Ch 1 Introduction

- A differential equation is an equation containing at least one derivative
- Ex. $\frac{dy}{dt} = t$, $\frac{d^3y}{dt^3} + y\frac{d^2y}{dt^2} = \sin t$, $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial u^2} = 0$ (Laplace Eqn)

ordinary differentials — partial differential

① Classification of DE's

Classification
ODE — PDE
ORDER — ORDER
LINEAR NON-LINEAR " "

- Ordinary Differential Equations (ODEs)
- Partial Differential Equations (PDEs)

② Order

- $\frac{dy}{dt} = t$ ← first order ODE
- $\frac{d^3y}{dt^3} + y\frac{dy}{dt} = e^t$ ← Third order ODE

③ Linear or Non-linear

- A linear ODE is an equation that can be written in the form
  - $a_n(t)\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} \dots + a_1\frac{dy}{dt} + a_0 y = g(t)$

Example

- $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$ — 2^{nd} order linear ODE
- $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$ — 2^{nd} order non-linear ODE
- $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 t)y = t^3$ — 3^{rd} order linear ODE
- $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$ — 2^{nd} order non-linear ODE
- $ty' - y = t^2$ where $y = 3t + t^2$ is solution → $t(3+2t) - 3t + t^2$
  $= \cancel{3t} + 2t^2 - 3t + t^2$
  $= t^2$ ✓

Ch 2 First Order Differential Equations

2.1 First Order Linear Equations: Method of Integrating Factors

- $a_1(t)\frac{dy}{dt} + a_0(t)y = g(t)$, $a_1(t) \neq 0$

- $\frac{dy}{dt} + \frac{a_0(t)}{a_1(t)}y = \frac{g(t)}{a_1(t)}$ ← Divide eqn by $a_1(t)$

- $\frac{dy}{dt} + P(t)y = Q(t)$

- Ex. $2y' + y = 3t \Rightarrow y' + \frac{y}{2} = \frac{3}{2}t$. ← mult. by $\mu(t)$

$$\mu y' + \frac{\mu y}{2} = \frac{3}{2}\mu t$$

$\frac{d(\mu y)}{dt} = \mu\frac{dy}{dt} - y\frac{d\mu}{dt}$

$$\frac{d(\mu y)}{dt} - y\frac{d\mu}{dt} + \frac{1}{2}\mu y = \frac{3}{2}\mu t$$

$$\frac{d(\mu y)}{dt} - y\left(\cancel{\frac{d\mu}{dt} - \frac{\mu}{2}}\right) = \frac{3}{2}\mu t$$

$\frac{d\mu}{dt} - \frac{1}{2}\mu = 0$

$\frac{d\mu}{dt} = \frac{1}{2}\mu \Rightarrow \mu = e^{\frac{t}{2}}$

$$e^{\frac{t}{2}}y = \frac{3}{2}\int t e^{\frac{t}{2}}\,dt$$

$$e^{\frac{t}{2}}y = \frac{3}{2}\left[t2e^{\frac{t}{2}} - \int 2e^{\frac{t}{2}}\,dt\right]$$

$$e^{\frac{t}{2}}y = \frac{3}{2}\left[2te^{\frac{t}{2}} - 4e^{\frac{t}{2}} + k\right]$$

$$e^{\frac{t}{2}}y = 3te^{\frac{t}{2}} - 6e^{\frac{t}{2}} + c$$

$$y = 3t - 6 + e^{-\frac{t}{2}}c$$

- The Method/Process in variable form:

  - $a_1\frac{dy}{dt} + a_0 y = g$

$$\frac{dy}{dt} + Py = Q$$

$\frac{d(\mu y)}{dt} = \frac{\mu dy}{dt} - \frac{y d\mu}{dt}$

$$\mu\frac{dy}{dt} + P\mu y = Q\mu$$

$$\left(\frac{d(\mu y)}{dt} - \frac{y\,d\mu}{dt}\right) + P\mu y = Q\mu$$

Integrating Factor

$$\frac{d(\mu y)}{dt} - y\left(\cancel{\frac{d\mu}{dt} - P\mu}\right) = Q\mu \Leftrightarrow \frac{d\mu}{dt} = P\mu \Rightarrow \int\frac{d\mu}{\mu} = \int P\,dt \Rightarrow \ln\mu = \int P\,dt \Rightarrow \mu = e^{\int P\,dt}$$

$$\frac{d(\mu y)}{dt} = Q\mu$$

$$\mu y = \int Q\mu\,dt$$

$$y = \frac{\int Q\mu\,dt}{\mu}$$

∘Ex. $y' + \left(\frac{1}{t}\right)y = 3\cos(2t), \quad t > 0$ $\qquad \mu = e^{\int P\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$

$y' + \left(\frac{1}{t}\right)y = 3\cos(2t)$

t is variable others are fixed $\frac{d(ty)}{dt} = ty' + t'y = ty' + y$

$ty' + y = 3t\cos(2t)$

$\frac{d(ty)}{dt} = 3t\cos(2t)$

$\int d(ty) = \int 3t\cos(2t)\,dt.$

$ty = 3\left[\frac{t\sin 2t}{2} - \int \frac{\sin 2t}{2}\,dt\right]$

$ty = 3\left[\frac{t\sin 2t}{2} + \frac{\cos 2t}{4}\right]$

$ty = \frac{3}{2}t\sin 2t + \frac{3}{4}\cos 2t \Rightarrow y = \frac{3}{2}\sin 2t + \frac{1}{t}\frac{3}{4}\cos 2t + \frac{c}{t}$

Initial Value Problem (IVP)

•Ex. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$ $\qquad \mu = e^{\int \frac{t+1}{t}\,dt} = e^{\int 1 + \frac{1}{t}\,dt}$

$y' + \frac{(t+1)}{t}y = 1$ $\qquad = e^{t + \ln t} = e^{t}e^{\ln t} = te^{t}.$

$\frac{d(te^{t}y)}{dt} = te^{t}y' + e^{t}(t+1)y$

$te^{t}y' + e^{t}(t+1)y = te^{t}$

$te^{t}y = \int te^{t}\,dt.$

$te^{t}y = te^{t} - \int e^{t}\,dt$

$te^{t}y = te^{t} - e^{t} + c.$

$y = 1 - \frac{1}{t} + \frac{c}{te^{t}}$

$y(\ln 2) = 1 - \frac{1}{\ln 2} + \frac{c}{\ln 2\, e^{\ln 2}}$

$1 = 1 - \frac{1}{\ln 2} + \frac{c}{2\ln 2}$

$\frac{1}{\ln 2} = \frac{c}{2}\frac{1}{\ln 2} \quad c = 2.$

$y = 1 - \frac{1}{t} + \frac{2}{te^{t}}.$

∘Ex. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$ $\qquad P = e^{\int -\frac{1}{2}\,dt} = e^{-\frac{t}{2}}.$

$\frac{d(e^{-\frac{t}{2}}y)}{dt} = e^{-\frac{t}{2}}y' - \frac{1}{2}e^{-\frac{t}{2}}y \rightarrow$

$e^{-\frac{t}{2}}y' - e^{-\frac{t}{2}}\frac{1}{2}y = e^{-\frac{t}{2}}2\cos t$

$\frac{d(e^{-\frac{t}{2}}y)}{dt} = e^{-\frac{t}{2}}2\cos t$

$e^{-\frac{t}{2}}y = \int e^{-\frac{t}{2}}2\cos t\,dt.$

$e^{-\frac{t}{2}}y = 2\int e^{-\frac{t}{2}}\cos t\,dt.$

$e^{-\frac{t}{2}}y = 2\left[\frac{4}{5}e^{-\frac{t}{2}}(\sin t - \cos t)\right]$

$y = \frac{4}{5}(2\sin t - \cos t) + ce^{\frac{t}{2}}$

$y(0) = \frac{4}{5}(2\sin(0) - \cos(0)) + ce^{\frac{0}{2}}$

$a = -\frac{4}{5} + c$

$a + \frac{4}{5} = c$

$y = \frac{4}{5}(2\sin t - \cos t) + \left(a + \frac{4}{5}\right)e^{\frac{t}{2}}$

