{\tfrac{1}{2}mq_1^2\sin^2 q_2}_{a_{33}}\dot{q}_3^2$$

- In more complicated cases, terms $\dot{q}_1, \dot{q}_2, \dot{q}_2\dot{q}_3$, may appear
- Euler's Theorem: If $f(\{y_k, k=1,2\ldots d\})$ is a homogeneous fxn of the variables y_k of degree n, then

$$\sum_{k=1}^{d} y_k\left(\frac{\partial f}{\partial y_k}\right) = nf.$$

- Examples: $F(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2$ is a homogeneous fxn of degree 2 in $x_1 x_2$. (A)
  - $F(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2 + dx_1 + ex_2 + f$ is not (B)
  - For A: $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} = 2f.$

    B: - - - - - - $\neq 2f$
  - from (ii), via Euler's Theorem

$$\sum_{l=1}^{s} \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2T.$$

  - From (ii)

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_{jk} a_{jk}\underbrace{\left(\frac{\partial \dot{q}_j}{\partial \dot{q}_l}\right)}_{\delta_{jl}}\dot{q}_k + \sum_{jk} a_{jk}\dot{q}_j\underbrace{\left(\frac{\partial \dot{q}_k}{\partial \dot{q}_l}\right)}_{\delta_{kl}}$$

$$= \sum_k a_{lk}\dot{q}_k + \sum_j a_{jl}\dot{q}_j$$

$$\sum_{l=1}^{s} \dot{q}_l\frac{\partial T}{\partial \dot{q}_l} = \underbrace{\sum_{kl} a_{lk}\dot{q}_k\dot{q}_l}_{T} + \underbrace{\sum_{jl} a_{jl}\dot{q}_j\dot{q}_l}_{T} = 2T \rightarrow$$

For t-independent constraints, T is a generalized quadratic function of gen velocities. For t-dependent (rheonomic) constraints, it is not.

Hamiltonian Formulation

- Define a fxn: $H = \sum_j \dot{q}_j \phi_j - L$, $\phi_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$, $V(\{q_j\}, t)$

$$= \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$= \underbrace{\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}}_{2T} - L$$

$$= 2T - L = 2T - T + V = T + V = E \leftarrow \text{total mechanical system}$$

- Even if L & H both depend explicitly on 't' via $V(\{q_j, j=1\ldots s\}, t)$, $H = E$, as long as the constraints on the position coordinates are t-independent.

- By using $\frac{\partial L}{\partial \dot{q}_j} = p_j$ in $H = \sum_j \dot{q}_j p_j - L$, we can write

$H = H(\{q_j, p_j\}, t)$

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \quad (1)$$

- Also since $H = \sum_j p_j \dot{q}_j - L(\{q_j, \dot{q}_j\}, t)$.

$$dH = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - \left(\sum_j \frac{\partial L}{\partial q_i} dq_i + \sum_j \overbrace{\frac{\partial L}{\partial \dot{q}_j}}^{p_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt\right).$$

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \quad (2)$$

- From (1) & (2)

$$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} \quad ; \quad -\frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial q_i} \overset{L\ eqn.}{=} \frac{d}{dt}\left(\overbrace{\frac{\partial L}{\partial \dot{q}_j}}^{p_j}\right)$$

- $\boxed{\frac{\partial H}{\partial p_j} = \dot{q}_j \quad -\frac{\partial H}{\partial q_j} = \dot{p}_j}$ ← Hamitonians equations of motion

- In addition $\frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$.

- The dynamics ~~of~~ of a systam and with 's' degrees of freedom, can be described via.

~~i) L eqn $L = L(\{q_j, \dot{q}_j\}, t)$~~

- i) L-eqn $L = L(\{q_j, \dot{q}_j, j = 1 \ldots s\}, t)$

→ 's' 2nd order d.e in q_js.

- or equivalently
- ii) H-eqns $H = H(\{q_j, p_j\}, t)$.

→ $\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \dot{p}_j = \frac{\partial H}{\partial q_j}$ → '2s' 1 order d.e.

- (A) $H = H(\{q_j, p_j\}, t)$

$$\frac{dH}{dt} = \sum_j \left(\frac{\partial H}{\partial q_j}\dot{q}_j + \frac{\partial H}{\partial p_j}\dot{p}_j\right) + \frac{\partial H}{\partial t}.$$

$$= \sum_j \left[(-\dot{p}_j)\dot{q}_j + \dot{q}_j \dot{p}_j\right] + \frac{\partial H}{\partial t}.$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

$\frac{dH}{dt} = 0$ if $\frac{\partial H}{\partial t} = 0$ ⇒ H is a const. of motion unless H(& L) depends explicity on T.

• Examples

- ① S.H.O in 1D.

$$T=\frac{1}{2}m\dot{x}^2 \quad , \quad V=\frac{1}{2}kx^2$$

$$L=T-V=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x}$$

$$\dot{x}=\frac{p_x}{m} \rightarrow T=\frac{1}{2}m\dot{x}^2=\frac{1}{2}m\left(\frac{p_x}{m}\right)^2=\frac{p_x^2}{2m}$$

$$H=\sum_i p_i\dot{q}_i-L$$

$$=p_x\dot{q}_x-L$$

$$=p_x\frac{p_x}{m}-\left(\frac{p_x^2}{2m}-\frac{1}{2}kx^2\right)$$

$$H=\frac{p_x^2}{2m}+\frac{1}{2}kx^2=T+V=E$$

H- eqns.

$$\dot{x}=\frac{\partial H}{\partial p_x}=\frac{p_x}{m} \quad ①$$

$$\dot{p}_x=-\frac{\partial H}{\partial x}=-kx \quad ②$$

$$① \Rightarrow \ddot{x}=\frac{\dot{p}_x}{m}=\frac{-kx}{m} \rightarrow \ddot{x}+\omega x=0 \quad , \omega=\sqrt{\frac{k}{m}}$$

- ② Motion ~~under~~ central force.

$$\vec{F}(\vec{r})=-\vec{\nabla}V$$

$$=-\vec{\nabla}V(r)=-\frac{\partial}{\partial r}V(r)\hat{r}$$

$\hat{r}$ → unit vector along the radial direction.

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right) \quad , \quad L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-V(r)$$

- Step 1: calculate p_j's

$$p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r}$$

$$p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$$

- Step 2: a) Calculate H

$$H=\sum_j p_j\dot{q}_j-L$$

$$=p_r\dot{r}\ p_\theta\dot{\theta}-\left[\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-V(r)\right]$$

b.) Write H in terms of (r,θ,p_r,p_θ)

$$H=p_r\frac{p_r}{m}+p_\theta\frac{p_\theta}{mr^2}-\left[\frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2+r^2\left(\frac{p_\theta}{mr^2}\right)^2\right)-V(r)\right]$$

$$H=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}+V(r)=E$$

- Step 3: Write H- eqns.

$$\dot{q}_j=\frac{\partial H}{\partial p_j} \quad \dot{p}_j=-\frac{\partial H}{\partial q_j}$$

$$\dot{r}=\frac{p_r}{m} \quad , \quad \dot{\theta}=\frac{p_\theta}{mr^2}$$

$$\dot{p}_r=-\frac{\partial H}{\partial r}=\frac{p_\theta^2}{mr^3}-\frac{\partial V}{\partial r}$$

$$\dot{p}_\theta=-\frac{\partial H}{\partial \theta}=0 \Rightarrow \theta \text{ is cyclic (ignorable).}$$

p_θ = const. of motion

$$=mr^2\dot{\theta}$$

Radial Eqn of Motion.

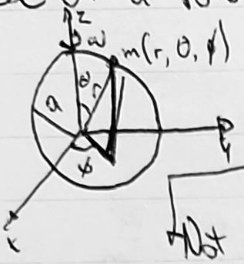
$$-\frac{\partial V}{\partial r}=\dot{p}_r-\frac{p_\theta^2}{mr^3}$$

$$=m\ddot{r}-\frac{1}{mr^3}m^2r^4\dot{\theta}^2$$

$$=m(\ddot{r}-r\dot{\theta}^2)$$

$$=ma_r$$

- An example where the H might not equal energy.
  - Bead on a rotating hoop.



$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$\dot{\phi} = \omega$, $\phi = t\omega$, $r = a$, $\dot{r}$ 0.

$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\omega^2$

Not a homogeneous quadratic function of $\dot{q}$'s

- T-dependent (rheonomic) constraint.
- Only one independent co-ordinate θ

$\frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}$

$\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}^2$

$\neq 2T.$

- $L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\omega^2 - V(\theta)$

$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}.$

$H = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L$

$= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{m}{2}a^2\sin^2\theta\,\omega^2 + V(\theta)$

$E = T + V(\theta) = \frac{m}{2}a^2\dot{\theta}^2 + \frac{m}{2}a^2\sin^2\theta\,\omega^2 + V(\theta).$

- $E \neq H$

$E - H = ma^2\sin^2\theta\,\omega^2$

- H is not an explicit function of t.

$\Rightarrow \frac{\partial H}{\partial t} = 0 \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow$ H is a constant of motion.

- Conversely, E is not a constant of motion.
- Notes: ① If V is an explicit function of 't' (e.g. in a situation where the centre of the hoop is driven up & down with time)

$H = H(t) \Rightarrow \frac{\partial H}{\partial t} = 0$, H will not be conserved & E will not be conserved.

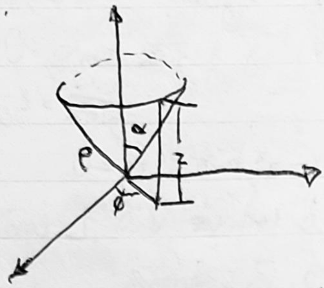
② Let $V = V(t)$, but not-dependent constraint.

$H = E$

$H(t) = H \Rightarrow$ is not conserved.

$\frac{\partial H}{\partial t} \neq 0.$

• Problem.



$\frac{\rho}{z} = \tan\alpha$

$\rho = z\tan\alpha$

$\dot{\rho} = \dot{z}\tan\alpha$

Work with Z independent variables ρ & ϕ.

$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2)$

- $T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \left(\frac{\dot{\rho}}{\tan\alpha}\right)^2\right)$
  $= \frac{1}{2}m(\rho^2\dot{\theta}^2 + \dot{\rho}^2\operatorname{cosec}^2\alpha)$

  $V = mgz$
  $= mg\frac{\rho}{\tan\alpha}$

  $L = T - V = \frac{1}{2}m(\rho^2\dot{\theta}^2 + \dot{\rho}^2\csc^2\alpha) - mg\frac{\rho}{\tan\alpha}$

- $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\csc^2\alpha$
  $\dot{\rho} = \frac{p_\rho}{m\csc^2\alpha} = \frac{p_\rho}{m}\sin^2\alpha$

  $p_\phi = \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta}$
  $\dot{\phi} = \frac{p_\phi}{m\rho^2}$

• $H = \sum_i p_i\dot{q}_i - L$
  $= p_\rho\dot{\rho} + p_\phi\dot{\phi} - L$

  $H(\rho, \phi, p_\rho, p_\phi) = \frac{p_\rho^2\sin^2\alpha}{2m} + \frac{p_\phi^2}{2m\rho^2} + mg\rho\cot\alpha$

- H-Eqns.
  $\dot{\rho} = \frac{\partial H}{\partial p_\rho}$, $\dot{p}_\rho = -\frac{\partial H}{\partial \rho}$
  $\left[\dot{q}_i = \frac{\partial H}{\partial p_i}\right.$, $\left.\dot{p}_i = -\frac{\partial H}{\partial q_i}\right]$
  $\dot{\phi} = \frac{\partial H}{\partial p_\phi}$, $\dot{p}_\phi = -\frac{\partial H}{\partial \phi}$

- $\frac{\partial H}{\partial p_\rho} = \frac{p_\rho\sin^2\alpha}{m} = \dot{\rho} \Rightarrow p_\rho = \frac{m\dot{\rho}}{\sin^2\alpha}$
  $\frac{\partial H}{\partial \rho} = -\frac{p_\phi^2}{m\rho^3} + mg\cot\alpha = -\dot{p}_\rho$
  $= -\frac{m\ddot{\rho}}{\sin^2\alpha}$ - ①

  $\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2} = \dot{\phi} \Rightarrow p_\phi = m\rho^2\dot{\phi}$
  $\frac{\partial H}{\partial \phi} = 0 \Rightarrow \dot{p}_\phi = 0 \Rightarrow p_\phi = m\rho^2\dot{\phi} =$ constant - ②
  ϕ is cyclic/ignorable.

- ② $m\rho^2\dot{\phi} = \text{constant} = c$.
  $\dot{\phi} = \frac{c/m}{\rho^2} = \frac{A}{z^2\tan^2\alpha}$. (derive by L-approach).

  ① $-\frac{p_\phi^2}{m\rho^3} + mg\cot\alpha + \frac{m\ddot{\rho}}{\sin^2\alpha} = 0$

  $\rho = z\tan\alpha$.
  $p_\phi = m\rho^2\dot{\phi}$
  $\ddot{\rho} = \ddot{z}\tan\alpha$

  $\frac{p_\phi^2}{m\rho^3} = \frac{m^2\rho^4\dot{\phi}^2}{m\rho^3} = m\rho\dot{\phi}^2 = mz\tan\alpha\,\dot{\phi}^2$

- ① $\Rightarrow$ $-mz\tan\alpha\,\dot{\phi}^2 + mg\cot\alpha + m\ddot{z}\tan\alpha/\sin^2\alpha = 0$

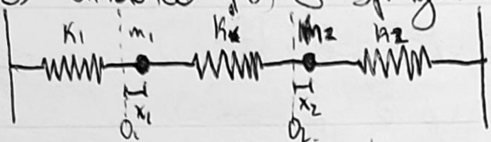
$$-z\tan^2\alpha\,\dot{\phi}^2 + g + \frac{\ddot{z}}{\sin^2\alpha}\tan^2\alpha = 0.$$

$$\frac{\tan^2\alpha}{\sin^2\alpha} = \sec^2\alpha.$$

$$\therefore\ \ddot{z}\sec^2\alpha - z^2\tan\alpha\,\dot{\phi}^2 + g = 0.$$

(As derived earlier via L-eqn).

- 2 masses attached to 3 springs



$x_{1,2}$: distance from equilibrium positions $O_{1,2}$ of $m_{1,2}$.

- $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'(x_1-x_2)^2$

$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'(x_1-x_2)^2$

- L-Eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0.$$

$$m_1\ddot{x}_1 + k_1x_1 + k'(x_2-x_1) = 0.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0.$$

$$m_2\ddot{x}_2 + k_2x_2 + k'(x_2-x_1) = 0.$$

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k'(x_2-x_1) \\ m_2\ddot{x}_2 &= -k_2x_2 + k'(x_2-x_1) \end{aligned}\right\}\ 2^{nd} \text{ order d.e's.}$$

- H-Eqns

$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad \dot{x}_1 = \frac{p_1}{m_1}$

$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 \qquad \dot{x}_2 = \frac{p_2}{m_2}$

$H = \sum p_i\dot{q}_i - L$

$= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k_1x_1^2}{2} + \frac{k_2x_2^2}{2} + \frac{k'(x_1-x_2)^2}{2}$

$= T + V.$

$\dot{q}_i = \frac{\partial H}{\partial p_i}\ ,\ \dot{p}_i = -\frac{\partial H}{\partial q_i}$

$$\left.\begin{aligned} \frac{\partial H}{\partial p_1} &= \frac{p_1}{m} = \dot{x}_1 \\ \frac{\partial H}{\partial p_2} &= \frac{p_2}{m_2} = \dot{x}_2 \\ \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = -k_1x_1 - k'(x_1-x_2). \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -k_2x_2 + k'(x_1-x_2). \end{aligned}\right\}\ 4\ 1^{st} \text{ order d.e.s.}$$

- Solution of the equations of motion.

