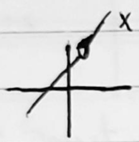
Appendix A and Sections 1.1-1.4

- To be Known
- Pre calc.

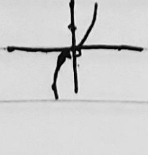
Section 2.2 Limits:

- $f(x) = \frac{x^2-4}{x-2}$, is considered undefined when $x=2$
-  when graph we see it is undefined at $x=2$.
- When f(x) is factored, we get x+2
- Since the graph approaches 2 on the x, then the y value approaches 4 ($y=x+2$)
- This means that 4 is the limit as x approaches 2.
- $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

Definition

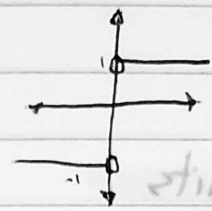
- Let a and L be real numbers
- Let f be a function defined near a, except possibly at a. We say that the limit of f(x) as x approaches a is L, and write $\lim_{x \to a} f(x) = L$, if f(x) can be arbitrarily close to L by keeping x sufficiently close to a

Example

- $g(x) = \begin{cases} \frac{x^3}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ $\begin{cases} x>0 \to x^2 \\ x<0 \to -x^2 \end{cases}$ $\lim_{x \to 0} g(x) = 0$
- 

Example

- $f(x) = \dfrac{|x|}{x}$ $\begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined at } 0 \end{cases}$



$\lim_{x \to 0} f(x) =$ doesn't exist.

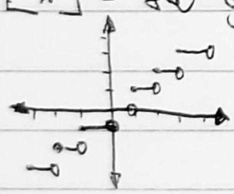
when approaching 0, there are 2 unique values

- $\lim_{x \to 0^+} f(x) = 1$ ← Approaches from the right.
- $\lim_{x \to 0^-} f(x) = -1$ ← Approaches from the left.

} One sided limits

Greatest Integer Function (Floor Function)

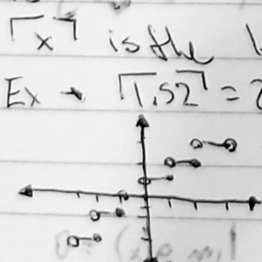
- This function is denoted by the symbol $\lfloor \ \rfloor$, and its domain is the set of all real numbers and the codomain is the set of all real integers.
  - Ex → $\lfloor 1.52 \rfloor = 1$; $\lfloor 3.91 \rfloor = 3$; $\lfloor \pi \rfloor = 3$
  - $\lfloor x \rfloor$ = the greatest integer less than or equal to x



$\lim_{x \to 2} \lfloor x \rfloor$ = doesn't exist

Least Integer Function (Ceiling Function)

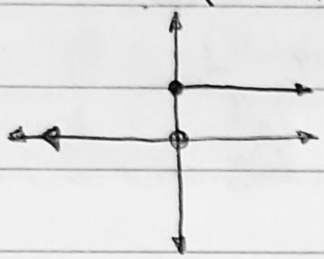
- Denoted by $\lceil \ \rceil$ and the domain is $\mathbb{R}$ and the codomain is $\mathbb{Z}$
  - $\lceil x \rceil$ is the least integer greater than or equal to x
  - Ex → $\lceil 1.52 \rceil = 2$; $\lceil 3.91 \rceil = 4$; $\lceil \pi \rceil = 4$; $\lceil 5 \rceil = 5$



$\lim_{x \to -1} \lceil x \rceil = \text{DNE}$

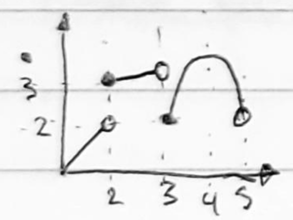
2.2

Heaviside Function

- Denoted by $H(x)$
- $H(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$
-  $\lim_{x \to 0^+} H(x) = 1$ $\lim_{x \to 0^-} H(x) = 0$ $\lim_{x \to 0} H(x) = \text{DNE}$

- Sometimes called the heaviside step function, unit step function, or Oliver Heaviside function.

Examples



a) $g(2) = 3$

b.) $\lim_{x \to 2^-} g(x) = 2$

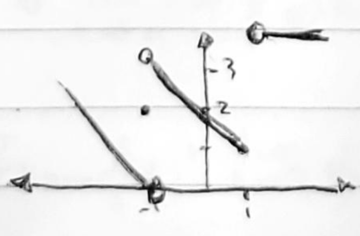
c.) $\lim_{x \to 2^+} g(x) = 3$

d.) $\lim_{x \to 2} g(x) = \text{DNE}$

e.) $g(3) = 2$

f.) $\lim_{x \to 3^-} g(x) = 3$

- $h(-1) = 2$, $\lim_{x \to -1^-} (h(x)) = 0$, $\lim_{x \to -1^+} h(x) = 3$, $h(1) = \lim_{x \to 1^-} h(x) = 1$, $\lim_{x \to 1^+} h(x) = 4$



2.3 Techniques for Computing Limits.

- Limit Laws
  - Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

① $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ ← Sum Rule.

② $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ ← Difference Rule.

③ $\lim_{x\to a}[cf(x)] = c\left(\lim_{x\to} f(x)\right)$ ← Constant Multiplier.

④ $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a}(g(x))\right)$ ← Product

⑤ $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)}$ ← Quotient ← Cannot be zero.

⑥ $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ ← Power, where n is a positive integer.

⑦ $\lim_{x\to a}[f(x)]^{\frac{m}{n}} = \left[\lim_{x\to a} f(x)\right]^{\frac{m}{n}}$ ← Fractional power, where m and n are positive in

* If $\frac{m}{n}$ is in its lowest terms, and n is even,

we require that $\lim_{x\to a} f(x) > 0$ *

- Example $p(x) = 3x^2-2x+10$ $\lim_{x\to 2} p(x) = \lim_{x\to 2}[3x^2-2x+10]$

$$= \lim_{x\to 2}[3x^2] - \lim_{x\to 2}[2x] + \lim_{x\to 2}[10]$$

$$= 3\lim_{x\to 2}[x^2] - 2\lim_{x\to 2}[x] + 10$$

$$= 12 - 4 + 10$$

$$= 18$$

① $\lim_{x\to a} p(x) = p(a)$ ← As $p(x)$ approaches a, sub in a to find the limit.

- Example: $\lim_{x\to 2} p(x) = 6x^3+2x+1$

$$= 6(2^3)+2(2)+1$$

$$= 48+4+1$$

$$= 52.$$

$\lim_{x\to -1} p(x) = 2x^6+2x-1$

$$= 2(-1^6)+2(-1)-1$$

$$= 2-2-1$$

$$= -1$$

- With Rational Functions
  - Example $r(x) = \frac{2x^2-x-1}{x+1}$

$$\lim_{x\to 2}\frac{2x^2-x-1}{x+1} = \frac{\lim_{x\to 2} 2x^2-x-1}{\lim_{x\to 2} x+1}$$ } Using Quotient Rule.

$$= \frac{2(2^2)-2-1}{2+1}$$

$$= \frac{5}{3}$$

Examples

- $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3}$, $\lim_{x\to 1} f(x) = 8$, $\lim_{x\to 1} f(x) = 3$

$= \lim_{x\to 1} (f(x)g(x)+3)^{\frac{1}{3}}$

$= \left(\lim_{x\to 1} f(x)g(x)+3\right)^{\frac{1}{3}}$

$= \left(\left(\lim_{x\to 1} f(x)\right)\left(\lim_{x\to 1} g(x)\right)+3\right)^{\frac{1}{3}}$

$= ((8)(3)+3)^{\frac{1}{3}}$

$= \sqrt[3]{27}$

$= 3$

$\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$ = DNE cause you cannot divide by 0. look to factor.

$= \frac{x^2-2x-3}{}$

$= \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}}$

$= \lim_{x\to 3} (x+1) = 4$

- $\lim_{b\to 2} \frac{3b}{\sqrt{4b+1}-1}$ $\rightarrow = \frac{\lim_{b\to 2} 3b}{\lim_{b\to 2} \sqrt{4b+1} - 1}$

$= \frac{6}{\left(\lim_{b\to 2} 4b + \lim_{b\to 2} 1\right)^{\frac{1}{2}} - 1}$

$= \frac{6}{(4(2)+1)^{\frac{1}{2}} - 1}$

$= \frac{6}{\sqrt{9}-1}$

$= \frac{6}{3-1}$

$= 3$

$\lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \frac{(5)(5+h)}{(5)(5+h)}$

$= \frac{5-(5+h)}{h(5)(5+h)}$

$= \frac{-\cancel{h}}{5h^2+25\cancel{h}}$

$= \frac{-1}{25+5h}$

$= \lim_{h\to 0} \frac{-1}{25+5h} = -\frac{1}{25}$

- $f(x) = \begin{cases} 0 & \text{if } x \le -5 \\ \sqrt{25-x^2} & \text{if } -5 < x < 5 \\ 3x & \text{if } x \ge 5 \end{cases}$ a) $\lim_{x\to -5^-} f(x) = 0$

b.) $\lim_{x\to -5^+} f(x) = \sqrt{25-x^2}$

$\lim_{x\to -5^+} f(x) = 0$

c.) $\lim_{x\to -5} f(x) = 0$.

d.) $\lim_{x\to 5^-} f(x) = \sqrt{25-x^2}$

$= 0$

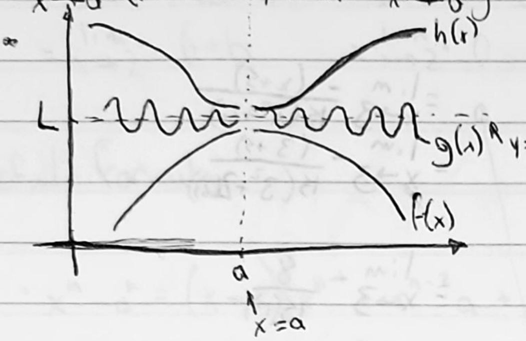
e.) $\lim_{x\to 5^+} f(x) = 3x$

$= 15$

f.) $\lim_{x\to 5} f(x) =$ DNE

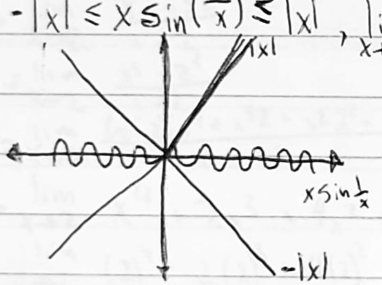
The Squeeze Theorem

- Let f, g, h be functions ~~st~~ satisfying $f(x) \le g(x) \le h(x)$ near a, except possibly at a. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.



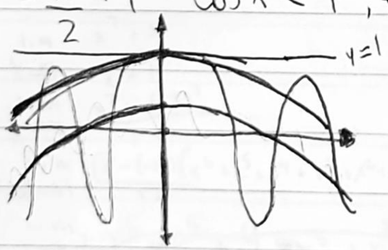
Examples.

- $-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|$, $\lim_{x \to 0} x \sin \frac{1}{x} = ?$



$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

- $-\frac{x^2}{2} + 1 \le \cos x \le 1$, for x near 0



$\lim_{x \to 0} 1 = 1$

$\lim_{x \to 0} -\frac{x^2}{2} + 1 = 1$

$\lim_{x \to 0} \cos x = 1$

- $\lim_{x \to a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}}$, $a > 0$

$= \frac{(x - a)(x + a)}{\sqrt{x} - \sqrt{a}}$

$= \frac{\cancel{(\sqrt{x} - \sqrt{a})}(\sqrt{x} + \sqrt{a})(x + a)}{\cancel{\sqrt{x} - \sqrt{a}}}$

$= (\sqrt{x} + \sqrt{a})(x + a)$

$= \lim_{x \to a} (\sqrt{x} + \sqrt{a})(x + a)$

$= \left(\lim_{x \to a} \sqrt{x} + \sqrt{a}\right)\left(\lim_{x \to a} x + a\right)$

$= \left(\lim_{x \to a} \sqrt{a} + \sqrt{a}\right)\left(\lim_{x \to a} a + a\right)$

$= (2\sqrt{a})(2a)$

$= 4a^{\frac{3}{2}}$

• $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$

$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^2 + ax + a^2)}{\cancel{x-a}}$

$= \lim_{x \to a} x^2 + ax + a^2$

$= \lim_{x \to a} a^2 + aa + a^2$

$= \lim_{x \to} 3a^2 = 3a^2$

• $\lim_{x \to 3} \frac{\left(\frac{1}{x^2+2x}\right) - \frac{1}{15}}{x - 3}$

$= \lim_{x \to 3} \frac{15 - (x^2 + 2x)}{(15)(x^2+2x)(x-3)}$

$= \lim_{x \to 3} \frac{-x^2 - 2x + 15}{(15)(x^2+2x)(x-3)}$

$\cancel{= \lim_{x \to 3} \frac{-(x^2 + 2x - 15)}{(15)(x^3 - 3x^2 + 2x^2 - 6x)}}$

$\cancel{= \lim_{x \to 3} -(x^2 + 2x - 15)}$

$= \lim_{x \to 3} - \frac{\cancel{(x-3)}(x+5)}{15\cancel{(x-3)}(x^2+2x)}$

$= \lim_{x \to 3} - \frac{(x+5)}{15(x^2+2x)}$

$= \lim_{x \to 3} - \frac{(3+5)}{15(3^2+2(3))}$

$= \lim_{x \to 3} - \frac{8}{15(15)}$

$= - \frac{8}{225}$

• $\lim_{x \to 1} \frac{\sqrt{10x-9} - 1}{x - 1}$

$= \lim_{x \to 1} \frac{(\sqrt{10x-9} - 1)(\sqrt{10x-9} + 1)}{(x-1)(\sqrt{10x-9} + 1)}$

$= \lim_{x \to} \frac{10x - 9 - 1}{\cancel{x\sqrt{10x-9}} + x - \cancel{\sqrt{10x-9}} - 1}$

$= \lim_{x \to 1} \frac{10\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{10x-9} - 1)}$

$= \lim_{x \to 1} \frac{10}{\sqrt{10x-9} - 1}$

$= \lim_{x \to 1} \frac{10}{\sqrt{10(1) - 9} - 1}$

$= \lim_{x \to 1} \frac{10}{2}$

$= \lim_{x \to 1} 5.$

- $f(x) = \begin{cases} 3x+b & \text{if } x \le 2 \\ x-2 & \text{if } x > 2 \end{cases}$ | Determine b as $\lim_{x\to 2}$ exists.

$= \lim_{x\to 2^-} f(x)$ $\qquad$ $= \lim_{x\to 2^+} f(x)$ $\qquad$ $b = -6.$

$= \lim_{x\to 2^-} 3(2)+b$ $\qquad$ $= \lim_{x\to 2^+} x-2$

$= \lim_{x\to 2^-} 6+b$ $\qquad$ $= \lim_{x\to 2^+} 0$

Useful Formula

- $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 \dots x a^{n-2} + a^{n-1})$

Examples

- $\lim_{x\to 2} \frac{x^5 - 32}{x-2}$

$= \lim_{x\to 2} \frac{x^5 - 2^5}{x-2}$

$= \lim_{x\to 2} \frac{(x-2)(x^4 + x^3 2 + x^2 2^2 + x 2^3 + 2^4)}{(x-2)}$

$= \lim_{x\to 2} x^4 + 2x^3 + 4x^2 + 8x + 16$

$= \lim_{x\to 2} (2)^4 + 2(2)^3 + 4(2)^2 + 8(2) + 16$

$= 80$

- $\lim_{x\to -1} \frac{x^7 + 1}{x+1}$

$= \lim_{x\to -1} \frac{x^7 - (-1)^7}{x+1}$

$= \lim_{x\to -1} \frac{(x-(-1))(x^6 + x^5 + x^4 + x^3 + x^2 + x 1)}{x+1}$

$= \lim_{x\to -1} \cancel{x^6 - x^5 + x^4 - x^3 + x^2 - x + 1}$

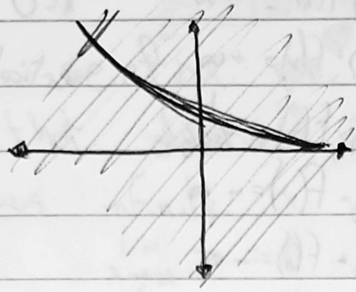
$= 7$

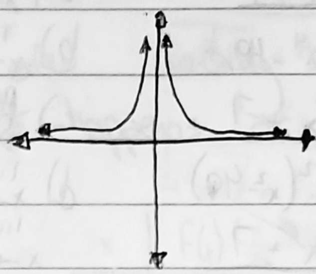
- $\lim_{x\to 1} \frac{f(x)}{x-1} = 2$

$f(x) = 2x - 2$

2.4 Infinite Limits

• $f(x) = \frac{1}{x^2}$

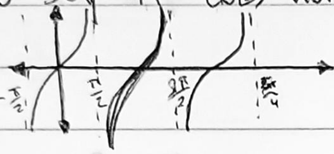




$\lim_{x\to 0}\left(\frac{1}{x^2}\right) = \infty$

• When a limit doesn't ~~approach~~ approach a finite number, it is correct to say it does not exist as well.

