PHYS 3P35 Electromagnetism I.

Recall vector calculus.

- This course studies the Maxwell Equations which describe all electromagnetic phenomenon.

1. $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ - $\vec{\nabla}\cdot$ - Divergence.
   $\vec{E}$ = electric field.
   - charge producese $\vec{E}$

   ρ = charge density
   $\epsilon_0 = 8.85\times10^{-12}\ \frac{C^2}{Nm^2} = \frac{F}{m}$

2. $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$ - $\vec{\nabla}\times$ - Curl   t = time.
   $\vec{B}$ = Magnetic field
   - changing $\vec{B}$ makes $\vec{E}$ (Faradays Law).
   - In this course, we study electrostatics so there is no time dependence thus $\vec{\nabla}\times\vec{E} = 0$.

   $\Big\}$ 3P35.

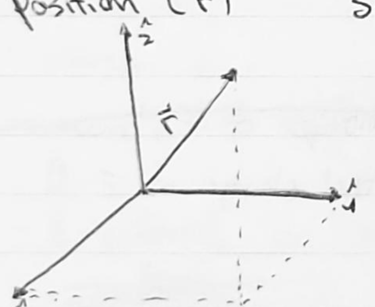
3. $\vec{\nabla}\cdot\vec{B} = 0$ - no isolated poles
4. $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$ - $\vec{J} = \frac{\text{current}}{\text{Area}}$
   - $\mu_0 = 4\pi\times10^{-7}\ \frac{Tm}{A}$

   $\Big\}$ 3P36.

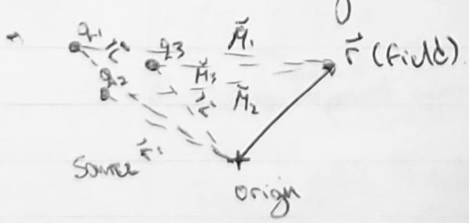
- First 2 eqn's in 3P35, 2nd 2 in 3P36.

Position & Seperation

- Position ($\vec{r}$)   separation ($\vec{\mathscr{r}}$) curly $\vec{r}$
-



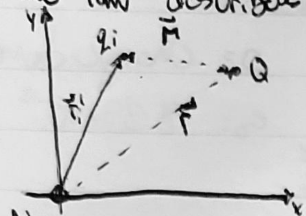
- Assume set of charges



Here we see that $\vec{\mathscr{r}}_n = \vec{r} - \vec{r}'_n$.

Coulomb's Law

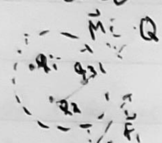
- The law describes the force between 2 charges.



$\vec{M} = \vec{r} - \vec{r}_i$ $\quad\quad$ $\hat{M}$ = unit vector $= \frac{\vec{M}}{|\vec{M}|}$ direction only.

$\vec{F}_0 = \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{M^2} \hat{M}$ $\quad\quad$ $M = |\vec{M}|$

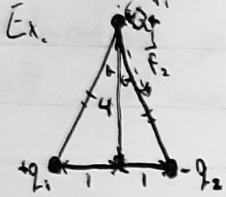
- NET FORCE on Q



$\vec{F}_{net} = \sum \vec{F}_i \leftarrow$ force of q_i on Q

$\vec{F}_{net} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{M^2} \hat{M}$

- Ex.



what is the net charge on Q (+)?

q_1 & Q have a repulsive force

q_2 & Q have an attractive force.

$|\vec{F}_1| = |\vec{F}_2|$, vertical components cancel

the sum of the horizontal component is in the positive direction

$\sum \vec{F}_{iy} = 0$

$\vec{\sum} F_{ix} = 2|F_1| \sin\theta$. $|\vec{F}_1| = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{q_1 Q}{(1^2+4^2)} \left(\frac{1}{\sqrt{17}}\right)$

