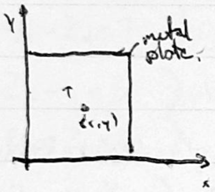
Advanced Differential Equations.

Assignments Due Fridays @ 12:30. in class

10 Partial Differential Equations and Fourier Series

- 

  - Metal plate heats up due to source at the origin, eventually reaches steady state.
  - $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

10.1 Two-Point Boundary Value Problems.

- $y'' + p(x)y' + q(x) = g(x)$
  $\underline{y(\alpha) = A}$, $y'(\beta) = B$ where $\alpha \neq \beta$
  ↳ Two-core boundary conditions.
- In Boundary Value Problems, the problem commonly resolve with no solutions, one solution, or infinite solutions.
- Ex (1) $y'' + 4y = \cos x$ $y(0) = 0$, $y(\pi) = 0$

  ① Find complementary solution

  $y'' + 4y = 0$ Try $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

  $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ (when $r = \lambda + i\mu$, $y = e^{\lambda x}[c_1\cos(\mu x) + c_2\sin(\mu x)]$)

  $\therefore y_c = c_1\cos(2x) + c_2\sin(2x)$

  ② Find particular solution.

  Try $y = A\cos x + B\sin x$.

  then $y' = -A\sin x + B\cos x$, $y'' = -A\cos x - B\sin x$

  $-A\cos x - B\sin x + 4A\sin x + 4B\cos x = \cos x$.

  $-3A\cos x + 3B\sin x = \cos x$. $3B = 0$, $B = 0$

  $3A = 1$, $A = \frac{1}{3}$

  $y_p = \frac{1}{3}\cos x$

  The general solution is $y_g = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\cos x$.

  $y(0)$: $0 = c_1(1) + c_2(0) + \frac{1}{3}$, $c_1 = -\frac{1}{3}$. c_1

  $y(\pi) = 0 = c_1(1) + c_2(0) + \frac{1}{3}$, $c_1 = \frac{1}{3}$ $\neq$ c_1

  Since we cannot find a c_1 where it is defined, the Boundary value problem has zero solutions.

$\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

• Ex (1) $y'' + 4y = \sin x \quad y(0) = 0, \quad y(\pi) = 0.$

① find complementary solution

$r^2 + 4 = 0, \quad r = \pm 2i$

$y_c = c_1\cos(2x) + c_2\sin(2x)$

② find particular solution.

Try $y = A\cos x + B\sin x, \quad y' = -A\sin x + B\cos x, \quad y'' = -A\cos x - B\sin x.$

$-A\cos x - B\sin x + 4(A\cos x + B\sin x) = \sin x$

$3A\cos x + 3B\sin x = \sin x. \quad A = 0, \ B = \frac{1}{3}.$

③ General Solution

$y_g = y_c + y_p = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x.$

④ Solve

$y(0) = 0 = c_1(1) + c_2(0) + 0, \quad c_1 = 0$

$y(\pi) = 0 = c_1(1) + c_2(0) + \frac{1}{3}(0) \quad c_1 = 0.$

$y = c_2\sin(2x) + \frac{1}{3}\sin x$ is the solution to the BVP (has infinitely many).

• Ex (2) $y'' + 2y = 0 \quad y'(0) = 1, \quad y'(\pi) = 0.$

① $r^2 + 2 = 0, \quad r = \pm i\sqrt{2}$

$y_c = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x).$

② Since it is homogeneous, no need to find particular solution

③ $y_g = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x).$

④ $y' = -\sqrt{2}c_1\sin(\sqrt{2}x) + \sqrt{2}c_2\cos(\sqrt{2}x)$

$y'(0) = 1 = 0 + \sqrt{2}c_2 \Rightarrow c_2 = \frac{1}{\sqrt{2}}$

$y'(\pi) = 0 = -\sqrt{2}c_1\sin(\sqrt{2}\pi) + \sqrt{2}c_2\cos(\sqrt{2}\pi).$

$\sqrt{2}c_1\sin(\sqrt{2}\pi) = \cos(\sqrt{2}\pi), \quad c_1 = \cot(\sqrt{2}\pi)\frac{1}{\sqrt{2}}$

$y = \left(\frac{1}{\sqrt{2}}\cot(\sqrt{2}\pi)\cos(\sqrt{2}x) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}x)\right)$

$\therefore$ the solution of the BVP is

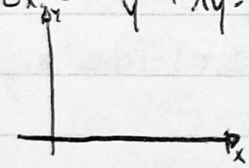
$y = \frac{1}{\sqrt{2}}\cot(\sqrt{2}\pi)\cos(\sqrt{2}x) + \frac{1}{2}\sin(\sqrt{2}x).$

Comparisons with Systems of Linear Equations.

- $\left.\begin{aligned} 2x - y &= 3 \\ 4x - 2y &= 7 \end{aligned}\right\}$ No solutions
- $\left.\begin{aligned} 2x - y &= 3 \\ 4x - 2y &= 6 \end{aligned}\right\}$ Infinitely many solutions.
- $\left.\begin{aligned} 2x - y &= 3 \\ x - y &= 0 \end{aligned}\right\}$ Exact one solution.

Homogeneous Two-Point BVPs.

- Homogeneous two-point BVPs is a problem of the type
  $y'' + p(x)y' + q(x)y = 0$, $y(\alpha) = 0$ & $y'(\beta) = 0$.
- We will study two-point BVPs of the type $y'' + \lambda y = 0$, $y(\alpha) = 0$, $y(\beta) = 0$.
  - where y' term is absent.
- Ex. $y'' + \lambda y = 0$ $\quad y(0) = 0$, $y'(\pi) = 0$.



<u>Interested in find non-zero solutions.</u> $(\underset{\text{square matrix } n\times n}{A}\vec{x} = \lambda \underset{\text{vector } n\times 1}{\vec{x}})$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$y = 0$ is a solution regardless of the value of λ. If for a particular λ, there is at least one non-zero solution (for y) then that λ is called an eigenvalue and that solution is called an eigenfunction. $\vec{x} = 0$ is a trivial solution

Do we have negative eigenvalues?

- Set $\lambda = -\mu^2$, where $\mu > 0$.
- $y'' - \mu^2 y = 0 \Rightarrow r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$

  General solution is, $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$

  then $\quad y' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x}$

  $y(0) = 0 \Rightarrow 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$ sub

  $y'(\pi) = 0 \Rightarrow 0 = \mu c_1 e^{\mu\pi} + \mu c_1 e^{-\mu\pi}$

  $0 = \mu c_1 (e^{\mu\pi} + e^{-\mu\pi})$ $\quad c_1 = 0$ & $\therefore c_2 = 0 \Rightarrow y = 0$.

-

Is zero an eigenvalue?

- $y''=0 \Rightarrow y''=A \Rightarrow y=Ax+B.$ $\quad y'(\pi)=0 \Rightarrow A=0$

  $y=B.$ $\quad y(0)=0 \Rightarrow B=0$

- The only solution is $y=0.$

Do we have positive eigenvalues?

- $y''+\mu^2 y=0$ where ~~$y\#$~~ $\mu>0.$

  $r^2+\mu^2=0 \Rightarrow r=\pm i\mu$

  $y=e^{\lambda x}(c_1\cos\mu x + c_2\sin\mu x)$

  then $y'=-\mu c_1\sin(\mu x)+\mu c_2\cos(\mu x)$

  $y(0)=0 \Rightarrow 0=c_1(1) \Rightarrow c_1=0$

  $y'(\pi)=0 \Rightarrow$ ~~$0=$~~ $\mu c_2\cos(\mu\pi)$ $\quad$ For $y\neq 0$, we must have $c_2\neq 0$ (want $\cos(\mu\pi)=0$)

  $\mu\pi=\frac{\pi}{2},\frac{3}{2}\pi,\frac{5}{2}\pi \dots$

  $\mu=\frac{1}{2},\frac{3}{2},\frac{5}{2}\dots=\frac{2n-1}{2}$, where $n=1,2,3\dots$

  $\lambda=\mu^2=\frac{1}{4},\frac{9}{4},\frac{25}{4}\dots=\frac{(2n-1)^2}{4}$, where $n=1,2,3\dots$ are eigenvalues.

What are the eigenfunctions?

- Fix an eigenvalue $\lambda=\mu^2=\frac{(2n-1)^2}{4}$ for some fixed positive integer, n

  $y=c_1\cos(\mu x)+c_2\sin(\mu x)$, $c_1=0$

  $y=c_2\sin\left(\left(\frac{2n-1}{2}\right)x\right).$

  $y=\sin\left(\left(\frac{2n-1}{2}\right)x\right)$ is an eigenfunction corresponding to the eigenvalue $\frac{(2n-1)^2}{4}$

10.2 Fourier Series.

- A fourier series is a series of the form

$$\frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{L}\right) + b_1\sin\left(\frac{\pi x}{L}\right) + a_2\cos\left(\frac{2\pi x}{L}\right) + b_2\sin\left(\frac{2\pi x}{L}\right) \dots$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right]$$

$a_1, a_2, a_3 \dots$ & $b_1, b_2, b_3 \dots$ are real constants and L is a positive constant.

Periodicity of functions.

- We say a function, f, is periodic with period $T>0$, if x is in the domain of, then $x+T$ is also in the domain an $f(T+x) = f(x)$
- Ex. $T = 2\pi$, $f(x) = \sin(x)$, $4\pi, 6\pi, 8\pi \dots$ $f(x+2\pi) = \sin x$.
- Smallest such T, if it exists, is called the fundamental period.
- Ex $f(x) = 2$, $f(x+T) = 2$, no fundamental period.
- Ex. $\cos\left(\frac{\pi x}{L}\right) = \cos\left(\frac{\pi x}{L} + 2\pi\right) = \cos\left[\frac{\pi}{L}(x+2L)\right]$ $2L$ is a period

$\cos\left(\frac{2\pi x}{L}\right) = \cos\left(\frac{2\pi x}{L} + 2\pi\right) = \cos\left[\frac{2\pi}{L}(x+L)\right]$ L is a period.

$\cos\left(\frac{3\pi x}{L}\right) = \cos\left(\frac{3\pi x}{L} + 2\pi\right) = \cos\left[\frac{3\pi}{L}\left(x+\frac{2}{3}L\right)\right]$ $\frac{2}{3}L$ is a period

In general, $\cos\left(\frac{m\pi x}{L}\right) = \cos\left[\frac{m\pi}{L}\left(x+\frac{2L}{m}\right)\right]$ $\frac{2L}{m}$ is a period.

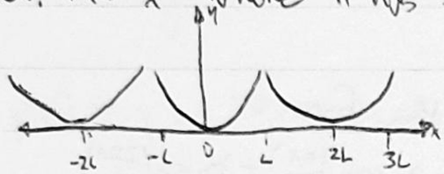
- If T is a period for some function, f, then $2T, 3T, 4T \dots$ are periods aswell

$$f(x+2T) = f(x+T+T) = f(x+T) = f(x)$$

- Theorem: Let f and g be periodic with the same period T. Then any linear combination $c_1f + c_2g$ is also periodic with period T
- Proof: $(c_1f + c_2g)(x+T) = c_1f(x+T) + c_2g(x+T) = c_1f(x) + c_2g(x) = (c_1f + c_2g)(x)$.
- If the fourier series converges to a function, then that function must be periodic with period $2L$.
-

function	fundamental period
$\cos\left(\frac{\pi x}{L}\right)$	$2L$
$\cos\left(\frac{2\pi x}{L}\right)$	L
$\cos\left(\frac{3\pi x}{L}\right)$	$\frac{2}{3}L$

• Ex. $f(x)=x^2$ where it has periodicity of $2L$.



• Want to find the Fourier Series of a periodic function with period $2L$.