• $f(x) = \tan x$



$\lim_{x\to \frac{\pi}{2}^-}(\tan x) = \infty$, $\lim_{x\to \frac{\pi}{2}^+}(\tan x) = -\infty$

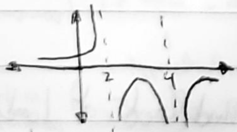
Vertical Asymptotes

• The line $x = a$ is a vertical asymptote to the graph of f, if at least one of the following holds:

① $\lim_{x\to a^-} f(x) = \infty$ ② $\lim_{x\to a^+} f(x) = \infty$

③ $\lim_{x\to a^-} f(x) = -\infty$ ④ $\lim_{x\to a^+} f(x) = -\infty$

Examples

• 

a.) $\lim_{x\to 2^-} f(x) = \infty$ c.) $\lim_{x\to 2} f(x) = \text{DNE}$

b.) $\lim_{x\to 2^+} f(x) = -\infty$

• $\lim_{x\to 1^+} \frac{(x-2)}{(x-1)^3} = -\infty$ $\lim_{x\to 1^-} \frac{(x-2)}{(x-1)^3} = \infty$ $\lim_{x\to 1} \frac{(x-2)}{(x-1)^3} = \text{DNE}$

• $\lim_{x\to -2^+}\left(\frac{x^3-5x^2+6x}{x^4-4x^2}\right)$

$= \lim_{x\to -2^+} \frac{\cancel{x}(x^2-5x+6)}{x^{\cancel{2}}(x^2-4)}$

$= \lim_{x\to -2^+} \frac{(\cancel{x-2})(x-3)}{x(\cancel{x-2})(x+2)}$

$= \lim_{x\to -2^+} \frac{(x-3)}{x(x+2)} = +\infty$

$\lim_{x\to -2^-} \frac{(x-3)}{x(x+2)} = -\infty$

$\lim_{x\to 2} \frac{(x-3)}{x(x+2)} = \frac{2-3}{2(2+2)} = \frac{-1}{8}$

• $\lim_{z\to 4} \frac{z-6}{(z^2-10z+\cancel{24})^2}$

$= \lim_{z\to 4} \frac{z-5}{((z-6)(z+6))^2} = -\infty$

- $f(x) = \dfrac{x+7}{x^4 - 49x^2}$

$= \dfrac{x+7}{x^2(x^2-49)}$

$= \dfrac{x+7}{x^2(x+7)(x-7)}$

$= \dfrac{1}{x^2(x-7)} \rightarrow \dfrac{\frac{1}{x^2}}{x-7}$

a.) $\lim\limits_{x \to 7^-} f(x) = -\infty$

b.) $\lim\limits_{x \to 7^+} f(x) = +\infty$

c.) $\lim\limits_{x \to -7} f(x) = -\dfrac{1}{686}$

d.) $\lim\limits_{x \to 0^-} f(x) = -\infty$

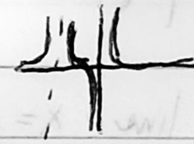
$\lim\limits_{x \to 0^+} f(x) = -\infty$

$\lim\limits_{x \to 0} f(x) = -\infty$

$x=0$, and $x=7$ are vertical asymptoto.

- $f(x) = \dfrac{\cos x}{x^2+2x} = \dfrac{\cos x}{x(x+2)}$

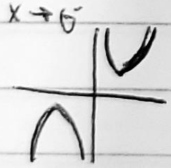
$x = 0$, -2 are VA's



$\lim\limits_{x \to 0^+} \dfrac{\frac{\cos x}{\cancel{x}(x+2)}}{x} = \infty$

$\lim\limits_{x \to -2^-} \dfrac{\frac{\cos x}{x}}{x+2} = +\infty$

- $\lim\limits_{x \to 0^-} \csc x = -\infty$ / $\lim\limits_{x \to 0^+} \csc x = \infty$



- $P(x)$ and $Q(x)$ such that $f(x) = \frac{P(x)}{Q(x)}$ is undefined at 1 and 2 but f has a VA at 2.

$f(x) = \dfrac{(x-1)}{(x-2)(x-1)}$

$= \dfrac{x-1}{x^2-3x+2}$

2.5 Limits at Infinity

- In this section we are dealing with limits that observe what happens as the graph approaches infinity.

- $f(x) = \dfrac{1}{x^2}$ $\lim_{x \to \infty} \dfrac{1}{x^2} = 0$ $\lim_{x \to -\infty} \dfrac{1}{x^2} = 0$

x	$f(x)$
1	1
10	0.01
100	0.0001
⋮	⋮

Horizontal Asymptotes.

- The line $y = L$ is a horizontal asymptote to the graph of f if $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$, or both

Examples.

- $\lim_{x \to \infty} \left(5 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 5$
- $\lim_{x \to -\infty} \left(5 + \dfrac{100}{x} + \dfrac{\sin^4(x^3)}{x^2}\right) = 5$

  Approaches0 Approaches 0.
- $\lim_{x \to \infty} \left(x^3 - x\right) = \infty$
- $\lim_{x \to -\infty} \left(3x^7 - x^2\right) = -\infty$
- $\lim_{x \to \infty} \dfrac{4x^2 - 7}{8x^2 + 5x + 2} \to \lim_{x \to \infty} \dfrac{\frac{4x^2 - 7}{x^2}}{\frac{8x^2 + 5x + 2}{x^2}}$

  $= \lim_{x \to \infty} \dfrac{4 - \frac{7}{x^2}}{8 + \frac{5}{x} + \frac{2}{x^2}}$

  $= \dfrac{1}{2}.$

  Degree of numerator is lessthan denominator = coefficients

- $f(x) = \dfrac{x^4 + 7}{x^5 + x^2 - x}$

  $= \dfrac{\frac{1}{x} + \frac{7}{x^5}}{1 + \frac{1}{x^3} - \frac{1}{x^4}}$

  $\lim_{x \to \infty} f(x) = 0$

  $\lim_{x \to -\infty} f(x) = 0$

  Degree of denominator is greater than numerator = 0.

- $f(x) = \dfrac{-x^3 + 1}{2x - 8}$

  $= \dfrac{-x^2 + \frac{1}{x}}{2 - \frac{8}{x}}$

  $\lim_{x \to \infty} f(x) = -\infty$

  $\lim_{x \to -\infty} f(x) = -\infty$

  Degree of numerator greater than denominator = $\pm\infty$

$\cdot\ f(x) = \dfrac{4x^3+1}{2x^3+\sqrt{16x^6+1}}$

$= \dfrac{4x^3+1}{2x^3+\sqrt{16x^6+1}}$

$= \dfrac{\frac{4x^3+1}{x^3}}{\frac{2x^3}{x^3}+\frac{\sqrt{16x^6+1}}{\sqrt{x^6}}}$

$= \dfrac{4+\frac{1}{x^3}}{2+\sqrt{16+\frac{1}{x^6}}}$

$= \dfrac{4}{2+\sqrt{16}}$

$= \frac{4}{6} = \frac{2}{3}$

$\lim_{x\to\infty} f(x) = \frac{2}{3}$

$\lim_{x\to-\infty} f(x) = -2$

$\cdot\ f(x) = 4x(3x-\sqrt{9x^2+1})$

$= \dfrac{4x(3x-\sqrt{9x^2+1})}{1}$

$= \dfrac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}}$

$= \dfrac{4x(9x^2-(9x^2+1))}{3x^2+\sqrt{9x^2+1}}$

$= \dfrac{4x(-1)}{3x+\sqrt{9x^2+1}}$

$= \dfrac{-\frac{4x}{x}}{\frac{3x}{x}+\frac{\sqrt{9x^2+1}}{\sqrt{x^2}}}$

$= \dfrac{-4}{3+\sqrt{9+\frac{1}{x^2}}}$

$= \dfrac{-4}{3\ \sqrt{9}}$

$= -\frac{2}{3}$

$\lim_{x\to\infty} f(x) = -\frac{2}{3}$

$\lim_{x\to-\infty} f(x) = \infty$

$\cdot\ f(x) = \frac{50}{e^{2x}}$ $\lim_{x \to \infty} f(x) = 0$ $\lim_{x \to -\infty} = \infty$

$\cdot\ f(x) = \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4}$ $\lim_{x \to \infty} f(x) = \frac{5}{2}$

$= \frac{\sqrt{\frac{16x^4}{x^4} + \frac{64x^2}{x^4}} + \frac{x^2}{x^2}}{\frac{2x^2}{x^2} - \frac{4}{x^2}}$ $\lim_{x \to -\infty} f(x) = \frac{5}{2}$

$= \frac{\sqrt{16 + \frac{64}{x^2}} + 1}{2 - \frac{4}{x^2}}$

$= \frac{4 + 1}{2}$

$= \frac{5}{2}$

$\bullet\ f(x) = \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$ $\longrightarrow$ $\bullet\ \dfrac{\sqrt{16x^4+64x^2}+x^2}{2(x-\sqrt{2})(x+\sqrt{2})}$

HA $\begin{cases} \lim_{x \to \infty} f(x) = \frac{5}{2} \\ \lim_{x \to -\infty} f(x) = \frac{5}{2} \end{cases}$ VA $\begin{cases} \lim_{x \to \sqrt{2}^+} f(x) = \infty & \lim_{x \to -\sqrt{2}^+} f(x) = -\infty \\ \lim_{x \to \sqrt{2}^-} f(x) = -\infty & \lim_{x \to -\sqrt{2}^-} f(x) = \infty \end{cases}$

$\bullet\ f(x) = \dfrac{|1-x^2|}{x(x+1)}$

HA $\rightarrow \lim_{x \to \infty} f(x) = \dfrac{x^2-1}{x^2+x}$

$= \dfrac{\frac{x^2}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2}}$

$= \dfrac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}}$

$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = \dfrac{x^2-1}{x(x+1)}$

$= \dfrac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}}$

$\lim_{x \to -\infty} f(x) = 1$

VA $\rightarrow$ $(x = 0, -1)$

$\lim_{x \to -1^+} f(x) = \dfrac{1-x^2}{x(x+1)}$

$= \dfrac{\cancel{(x+1)}(x-1)}{x\cancel{(x+1)}}$

$= \dfrac{x-1}{x}$

$\lim_{x \to -1^+} f(x) = -2$

$\lim_{x \to -1^-} f(x) = \dfrac{|1-x^2|}{x(x+1)}$

$= \dfrac{x^2-1}{x(x+1)}$

$= \dfrac{x-1}{x}$

$\lim_{x \to -1^+} f(x) = 2$

$x = -1$ isnt a vertical asymptote cause the limits dont go to infinity.

$\lim_{x \to 0^+} f(x) = \dfrac{1-x^2}{x(x+1)}$

$= \dfrac{(1-x)\cancel{(1+x)}}{x\cancel{(x+1)}}$

$= \dfrac{1-x}{x}$

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = -\infty$

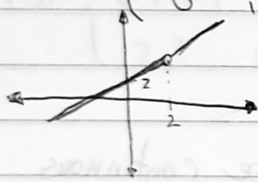
$x = 0$ is a VA

2.6 Continuity

- The function, f, is said to be continuous at a real number a if the following holds:
  1. The limit of the function as $x \to a$ exists ($\lim_{x \to a} f(x)$ exists)
  2. The function value at the limit exists ($f(a)$ exists)
  3. $\lim_{x \to a} f(x) = f(a)$

Example

- $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases} \rightarrow \begin{cases} x+2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$



$\lim_{x \to 2} f(x) = 4$ $\quad f(2) = 0$, not continuous

Rules of Continuity

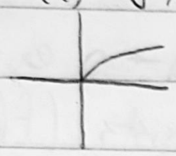
- Suppose functions, f & g, are continuous at a. Then so are the following:
  1. $f+g$ (addition of fxns)
  2. fg (multiplication of fxns)
  3. cf, where c is a constant
  4. f/g, (Division of fxns) where $g(a) \neq 0$
  5. $f-g$, (Subtraction of fxns)

One Side Continuity

- f is continuous from the left (or left-continuous) at a if
  1. $\lim_{x \to a^-} f(x)$ exists
  2. $f(a)$ exists
  3. $\lim_{x \to a^-} f(x) = f(a)$

Example

- $f(x) = \sqrt{x}$



$\lim_{x \to 0} \sqrt{0} = DNE$

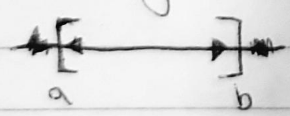
Not left continuous.

$\lim_{x \to 0^+} \sqrt{0} = 0 \rightarrow$ f is right continuous.

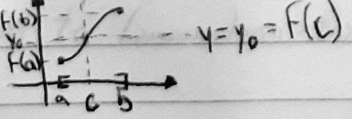
Theorem

- f is left-continuous and right continuous at a, then, f is continuous at a.

Continuity Over An Interval

- A function, f, is said to be continuous on an open interval (a,b) if f is continuous at each point of the interval
- A function, f, is said to be continuous on a closed interval $[a,b]$ if f is continuous on (a,b) and right-continuous at a and left continuous at b.
- 

Intermediate Value Theorem (IVT)

- Suppose f is continuous on a closed interval $[a,b]$. Let y_0 be a number strictly between $f(a)$ and $f(b)$. Then there is a number c in (a,b) such that $f(c) = y_0$.
- 

Examples.

- $f(x) = \dfrac{2x^2+3x+1}{x^2+5x}$, continuous at $a=-5$ $\quad f(-5) = \text{DNE}$, $\therefore$ f is not continuous at $f(-5)$

- $F(x) = \begin{cases} \dfrac{x^2-4x+3}{x-3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$, $a=3$, $\quad F(3) = 2$ ✓

$$\lim_{x\to3} f(x) = \lim_{x\to3} \frac{x^2-4x+3}{x-3} = \lim_{x\to3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to3} x-1 = 3-1 = 2 \checkmark$$

$\lim_{x\to3} f(x) = f(3) \to f$ is continuous

- $f(x) = \begin{cases} x^3+4x+1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0 \end{cases}$ $\quad f(0) = 1$

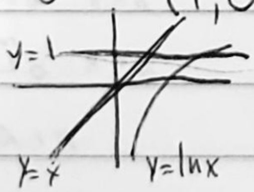
$\lim_{x\to0^-} x^3+4x+1 = 1$

$\lim_{x\to0^+} 2x^3 = 0$

$\lim_{x\to0} f(x) = \text{DNE}$ $\therefore$ the function is not continuous.

- $x\ln x - 1 = 0$ $\quad (1,e)$ $\quad e \simeq 2.718$

$[1,e]$



since ~~both~~ all are continuous, $x\ln x - 1$ is continuous.