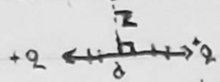
Electric Fields

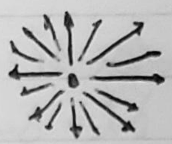
- Operation Definition ~~place a test charge at $\vec{r}$, measure~~
  - 1 place a (+) test charge at $\vec{r}$, Q
  - 2 measure $\vec{F}_{TOT}$
  - 3 $\vec{E} = \frac{\vec{F}_{TOT}}{Q}$
- $\vec{F}_{TOT} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{M_i^2} \hat{M}_i$
- Set of q_i creates $\vec{E}$
- $\vec{F}_{TOT} = Q\underbrace{\left[\sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i}{M_i^2} \hat{M}_i\right]}_{\vec{E}}$ → single charge produce $\vec{E}$
-

Q creates $\vec{E}$ at $\vec{r}$

$\vec{F} = Q\vec{E}$

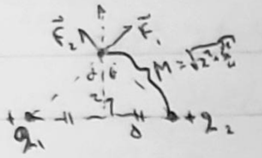
• Griffiths 2.2(a). Find $\vec{E}(z)$ produced by 2 charges.



field produced by (+)q  , vice versa for negative.

$\frac{1}{4\pi\epsilon_0}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}$

Resulting $\vec{E}$



$E_x^{TOT}=0$.

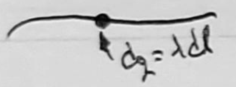
$E_y^{TOT}=2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}_i^2}\cos\theta = \frac{1}{2\pi\epsilon_0}\frac{q}{(z^2+\frac{d^2}{4})}\frac{z}{\sqrt{z^2+\frac{d^2}{4}}}$

$=\frac{1}{2\pi\epsilon_0}\frac{qz}{(z^2+\frac{d^2}{4})^{3/2}}\hat{y}$

• Continuous Charge distribution...

Continuous Charge Distribution.

• Linear $\lambda=\frac{\text{charge}}{\text{length}}$, λ can change or be a function of position $\vec{r}'$



• Surface $\sigma=\frac{\text{charge}}{\text{area}}$, σ can also be a function of $\vec{r}'$

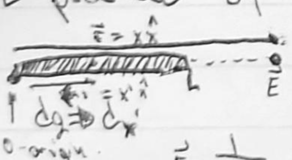
$dq=\sigma\, dx\, dy$.

• Volume ρ,

$dq=\rho\, d\tau$ or $\rho\, dV$

both mean volume.

• $\vec{E}$ produced by line of constant (+) charge density (λ)



$\vec{\mathscr{r}}=\vec{r}-\vec{r}'=(x-x')\hat{x}$

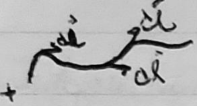
$\vec{E}=\frac{1}{4\pi\epsilon_0}\sum\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}$

$\vec{E}=\frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}_i^2}\hat{\mathscr{r}}$ depend on where dq is

$\vec{E}=\frac{1}{4\pi\epsilon_0}\int_0^L\frac{\lambda\, dx'}{(x-x')^2}\hat{x} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{x-x'}\right]_0^L\hat{x} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{x-L}-\frac{1}{x}\right]\hat{x}$

Cylindrical Coordinates,

Elements of Path ($d\vec{l}$).



- cartesian: $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

  cylindrical: $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

- Element of Volume: cartesian - $d\tau = dx\,dy\,dz$

  cylindrical - $d\tau = ds(s\,d\phi)dz = s\,ds\,d\phi\,dz$

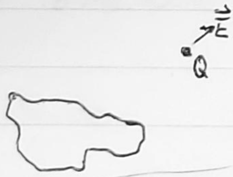
- Element of Surface: cartesian - $da = dx\,dy$, $dx\,dz$, $dy\,dz$

  cylindrical $da = dz\,s\,d\phi$, $s\,ds\,d\phi$,

Electrostatics.

- Main Ideas - ① Source Charges/Test Charges.
  ② Superposition.
  ③

Source vs Test Charges.



$f = Q\vec{E}$

$\{q_i\}$ sources → produce $\vec{E}$

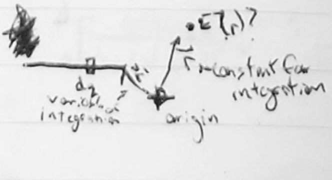
Superposition.