- Let $x_1 = a_1 \cos(\omega t + \delta_1)$

$x_2 = a_2 \cos(\omega t + \delta_2)$

$m\ddot{x}_1 = k'(x_2 - x_1) - k_1 x_1 = k' x_2 - (k' + k_1) x_1$

- $-m_1\omega^2 x_1 = k' x_2 - (k' + k_1) x_1$

$[(k' + k_1) - m_1\omega^2] x_1 - k' x_2 = 0$ ①

$m\ddot{x}_2 = k'(x_1 - x_2) - k_2 x_2 = k'(x_1) - (k' + k_2) x_2$

$-m_2\omega^2 x_2 = k' x_1 - (k' + k_2) x_2$

$-k' x_1 + [(k' + k_2) - m_2\omega^2] x_2 = 0$ ②

- For a nontrivial solution of ① & ②

$$\begin{vmatrix} (k' + k_1) - m_1\omega^2 & -k' \\ -k' & (k' + k_2) - m_2\omega^2 \end{vmatrix} = 0$$

Relativistic Lagrangian

- Free particle in cartesian coordinates

- $p_i = \dfrac{mv_i}{\sqrt{1 - \beta^2}}$ $\quad \beta = \dfrac{v}{c}$ $\quad v^2 = v_1^2 + v_2^2 + v_3^2$

$v_i = \dot{x}_i$

- $KE = \underbrace{\dfrac{mc^2}{\sqrt{1 - \beta^2}}}_{\text{Total energy of the free particle.}} - mc^2$ $\quad m = \text{rest mass}$

$= mc^2 (1 - \beta)^{-\frac{1}{2}} - mc^2$ $\quad$ when $\beta \ll 1 \Rightarrow (1 + \beta)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}\beta^2$ using $(1+x)^n$ for $|x| < 1$

- Non relativistic $KE = mc^2 + \frac{1}{2} mc^2\beta^2 - mc^2$

$= \frac{1}{2} m v^2$

Binomial Expansion

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$

$(1 - \beta^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots$

- $mc^2 (1 - \beta^2)^{-\frac{1}{2}} = mc^2 \left(1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4\right)$

$= mc^2 + \frac{1}{2} m v^2 + \frac{3}{8} m \frac{v^4}{c^2} + \dots$

- $\therefore KE = mc^2 (1 - \beta^2)^{-\frac{1}{2}} - mc^2$

$= \underbrace{\frac{1}{2} m v^2}_{\text{non-relativistic}} + \underbrace{\frac{3}{8} mc^2 \left(\frac{v}{c}\right)^4}_{\text{leading relativistic correction to K.E.}}$

- A suitable relativistic Lagrangian for a single particle acted on by a conservative force independent of velocity

- $L = -mc^2 \sqrt{1 - \beta^2} - \underbrace{V(\{x_i, i = 1, 2, 3\})}_{\text{dependent only on position} \Rightarrow \frac{\partial V}{\partial \dot{x}_i} = 0}$

- Note: $L \neq T-V$, but a function which, when substituted in L-eqn gives the correct equation of motion

- $\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial v_i} = -mc^2 \frac{1}{2\sqrt{1-\beta^2}}\left(-\frac{\partial \beta^2}{\partial v_i}\right)$
- $\frac{\partial \beta^2}{\partial v_i} = \frac{\partial \beta^2}{\partial v_i^2}\frac{\partial v_i^2}{\partial v_i} = \frac{\partial}{\partial v_i^2}\frac{(v_1^2+v_2^2+v_3^2)}{c^2} 2v_i$

$= \frac{\partial}{\partial v_i^2}\left(\sum_{j=1}^{3}\frac{v_j^2}{c^2}\right)2v_i$

$= \sum_{j=1}^{3}\frac{\delta_{ji}}{c^2}\cdot 2v_i \qquad \left(\frac{\partial v_i^2}{\partial v_j^2} = \delta_{ji}\right)$

$= 2\frac{v_i}{c^2}$

- $\frac{\partial L}{\partial \dot{x}_i} = \frac{mc^2}{\sqrt{1-\beta^2}}\frac{v_i}{c^2} = \frac{mv_i}{\sqrt{1-\beta^2}} = p_i \leftarrow$ momentum conjugate to x_i

- L-Equation:
  - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = \frac{d}{dt}(p_i) + \frac{\partial V}{\partial x_i} = 0$
  - $\therefore \frac{dp_i}{dt} = \dot{p}_i = -\frac{\partial V}{\partial x_i} = F$ (Newtons 2^{nd} law)

- For a system of N-particles in 3-D
  - $L = \sum_{j=1}^{N}\left(-mc^2\sqrt{1-\beta_j^2}\right) - V(\{x_{1j}, x_{2j}, x_{3j}\}, j=1,\dots N)$
  - $\beta_j^2 = \frac{v_j^2}{c^2} = \frac{1}{c^2}(v_{1j}^2 + v_{2j}^2 + v_{3j}^2)$
  - $\frac{\partial L}{\partial v_{ij}} = p_{ij} \Rightarrow$ ith cartesian component of the jth particle

- Motion of a single particle under a constant force (1D)
  - $F_x = ma = -\frac{\partial V}{\partial x} \Rightarrow V = -max$ (with $V=0$ for $x=0$)

    constant force per unit mass.
  - $L = -mc^2\sqrt{1-\beta^2} + max \qquad \beta = \frac{v}{c} = \frac{\dot{x}}{c}$
  - $\frac{\partial L}{\partial \dot{x}} = -mc^2\frac{1}{2\sqrt{1-\beta^2}}\left(-2\beta\frac{1}{c}\right)$

    $= \frac{m\dot{x}}{\sqrt{1-\beta^2}}$
  - $\frac{\partial L}{\partial x} = ma$
  - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\beta^2}}\right) - ma = 0 \quad , \beta = \frac{\dot{x}}{c}$

    $\frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{a}{c} \qquad \dot{x} = v = \beta c$

    $\Rightarrow \frac{\beta}{\sqrt{1-\beta^2}} = \frac{at+\alpha}{c} \leftarrow$ constant of integration.

    dimension of $\frac{\alpha}{a}$ = dimension of 't'
  - Simplify to get

    $\frac{\dot{x}}{c} = \beta = \frac{at+\alpha}{\sqrt{c^2+(at+\alpha)^2}} \rightarrow ①$
  - $x - x_0 = c\int_0^t \frac{at'+\alpha}{\sqrt{c^2+(at'+\alpha)^2}}dt'$
  - $\sqrt{c^2+(at'+\alpha)^2} = \omega, \ d\omega = \frac{2(at'+\alpha)}{2\sqrt{c^2+(at'+\alpha)^2}}a\,dt'$
  - $\int_{\sqrt{c^2+\alpha^2}}^{\sqrt{c^2+(at+\alpha)^2}}\frac{d\omega}{a} = \frac{c}{a}\left[\sqrt{c^2+(at+\alpha)^2} - \sqrt{c^2+\alpha^2}\right] \rightarrow ②$
  - Let $x=0$ at $t=0$, $x_0 = 0$

    $\dot{x}(0) = v_0 = 0$, from ① $\frac{c\alpha}{\sqrt{c^2+\alpha^2}} = 0$, $\therefore \alpha = 0$

- $x = \frac{c}{a}\left[\sqrt{c^2+(at)^2} - c\right]$ → (3)
- $\left(x+\frac{c^2}{a}\right)^2 = \frac{c^4}{a^2} + c^2t^2$

$\left(x+\frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2}$

$\frac{\left(x+\frac{c^2}{a}\right)^2}{a'^2} - \frac{t^2}{b'^2} = 1$ → $a' = \frac{c^2}{a}$, $b' = \frac{c}{a}$ correct dimensions?

- This is a hyperbola in (x, t)

- For a nonrelativistic particle under a constant force
  - $x = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}at^2$ $(x_0 = 0, v_0 = 0)$
  - when $at \ll c$, (3) → $x = \frac{c}{a}\left[c\left(1+\left(\frac{at}{c}\right)^2\right)^{\frac{1}{2}} - c\right]$

$= \frac{c}{a}\left[c\left(1+\frac{1}{2}\left(\frac{at}{c}\right)^2\right) - c\right]$

$= \frac{c}{a}\,\frac{1}{2}\,\frac{a^2t^2}{c^2}$

$= \frac{1}{2}at^2$, $\frac{at}{c} \ll 1$ or $t \ll \frac{c}{a}$

Poisson Bracket

- System with f degrees of freedom
- Hamiltonian,
  - $H = H(q_1, q_2, \dots, q_f, p_1, p_2 \dots p_f, t)$.
  - Let $A = A(q_1 \dots q_f, p_1 \dots p_f, t)$ be another dynamic variable
- Example: Angular Momentum
  - $\vec{L} = \vec{r} \times \vec{p}$

$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, $L_z = xp_y - yp_x$

$L_z = L_z(x, y, z, p_x, p_y, p_z)$

- Time rate ~~~~ of change of A:
  - $A = A(\{q_i\}, \{p_i\}, t)$
  - $\frac{dA}{dt} = \sum_{i=1}^{f}\left[\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i\right] + \frac{\partial A}{\partial t}$
- H-Eqns
  - $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
  - $\frac{dA}{dt} = \underbrace{\sum_{i=1}^{f}\left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i}\right)} + \frac{\partial A}{\partial t}$

$= \{A, H\} + \frac{\partial A}{\partial t}$

Poisson Brackets of A & H

Note: the order of (A & H) & (q_i & p_i)

- In general, if
  - $f_1 = f_1(\{q_i\}, \{p_i\}, i=1 \ldots f, t)$
  - $f_2 = f_2(\text{''} \ \ \text{''} \ \ \text{''} \ \ \text{''})$
  - $\{f_1, f_2\} = \sum_{i=1}^{f} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right)$
- Fundament Poisson Brackets
  - $\{q_i, q_j\} = \sum_{k=1}^{f} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right)$

$$\frac{\partial q_i}{\partial q_k} \to \delta_{ik}, \quad \frac{\partial q_j}{\partial p_k} = 0, \quad \frac{\partial q_i}{\partial p_k} \to 0 \ \ (\text{sic: } \delta_{jk}), \quad \frac{\partial q_j}{\partial q_k} = 0$$

$= 0$

  - $\{q_i, p_j\} = \sum_{k=1}^{f} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right)$

$$\frac{\partial q_i}{\partial q_k} \to \delta_{ik}, \quad \frac{\partial p_j}{\partial p_k} \to \delta_{jk}, \quad \frac{\partial q_i}{\partial p_k} \to 0, \quad \frac{\partial p_j}{\partial q_k} \to 0$$

$= \sum_{k=1}^{f} \delta_{ik} \delta_{jk}$

$= \delta_{ij}$

  - $\{p_i, p_j\} = 0$ (similar to $\{q_i, q_j\}$.

Review →

- Poisson Bracket Review
  - $\{f_1, f_2\} = \sum_{i=1}^{f} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right)$
- Fundamental Poisson Bracket.
  - $\{q_i, q_j\} = \{p_i, p_j\} = 0$
  - $\{q_i, p_j\} = \delta_{ij}$
- Suggests
  - $\underset{\text{Quantum Mechanics}}{\frac{1}{i\hbar}[\hat{A}, \hat{B}]} \to \underset{\text{Classical Mechanics}}{\{\hat{A}, \hat{B}\}}$.

- Arbitrary fxn A of p's & q's
  - $\{q_i, A\} = \sum_{k=1}^{f} \left(\underbrace{\frac{\partial q_i}{\partial q_k}}_{=\delta_{ik}} \frac{\partial A}{\partial p_k} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{=0} \frac{\partial A}{\partial q_k} \right) = \frac{\partial A}{\partial p_i}$
  - $\{p_i, A\} = \sum_{k=1}^{f} \left(\underbrace{\frac{\partial p_i}{\partial q_k}}_{0} \frac{\partial A}{\partial p_k} - \underbrace{\frac{\partial p_i}{\partial p_k}}_{=\delta_{ik}} \frac{\partial A}{\partial q_k} \right) = -\frac{\partial A}{\partial q_i}$
  - Using these rewrite the H-Equation to:
    - $\dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}$, $\{p_i, H\} = 0$, p_i = constant of momentum
    - $\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}$, $\{q_i, H\} = 0$, q_i = " " position.

(2)

- ② is a special case of the more general equation ①
  - $\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$ ①
  - If A is a constant of motion, $\frac{dA}{dt} = 0$,

    $$\frac{\partial A}{\partial t} = -\{A, H\}$$

    - If A doesn't depend on 't' explicitly, A is a constant of motion, if $\{A, H\} = 0$
- Theorem - If F_1 & F_2 are two integral (constants) of motion, then $\{F_1, F_2\}$ is also an integral ~~of~~ motion
- Proof - Since F_1 & F_2 are constants of motion

$$\frac{\partial F_1}{\partial t} = \{H, F_1\} \qquad \frac{\partial F_2}{\partial t} = \{H, F_2\}$$

$$\frac{\partial \{F_1 F_2\}}{\partial t} = \left\{\frac{\partial F_1}{\partial t}, F_2\right\} + \left\{F_1, \frac{\partial F_2}{\partial t}\right\}$$

$$= \{\{H, F_1\}, F_2\} + \{F_1, \{H, F_2\}\} \quad \text{(A1)}$$

$$= \{H, \{F_1, F_2\}\} + \{F_1, \{F_2, H\}\} + \{F_2, \{H, F_1\}\} = 0 \quad \leftarrow \text{(Jacobi Identity)}$$

$$\{H, \{F_1, F_2\}\} = -\{F_1, \{F_2, H\}\} - \{F_2, \{H, F_1\}\}$$

$$= \{F_1, \{H, F_2\}\} + \{\{H, F_1\}, F_2\} \quad \leftarrow \text{(A2)}$$

$$\therefore \frac{\partial \{F_1, F_2\}}{\partial t} = \{H, \{F_1, F_2\}\}$$

- This sequence cannot be repeated indefinitely, since in a dynamical problem involving f degrees of freedom, there are at most $2f$ constants.
- Some important dynamical variables:
  - Total linear momentum:

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = P_x \hat{e}_x + P_y \hat{e}_y + P_z \hat{e}_z$$

$$P_x = \sum_{i=1}^{N} p_{ix}, \quad P_y = \sum_{i=1}^{N} p_{iy}, \quad P_z = \sum_{i=1}^{N} p_{iz}$$

  - Total Angular Momentum:

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i$$

$$= L_x \hat{e}_x + L_y \hat{e}_y + L_z \hat{e}_z$$

$$L_x = \sum_{i=1}^{N} l_{ix} = \sum_{i=1}^{N} (y_i p_{iz} - z_i p_{iy})$$

$$L_y = \sum_{i=1}^{N} l_{iy} = \sum_{i=1}^{N} (z_i p_{ix} - x_i p_{iz})$$

$$L_z = \sum_{i=1}^{N} l_{iz} = \sum_{i=1}^{N} (x_i p_{iy} - y_i p_{ix})$$

• Let us assume that P_x & L_z are integrals of motion.

$$-\{p_x, L_x\} = \sum_{i=1}^{N}\Big(\underbrace{\frac{\partial p_x}{\partial x_i}\frac{\partial L_z}{\partial p_{ix}}}_{0} + \underbrace{\frac{\partial p_x}{\partial y_i}\frac{\partial L_z}{\partial p_{iy}}}_{0} + \underbrace{\frac{\partial p_x}{\partial z_i}\frac{\partial L_z}{\partial p_{iz}}}_{0} - \underbrace{\frac{\partial L_z}{\partial x_i}}_{p_{iy}}\underbrace{\frac{\partial p_x}{\partial p_{ix}}}_{1} - \underbrace{\frac{\partial L_z}{\partial y_i}\frac{\partial p_x}{\partial p_{iy}}}_{0} - \underbrace{\frac{\partial L_z}{\partial z_i}\frac{\partial p_x}{\partial p_{iz}}}_{0}\Big)$$

$$= -\sum_{i=1}^{N} p_{iy} = -P_y$$, ∴ P_y is also a constant of motion.

$$-\{p_x, p_y\} = \sum_{i=1}^{N}\Big(\underbrace{\frac{\partial p_x}{\partial x_i}\frac{\partial p_y}{\partial p_{ix}}}_{0} + \underbrace{\frac{\partial p_x}{\partial y_i}\frac{\partial p_y}{\partial p_{iy}}}_{0} + \underbrace{\frac{\partial p_x}{\partial z_i}\frac{\partial p_y}{\partial p_{iz}}}_{0} - \underbrace{\frac{\partial p_y}{\partial x_i}\frac{\partial p_x}{\partial p_{ix}}}_{0} - \underbrace{\frac{\partial p_y}{\partial y_i}\frac{\partial p_x}{\partial p_{iy}}}_{0} - \underbrace{\frac{\partial p_z}{\partial z_i}\frac{\partial p_x}{\partial p_{iz}}}_{0}\Big)$$

$= 0$, Since $\vec{P}$ does not involve position coordinates, no new constants of motion results.

$$-\{L_z, p_y\} = \sum_{i=1}^{N}\Big(\underbrace{\frac{\partial L_z}{\partial x_i}\frac{\partial p_y}{\partial p_{ix}}}_{0} + \underbrace{\frac{\partial L_z}{\partial y_i}}_{-p_{ix}}\underbrace{\frac{\partial p_y}{\partial p_{iy}}}_{1} + \frac{\partial L_z}{\partial z_i}\frac{\partial p_y}{\partial p_{iz}} + \dots\Big)$$

$$= -\sum_{i=1}^{N} p_{ix} = -P_x$$, which was already in the set of constants

- P_x, P_y, & L_z are involution, form a closed set under Poisson bracket operation.