Let $f(x) = x\ln x - 1$. Then f is continuous on $[1,e]$ because the function, x, $\ln$, and 1 are continuous on $[1,e]$

$f(1) = (1)\ln(1) - 1 = -1$

$f(e) = (e)\ln(e) - 1 \approx 1.718$

Since 0 is between $f(1)$ and $f(e)$ there must be a number c in $(1,e)$ such that $f(c) = 0$

$-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$

$f(0) = -(0)^5 - 4(0^2) + 2\sqrt{0} + 5$

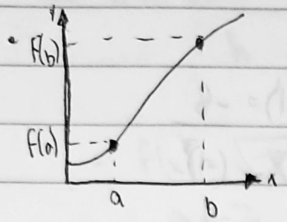
$= 5$

$f(3) = -(3^5) - 4(3^2) + 2\sqrt{3} + 5$

$= -270.5$

$f(c) = 0$ because the IVT gaurentees there is a solution when $f(0) = 5$ and $f(3) = -270.5$

2.1 and 3.1



average rate of change = $\dfrac{\text{increase of } f(x)}{\text{increase of } x}$

$= \dfrac{f(b) - f(a)}{b - a}$

- Instantaneous Rate of Change = $\lim\limits_{b \to a} \dfrac{f(b) - f(a)}{b - a}$

Slope of tangent line. = $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

- Instantaneous Rate of Change is the derivative of the function and is denoted by $f'(a)$

Example

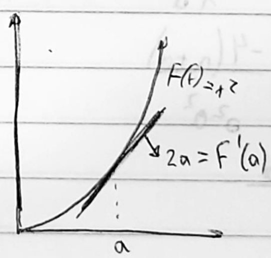
- $f(x) = x^2$
- $f'(a) = \lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x \to a} \dfrac{(x-a)(x+a)}{x - a}$

$= \lim\limits_{x \to a} x + a$

$= a + a$

$= 2a$



Note

- If the derivative exists at a, then we say the the function is differentiable at a. Otherwise, the function is not differentiable at a (limit exists → differentiable; DNE → Not differentiable)

Another Way to Write Derivatives

- $f'(a) = \dfrac{f(a+h) - f(a)}{a+h - a}$

$f'(a) = \dfrac{f(a+h) - f(a)}{h}$

$f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Example

- $f(x) = \frac{4}{x^2}$ $(-1, 4)$

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \frac{\frac{4}{x^2} - \frac{4}{a^2}}{x - a}$

$= \lim_{x \to a} \frac{4a^2 - 4x^2}{x^2 a^2 (x - a)}$

$= \lim_{x \to a} \frac{-4(-a+x)(a+x)}{a^2 x^2 \cancel{(x-a)}}$

$= \lim_{x \to a} \frac{-4(a+x)}{x^2 a^2}$

$= \lim_{x \to a} \frac{-4(a+a)}{a^2 a^2}$

$= \frac{-8a}{a^4}$

$= \frac{-8}{a^3}$

$f'(-1) = \frac{-8}{(-1)^3}$

$= -8/_{-1}$

$f'(-1) = 8$

tangent line $y = 8x + b$

$y = 8x + 12$

- $f(x) = \sqrt{x-1}$ $(2, 1)$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$= \lim_{h \to 0} \frac{(\sqrt{a+h-1} - \sqrt{a-1})(\sqrt{a+h-1} + \sqrt{a-1})}{h(\sqrt{a+h-1} + \sqrt{a-1})}$

$= \lim_{h \to 0} \frac{a+h-1 - (a-1)}{h(\sqrt{a+h-1} + \sqrt{a-1})}$

$= \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}\ \sqrt{a+h-1} + \sqrt{a-1}}$

$= \lim_{h \to 0} \frac{1}{\sqrt{a-1} + \sqrt{a-1}}$

$= \lim_{h \to 0} \frac{1}{2\sqrt{a-1}}$

$f'(2) = \frac{1}{2\sqrt{2-1}}$

$= \frac{1}{2}$

- $f(x) = \lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$ compare with $\frac{f(a+h) - f(a)}{h}$

$= f(x) = \sqrt{x}$

$a = 2.$

- $y = f(x)$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Notation

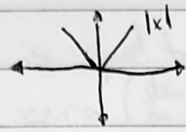
- $\underbrace{y' = f'(x)}_{\text{Newton}} = \frac{dy}{dx} = D_x f(x) = \frac{d[f(x)]}{dx}$

Leibniz

- $f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$

3.2 Working With Derivatives

- Consider, $f(x) = |x|$



- $f'(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{f(h)}{h} \rightarrow \lim_{h \to 0^+} \frac{f(h)}{h} \rightarrow \lim_{h \to 0^-} \frac{f(h)}{h}$

$= DNE \quad = \lim_{h \to 0^+} \frac{h}{h} \quad = \lim_{h \to 0^-} \frac{-h}{h}$

$\leftarrow = 1 \quad \leftarrow = -1$

Theorem

- If, f, is differentiable at a, then f is continuous at a.
- Proof: Since f is differentiable at a, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f'(a) \cdot (\lim_{x \to a} x - a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{\cancel{x-a}}\right)(\cancel{x-a})$ ← Cancel terms but on the other $x - a$, approaching 0.

$0 = \lim_{x \to a} f(x) - f(a)$

$= \lim_{x \to a} f(x) - \lim_{x \to a} f(a)$ ← $\lim_{x \to a} f(a)$ is just $f(a)$

$\lim_{x \to a} f(x) = f(a)$

Examples

- $y = \sqrt{x}$ $(4, 2)$ find an equation for the tangent at this point

$y' = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$

$= \lim_{h \to 0} \frac{(\sqrt{4+h} - \sqrt{4})(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)}$

$= \lim_{h \to 0} \frac{\cancel{4} + \cancel{h} - \cancel{4}}{\cancel{h}(\sqrt{4+h} + 2)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$

$= \lim_{h \to 0} \frac{1}{\sqrt{4} + 2}$

$y' = \frac{1}{4}$

$y = \frac{1}{4}x + b.$

3.3 Rules of Differentiation.

① $y = f(x) = c$, if the function is equal to some constant, the derivative is zero ($\frac{dy}{dx} = 0$)(Constant Rule).

② $y = f(x) = x^n$, if the function is some power, the derivative is nx^{n-1} (Power Rule)

③ $\frac{d(f(x)+g(x))}{dy} = \frac{d(f(x))}{dy} + \frac{d(f(x))}{dy}$, the derivative of two added functions is equal to the derivative of each added (Sum rule).
$(f(x)+g(x))' = f(x)' + g(x)'$

④ $\frac{d[cf(x)]}{dy} = c\frac{df(x)}{dy}$, a constant multiple to a function, is the same as differentiating the function and then multiplying by a constant (Constant Multiple Rule).

⑤ $(f(x)-g(x))' = f'(x) + (-g'(x))$, Derivative of a difference of functions is
$= f'(x) - g'(x)$ equal to the difference of the derivatives.
(Difference rule.)

⑥ e^x, the derivative of e^x is itself (e^x)

Example

- $g(x) = 6x^5 - x$
  $g'(x) = (6x^5)' - (x)'$
  $= 30x^4 - 1$

  $f(x) = 10x^4 - 32x + e^2$
  $f'(x) = (10x^4)' - (32x)' + (e^2)'$ ← not e^x, this is a constant
  $f'(x) = 40x^3 - 32$ ~~$+ e^2$~~

- $y = \frac{e^x}{4} - x$ find equation at the tangent line where $x = 0$.

  $y' = \frac{1}{4}(e^x)' - (x)'$

  $y' = \frac{e^x}{4} - 1$

  $f'(0) = \frac{e^0}{4} - 1$

  $f'(0) = -\frac{3}{4}$ Am

  $y = mx + b$

  $y = -\frac{3}{4}x + b$.

  $y = -\frac{3}{4}x + \frac{1}{4}$

- $f(t) = t^3 - 27t + 5$ (Find value of t such that tangent = 0)
  $f'(t) = 3t^2 - 27$
  $0 = 3t^2 - 27$
  $27 = 3t^2$
  $\frac{27}{3} = t^2$
  $\sqrt{9} = t$
  $\pm 3 = t$

  find when slope = 21
  $21 = 3t^2 - 27$
  $21 + 27 = 3t^2$
  $48 = 3t^2$
  $\frac{48}{3} = t^2$
  $\sqrt{16} = t$
  $\pm 4 = t$

Higher Order Derivatives

- $f(x) = 6x^5 - 3x^2 + 2e^x$

$\frac{dy}{dx}$ $f'(x) = 30x^4 - 6x + 2e^x$ ← 1st Derivative.

$\frac{d^2y}{dx^2} \Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dx}$ $f''(x) = 120x^3 - 6 + 2e^x$ ← 2nd Derivative

$\frac{d^3y}{dx^3}$ $f'''(x) = 360x^2 + 2e^x$ ← 3rd Derivative.

$f^{(4)}(x) = 720x + 2e^x$ ← 4th Derivative

$f^{(5)}(x) = 720 + 2e^x$ ← 5th Derivative

$\vdots$ $\vdots$

- In Leibnz notation, the n^{th} derivative is $\frac{d^n y}{dx^n}$
- In Newtons notation, the n^{th} derivative is $f^{(n)}(x)$,

Examples

- $f(x) = 3x^3 + 5x^2 + 6x$ Find the 1-3 derivatives.

$f'(x) = 9x^2 + 10x + 6$

$f''(x) = 18x + 10$

$f'''(x) = 18$

3.4 The Product and Quotient Rules

- $f(x) = x$, $g(x) = x^2$ $\qquad f(x)g(x) = x^3 \qquad f'(x)g'(x) = 2x$

  $f'(x) = 1$, $g'(x) = 2x \qquad (f(x)g(x))' = 3x^2 \neq f'(x)g'(x)$

The Product Rule

- $$\frac{d(f(x)g(x))}{dx} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)(f(x+h) - f(x))}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h)-g(x))}{h}$$

$$= \lim_{h\to 0} g(x+h) \frac{\lim_{h\to 0} f(x+h) - f(x)}{h} + \frac{f(x) \lim_{h\to 0} (g(x+h) - g(x))}{h}$$

$$= f'(x) \cdot g(x) + f(x)g'(x)$$

$$\boxed{= f'(x)\cdot g(x) + f(x)g'(x)}$$

- $(f(x)g(x))' = (1)(x^2) + x(2x)$

  $= x^2 + 2x^2$

  $= 3x^2$

$+\frac{1}{2}-1$

Example

- $g(x) = 6x - 2xe^x$

  $g'(x) = 6 - 2(xe^x)'$

  $g'(x) = 6 - 2(1(e^x) + xe^x)$

  $g'(x) = 6 - 2e^x - 2xe^x$

- $s = 4e^t\sqrt{t}$

  $s' = 4(e^t\sqrt{t})'$

  $s' = 4(e^x\sqrt{t} + e^x \frac{1}{2\sqrt{t}})$

  $s' = 4e^x(\sqrt{t} + \frac{1}{2\sqrt{t}})$

- $g(x) = e^w(5w^2 + 3w + 1)$

  $g'(x) = e^w(5w^2+3w+1) + e^w(10w+3)$

  $g'(x) = e^w(5w^2+3w+1+10w+3)$

  $g'(x) = e^w(5w^2+13w+4)$

The Quotient Rule

- $$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)g(x)\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right)}{g(x+h)g(x)\,h}$$

$$= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x+h)f(x)}{g(x+h)g(x)\,h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x)g(x+h)}$$

$$= \lim_{h\to 0} \frac{g(x)\left(f(x+h) - f(x)\right) - f(x)\left(g(x+h) - g(x)\right)}{h\,g(x)g(x+h)}$$

$$= \lim_{h\to 0} \left(\left(\frac{g(x)\left(f(x+h) - f(x)\right)}{h}\right) - \left(\frac{f(x)\left(g(x+h) - g(x)\right)}{h}\right)\right)\frac{1}{g(x)g(x+h)}$$

$$= \lim_{h\to 0} \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \boxed{\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

~~Example~~ $\frac{d(e^x)}{dx}$

- $$\frac{d(e^{kx})}{dx} = ke^{kx}$$

Examples.

- $f(x) = \dfrac{x^3 - 4x^2 + x}{x-2}$

$$f'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$= \frac{(x-2)(3x^2-8x+1) - (x^3-4x^2+x)(x)}{(x-2)(x-2)}$$

$$= \frac{3x^3 - 8x^2 + 1 - \cancel{2}x^2 + 16x - 2 - x^4 + 4x^3 - x^2}{x^2-2x-2x+4}$$

$$= \frac{-x^4 + 7x^3 - 19x^2 + 16x - 7}{x^2 - 4x + 4}$$

- $f(x) = \dfrac{2e^x - 1}{2e^x + 1}$

$$f'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$= \frac{(2e^x)(2e^x+1) - (2e^x-1)(2e^x)}{(2e^x+1)^2}$$

$$= \frac{4e^{2x} + 2e^x - 4e^{2x} + 2e^x}{(2e^x+1)(2e^x)^2}$$

$$= \frac{4e^x}{(2e^x+1)^2}.$$

- $f(x) = (1-2x)e^{-x}$

$= f'(x)g(x) + g'(x)(f(x))$

$= (2)(e^{-x}) + (-e^{-x})(1-2x)$

$= -2e^{-x} + e^{-x} + 2xe^{-x}$

$= -3e^{-x} + 2xe^{-x}$

$= 2xe^{-x} - 3e^{-x}$

- $f'(x) = \dfrac{2x^2}{3x-1}$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

$$= \frac{(4x)(3x-1) - (3)(2x^2)}{(3x-1)(3x-1)}$$

$$= \frac{12x^2 - 4x - 6x^2}{9x^2 - 6x + 1}$$

$$= \frac{6x^2 - 4x}{9x^2 - 6x + 1}$$

Slope at 1 $= \dfrac{6-4}{9-6+1}$

$= \dfrac{2}{4} \to \dfrac{1}{2}$

• $h(x) = \dfrac{(x-1)(2x^2-1)}{(x^3-1)}$

$h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$= \dfrac{[(x-1)(2x^2-1)]'(x^3-1) - [(x-1)(2x^2-1)](x^3-1)'}{(x^3-1)^2}$

$= \dfrac{[(x-1)'(2x^2-1)+(x-1)(2x^2-1)']'(x^3-1) - (x-1)(2x^2-1)(x^2)}{(x^3-1)^2}$

$= \dfrac{[(1)(2x^2-1) + (x-1)(4x)]'(x^3-1) - (x-1)(2x^2-1)(x^2)}{(x^3-1)^2}$

$= \dfrac{[(2x^2-1) + (4x)(x-1)](x^3-1) - (x-1)(2x^2-1)(x^2)}{(x^3-1)^2}$

• $h(x) = \dfrac{x+1}{x^2e^x}$

$= \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$= \dfrac{(1)(x^2e^x) - (x+1)(2xe^x)}{(x^2e^x)^2}$

$= \dfrac{x^2\cancel{e^x} - 2x^2\cancel{e^x} - 2x\cancel{e^x}}{x^4 e^{2x}}$

$= \cancel{e^x}\left\{\dfrac{x^2 - 2x^2 - 2x}{x^4 e^x}\right.$

$= \dfrac{x - 2x - 2}{x^3 e^x}$

$= \dfrac{-x-2}{x^3e^x}$

- $f(x) = \frac{1}{x} = x^{-1}$ $f'(x) = \frac{-1}{x^2} = x^{-2}$ $f''(x) = \frac{2}{x^3}$ $f'''(x) = \frac{-6}{x^4}$

3.5 Derivatives of Trigonometric Functions

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, where x is in radians.

- $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = \lim\limits_{x\to 0}\dfrac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to}\frac{\sin x}{1+\cos x}\right)$$

$$= (1)\ (0)$$

$$= 0.$$

$\lim\limits_{x\to 0}\dfrac{\sin x}{x}$	$= 1$
$\lim\limits_{x\to 0}\dfrac{\sin kx}{x}$	$= k$
$\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}$	$= \dfrac{m}{n}$
$\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$	$= 0$
$\lim\limits_{x\to 0}\dfrac{\tan x}{x}$	$= 1$
$\lim\limits_{x\to 0}\dfrac{\tan kx}{x}$	$= k.$
$\lim\limits_{x\to 0}\dfrac{\tan mx}{\tan nx}$	$= \dfrac{m}{n}$
$\lim\limits_{x\to 0}\dfrac{\cos x}{x}$	$=$ DNE

- $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{x}$, $3x=\theta$, $x=\dfrac{\theta}{3}$ $\Rightarrow \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\frac{\theta}{3}}$

$$= \lim_{\theta\to 0}\frac{3\sin\theta}{\theta}$$

$$= 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= 3(1)$$

$$= 3.$$

- $\boxed{\lim\limits_{x\to 0}\dfrac{\sin(kx)}{x} = k}$.