- $\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\left(\frac{(m+n)\pi x}{L}\right)+\sin\left(\frac{(n-m)\pi x}{L}\right)\right]dx$

$= \frac{1}{2}\left[-\cos\left(\frac{(m+n)\pi x}{L}\right)\frac{L}{(m+n)\pi} - \frac{L}{(n-m)\pi}\cos\left(\frac{(n-m)\pi x}{L}\right)\right]_{-L}^{L}$ if $n \neq m$

when $m \neq n$

$= 0$, by symmetry

- $\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi x}{L}\right)+\cos\left(\frac{(m-n)\pi x}{L}\right)\right]$

$= \frac{1}{2}\left[\frac{L}{\pi(m+n)}\sin\left(\frac{(m+n)\pi x}{L}\right)+\frac{L}{(n-m)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right)\right]_{-L}^{L} dx$

$= 0$, by $\sin(k\pi)$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

$\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$

$\cos(2x)=2\cos^2 x$

$\sin(2\theta)=2\sin\theta\cos\theta$

when $m=n$

- $\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{2}\int_{-L}^{L}\left[1+\cos\left(\frac{2n\pi x}{L}\right)\right]dx$ if $n \neq 0$:

$= \frac{1}{2}\left[x+\frac{L}{2\pi n}\sin\left(\frac{2n\pi x}{L}\right)\right]_{-L}^{L}$

$= L$

- $\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx =$

$u=\sin\left(\frac{n\pi x}{L}\right)$, $du=\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)dx$.

$\frac{L}{n\pi}\int_0^0 u\,du = \frac{L}{n\pi}\left[\frac{1}{2}u^2\right]_0^0 = 0.$

• $\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & \text{if } m\neq n \\ L & \text{if } m=n \end{cases}$

$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & \text{if } m\neq n \\ L & \text{if } m=n. \end{cases}$

$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = 0.$

- ~~…, with n≠0~~ Fixed n, $n \neq 0$.

$\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \Rightarrow \int \frac{a_0}{2}\sin\left(\frac{n\pi x}{L}\right)dx$

$= \left[-\frac{a_0}{2}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$

$= 0.$

$\int_{-L}^{L} a_m\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$

$\int_{-L}^{L} b_m\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$ but $\int_{-L}^{L} b_n\sin^2\left(\frac{n\pi x}{L}\right) = b_n L$

★ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$; $n = 1, 2, 3, \ldots$

- Fixed n, $n \neq 0$.

$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

… $\int_{-L}^{L}\frac{a_0}{2}\cos\left(\frac{n\pi x}{L}\right) = 0$

same process as above

★ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$; $n = 0, 1, 2, 3, \ldots$

- $\int_{-L}^{L} f(x)dx = \int_{-L}^{L}\frac{a_0}{2}dx = \frac{a_0}{2}\left[x\right]_{-L}^{L} = a_0 L$

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$ ★ but we can forget it cause it achievable by ($n=0$)

- These are Euler-Fourier formulas.

Linear Algebra View Point

- $C([-L,L]) = \{$ the set of all continuous real valued functions with domain $[-L,L]\}$.
  $= \{f : [-L,L] \to \mathbb{R} \mid f \text{ is continuous}\}$
- $C([-L,L])$ is closed under addition

① $\alpha f \in C([-L,L])$, where $\alpha \in \mathbb{R}$, $f \in C([-L,L])$

② $(\alpha+\beta)f = (\alpha)f + (\beta)f$.

③ $\alpha(f+g) = \alpha f + \alpha g$

④ $(\alpha\beta)f = \alpha(\beta f)$

⑤ $1f = f$.

⑥ $f + 0 = f$

⑦ $f + (-f) = 0$

⑧ $f + (g+h) = (f+g) + h$.

⑨ $f + g = g + f$.

⑩ Closed under addition.

- Satisfying all properties is a vector space.

• V is a real vector space

• Inner Product - $\langle , \rangle : V V \to \mathbb{R}$

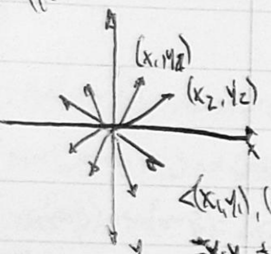
$\Rightarrow$ $\langle v, w \rangle = \langle w, v \rangle$

③ $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$.

④ $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$

⑤ $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

$\mathbb{R}^2$



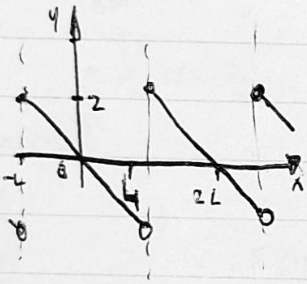
$\langle (x_1, y_1), (x_2, y_2) \rangle$

$= x_1 y_1 + x^2 y_2$

$(x_1\ y_1)(x_2\ y_2)$

• $\langle f, g \rangle := \int_{-L}^{L} f(x) g(x) dx$.

• Ex ③ $f(x) = -x$



$-L \leq x \leq L$

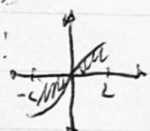
$f(x + 2L) = f(x)$.

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$a_n = -\frac{1}{L} \int_{-L}^{L} x \cos\left(\frac{n\pi x}{L}\right) dx$

$a_n = 0$, because $x\cos\left(\frac{n\pi x}{L}\right)$ is an odd fxn

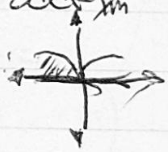
Odd fxn:



$\int = 0$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Even fxn:



$b_n = -\frac{1}{L} \int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx$

$b_n = -\frac{2}{L} \int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx$

$b_n = -\frac{2}{L} \left[-\frac{xL}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right) \right]_{-L}^{L}$

$b_n = -\frac{2}{L} \left[-\frac{L^2}{n\pi} \cos(n\pi) - 0 \right]$

$b_n = \frac{2L}{n\pi} \cos(n\pi)$

$b_n = \frac{2L}{n\pi} (-1)^n$

$\cos(n\pi) = \begin{cases} 1 & \text{when } n \text{ is even} \\ -1 & \text{when } n \text{ is odd} \end{cases}$

$= (-1)^n$

$\int x \sin\left(\frac{n\pi x}{L}\right) dx$

$= x\left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right] - \int \left(\frac{-L}{n\pi}\right) \cos\left(\frac{n\pi x}{L}\right) dx$

$= \frac{-xL}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)\left(\frac{L}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) + C$

$= \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right]$

$= \quad + \sum_{m=1}^{\infty} \left[\frac{2L}{m\pi} (-1)^m \sin\left(\frac{m\pi x}{L}\right) \right]$

$= \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin\left(\frac{m\pi x}{L}\right)$

• Ex ④ $\sin\left(\frac{\pi x}{L}\right) = \sin\left(\frac{\pi x}{L} + 2\pi\right)$

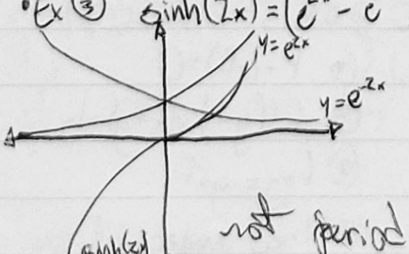
$= \sin\left(\frac{\pi}{L}(x + 2L)\right)$

$=$

2L is a period.

2L is the fundamental period

• Ex ⑤ $\sinh(2x) = \left(e^{2x} - e^{-2x}\right)/2$.



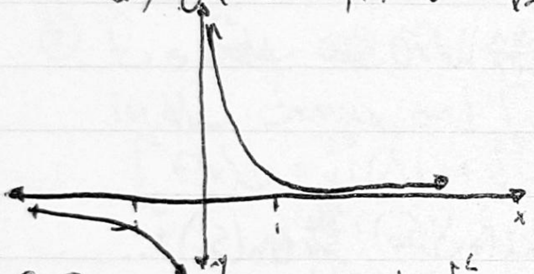
not periodic.

10.3 The Fourier Convergence Theorem.

~~Piecewise Con~~

Piecewise Continuous Functions

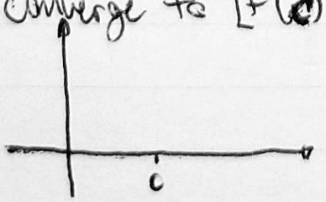
- An function $f:[a,b] \to \mathbb{R}$ is said to be piecewise continuous on $[a,b]$ if there is a partition. $a = x_0 < x_1 < x_2 \dots x_{n-1} < x_n = b$ such that for every i such that $1 \le i \le n$, the function is continuous on (x_{i-1}, x_i) and $\lim_{x \to x_{i-1}^+} f(x)$ exists, and $\lim_{x \to x_i^-} f(x)$ exists.
- $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$



- f function $\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right)$$

- For what values of x will this fourier Series converge to $f(x)$?
- Fourier Convergence Theorem
  - Suppose f and f' are piecewise continuous on $[-L,L]$, and f is defined outside this interval so that it is periodic with period $2L$. Then the fourier Series $\frac{a_0}{2} + \sum\left(a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right)$ where a_m & b_m are given by the Euler-Fourier formulas, converge to $[f(c^+) + f(c^-)]/2$, for all values of c.
  - 

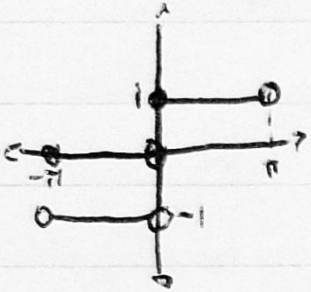
$$f(c^+) = \lim_{x \to c^+} f(x)$$
$$f(c^-) = \lim_{x \to c^-} f(x)$$

Ex ① $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < 0. \end{cases}$

$f(x+2\pi) = f(x)$

$2L = 2\pi$

$L = \pi$



$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \quad L=\pi$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[-\cos(nx)/n\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1-\cos(n\pi)}{n}\right] = \begin{cases} 0 & \text{if } n = \text{even} \\ \frac{4}{n\pi} & \text{if } n = \text{odd.} \end{cases}$$

$=$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = 0.$$

$\cos(n\pi) = 0$, when $n \neq$ even

$=$

Fourier Series.

↳ $\sum_{n=1}^{\infty} b_n \sin(nx)$

$= \frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \ldots$

n_1 $\quad n_2 = 0$ n_3 $\quad n_4 = 0.$ n_5

$$= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin((2n-1)x).$$

10.4 Even & Odd Functions.

- ① f is even and defined on $[-L,L]$.

  f is even if and only if ~~whenever~~ x is in the domain of f, $-x$ is also in the domain and $f(-x)=f(x)$

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$$
$$= \int_{L}^{0} f(-x)d(-x) + \int_{0}^{L} f(x)dx$$
$$= -\int_{L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$$
$$= \int_{0}^{L} f(x)dx + \int_{0}^{L} f(x)dx$$
$$= 2\int_{0}^{L} f(x)dx$$

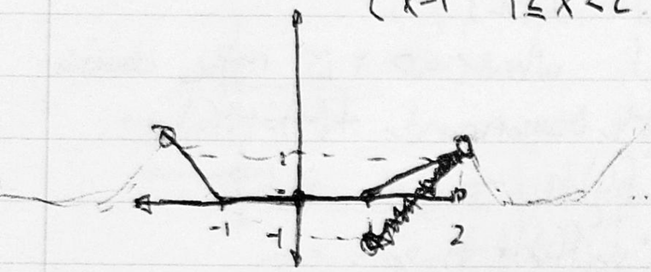
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right)dx$$

② f is ~~even~~ ~~odd~~ iff x, in is in domain of f, $-x$ is also in the domain, and $f(-x) = -f(x)$

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$$
$$= \int_{L}^{0} f(-x)d(-x) + \int_{0}^{L} f(x)dx$$
$$= \int_{L}^{0} -f(x)dx + \int_{0}^{L} f(x)dx$$
$$= -\int_{0}^{L} f(x)dx + \int_{0}^{L} f(x)dx$$
$$= 0.$$

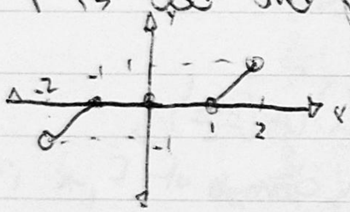
- Fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ $\quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
  - such a series is called a Fourier Cosine Series.
  - This is for an even function
- Fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $\quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$
  - such a series is called a Fourier sine Series.
  - This is ~~to~~ for an odd function.