$I = \int e^{-\frac{t}{2}}\cos t\,dt = e^{-\frac{t}{2}}\sin - \int -\frac{1}{2}e^{-\frac{t}{2}}\sin t\,dt.$

$= e^{-\frac{t}{2}}\sin t + \frac{1}{2}\int e^{-\frac{t}{2}}\sin t\,dt.$

$= e^{-\frac{t}{2}}\sin t + \frac{1}{2}\left[e^{-\frac{t}{2}}(-\cos t) - \int -\frac{1}{2}e^{-\frac{t}{2}} - \cos t\,dt\right]$

$= e^{-\frac{t}{2}}\sin t + \frac{1}{2}\left[-e^{-\frac{t}{2}}\cos t - \frac{1}{2}\int e^{-\frac{t}{2}}\cos t\,dt\right]$

$I = e^{-\frac{t}{2}}\sin t - \frac{1}{2}e^{-\frac{t}{2}}\cos t - \frac{1}{4}I$

$\frac{5}{4}I = e^{-\frac{t}{2}}(\sin t - \cos t) \Rightarrow I = \frac{4}{5}e^{-\frac{t}{2}}(\sin t - \cos t)$

critical value, $a_0 = -\frac{4}{5}$

- If $a + \frac{4}{5} > 0$ then $\lim_{t\to\infty} = \infty$

If $a + \frac{4}{5} < 0$ then $\lim_{t\to\infty} y = -\infty$

If $a + \frac{4}{5} = 0$ the $\lim_{t\to\infty} y$ = bounded and oscillates

$\circ\ ty' + (t+1)y = 2te^{-t}$, $y(1)=a$, $t>0$ $\qquad \mu = e^{\int P dt} = e^{\int (1+\frac{1}{t})dt} = te^{t}$

$y' + (1+\frac{1}{t})y = 2e^{-t}$

$te^{t}y' + (1+\frac{1}{t})yte^{t} = 2t$

$\frac{d(te^{t}y)}{dt} = 2t$

$\int d(te^{t}y) = \int 2t\,dt$

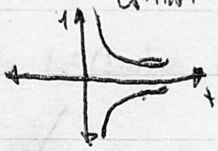
$te^{t}y = t^2 + c$

$y = te^{-t} + c\frac{1}{te^{t}}$

$y = te^{-t} + \frac{c}{t}e^{-t}$, $y(1) = e^{-1} + ce^{-1} = a \Rightarrow c = ae-1$

$y = te^{-t} + \frac{(ae-1)}{t}e^{-t}$. $\quad$ critical value when $ae-1=0$. $a_0 = \frac{1}{e}$.

Behavior when $t \to 0^+$



Bernoulli Equations

- The general form of the equation is $\frac{dy}{dt} + Py = Qy^n$, where n is a real number and P & Q are functions of t
- To achieve a first order differential divide by y^n
- $y^{-n}\frac{dy}{dt} + Py^{1-n} = Q$, Let $v = y^{1-n}$, then
- $\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt} \Rightarrow (1-n)y^{-n}\frac{dy}{dt} + (1-n)Py^{1-n} = (1-n)Q$

$= \frac{dv}{dt} + (1-n)Pv = (1-n)Q$ ★

- Ex. $\frac{dy}{dx} - y = xy^5 \Rightarrow y^{-5}\frac{dy}{dx} - y^{-4} = x$ $(v = y^{-4}) \Rightarrow$

$\Rightarrow -4y^{-5}\frac{dy}{dx} + 4y^{-4} = -4x \Rightarrow \frac{dv}{dx} + 4v = -4x$ $(\mu = e^{\int P dx}) \Rightarrow$

$\Rightarrow e^{4x}\frac{dv}{dx} + 4e^{4x}v = -4xe^{-4} \Rightarrow \frac{d(e^{4x}v)}{dx} = -4xe^{-4} \Rightarrow \int \frac{d(e^{4x}v)}{dx} = \int -4xe^{-4}\,dx$

$\Rightarrow e^{4x}v = -4\int xe^{4x}dx \Rightarrow e^{4x}v = -4\left[\frac{x}{4}e^{4x} - \int \frac{e^{4x}}{4}dx\right] \Rightarrow e^{4x}v = -xe^{4x} + \int e^{4x}dx$

$\Rightarrow -xe^{4x} + \frac{e^{4x}}{4} + c = e^{4x}v \Rightarrow v = -x + \frac{1}{4} + ce^{-4x} \Rightarrow y^{-4} = -x + \frac{1}{4} + ce^{-4x}$

$\Rightarrow y^{-4} = \frac{-4x + 1 + ce^{-4x}}{4} \Rightarrow y^{4} = \frac{4}{-4x + 1 + ce^{-4x}} \Rightarrow y = \frac{\pm\sqrt{2}}{(1 - 4x + ce^{-4x})^{\frac{1}{4}}}$

2.2 Separable Equations

- $\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow g(y)\frac{dy}{dx} = f(x) \Rightarrow g(y)\,dy = f(x)\,dx \Rightarrow \int g(y)\,dy = \int f(x)\,dx.$
- Ex. $y' = \frac{x^2}{y(1+x^3)} \Rightarrow \frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \Rightarrow \int y\,dy = \int \frac{x^2}{1+x^3}\,dx \Rightarrow \frac{y^2}{2} = \frac{1}{3}\int \frac{du}{u}$

  $\Rightarrow \frac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + c \Rightarrow y^2 = \frac{2}{3}\ln|1+x^3| + 2c.$
- Ex. $xy' = \sqrt{1-y^2} \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x} \Rightarrow \sin^{-1}(y) = \ln|x| + c. \Rightarrow y = \sin(\ln|x| + c)$
- Ex. $y(0)=1$ $y' = \frac{(e^{-x} - e^{x})}{3+4y} \Rightarrow \int (3+4y)\,dy = \int (e^{-x} - e^{x})\,dx \Rightarrow 3y + 2y^2 = -e^{-x} - e^{x} + c \Rightarrow 3+2 = -1-1+c\ (c=7)$

  $\Rightarrow 3y + 2y^2 = -e^{-x} - e^{x} + 7 \Rightarrow y^2 + \frac{3}{2}y = \frac{7 - e^{-x} - e^{x}}{2} \Rightarrow y^2 + \frac{3}{2}y + \frac{9}{16} = \frac{7 - e^{-x} - e^{x}}{2} + \frac{9}{16}$

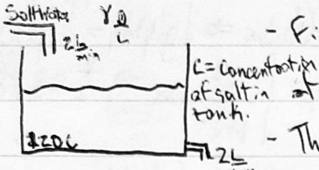
  $\Rightarrow \left(y + \frac{3}{4}\right)^2 = \frac{65 - 8e^{-x} - 8e^{x}}{16} \Rightarrow y + \frac{3}{4} = \frac{\sqrt{65 - 8e^{-x} - 8e^{x}}}{4} \Rightarrow y = \frac{-3 \pm \sqrt{65 - 8e^{-x} - 8e^{x}}}{4}$
- Ex $y' = \frac{(1-2x)}{y},\ y(1) = -2 \Rightarrow \int y\,dy = \int (1+2x)\,dx \Rightarrow \frac{y^2}{2} = x - x^2 + c \Rightarrow \frac{(-2)^2}{4} = 1 - 1^2 + c\ (c=2).$

  $\Rightarrow y^2 = 2x - 2x^2 + 4 \Rightarrow y = \pm\sqrt{2x - 2x^2 + 4} \Rightarrow y = -\sqrt{2x - 2x^2 - 4}$ with Initial Value.