- $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x} = \lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{x}\cdot\dfrac{x}{\sin 5x}\right)$

$$= \lim_{x\to}\left(\frac{\sin 3x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{x}{\sin 5x}\right)$$

$$= 3 \cdot \frac{1}{5}$$

$$= \frac{3}{5}.$$

- $\boxed{\lim\limits_{x\to 0}\dfrac{\sin(mx)}{\sin(nx)} = \dfrac{m}{n}}$

- $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$

$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0}\left(\frac{1}{\cos x}\right)$

$= (1) \cdot (1)$

$= 1$

- $\lim_{x\to 0} \frac{\tan mx}{\tan nx} = \lim_{x\to 0}\left(\frac{\sin mx}{\sin nx}\right)\left(\frac{\cos nx}{\cos mx}\right)$

$= \frac{m}{n}$

Examples

- $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{\cos\theta}\left(\frac{\sec\theta - 1}{\theta}\right)$

$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\cos\theta \cdot \theta}$

$= \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right)$

$= \lim_{\theta\to 0} (0)\,(1)$

$= 0$

- $\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15} = \lim_{x\to -3} \frac{\sin(x+3)}{(x+3)(x+5)}$ let θ be $x+3$

$= \lim_{x\to -3} \frac{\sin\theta}{\theta(\theta+2)}$

$= \lim_{x\to -3} \frac{\sin\theta}{\theta} \cdot \frac{1}{(\theta+2)}$

$= (1) \cdot \left(\frac{1}{2}\right)$

$= \frac{1}{2}$

- $\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \lim_{x\to 0} \frac{\sin 3x}{\sin 4x} \cdot \cos 4x$

$= \left(\frac{3}{4}\right) \cdot (1)$

$= \frac{3}{4}$

- $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \frac{\sin(x-2)}{(x-2)(x+2)}$ let $\theta = (x-2)$

$= \frac{\sin\theta}{\theta} \cdot \left(\frac{1}{\theta+4}\right)$

$= 1 \cdot \left(\frac{1}{4}\right)$

$= \frac{1}{4}$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x \cdot \left(\frac{1}{\cos^2 x}\right)$
$\sec x$	$\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} (\tan x \cdot \sec x)$
$\csc x$	$-\csc x \cdot \cot x \left(\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}\right)$
$\cot x$	$-\csc^2 x \left(-\frac{1}{\sin^2 x}\right)$

$f'(x) = \sin x \Rightarrow \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

$= \lim_{h\to 0} \frac{\sin x(\cosh - 1) + \cos x \sin h}{h}$

$= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}\right)$

$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$

$= \lim_{h\to 0} (\sin x)\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} (\cos x)\left(\frac{\sin h}{h}\right)$

$= \lim_{h\to 0} (\sin x)(0) + (\cos x)(1)$

$= \cos x.$

- $f'(x) = \cos x \Rightarrow \lim_{h\to 0} = \frac{\cos(x+h) - \cos x}{h}$

$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$

$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$

$= \lim_{h\to 0} \frac{\cos x \cos h - 1}{h} - \lim_{h\to 0} \sin x \frac{\sin h}{h}$

$= 0 - (\sin x)(1)$

$= -\sin x$

- $f'(x) = \tan x \Rightarrow \frac{f'(\sin x)}{f'(\cos x)} \Rightarrow \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{(\cos x)^2}$

$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}.$

- $f'(\sec x) = f'\left(\frac{1}{\cos x}\right) = \frac{(1)'(\cos x) - (1)(\cos x)'}{\cos^2 x}$

$= \frac{0 - (-\sin x)}{\cos^2 x}$

$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}.$

Examples

- $y = \dfrac{x \sin x}{1+\cos x} \Rightarrow y' = \dfrac{(x\sin x)'(1+\cos x) - (x\sin x)(1+\cos x)'}{(1+\cos x)^2}$

$$= \frac{(x\cos x + \sin x)(1+\cos x) - (x\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{x\cos x + x\cos^2 x + \sin x + \sin x\cos x + x\sin^2 x}{(1+\cos x)^2}$$

$$= \frac{x\cos x + \sin x + \sin x\cos x + (x\cos^2 x + x\sin^2 x)}{(1+\cos x)^2}$$

$$= x\cos x + \sin x + \sin x\cos x + x$$

$$= \frac{x + x\cos x + \sin x + \sin\cos x}{(1+\cos x)^2}$$

$$= (x + \sin x) + \cos x(x + \sin x)$$

$$= \frac{(x+\sin x)\cancel{(1+\cos x)}}{(1+\cos x)^{\cancel{2}}}$$

$$= \frac{x+\sin x}{1+\cos x}$$

Derivatives of Trigonometric Functions

-

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc x$

Examples

- $y = \dfrac{\tan x}{1 + \tan x}$ $\qquad \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$$y' = \frac{(\tan x)'(1+\tan x) - (\tan x)(1+\tan x)'}{(1+\tan x)^2}$$

$$y' = \frac{(\sec^2 x)(1+\tan x) - (\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$y' = \frac{(\sec^2 x)(1+\tan x - \tan x)}{(1+\tan x)^2} \rightarrow \frac{\sec^2 x}{(1+\tan x)^2}$$

- $y = \tan x$

$y' = \sec^2 x$

$y'' = (\sec x)(\sec x)$

$= (\sec x)'(\sec x) + (\sec x)(\sec x)'$

$= \sec x \tan x \sec x + \sec x \sec x \tan x$

$= 2\sec^2 x \tan x.$

3.7 The Chain Rule

- $y = f(g(x)) \Rightarrow (f \circ g)(x)$
- $y' = f'(g(x)) \cdot g(x)$

Example

- $y = (1+x^2)^7$

  $y' = 7(1+x^2)^6 \cdot (2x)$

  $y' = 14x(1+x^2)^6$

- $y = (\sin x)^5$

  $y' = 5(\sin x)^4 \cdot (\cos x)$

- $y = \sqrt{7x-1}$

  $y' = \dfrac{1}{2\sqrt{7x-1}} \cdot 7$

  $y' = \dfrac{7}{2\sqrt{7x-1}}$

- $y = \csc(e^x)$

  $y' = -\csc(e^x) \cdot \cot(e^x) \cdot e^x$

-

3.7 The Chain Rule

- $f(g(x)) = (f(g(x)))' = f'g(x) \cdot g'(x)$ ← Newton notation

$$f(g(x))' = f'g(x) \cdot g'(x)$$

Examples

- $y = \sin(4\cos z)$ let $u = 4\cos z$

$y' = \cos f(g(x)) \cdot g'(x)$

$= \cos(4\cos z) \cdot (-4\sin z)$

- $y = \cos^4(7x^3) \rightarrow (\cos(7x^3))^4$

$f'(g(x)) \cdot g'(x) \rightarrow (4\cos(7x^3))^3 \cdot (\cos 7x^3)'$

$= (4\cos(7x^3))^3 \cdot (-\sin(7x^3) \cdot (14x^2))$

$= (4\cos^3(7x^3))(-\sin(7x^3))(21x^2)$

$= 84(\cos^3(7x^3))(-\sin 7x^3)(x^2)$

- $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

$y' = (x + (x + x^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$

$f'(g) \cdot g'(x)$

$= \frac{1}{2}(x + (x + x^{\frac{1}{2}})^{\frac{1}{2}})^{-\frac{1}{2}} \cdot (x + (x + x^{\frac{1}{2}})^{\frac{1}{2}})'$

$= \frac{1}{2(x + (x + x^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}} \cdot \left(1 + \left(\frac{1}{2(x + x^{\frac{1}{2}})^{\frac{1}{2}}}\right)(x + x^{\frac{1}{2}})'\right)$

$= \frac{1}{2(x + (x + x^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}} \cdot \left(1 + \left(\frac{1}{2(x + x^{\frac{1}{2}})^{\frac{1}{2}}}\right)\left(1 + \frac{1}{2x^{\frac{1}{2}}}\right)\right)$

$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot \left(1 + \left(\frac{1}{2\sqrt{x + \sqrt{x}}}\right)\left(1 + \left(\frac{1}{2\sqrt{x}}\right)\right)\right)$

- $y = \cancel{\$}\tan xe^x$

$f' g(x) \cdot g'(x)$

$= \sec^2(\cancel{x}e^x) \cdot (xe^x)'$

$= \sec^2(xe^x) \cdot (xe^x + e^x)$

$= \sec^2(xe^x)(e^x)(x+1)$

- $y'' = \cancel{x}\cos x^2$ $\quad f(x)g'(x) + f'(x)g(x)$

~~$y' = x(-\sin x^2) + (1)\cos x^2$~~

$y' = (1)\cos x^2 + x(\cos x^2)'$

$y' = \cos x^2 + x(-\sin x^2 \cdot 2x)$

$= \cos x^2 + 2x^2(-\sin x^2)$

$y'' = \cos x^2 - 2x^2(\sin x^2)$

$= (-\sin x^2)(2x) - ((4x)(\sin x^2) + (2x^2)(\sin x^2)')$

$= -(\sin x^2)(2x) - ((4x)(\sin x^2) + (2x^2)(x\cos x^2))$

$= -(\sin x^2)(2x) - (4x)(\sin x^2) + 2x^3\cos x^2$

$= -2x\sin x^2 - 4x\sin x^2 + 4x^3\cos x^2$

$= -6x\sin x^2 + 4x^3\cos x^2$

3.8 Implicit Differentiation

- $\cos(y^2) + x = e^y$
- ① Take the derivative of both sides

$$= \frac{d(\cos(y^2) + x)}{dx} = \frac{d(e^y)}{dx}$$

$$= \frac{d(\cos y^2)}{dx} + 1 = \frac{d(e^y)}{dx}$$ → Let $\cos y^2 = z$.

$$= \frac{dz}{dy} \cdot \frac{dy}{dx} + 1 = \frac{d(e^y)}{dx}$$

$$= (-\sin y^2 \cdot 2y) \cdot \frac{dy}{dx} + 1 = \frac{d(e^y)}{dx}$$

$e^y = z \Rightarrow \frac{dz}{dy} \cdot \frac{dy}{dx}$

$= e^y \cdot \frac{dy}{dx}$

$$= (-2y\sin y^2) \cdot \frac{dy}{dx} + 1 = e^y \cdot \frac{dy}{dx}$$

$$1 = e^y \cdot \frac{dy}{dx} + 2y\sin y^2 \frac{dy}{dx}$$

$$1 = \frac{dy}{dx}\left(e^y + 2y\sin y^2\right)$$

$$\frac{1}{e^y + 2y\sin y^2} = \frac{dy}{dx}$$

3.8 Implicit Differentiation

- $\tan(xy) = x + y$

$$\frac{d \tan(xy)}{dx} = 1 + \frac{dy}{dx}$$

$$\sec^2(xy) \cdot \frac{d(xy)}{dx} = 1 + \frac{dy}{dx}$$

$$\sec^2(xy) \cdot \left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$$

$$\sec^2(xy)x\frac{dy}{dx} + \sec^2(xy)y = 1 + \frac{dy}{dx}$$

$$\sec^2(xy)y - 1 = \frac{dy}{dx} - \sec^2(xy)x\frac{dy}{dx}$$

$$\sec^2(xy)y - 1 = \frac{dy}{dx}\left(1 - \sec^2(xy)x\right)$$

$$\frac{\sec^2(xy)y - 1}{1 - \sec^2(xy)x} = \frac{dy}{dx} \quad \text{at } (0,0) \text{ Slope} = \frac{0-1}{1-0} = -1$$

- $f(x) = x^4 - x^2y + y^4 = 1$ a.) verify $(-1,1)$ exists
  b.) Find tangent line

a.) $(-1)^4 - (-1)^2 1 + 1^4 = 1$

$1 - 1 + 1 = 1$

$1 = 1$

It is a point

b.) $f'(x) = x^4 - x^2y + y^4 = 0$

$\cancel{4x^3 - (2xy + x^2) + 4y^3 = 0}$

$$4x^3 - \left(2xy + x^2\frac{dy}{dx}\right) + \frac{d(y^4)}{dy} \cdot \frac{dy}{dx} = 0$$

$$4x^3 - \left(2xy + x^2\frac{dy}{dx}\right) + 4y^3 \cdot \frac{dy}{dx} = 0$$

$$4x^3 - x^2\frac{dy}{dx} - 2xy + 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(4y^3 - x^2\right) = 2xy - 4x^3$$

$$\frac{dy}{dx} = \frac{2xy - 4x^3}{4y^3 - x^2}$$

$$\frac{dy}{dx} = \frac{2xy - 4x^3}{4y^3 - x^2}$$

$$= \frac{2(-1)(1) - 4(-1)^3}{4(1)^3 - (-1)^2}$$

$$= \frac{-2 + 4}{4 - 1} = \frac{2}{3}.$$

$\cdot\ f''(x) = x^4 + y^4 = 64$

$$\frac{dx^4}{dy} + \frac{dy^4}{dy} \cdot \frac{dy}{dx} = 0$$

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$4x^3 = -4y^3 \frac{dy}{dx}$$

$$\frac{4x^3}{-4y^3} = \frac{dy}{dx}$$

$$-\frac{x^3}{y^3} = \frac{dy}{dx}$$

Easier for 2nd Derivative.

$$x^3 + y^3 \frac{dy}{dx} = 0$$

Product Rule.

$$3x^2 + \frac{dy^3}{dx} \cdot \frac{dy}{dx} + y^3 \frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + y^3 \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2 - 3y^2\left(\frac{dy}{dx}\right)^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2 - 3y^2\left(\frac{x^3}{y^3}\right)^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2 - 3y^2\left(\frac{x^6}{y^6}\right)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-3\left(x^2 + \frac{x^6}{y^4}\right)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2\left(\frac{y^4 + x^4}{y^4}\right)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2\left(x^4 + y^4\right)}{y^7}$$

$$\frac{d^2y}{dx^2} = -\frac{192x^2}{y^7}$$

• $f(x) = xy^{\frac{5}{2}} + x^{\frac{3}{2}}y = 12$ Slope at $(4,1)$

$$\frac{d(xy^{\frac{5}{2}})}{dx} + \frac{d(x^{\frac{3}{2}}y)}{dx} = 0$$

$$\left(y^{\frac{5}{2}} + x\frac{dy^{\frac{5}{2}}}{dy}\frac{dy}{dx}\right) + \left(\frac{3}{2}x^{\frac{1}{2}}y + x^{\frac{3}{2}}\frac{dy}{dy}\cdot\frac{dy}{dx}\right) = 0$$

$$y^{\frac{5}{2}} + x\,\frac{5}{2}y^{\frac{3}{2}}\frac{dy}{dx} + \frac{3}{2}x^{\frac{1}{2}}y + x^{\frac{3}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(x\cdot\frac{5}{2}y^{\frac{3}{2}} + x^{\frac{3}{2}}\right) = \left(-\frac{3}{2}x^{\frac{1}{2}}y - y^{\frac{5}{2}}\right)$$

$$\frac{dy}{dx} = \frac{-\frac{3}{2}x^{\frac{1}{2}}y - y^{\frac{5}{2}}}{x\frac{5}{2}y^{\frac{3}{2}} + x^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{-y\left(\frac{3}{2}x^{\frac{1}{2}} + y^{\frac{3}{2}}\right)}{x\left(\frac{5}{2}y^{\frac{3}{2}} + x^{\frac{3}{2}}\right)}$$

$$= \frac{-(1)\left(\frac{3}{2}(4)^{\frac{1}{2}} + (1)^{\frac{3}{2}}\right)}{4\left(\frac{5}{2}(1)^{\frac{3}{2}} - (4)^{\frac{1}{2}}\right)}$$

$$= \frac{-4}{9}.$$

3.9 Derivative of Logarithmic and Exponential Functions.

- $\frac{d(e^x)}{dy} = e^x$
- Derivative of $y = \ln x$, $x > 0$

$y = \ln x \Rightarrow x = e^y$

$\frac{d(x)}{dx} = \frac{d e^y}{dy} \cdot \frac{dy}{dx}$

$1 = e^y \cdot \frac{dy}{dx}$

$\frac{1}{e^y} = \frac{dy}{dx}$

$\frac{1}{x} = \frac{dy}{dx}$

$y' = \ln x \Rightarrow \frac{1}{x}$

$f(x)$	$f'(x)$
$\ln x$	$\frac{1}{x}$, $x > 0$
$\ln\lvert x\rvert$	$\frac{1}{x}$, $x \neq 0$
e^x	e^x
a^x	$a^x \ln a$, $x > 0$
$\log_a x$	$\frac{1}{x \ln a}$, $a > 0, a \neq 1$, $x > 0$
x^x	$x^x(\ln x + 1)$

- Derivative of $y = \ln\lvert x\rvert$, $x \neq 0$
  - If $x > 0$, $\frac{dy}{dx} = \frac{1}{x}$
  - If $x < 0$, $y = \ln\lvert x\rvert \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

$y = \ln(-x) \Rightarrow = f'(g(x)) \cdot g'(x)$

$= \frac{1}{-x} \cdot -1$

$= \frac{1}{x}$

- $f(x) = a^x$ $f'(x) = a^x$ where a is positive.

$y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

$\frac{d(\ln y)}{dy} \cdot \frac{dy}{dx} =$

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d\, x(\ln a)}{dx}$

$\ln a$ is a constant.
derivative of a fxn times a constant is the derivative of the fxn and then multiplyed by the contant.