• Ex ⑤ $f(x)=\begin{cases}0 & 0\le x<1\\ x-1 & 1\le x<2\end{cases}$ $L=2$ $[0,2]$ Extended fxn so that it is periodic with period 4, and f is even



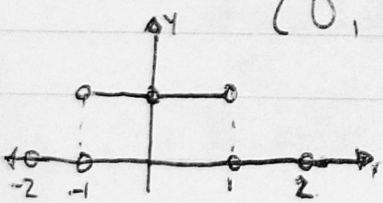
$$a_0=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$=\frac{2}{2}\int_0^2 f(x)\,(1)\,dx$$
$$=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx$$
$$=\int_1^2 (x-1)\,dx=\left[\frac{x^2}{2}-x\right]_1^2=\left[2-2-\frac{1}{2}-1\right]=\frac{1}{2}$$

f is odd and periodic with period 4.



$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

• Ex ⑮ $f(x)=\begin{cases}1, & \text{if } 0<x<1\\ 0, & \text{if } 1<x<2\end{cases}$ cosine series, period 4.

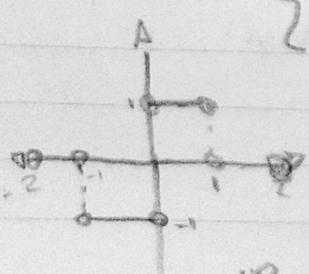


$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n x\pi}{L}\right)dx$$
$$b_n=0 \qquad L=\frac{\text{period}}{2}$$
$$a_n=\frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$
$$=\int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx+\overbrace{\int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx}^{0}$$
$$=\int_0^1\cos\left(\frac{n\pi x}{2}\right)dx=\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^1$$
$$=\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$a_0=\dots=1$

∴ the Fourier Series is $\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$

• Ex $f(x)=\begin{cases}1, & \text{if } 0<x<1\\ 0, & \text{if } 1<x<1\end{cases}$ sine series, period 4



$$a_n=0$$
$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$=\frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\right)\sin\left(\frac{n\pi x}{2}\right)$$

• (3) $\tan(2x)$ odd fxn

$|x|^3$ - even fxn

x^3-2x - odd fxn

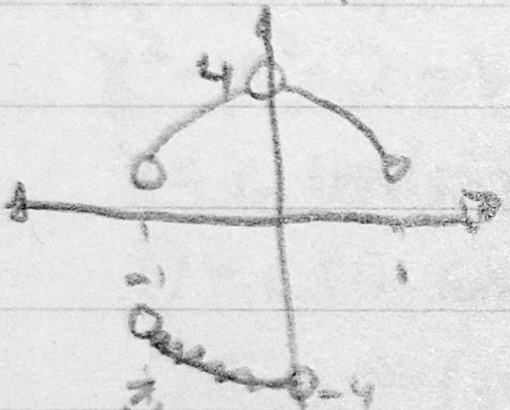
x^2-2x+1 = neither odd nor even.

• (12) $f(x)=4-x^2$ $L=1$

$0<x<1$

even extension

odd extension



odd extension

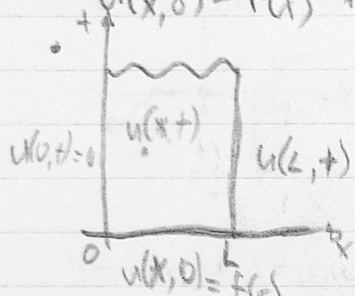
10.5 Separation of Variables

Heat Conduction in Rod

- 

Temp - $u(x,t)$ at distance x from left end at time, t

$u(x,0) = f(x)$ ← initial temp distribution

- 

- Heat Conduction Equation
  - $\alpha^2 u_{xx} = u_t$ $\qquad u_t = \frac{\partial u}{\partial t},\ u_{xx} = \frac{\partial^2 u}{\partial x^2}$
  - α - thermal diffusivity (constant for material of rod)
- $u = XT$, X is function of x only
  T " " " " t only
- $\alpha^2 X'' T = X T'$ divide by $\alpha^2 X T$

$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -\lambda$, where λ is constant

$X'' + \lambda X = 0$

$T' + \alpha^2 \lambda T = 0$

$u(x,t) = X(x)\,T(t)$

$u(x,0) = X(x)\,T(0) = f(x)$

$u(0,t) = X(0)\,T(t) = 0$, $X(0) = 0$

$u(L,t) = X(L)\,T(t) = 0$, $X(L) = 0$

$X'' + \lambda X = 0 \quad X(0) = 0, X(L) = 0$, TWO POINT BVP

- Are there negative eigenvalues?

$\lambda = -\mu^2$, where $\mu > 0$.

$X'' - \mu^2 X = 0 \Rightarrow r^2 - \mu^2 = 0 \Rightarrow r = \pm \mu$ $\qquad$ No negative eigenvalues.

$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$X(0) = 0 \Rightarrow 0 = c_1 + c_2$

$X(L) = 0 \Rightarrow 0 = c_1 e^{\mu L} + c_2 e^{-\mu L}$ $\quad e^{2\mu L} = 0$, $c_1 = 0 \Rightarrow \mu = 0$ are this is not we care

- Is zero an eigenvalue?

$X'' = 0 \Rightarrow X' = c_1 \Rightarrow X = c_1 x + c_2$

$\left.\begin{array}{l} X(0) = 0 \Rightarrow \therefore c_2 = 0 \\ X(L) = 0 \Rightarrow c_1 = 0 \end{array}\right\} X = 0$ only trivial solution.

- Is there positive eigenvalues?

$X'' + \mu^2 X = 0$ when $\mu > 0$

$r^2 - \mu^2 = 0 \Rightarrow \pm i\mu$

$X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$0 = c_1(1), \Rightarrow c_1 = 0$

$0 = c_2 \sin(\mu L) \Rightarrow \sin(\mu L) = 0 \Rightarrow \mu L = n\pi$

$\mu = \frac{n\pi}{L}$, when $n \in \mathbb{Z}^+$

$\lambda = \mu^2 = \frac{n^2\pi^2}{L^2}$ $n \in \mathbb{Z}^+$

Eigenfunctions corresponding to the eigenvalue $\lambda_n = \frac{n^2\pi^2}{L^2}$ are, $\sin\left(\frac{n\pi x}{L}\right)$.

- $T' + \alpha^2 \lambda T = 0$

$T' + \alpha^2 \frac{n^2\pi^2}{L^2} T = 0$

$\frac{dT}{dt} = -\frac{\alpha^2 n^2 \pi^2}{L^2} T$

$\int \frac{dT}{T} = \int -\frac{\alpha^2 n^2 \pi^2}{L^2} dt$

$\ln|T| = -\frac{\alpha^2 n^2 \pi^2}{L^2} t + C$

$|T| = \pm e^c \cdot e^{-\left(\frac{\alpha^2 n^2 \pi^2}{L^2} t\right)}$ $A = \pm e^c$

$T = A \cdot e^{-\left(\frac{\alpha^2 n^2 \pi^2}{L^2} t\right)}$

- $XT = e^{-\left(\frac{\alpha^2 n^2 \pi^2}{L^2} t\right)} \sin\left(\frac{n\pi x}{L}\right)$ is a solution

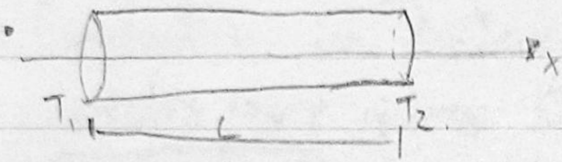
- General Solution

$\therefore u(x,t) = X_1 T_1 + X_2 T_2 + X_3 T_4 + \cdots$

$= \sum_{n=1}^{\infty} c_n X_n T_n = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\alpha^2 n^2 \pi^2}{L^2}\right) t} \sin\left(\frac{n\pi x}{L}\right)$

$\Rightarrow u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$, $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

10.6 Other Heat Conduction Problems

- 

$u(x,0) = f(x)$
$u(0,t) = T_1$
$u(L,t) = T_2$

- $w(x,t) = u(x,t) - \left(T_1 + \frac{(T_2 - T_1)}{L} x\right)$ $\qquad T_1 + \frac{(T_2 - T_1)}{L} x$
- $\alpha^2 u_{xx} = u_t$

  $u(x,t) = w(x,t) + T_1 + \frac{T_2 - T_1}{L} x$

  $u_{xx} = w_{xx} + 0 + 0 \Rightarrow w_t = u_t$

  $\alpha^2 w_{xx} = w_t$
- $w(0,t) = u(0,t) - T_1 = T_1 - T_1 = 0$

  $w(L,t) = u(L,t) - \left(T_1 + \frac{(T_2 - T_1)}{L} L\right) = u(L,t) - T_2 = T_2 - T_2 = 0$

$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right)$

  $w(x,0) = u(x,0) - \left(T_1 + \frac{(T_2 - T_1)}{L} x\right) = f(x) - \left(T_1 + \frac{(T_2 - T_1)}{L} x\right) =$

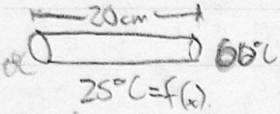
  $w(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-\alpha^2 n^2 \pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$
- $c_n = \frac{2}{L} \int_0^L w(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$

  $= \frac{2}{L} \int_0^L \left[f(x) - \left(T_1 + \frac{(T_2 - T_1)}{L} x\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx$
- $u(x,t) = T_1 + \frac{(T_2 - T_1)}{L} x + \sum_{n=1}^{\infty} c_n e^{\frac{-\alpha^2 n^2 \pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$
- $\lim_{t \to \infty} u(x,t) = \underbrace{T_1 + \frac{(T_2 - T_1)}{L} x}_{\text{Steady state temperature distribution}}$ $\qquad$ → transient temperature distribution

- $\alpha^2 u_{xx} = 0$, $u_{xx} = 0$, $u_x = c_1$, $u = c_1 x + c_2$

  $T_1 = c_1(0) + c_2$, $T_2 = c_1 L + T_1 \Rightarrow c_1 = \frac{T_2 - T_1}{L} \Rightarrow u(x,t) = \frac{(T_2 - T_1)}{L} x + T_1$

• (9)



$u(x,t)$ $\alpha^2 = 0.8418 \frac{cm^2}{s}$

$T_1 = 0$, $L = 20cm$ $C_n = \frac{2}{20}\int_0^{20}\left[25 - \left(\frac{60}{20}x\right)\right]\sin\left(\frac{n\pi x}{20}\right)dx$

$T_2 = 60°C$

• (3) $\alpha^2 u_{xx} = u_t$

$u_x(0,t) = 0$

$u_{xx} = 0$

$u_x = c_1$

$u = c_1 x + c_2$

$u(L,t) = 0$

$0 = c_1 L + c_2$

$u_x = c_1 = 0$

$c_2 \to 0$

$u = 0$

• (8) $\alpha^2 u_{xx} = u_t$

$u(0,t) = T$ $u_x(L,t) + u(L,t) = 0$

$u_{xx} = 0$

$u(x,t) = c_1 x + c_2 \Rightarrow T = c_1(0) + c_2 \Rightarrow T = c_2$

$c_1 + c_1 L + c_2 = 0$

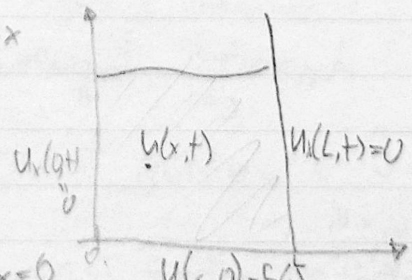
$c_1(1+L) = -T$

$c = \frac{-T}{(1+L)}$

Steady State Temperature distribution is

$u(x,t) = T - \frac{T}{(1+L)}x$

•



$u(x,0) = f(x)$

$u_x(0,t) = 0$

$u_x(L,t) = 0$

$\alpha^2 u_{xx} = u_t$ $X'' + \lambda X = 0$

$u = XT$ $u_x(x,t) = X'T$

$\alpha^2 X''T = XT'$ $u_x(0,t) = 0 \Rightarrow 0 = X'(0)T(t) \Rightarrow X'(0) = 0$

$\frac{X''}{X} = \frac{T'}{\alpha^2 T}$ $u_x(L,t) = 0 \Rightarrow 0 = X'(L)T(t) \Rightarrow X'(L) = 0$

$X'' + \lambda X = 0$

$T' + \alpha^2 \lambda T = 0$

Negative eigenvalues?