Homogeneous Equations

- $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$ let $v = \frac{y}{x}$ and divide equation by v.
- $\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2\frac{y}{x}} \Rightarrow \frac{dy}{dx} = \frac{1+3v^2}{2v} \Rightarrow$ Find $\frac{dv}{dx}$. $v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + \frac{x\,dv}{dx}$
- $v + x\frac{dv}{dx} = \frac{1+3v^2}{2v} \Rightarrow \frac{x\,dv}{dx} = \frac{1+3v^2-2v^2}{2v} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} \Rightarrow \int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x}$
- $\ln|1+v^2| = \ln|x| + c. \Rightarrow \ln\left|\frac{1+v^2}{x}\right| = c \Rightarrow \frac{1+v^2}{x} = e^c \Rightarrow 1 + v^2 = xe^c$
- $\frac{y^2}{x^2} = xe^c - 1 \Rightarrow y^2 = x^3e^c - x^2 \Rightarrow y = \sqrt{x^3e^c - x^2}$

2.3 Modeling with First Order Equations.

- 

  - Find the amount of salt in the tank as a function of t
  - The amount of salt in the tank at time $t = y$

    (The rate of change of mass of salt = (the rate of salt coming in) − (the rate of salt going out).

$$\frac{dy}{dt} = 2\gamma - 2c \quad \text{where } c = \frac{y}{120}$$

$$\frac{dy}{dt} = 2\gamma - 2\frac{y}{120}$$

$$\frac{dy}{dt} + \frac{y}{60} = 2\gamma \qquad \mu = e^{\int \frac{1}{60} dt} = e^{\frac{t}{60}}$$

$$e^{\frac{t}{60}}\frac{dy}{dt} + e^{\frac{t}{60}}\frac{y}{60} = e^{\frac{t}{60}} 2\gamma$$

$$\frac{d(e^{\frac{t}{60}} y)}{dt} = 2\gamma e^{\frac{t}{60}}$$

$$e^{\frac{t}{60}} y = 2\gamma e^{\frac{t}{60}} 60 + c$$

$$y = 2\gamma \cancel{e^{\frac{t}{60}}} 60 + ce^{-\frac{t}{60}}$$

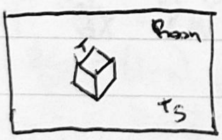
$$y = 120\gamma + ce^{\frac{t}{60}}$$

$$y(0) = 120\gamma + c$$

$$-120\gamma = c$$

$$y = 120\gamma\left(1 - e^{-\frac{t}{60}}\right)$$

- Newtons Law of Cooling



T_s - Temperature of Surroundings.
T - Temperature of object.

$$\frac{dT}{dt} = k(T - T_s)$$

$$\int \frac{dT}{T - T_s} = \int k\,dt,$$

$$\ln(T - T_s) = kt + c$$

$$T - T_s = e^{kt + c}$$

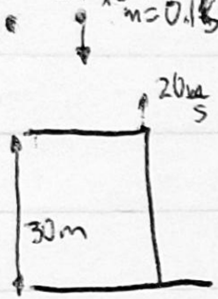
$$T(0) = T_0 \Rightarrow e^c \; T_0 - T_s$$

$$T - T_s = e^{kt}(T_0 - T_s)$$

$$T = e^{kt}(T_0 - T_s) + T_s$$

ball falls

$m = 0.18\,kg$ $\quad \frac{|v|}{30}$



$$ma = \Sigma F$$

$$m\frac{d^2y}{dt^2} = -mg - \frac{|v|}{30} = -mg - \frac{v}{30}$$

Max height = $v = \frac{dy}{dt}$

$$m\frac{dv}{dt} = -mg - \frac{v}{30}$$

$$m\frac{dv}{dt} + \frac{v}{30} = -mg$$

$$\frac{dv}{dt} + \frac{1}{30}v = -g \qquad \mu = e^{\int \frac{1}{30}dt}$$

$$e^{\frac{t}{30}}\frac{dv}{dt} + \frac{e^{\frac{t}{30}}}{30}v = -e^{\frac{t}{30}}g$$

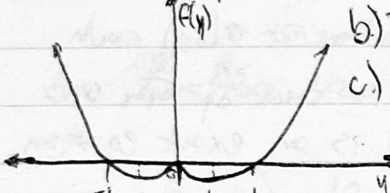
$$\frac{d(e^{\frac{t}{30}}v)}{dt} = -e^{\frac{t}{30}}g$$

$$e^{\frac{t}{30}}v = -g\,30\,e^{\frac{t}{30}} + C$$

2.5 Autonomous Equations

- An equation of the form $\frac{dy}{dt}=f(y)$ is called an autonomous equation
- $K \to f(K)=0$
- $y=K \leftarrow$ Equilibrium Solution (Critical Point)

- Ex. $\frac{dy}{dt}=y^2(y^2-1)$
  a.) Sketch $f(y)$ against y
  b.) Determine critical points
  c.) classify equilibrium solution, draw phaseline, and sketch graphs in the t-y plane.



$f'(y)=(y^4-y^2)'$

$=4y^3-2y$

$=2y(2y^2-1)$

Critical points $\to \pm\frac{1}{\sqrt{2}}, 0$

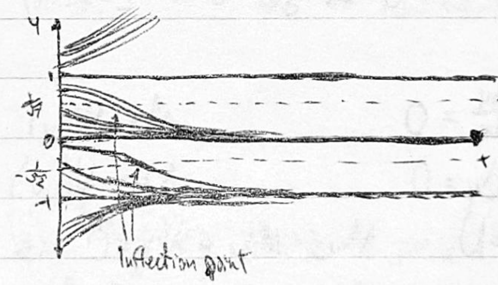
Equilibrium Points $\to \pm 1, 0$

phaseline $\to$

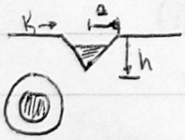
$y=-1 \leftarrow$ asymptotically stable solution

$y=1 \leftarrow$ unstable solution

$y=0 \leftarrow$ semistable solution



- Ex Water is lost at a rate proportional to the surface area.



a) show $\frac{dV}{dt}=k-\alpha\pi\left(\frac{3a}{\pi h}\right)^{\frac{2}{3}}V^{\frac{2}{3}}$

b.) find equilibrium depth of water, Is the equilibrium asymptotically stable?

c.) find a condition for the pond not to overflow.

$\frac{dV}{dt}=k-\alpha\pi r^2$

$\frac{dV}{dt}=k-\alpha\pi\left(\frac{3aV}{\pi h}\right)^{\frac{2}{3}}$

$\frac{dV}{dt}=k-\alpha\pi\left(\frac{3a}{\pi h}\right)^{\frac{2}{3}}V^{\frac{2}{3}}$

2.6 Exact Equations

- $z = \phi(x,y)$ x, y are functions of t
- $\frac{dz}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$
- Case $t = x$:
  - $\frac{\partial z}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$
- Consider a DE of the form $M + N\frac{dy}{dx} = 0$, where M and N are functions of x and y. Is there a function $z = \phi(x,y)$ such that $M = \frac{\partial \phi}{\partial x}$ and $N = \frac{\partial \phi}{\partial y}$? If there is then $\frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$. We say that $M + N\frac{dy}{dx} = 0$ is an exact equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- $M dx + N dy = 0 \iff M + N\frac{dy}{dx} = 0 \iff M_y = N_x$
- $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{\partial z}{\partial x} = 0 \Rightarrow z = \phi(x,y) = K$.

New →

- $M + N\frac{dy}{dx} = 0$ $\qquad M = M(x,y)$
  $M dx + N dy = 0$ $\qquad N = N(x,y)$
- If $M_y = N_x$, the $M dx + N dy = 0$ is an exact equation
- In that case, there is a function $z = \phi(x,y)$ such that $M = \frac{\partial z}{\partial x}$ and $N = \frac{\partial z}{\partial y}$.
- Then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = 0$ (z = constant).
- Example $(2xy^2 + 2y) + (2x^2y + 2y)y' = 0$

  $(2xy^2 + 2y)dx + (2x^2y + 2y)dy = 0$

  $\frac{\partial(2xy^2 + 2y)}{\partial y} = 4xy + 2$ $\qquad \frac{\partial(2x^2y + 2y)}{\partial y} = 4xy + 2$

  This implies that there is a function $z = \phi(x,y)$ such that $\frac{\partial z}{\partial x} = 2x^2y + 2y$ ① and $\frac{\partial z}{\partial y} = 2x^2y + 2x$ ②

① ~~$z = x^2y^2 + 2xy$~~

$= \int 2x^2y + 2y \, dx$

$z = x^2y^2 + 2xy + f(y)$

$\frac{\partial z}{\partial y} = 2x^2y + 2x + f'(y)$ ← ③

from ② & ③, we see that $f'(y) = 0$ so $f(y)$ = constant. Therefore,

$z = x^2y^2 + 2xy + c$

Since $\frac{\partial z}{\partial x} = 0$, z = constant

$x^2y^2 + 2xy + c = K$

$x^2y^2 + 2xy = C_2$ ← This is the solution

• Example $\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0$, $x > 0$.