$\frac{dy}{dx} = y \ln a$

$\frac{dy}{dx} = a^x \ln a.$

- $y = \log_a x \quad , a > 0, a \neq 1 \quad , x > 0$

$$= \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$$

Constant, no need for derivative.

$$y' = \left(\frac{1}{\ln a}\right)\frac{1}{x}$$

$$y' = \frac{1}{x \ln a}$$

- $y = x^x$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Technique used is logarithmic differentiation.

$$\frac{d(\ln y)}{dy} \cdot \frac{dy}{dx} = \frac{d(x \ln x)}{dx}$$

$$\frac{1}{y} \cdot \frac{dy}{dy} = (x)'(\ln x) + (x)(\ln x)'$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dy} = x^x(\ln x + 1)$$

- $y = x^{\ln x}$

$$\ln y = \ln x^{\ln x}$$

$$\ln y = \ln x \cdot \ln x$$

$$\frac{d(\ln y)}{dy}\frac{dy}{dx} = \frac{d((\ln x)^2)}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = f'(g(x)) \cdot g(x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2(\ln x)\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{y\,2(\ln x)'}{x}$$

$$\frac{dy}{dx} = \frac{x^{\ln x}\,2(\ln x)}{x}$$

$x = e$

$$\hookrightarrow = \frac{e^{\ln x}\,2(\ln e)}{e}$$

$$= 2.$$

• $f(x) = (\tan x)^{x-1}$

$\ln f(x) = \ln \tan x^{x-1}$

$\frac{1}{f(x)} \cdot f'(x) = (x-1)\ln(\tan x)$

$\frac{1}{f(x)} \cdot f'(x) = (x-1)' \ln(\tan x) + (x-1)(\ln \tan x)'$

$\frac{1}{f(x)} \cdot f'(x) = \ln \tan x + (x-1)\left(\frac{1}{\tan x} \cdot \sec^2 x\right)$

$\frac{1}{f(x)} \cdot f'(x) = \ln \tan x + \frac{(x-1)(\sec^2 x)}{\tan x}$

$f'(x) = \left((\tan x)^{x-1}\right)\left(\ln \tan x + \frac{(x-1)(\sec^2 x)}{\tan x}\right)$

$f'\left(\frac{\pi}{4}\right) = \left(\tan \frac{\pi}{4}^{\frac{\pi}{4}-1}\right)\left(\ln \tan \frac{\pi}{4} + \frac{\left(\frac{\pi}{4}-1\right)\left(\sec^2 \frac{\pi}{4}\right)}{\tan \frac{\pi}{4}}\right)$

$\tan \frac{\pi}{4} = 1$

$\sec \frac{\pi}{4} = \sqrt{2}$

$\sec^2 \frac{\pi}{4} = 2.$

$f'\left(\frac{\pi}{4}\right) = \left(1^{\frac{\pi}{4}-1}\right)\left(\ln(1) + \frac{\left(\frac{\pi}{4}-1\right)(2)}{1}\right)$

$=$

• $f(x) = \frac{x^8 \cos^3 x}{\sqrt{x-1}}$

* using logarithmic differentiation helps with simplifying fxns *

$\ln f(x) = \ln\left(\frac{x^8 \cos^3 x}{\sqrt{x-1}}\right)$

$= \ln x^8 + \ln \cos^3 x - \ln \sqrt{x-1}$

$\ln f(x) = 8 \ln x + 3 \ln \cos x - \frac{1}{2} \ln(x-1)$

$\frac{1}{f(x)} \cdot f'(x) = \frac{8}{x} + \frac{3 \cdot -\sin x}{\cos x} - \frac{1}{2(x-1)}$

$f'(x) = f(x)\left(\frac{8}{x} - \frac{3\sin x}{\cos x} - \frac{1}{2(x-1)}\right)$

$= f(x)\left(\frac{8}{x} - 3\tan x - \frac{1}{2(x-1)}\right)$

$$\cdot\ f(x) = \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}$$

$$\ln f(x) = \ln(x+1)^{\frac{3}{2}} + \ln(x-4)^{\frac{5}{2}} - \ln(5x+3)^{\frac{2}{3}}$$

$$\ln f(x) = \frac{3}{2}\ln(x+1) + \frac{5}{2}\ln(x-4) - \frac{2}{3}\ln(5y+3)$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}$$

$$f'(x) = \left[\frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}\right]\left[\frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}\right]$$

3.9 Log and Exponential Functions

-

$f(x)$	$f'(x)$
e^x	e^x
$\ln x$	$\frac{1}{x}, x>0$
$\ln\lvert x\rvert$	$\frac{1}{x}, x\neq 0$
a^x	$a^x \ln a, a>0$
$\log_b x$	$\frac{1}{\ln b\, x}, b>0, x>0$

- $f'(x) = x^2 \ln x$

$$= f'(x)g(x) + g'(x)f(x)$$
$$= (2x)(\ln x) + \left(\frac{1}{x}\right)(x^2)$$
$$= 2x\ln x + x$$

- $f''(x) = \dfrac{\ln x}{\ln x + 1}$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$
$$= \frac{\frac{1}{x}(\ln x + 1) - \ln x\left(\frac{1}{x}\right)}{(\ln x + 1)^2}$$
$$= \frac{\frac{1}{x}}{(\ln x + 1)^2}$$
$$= \frac{1}{x(\ln x + 1)^2}$$

- $f'(x) = e^x \ln x$

$$= f'(x)g(x) + g'(x)f(x)$$
$$= e^x \ln x + \frac{1}{x}e^x$$
$$= e^x \ln x + \frac{e^x}{x}$$

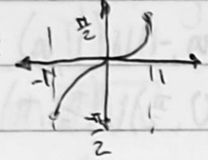
- $f(x) = \ln\left[\dfrac{(2x-1)(x+2)^3}{(1-4x)^2}\right]$

use chain rule, ln & 2x-1 ..

$$= \ln(2x-1) + 3\ln(x+2) - 2\ln(1-4x)$$
$$=$$
$$= f'(g(x))\cdot g'(x) + \ldots$$
$$= \frac{1}{(2x-1)}\cdot 2 + 3\left(\frac{1}{x+2}\right)(1) - 2\left(\frac{1}{1-4x}\right)(-4)$$
$$= \frac{2}{2x-1} + \frac{3}{x+2} + \frac{8}{1-4x}$$

3.10 Derivatives of Inverse Trigonometric Functions.

- Derivative of $\sin^{-1}x$ or $\arcsin x$
- $y = \sin^{-1}x$



$f(x)$	$f'(x)$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$, $1 < x < 1$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$, $1 \le x < 1$
$\tan^{-1}x$	$\frac{1}{x^2+1}$,
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$, $1 > x > -1$
$\csc^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$, $\lvert x\rvert > 1$
$\cot^{-1}x$	$\frac{-1}{1+x^2}$ $-\infty < x < \infty$

- $f(x) = y = \sin^{-1}x$

$$\sin y = \sin(\sin^{-1}x)$$

$$\sin y = x$$

$$\frac{d(\sin y)}{dx} \cdot \frac{dy}{dx} = \frac{d(x)}{dx}$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

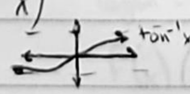
$$\frac{dy}{dx} = \frac{1}{+\sqrt{1-x^2}}$$

$$\sin^2 y + \cos y^2 = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = +\sqrt{1-x^2}$$

- $f(x) = \cos^{-1}x$ (same as $\sin^{-1}x$)
- $f(x) = y = \tan^{-1}x$



$$\tan y = \tan(\tan^{-1}x)$$

$$(\tan y)' = (x)'$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{\quad}}$$

$$\frac{dy}{dx} = \frac{1}{x^2+1}$$

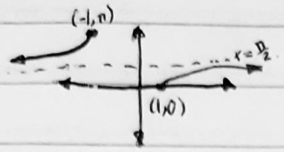
$$\sin^2 y + \cos^2 y = 1$$

$$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

$$\tan^2 y + 1 = \sec^2 y$$

Inverse Secant Function

• $\sec^{-1}x$



Domain ⇒ $(-\infty,-1)\cup(1,\infty)$

Range ⇒ $(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$

• $\sec^{-1}x = y$

$\sec(\sec^{-1}x)=\sec y$

$(x)'=(\sec(y))'$

$1=\sec y\tan y\,\frac{dy}{dx}$

$\frac{1}{\sec y\tan y}=\frac{dy}{dx}$

$1+\tan^2 y=\sec^2 y$

$\tan y=\sqrt{x^2-1}$

$\frac{1}{x\tan y}=\frac{dy}{dx}$

$\frac{1}{x\sqrt{x^2-1}}=\frac{dy}{dx}$

Examples

• $f(x)=x\sin^{-1}(x)$

$=f'(x)g(x)+f(x)g'(x)$

$=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$

• $f(x)=\sin^{-1}(e^{\sin x})$

$=f'(g(x))\cdot g'(x)$

$=\frac{1}{1-(e^{\sin x})^2}\cdot(e^{\sin x})'$

$=\frac{1}{1-e^{2\sin x}}\cdot e^{\sin x}\cdot(\sin x)'$

$=\frac{1}{1-e^{2\sin x}}\cdot e^{\sin x}\cdot\cos x$

$=\frac{e^{\sin x}\cos x}{1-e^{2\sin x}}$

- $f(x) = \sec^{-1}(e^x)$ Find tangent line at $(\ln 2, \frac{\pi}{3})$

$= f'(g(x)) \cdot g'(x)$

$= \dfrac{1}{|e^x|\sqrt{(e^x)^2 - 1}} \cdot e^x$

$= \dfrac{e^x}{|e^x|\sqrt{e^{2x} - 1}}$

$f'(\ln 2) = \dfrac{1}{\sqrt{e^{2x} - 1}}$

$f'(\ln 2) = \dfrac{1}{\sqrt{e^{2\ln 2} - 1}}$ $\quad (e^{\ln 2})^2 \to (2)^2 \to 4$

$= \dfrac{1}{\sqrt{4 - 1}}$

$= \dfrac{1}{\sqrt{3}}$

$y = \dfrac{x}{\sqrt{3}} + b$

$\dfrac{\pi}{3} - \dfrac{\ln 2}{\sqrt{3}} + b$

$y = \dfrac{x}{\sqrt{3}} + \left(\dfrac{\pi}{3} - \dfrac{\ln 2}{\sqrt{3}}\right)$

- $f(x) = \sin(\tan^{-1}(\ln x))$

$= \cos(\tan^{-1}(\ln x)) \cdot (\tan^{-1}(\ln x))'$

$= \cos(\tan^{-1}(\ln x)) \cdot \dfrac{1}{(\ln x)^2 + 1} \cdot \dfrac{1}{x}$

$= \cos(\tan^{-1}(\ln x)) \cdot \dfrac{1}{x(\ln x^2 + 1)}$

$= \dfrac{\cos(\tan^{-1}(\ln x))}{x(\ln x^2 + 1)}$

- $f(x) = \ln(\tan^{-1}(t))$

$= \dfrac{1}{\tan^{-1}(t)} \cdot (\tan^{-1}(t))'$

$= \dfrac{1}{\tan^{-1}(t)} \cdot \dfrac{1}{t^2 + 1} \cdot 1$

$= \dfrac{1}{\tan^{-1}(t)(t^2 + 1)}$

Tangent Line $(2, \frac{\pi}{16})$

- $f(x) = \sin^{-1}\left(\frac{x}{4}\right)$

$= \dfrac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \cdot \dfrac{1}{4}$

$f'(2) = \dfrac{1}{4\sqrt{1 - \frac{x^2}{16}}}$

$= \dfrac{1}{4\sqrt{1 - \frac{4}{16}}}$

$= \dfrac{1}{4\sqrt{\frac{3}{4}}}$

$= \dfrac{1}{4 \frac{\sqrt{3}}{2}}$

$= \dfrac{1}{2\sqrt{3}}$

$y = \dfrac{x}{2\sqrt{3}} + b$

$\dfrac{\pi}{6} - \dfrac{1}{\sqrt{3}} = b$

$y = \dfrac{x}{2\sqrt{3}} + \left(\dfrac{\pi}{6} - \dfrac{1}{\sqrt{3}}\right)$

3.11 Related Rates.

- Start with an exercise.
  - Ballons - ballons volume changes at $15 \frac{in^3}{min}$, whats the rate of change of the radius.

$V = \frac{4}{3}\pi r^3$ find $\left.\frac{dr}{dt}\right|_{r=10\,in}$

$\frac{dV}{dt} = 15 \frac{in^3}{min}$

$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d(r^3)}{dt}$

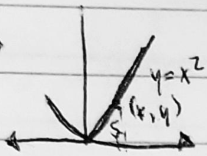
$= \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$15 = 4\pi(10^2)\frac{dr}{dt}$

$\frac{15}{400\pi} = \frac{dr}{dt} \rightarrow \frac{3}{80\pi}\frac{in}{min} = \frac{dr}{dt}$

- Guideline
  1. Read the problem carefully and draw diagrams.
  2. Introduce variables
  3. Write the given rate(s) as an equation
  4. Recognize whats to be found.
  5. Find relation(s) between the independent variables and dependent variables.
  6. Use implicit differentiation with respect to time to obtain a relation between rates.
  7. Substitute numerical values and solve for the missing variable.



Have:

$$\frac{ds}{dt} = \frac{1\ cm}{min}$$

Want:

$$\frac{dx}{dt} \ \& \ \frac{dy}{dt}$$

$$\left.\frac{dx}{dt}\right|_{\substack{x=2\\y=4}} \quad \left.\frac{dy}{dt}\right|_{\substack{x=2\\y=4}}$$

$s^2 = x^2 + y^2 \qquad y = x^2$

$s^2 = x^2 + x^4$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 4x^3\frac{dx}{dt}$$

$$s\frac{ds}{dt} = \frac{dx}{dt}\left(x + 2x^3\right)$$

$s = \sqrt{x^2+x^4}$

$$\frac{s}{x+2x^3} = \frac{dx}{dt}$$

$$\frac{\sqrt{20}}{2+2(2)^3} = \frac{dx}{dt}$$

$$\frac{\sqrt{20}}{18} = \frac{dx}{dt}$$

$$\frac{2\sqrt{5}}{18} = \frac{dx}{dt}$$

$$\frac{\sqrt{5}\ cm}{9\ min} = \frac{dx}{dt}$$

$y = x^2$

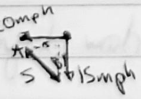
$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(2)\left(\frac{\sqrt{5}}{9}\right)$$

$$\frac{dy}{dt} = \frac{4\sqrt{5}}{9}\ \frac{cm}{min}$$

∴ the x coordinate changes at a rate of $\frac{\sqrt{5}}{9}\frac{cm}{min}$ and the y coordinate changes at a rate of $\frac{4\sqrt{5}}{9}\frac{cm}{min}$.

- Boats -



$$\frac{dx}{dt} = 20\,mph \qquad \frac{dy}{dt} = 15\,mph$$

Find $\left.\frac{ds}{dt}\right|_{t=\frac{1}{2}h}$

$s^2 = x^2 + y^2$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$s\frac{ds}{dt} = x(20) + y(15)$$

$v \cdot t = d$

$20 \cdot t =$

$s = \frac{25}{2}$

$$s\frac{ds}{dt} = (10)(20) + (7.5)(15)$$

$$\frac{25}{2}\frac{ds}{dt} = 200 + \frac{225}{2}$$

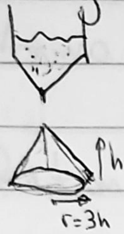
$$\frac{ds}{dt} = \frac{200 + \frac{225}{2}}{\frac{25}{2}}$$

$$\frac{ds}{dt} = \frac{400+225}{25}$$

$$\frac{ds}{dt} = \frac{625}{25}$$

$$\frac{ds}{dt} = 25\ mph$$

- Growing Sandpile



Sand falls from the bin into a conical pile with a radius 3x its height. Height increases at 2cm/s when the height is 12. At what rate is sand leaving the bin?

$\left.\frac{dh}{dt}\right|_{h=12} = 2\frac{cm}{s}$ Find $\left.\frac{dV}{dt}\right|_{h=12}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi (3h)^2 h$

$= \frac{1}{3}\pi 9h^3$

$\frac{dV}{dt} = \frac{d(\pi 3h^3)}{dt}$

$\frac{dV}{dt} = 3\pi(3h^2)\frac{dh}{dt}$

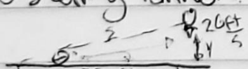
$\frac{dV}{dt} = 9\pi h^2 (2)$

$\frac{dV}{dt} = 18\pi h^2$

$\left.\frac{dV}{dt}\right|_{h=12} = 18\pi(12)^2$

$\left.\frac{dV}{dt}\right|_{h=12} = (18)(144)(\pi)\frac{cm^3}{s}$

- Observing launch. Balloon rises at $20\frac{ft}{s}$. Observer 300ft away what is the rate of change in the angle when Balloon = 400ft?



Find: $\left.\frac{d\theta}{dt}\right|_{y=400} =$ $\frac{dy}{dt} = 20\frac{ft}{s}$ $\tan\theta = \frac{y}{300}$

$\tan\theta = \frac{y}{300}$

$\theta = \tan^{-1}\left(\frac{y}{300}\right)$

$\cancel{\frac{d\theta}{dt} = \frac{d(\tan^{-1}(\frac{y}{300}))}{dt}}$

$\frac{d(\tan\theta)}{dt} = \frac{d(\frac{y}{300})}{dt}$

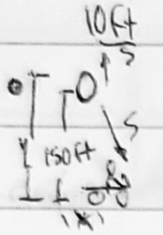
$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{300}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{\cos^2\theta}{300}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{\cos^2(\tan^{-1}(\frac{y}{300}))(20)}{300}$

$\frac{d\theta}{dt} = \frac{\cos^2(\tan^{-1}(\frac{400}{300}))(20)}{300}$

$\frac{d\theta}{dt} = \frac{6}{250} = \frac{3}{125}\frac{\text{radians}}{s}$



Ballon rises at $10\frac{ft}{s}$. Cyclist moves a $40\frac{mph}{s}$. Find the R of of distance.

$y = \cancel{10t+150}$ $\quad s=\sqrt{x^2+y^2}$ $\quad \frac{ds}{dt}\Big|_{t=10} = ?$

$x = \cancel{40t}$

$\frac{dy}{dt} = 10\frac{ft}{s}$ $\quad \frac{dx}{dt} = 40\frac{mi}{h} \to 58.67\frac{ft}{s}$.