If so, $\lambda = -\mu^2$, $\mu > 0$.

Characteristic equation is $r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$.

$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$X' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x}$

$X'(0) = 0 \Rightarrow 0 = \mu(c_1 - c_2) \Rightarrow c_1 = c_2$

$X'(L) = 0 \Rightarrow 0 = \mu c_1 (e^{\mu L} - e^{-\mu L})$

$c_1 = 0,\ c_2 = 0.$

Zero eigenvalues?

$X'' = 0 \Rightarrow X = c_1 x + c_2$ $\quad X(x) = 0$

$X' = c_1$

$X'(0) = 0 \Rightarrow c_1 = 0.$ $\quad$ $\lambda = 0$ is an eigenvalue

$X'(L) = 0$ $\quad$ $X = 1$ is an eigenfunction

Positive eigenvalues?

$\lambda = \mu^2$, $\mu > 0$

$X'' + \mu^2 X = 0 \Rightarrow r^2 - \mu^2 = 0 \Rightarrow r = \pm i\mu$

$X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

$X' = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x)$.

$X'(0) = 0 \Rightarrow 0 = \mu c_2 \cos(0) \Rightarrow c_2 = 0$

$X'(L) = 0 \Rightarrow 0 = -\mu c_1 \sin(\mu L)$.

$\Rightarrow \sin(\mu L) = 0$ $\quad \mu L = n\pi \Rightarrow \mu = \frac{n\pi}{L}$

Eigenvalues, $\lambda = \mu^2 = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$

$\lambda_n = \frac{n^2\pi^2}{L^2}$, Eigenfunction ...

$\lambda = 0$ $\quad T' + \alpha^2(0)T = 0 \Rightarrow T' = 0$, $T = K$.

$u = K$

$\lambda = \frac{n^2\pi^2}{L^2} \Rightarrow T' + \frac{\alpha^2 n^2\pi^2}{L^2} T = 0 \Rightarrow \frac{dT}{dt} = -\frac{\alpha^2\pi^2 n^2}{L^2} T$

$\int \frac{dT}{T} = \int -\frac{\alpha^2 n^2\pi^2}{L^2} dt$

$\ln|T| = -\frac{\alpha^2 n^2\pi^2}{L^2} t + K$.

$T = A e^{-\frac{\alpha^2 n^2\pi^2 t}{L^2}}$

$u_n = XT = e^{-\frac{\alpha^2 n^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right)$

$u(x,t) = K + \sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right)$

$K = \frac{a_0}{2}$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{\left(\frac{-\alpha^2 n^2 \pi^2}{L^2}\right)t} \cos\left(\frac{n\pi x}{L}\right)$$

$$u(x,0) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 0, 1, 2, 3 \ldots$$

steadystate temp distribution

transient temprature distribution.

- The amount heat energy at steady state = (mass)(specific heat)(temp)

$= mc\frac{a_0}{2} = \rho ALc\left(\frac{a_0}{2}\right)$

$= (\rho ALc)\left(\frac{1}{2}\right)\left(\frac{2}{L}\right)\int_0^L f(x)dx$

volume = $A\,dx$

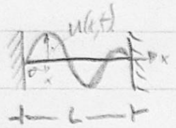
mass = $\rho A\,dx$

The amount of heat energy in the disk = $\rho A c f(x) dx$

The total heat energy in the rod initialy = $\int_0^L \rho A c f(x) dx$

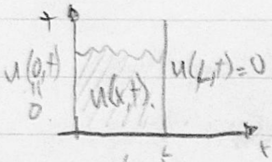
$= \int_0^L \rho A c f(x) dx = \rho A c \int_0^L f(x) dx$.

10.7 The Wave Equation: Vibrations on an Elastic String.



$a^2 u_{xx} = u_{tt}$ $\quad u(0,t)=0$, $u(L,t)=0$.

$u(x,0)=f(x)$



$u(x,0)=f(x)$

$u_t(x,0)=0$

Assume $u = XT$. $\qquad X'' + \lambda X = 0$.

$a^2 X'' T = X T''$ $\qquad T'' + a^2 \lambda T = 0$

$\dfrac{X''}{X} = \dfrac{T''}{a^2 T} = -\lambda$. $\qquad u(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0$

$u(L,t)=0 \Rightarrow X(L)=0$

Show there are no negative eigenvalues.

" " zero is not an "

Are there positive eigenvalues? $\lambda = \mu^2$ where $\mu > 0$.

$X'' + \mu^2 X = 0 \Rightarrow r^2 + \mu^2 = 0 \Rightarrow r = \pm i\mu$

$X = C_1 \cos(\mu x) + C_2 \sin(\mu x)$ $\Big\} \Rightarrow X = C_2 \sin(\mu x)$

$X(0)=0 \Rightarrow 0 = C_1$ $\Big\}$ $X(L)=0 \Rightarrow 0 = C_2 \sin(\mu L) \Rightarrow \mu L = n\pi$

$\mu = \frac{n\pi}{L}$, $n=1,2$

$\lambda = \mu^2 = n^2\pi^2/L^2$

- Eigenfunctions are $\sin\left(\frac{n\pi x}{L}\right)$, $n=1,2,\ldots$

$T'' + \frac{n^2\pi^2 a^2}{L^2} T = 0$ characteristic equation is $r^2 + \frac{n^2\pi^2 a^2}{L^2} = 0$

$\Rightarrow r = \pm \frac{i n\pi a}{L}$

$T = K_1 \cos\left(\frac{n\pi a}{L} t\right) + K_2 \sin\left(\frac{n\pi a t}{L}\right)$

- $u_t(x,t) = XT' \Rightarrow u_t(x,0) = 0 = X(x)T'(0) \Rightarrow T'(0) = 0$

$T' = -K_1\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi a t}{L}\right) + K_2\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)$

$0 = K_2\left(\frac{n\pi a}{L}\right) \Rightarrow K_2 = 0$

$T = \cos\left(\frac{n\pi a t}{L}\right)$.

- $u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$

$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

initial displacement
is zero

• Same case but $u(x,0)=0$, $u_t(x,0)=g(x)$.

$$u(x,t)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right)$$

$$\left(\frac{n\pi a}{L}\right)c_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Nonzero Initial Displacement & Nonzero Initial Velocity.

• $a^2 u_{xx}=u_{tt}$, $u(0,t)=0=u(L,t)$, $u(x,0)=f(x)$, $u_t(x,0)=g(x)$.

• Suppose the solution for (initial displacement is zero case) is $v(x,t)$ &

" " " (initial velocity is zero case) is $w(x,t)$

• Let $u(x,t)=v(x,t)+w(x,t)$

$$a^2u_{xx}=a^2(v_{xx}+w_{xx})=a^2v_{xx}+a^2w_{xx}=v_{tt}+w_{tt}=(v+w)_{tt}=u_{tt}$$

$$u(x,0)=v(x,0)+w(x,0)=f(x)+0=f(x).$$

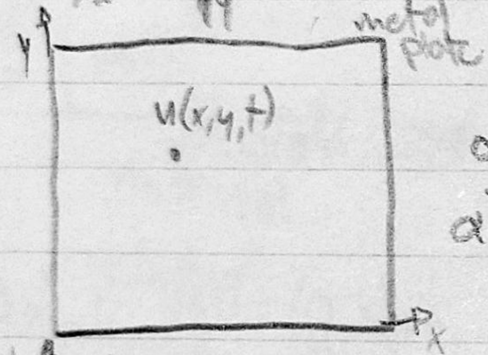
$$u_t(x,t)=v_t(x,t)+w_t(x,t)$$

$$u_t(x,0)=v_t(x,0)+w_t(x,0)=0+g(x)=g(x)$$

10.8 Laplace's Equation

- $u(x,y)$ $\quad$ $u(x,y,z)$

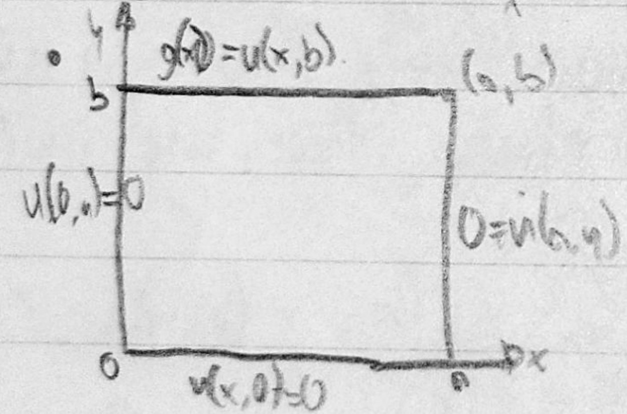
  $u_{xx}+u_{yy}=0$ $\quad$ $u_{xx}+u_{yy}+u_{zz}=0$

- 

  steady state case: $u_t=0$

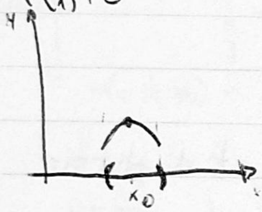
  $\alpha^2 u_{xx}=u_t$ $\quad$ $\alpha^2(u_{xx}+u_{yy})=0$

  $\alpha^2(u_{xx}+u_{yy})=u_t$ $\quad$ $u_{xx}+u_{yy}=0.$

- 

  $u_{xx}+u_{yy}=0$

5.2 Series Solutions Near Ordinary Point Part 1.

- $P(x)y'' + Q(x)y' + R(x)y = 0$, where P, Q, R are polynomials
- A real number x_0 is called an ordinary point if $P(x_0) \neq 0$. In that case, there is an open interval containing x_0, on which $P(x) \neq 0$
- 

- $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$ $\quad p(x) = \frac{Q(x)}{P(x)}$

  $y'' + p(x)y' + q(x)y = 0$ $\quad q(x) = \frac{R(x)}{P(x)}$

  $y(x_0) = A, \quad y'(x_0) = B$

- Ex ③ $y'' - xy' - y = 0$, $x_0 = 0$

  seek solution of the form $y = a_0 + a_1 x + a_2 x^2 \ldots = \sum_{n=0}^{\infty} a_n x^n$

  $y' = 0 + a_1 + 2a_2 x \ldots = \sum_{n=0}^{\infty} n a_n x^{n-1}$

  $y'' = 2a_2 + 6a_3 x + 12a_4 x^2 \ldots = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0 \quad \text{let } m = n-2, \ n = m+2.$$

$$\sum_{m=0}^{\infty} a(m+2)(m+1)x^{m-2}$$

$$\sum_{n=0}^{\infty} a_n (n+2)(n+1)x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n a_n - a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0, \quad n = 0, 1, 2, 3 \ldots$$

$$a_{n+2} = \frac{n+1}{(n+1)(n+2)}a_n = \frac{1}{n+2}a_n$$

$$a_{n+2} = \frac{a_n}{n+2} \quad \text{for } n = 0, 1, 2, \ldots \quad \text{- recurrence rule}$$

ⓑ $a_2 = \frac{a_0}{2}$ $\qquad a_3 = \frac{a_1}{3}$

$a_4 = \frac{a_2}{4} = \frac{a_0}{8}$ $\qquad a_5 = \frac{a_3}{5} = \frac{a_1}{15}$

$a_6 = \frac{a_4}{6} = \frac{a_2}{4 \cdot 6} = \frac{a_0}{2 \cdot 4 \cdot 6} = \frac{a_0}{48}$ $\qquad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7} = \frac{a_1}{105}$

$\vdots$

21
21
21
21
$+21$
105

General Solution: $y = a_0 + \left(\frac{a_0}{2}\right)x^2 + \left(\frac{a_0}{8}\right)x^4 + \left(\frac{a_0}{48}\right)x^6 + \ldots$

$+ a_1 x + \left(\frac{a_1}{3}\right)x^3 + \left(\frac{a_1}{15}\right)x^5 + \left(\frac{a_1}{105}\right)x^7 \ldots$

$= \left[a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{8} \ldots\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} \ldots\right)\right]$

Let $y_1 = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots$

$y_2 = x + \frac{x^3}{3} + \frac{x^5}{3} + \dots$

Wronksian, $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$

$f:(a,b) \to \mathbb{R}$

$g:(a,b) \to \mathbb{R}$

if $W \neq 0$, there is at least one x value that is not zero there is at least one solution.