$\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$

$\frac{\partial(\frac{y}{x} + 6x)}{\partial y} = \frac{1}{x}$ $\quad$ $\frac{\partial(\ln x - 2)}{\partial x} = \frac{1}{x}$ ← The equation is exact. Find z such that $\frac{\partial z}{\partial x} = \frac{y}{x} + 6x$ and $\frac{\partial z}{\partial y} = \ln x - 2$.

- Since, $\frac{\partial z}{\partial y} = \ln x - 2$, we have $z = y\ln x - 2y + f(x)$
- Differentiating WRT x, we get $\frac{\partial z}{\partial x} = \frac{y}{x} + f'(x)$.
- From that we see $f'(x) = 6x$ and subbed into z,

$z = y\ln x - 2y + 3x^2$

- Therefore $z = y\ln x - 2y + 3x^2 = C$.

• Ex. $(2x + 4y) + (2x - 2y)y' = 0$.

$\frac{\partial(2x + 4y)}{\partial y} = 4$ $\quad$ $\frac{\partial(2x - 2y)}{\partial x} = 2$. ← Not exact.

• Ex. $(ye^{2xy} + x) + bxe^{2xy}y' = 0$. Find b such that the equation is exact.

Want ⇒ $\frac{\partial}{\partial y}(ye^{2xy} + x) = \frac{\partial(bxe^{2xy})}{\partial x}$

⇒ $ye^{2xy}(2x) + e^{2xy}(1) = b[xe^{2xy}(2y) + e^{2xy}]$

⇒ $b = 1$

Solve $(ye^{2xy} + x) + xe^{2xy}y' = 0$. Find z such that $\frac{\partial z}{\partial x} = ye^{2xy} + x$ and $\frac{\partial z}{\partial y} = xe^{2xy}$

$z = x\frac{e^{2xy}}{2x} + f(x)$

$z = \frac{1}{2}e^{2xy} + f(x)$

$\frac{\partial z}{\partial y} = xe^{2xy}$ $\quad$ $\frac{\partial z}{\partial x} = ye^{2xy} + f'(x)$ ⇐ $f'(x) = x$

$f(x) = \frac{x^2}{2}$

$z = \frac{1}{2}e^{2xy} + \frac{x^2}{2} = C$

- What happens if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$?
- we multiply the egation by $\mu = \mu(x,y)$ so the equation becomes satisfied.
- Ex $(x+2)\sin y + (x\cos y)y' = 0$.

$\frac{\partial (x+2)\sin y}{\partial y} = (x+2)\cos y \qquad \frac{\partial (x\cos y)}{\partial x} = \cos y \neq$ Not exact.

- Integrating factor $\mu = x e^x$

$(x+2)xe^x \sin y + (x^2 e^x \cos y)y' = 0.$

$\frac{\partial (x+2)xe^x \sin y}{\partial y} = (x+2)xe^x \cos y \qquad \frac{\partial (x^2 e^x \cos y)}{\partial x} = \cos y\, xe^x (x+2).$

- The new egation is now exact.
- $\frac{\partial z}{\partial x} = (x+2)xe^x \sin y$ and $\frac{\partial z}{\partial y} = x^2 e^x \cos y$
- $z = x^2 e^x \sin y + f(x)$
- $\frac{\partial z}{\partial x} = (x+2)\,xe^x (x+2) + f'(x)$.
- $f'(x) = 0 \Rightarrow f(x) = 0$
- $z = x^2 e^x \sin y = C$.

- Ex $(3x^2 y + 2xy + y^3) + (x^2+y^2)y' = 0$.

$\frac{\partial (3x^2y+2xy+y^3)}{\partial y} = 3x^2 + 2x + 3y^2 \quad , \frac{\partial}{\partial x}(x^2+y^2) = 2x$

- Not exact.
- $(3x^2y + 2xy + y)\mu + (x^2+y^2)\mu y' = 0$
- We want to find μ such that

$[(3x^2y + 2xy + y^3)\mu]_y = [(x^2+y^2)\mu]_x$

$\Rightarrow (3x^2y + 2xy + y^3)\mu_y + (3x^2 + 2x + 3y^2)\mu = (x^2+y^2)\mu_x + (2x)\mu$

- Is there a μ such that μ is a function of x only?

$(3x^2 + \cancel{2x} + 3y^2)\mu = (x^2+y^2)\frac{\partial \mu}{\partial x} + \cancel{(2x)\mu}$.

$3\cancel{(x^2+y^2)}\mu = \frac{d\mu}{dx}\cancel{(x^2+y^2)}$

$3\mu = d\mu/dx \Rightarrow \int \frac{d\mu}{\mu} = \int 3\,dx \Rightarrow \mu = e^{3x}$

There is a function $z = z(x,y)$ such that

$\frac{\partial z}{\partial x} = e^{3x}(3x^2y + 2xy + y^3)$ and $\frac{\partial z}{\partial y} = e^{3x}(x^2+y^2)$

$z = e^{3x}\left(x^2 y + \frac{y^3}{3}\right) + f(x)$

$\frac{\partial\left(e^{3x}\left(x^2y + \frac{y^3}{3}\right)\right)}{\partial x} = 3e^{3x}\left(x^2y + \frac{y^3}{3}\right) + e^{3x}(2xy) + f'(x)$

$= e^{3x}(3x^2y + y^3 + 2xy)$

$f'(x) = 0 \Rightarrow f(x) = 0$

solution

$z = e^{3x}\left(x^2 y + \frac{y^3}{3}\right) = \text{constant}$

• Ex $\quad y + (2xy - e^{-2y})y' = 0$

$\frac{\partial y}{\partial y} = 1 \qquad \frac{\partial(2xy - e^{-2y})}{\partial x} = 2y.$ ⇐ Not exact equation

$\mu y + \mu(2xy - e^{-2y})y' = 0$

$(\mu y)_y = [\mu(2xy - e^{-2y})y']_x$

$(\mu_y) y + \mu = \mu_x(2xy - e^{-2y}) + \mu(2y)$

Is there a μ such that μ is a function of x only? In that case $\mu_y = 0$.

$\mu = \frac{d\mu}{dx}(2xy - e^{-2y}) + 2\mu y. \Rightarrow \frac{d\mu}{dx} = \frac{\mu(1-2y)}{(2xy - e^{-2y})}$

There is no function of x only that is an integrating factor

Is there a function of y only that satisfies the equation? $(\mu_x = 0)$

$\mu_y y + \mu = 2\mu y. \Rightarrow \frac{d\mu}{dy} y + \mu - 2\mu y = 0 \Rightarrow \frac{d\mu}{dy} = \frac{2\mu y - \mu}{y} \Rightarrow \frac{d\mu}{\mu} = \frac{2y-1}{y} dy$

$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{2y-1}{y} dy \Rightarrow \ln\mu = 2y - \ln y \Rightarrow \ln\mu + \ln y = 2y \Rightarrow \ln \mu y = 2y$

$\Rightarrow \mu y = e^{2y} \Rightarrow \mu = \frac{e^{2y}}{y}$

$e^{2y} + (2xe^{2y} - \frac{1}{y})y' = 0$

$\frac{\partial(e^{2y})}{\partial y} = 2e^{2y} \qquad \frac{\partial(2xe^{2y} - \frac{1}{y})}{\partial x} = 2e^{2y}$ ⇐ Exact

$z = xe^{2y} + f(y)$

$\frac{\partial z}{\partial y} = 2xe^{2y} + f'(y) \Rightarrow f'(y) = \frac{1}{y} \Rightarrow f(x) = -\ln|y|$