$s^2 = x^2+y^2$

$\frac{d(s^2)}{dt} = \frac{d(x^2)}{dt} + \frac{d(y^2)}{dt}$.

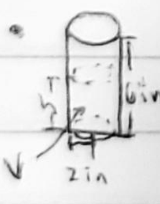
$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

$s\frac{ds}{dt} = (586.7)(58.67) + (250)(10)$

$\frac{ds}{dt} = \frac{(586.7)(58.67) + (250)(10)}{\sqrt{586.7^2 + 250^2}}\ \frac{ft}{s}$

$\frac{ds}{dt} =$



Drinking pop. pop decreases at $-0.25\frac{in}{s}$

Find $\frac{dV}{dt}$.

$V = \pi(2^2)h$

$\frac{dV}{dt} = 4\pi\frac{dh}{dt}$

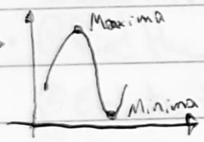
$\frac{dV}{dt} = 4\pi(-.25)$

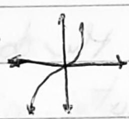
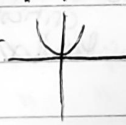
$\frac{dV}{dt} = -\pi$

Chapter 4. Applications of the Derivative.

4.1 Maxima and Minima

Definition: Let f be a function defined on a domain D. Let c be a point in D such that $f(c) \geq f(x)$ for all x in D. Then $f(c)$ is the absolute maximum value of f on D, and we say that the absolute maximum value occurs at $x=c$.
Let b be a point in D such that $f(b) \leq f(x)$ for all x in D. The $f(b)$ is the absolute minimum value of f on D.

- 

- $f(x) = x^3$  No min/max
- $f(x) = x^2$  Min but no max

- Functions that are continuous and are on a closed interval will have a max on min.
- Extreme Value Theorem - Let f be defined on a closed interval $[a,b]$ and f is continuous on that interval, Then f attains an absolute maximum and and absolute minimum on $[a,b]$.
- Local Extrema - Let f be defined on a domain D. Let c be a point in D and suppose there is an open interval I containing c where I is a subset of D. If $f(x)$ is bigger than or equal to $f(c)$, $f(x) \geq f(c)$. for all x in I then $f(c)$ is said to be a local minimum value of f. If $f(x) \leq f(c)$ for all x in I then $f(c)$ is said to be a local maximum value of f.

Fermat's Theorem

- If f has a local extreme value at c, the $f'(c)=0$ or $f'(c)$ is undefined.
- Definition: An interior point of the domain of f at which the derivative is 0 or undefined is called a critical point or critical number.

Examples:

- $f(x) = \frac{1}{8}x^3 - \frac{1}{2}x$ $D=[-1,3]$

$f'(x) = \frac{3x^2}{8} - \frac{1}{2}$

$0 = \frac{3x^2}{8} - \frac{1}{2}$

$\frac{1}{2} = \frac{3x^2}{8}$

$\frac{8}{6} = x^2$

$\sqrt{\frac{4}{3}} = x$

$\pm \frac{2}{\sqrt{3}} = x$

$+\frac{2}{\sqrt{3}}$ is in the domain

$-\frac{2}{\sqrt{3}}$ is not.

$\frac{2}{\sqrt{3}}$ is a ~~[illegible]~~ critical point.

- $f(x) = \frac{4x^5}{5} - 3x^3 + 5$ $[-2,2]$

$f'(x) = 4x^4 - 9x^2$

$f'(x) = 0 = 4x^4 - 9x^2$

$0 = x^2(4x^2 - 9)$

$0 = x^2(2x-3)(2x+3)$

$x = 0, \frac{3}{2}, -\frac{3}{2}$

$\frac{3}{2}, -\frac{3}{2}, 0$ are all critical points

- $f(x) = x^2 - 2\ln(x^2+1)$

$f'(x) = 2x - (2)\frac{1}{x^2+1}(2x)$

$f'(x) = 2x - \frac{4x}{x^2+1}$

$f'(x) = 0 = 2x - \frac{4x}{x^2+1} \Rightarrow 2x = \frac{4x}{x^2+1} \Rightarrow x(x^2+1) = 2x \Rightarrow x(x^2+1-2) = 0$

$x(x-1)(x+1) = 0$

- The critical points are $0, -1, 1$.

Closed Interval Method

- To find absolute extrema of a continuous function on a closed interval
- ① Find the critical points
  ② Evaluate the function at the critical points and at the endpoints
  ③ The largest value in step 2 is the absolute maximum and the smallest value " " " " " minimum.

Examples

- $f(x) = \frac{x}{(x^2+3)^2}$

$D = [-2, 2]$

$\rightarrow f'(x) = \frac{\overbrace{(1)(x^2+3)^2}^{f'(x)g(x)} - \overbrace{x\,2(x^2+3)(2x)}^{f(x)\,g'(x)}}{(x^2+3)^4}$

$= \frac{\cancel{(x^2+3)}((x^2+3) - 4x^2)}{(x^2+3)^4}$

$= \frac{-3x^2+3}{(x^2+3)^2}$

$= \frac{3(1-x)(1+x)}{(x^2+3)^2}$

Step 1 - Critical points are ± 1.

Step 2 - $f(1) = \frac{1}{16}$, $f(-1) = \frac{-1}{16}$, $f(2) = \frac{2}{49}$, $f(-2) = \frac{-2}{49}$

Step 3 - max ($f(1)$), min ($f(-1)$)

$f(1) = \frac{1}{16}$

$f(-1) = \frac{-1}{16}$

$f(2) = \frac{2}{49}$

$f(-2) = -\frac{2}{49}$

• $f(x) = x\ln\left(\frac{x}{5}\right)$

$f'(x) = f'(x)g(x) + f(x)g'(x)$

$= \ln\left(\frac{x}{5}\right) + x\left(\frac{5}{x} \cdot \frac{1}{5}\right)$

$= \ln\left(\frac{x}{5}\right) + x\left(\frac{1}{x}\right)$

$= \ln\left(\frac{x}{5}\right) + 1$

$0 = \ln\left(\frac{x}{5}\right) + 1$

$-1 = \ln\left(\frac{x}{5}\right)$

$e^{-1} = \frac{x}{5}$

$\frac{5}{e} = x$

$f\left(\frac{5}{e}\right) = \frac{5}{e}\ln\left(\frac{\frac{5}{e}}{5}\right)$

$= -\frac{5}{e}$

$f(.1) = (.1)\ln(0.02)$ ⇒ will be negative.

$f(5) = (5)\ln(1) = 0$

① Critical Point ⇒ $x = \frac{5}{e}$

② $f\left(\frac{5}{e}\right) = -\frac{5}{e}$, $f(.1) = (.1)\ln(0.02)$, $f(5) = 0$

③ max ⇒ $f(5) = 0$, min $f\left(\frac{5}{e}\right) = -\frac{5}{e}$

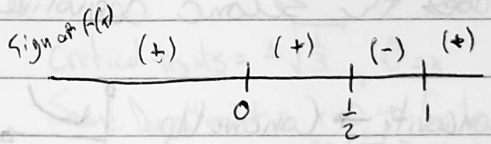
4.2 What derivatives tell us

- Definition - let f be defined on an interval I.
  1. We say f is increasing on I, if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$
  2. We say f is decreasing on I, if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

- Increasing/Decreasing Test - 1. If $f'(x) > 0$ on an interval I, then f is increasing on I.
  2. if $f'(x) < 0$ on an interval I, then f is decreasing on I.

Example

- $f(x) = 2x^5 - \frac{15x^4}{4} + \frac{5x^3}{3}$



$f'(x) = 10x^4 - 15x^3 + 5x^2$
$= 5x^2(2x^2 - 3x + 1)$ f is increasing on $(-\infty, \frac{1}{2}] \cup [1, \infty)$
$= 5x^2(2x-1)(x-1)$ f is decreasing on $[\frac{1}{2}, 1]$.

First Derivative Test

- f is defined on an open interval containing critical point c and f is differentiable on the interval except at c itself
1. If f' changes sign from $+$ to $-$ at c, then f has a local maximum at c.
2. If f' changes sign from $-$ to $+$ at c, then f has a local minimum at c.
3. If f' doesn't change sign at c then there is neither a maximum nor minimum at c.

Examples

- $f(x) = -x^2 - x + 2$ $[-4, 4]$

$f'(x) = -2x - 1$

Critical Points $= -\frac{1}{2}$

Sign of $f'(x)$

$\oplus$ | $\ominus$

$-\frac{1}{2}$

Local maximum.

$f(-\frac{1}{2}) = \frac{9}{4}$, $f(-4) = -10$

$f(4) = -18$ + Local min

- $f(x) = \dfrac{x^2}{x^2-1}$ $[-4, 4]$

$f'(x) = \dfrac{(x^2-1)'(x^2) - (x^2)'(x^2-1)}{(x^2-1)^2}$

$= \dfrac{-2x}{(x^2-1)^2}$

Critical Point $= 0$

$\oplus$ | $\ominus$

0

local max at zero

$f(0) = 0$

$f(4) = \frac{16}{15}$, $f(-4) = \frac{16}{15}$

no absolute max because it approaches infinity at points.

What does the second derivative say about the function?

- Concavity → Concave Up 
  → Concave down 
- The graph of the function f is concave down on an interval i if f' decreases on I.
- The graph of the function F is concave up on an interval i if f' increases on I.
-

Test for Concavity ~~(The Second Derivative Test)~~.

- ~~Let f be differentiable~~ Suppose f'' exists on an open interval I.
  - If $f'' > 0$ on I then f is concave up
  - If $f'' < 0$ on I then F is concave down

- $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

sign of $f''(x)$

$\ominus$ | $\oplus$

0

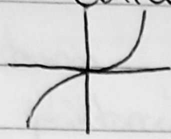
concave down — concave up.

graph is concave up $(0, \infty)$

graph is concave down $(-\infty, 0)$.

• Definition - Suppose f is continuous at c and f changes concavity at c. Then $(c, f(c))$ is called an inflection point of f.

• $f(x) = x^3$



Concave down $(-\infty, 0)$ 0 is the inflection point.
Concave Up $(0, \infty)$

The Second Derivative Test.

• If $f''(c) > 0$ there is a local minimum at c
• If $f''(c) < 0$ there is a local maximum at c.
• If $f''(c) = 0$ then the test fails (incomplete).

Examples

• $g(x) = \dfrac{x^4}{2-12x^2}$

Critical Points $= \pm\sqrt{\frac{1}{3}}$, $0 = x$
Second Derivative Test $=$ $x = 0 \Rightarrow$ fails.
$x = +\frac{1}{\sqrt{3}} \Rightarrow$ negative $\Rightarrow$ concave down
$x = -\frac{1}{\sqrt{3}} \Rightarrow$ negative $\Rightarrow$ concave down.

$$g'(x) = \frac{(x^4)'(2-12x^2) - (x^4)(2-12x^2)'}{(2-12x^2)^2}$$

$$= \frac{4x^3(2-12x^2) - x^4(-24x)}{(2-12x^2)^2}$$

$$= \frac{8x^3 - 48x^5 - 24x^5}{(2-12x^2)^2}$$

$$= \frac{8x^3(1-6x^2+3x^2)}{(2-12x^2)^2}$$

$$= \frac{2x^3(1-3x^2)}{(1-6x^2)^2}$$

$$g''(x) = \frac{2(x^3-3x^5)}{(1-6x^2)^2}$$

$$= 2\left(\frac{(3x^2-15x^4)(1-6x^2)^2 - (x^3-3x^5)(2)(1-6x^2)(-12x)}{(1-6x^2)^4}\right)$$

$$= \frac{2x^2(1-6x^2)}{(1-6x^2)^4}\left((3-15x^2)(1-6x^2) + (x-3x^3)(-24x)\right)$$

$$= \frac{2x^2}{(1-6x^2)^3}(18x^4-9x^2+3)$$

$$= \frac{6x^2}{(1-6x^2)^3}(6x^4-3x^2+1)$$

4.3. Graphing function.

Guideline

1. Find the domain
2. Find the y-intercept (if it exists) $(x=0)$
   Find the x-intercepts (if not too difficult) $(y=0)$.
3. Find the horizontal and vertical asymptotes (if any).
4. Compute first derivative to
   - find intervals of increase/decrease
   - find local maxima/minima.
5. Find the second derivative to
   - find intervals of concave up/down
   - find inflection points
6. Sketch (and check if it agrees with any symmetries present)

Examples.

- $f(x) = 2x^6 - 3x^4$

① $D = (-\infty, \infty)$, ② y-int $= 0$, x-int $= 0, \pm\sqrt{\frac{3}{2}}$, ③ no VA, no HA, end behaviours $x \to -\infty, y \to \infty$; $x \to \infty, y \to \infty$

④ $f'(x) = 12x^5 - 12x^3$

$= 12x^3(x^2-1)$

$= 12x^3(x+1)(x-1)$

$\ominus \;\; -1 \;\; \oplus \;\; 0 \;\; \ominus \;\; 1 \;\; \oplus$

local min $= f(-1) = -1$, local max $= f(0) = 0$

⑤ $f''(x) = 60x^4 - 36x^2$

$= 12x^2(5x^2-3)$

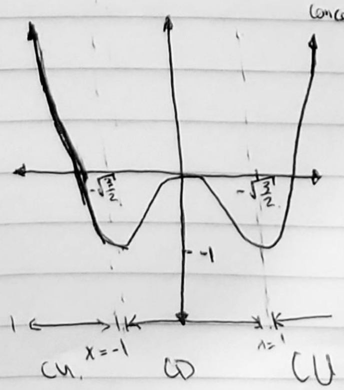
$\oplus \;\; \frac{\sqrt{3}}{\sqrt{5}} \;\; \ominus \;\; 0 \;\; \ominus \;\; \frac{\sqrt{3}}{\sqrt{5}} \;\; \oplus$

concave down

concave up

Inflection points $\Rightarrow x = \pm\frac{\sqrt{3}}{\sqrt{5}}$

⑥



• $f(x) = \dfrac{2x-3}{2x-8}$ $\quad D = (-\infty, 4) \cup (4, \infty)$, $\quad$ y-int $= \frac{3}{8}$, $\quad$ x-int $= \frac{3}{2}$.

VA ⇒ $x = 4$, $\quad$ HA. $\lim_{x \to \infty} f(x) = 1$ $\quad$ HA $= 1$,

$x \to 4^- \to -\infty$ $\qquad \lim_{x \to -\infty} f(x) = 1$

$4^+ \to \infty$

• $f'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$= \dfrac{(2)(2x-8) - (2x-3)(2)}{(2x-8)^2}$

$= \dfrac{4x - 16 - 4x + 6}{(2x-8)^2}$

$= \dfrac{-10}{(2x-8)^2}$

decreasing from $(-\infty, 4) \cup (4, \infty)$

• $f''(x) = -10(2x-8)^{-2}$

$= +20(2x-8)^{-3}(2)$

$= \dfrac{40}{(2x-8)^3}$

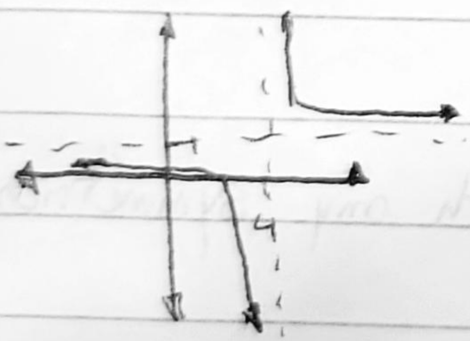
$= \dfrac{40}{8(x-4)^3}$

$= \dfrac{5}{(x-4)^3}$

⊖ | ⊕

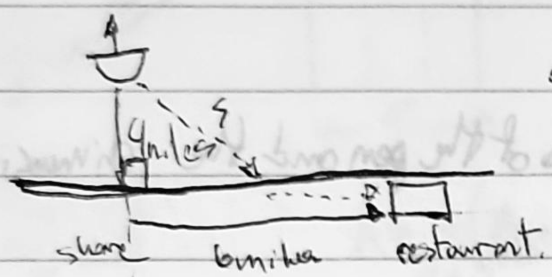
4

no inflection points.

• 

1 Optimization Problems.

• Ex



a) Walk = 3 mph

Row = 2 mph.

T = time needed to arrive at restaurant.

T = rowing time + walking time.