- $\vec{E}_{net} = \sum \vec{E}_i$ produced by q_i
- $\vec{E} = \sum_i \frac{q_i}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$

  or

  $\vec{E} = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$

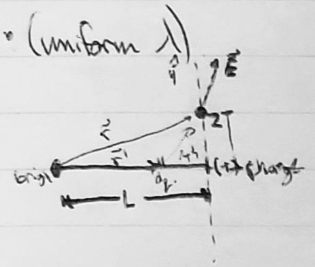
- Line of Charge:



new cartesian.

Problem 2.3.

• (uniform λ)



$$\cancel{\vec{E}(r) = \frac{\lambda}{4\pi\epsilon_0}\int \frac{dx'}{\mathscr{r}^2} = \frac{\lambda}{4\pi\epsilon_0}}$$

$\hat{x}$

$\vec{r} = L\hat{x} + z\hat{y}$ $\quad \vec{\mathscr{r}} = (L-x')\hat{x} + z\hat{y}$ $\quad \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$ $\quad |\vec{\mathscr{r}}| = \mathscr{r} = \sqrt{(L-x')^2 + z^2}$

$\vec{r}' = x'\hat{x}$

constants

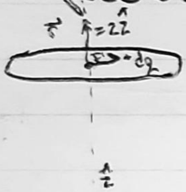
$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0}\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \int \frac{\lambda dx'}{4\pi\epsilon_0}\frac{[(L-x')\hat{x} + z\hat{z}]}{[(L-x')^2+z^2]^{\frac{3}{2}}}$$

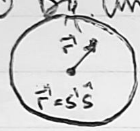
$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\hat{x}\int\frac{(L-x')dx}{[(L-x')^2+z^2]^{\frac{3}{2}}} + \hat{z}z\int\frac{dx'}{[(L-x')^2+z^2]^{\frac{3}{2}}}\right]$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\left[\frac{1}{z} - \frac{1}{\sqrt{L^2+z^2}}\right]\hat{x} + \left[\frac{1}{z}\frac{L}{\sqrt{L^2+z^2}}\right]\hat{z}\right].$$

Problem 2.6

• $\vec{E}$ produced by uniform disk along axis of symmetry.





Two methods.

① $dq = s\,ds\,d\phi$

$\vec{E} = \iint \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$ $\qquad \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$

② consider disk to be a set of rings.

ring charge $Q_{ring} = \sigma 2\pi s' ds'$

Problem 2.5.

• $\vec{E}$ produced by ring.

by symmetry $\Rightarrow$ $\vec{E} \parallel \hat{z}$

$$\vec{E} = \int dE_z\,\hat{z}$$

$$dE_z = dE\cos\theta$$

$$dE = \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2} = \frac{\lambda dl'}{4\pi\epsilon_0 \mathscr{r}^2} \qquad dl = R\,d\phi' \qquad \mathscr{r} = \sqrt{R^2+z^2}.$$

$$\cos\theta = \frac{z}{\mathscr{r}} = \frac{z}{\sqrt{R^2+z^2}}$$

$$E_{TOT} = \int \frac{\lambda R\,d\phi'}{4\pi\epsilon_0 [R^2+z^2]}\cdot\frac{z}{\sqrt{R^2+z^2}}$$

$$\vec{E} = \frac{\lambda R z}{4\pi\epsilon_0}\frac{1}{[R^2+z^2]^{\frac{3}{2}}}\cdot\int d\phi' = \frac{\lambda R z}{4\pi\epsilon_0}\frac{2\pi}{(R^2+z^2)^{\frac{3}{2}}}\hat{z} = \frac{\lambda R z}{2\epsilon_0}\frac{1}{[R^2+z^2]^{\frac{3}{2}}}\hat{z}.$$

$$Q = 2\pi R\lambda$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{z}{[R^2+z^2]^{\frac{3}{2}}}\hat{z}$$

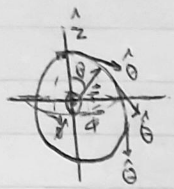
Problem 2.6. cont.

change ones ing for 2.6

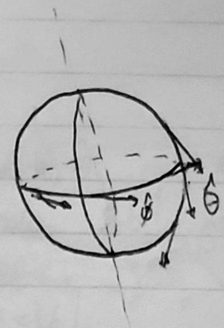
$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{z}{[R^2+z^2]^{3/2}} \hat{z}$$

s for 2.6.