The solution is $xe^{2y} - \ln|y| = c$

• $\frac{dy}{dx} + Py = Q$

$\mu \frac{dy}{dx} + \mu P y = Q\mu$

$\frac{d(\mu y)}{dx} = Q\mu \qquad \mu = e^{\int P dx}$

$Py - Q + \frac{dy}{dx} = 0$

$(Py - Q)dx + dy = 0$

$\mu(Py - Q)dx + \mu dy = 0$

$\mu_y(Py - Q) + \mu P = \frac{d\mu}{dx}$

$\mu P = \frac{d\mu}{dx}$

$\int \frac{d\mu}{\mu} = \int P dx \Rightarrow \mu = e^{\int P dx}$

• $\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow f(x)dx - g(y)dy = 0$

- Example $y' = 3 + t - y$, $y(0) = 1$ Find approximate values of the solution at $t = 0.1, 0.2, 0.3, \& \ 0.4$ with $h = 0.1$

n	t_n	y_n	$f(t_n, y_n) = 3 + t_n - y_n$
0	$t_0 = 0$	$y_0 = 1$	$f(t_0, y_0) = 1$
1	$t_1 = t_0 + h = 0.1$	$y_1 = y_0 + h f(t_0, y_0) = 1.2$	$f(t_1, y_1) = 1.9$
2	$t_2 = t_1 + h = 0.2$	$y_2 = y_1 + h f(t_1, y_1) = 1.39$	$f(t_2, y_2) = 1.81$
3	$t_3 = t_2 + h = 0.3$	$y_3 = y_2 + h f(t_2, y_2) = 1.571$	$f(t_3, y_3) = 1.729$
4	$t_4 = t_3 + h = 0.4$	$y_4 = y_3 + h f(t_3, y_3) = 1.7439$	$f(t_4, y_4) = 1.6561$

Systems of Differential Equations

- t - independent variable
- $x_1, x_2, \dots x_n$ - dependent variables
- $x_1' = a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n + g_1(t)$

  $x_2' = a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n + g_2(t)$

  $\dots$

  $x_n' = a_{n1}x_1 + a_{2n}x_2 + \dots a_{nn}x_n + g_n(t)$
- if $g_1(t) = g_2(t) = \dots = g_n(t) = 0$, then the system is homogeneous.
- We will consider the homogeneous case with $n=2$.
  - $x_1' = ax_1 + bx_2$

    $x_2' = cx_1 + dx_2$
  - Can be written as $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \vec{x}' = A\vec{x}$
  - Solving a similar case $\Rightarrow x' = rx$

$$\frac{dx}{dt} = rx$$

$$\int \frac{dx}{x} = \int r\,dt$$

$$\ln|x| = rt + c$$

$$x = e^{rt+c}$$

$$x = e^{rt}e^{c} \quad \leftarrow K = e^{c}$$

$$x = Ke^{rt}$$

  - Similar Solution $\vec{x}' = A\vec{x} \Rightarrow \vec{x} = \vec{e}_1 e^{rt} \Rightarrow r\vec{e}_1 e^{rt} = A\vec{e}_1 e^{rt}$

$$r\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} e^{rt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} e^{rt}$$

$$r e_1 e^{rt} = (ae_1 + be_2)e^{rt}$$

  - If that is a solution then, $A\vec{e}_1 = r\vec{e}_1$, which implies r is an eigenvalue and $\vec{e}_1$ is an eigenvector of A corresponding to r
  - Suppose r_1 & r_2 are two distinct eigenvalues of A, and $\vec{v}$ & $\vec{w}$ are eigenvectors corresponding to r_1 & r_2 respectively
    - Then, $\vec{x} = \vec{v}e^{r_1 t}$ & $\vec{x} = \vec{w}e^{r_2 t}$ are solutions
    - Aside, suppose $\vec{y}$ is a solution of $\vec{x}' = A\vec{x}$, then $c\vec{y}$ is also a solution
    - We have $\vec{y}' = A\vec{y}$, LHS $= c\vec{y}'$ & RHS $A(c\vec{y}) = cA\vec{y}$
    - LHS = RHS

- Suppose $\vec{y}$ & $\vec{z}$ are solutions of $\vec{x}' = A\vec{x}$
- Then, $\vec{y} + \vec{z}$ is also a solution, LHS $= \vec{y}' + \vec{z}'$ & RHS $= A(\vec{y} + \vec{z}) = A\vec{y} + A\vec{z}$
- LHS = RHS, since $\vec{y}' = A\vec{y}$ & $\vec{z}' = A\vec{z}$

- Following the aside, the general solution is
  - $\vec{x} = c_1 \vec{v} e^{r_1 t} + c_2 \vec{w} e^{r_2 t}$

- Brief review of eigenvectors & values
  - Let A be a 2x2 matrix
  - A non-zero vector $\vec{e}_1 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is called an eigenvector if $A\vec{e}_1 = r\vec{e}_1$ for some scalar r
  - Then, the scalar r is called an eigenvalue and $\vec{e}_1$ is an eigenvector corresponding to r
  - $\vec{e}_1 = \vec{0}$ is always a solution
  - $A\vec{e}_1 - r\vec{e}_1 = \vec{0} \Rightarrow (A - rI)\vec{e} = 0$, I = identity matrix.
  - Characteristic Equation - $\det(A - rI) = 0$

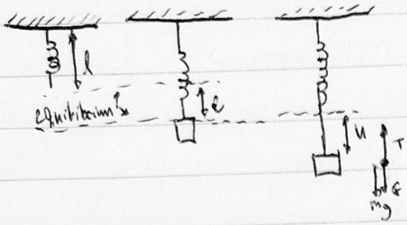
Solving Ordinary Differential Equations

① Of 1st Order - Integrating Factor ($e^{\int P dt}$)
- Bernoulli Equations ($\frac{dy}{dt} + Py = Qy^n \Rightarrow \frac{dv}{dt} + (1-n)Pv = (1-n)Q \Rightarrow v = y^{1-n}$)
- Separable Equations ($\frac{dy}{dx} = \frac{f(x)}{g(y)}$)
- Homogeneous Equations ($v = \frac{y}{x}$).

② Autonomous Equations.
③ Exact Equations + Integrating Factors ($M + N\frac{dy}{dx}$) $\Rightarrow$ ($M_y = N_x$).
④ Numerical Integration (Eulers Method). (
⑤ System
⑥ Homogeneous 2nd Order
⑦ Wronskian
⑧ Eulers Equations.
⑨ Non-homogeneous Second Order - Undetermined Coefficients (Particular Sol)
- Differential Operator Method
- Variation of parameters.
- Reduction

3.7 Mechanical and Electrical Systems

Spring-Mass Systems



$f = mu''$ $\quad T = k(e+u)$

$mg + f - T - \gamma u' = mu''$ $\quad mg = ke$

$mu'' + \gamma u' = mg + f - k(u+e)$

$mu'' + \gamma u' + ku = f$

- Undamped ($\gamma = 0$)

$mu'' + ku = 0$ $\qquad r = \pm i\omega_0 = \lambda \pm i\mu$

$u'' + \frac{k}{m}u = 0$ $\qquad u = e^{\lambda t}[c_1 \cos\mu t + c_2 \sin\mu t]$

$u'' + \omega_0^2 u = 0$ $\qquad u = e^{\lambda t}[c_1 \cos\omega_0 t + c_2 \sin\omega_0 t]$

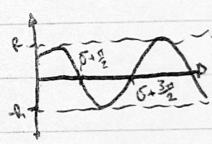
$r^2 + \omega_0^2 = 0$ ← characteristic eqn. $\quad u = \sqrt{c_1^2+c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega_0 t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega_0 t\right]$

Choose δ in $[0, 2\pi)$ such that $\cos\delta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\sin\delta = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$

Then, $u = \sqrt{c_1^2+c_2^2}\,[\cos\omega_0 t \cos\delta + \sin\omega_0 t \sin\delta]$

$= \sqrt{c_1^2+c_2^2}\cos(\omega_0 t - \delta)$

$= R\cos(\omega_0 t - \delta)$, $R = \sqrt{c_1^2+c_2^2}$



$\omega_0 t$ increases by 2π during one cycle

∴ the time taken for a cycle is $\frac{2\pi}{\omega_0}$ period of oscillation is $\frac{2\pi}{\omega_0}$