$= \frac{d_R}{v_R} + \frac{d_W}{v_W}$

$= \frac{s}{v_R} - \frac{6-x}{v_W}$ $\quad s^2 = x^2 + 4^2.$

$T = \frac{s}{2} - \frac{6-x}{3}$ $\leftarrow$ $s = \sqrt{x^2+16}$

$T = \frac{\sqrt{x^2+16}}{2} - \frac{6-x}{3}$ $\quad 0 \le x \le 6$

$\frac{dT}{dx} = \frac{d\left(\frac{\sqrt{x^2+16}}{2}\right)}{dx} - \frac{d\left(\frac{6-x}{3}\right)}{dx}$

$\frac{dT}{dx} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+16}} \cdot 2x - \frac{1}{3}$

$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3} \Rightarrow \frac{dT}{dx} = 0$ when $\frac{x}{2\sqrt{x^2+16}} = \frac{1}{3}.$

$3x = 2\sqrt{x^2+16}$

$9x^2 = 4(x^2+16)$

$9x^2 - 4x^2 = 64$

$5x^2 = 64$

$x = \sqrt{\frac{64}{5}} \Rightarrow x = \frac{8}{\sqrt{5}}$

$T = \frac{\sqrt{\left(\frac{8}{\sqrt{5}}\right)^2 + 16}}{2} - \frac{6 - \frac{8}{\sqrt{5}}}{3}$

$T = \frac{\sqrt{\frac{64}{5} + 16}}{2} - \frac{6 - \frac{8}{\sqrt{5}}}{3}$

$T = \frac{\sqrt{\frac{64}{5} + \frac{80}{5}}}{2} - \frac{6\sqrt{5} - 8}{3\sqrt{5}}$

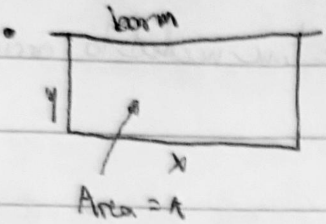
$T = \frac{\sqrt{\frac{144}{5}}}{2} - \frac{6\sqrt{5} - 8}{3\sqrt{5}}$

$T = \frac{10 + 6\sqrt{5}}{3\sqrt{5}}$ hours

$T(0) = 4$ hours

$T(6) = \sqrt{13}$ hours.

4.4 Optimization Problems.

- 

200 yrds of fence
Find the max area of the pen and the dimensions.

$A = xy$ $\qquad 2y + x = 200$

$y = \dfrac{200-x}{2}$

$A = x\left(\dfrac{200-x}{2}\right)$

$A = \dfrac{200x - x^2}{2} \quad 0 < x \le 200$

$\dfrac{dA}{dx} = \dfrac{d\left(\frac{200x - x^2}{2}\right)}{dx}$

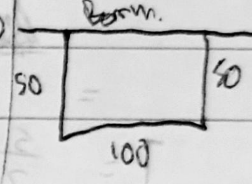
$\dfrac{dA}{dx} = \dfrac{200-2x}{2}$ $\Rightarrow x = 100$ is the critical point

$2y + \cancel{x} = 200$

$2y = 200 - 100$

$y = \dfrac{100}{2}$

$y = 50.$



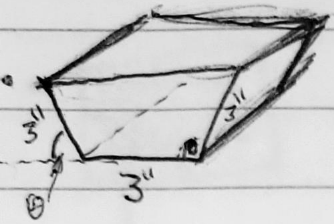
$A(0) = 0$

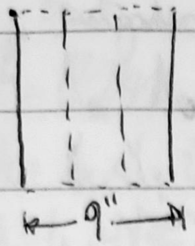
$A(100) = 5000 \leftarrow$ max pen.

$\Rightarrow A(200) = 0.$

Guideline.

- ① Read the problem carefully and draw a diagram
  ② Introduce variables and identify the objective function (the function to be optimized)
  ③ Write the objective function in terms of other variables
  ④ Write the objective function as a function of a single variable (Using relations between variables).
  ⑤ Determine the domain of the function.
  ⑥ Use calculus methods to maximize or minimize (optimize).





What angle, θ, maximizes the cross sectional area?

$A = \frac{1}{2}(6 + 6\cos\theta)(3\sin\theta)$

$A = \frac{18}{2}(1 + 2\cos\theta)(\sin\theta)$

$A = \frac{18}{2}(\sin\theta + 2\cos\theta\sin\theta)$

$A = \frac{18}{2}(\sin\theta + \sin 2\theta)$ $\qquad 0 \leq \theta \leq \frac{\pi}{2}$

Closed Interval Method $\quad \frac{dA}{d\theta} = \frac{d\left(9\left(\sin\theta + \frac{1}{2}\sin 2\theta\right)\right)}{d\theta}$

$\frac{dA}{d\theta} = 9(\cos\theta + \cos 2\theta)$

$\frac{dA}{d\theta} = 0$, when $\cos\theta + \cos 2\theta = 0$.

$\cos 2\theta = 2\cos^2\theta - 1 \qquad (2\cos^2\theta - 1) + \cos\theta = 0$

$2\cos^2\theta - 1 + \cos\theta = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$. $\quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.

Critical Points $\quad A\left(\frac{\pi}{3}\right) = 9\left(1 + \cos\left(\frac{\pi}{3}\right)\right)\sin\left(\frac{\pi}{3}\right) \qquad A(0) = 0$

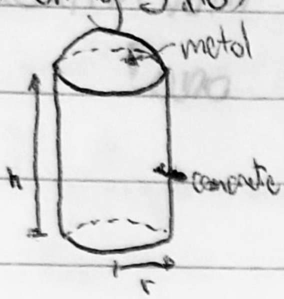
$= 9\left(1 + \frac{1}{2}\right)\frac{\sqrt{3}}{2} \qquad A\left(\frac{\pi}{2}\right) = 9$

$= \frac{27}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{27\sqrt{3}}{4}$

Max area when $\theta = \frac{\pi}{3}$.

• Making Silos



cost of metal $= c_m = 1.5\, c_c$

cost of concrete $= c_c$

Volume $= 750\ m^3$

T = total cost $= c_c + c_m$

$= \cancel{c_c + 1.5c_c}$

$= (2\pi r h)c_c + (2\pi r^2 c_m)$

$= 2\pi r h c_c + 2\pi r^2 (1.5 c_c)$

$= \pi c_c (2rh + 3\pi r^2)$

$= \pi c_c \left(2r\left(\frac{750 - \frac{2}{3}\pi r^3}{\pi r^2}\right) + 3\pi r^2\right)$

$= \pi c_c \left(\frac{1500 - \frac{4}{3}\pi r^3}{\pi r}\right) + 3\pi r^2)$

$= \pi c_c \left(\frac{1500}{\pi r} - \frac{4\pi r^3}{3\pi r} + 3\pi r^2\right)$

$= \pi c_c \left(\frac{1500}{\pi r} - \frac{4}{3} r^2 + 3\pi r^2\right)$

$= \pi c_c \left(\frac{1500}{\pi r} + \frac{5}{3} r^2\right)$

$V_c + V_H = 750$

$\pi r^2 h + \frac{2}{3}\pi r^3 = 750$

$h = \dfrac{750 - \frac{2}{3}\pi r^3}{\pi r^2}$

$0 < r < \sqrt[3]{\frac{3(750)}{2\pi}}$

$\frac{750}{+750} \over 1500$

$\dfrac{dT}{dr} = \pi c_c \left(\frac{1500}{\pi r} + \frac{5 r^2}{3}\right)$

$T'(r) = \pi c_c \left(-\frac{1500}{\pi r^2} + \frac{10 r}{3}\right) \to T''(r) = \pi c_c \left(-\frac{1500}{\pi r^2} + \frac{10r}{3}\right)$

$= \pi c_c \left(\frac{3000}{\pi r^3} + \frac{10}{3}\right)$

$\dfrac{1500}{\pi r^2} = \dfrac{10 r}{3}$

$\dfrac{450}{\pi} = r^3$

$\sqrt[3]{\frac{450}{\pi}} = r$ ← Critical Point.

$h = \dfrac{750 - \frac{2}{3}\pi r^3}{\pi r^2}$

$h = \dfrac{750 - \frac{2}{3}\pi \left(\sqrt[3]{\frac{450}{\pi}}\right)^3}{\pi \left(\sqrt[3]{\frac{450}{\pi}}\right)^2}$

$h = \dfrac{450}{\pi \left(\frac{450}{\pi}\right)^{\frac{2}{3}}}$

Area of garden + Area of borders

30 m^2

$A = \cancel{A_G + A_B}\ A_{G+B}$

$A = (x+4)(y+2)$

$A = (x+4)\left(\frac{30}{x}+2\right)$

$xy = 30$

$y = \frac{30}{x}$

$A = 30 + 2x + \frac{120}{x} + 8.$

$A = 2x + \frac{120}{x} + 38.$ $\quad 0 < x < \infty$

$A'(x) = 2 - \frac{120}{x^2}$ $\quad$ Critical Point $= \sqrt{60}$

$2 = \frac{120}{x^2}$ $\quad$ Concave Up.

$x = \sqrt{60}$

$A''(x) = \frac{240}{x^3}$

$x = \sqrt{60} = 2\sqrt{15}$

$y = \frac{30}{2\sqrt{15}} = \frac{15}{\sqrt{15}}$

4.7 L'Hôpital's Rule.

Indeterminant forms

- $\lim_{x\to c} \frac{f(x)}{g(x)}$, If the ~~limit of~~ $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$, then the $\lim_{x\to c}\frac{f(x)}{g(x)}$ is said to be an indeterminant form of type $\frac{0}{0}$.
- Ex $\lim_{x\to 0} \frac{\sin x}{x} \Rightarrow \lim_{x\to 0} \frac{0}{0} \Rightarrow$ indeterminant form of $\frac{0}{0}$
- $\lim_{x\to} \frac{f(x)}{g(x)}$, If the $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$, then the $\lim_{x\to} \frac{f(x)}{g(x)}$ is said to be an indeterminant form of type $\frac{\infty}{\infty}$.
- Ex $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \Rightarrow \frac{-\infty}{\infty} \Rightarrow$ indeterminant form of $\frac{\infty}{\infty}$.

Rule

- Let $\lim_{x\to c} \frac{f(x)}{g(x)}$ be an indeterminant form (of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

- $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1}$

$= \lim_{x\to 0} \cos x$

$= 1$

- Always remember to check if it is an indeterminant form.

Examples

- $\lim_{x\to 2} \frac{x^2-2x}{8-6x+x^2}$

$= \lim_{x\to 2} \frac{x(\cancel{x-2})}{(\cancel{x-2})(x-4)}$

$= \lim_{x\to 2} \frac{x}{x-4}$

$= \lim_{x\to 2} \frac{2}{2-4}$

$= -1$

- $\lim_{x\to e} \frac{\ln x - 1}{x-e}$

$= \lim_{x\to e} \frac{\frac{1}{x}}{1}$

$= \lim_{x\to e} \frac{1}{x}$

$= \frac{1}{e}$.

- $\lim_{x\to 2\pi} \frac{x\sin x + x^2 - 4\pi^2}{x-2\pi}$

$= \lim_{x\to 2\pi} \cancel{\frac{\cos x + 2x}{1}} = \frac{x\cos x + \sin x - 2x}{1}$

$= \lim_{x\to 2\pi} \cancel{\cos x}\ x\cos x + \sin x - 2x$

$= 2\pi + 4\pi$

$= 6\pi$.

- $\lim_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5}$

$= \lim_{x\to\infty} \frac{3e^{3x}}{9e^{3x}}$

$= \frac{1}{3}$

- $\lim_{x\to \frac{\pi}{2}} \frac{2\tan x}{\sec^2 x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{2\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}}$

$= \lim_{x\to\frac{\pi}{2}} 2\sin x\cos x$

$= 0$.

- $\lim_{x\to\infty} \frac{\sqrt{x-2}}{\sqrt{x-1}}$

$= \lim_{x\to\infty} \frac{\frac{1}{2}(x-2)^{-\frac{1}{2}}}{\frac{1}{2}(x-1)^{-\frac{1}{2}}}$

$= \lim_{x\to\infty} \frac{(x-1)^{\frac{1}{2}}}{(x-2)^{\frac{1}{2}}}$

$= \lim_{x\to} \frac{\frac{1}{2}(x-1)^{-\frac{1}{2}}}{\frac{1}{2}(x-2)^{-\frac{1}{2}}}$

$= \lim_{x\to\infty} \frac{\sqrt{x-2}}{\sqrt{x-1}}$

- $\lim_{x\to\infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

$= \lim_{x\to\infty} \dfrac{12x^2 - 4x}{3\pi x^2}$

$= \lim_{x\to\infty} \dfrac{24x - 4}{6\pi x}$

$= \lim_{x\to} \dfrac{24 - \frac{4}{x}}{6\pi}$

$= \dfrac{24}{6\pi} \to \dfrac{4}{\pi}$

- $\lim_{x\to\infty} \dfrac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{3x+5e^x} \cdot 3+5e^x}{\frac{1}{7x+3e^{2x}} \cdot 7+6e^{2x}}$

$= \lim_{x\to\infty} \dfrac{(3+5e^x)(7x+3e^{2x})}{(3x+5e^x)(7+6e^{2x})}$

$= \lim_{x\to\infty} \dfrac{(7x+3e^{2x})}{(7+6e^{2x})} \cdot \dfrac{(3+5e^x)}{(3x+5e^x)}$

$= \lim_{x\to\infty} \dfrac{(7x+3e^{2x})}{(7+6e^{2x})} \cdot \lim_{x\to\infty} \dfrac{(3+5e^x)}{(3x+5e^x)}$

$= \lim_{x\to\infty} \dfrac{\frac{7x}{x} + 3e^{2x}}{\frac{7}{x}\ 6e^{2x}} \cdot \lim_{x\to\infty} \dfrac{\frac{3}{x} + 5e^x}{\frac{3x}{x}\ 5e^x}$

$= \left(\dfrac{1}{2}\right) \cdot (1)$

$= \dfrac{1}{2}$

- $\lim_{x\to\infty} \dfrac{\sqrt{x-2}}{\sqrt{x-1}}$

Indeterminant form of type $0\cdot\infty$

- $\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right)$

$= \lim_{x\to 1^-} (1-x) \cdot \lim_{x\to 1^-} \tan\left(\frac{\pi x}{2}\right)$

$= 0 \cdot \infty$

$\to \lim_{x\to 1^-} \dfrac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)}$

$= \lim_{x\to 1^-} \dfrac{-1}{-\csc^2\left(\frac{\pi x}{2}\right)\frac{\pi}{2}}$

$= \lim_{x\to 1^-} \dfrac{2}{\pi} \sin^2\left(\frac{\pi x}{2}\right)$

$= \dfrac{2}{\pi}$

Indeterminate form of type $\infty-\infty$ $\to$

- $\lim_{x\to\infty} \left(x - \sqrt{x^2+4x}\right)$

$= \lim_{x\to\infty} \dfrac{(x - \sqrt{x^2+4x})(x + \sqrt{x^2+4x})}{x + \sqrt{x^2+4x}}$

$= \lim_{x\to\infty} \dfrac{x^2 - (x^2+4x)}{x + \sqrt{x^2+4x}}$

$= \lim_{x\to\infty} \dfrac{-4x}{x + \sqrt{x^2+4x}}$

$= \lim_{x\to\infty} \dfrac{-4}{1 + \sqrt{\frac{x^2}{x^2} + \frac{4x}{x^2}}}$

$= \lim_{x\to\infty} \dfrac{-4}{1 + \sqrt{1 + \frac{4}{x}}}$

$= \lim_{x\to\infty} \dfrac{-4}{1+\sqrt{1}}$

$= \dfrac{-4}{2} \to -2$

4.9 Antiderivatives

- Definition: Suppose $F(x)$ and $f(x)$ are two functions such that $F'(x) = f(x)$ on an interval I. Then F is called an antiderivative of f on I.
- Examples ① $f(x) = x$ $\quad F(x) = \frac{x^2}{2} + c$

  ② $f(x) = \cos x$ $\quad f(x) = \sin x + c$
- Theorem: Let $F(x)$ be an antiderivative of $f(x)$ on an interval I. Then each antiderivative of $f(x)$ is of the form $F(x) + C$, where C is a constant.
- Notation: The set of all antiderivatives of $f(x)$ is denoted by $\int f(x)\,dx$ (integral sign)(the x is the variable of integration) ($f(x)$ is the integrand.
  - $\int f(x)\,dx \pm F(x) + c$ (c is the constant of integration).

Examples.

$\int f(x)\,dx \to$ indefinite integral of $f(x)$

- $\int x^5\,dx = \frac{x^6}{6} + c$
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
- $\int \left(\frac{12t^8 - t}{t^3}\right) dt = \int \left(12t^5 - \frac{1}{t^2}\right) dt$

  $= 2t^6 \int (-t^{-2})\,dt$

  $= 2t^6 + \frac{1}{t} + c$
- $\int \left(\frac{\sin\theta - 1}{\cos^2\theta}\right) d\theta = \int (\tan\theta \sec\theta - \sec^2\theta)\,dx$

  $= \sec\theta - \tan\theta + c$
- $\int x^{-1}\,dx = \int \left(\frac{1}{x}\right) dx$

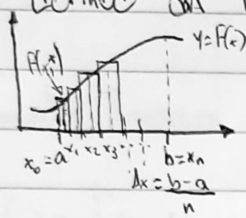
  $= \ln x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$
- $\int (\sec^2 x - 1)\,dx = \tan x - x + c$
- $\int (\tan^2 x)\,dx = \int (\sec^2 x - 1)\,dx$

  $= \tan x - x + c$
- $\int \frac{1}{4+x^2}\,dx = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$

Chapter 5 Integration

5.2 Definite Integrals.

- f defined on $[a,b]$



Area Under Curve $\approx$ Sum of the rectangles

$$\approx f(x_1^*)\Delta x + f(x_2^*)\Delta x \dots$$

$$\approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

- Exact Area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$
- The definite integral of f on $[a,b]$ is, $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$

If this limit exists we say that f is integrable over $[a,b]$.