$ay_1 + by_2 = 0 \Rightarrow a=0$ & $b=0$ then linear independent.

⑯ $W(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\Rightarrow \{y_1, y_2\}$ form a fundamental set of solutions.

$y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n(n!)}$

$y_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1\cdot 3\cdot 5 \cdots (2n+1)} \Rightarrow \sum_{n=0}^{\infty} \frac{2^n(n!)x^{2n+1}}{(2n+1)!}$

- On which intervals do y_1 & y_2 converge?

y_1, ratio test, $\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{2^{n+1}(n+1)!} \cdot \frac{2^n(n!)}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2(n+1)} \right| = 0 < 1$

$y_1 = e^{\frac{x^2}{2}}$

y_2 ratio test, $\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!\, x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^n(n!)x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{2x^2(n+1)}{(2n+3)(2n+2)} \right| = 0$

The Hermite Equation of Order λ

⑰

- $y'' - 2xy' + \lambda y = 0 \quad , -\infty < x < \infty$

Charles Hermite

$A^* = (\bar{A})^T$

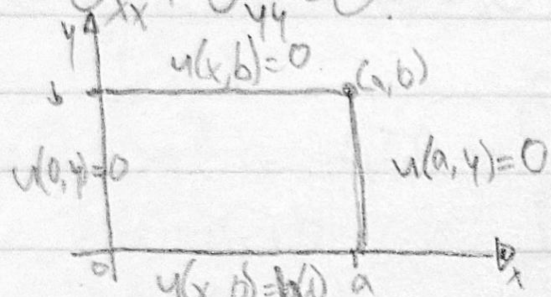
$A = \begin{bmatrix} 1+i & 2 \\ 3 & 1-i \end{bmatrix}$

$A^* = \begin{bmatrix} 1-i & 2 \\ 3 & 1+i \end{bmatrix}^T = \begin{bmatrix} 1-i & 3 \\ 2 & 1+i \end{bmatrix}$

e is a transcendental number.

Q.2 of 10.8

• $U_{xx} + U_{yy} = 0$.



$u(x,y) = XY$

$X''Y + XY'' = 0$

$\frac{X''}{X} + \frac{Y''}{Y} = 0$.

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$X'' + \lambda X = 0$, No negative eigenvalues, zero eigenvalue.

$\mu = \frac{n\pi}{a}$, $\lambda = \frac{n^2\pi^2}{a^2}$

$X_n = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \dots$

$Y'' - \lambda Y = 0$

$Y'' - \frac{n^2\pi^2}{a^2} Y = 0 \Rightarrow r^2 = \frac{n^2\pi^2}{a^2}$, $r = \pm \frac{n\pi}{a}$.

$Y = K_1 e^{\frac{n\pi}{a} y} + K_2 e^{-\frac{n\pi}{a} y}$

• $u(x,b) = 0$

$X(x)Y(b) = 0 \Rightarrow Y(b) = 0$

$0 = K_{1,n} e^{\frac{n\pi b}{a}} + K_{2,n} e^{-\frac{n\pi b}{a}}$

$K_{2,n} e^{-\frac{n\pi b}{a}} = -K_{1,n} e^{\frac{n\pi b}{a}}$

$K_{2n} = -K_{1n} e^{\frac{2n\pi b}{a}}$

$Y_n = K_{1,n}\left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right)$

• $u(x,y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} K_{1n}\left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right) \sin\left(\frac{n\pi x}{a}\right)$

$u(x,0) = h(x)$.

$h(x) = \sum_{n=1}^{\infty} K_{1n}\left(1 - e^{\frac{2n\pi b}{a}}\right) \sin\left(\frac{n\pi x}{a}\right)$ ← Fourier Sine Series.

$K_{1n}\left(1 - e^{\frac{2n\pi b}{a}}\right) = \frac{2}{a}\int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx$.

5.2.

- #14 $(1-x)y'' + xy' - y = 0$, $y(0) = -3$, $y'(0) = 2$.

$y = \sum_{n=0}^{\infty} a_n x^n$ + recurrence relation

$$a_{n+2} = \frac{n(n+1)a_{n+1} - (n-1)a_n}{(n+2)(n+1)} \quad n = 0, 1, 2, \ldots$$

$n=0$, $a_2 = \frac{a_0}{2}$

$n=1$, $a_3 = \frac{2a_2}{3(2)} \Rightarrow \frac{a_0}{6}$

$n=2$, $a_4 = \frac{6a_3 - a_2}{12} \Rightarrow \frac{a_0 - \frac{a_0}{2}}{12} = \frac{a_0}{24}$

$y = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 + \frac{a_0}{24}x^4 + \ldots$

$y(0) = -3 \Rightarrow -3 = a_0$,

$y' = a_1 + a_0 x + \ldots$

$y'(0) = 2 = a_1 \Rightarrow a_1 = 2$

$y = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 + \ldots$

Legendre Polynomials.

- Legendre polynomials of degree n, denoted by $P_n(x)$ is the n^{th} degree polynomial such that
  1. $\int_{-1}^{1} P_n(x) f(x)\,dx = 0$ for all polynomials $f(x)$ with degree strictly less than n
  2. $P_n(1) = 1$
- Let $V = C[-1,1] = \{f : [-1,1] \to \mathbb{R} \mid f \text{ is continuous}\}$.

  V is a real vector space

  $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

  If $\langle f, g \rangle = 0$, then we say that f & g are orthogonal

- $P_0(x) = ?$

$P_0(x) = c$

$P_0(1) = c = 1.$

$P_0(x) = 1$

- $P_1(x) = ?$

$P_1(x) = ax + b$

$P_1(1) = 1 = a + b$

$a = 1.$

$P_1(x) = ax = x$

$\int_{-1}^{1} P_1(x)(1)\,dx = \int_{-1}^{1} (ax+b)\,dx = 0$

$\left[\frac{ax^2}{2} + bx\right]_{-1}^{1} = 0 \Rightarrow b = 0.$

n	$P_n(x)$
0	1
1	x
2	$\frac{3}{2}x^2 - \frac{1}{2}$
3	$\frac{5}{2}x^3 - \frac{3}{2}x$

- $P_2(x) = ?$

$P_2 = ax^2 + bx + c$

$P(1) = a + b + c = 1$

$\int_{-1}^{1} P_2(x)\cdot(1)\,dx = 0.$

$\int_{-1}^{1} (ax^2 + bx + c)\,dx = 0$

$\left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-1}^{1} = 0$

$\frac{2a}{3} + 2c = 0 \Rightarrow a + 3c = 0$

$\int_{-1}^{1} P_2(x)(x)\,dx = 0$

$\int_{-1}^{1} (ax^3 + bx^2 + cx)\,dx = 0$

$\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2}\right]_{-1}^{1} = 0$

$b = 0.$

$\Rightarrow a + c = 1$

$a + c = 1$

$-\underline{a + 3c = 0}$

$c = -\frac{1}{2}, \quad a = \frac{3}{2}$

- $P_3(x) = ?$

$P_3(x) = ax^3 + bx^2 + cx + d$

$1 = P_3(1) = a + b + c + d$

$\int_{-1}^{1} P_3(x)\cdot 1\,dx = 0.$

$\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + \frac{dx}{1}\right]_{-1}^{1} = 0$

$\frac{2}{3}b + 2d = 0$

$b + 3d = 0$

$\int_{-1}^{1} P_3(x)\cdot x\,dx = 0$

$\left[\frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2}\right]_{-1}^{1} = 0.$

$\frac{2a}{5} + \frac{cx}{2} = 0.$

$3a + 5c = 0.$

$\int_{-1}^{1} P_3(x)\cdot x^2\,dx = 0$

$3b + 5d = 0.$

$c = \frac{-3}{2}, \quad a = \frac{5}{2}.$

$-3b + 5d = 0$

$-3b + 9d = 0$

$b = 0$

$d = 0$

Rodrigues's Formula.

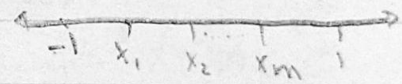
- $P_n(x) = \frac{(-1)^n}{2^n(n!)} \frac{d^n\left[(1-x^2)^2\right]}{dx^n}$

$P_n(x) = \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \frac{1\cdot 3\cdot 5\ldots(2n-2i-1)}{2^i\,(i!)\left[(n-2i)!\right]} x^{n-2i}$ + flow function.

- Recurrence Relation – $(n+1)P_{n+1}(x) - (2n-1)xP_n(x) + nP_{n-1}(x) = 0.$

Theorem

- The zeros of Legendre polynomials are real distinct and lie in $(-1,1)$
- The theorem is true for $n=0$, vacuously. If $n \in \mathbb{Z}^+$, then $\int_{-1}^{1} P_n(x)\cdot 1\,dx = 0$. Therefore, the graph $P_n(x)$ must intersect (must cross) the x-axis at least once in $(-1,1)$. Suppose the graph $P_n(x)$ crosses the x-axis at $x_1, x_2, \ldots, x_m$ where $x_1, x_2, \ldots, x_m \in (-1,1)$. Then $m \leq n$.
- $P_n(x)(x-x_1)(x-x_2)\ldots(x-x_m)$



$\int_{-1}^{1} P_n(x)(x-x_1)(x-x_2)\ldots(x-x_m)\,dx \neq 0$

$\Downarrow$ (This implies)

$m \geq n$.

- This leads to $m=n$

The Legendre Equation of α

- The Legendre equation of order α is the ODE $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ (α is a real constant)
- If $\alpha = 0$ or $\alpha = -1$, then this is easy to solve!

$$(1-x^2)y'' - 2xy' = 0$$
$$\left((1-x^2)y'\right)' = 0$$
$$(1-x^2)y' = C_1$$

$$y' = \frac{C_1}{1-x^2} = C_1\left(\frac{1}{1-x^2}\right)$$
$$= C_1\left(\frac{1}{(1-x)(1+x)}\right)$$
$$= C_1\left(\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right)$$
$$y = C_1\left(\tfrac{1}{2}(-\ln(1-x)) + \tfrac{1}{2}\ln(1+x)\right) + c$$
$$= \frac{C_1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$
$$= C_1\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right] \text{ for } + C_2\,(1)$$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots$

$\ln = C + x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots$

Laplace

- ~~Legendre~~ in Spherical Coordinates

$$\frac{d^2}{d\phi^2}F(\phi) + \cot(\phi)\frac{d}{d\phi}F(\phi) + n(n+1)F(\phi) = 0 \quad 0<\phi<\pi$$

$x = \cos\phi \quad y = F(\phi) = F(\cos^{-1}(x))$

$$\frac{dy}{dx} = \frac{dF}{d\phi}\frac{d\phi}{dx} \Rightarrow \frac{dF}{d\phi} = \frac{dy}{dx}\frac{dx}{d\phi} \qquad \frac{dx}{d\phi} = -\sin\phi$$

$$= \frac{dy}{dx}\left(-\sqrt{1-x^2}\right) \qquad = \sqrt{1-\cos^2\theta}$$

$$\frac{d^2F}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\cdot\frac{dx}{d\phi}\right)\frac{d\phi}{dx} = \left[\frac{dy}{dx}\frac{d}{dx}\left(\frac{dx}{d\phi}\right) + \frac{dy}{d\phi}\frac{d\phi}{dx^2}\right]\frac{1}{-\sin\theta} \qquad \sqrt{1-x^2}$$

$$= \left(\frac{dy}{dx}(-\cos\phi)\frac{d\phi}{dx} + (-\sin\phi)\frac{d^2y}{dx^2}\right)(-\sin\theta)$$

$$= \sin^2\phi\frac{d^2y}{dx^2} + (\sin\phi)(\cos\phi)\frac{1}{(-\sin\phi)}\frac{dy}{dx}$$

$$= (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{x}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)\frac{dy}{dx} + n(n+1)y = 0$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

- $P(x)y'' + Q(x)y' + R(x)y = 0, \quad P(x_0) \neq 0$

$$y = \sum_{n=0}^{\infty} a_n(x_1 - x_0)^n$$

The point is ordinary if $P(x_0) \neq 0$. Otherwise x_0 is singular.