~~$\omega_0 = \sqrt{k/m}$~~

- Damped free vibrations ($\gamma > 0$)

$mu'' + \gamma u' + ku = 0$

$mr^2 + \gamma r + k = 0$

$\gamma^2 - 4mk$

Overdamped case $\gamma^2 > 4mk$

$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

$u_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1, r_2 < 0$

critically damped

Case $\gamma^2 = 4mk$

$r = -\frac{\gamma}{2m}, -\frac{\gamma}{2m}$

$u = (c_1 + c_2)e^{-\frac{\gamma}{2m}t}$

underdamped

case $\gamma^2 < 4mk$

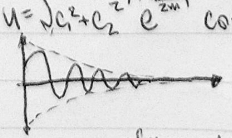
$r_{1,2} = \frac{-\gamma \pm \sqrt{(-1)(4mk - \gamma^2)}}{2m}$

$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$, $\lambda = -\frac{\gamma}{2m}$, $r = \lambda \pm i\mu$

$u = \sqrt{c_1^2+c_2^2}\, e^{-\frac{\gamma}{2m}t}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\mu t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\mu t\right]$

⋮

$u = \sqrt{c_1^2+c_2^2}\, e^{-\frac{\gamma}{2m}t}\cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t - \delta\right)$



μ = quasi frequency.

• Ex $mu'' + \gamma u' + ku = 0$, $\gamma^2 \geq 4mk$. show mass passes equilibrium once.

Overdamped case

$u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $r_1, r_2 < 0$

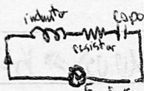
If $u = 0$ then $c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0$. Assume without loss of generality that $r_2 < r_1$

$$c_1 e^{r_1 t} = -c_2 e^{r_2 t}$$

$$c_1 e^{(r_1 - r_2)t} = -c_2$$

$$e^{(r_1 - r_2)t} = -\frac{c_2}{c_1} \Rightarrow t = \frac{1}{r_1 - r_2} \ln\left(-\frac{c_2}{c_1}\right)$$

Electrical Vibrations.

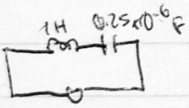
• LRC Circuits. - 

$V = IR$ - resistor; $V = L\frac{dI}{dt}$ - inductor; $V = \frac{Q}{C}$ - capacitor

• Kirchoff's Second Law - $L\frac{dI}{dt} + IR + \frac{Q}{C} = E$ $I = \frac{dQ}{dt}$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \left(\frac{1}{C}\right)I = \frac{dE}{dt}.$$

• Ex 

$Q(0) = 10^{-6}$ C

$L\frac{dI}{dt} + \frac{Q}{C} = 0$. $I(0) = 0$

$$L\frac{d^2Q}{dt} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt} + \frac{1}{0.25\times10^{-6}}Q = 0$$

$$\frac{d^2Q}{dt} + 4\times10^{6}Q = 0.$$

$Q = e^{rt}$

$r^2 + 4\times10^{6} = 0 \Rightarrow r = \pm i(2000)$

$Q = c_1\cos(2000t) + c_2\sin(2000t)$.

$Q(0) = c_1 \Rightarrow 10^{-6} = c_1$

$\frac{dQ}{dt} = I = -c_1 2000\sin(2000t) + c_2 2000\cos(2000t)$

$0 = c_2$

$$Q = 10^{-6}\cos(2000t).$$

• Ex. $C = 0.8\times10^{-6}$ f, $L = 0.2$H. Find R for critically damped.

$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0$

$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}(I) = 0$

$I = e^{rt}$

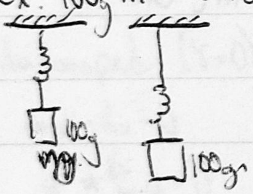
$Lr^2 + Rr + \frac{1}{C} = 0$

$R^2 - 4L\left(\frac{1}{C}\right) = 0$

$R^2 = 4\frac{L}{C}$

$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{0.2}{0.8\times10^{-6}}} = 1000\,\Omega$

• Ex. 100g m (0.1kg) Stretches spring by 5cm



$mu'' + \gamma u' + ku = 0$

$\gamma = 0$, $u(0) = 0$, $u'(0) = 0.1$ m/s.

$0.1u'' + ku = 0$ $\quad f = -ku \Rightarrow 0.1(9.8) = k(0.05) \Rightarrow k = \frac{98}{5}$

$0.1u'' + \frac{98}{5}u = 0$ $\quad u = e^{rt} \Rightarrow r^2 + 196 = 0 \Rightarrow r = \pm i\sqrt{196} = \pm i14$

$u'' + 2(98)u = 0$ $\quad u = C_1\cos(14t) + C_2\sin(14t)$

$u'' + 196u = 0$ $\quad u(0) = 0 = C_1$

$u'(0) = 0.1 = 14c_2\cos(14t)$ $\quad u = \frac{1}{140}\sin(14t)$

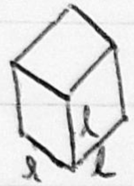
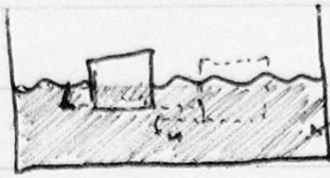
$\frac{0.1}{14} = C_2$

When does it 1st return to equilibrium?

$\sin(14t) = 0$

$14t = \pi$

$t = \frac{\pi}{14}$

• Ex  density $= \rho$  ρ_0: density $\rho_0 > \rho$

Archimedes Principle - Upthrust on the object = weight of the displaced fluid by object.

$$l^3 \rho g = l^2 L \rho_0 g \qquad ma = F$$

$$l \rho g = L \rho_0 g \qquad mu'' = mg - (u+L)l^2 \rho_0 g$$

$$l \rho = L \rho_0 \qquad = mg - Ll^2 \rho_0 g - u l^2 \rho_0 g$$

$$mu'' = \cancel{mg} - \cancel{l^3 \rho g} - u l^2 \rho_0 g$$

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \qquad mu'' + u l^2 \rho_0 g = 0$$

$$u'(0) = 0 = -\omega_0 c_1 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega t) \qquad l^3 \rho u'' + u l^2 \rho_0 g = 0$$

$$0 = 0 + c_2 \qquad u'' + \frac{u \rho_0 g}{l \rho} = 0 \qquad \omega_0^2 = \frac{\rho_0 g}{l \rho}$$

$$u = c_1 \cos(\omega_0 t) \qquad u'' + \omega_0^2 u = 0$$

$$r = \pm i \omega_0$$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{\rho_0 g}{l\rho}}} = 2\pi \sqrt{\frac{l \rho}{\rho_0 g}}$$

3.6 Variation of Parameters

- $y'' + p(t)y' + q(t)y = g(t)$
- Complementary Solution

  $y = c_1 y_1(t) + c_2 y_2(t)$
- Particular Solution

  Try $Y = u_1 y_1 + u_2 y_2$ where u_1 & u_2 are functions of t.