• Time evolution of dynamical variables via Poisson brackets (unfolding theorem). Consider $\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$

For q & p, $\frac{\partial q}{\partial t} = \frac{\partial p}{\partial t} = 0$

- $\frac{dq}{dt} = \{q, H\}$

  $\frac{dp}{dt} = \{p, H\}$

• Consider Taylor (MacLaurin) series

- $q(t) = q(0) + \frac{dq}{dt}\Big)_{t=0} t + \frac{d^2q}{dt^2}\Big)_{t=0}\frac{t^2}{2!} + \dots \frac{d^nq}{dt^n}\Big)_{t=0}\frac{t^n}{n!}$
- $\frac{dq}{dt} = \{q, H\}$, $\frac{d^2q}{dt^2} = \frac{d}{dt}\{q, H\}$

  $= \{\{q, H\}, H\}$

- $q(t) = q_0 + \{q, H\}_{t=0}\, t + \{\{q, H\}, H\}_{t=0}\frac{t^2}{2!} + \dots$

• Define an operator T in general

- $T = \frac{d}{dt} = \{\ , H\} + \frac{\partial}{\partial t}$

• In general

- $A(t) = A(t=0) + t(TA)_{t=0} + \frac{t^2}{2!}(T^2A)_{t=0} + \dots$

  $= e^{tT}(A)|_{t=0}$

$q(t) = e^{tT}(q)|_{t=0}$, The Hamiltonian is seen as the generator of motion in (q,p) space.

$p(t) = e^{tT}(p)|_{t=0}$

- Linear motion under a constant force $F = ma = -\frac{\partial V}{\partial x}$

  - $V(x) - V(0) = -ma\int_0^x dx$

  $= -max$

  $V(x) = -max + V(0)$, set $V(0) = 0$.

  - $L = T - V = \frac{1}{2}m\dot{x}^2 + max$ $\quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$.

  $H = \dot{x}p - L$

  $= \frac{p^2}{m} - \frac{m}{2}\frac{p^2}{m^2} - max$

  $= \frac{p^2}{2m} - max \quad (H = T + V)$

  - $\frac{dx}{dt} = \{x, H\}$, $\frac{\partial p}{\partial t} = \{p, H\}$, $\Rightarrow \frac{\partial x}{\partial t} = \frac{\partial p}{\partial t} = 0$

  - ① $x(t) = x(0) + t\{x, H\}_{t=0} + \frac{t^2}{2!}\{\{x, H\}, H\} + \dots$

  $\Rightarrow \{x, H\} = \underset{1}{\frac{\partial x}{\partial x}} \cdot \underset{\frac{p}{m}}{\frac{\partial H}{\partial p}} - \frac{\partial H}{\partial x}\underset{0}{\frac{\partial x}{\partial p}} = \frac{p}{m}$

  $\{\{x, H\}, H\} = \frac{1}{m}\{p, H\}$

  $= \frac{1}{m}\left(\underset{0}{\frac{\partial p}{\partial x}}\frac{\partial H}{\partial p} - \underset{-ma}{\frac{\partial H}{\partial x}}\underset{1}{\frac{\partial p}{\partial p}}\right)$

  $= a \leftarrow$ constant.

  $\{a, H\} = 0 \Rightarrow$ all higher order terms in the expansion vanish.

  - $x(t) = x_0 + \frac{p}{m}t + a\frac{t^2}{2}$.

  - ② $\{p, H\} = \frac{\partial p}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial x}\frac{\partial p}{\partial p} = -(-ma) = ma \Rightarrow$ all higher order terms vanish

  $p(t) = \underset{p_0}{p(0)} + mat.$

Baker Campbell Relation.

- Canonical Transformation.

  - $H = H(q, p, t)$

  $q = q_1, q_2 \dots q_n$, $\quad \dot{q}_i = \frac{\partial H}{\partial p_i}$ —①

  $p = p_1, p_2 \dots p_n$, $\quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$

  - We may want to transform to a new set of generalized coordinates of

  $Q_j = Q_j(q, p, t)$ —②

  $P_j = P_j(q, p, t)$

  - and demand that there is a function $K(Q, P, t)$,

  $Q = Q_1, Q_2 \dots Q_n$, such that

  $P = P_1, P_2 \dots P_n$

  $\dot{Q}_j = \frac{\partial K}{\partial p_j}$, $\dot{P}_j = -\frac{\partial K}{\partial q_i}$ —③

- the K is a new Hamiltonian, with $\{(Q_i, P_i)\}$ constituting alternate canonically conjugate variables to describe the system
- Transformation ② satisfying ③ is called canonical
- To be canonical a transformation, must preserve the Poisson Brackets between two variables. Canonical transformations must preserve fundamental Poisson brackets:

$$\{Q_i, Q_j\} = 0 \quad , \quad \{P_i, P_j\} = 0$$
$$\{Q_i, P_j\} = \delta_{ij}$$

$y = \cos^{-1} x$
$\cos y = x$
$-\sin y \frac{dy}{dx} = 1$
$\frac{dy}{dx} = \frac{-1}{\sin y}$
$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

• Consider ~~the transformation~~
- $Q = \sqrt{e^{-2q} - p^2}$, canonical?
$P = \cos^{-1}(pe^{q})$
- $\{Q, Q\} = \{P, P\} = 0$
- $\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$
$= \left(\frac{-e^{-2q}}{\sqrt{e^{-2q} - p^2}}\right)\left(\frac{-e^{q}}{\sqrt{e^{-2q} - p^2}}\right)^{e} - \left(\frac{-p}{\sqrt{e^{-2q} - p^2}}\right)\left(\frac{-p}{\sqrt{e^{-2q} - p^2}}\right)$
$= 1.$

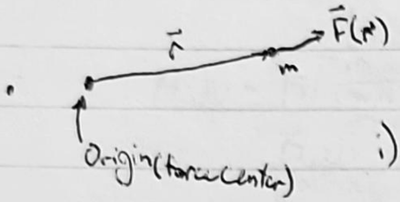
• Consider the transformation
- $Q_1 = q_1^2 + \lambda p_1^2$
$Q_2 = \frac{1}{2\lambda^2}\left[q_1^2 + q_2^2 + \lambda^2(p_1^2 + p_2^2)\right]$
$P_1 = \frac{1}{2\lambda}\left[\tan^{-1}\left(\frac{q_2}{\lambda p_2}\right) - \tan^{-1}\left(\frac{q_1}{\lambda p_1}\right)\right]$
$P_2 = \lambda \tan^{-1}\left(\frac{-q_2}{\lambda p_2}\right)$
- Consider the sets $\{Q_1, Q_1\}$, $\{Q_2, Q_2\}$, $\{P_1, P_1\}$, $\{P_2, P_2\}$, $\{Q_1, Q_2\}$, $\{P_1, P_2\}$, $\{Q_1, P_1\}$, $\{Q_2, P_2\}$, $\{Q_1, P_2\}$, $\{Q_2, P_1\}$
$= 0$ (each)

- $\{Q_1, P_2\} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_2}{\partial q_2}$
$= 0.$

• Consider
- $H = \frac{1}{2}(p^2 + q^2)$, let $m=1$, $k=1$.
$Q = \frac{1}{\sqrt{2i}}(q + ip)$, $H = -iQP$
$P = \frac{1}{\sqrt{2i}}(q - ip)$
- $H = -iQP$ (New Hamiltonian)
$\dot{Q} = \frac{\partial K}{\partial P} = iQ$
$\dot{P} = -\frac{\partial K}{\partial Q} = iP$

- $\{Q,P\}=1$ (exercise)
- $\dot{Q}=\frac{\partial K}{\partial P}=-iQ$, $\dot{P}=\frac{\partial K}{\partial Q}=iP$ $\Rightarrow$ $Q=Q_0e^{-it}$, $P=P_0e^{+it}$

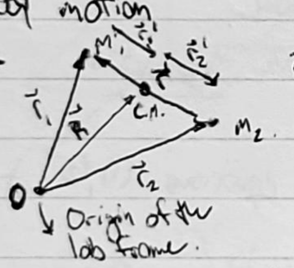
Central Force Motion

- 

$\vec{F}(\vec{r})=F(r)\hat{e}_r$

i) line of action of the force passes through a single point (Force center; fixed or in motion with constant velocity)

ii) Magnitude of the force depends only on the distance from the center.

- $\vec{F}=-\vec{\nabla}V$ (P.E)
- In general, $\vec{F}=\vec{F}(r,\theta,\phi)$
  $V=V(r,\theta,\phi)$
- For centered force $\vec{F}=\vec{F}(r)\Rightarrow V(r)$
  $\vec{\nabla}V=\frac{\partial}{\partial r}V(r)\Rightarrow F(r)=-\frac{\partial V}{\partial r}$
- Two body motion
  - 

$\vec{r}_1,\vec{r}_2$ - position vectors of m_1 & m_2 in the lab frame.

$\vec{R}$ - Center of Mass position in lab frame

$\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$

$\vec{r}_1',\vec{r}_2'$ - position vectors of m_1 & m_2 in the C.M. frame.

- In the C.M. frame $\vec{R}=0$, $m_1\vec{r}_1'+m_2\vec{r}_2'=0$
  $\vec{r}_1'=-\frac{m_2}{m_1}\vec{r}_2'$
  $\vec{r}_2'=-\frac{m_1}{2}\vec{r}_1'$
- $\vec{r}$ - relative position vector
- $\vec{r}_2+\vec{r}=\vec{r}_1$
  $\vec{r}=\vec{r}_1-\vec{r}_2=\vec{r}_1+\frac{m_1}{m_2}\vec{r}_1=\left(\frac{m_1+m_2}{m_2}\right)\vec{r}_1'\Rightarrow \left.\begin{array}{l}\vec{r}_1'=\left(\frac{m_2}{m_1+m_2}\right)\vec{r}\\ \vec{r}_2'=-\left(\frac{m_1}{m_1+m_2}\right)\vec{r}\end{array}\right\}$ ①

$$-\ \vec r_1 = \vec R + \vec r_1' = \vec R + \frac{m_2}{m_1+m_2}\vec r \qquad \textcircled{2}$$
$$\vec r_2 = \vec R + \vec r_2' = \vec R - \frac{m_1}{m_1+m_2}\vec r$$

- ~~PE~~ PE $= V(r,t) = V(|\vec r_1' - \vec r_2'|, t) = V(|\vec r_1 - \vec r_2|, t)$

$\dot{\vec r}_1, \dot{\vec r}_1'$

- $L = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} - V(|\vec r_1 - \vec r_2|, t)$

$$= \frac{m_1}{2}\underbrace{\left[\dot{\vec R} + \frac{m_2}{m_1+m_2}\dot{\vec r}\right]^2}_{\left[\dot{\vec R} + \frac{m_2}{m_1+m_2}\dot{\vec r}\right]\cdot\left[\dot{\vec R} + \frac{m_2}{m_1+m_2}\dot{\vec r}\right]} + \frac{m_2}{2}\left[\dot{\vec R} - \frac{m_1}{m_1+m_2}\dot{\vec r}\right]^2 - V(r,t)$$

$$= \frac{1}{2}(m_1+m_2)|\dot{\vec R}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}|\dot{\vec r}|^2 - V(r,t)$$

$$= \frac{1}{2} M |\dot{\vec R}|^2 + \frac{1}{2}\mu|\dot{\vec r}|^2 - V(r,t).$$

$\mu = \frac{m_1 m_2}{m_1+m_2} \to$ reduced Mass

$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, if $m_1 >> m_2$

$\frac{1}{\mu} \approx \frac{1}{m_1}$, $\mu \approx m_1$

- In cartesian coordinates

$\vec R(X,Y,Z)$, $\vec r =$ ~~...~~ (x,y,z), $r = \sqrt{x^2+y^2+z^2}$

$L = \frac{1}{2}M(\dot X^2 + \dot Y^2 + \dot Z^2) + \frac{1}{2}\mu(\dot x^2 + \dot y^2 + \dot z^2) - V(x,y,z,t)$

- $\frac{\partial L}{\partial X}, \frac{\partial L}{\partial Y}, \frac{\partial L}{\partial Z} = 0$

$$\frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot X}\right)}_{p_X} - \underbrace{\frac{\partial L}{\partial X}}_{0} = 0$$

- p_X, p_Y, p_Z : constants of motion.
- $M\dot X, M\dot Y, M\dot Z$: constants of motion

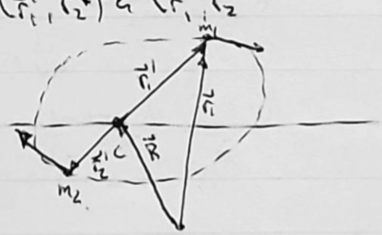
$\Longrightarrow$ CM moves with a constant velocity or at rest in the lab frame

- We can effectively solve the motion:

$L = \frac{1}{2}\mu|\dot{\vec r}|^2 - V(r,t)$

- find $\vec r(t)$ from L use equations (1), (2) to find $(\vec r_1', \vec r_2')$ & $(\vec r_1, \vec r_2)$

-



• General Properties of Central force motion

- ① L (angular momentum) is conserved

$L = T - V = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2) - V(r,t)$

ϕ is cyclic/ignorable

$\frac{\partial L}{\partial \phi} = p_\phi$ is conserved

$p_\phi = \frac{\partial L}{\partial \dot\phi} = \mu r^2 \sin^2\theta\,\dot\phi$

$\vec L = \vec r \times \vec p$

$= \mu r \hat e_r \times (\dot r \hat e_r + r\dot\theta \hat e_\theta + r\sin\theta\,\dot\phi\,\hat e_\phi)$

$= -\mu r^2\dot\theta\sin\phi\,\hat i + \mu r^2\dot\theta\cos\phi\,\hat j - \mu r^2\sin\theta\cos\theta\cos\phi\,\dot\phi\,\hat i - \mu r^2\sin\theta\cos\theta\sin\phi\,\dot\phi\,\hat j + \mu r^2\sin^2\theta\,\dot\phi\,\hat k$

because z direction can be chosen arbitrarily, L is conserved.

$L_z = \mu r^2\sin^2\theta\,\dot\phi = p_\phi \to$ conserved!

$\hat e_\phi \times \hat e_\theta = \hat e_r$

$\hat e_\phi \times \hat e_r = \hat e_\theta$

$\hat e_r \times \hat e_\theta = \hat e_\phi$

- Alternatively, $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \mu \dot{\vec{r}}$

$$\frac{d}{dt}\vec{L} = \underbrace{\dot{\vec{r}} \times \mu \dot{\vec{r}}}_{0} + \vec{r} \times \mu \ddot{\vec{r}}$$

$$= \vec{r} \times \vec{f}(\vec{r}) \qquad \vec{f}(\vec{r}) = |\vec{f}(\vec{r})| \hat{e}_r$$

$$= \vec{r} \times g(r)\vec{r} \qquad = |\vec{f}(\vec{r})| \cdot \vec{r}\hat{e}_r$$

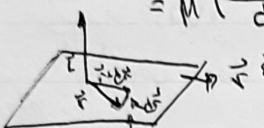
$$= 0 \qquad = g(r)\vec{r}$$

L is conserved

- ② The motion lies entirely in one plane.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \mu \frac{d}{dt}(\vec{r})$$

$$= \mu \left(\frac{\vec{r} \times d\vec{r}}{dt}\right) \Rightarrow \perp \vec{r} \ \& \ d\vec{r}$$

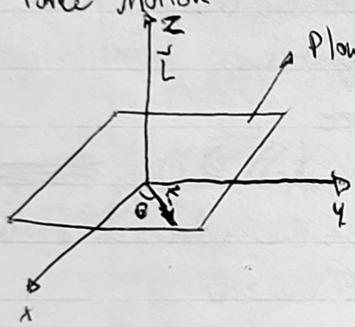


$\Rightarrow$ $\vec{r}$ & $d\vec{r}$ must lie in the same plane to be $\perp \vec{L}$.

change position between time t + dt & t.

$\therefore$ Motion is planar we can use plane polar coordinates (r, θ) to describe the motion.