- Our technique seeking solutions of the forms $y = \sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ may ~~no~~ not work is x_0 is singular.
- Consider: $x^2y'' + (x^2-x)y' + 2y = 0$

seek solutions of the form $y = \sum a_n x^n$, $y' = \sum n a_n x^{n-1}$,

$y'' = \sum^{\infty} n(n-1)a_n x^{n-2}$ and sub.

It ends up failing through math.

- $x^2y'' + (x^2-x)y' + 2y = 0$

$y = \sum a_n x^n \quad y', y''$

$P(x)y'' + Q(x)y' + R(x)y = 0$

$P(x_0) = 0$

- Forbenius Theorem - $P(x)y'' + Q(x)y' + R(x)y = 0$, P, Q, R are polynomials
  A point x_0 is a singular point if $P(x_0) = 0$. We say that the similar point x_0 is a regular singular point if $\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)}$ and $\lim_{x \to x_0} \frac{R(x)}{P(x)}$ exist

- Let x_0 be a regular singular point of $P(x)y'' + Q(x)y' + R(x)y = 0$

$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ $\quad x=1$ singular point, $t = x-1$

$(-t)(2+t)\frac{d^2y}{dt^2} + (t+1)\frac{dy}{dt} + \alpha(\alpha+1)y = 0$

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} \quad \& \quad \frac{d^2y}{dy^2}$

- Let 0 be a regular singular point of $P(x)y'' + Q(x)y' + R(x)y = 0$. The equation $r(r-1) + p_0 r + q_0 = 0$ is called the incidical equation, where $p_0 = \lim_{x \to 0} x\frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x \to 0} x^2\frac{R(x)}{P(x)}$. The roots of the incidical equation are called the exponents of the ODE at the regular singular point 0.
- Suppose for real exponents, say, r_1 & r_2 with $r_1 \geq r_2$
- ① For $x > 0$ the ODE has a $\overset{\text{no}}{\vee}$ solution
- ② If $r_1 - r_2$ is not an integer, then there is a second linearly independent solution, for $x > 0$ $y_2 = x^{r_2}\sum b_n x^n$, where $b_0 \neq 0$

Forbenius Theory

- $P(x)y'' + Q(x)y' + R(x)y = 0$
- If $P(0)=0$, then $x=0$ is a singular point. $x=0$ is a regular singular point, if moreover, $\lim_{x\to 0} x\frac{Q(x)}{P(x)}$ and $\lim_{x\to 0} x^2\frac{Q(x)}{P(x)}$ exists
- Let $\lim_{x\to 0} x\frac{Q(x)}{P(x)} = p_0$ and $\lim_{x\to 0} \frac{x^2 R(x)}{P(x)} = q_0$, then the indicial equation $r(r-1) + p_0 r + q_0 = 0$
- Suppose that the roots of the indicial equation are real. Say r_1 & r_2 with $r_1 \geq r_2$.
  - ① For $x>0$, there is a solution of the form
    $y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$
  - ② If $r_1 - r_2$ is not an integer, then for $x>0$, there is a 2nd solution
    $y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$, with $b_0 \neq 0$
  - ③ If $r_1 = r_1$, then the second solution for $x>0$ is
    $y_2 = y_1\ln(x) + x^{r_1}\sum_{n=1}^{\infty} b_n x^n$
  - ④ If $r_1 - r_2$ is a positive integer, then the second solution for $x>0$ is
    $y_2 = a y_1 \ln(x) + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$

Bessel's Equation

- Bessels Equation of order ν is the ODE $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where ν is a nonnegative constant.
- Bessels Equation of order 0
  - $x^2y'' + xy' + (x^2)y = 0$.
  - We see zero is a singular point.
  - $\lim_{x\to 0} x\left(\frac{x}{x^2}\right) = 1 = p_0$, $\lim_{x\to 0} x^2\left(\frac{x}{x^2}\right) = 0 = q_0$, zero in regular singular.
- The indicial equation is $r(r-1) + p_0 r + q_0 = 0$.
  - $r(r-1) + r = 0 \Rightarrow r^2 = 0$, $r = 0, 0$.
- Seek solutions of the form $y = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
- Substitute
  - $\sum_{n=2}^{\infty} n(n-1)a_n x^{n} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$.

  $\sum_{n=1}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$

$$= a_1 x + \sum_{n=2}^{\infty} [n(n-1)a_n \overset{+na_n}{\vee} + a_{n-2}] x^2 = 0.$$

- $a_1 = 0$
- $n(n-1)a_n + na_n = -a_{n-2}$

$$n^2 a_n = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{n^2}, \quad n = 2, 3, 4, \dots$$

- $n=2 \quad a_2 = -\frac{a_0}{4}$

$$n=3 \quad a_3 = \frac{-a_1}{9} = \frac{-a_0}{\cancel{36}}$$

$\vdots$

- $y_1 = \sum \frac{x^{2n}}{2^{2n}(n!)^2}(-1)^n$ ← Bessels function of the first kind of order 0.

- Case $\nu = \frac{1}{2}$
- $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$, we see that 0 is a singular point

$$\lim_{x \to 0} x\left(\frac{x}{x^2}\right) = 1, \quad \lim_{x \to 0} x^2\left(\frac{x^2 - \frac{1}{4}}{x^2}\right) = -\frac{1}{4}.$$

- 0 is a regular singular point.
- Indicial Equation - $r(r-1) + p_0 r + q_0 = 0$

$$r(r-1) + r - \frac{1}{4} = 0$$

$$r^2 - \frac{1}{4} = [\text{scribbled out}] \Rightarrow r = \pm\frac{1}{2}$$

- Seek solution of the form $y = x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \Rightarrow y' = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) a_n x^{n-\frac{1}{2}} \text{ and } \Rightarrow y'' = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})(n-\tfrac{1}{2}) a_n x^{n-\frac{3}{2}}$$

- Substitute ODE

$$\sum^{\infty} (n+\tfrac{1}{2})(n-\tfrac{1}{2}) a_n x^{n+\frac{1}{2}} + \sum^{\infty} (n+\tfrac{1}{2}) a_n x^{n+\frac{1}{2}} + \sum^{\infty} a_n x^{n+\frac{5}{2}} - \frac{1}{4}\sum^{\infty} a_n x^{n+\frac{1}{2}} = 0$$

$$\cancel{\sum_{n=0} a_{n-2} x^{n+\frac{1}{2}}} \quad '' \quad + \quad '' \quad + \sum^{\infty} a_{n-2} x^{n+\frac{1}{2}} - \frac{1}{4}\sum^{\infty} a_n x^{n+\frac{1}{2}} = 0$$

- $n=0$, $-\frac{1}{4} a_0 x^{\frac{1}{2}} + \frac{1}{2} a_0 x^{\frac{1}{2}} - \frac{1}{4} a_0 x^{\frac{1}{2}} = 0$

coefficient $x^{\frac{1}{2}} = -\frac{1}{4} a_0 + \frac{1}{2} a_0 - \frac{1}{4} a_0 = 0$.

$n=1$, coefficient of $x^{\frac{3}{2}} = \frac{3}{4} a_1 + \frac{3}{2} a_1 - \frac{1}{4} a_1 = 0$, $a_1 = 0$.

$n \geq 2$, coefficient of $x^{n+\frac{1}{2}} = (n+\frac{1}{2})(n-\frac{1}{2}) a_n + (n+\frac{1}{2}) a_n + a_{n-2} - \frac{1}{4} a_n = 0$.

$$(n^2 + n) a_n = a_{n-2} \Rightarrow a_n = \left(\frac{a_{n-2}}{n(n+1)}\right) \quad n = 2, 3, 4.$$

- $a_0 = 1$, $a_2 = \frac{a_0}{6}$, $a_4 = \frac{a_0}{24}$, $a_6 = \frac{-a_0}{7!}$

$a_1 = 0$, $a_3 = 0$, $a_5 = 0$.

- One solution of Bessel's Equation of the order $\frac{1}{2}$ is

$$y = x^{\frac{1}{2}}[a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 \dots]$$

$$= x^{\frac{1}{2}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \dots\right] = \frac{x}{x^{\frac{1}{2}}}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} \dots\right) = \frac{1}{\sqrt{x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}\right)$$

$$= \frac{\sin x}{\sqrt{x}} \qquad \text{General Soln } y = C_1\left(\frac{\sin x}{\sqrt{x}}\right) + C_2\left(\frac{\cos x}{\sqrt{x}}\right)$$

• If $x<0$, let $t=-x$. Then $t>0$.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}(-1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{d^2y}{dt^2}(-1) = \frac{d^2y}{dt^2}$$

• $$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \left(t^2 - \frac{1}{4}\right)y = 0.$$

$$y = c_1\frac{\sin(-x)}{\sqrt{-x}} + c_2\frac{\cos x}{\sqrt{-x}}$$

• What happens as $x \to 0$?

$$y = c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}$$ valid for $x>0$

If $c_2 \neq 0$

$$\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}} = 0.$$

If $c_2 = 0$, then $y \to 0$ as $x \to 0^+$.

Chapter 6. Laplace Transforms

• IVP: $\left.\begin{aligned} &y'' - 2y' + y = \sin t \\ &y(0) = 1 \\ &y'(0) = 2 \end{aligned}\right\} \xrightarrow[\text{Transf.}]{\text{Laplace}}$ $\boxed{\text{Algebra Equation} \downarrow \text{Solution}}$

$\xleftarrow{}$ inverse Laplace Transform

• Definition: Let $f(t)$ be a function of t. The Laplace transformation of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$ or simply $F(s)$, is defined by $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

• 6.2 (8)

$y'' - y' - 6y = 0$

$y(0) = 1,\ y'(0) = -1$

Let $Y(s)$ be the Laplace transform of $y(t)$.

$Y(s) = \int_0^\infty e^{-st} y(t)\,dt$

$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st} y'(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} y(t)\,dt$

$= \lim_{b\to\infty} \left[\left(e^{-st} y(t)\right)_0^b + \int_0^b -se^{-st} y(t)\,dt \right]$

$= \lim_{b\to\infty} \left[e^{-sb} y(b) - y(0) + s\int_0^b e^{-st} y(t)\,dt \right]$

$= -y(0) + s\int_0^\infty e^{-st} y(t)\,dt = -y(0) + sY(s)$

$\mathcal{L}\{y''(t)\} = -y'(0) + s^2Y(s) - sy(0)$

$f(t)$	$F(s)$
y'	$sY(s) - y(0)$
y''	$s^2Y(s) - sy(0) - y'(0)$

6.1 Definition of the Laplace Transform

- Let $f:[0,\infty) \to \mathbb{R}$ be a continuous function. Then the Laplace transform of f, denoted by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

$f(t)$	$F(s)$				
y'	$sY - y(0)$	C	$\frac{C}{s}$, $s>0$	e^{at}	$\frac{1}{s-a}$, $s>a$
y''	$s^2Y - sy(0) - y'(0)$	t	$\frac{1}{s^2}$, $s>0$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$, $s>\lvert b\rvert$
1	$\frac{1}{s}$, $s>0$	t^2	$\frac{2}{s^3}$, $s>0$	$\cos(t)$	$\frac{s}{s^2+1}$
		t^n	$\frac{n!}{s^{n+1}}$, $s>0$, $n\in\mathbb{Z}_{\geq 0}$		

- Let $f(t)=1$.

  $\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[-e^{-st}/s\right]_0^b = \lim_{b\to\infty}\left[-\frac{e^{-sb}}{s} - \left(-\frac{1}{s}\right)\right]$

  if s is positive, $\mathcal{L}\{1\} = \frac{1}{s}$.

- Let $cf(t)$

  $\mathcal{L}\{cf(t)\} = c\int_0^\infty e^{-st} f(t)\,dt = c\mathcal{L}\{f(t)\}$.