If this limit does not exist we say that f is not integrable over $[a,b]$.

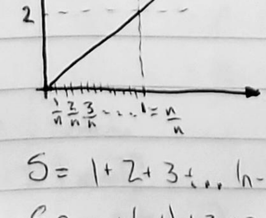
- $\int_a^b f(x)\,dx$

upper limit of integration → b; integral sign → $\int$; lower limit of integration → a; integrand → $f(x)$; dx ← Variable of integration

Properties of the Definite Integral

- ① $\int_a^a f(x)\,dx = 0$
- ② $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- ③ $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ ← Addition/Subtraction rules
- ④ $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ ← Constant Multiple
- ⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c is between $[a,b]$.

Example



$$\int_0^1 f(x)\,dx = \int_0^1 2x\,dx = 2\left(\tfrac{1}{n}\right)\Delta x + 2\left(\tfrac{2}{n}\right)\Delta x \dots$$

$$= \frac{2}{n}\Delta x(1+2+3+\dots n)$$

$$= \lim_{n\to\infty} \frac{2}{n}\Delta x(1+2 \dots n)$$

$$= \lim_{n\to\infty} \frac{\cancel{2}}{n}\left(\frac{1}{n}\right)\left(\frac{n(n+1)}{\cancel{2}}\right)$$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1}$$

$$= 1$$

$S = 1+2+3+\dots(n-1)+n$

$S = n+(n-1)+\dots 3+2+1$

$2S = (n+1)+(n+1)+(n+1)\dots$

$2S = n(n+1)$

$S = \frac{n(n+1)}{2}$

- $\int_{-4}^{2}(2x+4)dx = \int_{-4}^{-2} f(x)dx + \int_{-2}^{2} f(x)dx$

$= \frac{1}{2}bh + \frac{1}{2}bh$

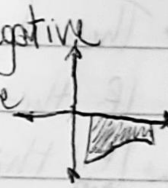
$= 16 + (-4)$

$= 12.$

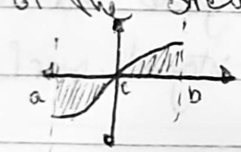
Note

- If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx$ is the area under the curve

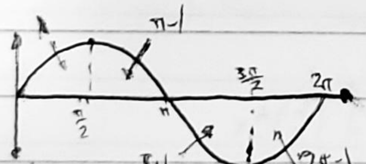
- If $f(x) \leq 0$ on $[a,b]$ then $\int_a^b f(x)dx$ is negative of the area between the x axis and the curve



- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$



Example

- 

$\int_0^{\pi} x\sin x\,dx = \pi$ $\int_0^{2\pi} x\sin x\,dx = -2\pi$

$\int_0^{\frac{3\pi}{2}} x\sin x\,dx = -1$ $\int_{\frac{\pi}{2}}^{2\pi} x\sin x\,dx = -2\pi-1$

- $\int_1^4 f(x)dx = 8$, $\int_1^6 f(x)dx = 5$

a.) $\int_1^4 (-3)f(x)dx = -24$ b.) $\int_1^4 (3)f(x)dx = 24$ c.) $\int_6^4 12f(x)dx = 36$ d.) $\int_4^6 3f(x)dx = -9$

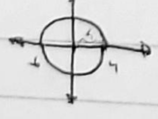
- $f(x) \geq 0$, $[0,2]$ $f(x) \leq 0$, $[2,5]$, $\int_0^2 f(x)dx = 6$, $\int_2^5 f(x)dx = -8$

a.) $\int_0^5 f(x)dx = -2$ b.) $\int_0^5 |f(x)|dx = 14$ c.) $\int_2^5 4|f(x)|dx = 32$ d.) $\int_0^5 (f(x)+|f(x)|)dx =$

$y^2 = 24-2x-x^2 \rightarrow$

$x^2+2x+1+y^2 = 24+1$

$(x+1)^2+y^2 = 25$

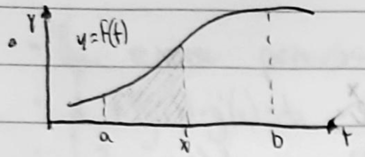


- $\int_{-6}^{4}\sqrt{24-2x-x^2}\,dx = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(5)^2$

$= \frac{25\pi}{2}$

5.3 Fundamental Theorem of Calculus



Area under curve from $[a, x] = A(x)$

$= \int_a^x f(t)dt$

Fundamental Theorem (Part 1)

- $A(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and, $A'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$. When you integrate and then differentiate you get the original function back (reverse process).

Fundamental Theorem (Part 2)

- Let $F(t)$ be an antiderivative of $f(t)$ on $[a,b]$. Then $\int_a^b f(t)dt = F(b) - F(a)$
- Let F be an antiderivate of f. Then, since $\int_a^x f(t)dt$ is also an antiderivate of f,

$$\int_a^x f(t)dt - F(x) = c, \text{ where } c \text{ is a constant.}$$

they have the same derivative lol its so awkward and quite

Now, $\int_a^b f(t)dt - F(b) = c$, and $\int_a^a f(t)dt - F(a) = c \Rightarrow -F(a) = c$.

$$\therefore \int_a^b f(t)dt = F(b) + c$$

$$\int_a^b f(t)dt = F(b) - F(a)$$

- This can also be written as $F(x)\Big|_a^b$ or $[F(x)]_a^b$.

Examples.

- $\int_0^2 (3x^2 + 2x)dx = [x^3 + x^2]_0^2$
  $= 2^3 + 2^{2} - 0^3 + 0^2$
  $= 12.$
- $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx$
  $= [\sin^{-1}(\frac{1}{2}) - \sin^{-1}(x)]$
  $= \frac{\pi}{6}.$
- $\int_0^{\frac{\pi}{4}} 2\cos x\,dx$
  $= 2[\sin x]_0^{\frac{\pi}{4}}$
  $= 2[\sin\frac{\pi}{4} + \sin 0]$
  $= 2(\frac{1}{\sqrt{2}})$
  $= \frac{2}{\sqrt{2}}$

Odd functions

$\int_{-a}^{a} f(x)dx = 0$

Even functions

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

Examples

- $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_1^{\sqrt{3}}$
$= \tan^{-1}\sqrt{3} - \tan^{-1}(1)$
$= \frac{\pi}{3} - \frac{\pi}{4}$
$= \frac{\pi}{12}.$

- $\displaystyle\frac{d}{dx}\int_0^x e^t\,dt = e^x.$

- $\displaystyle\frac{d}{dx}\left[\int_x^0 \frac{dp}{p^2+1}\right] = \frac{d}{dx}\left[-\int_0^x \frac{dp}{p^2+1}\right]$
$= -\frac{1}{x^2+1}$

- $\displaystyle\frac{d}{dx}\int_{10}^{x^2} \frac{dz}{z^2+1}$ = let $y = \displaystyle\int_{10}^{x^2} \frac{dz}{z^2+10}$ let u be x^2

$\displaystyle\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$\displaystyle\frac{dy}{du} = \frac{d}{du}\int_0^u \frac{dz}{z^2+1}$

$= \frac{1}{u^2+1}$

$= \cancel{\ } \cdot \frac{2x}{x^4+1}$

- $\displaystyle\frac{d}{dx}\int_2^{x^3} \frac{dp}{p^2}$ let u be x^3, let $y = \displaystyle\int_2^{x^3} \frac{dp}{p^2}$

$\frac{du}{dx} = 3x^2$

$\displaystyle\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \rightarrow \frac{dy}{du} = \frac{d}{du}\int_2^u \frac{dp}{p^2}$

$= \frac{1}{u^2}(3x^2)$

$= \frac{3x^2}{x^6} = \frac{3}{x^4}$

- $\displaystyle\frac{d}{dx}\int_1^{\sin x}(t^2+1)dt = (\sin^2 x+1)\cos x$
$= \sin^2 x\cos x + \cos x$ } short way

$u = \sin x$, $\frac{du}{dx} = \cos x$

$\displaystyle\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= \displaystyle\frac{d}{du}\int_1^u (t^2+1)dt \cdot \cos x$

$= (u^2+1)\cos x$

$= (\sin^2 x+1)\cos x.$

- $\displaystyle\int_0^{\sqrt{3}} \left(\frac{3}{9+x^2}\right)dx$
$= \left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^{\sqrt{3}}$
$= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \tan^{-1}(0)$
$= \frac{\pi}{6}.$

- $\displaystyle\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln(t^2)dt = \frac{d}{dx}\left[\int_{e^x}^1 \ln(t^2)dt + \int_1^{e^{2x}} \ln(t^2)dt\right]$

$= \displaystyle\frac{d}{dx} -\int_1^{e^x} \ln t^2\,dt + \frac{d}{dx}\int_1^{e^{2x}} \ln t^2\,dt$

$= -\ln e^{2x}\cdot e^x + \ln e^{4x}\cdot 2e^{2x}$

$= -2xe^x + 4x\cdot 2e^{2x}$

$= -2xe^x + 8xe^{2x}$

$= 2xe^x(4e^x - 1)$

5.5 The Substitution Rule (Integration by substitution)

- The reverse process to the chain rule.
- $\int f'(g(x)) \cdot g'(x) \cdot dx = f(g(x)) + c$.
- $\int 2x\cos(x^2)\,dx = \sin(x^2) + c$.
- To solve by substitution
  - Let $x^2 = u$, and $\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$
  - $\int 2x\cos(x^2)\,dx$

    $= \int \cos u\,du$

    $= \sin u + c$

    $= \sin(x^2) + c$
- Generalized
  - Let $g(x) = u$, and $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$
  - $\int f'(g(x)) \cdot g'(x)dx$

    $= \int f'(u) \cdot du$

    $= f(u) + c$

Examples

- $\int (x^6 - 3x^2)^4 (x^5 - x)\,dx$

  Let u be $x^6 - 3x^2$, $\frac{du}{dx} = 6x^5 - 6x = 6(x^5 - x)$

  $= \int u^4 \left(\frac{1}{6}du\right)$

  $= \frac{1}{6}\int u^4 du$

  $= \frac{1}{6}\frac{u^5}{5} + c$

  $= \frac{u^5}{30} + c$

  $= \frac{(x^6 - 3x^2)^5}{30} + c$.

- $\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx$

  Let $u = \sqrt{x} + 1$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

  $= \int u^4 du$.

  $= \frac{u^5}{5} + c$

  $= \frac{(\sqrt{x}+1)^5}{5} + c$.

- $\int x e^{x^2} dx$

  Let $u = x^2$, $\frac{du}{dx} = 2x$

  $= \int e^u \cdot \frac{1}{2}du$

  $= \frac{1}{2}\int e^u du$

  $= \frac{1}{2}e^u$

  $= \frac{1}{2}e^{x^2} + c$.

- $\int \left(\frac{x}{x-2}\right)dx$

  Let $u = x - 2$, $\frac{du}{dx} = 1 \Rightarrow du = dx$

  $= \int \frac{u+2}{u}\,du$.

  $= \int 1 + \frac{2}{u}\,du$.

  $= u + 2\ln|u| + c$.

- $\int \frac{3}{1+25y^2}\,dy$

  ~~Let $u = 25y^2$, $\frac{du}{dy} = 50y$.~~

  ~~$= \int \frac{3}{1+u}\frac{du}{50y}$~~

  ~~$= \int$~~

  Let $u = 5y$, $du = 5dy$

  $= \int \frac{3}{1+u^2}\frac{du}{5}$

  $= \frac{3}{5}\int \frac{1}{1+u^2}du$

  $= \frac{3}{5}\tan^{-1}u + c$.

  $= \frac{3}{5}\tan^{-1}(5y) + c$.

- $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Let $u = e^x + e^{-x}$, $\frac{du}{dx} = e^x - e^{-x}$

$= \int \frac{1}{u}\,dx$

$= \ln|u| + c$

$= \ln|e^x + e^{-x}| + c.$

- $\int \frac{\cos x}{\sin x}\,dx$

Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$= \int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln|\sin x| + c.$

- $\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$, $\frac{du}{dx} = \sin x$

$= \int \frac{du}{-u}$

$= -\ln|u| + c$

$= -\ln|\cos x| + c.$

$= \ln|\cos x^{-1}| + c$

$= \ln|\sec x| + c.$

- $\int (z+1)\sqrt{3z+2}\,dz$

Let $u = \sqrt{3z+2} \Rightarrow u^2 = 3z+2 \Rightarrow 2u\,du = 3\,dz$

$= \int$

Integration by Substitution for Definite Integrals

definite

Use the indefinite integral to evaluate the definite integral.

- $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\,dx \Rightarrow \int \frac{\sin}{\cos^2 x}\,dx$

let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$= \int \frac{-du}{u^2}$

$= \frac{1}{u} + c$

$= \frac{1}{\cos x} + c$

$= \left[\frac{1}{\cos x}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{\cos(\frac{\pi}{4})} - \frac{1}{\cos(0)}$

$= \sqrt{2} - 1$

longer

$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\,dx$

$= \int_1^{\frac{\sqrt{2}}{2}} \frac{-du}{u^2}$

$= \int_1^{\sqrt{2}} \frac{du}{u^2}$

$= \left[\frac{1}{u}\right]_1^{\sqrt{2}}$

Shorter

relies on converting the $\int$.

- $\int_0^{\ln 4} \left(\frac{e^x}{3+2e^x}\right)dx$

Let $u = 3 + 2e^x$, $du = 2e^x\,dx$

$= \frac{1}{2}\int_0^{\ln 4} \frac{du}{u}$

$= \frac{1}{2}\int^{u} \frac{du}{u}$

$= \frac{1}{2}\cancel{[\ldots]}$

$= \frac{1}{2}\left[\ln|u|\right]_5^{11}$

$= \frac{1}{2}\left[\ln|11| - \ln|5|\right]$

$= \frac{1}{2}\left[\ln\frac{11}{5}\right]$

$= \int_0^4 \left(\frac{x}{x^2+1}\right)dx$

Let $u = x^2+1$, $du = 2x\,dx$

$= \int_0^4 \frac{\frac{1}{2}du}{u}$

$= \frac{1}{2}\int_0^4 \frac{du}{u}$

$= \frac{1}{2}\int_1^{17} \frac{du}{u}$

$= \frac{1}{2}\left[\ln|u|\right]_1^{17}$

$= \frac{1}{2}\left[\ln|17| - \ln|1|\right]$

$= \frac{1}{2}\ln(17)$

uses double angle & pythagorean identities.

- $\int \sin^2 x \, dx$

$= \frac{1}{2}\int 1 - \cos 2x \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$\int \cos^2 x \, dx$

$= \frac{1}{2}\int 1 + \cos 2x \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

- $\int \frac{x}{1+x^4} dx.$

Let $u = x^2$, $\frac{du}{dx} = 2x$

$= \int \frac{\frac{1}{2}du}{1+u^2}$

$= \frac{1}{2}\int \frac{du}{1+u^2}$

$= \frac{1}{2}(\tan^{-1} u)$

$= \frac{1}{2}(\tan^{-1} x^2) + c.$

- $\int_0^2 x^3 \sqrt{16 - x^4} \, dx.$

Let $u^2 = 16 - x^4$, ~~...~~ $2u\frac{du}{dx} = -4x^3 \Rightarrow du = \frac{-4x^3 dx}{2u}$

$u \, du = -2x^3 dx$

$= \int_0^2 -\frac{1}{2} u \, du \cdot u$

$= -\frac{1}{2}\int_0^2 u^2 \, du.$

$= -\frac{1}{2}\int_4^0 u^2 \, du.$

$= \frac{1}{2}\int_0^4 u^2 \, du$

$= \frac{1}{2}\left[\frac{u^3}{3}\right]_0^4$

$= \frac{1}{2}\frac{4^3}{3}$

$= \frac{64}{6} \Rightarrow \frac{32}{3}.$

- $\int_{\pi}^{0} \left(\frac{\sin x}{2 + \cos x}\right) dx$

Let $u = 2 + \cos x$, $du = -\sin x \, dx$

$\int_{\pi}^{0} \frac{-du}{u}$

$= -\int_1^3 \frac{du}{u}$

$= -[\ln|u|]_1^3$

$= ~~[\ldots]~~$

$= -[\ln|3| - \ln|1|]$

$= -\ln 3.$