- Central Force Motion



- Plane of motion $\perp \vec{L}$ (constant of motion)

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,t)$$

$$\vec{F} = \vec{F}(r,t)\,\hat{e}_r = -\vec{\nabla} V(r,t) = -\frac{\partial V(r,t)}{\partial r}\hat{e}_r$$

$$F(r,t) = -\frac{\partial V}{\partial r} \Rightarrow V(a,t) - V(b,t) = -\int_a^b F(r,t)\,dr.$$

$$V(r,t) - \underbrace{V(\infty,t)}_{0} = -\int_{\infty}^{r} F(r',t)$$

- $V(r,t) = -\int_{\infty}^{r} F(r',t)\,dr'$

- L-Equations:

$$\frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \qquad \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \text{constant}$$

$$\Big(\underbrace{\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big)}_{\text{Constant}} - \frac{\partial L}{\partial \theta} = 0\Big) \qquad \mu r^2\dot{\theta} = \underbrace{l}_{\text{constant}} \ (L_z)$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{r}}\Big) - \frac{\partial L}{\partial r} = 0 \Rightarrow \mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\mu r^2\dot{\theta} = l$$

$$\dot{\theta} = \frac{l}{\mu r^2}$$

$$\mu\ddot{r} - \mu r\Big(\frac{l}{\mu r^2}\Big)^2 - f(r,t) = 0$$

- $\mu \ddot{r} = \underbrace{\frac{l^2}{\mu r^3}}_{\text{centrifugal barrier force}} + f(r,t) \quad -①$

applied force

centrifugal barrier force ($>0 \Rightarrow$ repulsive) tries to keep particle away from the centre

- Let $f(r,t) = f(r)$

$u = \dot{r}, \quad \ddot{r} = \frac{du}{dt} = \frac{du}{dr}\frac{dr}{dt} = \dot{r}\frac{du}{dr} = u\frac{du}{dr} = \frac{1}{2}\frac{d(u^2)}{dr}$

① $\Rightarrow$

$$\frac{d}{dr}\left(\frac{1}{2}\mu u^2\right) = -\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) - \frac{dV}{dr}$$
$$= -\frac{d}{dr}\left(\frac{l^2}{2\mu r^2} + V(r)\right)$$

$$\frac{d}{dr}\left[\frac{1}{2}\mu u^2 + \frac{l^2}{2\mu r^2} + V(r)\right] = 0$$

- $\therefore \underbrace{\frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2}}_{T} + V(r) = \text{constant}$ (independent of r)

$T + V = E$ (energy is conserved).

- Thus, i) $\frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) = E$ (constant)

ii) $\mu r^2 \dot{\theta} = l$ (constant)

Let $V(r) + \underbrace{\frac{l^2}{2\mu r^2}}_{\text{centrifugal barrier potential}} = V_{eff}(r)$

$\frac{1}{2}\mu \dot{r}^2 + V_{eff}(r) = E$

$\dot{r}^2 = \frac{2}{\mu}(E - V_{eff}(r)) \geq 0$, ~~All possible values~~

- The only possible values of r are those for which $E - V_{eff} \geq 0$

• Example: $f(r) = \frac{k}{r^2}$, $k = \begin{cases} >0 \Rightarrow \text{repulsion} \\ <0 \Rightarrow \text{attraction.} \end{cases}$

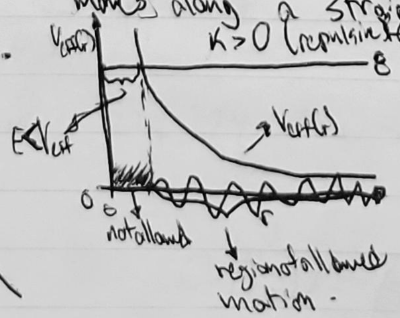
- $V(r) = -\int_{\infty}^{r} f(r')dr' = -\int_{\infty}^{r} \frac{k}{r'^2}dr' = \frac{k}{r}$

$V_{eff}(r) = \frac{k}{r} + \frac{l^2}{2\mu r^2}$

- i) If $l = 0$, $\mu r^2 \dot{\theta} = l = 0 \Rightarrow \dot{\theta} = 0$, $\theta = \text{constant}$, particle moves along a straight line.

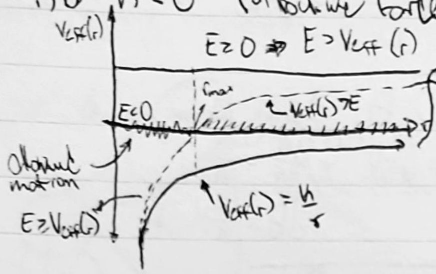
a) or $k>0$

• $k>0$ (repulsive force).



$r_{min} < r < \infty$ motion is unbounded with a lower limit r_{min}.

- i) b) $k<0$ (attractive force).



motion is unbounded $0 \leq r \leq \infty$

$\Rightarrow$ Not allowed motion.

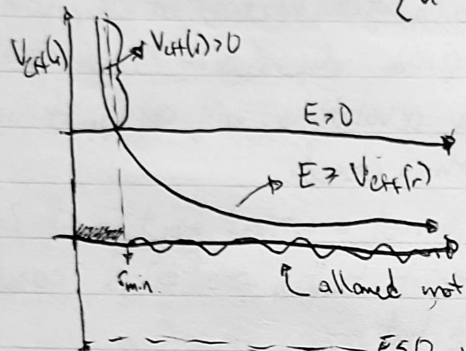
for $E>0$, $0 \leq r \leq r_{max}$

$\Rightarrow$ motion is bounded

- ii) $l>0$

ii) a) $k>0$ (repulsive force).

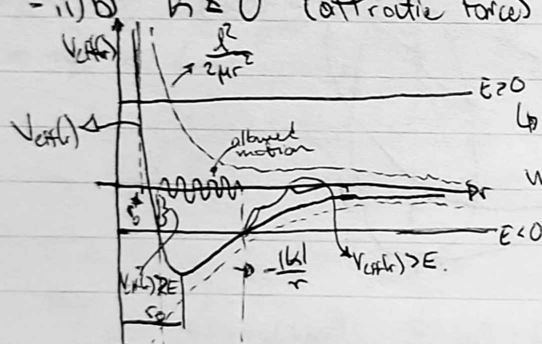
$$V_{eff}(r) = \frac{k}{r} + \frac{l^2}{2\mu r^2} \cdot \begin{cases} \sim \frac{l^2}{2\mu r^2}, & \text{as } r \to 0^+ \\ \sim \frac{k}{r}, & \text{as } r \to \infty \end{cases}$$



allowed motion : $r_{min} \leq r \leq \infty$ (hyperbolic orbit)

$E \leq 0$, no motion

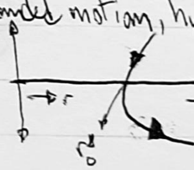
- ii) b) $k \leq 0$ (attractive force), $V_{eff}(r) = \frac{-|k|}{r} + \frac{l^2}{2\mu r^2}$



$\Rightarrow$ allowed motion for $r_0^* \leq r \leq \infty$

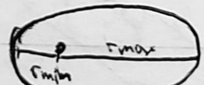
unbounded motion, hyperbolic orbit

for $E>0$



$\Rightarrow$ for $E<0$, allowed motion is bounded $r_{min} \leq r \leq r_{min}$,

elliptic orbit



$\Rightarrow$ For $E = V_{eff\,MIN}$, $r = r_0$ = constant $\Rightarrow$ orbit is a circle

$r = r_0$, $\dot{r} = 0$ at $r = r_0$ $\quad \frac{d(V_{eff}(r))}{dr}\Big|_{r=r_0} = \frac{|k|}{r^2} - \frac{l^2}{\mu r^3}\Big|_{r_0} = 0$

$r_0 = \frac{l^2}{\mu |k|}$ larger $l \Rightarrow$ larger the radius of the circular orbit

$E = V_{eff\,Min} = \frac{-|k|}{r_0} + \frac{l^2}{2\mu r_0^2} = \frac{\mu |k|^2}{2 l^2}$ with $r_0 = \frac{l^2}{\mu |k|}$

Closed & open orbits for the bounded motion:

$$\frac{1}{2}\mu \dot{r}^2 + \frac{-l^2}{2\mu r^2} + V(r) = E$$

$$\left(\frac{\partial r}{\partial t}\right)^2 = \frac{2}{\mu}\left(E - V(r) - \frac{l^2}{2\mu r^2}\right)$$

$\Rightarrow$

τ = period of motion.

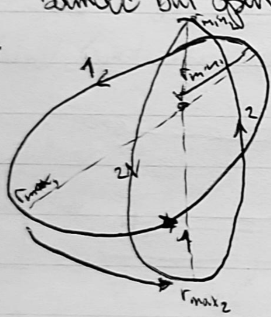
$$-2\int_0^{\tau/2} dt = \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{\frac{2}{\mu}(E-V(r)) - \frac{l^2}{2\mu^2 r^2}}}$$

- The change θ during time period τ, can be obtained from $\mu r^2 \dot{\theta} = l$ $\quad \frac{d\theta}{dt} = \frac{l}{\mu r^2}$

$$d\theta = \frac{l}{\mu r^2} dt$$

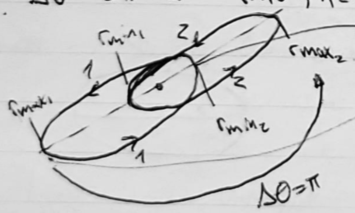
- $d\theta = \frac{l}{\mu r^2} \frac{dr}{\sqrt{\frac{2}{\mu}(E - V(r) - \frac{l^2}{2\mu r^2})}}$
- $\Delta\theta = 2\int_{r_{min}}^{r_{max}} d\theta = 2\int_{r_{min}}^{r_{max}} \frac{\frac{l}{r^2}\, dr}{\sqrt{2\mu\left(E - V(r) - \frac{l^2}{r^2}\right)}}$

change in polar angle during a complete period

- If $\Delta\theta$ is a rational fraction of 2π, $\Delta\theta = 2\pi\left(\frac{m}{n}\right)$ or $\Delta\theta n = 2\pi m$
- After n periods the radius vector will have made 'm' complete revolutions, will occupy its original position $\Rightarrow$ path is closed
- If $\Delta\theta$ is not a rational fraction of 2π, the orbit will never repeat itself, orbit is a rosette - path is bounded but open.
- 

- rosette path of the particle "bound but open orbit"

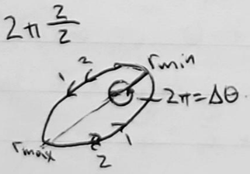
- $\Delta\theta = 2\pi \frac{1}{2}$ $\quad (m=1, n=2)$



pericentre, perihelion
apocentre, aphelion

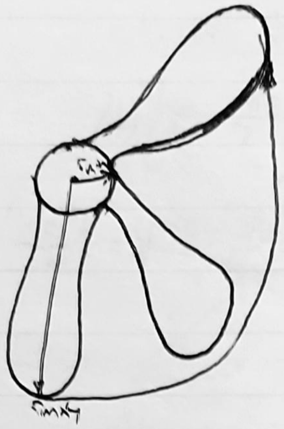
for earth's orbit: perigee, apogee.

- $\Delta\theta = 2\pi \frac{2}{2}$



Advancement of perihelion of Mercury
575''/century (observed) ('' = arc seconds)
Newtonian theory from sun's attraction
532''/century
simulations, including other nearby planets.
528''/century
General Relativity Correction
43''/century

Schematic Illustration of Classical Orbit Between Two Atoms/Molecules Short Range Repulsion, Long Range Attraction.



- Orbit is closed, if it eventually overlaps with itself otherwise open.

- Orbit Equation (Binet's Formula)
  - $\mu r^2 \dot\theta = l = $ (constant).
  - $\dot\theta = \frac{l}{\mu r^2} = \frac{l u^2}{\mu}$, $u = \frac{1}{r}$
  - L-Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$
$$\Rightarrow \mu\ddot r - \mu r\dot\theta^2 + \frac{\partial V}{\partial r} = 0$$
$$\mu(\ddot r - r\dot\theta^2) = -\frac{\partial V}{\partial r} = f(r)$$

  - $r = \frac{1}{u}$, $\dot r = -\frac{1}{u^2}\dot u \Rightarrow -\frac{1}{u^2}\frac{du}{d\theta}\dot\theta = -\frac{1}{u^2}\frac{du}{d\theta}\frac{lu^2}{\mu}$
$= -l\mu^{-1}\frac{du}{d\theta}$
  - $\ddot r = -lu^{-1}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -l\mu^{-1}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\dot\theta = -l\mu^{-1}\frac{d^2u}{d\theta^2}\frac{l^2u^2}{\mu}$
$= -\left(\frac{lu}{\mu}\right)^2\frac{d^2u}{d\theta^2}$
  - Now,

$$\mu(\ddot r - r\dot\theta^2) = f(r)$$
$$\mu\left[-\left(\frac{lu}{\mu}\right)^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\frac{l^2\mu^4}{\mu^2}\right] = f\left(\frac{1}{u}\right)$$
$$\mu\left(\frac{lu}{\mu}\right)^2\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right)$$
$$u + \frac{d^2u}{d\theta^2} = -\frac{\mu}{l^2}\frac{1}{u^2}f\left(\frac{1}{u}\right)$$
$$u + u_2 = -\frac{\mu}{l^2}\frac{1}{u^2}f\left(\frac{1}{u}\right) \quad \text{—(1) orbit Equation}$$

  - Let $l = \mu h \leftarrow$ angular momentum per reduced mass

$$u + u_2 = \frac{\mu}{l^2h^2u^2}f\left(\frac{1}{u}\right)$$
$$= -\frac{\cancel{\mu}}{\mu h^2}\frac{1}{u^2}f\left(\frac{1}{u}\right)$$
, force per reduced mass
$$= -\frac{P(\frac{1}{u})}{h^2u^2} \quad \text{—(2)} \rightarrow \frac{f(\frac{1}{u})}{\mu} = P\left(\frac{1}{u}\right)$$

- Example: $ae^{b\theta} = r$ (what force law would generate this orbit?)

$$u = \frac{1}{a}e^{-b\theta}$$
$$\frac{du}{d\theta} = \frac{-b}{a}e^{-b\theta}$$
$$\frac{d^2u}{d\theta^2} = \frac{b^2}{a}e^{-b\theta} = b^2 u$$
$$u + u_2 = u + \frac{d^2u}{d\theta^2} = u(1+b^2) = -\frac{\mu}{l^2}\frac{1}{u^2}f\left(\frac{1}{u}\right)$$
$$f\left(\frac{1}{u}\right) = f(r) = -\frac{l^2}{\mu}u^3(b^2+1)$$
$$= -\frac{l^2}{\mu}(b^2+1)\frac{1}{r^3} = \frac{-k}{r^3}, \quad k = \frac{l^2(b^2+1)}{\mu}$$

- Orbit Equation (3rd form)
  - $f(r) = \mu(\ddot{r} - r\dot{\theta}^2)$

$$\mu r^2\dot{\theta} = l \Rightarrow \dot{\theta} = \frac{l}{\mu r^2}$$
$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{d\theta}\dot{\theta}\right) = \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{l}{\mu r^2}\right)$$
$$= \frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{l}{\mu r^2}\right)\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{l}{\mu r^2}\right)\frac{l}{\mu r^2}$$
$$= \left[\frac{d^2r}{d\theta^2}\frac{l}{\mu r^2} + \frac{dr}{d\theta}\frac{l}{\mu}\frac{d}{d\theta}\left(\frac{1}{r^2}\right)\right]\frac{l}{\mu r^2}$$
$$= \left[\frac{l}{\mu r^2}\frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\frac{l}{\mu}\left(\frac{-2}{r^3}\frac{dr}{d\theta}\right)\right]\frac{l}{\mu r^2}$$
$$= \frac{l^2}{\mu^2 r^4}\frac{d^2r}{d\theta^2} - \frac{2l^2}{\mu^2 r^5}\left(\frac{dr}{d\theta}\right)^2 \quad \text{— ①}$$

  - $f(r) = \mu(\ddot{r} - r\dot{\theta}^2) = \mu\left(\ddot{r} - r\frac{l^2}{\mu^2 r^4}\right)$

$$= \mu\left[\frac{l^2}{\mu^2 r^4}\frac{d^2r}{d\theta^2} - \frac{2l^2}{\mu^2 r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{l^2}{\mu^2 r^3}\right]$$
$$= \frac{l^2}{\mu r^4}\left[\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r\right]$$

  - $l = \mu h$ ← angular momentum/rest mass
  - $P(r) = \frac{f(r)}{\mu} = \frac{h^2}{r^4}\left[\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r\right]$ — ②

- Orbits for inverse square force:
  - Attractive Case:

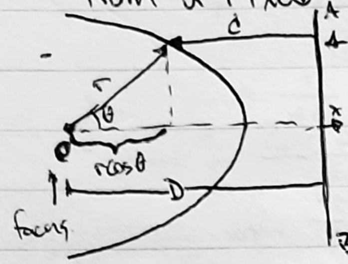
$$f(r) = f\left(\frac{1}{u}\right) = \frac{-k}{r^2} = -ku^2$$
$$u + u_2 = u + \frac{d^2u}{d\theta^2} = -\frac{\mu}{l^2}\frac{1}{u^2}\underbrace{(-ku^2)}_{f\left(\frac{1}{u}\right)} \Rightarrow \text{Binet's Formula}$$
$$\underbrace{\frac{d^2u}{d\theta^2}}_{=\frac{d^2u'}{d\theta^2}} + \underbrace{\left(u - \frac{\mu k}{l^2}\right)}_{u'} = 0 \rightarrow \text{constant}$$
$$\frac{d^2u'}{d\theta^2} + u' = 0, \quad u' = u - \frac{\mu k}{l^2}$$

  - Solution: $u' = b\cos(\theta - \theta_0)$, $u = \frac{\mu k}{l^2} + b\cos(\theta - \theta_0)$

$$r = \frac{\frac{l^2}{\mu k}}{\left(1 + \frac{l^2 b}{\mu k}\cos(\theta - \theta_0)\right)} = \frac{p}{1 + \epsilon\cos(\theta - \theta_0)} \quad \Bigg| \quad \frac{1}{r} = \frac{\mu k}{l^2} + \left(1 + \frac{l^2 b}{\mu k}\cos(\theta - \theta_0)\right)$$

$p = \frac{l^2}{\mu k}$, $\epsilon = \frac{l^2 b}{\mu k}$. Equation of a conic section in polar form (r, θ)

- Conic Section - Intersection of a plane and a cone at different angles
  - Locus of a point P which moves in a way so that its distance from a fixed point, O (focus), bears a constant ~~ration~~ ratio, ϵ, to its distance from a fixed line AB (directrix)



$\frac{r}{d} = \epsilon$ ← eccentricity

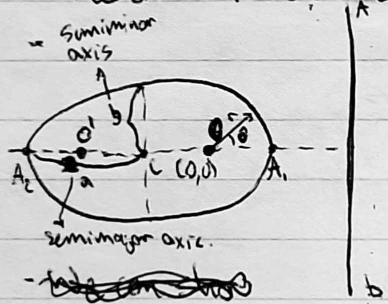
$r\cos\theta + d = D$

$r\cos\theta + \frac{r}{\epsilon} = D$

$r(\epsilon\cos\theta + 1) = D\epsilon \Rightarrow r = \frac{D\epsilon}{(1+\epsilon\cos\theta)} = \frac{p}{(1+\epsilon\cos\theta)}$, $p = \epsilon D$

  - Same equation under the attraction ~~of an~~ inverse square force with x-axis chosen such that $\theta_0 = 0$.

- Ellipse: $0 \le \epsilon < 1$ $\quad r = \frac{p}{1+\epsilon\cos\theta}$
  - $\Rightarrow 1 + \epsilon\cos\theta \neq 0$, $r \neq \infty$
  - The orbit is bounded



~~At~~ At A_1, $\theta = 0$, $r = OA_1 = \frac{p}{1+\epsilon} > 0$, $OA_1 = r_{min}$

At A_2, $\theta = \pi$, $r = OA_2 = \frac{p}{1-\epsilon} > 0$, $OA_2 = r_{max}$

  - ~~[illegible]~~
  - We can show $a = \frac{p}{1-\epsilon^2}$, $b = \frac{p}{\sqrt{1-\epsilon^2}} = a\sqrt{1-\epsilon^2}$

    $p = a(1-\epsilon^2)$

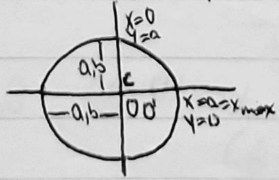
    X_C: x-coordinate of $C = \frac{-p\epsilon}{1-\epsilon^2}$

    $y_C := 0$

    $x_{min} = X_C - a$, $x_{max} = x_C + a$

    y is maximum when $x = x_C$ $\quad y_{max} := b$, $y_{min} := -b$
  - When $\epsilon = 0$, $r = \frac{p}{1+\epsilon\cos\theta} = p$, $\epsilon = 0 \Rightarrow$ circle

    $a = \frac{p}{1+\epsilon\cos\theta} = p$, $b = \frac{p}{1+\epsilon\cos\theta} = p$, $x_C = \frac{-p\epsilon}{1-\epsilon^2} = 0$



$\frac{r}{\epsilon} = d$, $\epsilon = 0$, $d = \infty$, the directrix has moved infinitely away.

$$r = \frac{p}{1+\epsilon\cos(\theta-\theta_0)} \quad (1)$$
$$p = \frac{l^2}{\mu k}$$
$$\epsilon = \frac{l^2 b}{\mu k}.$$

- Central Force Motion:
  - Relation between ϵ & the total energy E:
  - $E = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) + V(r)$
  - Inverse square force

$$V(r) - V(\infty) = -\int_\infty^r f(r')\,dr' = -\int_\infty^r \left(-\frac{k}{r'^2}\right)dr' = -\frac{k}{r}$$

$$l = \mu r^2\dot\theta, \quad E = \frac{1}{2}\mu\dot r^2 + \frac{l^2}{2\mu r^2} - \frac{k}{r} \quad \text{at } r = r_{min},\ \dot r = \frac{dr}{dt} = 0$$

$$E = \frac{l^2}{2\mu r_{min}^2} - \frac{k}{r_{min}} \quad (2)$$

- From (1), $r_{min} = \frac{p}{1+\epsilon}$

$$E = \frac{l^2(1+\epsilon)^2}{2\mu p^2} - \frac{k(1+\epsilon)}{p}$$

$$\frac{(1+\epsilon)}{p} = x, \quad E = \frac{l^2x^2}{2\mu} - kx.$$

$$\frac{l^2x^2}{2\mu} - kx - E = 0$$

$$x = \frac{k \pm\sqrt{k^2 + 4\frac{l^2}{2\mu}E}}{2\frac{l^2}{2\mu}}$$

inadmissable.

$$x = \frac{1+\epsilon}{p} = \frac{k\mu}{l^2} + \frac{k\mu}{l^2}\sqrt{1 + \frac{2l^2}{k\mu^2}E}$$

$$1+\epsilon = 1 + \sqrt{1 + \frac{2l^2}{\mu k^2}E}$$

$$\epsilon = \sqrt{1 + \frac{2l^2}{\mu k^2}E}$$

$$\epsilon = \sqrt{1 - \frac{E}{E_{min}}}$$

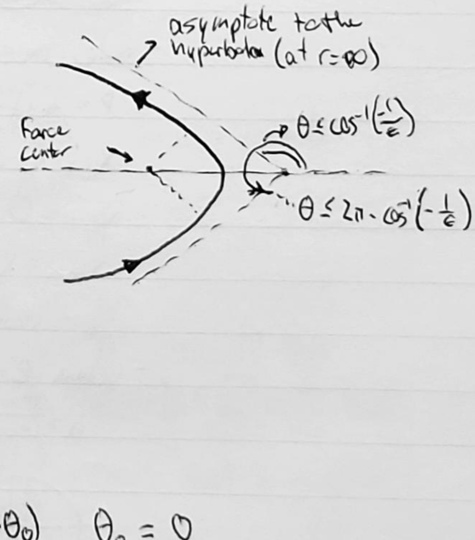
- At $r = r_{min}$, $\frac{dE}{dr_{min}} = 0$

$$\Rightarrow \frac{l^2}{-\mu r_{min}^3} + \frac{k}{r_{min}^2} = 0$$

$$r_{min} = \frac{l^2}{\mu k}$$

$$E_{min} = E(r = r_{min})$$
$$= \frac{-\mu k^2}{2l^2}$$

- Hyperbolic Orbit $(\epsilon > 1,\ E > 0)$



Attractive force - $r = \frac{p}{1+\epsilon\cos\theta}$, $p = \frac{l^2}{\mu k}$

$$r \geq 0, \quad 1 + \epsilon\cos\theta \geq 0$$
$$\cos\theta \geq -\frac{1}{\epsilon}$$

Repulsive force - $f(r)\ f\left(\frac{1}{u}\right) = \frac{k}{r^2} = ku^2$

$$u + u_2 \Rightarrow u_2 = \frac{d^2u}{d\theta^2}$$

$$u + \frac{d^2u}{d\theta^2} = \frac{-\mu}{l^2u^2}\ f\left(\frac{1}{u}\right) = \frac{-\mu k}{l^2}$$

$$\frac{d^2u}{d\theta^2} + \underbrace{\left(u + \frac{\mu k}{l^2}\right)}_{u'} = 0$$

$$\frac{d^2u'}{d\theta^2} + u' = 0, \quad u' = -b\cos(\theta - \theta_0), \quad \theta_0 = 0.$$

$$u = -\frac{\mu k}{l^2} - b\cos(\theta - \theta_0)$$

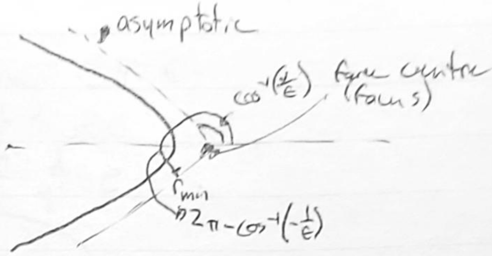
$$u = \frac{1}{r} = -\frac{\mu k}{l^2}\left(1 + \frac{bl^2}{\mu k}\cos\theta\right)$$

$$r = \frac{l^2}{\mu k(1+\epsilon\cos\theta)}, \quad \epsilon = \frac{bl^2}{\mu k}$$

$\Longrightarrow$

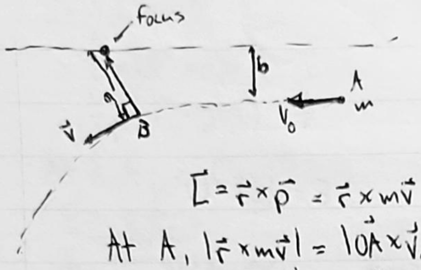
- $r \geq 0$, $1+\epsilon\cos\theta \leq 0$

$$\epsilon\cos\theta \leq -1$$
$$\theta \geq \cos^{-1}\left(-\frac{1}{\epsilon}\right)$$



• Ex. A body of mass, m, moves in a central repulsive force field that varies as $\frac{1}{r^n}$:

$$f(r) = \frac{k}{r^n}, \quad k>0$$



The body moves in, with an initial velocity of v_0 and impact parameter 'b'. Find the closest distance of approach to the force center.

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v}$$

At A, $|\vec{r}\times m\vec{v}| = |\vec{OA}\times\vec{v}_0|\,m = b v_0 m$ $\qquad m v_0 b = mav$ — (1)

At B, $|\vec{r}\times m\vec{v}| = |\vec{OB}\times\vec{v}| = mav$

$f(r) = \frac{k}{r^n} = -\frac{\partial V}{\partial r}$ $\therefore$ $dV = -\frac{k}{r^n}dr$

$$V(r)-V(\infty) = -\int_\infty^r \frac{k}{r^n}dr = -k\frac{r^{(-n+1)}}{-n+1}\Big|_\infty^r = k\left(\frac{r^{-n+1}}{n-1}\right)$$

Energy Conservation:

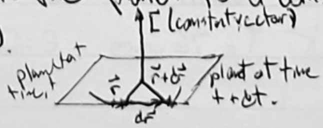
$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{k}{n-1}\frac{1}{a^{n-1}} \quad -(2)$$
$$= \frac{1}{2}m\left(\frac{v_0 b}{a}\right)^2 + \frac{k}{(n-1)}\frac{1}{a^{n-1}}$$
$$\frac{1}{2}mv_0^2\left(1-\left(\frac{b}{a}\right)^2\right) = \frac{k}{(n-1)\,a^{n-1}} \quad -(3)$$

If $n=3$, $\quad a^2 = b^2 + \frac{k}{mv_0^2}$

Kepler's Laws of Planetary Motion (Based on observations by Tycho Brahe)

• (1) Planets move in elliptical orbits about the sun with the sun at the focus (consequence of inverse square law of gravitation)

• (2) The area per unit time swept out by a radius vector from the sun to the planet is a constant (a consequence of central force motion).



Area swept in time dt.

$$dA = \frac{1}{2}|\vec{r}\times d\vec{r}|$$

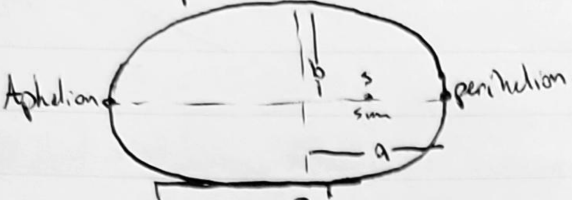
Areal Velocity

$$S = \frac{dA}{dt} = \frac{1}{2}\left|\vec{r}\times\frac{d\vec{r}}{dt}\right|$$
$$= \frac{1}{2}|\vec{r}\times\vec{v}|$$
$$= \frac{1}{2\mu}|\vec{r}\times\vec{p}|$$
$$= \frac{1}{2\mu}|\vec{L}| = \frac{l}{2\mu} \text{ (constant).}$$

$$dA = \frac{1}{2}r\,dr = \frac{1}{2}r\,r\,d\theta = \frac{r^2}{2}d\theta$$
$$S = \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2\mu}$$
$$\mu r^2\dot{\theta} = l$$

③ The square of a planets orbital period is proportional to the cube of the major axis of the planet's orbit.

$S = \frac{l}{2\mu} = \frac{A}{T} = \frac{\pi ab}{T}$, A: Area of elipse, T: time period of the planets revolution around the sun.



$S^2 = \left(\frac{\pi ab}{T}\right)^2$ —① $r = \frac{p}{1+\epsilon\cos\theta}$, $p = \frac{l^2}{\mu k}$ —②

$b = a\sqrt{1-\epsilon^2}$ —③ $p = a(1-\epsilon^2)$ —④

$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$ $E < 0$, $\epsilon > 1$

$\epsilon^2 = 1 + \frac{2El^2}{\mu k^2}$

$1-\epsilon^2 = -\frac{2El^2}{\mu k^2} = \frac{2|E|l^2}{\mu k^2}$

$= \frac{2|E|l^2}{\mu k}\frac{1}{k} = 2|E|\frac{p}{k}$. $E = -|E| < 0$

$\therefore\ a = \frac{p}{1-\epsilon^2} = \frac{k}{2|E|}$ —⑤

$b = a\sqrt{1-\epsilon^2} = \frac{k}{2|E|}\left(\frac{2|E|p}{k}\right)^{\frac{1}{2}} = \frac{k}{\sqrt{2|E|}}\left(\frac{l^2}{\mu k^2}\right)^{\frac{1}{2}}$.

$= \frac{l}{\sqrt{2|E|\mu}}$ —⑥

From (i) $T^2 = \pi^2 a^2 b^2 / S^2$, $S = \frac{l}{2\mu}$.

$= \frac{\pi^2 a^2 b^2 4\mu^2}{l^2}$

$= \pi^2 \frac{k^2}{4|E|^2} \frac{l^2}{2\mu|E|} \frac{4\mu^2}{l^2} = \pi^2 k^2 \frac{\mu}{2} \frac{1}{|E|^3}$ $|E| = \frac{k}{2a}$ from ⑤

$= \pi^2 k^2 \frac{\mu}{2} \frac{8a^3}{k^3}$

$= 4\pi^2 \frac{\mu}{k} a^3$

$T = 2\pi\sqrt{\frac{\mu}{k}}\, a^{\frac{3}{2}}$, $T^2 \propto a^3$

For planetary motion around the sun:

$k = GMm$ → solar mass

$\mu = \frac{Mm}{M+m} \simeq \frac{Mm}{M} = m$.

$\frac{\mu}{k} = \frac{1}{GM}$

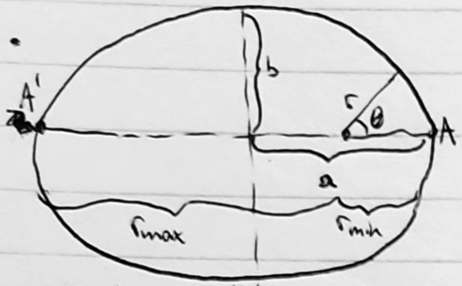
$T^2 = \frac{4\pi^2 Mm a^3}{(M+m)GMm} = \frac{4\pi^2 a^3}{G(M+m)}$

Most planetary orbits are almost circular.

ϵ = 0.007 (Venus), 0.017 (Earth) [0.249 (Pluto), 0.21 (Mercury)] ← highest.

Halley's Comet: T = 76 years, ϵ = 0.967

Speeds & Radius Vector.