- Let $f(t)=c$

  $\mathcal{L}\{f(t)\} = \mathcal{L}\{c(1)\} = c\mathcal{L}\{1\} = \frac{c}{s}$, $s>0$

- Let $h(t) = f(t)+g(t)$

  $\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt = \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

- Sequentially

  $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

- Let $f(t)=t$

  $\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \lim_{b\to\infty}\int_0^b e^{-st}t\,dt = \lim_{b\to\infty}\left[\left[\frac{e^{-st}t}{-s}\right]_0^b - \int_0^b \frac{e^{-st}}{-s}(1)\,dt\right]$

  $= \lim_{b\to\infty}\left[\frac{e^{-sb}b}{-s} - 0 + \left[\frac{e^{-st}}{-s^2}\right]_0^b\right] = \lim_{b\to\infty}\left[\cancel{-\frac{e^{-sb}b}{s}} - \cancel{\frac{e^{-sb}}{s^2}} + \frac{1}{s^2}\right]$

  $= \frac{1}{s^2}$, $s>0$

- Let $f(t)=t^2$

  $\mathcal{L}\{t^2\} \Rightarrow \ldots = \frac{2}{s^3}$ $\qquad$ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

- Let $f(t)=t^3$

  $\mathcal{L}\{t^3\} \Rightarrow \ldots = \frac{3!}{s^4}$

  $\int_0^\infty e^{-st}dt = \frac{1}{s} = F(s)$

- Let $f(t)=e^{at}$

  $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \Rightarrow \ldots = \frac{1}{s-a}$. $\quad F(s-a)$

- $\mathcal{L}\{\cosh(bt)\} = \mathcal{L}\{\frac{1}{2}(e^{bt}+e^{-bt})\} = \frac{1}{2}\mathcal{L}\{e^{bt}+e^{-bt}\} = \frac{1}{2}\left(\mathcal{L}\{e^{bt}\} + \mathcal{L}\{e^{-bt}\}\right)$

  $= \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right) \Rightarrow \ldots = \frac{s}{s^2-b^2}$

- $\mathcal{L}\{\cos(t)\} = \ldots = \frac{s}{s^2+1}$

$\bullet\ \mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\bullet$ 6.2 ⑦ $y'' - y' - 6y = 0 \quad y(0)=1,\ y'(0)=-1$

$s^2Y - s(1) - (-1) - [sY - 1] - 6Y = 0$

$[s^2 - s - 6]Y = -2+s$

$$Y = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s+2)(s-3)} = \frac{1/5}{s-3} \cdot \frac{4/5}{s+2} = \frac{1}{5(s-3)} + \frac{4}{5(s+2)}$$

$y = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$

$\bullet$ ⑬ $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0 \quad y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y'''(0)=1$

$= s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4[s^3Y - s^2y(0) - sy'(0) - y''(0)]$
$+ 6[s^2Y - sy(0) - y'(0)] - 4[sY - y(0)] + Y = 0$

$= s^4Y - s^2 - 1 - 4[s^3Y - s] + 6[s^2Y - 1] - 4s + Y = 0$

$= Y[s^4 - 4s^3 + Y] = s^2 + s + 1 - 1 - 4s$

$Y[s^4 - 4s^3 + 6s^2 - 4s + 1] = s^2 - 4s + 7$

$$Y = \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1} = \frac{s^2-4s+7}{(s-1)^4}$$

$$= \frac{s^2 - 2s + 1 - 2s + 6}{(s-1)^4}$$

$$= \frac{(s-1)^2 - 2s + 6}{(s-1)^4} = \frac{1}{(s-1)^2} - 2\left[\frac{s-1-2}{(s-3)^4}\right]$$

$$= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}$$

$$= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{6}\left[\frac{6}{(s-1)^4}\right]$$

$y = te^t - t^2e^t + \frac{4}{6}t^3e^t$

$\bullet\ F(s) = \int_0^\infty e^{-st}f(t)\,dt$

$\frac{dF(s)}{ds} = \int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt = -\int_0^\infty te^{-st}f(t)\,dt$

$-\frac{dF(s)}{ds} = \int_0^\infty e^{-st}t f(t)\,dt = \mathcal{L}\{tf(t)\}$

$\bullet\ \mathcal{L}\{t^2f(t)\} = \frac{d^2F(s)}{ds^2}$

The Gamma Function.

- $\Gamma(p+1) = \int_0^\infty e^{-x} x^{p} dx = \int_0^1 e^{-x} x^p dx + \int_1^\infty e^{-x} x^p dx$.
- Consider $\int_1^\infty e^{-x} x^p dx$.

$$\int_1^\infty e^{-x} x^p dx = \lim_{b\to\infty} \int_1^b e^{-x} x^p dx = \lim_{b\to\infty} \int_1^b \frac{x^p}{e^x} dx$$

$$\frac{x^p}{e^x} = \frac{x^p}{1+x+\frac{x^2}{2!}+\cdots} < \frac{x^p}{\left(\frac{x^n}{n!}\right)}$$

$$\lim_{b\to\infty} \int_1^b \frac{n!}{x^{n-p}} dx = \lim_{b\to\infty} n! \left[\frac{x^{-n+p+1}}{-n+p+1}\right]_1^b$$

$$= \lim_{b\to\infty} \frac{n!}{(-n+p+1)} \left[b^{-n+p+1} - 1\right]$$

$\frac{n!}{x^{n-p}}$ choose $n > p+2$

$n-p > 2$.

- $\int_0^1 e^{-x}/x^{-p}$

$0 > p > -1$	$\int_0^1 \frac{e^{-x}}{x^m} dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{e^x x^m} dx$.
$p = -m$	$1 < e^x < e$.
$0 < -p < 1$	$\le \lim_{a\to 0^+} \int \frac{1}{x^m} dx$
$0 < m < 1$	$= \lim_{a\to 0^+} \left[\frac{x^{-m+1}}{1-m}\right]_a^1 = \lim_{a\to 0^+} \left[\frac{1}{1-m} - \frac{a^{1-m}}{1-m}\right] = \frac{1}{1-m}$

- $\Gamma(t)$ is defined when $t > 0$.
- 6.1 Show that for $p > 0$, $\Gamma(p+1) = p\Gamma(p)$.

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx = \lim_{b\to\infty} \int_0^b e^{-x} x^p$$

$$= \lim_{b\to\infty} \left[\underbrace{\left[\frac{e^{-x}}{(-1)} \cdot x^p\right]_0^b}_{\text{zero}} - \int_0^b \frac{e^{-x}}{(-1)} p x^{p-1} dx\right]$$

$$= p\int_0^\infty e^{-x} x^{p-1} dx = p\Gamma(p).$$

Show $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = \ldots\ldots\ldots = 1$$

- If $p = n$, where n is a positive integer.

$\Gamma(n+1) = n\Gamma(x)$ $\quad$ $\Gamma(n) =$

$\Gamma(2) = 1\Gamma(1) = 1$ $\quad$ $\Gamma(n+1)! = n!$

$\Gamma(3) = 2\Gamma(1) = 2$.

2. a) $\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p \, dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx$

$= \frac{\Gamma(p+1)}{s^{p+1}}$

Let $x = st \Rightarrow dx = s\,dt$.

Then $\int_0^\infty e^{-st} t^p \, dt = \int_0^\infty \frac{e^{-x}}{s} \left(\frac{x}{s}\right)^p dx \quad s>0.$

$= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx = \frac{\Gamma(p+1)}{s^{p+1}} \quad s>0$

b) $p = n \in \mathbb{Z}^+$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

c) $\mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx.$

$\mathcal{L}\{t^{-\frac{1}{2}}\} = \int_0^\infty e^{-st} t^{-\frac{1}{2}} dt \qquad$ Let $st = x^2 \Rightarrow 2x\,dx = s\,dt$

$= \int_0^\infty e^{-x^2} \frac{\sqrt{s}}{x} \frac{2x}{s} dx.$

$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx.$

$\left[\int_0^\infty e^{-x^2} dx\right]^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$

$= \iint_0^\infty e^{-(x^2+y^2)} dx\,dy$

$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\,dr\,d\theta$

area of integration.

$= \int_0^{\frac{\pi}{2}} \left[\left(-\frac{1}{2}\right) e^{-r^2}\right]_0^\infty d\theta.$

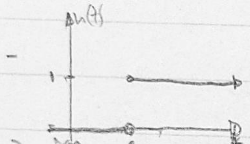
$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ so

$\int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{4}}$

$= \frac{2}{\sqrt{s}} \sqrt{\frac{\pi}{4}} = \sqrt{\frac{\pi}{s}}$

6.3 Step Functions

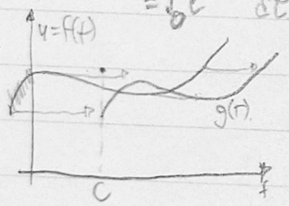
- Unit step functions (Heaviside Functions)
  - For $c \geq 0$, we define, $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$



- $\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^c e^{-st} u_c(t)\,dt + \int_c^\infty e^{-st} u_c(t)\,dt$

$= 0 + \int_c^\infty e^{-st}\,dt$. Let $\tau = t - c$, then $d\tau = dt$

$= \int_0^\infty e^{-s(\tau + c)}\,d\tau = \int_0^\infty e^{-s\tau} e^{-sc}\,d\tau = e^{-sc}\int_0^\infty e^{-s\tau}\,d\tau = \frac{e^{-cs}}{s}$

- 

$\mathcal{L}\{f(t)\} = F(s)$

$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c \end{cases}$

$= u_c(t) f(t-c)$

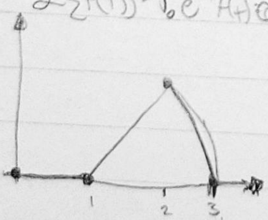
$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt = \int_0^c e^{-st} u_c(t) f(t-c)\,dt + \int_c^\infty e^{-st} u_c(t) f(t-c)\,dt$

$= 0 + \int_c^\infty e^{-st} f(t-c)\,dt$ Let $\tau = t-c$, then $d\tau = dt$.

$= \int_0^\infty e^{-s(\tau+c)} f(\tau)\,d\tau = e^{-cs}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-cs} F(s)$

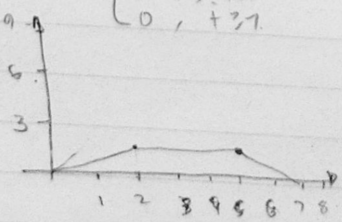
- (1) $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$ for $t \geq 0$.

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$.



Let $g(t) = \begin{cases} 0 & , 0 \leq t < 1 \\ t-1 & , 1 \leq t < 1 \\ 3-t & , 2 \leq t < 3 \\ 0 & , 3 \leq t \end{cases}$

(2) $f(t) = \begin{cases} t & , 0 \leq t < 2 \\ 2 & , 2 \leq t < 5 \\ 7-t & , 5 \leq t < 7 \\ 0 & , t \geq 7 \end{cases}$



$= t - tu_2(t) + 2u_2(t) - 2u_5(t) + (7-t)u_5(t) - (7-t)u_7(t)$

$= t + u_2(t)(t-2) + u_5(t)(5-t) - u_7(t)(7-t)$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} G(s)$

$F(s) = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2}\right) + e^{-7s}\left(\frac{1}{s^2}\right)$

$= \frac{1}{s^2}\left(1 - e^{-2s} - e^{-5s} + 7e^{-7s}\right)$

- $f(t) = \begin{cases} 0 & , 0 \leq t < \pi \\ t-\pi & , 0 \leq t < 2\pi \\ 0 & , t > 2\pi \end{cases}$

$= 0 + (t-\pi)u_\pi(t) - (t-\pi)u_{2\pi}(t) = u_\pi(t)(t-\pi)\ u_{2\pi}(t)(t-\pi)$

$g(t) = t$, $c = \pi$, $G(s) = \frac{1}{s^2}$

$F(s) = e^{-\pi s}\left(\frac{1}{s^2}\right) - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)$

• (14) $F(s) = \frac{e^{-2s}}{s^2+s-2}$, Let $G(s) = \frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2} = \frac{1}{3(s-1)} - \frac{1}{3(s+2)}$

$F(s) = e^{-2s} G(s)$ $\qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$f(t) = u_c(t) g(t-c) = u_2(t) g(t-2)$ $\qquad g(t) = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$

$= u_2(t)\left[\frac{1}{3}e^{(t-2)} - \frac{1}{3}e^{-2(t-2)}\right]$

• (15) $F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s} = e^{-s}\left(\frac{1}{s}\right) + e^{-2s}\left(\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s}\right) - e^{-4s}\left(\frac{1}{s}\right)$ Let $G(s) = \frac{1}{s}$.