  Compute Y'' & Y' and substitute in this equation

- Ex. $y'' - y' - 2y = 2e^{-t}$

  C.S. $y'' - y' - 2y = 0$

  $r^2 - r - 2 = 0$

  $(r+1)(r-1)$

  $y = c_1 e^{-t} + c_2 e^{2t}$

Try $Y = u_1 e^{-t} + u_2 e^{2t}$

impose the condition $u_1' e^{-t} + u_2' e^{2t} = 0$.

then $Y' = u_1' e^{-t} - u_1 e^{-t} + u_2' e^{2t} + 2u_2 e^{2t}$

$Y' = -u_1 e^{-t} + 2u_2 e^{2t} + \underbrace{u_1' e^{-t} + u_2' e^{2t}}$

then $Y'' = -u_1' e^{-t} + u_1 e^{-t} + 2u_2' e^{2t} + 4u_2 e^{2t}$

substitute

$-u_1' e^{-t} + \cancel{u_1 e^{-t}} + 2u_2' e^{2t} + 4u_2 e^{2t} - (-\cancel{u_1 e^{-t}} + 2u_2 e^{2t}) - 2(u_1 e^{-t} + u_2 e^{2t}) = 2e^{-t}$

$-u_1' e^{-t} + 2u_2' e^{2t} = 2e^{-t}$

$-u_1' + 2u_2' e^{3t} = 2$ (1)

$u_1' e^{-t} + u_2' e^{2t} = 0$

$u_1' + u_2' e^{3t} = 0$ (2)

$u_1' = -u_2' e^{3t}$

$u_1' = -\frac{2}{3} e^{-3t} e^{3t}$

$u_1' = -\frac{2}{3}$

$u_1 = -\frac{2}{3} t$

$-(1) + (2) = 3u_2' e^{3t} = 2$

$u_2' = \frac{2}{3} e^{-3t}$

$\int u_2 = -\frac{2}{9} e^{-3t}$.

on

3.2 Solutions of Linear Homogeneous Equation Wronskian

Existence and Uniqueness Theorem

- Consider the IVP $y''+p(t)y'+q(t)y=g(t)$, $y(t_0)=A$, $y'(t_0)=B$, where A & B are real constants. $p(t)$, $q(t)$, and $g(t)$ are continuous functions on an open interval, I, containing t_0. Then there exists exactly one $y=\phi(t)$ to the IVP and it exists throughout, I.

Example

- $y''+(t-4)+3ty'+4y=2$

$$y''+\underbrace{\frac{3}{t-4}}_{p(t)}y'+\underbrace{\frac{4}{t(t-4)}}_{q(t)}y=\underbrace{\frac{2}{t(t-4)}}_{g(t)}$$

Wronskian of 'n' functions

- $f_1, f_2, \dots f_n$
- domain $I \subseteq \mathbb{R}$
- The Wronskian of $f_1, f_2, \dots f_n$ is

$$W(f_1,f_2,\dots f_n)=\begin{vmatrix} f_1(t) & f_2(t) & \dots & f_n(t) \\ f_1'(t) & f_2'(t) & \dots & f_n'(t) \\ f_1''(t) & f_2''(t) & \dots & f_n''(t) \\ \vdots & \vdots & \dots & \vdots \\ f_1^{n}(t) & f_2^{n}(t) & & f_n^{n}(t) \end{vmatrix}$$

Note: The functions $(f_1, f_2, \dots f_n)$ are linearly independent if $W(f_1, f_2, \dots f_n)$ is not identically zero

Example

- $e^t\sin t$, $e^t\cos t$
- $W=\begin{vmatrix} e^t\sin t & e^t\cos t \\ e^t\sin t+e^t\cos t & e^t\cos t-e^t\sin t \end{vmatrix}$

$= e^t\sin t[e^t\cos t - e^t\sin t] - e^t\cos t[e^t\sin t+e^t\cos t]$

$= -e^{2t}$

Cont. Solutions of Linear Homogeneous Equations: The Wronskian

- $y^{(n)} + a_{n-1}(t)y^{n-1} + a_{n-2}(t)y^{n-2} + \dots a_1 y' + a_0 y = 0$
- Suppose $y = y_1(t)$, $y = y_2(t)$, ..., $y = y_n(t)$ are solutions, if these functions are linearly independent then ~~they~~ the general solution is $y = c_1 y_1 + c_2 y_2 + \dots c_n y_n$
- In that case, we say that $y_1(t), y_2(t) \dots y_n(t)$ form a fundamental set of solutions
- $y_1, y_2 \dots y_n$ are linearly independent, if $W(y_1 \dots y_n) \neq 0$, if $y_1, y_2 \dots y_n$ form a fundamental set of solutions
- Example: $(1 - x\cot x)y'' - xy' + y = 0$ $\quad 0 < x < \pi$, $y_1(x) = x$, $y_2(x) = \sin x$
  - $y_1 = x$, $y_1' = 1$, $y_1'' = 0$

    RHS $= 0$ LHS $= 0$ sub n solve
  - $y_2 = \sin x$, $y_2' = \cos x$ $y_2'' = -\sin x$.

    RHS $= 0$ LHS $= 0$.
  - $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$
  - $\therefore$, y_1 & y_2 are solutions
  - General solution $y = c_1 x + c_2 \sin x$
- Ex $y''' + 2y'' - y' - 2y = 0$ $\quad y = e^t$, $y = e^{-t}$, $y = e^{-2t}$
  - $y''' = e^t$, $y'' = e^t$, $y' = e^t$, $y = e^t$

    LHS $= 0 =$ RHS
  - $y' = -e^{-t}$, $y'' = e^{-t}$, $y''' = -e^{-t}$

    LHS $= 0 =$ RHS
  - $y' = -2e^{-2t}$, $y'' = 4e^{-2t}$, $y''' = -8e^{-2t}$

    LHS $= 0 =$ RHS
  - $W(y_1, y_2, y_3) = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = e^t e^{-t} e^{-2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = e^{-2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix}$

    $= e^{-2t} \begin{vmatrix} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & 3 \end{vmatrix} = -6e^{-2t}$. General solution

    $y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}$

3.3 2^{nd} Order Linear Homogeneous Equations with Constant Coefficients: Roots of the Characteristic Equation are Complex

- $ay'' + by' + cy = 0$
- Try $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2 e^{rt}$
  - $ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$
    $ar^2 + br + c = 0$ ← Characteristic Equation
  - $b^2 - 4ac < 0$
- $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} = -\dfrac{b}{2a} \pm i\dfrac{\sqrt{4ac - b^2}}{2a} = \lambda \pm i\mu,\ \mu > 0$

$e^{i\theta} = \cos\theta + i\sin\theta$
$\pi: e^{i\pi} = -1$
$e^{i\pi} + 1 = 0$

- $r = \lambda \pm i\mu$
  - $y = e^{(\lambda + i\mu)t}$ is a solution
  - $y = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos\mu t + i\sin\mu t)$
  - $y = \underbrace{e^{\lambda t}\cos\mu t}_{y_1} + \underbrace{ie^{\lambda t}\sin\mu t}_{y_2}$

- $ay'' + by' + cy = 0$
  $a(y_1'' + iy_2'') + b(y_1' + iy_2') + c(y_1 + iy_2) = 0$
  $ay_1'' + by_1' + cy_1 + i[ay_2'' + by_2' + cy] = 0 = 0 + i0$
  $ay_1'' + by_1' + cy_1 = 0$ & $ay_2'' + by_2' + cy = 0$, $y = y_1$ & $y = y_2$ are solutions of $ay'' + by' + cy = 0$
- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$
- $y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t = e^{\lambda t}[\cos\mu t\, c_1 + \sin\mu t\, c_2]$
  where c_1 & c_2 are real constants.
- Ex $y'' + 4y = 0$ $y(0) = 0$, $y'(0) = 1$
  $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$
  $\lambda \pm i\mu = 0 \pm 2i$, $\lambda = 0$, $\mu = 2$.

  General Solution

  $y = c_1\cos(2t) + c_2\sin(2t)$
  $y' = -2c_1\sin(2t) + 2c_2\cos(2t)$
  $y(0) = 0 \rightarrow 0 = c_1(1) + c_2(0) \Rightarrow c_1 = 0$
  $y' = 2c_2\cos(2t)$
  $y'(0) = 1 \Rightarrow 1 = 2c_2\cos(0) \Rightarrow \frac{1}{2}$
  $y = \frac{1}{2}\sin(2t)$

Summary

- $ay'' + by' + cy = 0$
- Try e^{rt}

$ar^2 + br + c = 0$

$r = \lambda \pm i\mu$

$y = e^{(\lambda \pm i\mu)t}$

$y = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$

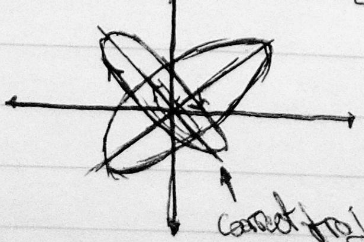
- General Solution $\to$ $y = e^{\lambda t}[c_1\cos\mu t + c_2\sin\mu t]$