$r_{min} = \frac{p}{1+\epsilon}$

$r_{max} = \frac{p}{1-\epsilon}$

$E = \frac{1}{2}\mu u^2 - \frac{k}{r}$ $\qquad \mu r^2 \dot\theta = l.$

$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - \frac{k}{r}$

$= \frac{1}{2}\mu\left(\dot r^2 + \frac{l^2}{\mu^2 r^2}\right) - \frac{k}{r}.$

At A, A', $\dot r = 0$.

$E = \frac{1}{2}\frac{l^2}{\mu r_{min}^2} - \frac{k}{r_{min}}$ ⑤

$\frac{E}{k} = \frac{l^2}{2\mu k r_{min}^2} - \frac{1}{r_{min}} = \frac{1}{2}\frac{(1+\epsilon)}{r_{min}} - \frac{1}{r_{min}}$ ④

$= \frac{1}{2r_m}(1+\epsilon-2) = -\frac{1}{2r_m}(1-\epsilon) = \frac{-(1+\epsilon)(1-\epsilon)}{2p} = \frac{-(1-\epsilon^2)}{2p} = -\frac{1}{2a}.$

For circular orbit, $r=a=b$.

$u^2 = \frac{k}{\mu a} \Rightarrow T = \frac{1}{2}\mu u^2 = \frac{k}{2a}$, $V = -\frac{k}{a}$, $E = T+V$ $\qquad T = |E|$

$E = \frac{-k}{2a}.$ $\qquad V = -2T = -2|E|$

From ①:

$\frac{1}{2}\mu u^2 = E + \frac{k}{r} = \frac{-k}{2a} + \frac{k}{r}$

$= k\left(\frac{1}{r} - \frac{1}{2a}\right)$

$u^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) \leftarrow$ ③

Comparing velocities

From (3), $V_{max}^2 = \frac{k}{\mu}\left(\frac{2}{r_{min}} - \frac{1}{a}\right)$, $r_{min} = \frac{p}{1+\epsilon}$

$V_{min}^2 = \frac{k}{\mu}\left(\frac{2}{r_{max}} - \frac{1}{a}\right)$, $r_{max} = \frac{p}{1-\epsilon}$

$V_{max}^2 = \frac{k}{\mu}\left(\frac{2(1+\epsilon)}{p} - \frac{(1-\epsilon)^2}{p}\right)$

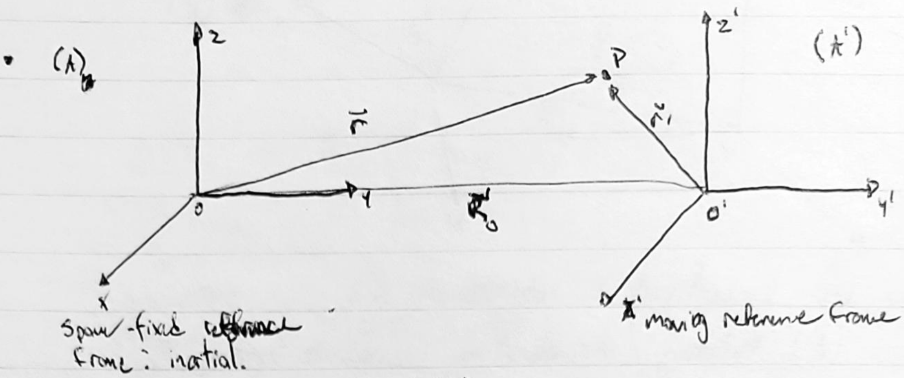
$V_{min}^2 = \frac{k}{\mu}\left(\frac{2(1+\epsilon)}{p} - \frac{(1-\epsilon)^2}{p}\right)$

$\frac{V_{max}^2}{V_{min}^2} = \frac{2+2\epsilon-1+\epsilon^2}{2-2\epsilon-1+\epsilon^2} = \frac{(1+\epsilon)^2}{(-1+\epsilon)^2}$

$\frac{V_{max}}{V_{min}} = \frac{1+\epsilon}{-1+\epsilon}.$

Noninertial Reference Systems:

In an inertial frame, Newtons Second Law is valid.



$\vec{r} = \vec{R}_0 + \vec{r}'$

$\dot{\vec{r}} = \vec{v} = \vec{v}_0 + \vec{v}'$ $\qquad$ $\vec{v}_0 = \dot{\vec{R}}_0,\ \vec{v}' = \dot{\vec{r}}'$

$\vec{a} = \vec{A}_0 + \vec{a}'$ $\qquad$ $\vec{A}_0 = \dot{\vec{v}}_0,\ \vec{a}' = \dot{\vec{v}}'$

- $\vec{R}_0, \vec{v}_0, \vec{A}_0$: pos, vel, acc of the moving system w.r.t. fixed system
- If $\vec{v}_0 =$ constant, $\vec{A}_0 = 0$

  $\vec{a} = \vec{a}'$

  2nd Law - $\vec{F} = m\vec{a} = m\vec{a}'$ → is valid in both frames

  (A) & (A') both are inertial, all frames related by constant velocities are equivalent.

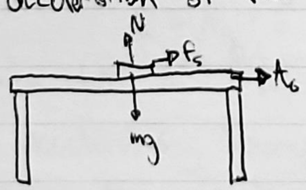
- For $\vec{A}_0 \neq 0$

  $\vec{F} = m\vec{a} = m\vec{A}_0 + m\vec{a}'$

  $\therefore m\vec{a}' = \vec{F} - m\vec{A}_0$

  Observations in (A') will reveal that in addition to applied force $\vec{F}$ there is an additional force $-m\vec{A}_0$ (inertial or fictitious force)

- Block of wood resting on an accelerating table. Find the horizontal acceleration of the table for which the block will slip



$f_s = m\vec{a} = m\vec{a}' + m\vec{A}_0$

$N = mg$, $f_s = \mu N = \mu mg$

$f_s^{max} = \mu_s mg$ ← coefficient of static friction

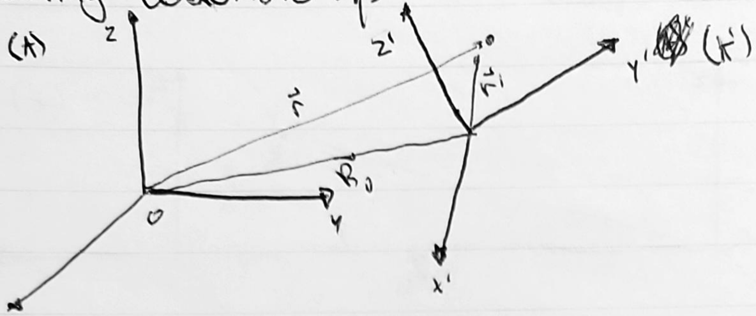
$m\vec{a}' = f_s - m\vec{A}_0$

At the point of slipping

$m\vec{a}' = f_s^{max} - m\vec{A}_0 = \mu_s mg - m\vec{A}_0$

If $\vec{A}_0 > \mu g$, then there is an acceleration of the block w.r.t. table. i.e. block slips.

• Rotating Coordinate System



(A): Space fixed Co-ordinate

(A'): Rotating / Accelerating Co-ordinate System

$\vec r = \vec R_0 + \vec r'$ —①

$\vec r = x\hat e_x + y\hat e_y + z\hat e_z$ —②

$\vec r' = x'\hat e_{x'} + y'\hat e_{y'} + z'\hat e_{z'}$ —③

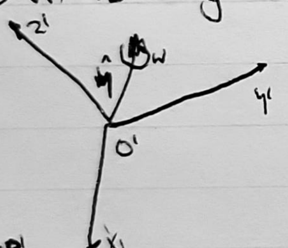
$\frac{d\vec r}{dt} = \frac{dx}{dt}\hat e_x + \frac{dy}{dt}\hat e_y + \frac{dz}{dt}\hat e_z$ ($\hat e_i$ ($i = x, y, z$) are fixed vectors, $\frac{d\hat e_i}{dt} = 0$)

$= \sum_{i=1}^{3} \frac{dx_i}{dt}\hat e_i$ —④

$\frac{d\vec r'}{dt} = \frac{dx'}{dt}\hat e_{x'} + \frac{dy'}{dt}\hat e_{y'} + \frac{dz'}{dt}\hat e_{z'} + x'\frac{d\hat e_{x'}}{dt} + y'\frac{d\hat e_{y'}}{dt} + z'\frac{d\hat e_{z'}}{dt}$

$= \sum_i \frac{dx'_i}{dt}\hat e'_i + \sum_i x'_i\frac{d\hat e_i}{dt}$ —⑤

• Suppose (A) rotating about an axis $\hat\eta$ with on other frame.



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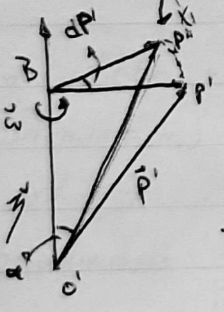


$\overrightarrow{O'P'} = \vec p'$

$\vec p'' = \overrightarrow{O'P''}$, $\overrightarrow{O'P''} = \overrightarrow{O'P'} + \overrightarrow{P'P''}$

$= \overrightarrow{O'P'} + \overrightarrow{dP'}$ → change in vector $\vec p'$ in time dt.

$\overrightarrow{P'P''} = \overrightarrow{dP'} \perp \hat\eta$ & $\overrightarrow{BP'}$

$|\overrightarrow{dP'}| = BP'\,d\phi = O'P'\sin\alpha\,d\phi$

$d\vec\phi = d\phi\,\hat\eta$

$\overrightarrow{dP'} = d\vec\phi \times \overrightarrow{O'P'} = d\vec\phi \times \vec P$

$\frac{d\vec P'}{dt} = \frac{d\vec\phi}{dt} \times \vec p' \Rightarrow \frac{d\vec P'}{dt} = \vec\omega \times \vec p'$ —⑥ → is valid for any vector (position, vel, acceleration) including vectors!

$\frac{d\hat e_i'}{dt} = \vec\omega \times \hat e_i'$

• ③ $\Rightarrow$ $\dfrac{d\vec{r}'}{dt} = \sum_i \dfrac{dx_i'}{dt}\hat{e}_i' + \underbrace{\sum_i x_i' \, \vec{\omega}\times\hat{e}_i'}_{\vec{\omega}\times\underbrace{\sum_i x_i' \hat{e}_i'}_{\vec{r}'}}$

$$\underbrace{\left.\frac{d\vec{r}'}{dt}\right)_{fixed}}_{\text{time rate of change of } \vec{r}' \text{ in fixed frame}} = \underbrace{\left.\frac{d\vec{r}'}{dt}\right)_{rot}}_{\text{time rate of change of } \vec{r}' \text{ in rotating frame}} + \vec{\omega}\times\vec{r}'$$

For ① $\vec{r} = \vec{R}_0 + \vec{r}'$ — (A1)

$$\left.\frac{d\vec{r}}{dt}\right)_{fixed} = \left.\frac{d\vec{R}_0}{dt}\right)_{fixed} + \left.\frac{d\vec{r}'}{dt}\right)_{fixed}$$

$$\vec{V}_0 + \underbrace{\left.\frac{d\vec{r}'}{dt}\right)_{rot}}_{\vec{v}'} + \vec{\omega}\times\vec{r}'$$

$\vec{v} = \vec{V}_0 + \vec{v}' + \vec{\omega}\times\vec{r}'$ — (A2)

$$\left.\frac{d\vec{v}}{dt}\right)_{fixed} = \left.\frac{d\vec{V}_0}{dt}\right)_{fixed} + \left.\frac{d\vec{v}'}{dt}\right)_{fixed} + \left.\frac{d}{dt}(\vec{\omega}\times\vec{r}')\right)_{fixed}.$$

$$\vec{a} \qquad \vec{A}_0 \qquad \Big[\underbrace{\left.\frac{d\vec{v}'}{dt}\right)_{rot}}_{\vec{a}'} + \vec{\omega}\times\vec{v}'\Big] \qquad \left.\frac{d}{dt}(\vec{\omega}\times\vec{r}')\right)_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

Note: $\left.\vec{\omega}\right)_{fixed} = \left.\vec{\omega}\right)_{rot} + \underbrace{\vec{\omega}\times\vec{\omega}}_{0}$

$\Rightarrow \vec{\omega}$ is the same in both frames

$\therefore$ $\vec{a} = \vec{A}_0 + \vec{a}' + \vec{\omega}\times\vec{v}' + \underbrace{\dot{\vec{\omega}}}_{\left.\frac{d\vec{\omega}}{dt}\right)_{rot}}\times\vec{r}' + \vec{\omega}\times\underbrace{\dot{\vec{r}}'}_{\left.\frac{d\vec{r}'}{dt}\right)_{rot}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$

$\vec{a} = \vec{A}_0 + \vec{a}' + \dot{\vec{\omega}}\times\vec{r}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$ — (A3)

- In (A1–A3): All primed quantities are as measured in rotating/accelerating frames. Unprimed quantities w.r.t fixed frames.
- $2\vec{\omega}\times\vec{v}'$: coriolis ~~force~~ acceleration (independent of position $\vec{r}'$)
  $\vec{\omega}\times(\vec{\omega}\times\vec{r}')$: Centripetal acceleration (ind. of $\vec{v}'$)
  $\dot{\vec{\omega}}\times\vec{r}'$: transverse or azimuthal acceleration.
- Newtons Second Law – $\vec{F}_{net} = \dot{\vec{p}} = m\vec{a} = m\vec{A}_0 + m\vec{a}' + m\dot{\vec{\omega}}\times\vec{r}' + 2m\vec{\omega}\times\vec{v}' + m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

$\Longrightarrow$

• If we would like to write

$m\vec{a}' = \vec{F}_{eff}$

$\vec{F}_{eff} = m\vec{a}' = \vec{F} - m\vec{A}_0 - m\dot{\vec{\omega}}\times\vec{r}' - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

$= \vec{F} + \vec{F}_{tr} + \vec{F}_{az} + \vec{F}_{cor} + \vec{F}_{cf}$

$\vec{F}_{tr} = -m\vec{A}_0$ = translational force

$\vec{F}_{az} = -m\dot{\vec{\omega}}\times\vec{r}$ = azimuthal force

$\vec{F}_{cor} = -2m\vec{\omega}\times\vec{v}'$ = coriolis force

$\vec{F}_{cf} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$ = centrifugal force

• $\vec{F}_{tr}, \vec{F}_{az}, \vec{F}_{cor}, \vec{F}_{cf}$: inertial / fictitious forces.

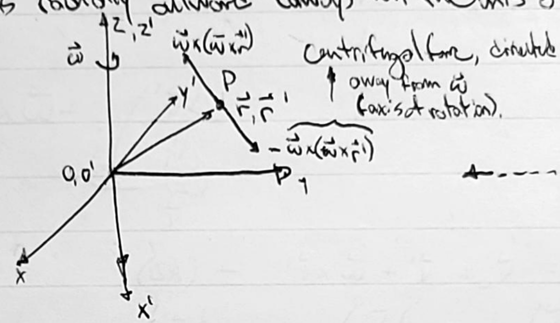
• Centrifugal force: $-m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

① $\perp \vec{\omega}$ or axis of rotation

② Acts radially outward (away) from the axis of rotation

Lets take the z-axis of (A) & (A') both about $\vec{\omega}$, with a common origin, $\vec{R}_0 = 0$

$\vec{r} = \vec{r}'$



• Centrifugal force (continued).

- $v' = 0$

↳ velocity of w.r.t (A')

- $\dot{\vec{\omega}} = 0$ (constant $\vec{\omega}$)

$\vec{A}_0 = 0$ O' is at rest w.r.t O

- $m\vec{a} = \vec{F} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

In equilibrium, $\vec{a}' = 0$

$\vec{F} + F_{cf} = 0$, applied force $\vec{F}$ is cancelled by centrifugal force

• In the fixed frame

$\vec{F} = m\vec{a} = \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r}')}$

$\vec{F} = m\vec{a}_c$ ← centrifugal acceleration

$\vec{F}_{cf} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

$= -m[(\vec{\omega}\cdot\vec{r}')\vec{\omega} - \omega^2\vec{r}'] = -\vec{\nabla}V_{cf}$

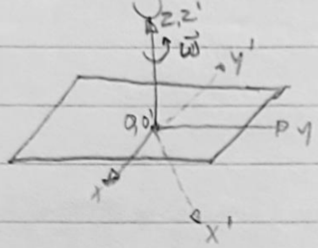
$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$V_{cf} = \frac{1}{2}m[(\vec{\omega}\cdot\vec{r}')^2 - \omega^2 \vec{r}'^2]$

↳ centrifugal P.E.