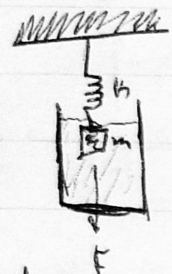
$= e^{-s}G(t) + e^{-2s}G(s) - e^{-3s}G(s) - e^{-4s}G(s)$

$= u_1(t)g(t-1) + u_2(t)g(t-2) - u_3(t)g(t-3) - u_4(t)g(t-4)$.

$g(t) = 1$ $\qquad = u_1(t) + u_2(t) - u_3(t) - u_4(t)$.

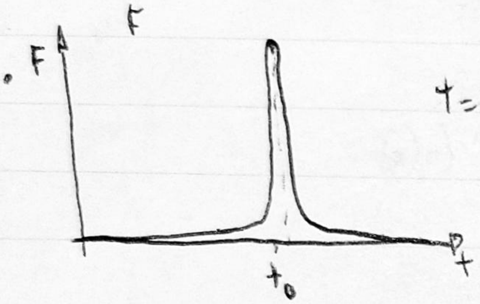
$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t < 2 \\ 2, & \text{if } 2 \le t < 3 \\ 1, & \text{if } 3 \le t < 4 \\ 0, & \text{if } t \ge 4 \end{cases}$ $\qquad f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

6.5 Impulse Functions

- 

  $my'' + \gamma y' + ky = F.$

  $my'' + \quad ky = F$ ← for case when not in medium or air resistance etc.

- 

  $t = t_0$

- Dirac Delta "functions".

  - $f_1 : \mathbb{R} \to \mathbb{R}$

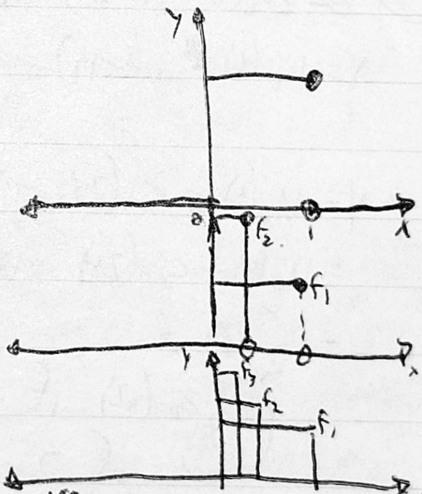
  $f_1(x) := \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

  $f_2(x) := \begin{cases} 2 & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

  $f_3(x) := \begin{cases} 3 & \text{if } 0 \le x \le \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$

  $f_n(x) := \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

  $f_\infty(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$

  $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$

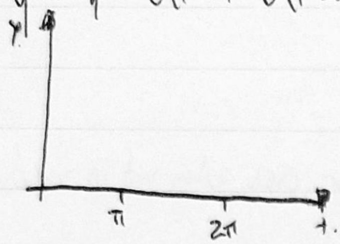
  - $\delta(x) = \begin{cases} 0 & \text{if } x \ne 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$ $\quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1.$

- $\mathcal{L}\{\delta(t)\} = 1$

  $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$

Force up at π — Force down

#12 $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi) \quad y(0)=0 \quad y'(0)=0$

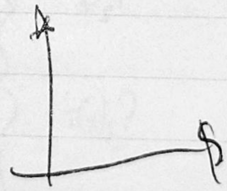


$$s^2Y - sy(0) - y'(0) + 4Y = e^{-\pi s} - e^{-2\pi s}$$
$$(s^2+4)Y = e^{-\pi s} - e^{-2\pi s}$$
$$Y = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{(s^2+4)} = e^{-\pi s}G(s) - e^{-2\pi s}G(s)$$

#27 $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$

$$y = u_\pi(t)\, g(t-\pi) - u_{2\pi}(t)\, g(t-2\pi)$$

$$y = u_\pi(t)\tfrac{1}{2}\sin(2(t-\pi)) - u_{2\pi}(t)\tfrac{1}{2}\sin(2(t-2\pi))$$
$$= u_\pi(t)\tfrac{1}{2}\sin(2t) - u_{2\pi}(t)(\tfrac{1}{2})\sin(2t)$$
$$= \begin{cases} 0 & \text{if } x \\ \tfrac{1}{2}\sin(x2) & \text{if } \pi \le x < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases}$$



#4 $y'' + 2y' + 3y = \sin(t) + \delta(t-3\pi) \quad y(0)=0, \ y'(0)=0$

$$s^2Y + sy(0) - y'(0) + 2[sY - y(0)] + 3Y = \frac{1}{s^2+1} + e^{-3\pi s}$$
$$(s^2+2s+3)Y = \frac{1}{s^2+1} + e^{-3\pi s}$$
$$Y = \frac{1}{(s^2+1)(s^2+2s+3)} + \frac{e^{-3\pi s}}{(s^2+2s+3)}$$

split & use laplace transforms

$$\frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3}$$

$$1 = (As+B)(s^2+2s+3) + (Cs+D)(s^2+1)$$

s^0: $3B + D = 1$

s^1: $3A + 2B + C = 0$

s^2: $B + 2A + D = 0$

s^3: $A + C = 0$

6.5 Impulse functions

- $\delta(t) = \begin{cases} 0 \text{ if } x \neq 0 \\ \text{undefined if } x = 0 \end{cases}$, $\int_{-\infty}^{\infty} \delta(t)dt = 1$
- $\delta(t-t_0) = \begin{cases} 0 \text{ if } x \neq t_0 \\ \text{undf if } x = t_0 \end{cases}$, $\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$
- $\mathcal{L}\{\delta(t)\} = 1$, $\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$
- (2) $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$
- Ex (3) $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$, $y(0) = 0, y'(0) = \frac{1}{2}$.

$$s^2Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = e^{-5s} - \frac{e^{-10s}}{s}$$

$$(s^2+3s+2)Y = e^{-5s} - \frac{e^{-10s}}{s} + \frac{1}{2}$$

$$Y = \frac{e^{-5s}}{s^2+3s+2} - \frac{e^{-10s}}{s(s^2+3s+2)} + \frac{1}{2(s^2+3s+2)}$$

let $G(s) = \frac{1}{s^2+3s+2}$.

$H(s) = \frac{1}{s(s^2+3s+2)}$

$$Y = e^{-5s}G(s) - e^{-10s}H(s) + \frac{1}{2}G(s).$$

$$y = u_5(t)g(t-5) + u_{10}(t)h(t-10) + \frac{1}{2}g(t)$$

$$y = u_5(t)e^{-(t-5)} + u_{10}(t)\left(\frac{1}{2} - e^{-t+10} + \frac{1}{2}e^{-2(t-10)}\right) + \frac{1}{2}\left[e^{-t} - e^{-2t}\right].$$

$$G(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$g(t) = e^{-t} + e^{-2t}.$$

$$H(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{(s+2)2}$$

$$h(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

6.6 The Convolution Integral

- $\mathcal{L}\{f(t)g(t)\} \neq F(s)G(s)$
- Let f, g be two functions. The convolution integral of f and g, denoted by $f*g$, is the function $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$
- $(f*g)(t) =$ $\quad t-\tau = u$ (t is fixed)

$= \int_t^0 f(u)g(t-u)(-du)$ $\quad t-u=\tau$, $-d\tau = du$

$= \int_0^t f(u)g(t-u)(du)$

$= \int_0^t g(t-u)f(u)\,du$

$= \int_0^t g(t-\tau)f(\tau)\,d\tau = (g*f)(t)$, the convolution integral is commutative

- $f*g = g*f$, $(f*g)*h = f*(g*h)$
- $f*1 \neq f$ $\quad g(t)=1$

$(f*1)(t) = \int_0^t f(t-\tau)\,d\tau$ $\quad f(t)=t$

$= \int_0^t (t-\tau)\,d\tau \Rightarrow \frac{t^2}{2}$

- $F(s)G(s) = \left(\int_0^\infty e^{-st}f(t)\,dt\right)\left(\int_0^\infty e^{-st}g(t)\,dt\right)$

$= \left(\int_0^\infty e^{-st}f(t)\,dt\right)\left(\int_0^\infty e^{-su}g(u)\,du\right)$

$= \iint_0^\infty e^{-st}e^{-su}f(t)g(u)\,dt\,du$

$= \int_0^\infty\int_0^\infty e^{-s(t+u)}f(t)g(u)\,dt\,du$ $\quad$ let $t+u=v$ with fixed t

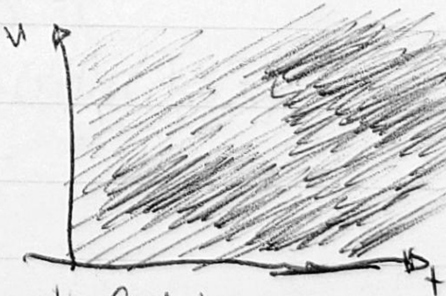
$= \int_0^\infty\int_t^\infty e^{-sv}f(t)g(v-t)\,dv\,dt$ $\quad du = dv$

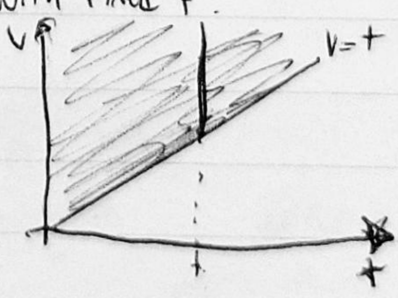
$= \int_0^\infty\int_0^v e^{-sv}f(t)g(v-t)\,dt\,dv$

$= \int_0^\infty e^{-sv}\left[\int_0^v f(t)g(v-t)\,dt\right]dv$ $\quad t \to \tau$

$= \int_0^\infty e^{-sv}\left[(f*g)(v)\right]dv$ $\quad v \to t$

$= \mathcal{L}\{(f*g)(v)\}$





• ⓐ $f(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\,d\tau$

$g(t) = t^2, \quad h(t) = \cos(2t)$.

$f(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau$

$= (h * g)(t)$

$F(s) = H(s)G(s)$. $\qquad H(s) = \frac{s}{s^2+4}, \quad G(s) = \frac{2}{s^3}$

$= \frac{2s}{s^3(s^2+4)} = \frac{2s}{s^5+4s^3}$

• ⓑ $f(t) = \int_0^t e^{-(t-\tau)} \sin \tau \, d\tau$

$g(t) = e^{-t}, \quad h(t) = \sin(t)$

$f(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau$.

$= (g * h)(t)$.

$F(s) = H(s)\, G(s) \qquad H(s) = \frac{1}{s+1} \quad G(s) = \frac{1}{s^2+1}$.

$= \frac{1}{(s+1)(s^2+1)}$

• ⓒ $G(s) = \frac{1}{s^4(s^2+1)}$

$= \frac{1}{s^4} \, \frac{1}{s^2+1} \qquad G(s) = \frac{1}{s^4}, \quad g(t) = \frac{1}{3!} t^3 = \frac{1}{6}t^3$

$\qquad H(s) = \frac{1}{s^2+1}, \quad h(t) = \sin(t)$.

$f(t) = (g * h)(t)$

$= \int_0^t g(t-\tau) h(\tau)\, d\tau$.

$= \int_0^t \frac{1}{6}(t-\tau)^3 \sin(\tau)\, d\tau$.

$=$

• ⓓ $F(s) = \frac{1}{(s+1)^2(s^2+4)}$

$= \frac{1}{(s+1)^2} \, \frac{1}{s^2+4} \qquad G(s) = \frac{1}{(s+1)^2} \quad g(t) = te^{-t}$

$= G(s) H(s) \qquad H(s) = \frac{1}{s^2+4}, \quad h(t) = \frac{1}{2}\sin(2t)$

$f(t) = (g * h)(t)$

$f(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau$.

$= \int_0^t (t-\tau) e^{-(t-\tau)} \frac{1}{2}\sin(2\tau)\, d\tau$

$=$