Systems of Differential Equations

- $\vec{x}' = A\vec{x}$ $\quad x_1' = ax_1 + bx_2$
  $\quad x_2' = cx_1 + dx_2$
- Try $\vec{x} = e^{rt}\vec{e}$
- $A\vec{e} = r\vec{e}$
- $r = \lambda \pm i\mu$, $\mu > 0$ + $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, $\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$
- $\vec{e} = \vec{p} + i\vec{q}$
- A solution is $\vec{x} = (\vec{p} + i\vec{q})e^{(\lambda + i\mu)t}$
- Eulers formula → $e^{i\theta} = \cos\theta + i\sin\theta$
- $\vec{x} = (\vec{p} + i\vec{q})e^{\lambda t}(\cos\mu t + i\sin\mu t)$
  $\vec{x} = e^{\lambda t}\left[(\vec{p}\cos\mu t - \vec{q}\sin\mu t) + i(\vec{p}\sin\mu t + \vec{q}\cos\mu t)\right]$
- $\vec{x} = e^{\lambda t}\left[(\vec{p}\cos\mu t - \vec{q}\sin\mu t) + i(\vec{p}\sin\mu t + \vec{q}\cos\mu t)\right]$ is a solution as well as
  $\vec{x} = e^{\lambda t}(\vec{p}\cos\mu t - \vec{q}\sin\mu t)$ and $\vec{x} = e^{\lambda t}(\vec{p}\sin\mu t + \vec{q}\cos\mu t)$

Example

- $x_1' = 3x_1 + x_2$
  $x_2' = -13x_1 - 3x_2$ $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -13 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \vec{x}' = A\vec{x}$
- $\det\begin{bmatrix} 3-r & 1 \\ -13 & -3-r \end{bmatrix} = 0 \Rightarrow (3-r)(-3-r) + 13 = 0$
  $r^2 = -4 \Rightarrow r = \pm i2$
- $r = 2i$ $\quad \begin{bmatrix} 3-r & 1 \\ -13 & -3-r \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -13e_1 + (-3-2i)e_2 = 0$
- $\vec{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3+2i \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\quad \vec{e} = \vec{p} + i\vec{q}$
- The general solution is $\vec{x} = c_1\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin(2t)\right) + c_2\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(2t)\right)$
- $x_1 = c_1\cos(2t) + c_2\sin 2t$
  $x_2 = c_1(-3\cos(2t) - 2\sin(2t)) + c_2(-3\sin 2t + 2\cos 2t)$
-



correct trajectory

Complex Eigenvalues with Real Part Negative

- $x_1' = -3x_1 - 2x_2$ $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \vec{x}' = A\vec{x}$

  $x_2' = 4x_1 + x_2$
- $\vec{x} = \vec{\xi} e^{rt}$
- $\det \begin{bmatrix} -3-r & -2 \\ 4 & 1-r \end{bmatrix} = 0 \Rightarrow (-3-r)(1-r) + 8 = 0$

  $r = -1 \pm 2i \qquad \lambda = -1, \; \mu = 2$
- $r = -1 + 2i$

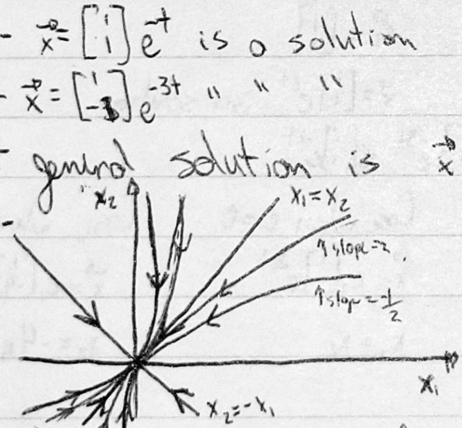
  $\begin{bmatrix} -3-(-1+2i) & -2 \\ 4 & 1-(-1+2i) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} (-2-2i)\xi_1 - 2\xi_2 = 0 \\ (1+i)\xi_1 + \xi_2 = 0 \end{matrix}$
- $2\xi_1 + (1-i)\xi_2 = 0$
- $\vec{x} = e^{\lambda t}\left(c_1[\vec{p}\cos\mu t - \vec{q}\sin\mu t] + c_2[\vec{p}\sin\mu t + \vec{q}\cos\mu t]\right)$
- $\vec{x} = e^{-t}\left(c_1\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin 2t\right] + c_2\left(\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos 2t\right]\right)\right)$

Systems of Differential Equations

- x_1, x_2, t
- $x_1' = ax_1 + bx_2$

  $x_2' = cx_1 + dx_2$
- $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $\vec{x}' = A\vec{x}$
- Look for solutions of the form $\vec{x} = \vec{e}_{\xi} e^{rt} \Rightarrow \vec{x}' = r\vec{e}_{\xi} e^{rt}$
- $r\vec{e}_{\xi} e^{rt} = A\vec{e}_{\xi} e^{rt} \Rightarrow A\vec{e}_{\xi} = r\vec{e}_{\xi}$
- $x_1' = -2x_1 + x_2$ $\qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \vec{x}' = A x$

  $x_2' = x_1 - 2x_2$
  - (-1) is an eigen value and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector
  - (-3) " " " " " $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ " " "
  - $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$ is a solution
  - $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$ " " "
- The general solution is $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$



Phase Plane / Phase Portrait.

$x_1' = \frac{dx_1}{dt}$

$x_2' = \frac{dx_2}{dt}$

- Case when $c_2 = 0$

  $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

  $x_1 = c_1 e^{-t}, \ x_2 = c_1 e^{-t}$

  $x_1 = x_2$

Case when $c_1 = 0$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$

$x_1 = c_2 e^{-3t}, \ x_2 = -c_2 e^{-3t}$

$x_2 = -x_1$

$\vec{x} = e^{-t}\left[c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}\right]$

Slope: $\frac{dx_2}{dx_1} = \frac{d\left(\frac{x_2}{t}\right)}{d\left(\frac{x_1}{t}\right)} = \frac{x_2'}{x_1'} = \frac{x_1 - 2x_2}{-2x_1 + x_2}$

- As $t \to -\infty$, we see that $\frac{dx_2}{dx_1} \to -1$
- The critical point is asymptotically stable, and the type is called a nodal sink.

Ex

- $x_1' = x_1 + x_2$, $x_2' = 4x_1 - 2x_2$ $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$
- seek solutions of the form $\vec{x} = \vec{e}_1 e^{rt}$

I = identity matrix

- $A\vec{e}_1 = r\vec{e}_1 \Rightarrow A\vec{e}_1 - rI\vec{e}_1 = \vec{0} \Rightarrow (A - rI)\vec{e}_1 = \vec{0}$
- Set determinant of $A - rI = 0$.
- $\det\left(\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\right) = 0$

  $\det\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} = 0 \Rightarrow (1-r)(-2-r) - 4 = 0 \Rightarrow r^2 + r - 6 \Rightarrow (r-2)(r+3)$ $r = -3, 2$.
- Eigen vectors

$r = -3$

$\begin{bmatrix} 1-r & 1 \\ 4 & -2-r \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4e_1 + e_2 = 0$

$\vec{e}_1 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$ is a solution

$r = 2$

$\begin{bmatrix} 1-r & 1 \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

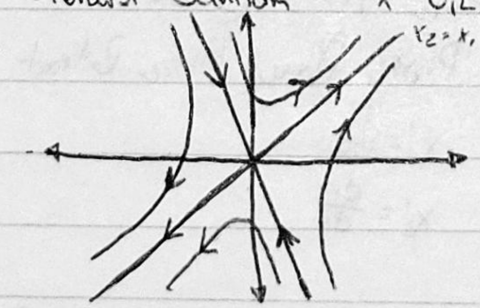
$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} -e_1 + e_2 = 0 \\ 4e_1 - 4e_2 = 0 \end{matrix}\right\} \Rightarrow e_1 = e_2$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ is a solution

- General Solution $\rightarrow$ $\vec{x} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$
-



Case when $c_1 = 0$

$\vec{x} = c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$x_1 = x_2$

Case when $c_2 = 0$

$\vec{x} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

$x_2 = -4x_1$

- The critical point is a saddle point and unstable.