$\vec{\nabla} = \vec{\nabla}_{r'}$

- Coriolis Force: $-2m\vec{\omega}\times\vec{v}'$
  - ① $\perp \vec{\omega}$ (axis of rotation)
  - ② $\perp \vec{v}'$ ← velocity of the body in the rotating frame
  - Rotating Table:



- Horizontal table rotating with a constant angular velocity $\vec{\omega}$ w.r.t. an inertial frame. A body of mass, m, moves with a velocity, $\vec{v}'$, w.r.t. the rotating table

- No friction - $\vec{\omega} = \omega\vec{k}$ $(\omega\vec{k}')$

$\dot{\vec{\omega}} = 0$

- (A), (A') have the same origin $\Rightarrow \ddot{h}_o = 0$

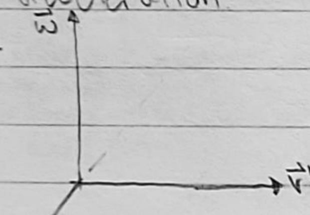
$\vec{a}' = \vec{F} - 2m\,\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

- Centrifugal force $\propto -\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

$= -[(\vec{\omega}\cdot\vec{r})\vec{\omega} - \omega^2\vec{r}'] = \omega^2\vec{r}'$, $\vec{r}'$ is in the plane of the table

- Centrifugal force, F_{cf}, generates a radially outward acceleration

-



$-\vec{\omega}\times\vec{r}$ ⊙ $\Rightarrow$ looking (walking) along $\vec{v}'$, the Coriolis Force acts to the right

-

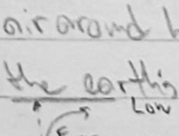
path of the body deflected to the right.

- Effects of Coriolis Force:

a) Deflection of falling body
- $\vec{\omega}$ is in N-S direction
- body's velocity is nearly vertical, $-2m\vec{\omega}\times\vec{v}'$ is in E-W direction
- A falling body in N-hemisphere is, in general, deflected eastward

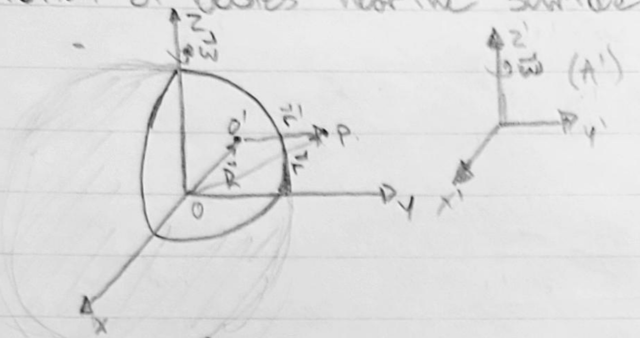
b) Must be considered in long-range gunnery & missile projection.

c) Responsible for the circulation of air around high-low pressure areas near the earth's surface



Counterclockwise cyclonic motion in N-Hemisphere.

• Motion of bodies near the surface of the Earth



- $\vec{r}' + \vec{R} = \vec{r}$

$m\vec{a}' = \vec{F} + m\vec{g} - m\vec{A}_0 - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$ —①

↳ any physical force other than gravity acting on the body.

- Velocity of O' in the fixed frame $\vec{v}_0 = \vec{\omega}\times\vec{R}$

$\vec{A}_0$ = acceleration of O' in the fixed frame $= \left.\frac{d\vec{v}_0}{dt}\right)_{fixed}$

$= \underbrace{\left.\frac{d\vec{v}_0}{dt}\right)_{rot}}_{0} + \vec{\omega}\times\vec{v}_0 \rightarrow \vec{\omega}\times(\vec{\omega}\times\vec{R})$

(in the rotating frame, O' has no acceleration)

- ∴ $m\vec{a}' = \vec{F} + m\vec{g} - m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r}')) - 2m(\vec{\omega}\times\vec{v}')$
- a) Effective Gravity near or on the surface of the Earth

$r \ll R$, set $\vec{v} = 0$, assuming the body to be sitting on the surface (F: normal forces of the Earth or hanging from a plumb line ($\vec{F}$: tension in the line).

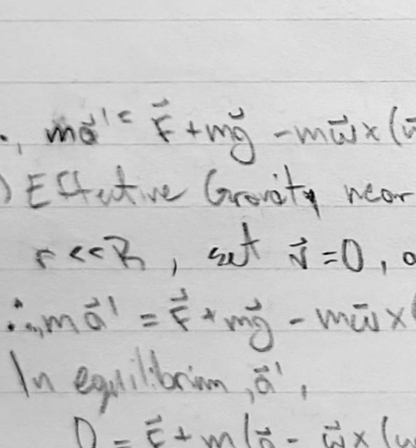
∴ $m\vec{a}' = \vec{F} + m\vec{g} - m\vec{\omega}\times(\vec{\omega}\times\vec{R})$

In equilibrium, $\vec{a}'$,

$0 = \vec{F} + m(\vec{g} - \vec{\omega}\times(\omega\times\vec{R}))$

$= \vec{F} + m\vec{g}_{eff}$

$\vec{g}_{eff} = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{R})$ — (A)



λ: latitude of P

$\theta = \frac{\pi}{2} - \lambda$

$|\vec{\omega}\times(\vec{\omega}\times\vec{R})| = \omega^2 R\sin\theta.$

$\vec{g}_{eff} = -g_{eff}\cos\alpha\,\hat{i} + g_{eff}\sin\alpha\,\hat{j}$ —(B-0)

$\vec{g} = -g\hat{j}$ —(B-1)

$-\vec{\omega}\times(\vec{\omega}\times\vec{R}) = \omega^2 R\sin\theta\sin\lambda\,\hat{i} + \omega^2 R\sin\theta\cos\lambda\,\hat{j}$ —(B-2)

- Sub (B-0) - (B-2) into (A)

$-g_{eff}\cos\alpha\,\hat{j} + g_{eff}\sin\alpha\,\hat{i} = -(g - \omega^2 R\sin\theta\cos\lambda)\hat{j} + \omega^2 R\sin\theta\cos\lambda\,\hat{i}$

$g_{eff}\sin\alpha = \omega^2 R\sin\lambda\sin\theta$

$g_{eff}\cos\alpha = g - \omega^2 R\cos\lambda\sin\theta$

$$\tan\alpha = \frac{\omega^2 R\sin\lambda\sin\theta}{g - \omega^2 R\cos\lambda\sin\theta} = \frac{\omega^2 R\sin\theta\cos\theta}{g - \omega^2 R\sin^2\theta} \simeq \frac{\omega^2 R\sin\theta\cos\theta}{g}$$

- $\omega = \frac{2\pi}{T} = \frac{2\pi}{24h\, 3600s} = 0.73\times10^{-4}\ \frac{rad}{s}$

$R = 371\,000\ \frac{m}{s}$, $\omega^2 R = 0.03\ \frac{m}{s^2}$

- $\tan\alpha = 0.03\,\frac{m}{s}\,\frac{\overbrace{\sin\theta\cos\theta}^{\frac{1}{2}\sin 2\theta}}{g}$

↳ max when $2\theta = 90°$, $\theta = 45°$

- $\tan\alpha_{max} \simeq \alpha_{max}$

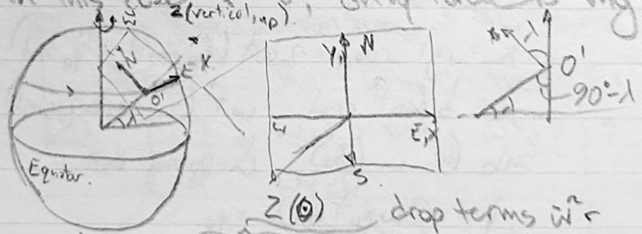
$= \frac{0.03}{9.8}\,\frac{1}{2} = 1.53\times10^{-3}$ rad $(0.09°)$

$= 1.53\times\frac{180}{\pi}\times 60\times10^{-3} \simeq 5'$ or 5 minutes

• Projectile Motion over the rotating Earth:

- In this case $\vec{F} = 0$, only force is $m\vec{g}$

-



- $\vec{a} = \frac{\vec{F}}{m} + \vec{g} - \underbrace{\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r}))}_{\vec{g}_{eff}\sim g} - \overbrace{2(\vec{\omega}\times\vec{v})}^{Z\,(0)}$ drop terms $\omega^2 r$

$\vec{a} = \vec{g} - 2\,\vec{\omega}\times\vec{v}$ - ①

- $m\vec{a}' = \vec{F} + m\vec{g} - m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r}')) - 2m\,\vec{\omega}\times\vec{v}'$

Drop all ' signs since all quantities are measured wrt Earth frame

$\vec{a} = \frac{\vec{F}}{m} + \vec{g} - \underbrace{\vec{\omega}\times(\vec{\omega}\times(\vec{R}\times\vec{r}))}_{\vec{g}_{eff}\simeq\vec{g}\ \text{(neglecting terms } O(\omega^2))} - 2\,\vec{\omega}\times\vec{v}$

For a free projectile $\vec{F} = 0$.

$\vec{a} = \vec{g} - 2\vec{\omega}\times\vec{v}$ -①

$\vec{\omega} = \omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k}$, $\vec{g} = -g\hat{k}$

$$\vec{\omega}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = (\omega\dot{z}\cos\lambda - \omega\dot{y}\sin\lambda)\hat{i} + (\omega\dot{x}\sin\lambda)\hat{j} + (-\omega\dot{x}\cos\lambda)\hat{k}$$

⇒

- ① $\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} = -g\hat{k} - 2(\omega\dot{z}\cos\lambda - \omega\dot{y}\sin\lambda)\hat{i} - 2(\omega\dot{x}\sin\lambda)\hat{j}$
$+2\omega\dot{x}\cos\lambda\,\hat{k}$

$$\ddot{x} = -2\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) \quad - ②$$
$$\ddot{y} = -2\omega\dot{x}\sin\lambda \quad - ③$$
$$\ddot{z} = -g - 2\omega\dot{x}\cos\lambda \quad - ④$$

- Integrate ② - ④ wrt t with,
$$\int_0^t \ddot{x}(t)dt = \dot{x}(t) - \dot{x}(t=0) = \dot{x} - \dot{x}_0$$
$$\dot{x} = -2\omega(z\cos\lambda - y\sin\lambda) + \dot{x}_0 \quad - ②a$$
$$\dot{y} = -2\omega x\sin\lambda + \dot{y}_0 \quad - ③a$$
$$\dot{z} = -gt + 2\omega x\cos\lambda + \dot{z}_0 \quad - ④a$$

- Note: $\int_0^t \dot{z}(t)dt = z(t) - z(t=0) =$
$$x_0 = y_0 = z_0 = 0.$$

- Sub ③a & ④a in ②
$$\ddot{x} = -2\omega(-gt\cos\lambda + 2\omega x\cos^2\lambda + \dot{z}_0\cos\lambda + 2\omega x\sin\lambda - \dot{y}_0\sin\lambda)$$

- Drop $\sim\omega^2$ terms.
$$\ddot{x} = 2\omega gt\cos\lambda - 2\omega(\dot{z}_0\cos\lambda - \dot{y}_0\sin\lambda) \quad ⑤$$

- Int ⑤ wrt
$$\dot{x} = \omega gt^2\cos\lambda - 2\omega t(\dot{z}_0\cos\lambda - \dot{y}_0\sin\lambda) + \dot{x}_0$$
$$x = \frac{1}{3}\omega gt^2\cos\lambda - \omega t^2(\dot{z}_0\cos\lambda - \dot{y}_0\sin\lambda) + \dot{x}_0 t \quad - ⑥$$

- Sub ⑥ in (3a), dropping ω^2
$$\dot{y} = -2\omega\dot{x}_0 t\sin\lambda + \dot{y}_0$$
$$y = -\omega\dot{x}_0 t^2\sin\lambda + \dot{y}_0 t \quad - ⑦$$

- Sub in ⑥ to ④a, dropping ω^2
$$\dot{z} = -gt + 2\omega\dot{x}_0 t\cos\lambda + \dot{z}_0$$
$$z = -\frac{1}{2}gt^2 + \omega\dot{x}_0 t^2\cos\lambda + \dot{z}_0 t$$

- ① Body falling from rest $(\dot{x}_0, \dot{y}_0, \dot{z}_0 = 0)$
$$x = \frac{1}{3}\omega gt^2\cos\lambda \quad (\text{from } ⑥)$$
$$y = 0$$
$$z = -\frac{1}{2}gt^2$$

If the body falls through a vertical ht 'h', then $z = -h = -\frac{1}{2}gt^2$, $t = \sqrt{\frac{2h}{g}}$.
$x = \frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{\frac{3}{2}}\cos\lambda$ (body drifts eastward as a consequence of Earths Rotation).

• ② An Upward Projectile with $x_0 = y_0 = 0$, $\dot{z}_0 = v_0$
- Sideways deviating, direction & magnitude.

• ③ Projectile fired at an elev in an easterly direction
- $\dot{x}_0, \dot{z}_0 \neq 0$, $\dot{y}_0 = 0$.

$x = \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \dot{z}_0 \cos\lambda + \dot{x}_0 t$ from ⑥

$y = -\omega \dot{x}_0 t^2 \sin\lambda$ from ⑦

$z = -\frac{1}{2} g t^2 + \dot{z}_0 t + \omega \dot{x}_0 t^2 \cos\lambda$ from ⑧

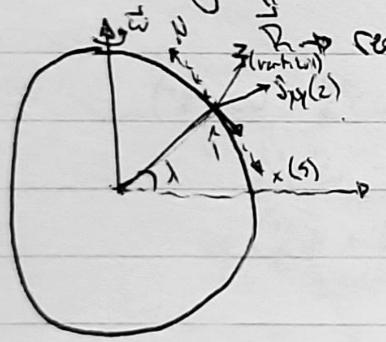
- looking east, projectile drifts to the right, there is a drift in negative y or southerly directly.
- For a projectile in a northerly direction $\dot{x}_0 = 0$, $\dot{y}_0, \dot{z}_0 \neq 0$

$x = \frac{1}{3}\omega g t^3 \cos\lambda + \omega t^2 (\dot{y}_0 \sin\lambda - \dot{z}_0 \cos\lambda)$

For large 't', and/or small $\dot{z}_0$, projectile drifts eastward (to the right).

• A particle that slides over a smooth plane tangent to the Earths surface at latitude λ. Determine the reaction of the plane on the body. Neglect terms $\sim \omega^2$ $\vec{\omega} \times \vec{\omega} \times (\vec{R} \times \vec{r}) \approx 0$

$m\vec{a} = m\vec{g} + \vec{F} - 2m(\vec{\omega} \times \vec{v})$



$\vec{R}$ → reaction force of the surface on the body.

$R_x = 0$, $R_y = 0$, $R_z = R$.

$\omega_z = \omega \sin\lambda$

$\omega_y = 0$

$\omega_x = -\omega\cos\lambda$

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

$= (-\dot{y}\omega\sin\lambda)\hat{i} + (\dot{x}\omega\sin\lambda)\hat{j} + (\dot{y}\omega\cos\lambda)\hat{k}$

$m\ddot{z} = -mg + R + 2m\dot{y}\omega\cos\lambda$ — ①

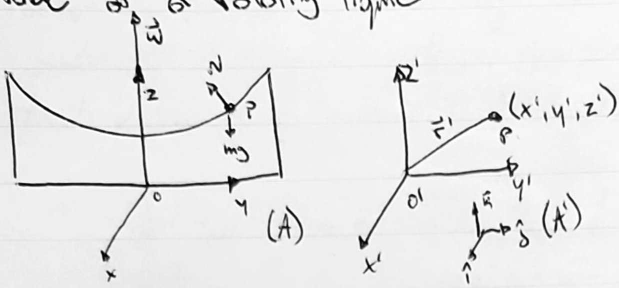
$m\ddot{x} = 2m\dot{y}\omega\sin\lambda$ — ②

$m\ddot{y} = -2m\omega\dot{x}\sin\lambda$ — ③

$\ddot{z} = 0$, $R = mg - 2m\dot{y}\omega\cos\lambda$ — ④

$\dot{y}$ is the solution of ② & ③

- Surface of a rotating liquid



- (A') rotating with the fluid.
- Forces on 'm', an element of the liquid:
  1. $\vec{N}$ = force exerted by the rest of the liquid on 'm'
  2. $m\vec{g}$
- 'm' moves with the rotating frame
- $\Rightarrow \vec{v}' = 0$, $\vec{a}' = 0$, $\vec{\omega}$ = constant, $\dot{\vec{\omega}} = 0$
- With O & O' coinciding, $\vec{A}_0 = 0$

$$m\vec{a}' = \vec{F}_{net} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

$$0 = \vec{N} + m\vec{g} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

$$\vec{\omega} = \omega\hat{k} = \omega\hat{k}'$$

$$\vec{r}' = x'\hat{i}' + y'\hat{j}' + z\hat{k}'$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}') = \omega\hat{k}' \times (\omega x'\hat{j}' - \omega y'\hat{i}')$$

$$= -\omega^2 x'\hat{i}' - \omega^2 y'\hat{j}'$$

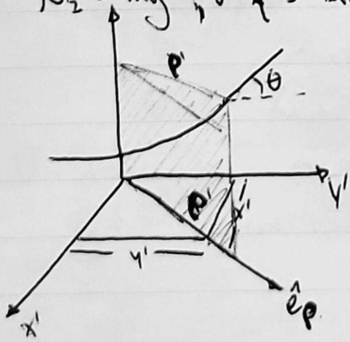
$$= -\omega^2 (x'\hat{i}' + y'\hat{j}')$$

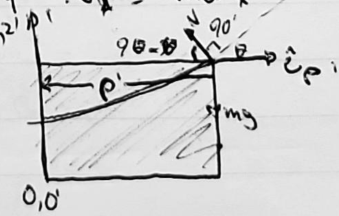
- From 1:

$$N_{x'}\hat{i}' + N_{y'}\hat{j}' + N_{z'}\hat{k}' - mg\hat{k}' + m\omega^2 (x'\hat{i}' + y'\hat{j}') = 0$$

$$N_{z'} = mg, \quad N_{y'} = -m\omega^2 y', \quad N_{x'} = -m\omega^2 x'$$

-





$$\tan\theta = \frac{dz'}{d\rho'} \quad -(2)$$

$$N_{z'} = mg \Rightarrow N\sin(90° - \theta) = N\cos\theta = mg \quad -(3a)$$

$$-N_{\rho'}\hat{e}_\rho = N_{y'}\hat{j} + N_{x'}\hat{i}$$

$$-N\cos(90-\theta)\hat{e}_\rho = -m\omega^2 (x'\hat{i}' + y'\hat{j}') = -m\omega^2\rho'$$

$$N\sin\theta = m\omega^2\rho' \rightarrow (3b)$$

- (3 a, b) $\Rightarrow \tan\theta = \frac{dz'}{d\rho'} = \frac{\omega^2 \rho'}{g}$

$\therefore \frac{dz'}{d\rho'} = \frac{\omega^2\rho'}{g}$

Integrating wrt ρ':

$z' = \frac{1}{2}\omega^2 \frac{\rho'^2}{g} + z_0'$ → constant of integration z' at $\rho' = 0$.

$$z = z' = \frac{1}{2}\frac{\omega^2}{g}\overbrace{(x'^2 + y'^2)}^{x^2+y^2} + z_0' \quad \text{(paraboloid of revolution about } z'\text{-axis)}$$

- In the fixed ~~frame~~

$$z = \frac{1}{2}\frac{\omega^2}{g}(x^2+y^2) + z_0$$

$N\cos\theta = mg$

$N\sin\theta = m\omega^2\rho$ ← Centrifugal deceleration

$\tan\theta = \frac{\omega^2\rho}{g} = \frac{dz}{d\rho}$.

- Energy Conservation

In the rotating frame

$\vec{v}' = 0 \Rightarrow K.E = 0$.

$P.E = $ gravitional P.E. $+ V_{cf}$ ← Centrifugal P.E.

$= mgz' + V_{cf}$

$V_{cf} = \frac{1}{2}m[(\vec{\omega}\times\vec{r}')^2 - \omega^2 \vec{r}'^2]$

(show $\vec{\nabla}V_{cf} = -\vec{F}_{cf}$, $\vec{\nabla}_{\vec{r}'}V_{cf} = -m(\vec{\omega}\times(\vec{\omega}\times\vec{r}'))$.

$\vec{\omega}\cdot\vec{r}' = \omega\hat{k}'(x'\hat{i}' + y'\hat{j}' + z'\hat{k}')$

$= \omega z'$

$V_{cf} = \frac{1}{2}m(\omega^2 z'^2 - \omega^2(x'^2 + y'^2 + z'^2))$

$= -\frac{1}{2}m\omega^2(x^2+y^2)$

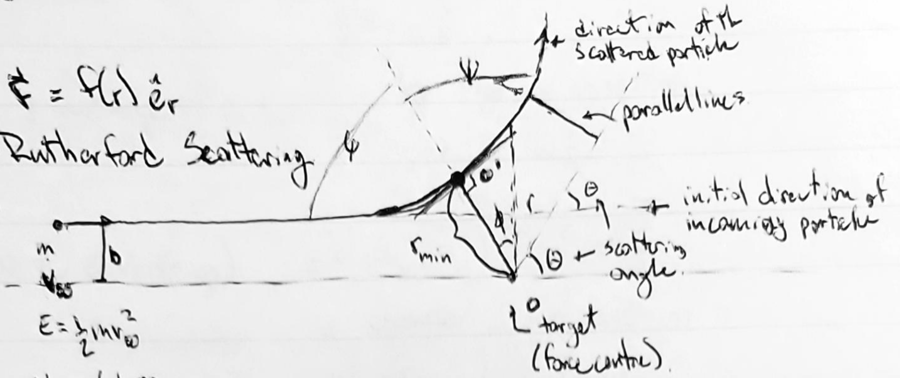
$KE + PE = $ Constant $\Rightarrow mgz' - \frac{1}{2}m(x^2+y^2)\omega^2$

$z' = \frac{\omega^2}{2g}(x'^2 + y'^2) + $ Constant.

$\vec{\omega}\times(\vec{\omega}\times\vec{r}') = \quad = \hat{k}' \times$

Scattering in a Central Repulsive Force Field

- $\vec{F} = f(r)\,\hat{e}_r$
- Rutherford Scattering
-



$E = \frac{1}{2}mv_\infty^2$

$v_\infty = v$ at ∞

b = impact parameter

$2\psi + \theta = \pi$; $\phi(r_{min}) = 0$, $\phi(r \to \infty) = \psi$

$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\phi}\,\hat{e}_\phi$

At O', $\dot{r} = 0 \Rightarrow \vec{v} = r\dot{\phi}\,\hat{e}_\phi$, $OO' \perp \vec{v}$ at O'

- Energy Conservation:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = E \quad -①$$

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v} = mr\hat{e}_r \times (\dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi)$$

$$= mr^2\dot{\phi}\,\hat{e}_z$$

$$|\vec{L}| = l = mr^2\dot{\phi} = mv_\infty b = \sqrt{2Em}\,b \quad -②$$

$$① \Rightarrow \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{l^2}{m^2r^4} + V(r) = E$$

$$\frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = E$$

$$\frac{m}{2}\dot{r}^2 + V_{eff}(r) = E$$

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r) = \frac{2Emb^2}{2mr^2} + V(r)$$

$$= \frac{Eb^2}{r^2} + V(r)$$

$$\frac{m}{2}\dot{r}^2 = E - V_{eff}(r)$$

$$\dot{r} = \sqrt{\frac{2}{m}(E - V_{eff}(r))} \quad -③$$

From ②, $\dot{\phi} = \sqrt{\frac{2E}{m}}\,\frac{b}{r^2} \quad -④$

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \frac{\sqrt{\frac{2E}{m}}\,\frac{b}{r^2}}{\sqrt{\frac{2E}{m} - \frac{2}{m}V_{eff}(r)}} = \frac{\frac{b}{r^2}}{\sqrt{1 - \frac{V_{eff}(r)}{E}}} = \frac{\frac{b}{r^2}}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}$$

$$d\phi = \frac{\frac{b}{r^2}\,dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}$$

$$\psi = \int_{r_{min}}^{\infty} d\phi = \int_{r_{min}}^{\infty} \frac{\frac{b}{r^2}\,dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}$$

- To determine r_{min}, note that for $r = r_{min}$, $\dot{r} = 0$

- $V_{eff}(r_{min}) = E = \frac{Eb^2}{r_{min}^2} + V(r_{min})$
- Example: $V(r) = \frac{k}{r}$ $\quad k > 0$

  $k = cZZ'e^2$ $\quad$ (Coulomb Scattering Case)

  $c = 1$ (CGS), $c = \frac{1}{4\pi\epsilon_0}$ (SI)

  Z, Z': Atomic numbers of incident and target particles.
- Gravitational Case $\quad k < 0$

  $k = GMm$ (Kepler Problem).
- $\Psi = \int_{r_{min}}^{\infty} \frac{\frac{b}{r^2}\, dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{k}{Er}}} = \int_{r_{min}}^{\infty} \frac{\frac{b}{r^2}\, dr}{\sqrt{\left(1 + \frac{k}{2Eb}\right)^2 - \left(\frac{b}{r} + \frac{k}{2Eb}\right)^2}}$

  $= \int_{r_{min}}^{\infty} \frac{d\left(\frac{b}{r} + \frac{k}{2Eb}\right)}{\sqrt{(1) + \left(\frac{k}{2Eb}\right)^2 - \left(\frac{b}{r} + \frac{k}{2Eb}\right)^2}} \Rightarrow \frac{b}{r} + \frac{k}{2Eb} = \sqrt{1 + \left(\frac{k}{2Eb}\right)^2}\, x.$

  $\frac{b}{r_{min}} + \frac{k}{2Eb} = \sqrt{1 + \left(\frac{k}{2Eb}\right)^2}\ x_{min}$
- When $r = \infty$, $x = \frac{\frac{k}{2Eb}}{\sqrt{1 + \left(\frac{k}{2Eb}\right)^2}} = x_\infty$

$\Psi = -\int_{x_m}^{x_\infty} \frac{dx}{\sqrt{1 - x^2}} = \cos^{-1}(x)\Big|_{x_{min}}^{x_\infty} = \cos^{-1}\left\{\frac{\frac{k}{2Eb}}{\sqrt{1 + \left(\frac{k}{2Eb}\right)^2}}\right\} - \cos^{-1}\left\{\frac{\left(\frac{b}{r_{min}} + \frac{k}{2Eb}\right)}{\sqrt{1 + \left(\frac{k}{2Eb}\right)^2}}\right\}$

$E = \frac{l^2}{2mr_{min}^2} + \frac{k}{r_{min}} = \frac{2mEb^2}{2mr_{min}^2} + \frac{k}{r_{min}}$

$\left(\frac{b}{r_{min}}\right)^2 + \frac{k}{Er_{min}} = 1$ from $\quad \longleftarrow \cos^{-1}(1) = 0.$

$\left(\frac{b}{r_{min}}\right)^2 + 2\frac{b}{r_{min}}\left(\frac{k}{2bE}\right) + \left(\frac{k}{2bE}\right)^2 = 1 + \left(\frac{k}{2bE}\right)^2.$

$\left(\frac{b}{r_{min}} + \frac{k}{2bE}\right)^2 = 1 + \left(\frac{k}{2Eb}\right)^2$

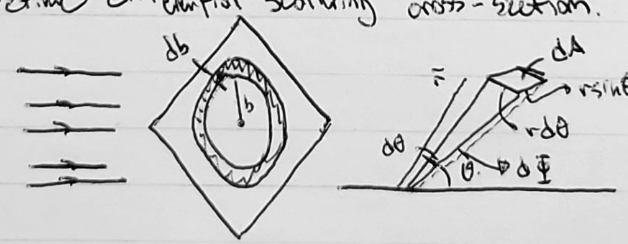
$\frac{b}{r_{min}} + \frac{k}{2Eb} = \sqrt{1 + \left(\frac{k}{2Eb}\right)^2}$ — (5)

- $\cos\Psi = \frac{\frac{k}{2Eb}}{\sqrt{1 + \left(\frac{k}{2Eb}\right)^2}}$, $\quad \Theta + 2\Psi = \pi \Rightarrow \Psi = \frac{\pi}{2} - \frac{\Theta}{2}$, $\cos\Psi = \sin\left(\frac{\pi}{2} - \frac{\Theta}{2}\right)$ $= \sin\left(\frac{\Theta}{2}\right)$

  $\therefore$ $\sin\left(\frac{\Theta}{2}\right) = \frac{\frac{k}{2Eb}}{\sqrt{1 + \left(\frac{k}{2Eb}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{2Eb}{k}\right)^2}}$

I = flux density: # of particles crossing unit area normal to the beam per unit time.

- In an actual experiment, we have a distribution of b. Define differential scattering cross-section.



$dA = r^2 \sin\theta\, d\theta\, d\Phi$

$d\Omega = \frac{dA}{r^2}$

$= \sin\theta\, d\theta\, d\Phi$

- Differential Scattering Cross-Section

$$\sigma(\theta,\Phi)\,d\Omega = \frac{\#\text{ of particles scattered into solid angle } d\Omega \text{ about } (\theta,\Phi) \text{ per unit}}{I}$$

$\rightarrow$ Probability that any given particle in the beam gets scattered into solid angle $d\Omega$ at Ω.

- For central force $\sigma(\theta,\Phi)$ should be independent of Φ

$$\int_{\Phi=0}^{2\pi} \underbrace{\sigma(\theta,\Phi)}_{=\sigma(\theta)} \sin\theta \, d\theta \, d\Phi = 2\pi\, \sigma(\theta) \sin\theta\, d\theta.$$

= # of particles scattered into the solid angle defined by cones θ & $\theta + d\theta$.

- Let only the particles with impact parameter in the range $|db|$ around 'b' get scattered into solid angle $2\pi \sin\theta\, |d\theta|$, then,

$$I\, 2\pi b\, |db| = I\, \sigma(\theta)\, 2\pi \sin\theta\, |d\theta|$$

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

- For the Coulomb (Rutherford).

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{1+\left(\frac{2Eb}{k}\right)^2}$$

$$\left(2\frac{Eb}{k}\right)^2 = \frac{1}{\sin^2\frac{\theta}{2}} - 1 = \frac{1-\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2}$$

$b = \frac{k}{2E}\cot\frac{\theta}{2}$ (we take + root because for $0 \le \theta \le \pi$ $\cot\frac{\theta}{2} > 0$).

$$\left|\frac{db}{d\theta}\right| = \frac{k}{4E}\left|-\csc^2\frac{\theta}{2}\right| = \frac{k}{4E}\csc^2\frac{\theta}{2}.$$

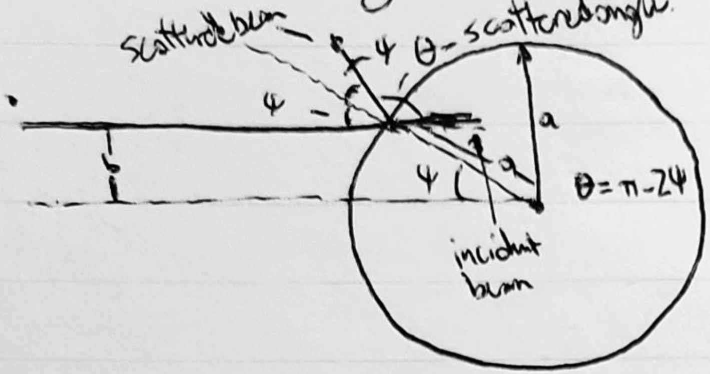
$$\therefore\ \sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{k}{2E}\,\frac{\frac{\cos\theta/2}{\sin\theta/2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,\frac{k}{4E}\,\frac{1}{\sin^2\frac{\theta}{2}}.$$

$$= \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4\theta/2}, \quad \text{with } k = czz'e^2.$$

$$\sigma(\theta) = \left(\frac{czz'e^2}{4E}\right)^2 \csc^4\left(\frac{\theta}{2}\right) \leftarrow \text{Rutherford Formula}$$

Hard Sphere Scattering



Hard Sphere Potential:

$V(r) = \infty \quad r \le a$

$\quad = 0 \quad r > a$.

For $b < a$, $\sin\psi = \frac{b}{a} \Rightarrow b = a\cos\frac{\theta}{2}$

$b > a$, $\theta = 0$

$\left|\frac{db}{d\theta}\right| = \frac{a}{2}\sin\frac{\theta}{2}$

$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{b}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \frac{a}{2}\sin\frac{\theta}{2}$.

$= \frac{ab}{4}\frac{1}{\cos\frac{\theta}{2}} = \frac{ab}{4}\frac{1}{b/a} = \frac{a^2}{4}$.

- Total Scattering Cross Section

$\sigma_t = \int_0^\pi \sigma(\theta)(\sin\theta)2\pi d\theta$.

$= 2\pi\frac{a^2}{4}\int_0^\pi \sin\theta d\theta$.

$= \cancel{\pi a^2}\cancel{[\cos\theta]} \ldots = \frac{4\pi a^2}{4} = \pi a^2$.