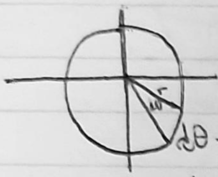
Spherical Coordinates

- 

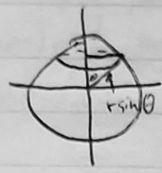
$\vec{r} \parallel \hat{r}$
$\hat{\theta}, \hat{\phi} \perp \hat{r}$



- $dl = dr + r\,d\theta + r\sin\theta\, d\phi$

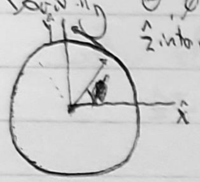


$d\theta = dl_\theta$



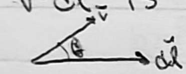
$r\sin\theta\, d\phi = dl_\phi$

- Deriving $\hat{\theta}, \hat{\phi}$
$\hat{z}$ into page.



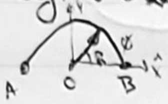
$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

Other important ideas

- Line Integral $\int \vec{v} \cdot d\vec{l}$ ← result will be scalar.
  - $\vec{v} \cdot d\vec{l}$ is "projection" of $\vec{v}$ onto path
  - 
  $\vec{v}\,d\vec{l} = |\vec{v}||d\vec{l}|\cos\theta = |\vec{v}|\cos\theta|d\vec{l}|$ ← this $\theta \neq$ spherical θ.
- Flux $\iint \vec{v} \cdot d\vec{a}$
  - how much of $\vec{v}$ is $\parallel$ $d\vec{a}$
  - Recall Faraday's Law → $\mathcal{E} = \frac{d\Phi}{dt}$, $\Phi = \iint \vec{B} \cdot d\vec{a}$
- Volume Integrals $\iiint T\, d\tau$ ← scalar.
  - T ← scalar fxn
  - $d\vec{\tau}$ doesn't exist in 3D space (volume vector)
  - 
  line
  surface.
  - $\iiint \vec{v}\, d\vec{\tau}$ doesn't exist, $\iiint T\,d\tau$ does, $\iiint \vec{v}\,d\tau$ does.
  what is the physical meaning? $\langle \vec{v} \rangle = \frac{1}{\tau}\iiint \vec{v}\, d\tau$
  τ = total volume
  $= \iiint d\tau$

Extra Problem.

- Important Idea $\hat{\theta}, \hat{\phi}$ not constant.
  - eg. $\int d\vec{l}$ along semicircle.



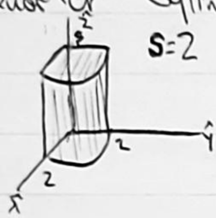
$\int d\vec{l} = 2R\hat{x}$ by inspection

- In cylindrical, $\int_{\pi}^{0} s\,d\phi\,\hat{\theta} \Rightarrow \int_{\pi}^{0} s\left[-\sin\phi\hat{x} + \cos\phi\hat{y}\right]$ $\rightarrow 0$

$= \hat{x}R\int_{0}^{\pi}\sin\phi\,d\theta + R\hat{y}\int_{0}^{\pi}(-\cos\theta)\,d\theta$

$= 2R\hat{x}$.

- Quarter Cylinder.



$s=2$ Calculate $\iint \vec{A}\cdot d\vec{a}$ on top flat surface.

$\vec{A} = sz^2\cos\phi\,\hat{s} + s^2 z \sin\phi\,\hat{\phi} + z^3\hat{z}$

$d\vec{a} = s\,d\phi\,ds\,\hat{z}$.

$\iint \vec{A}\cdot d\vec{a} = \iint A_z \cdot dz$ since $A_\phi \cdot d\vec{a} = 0$

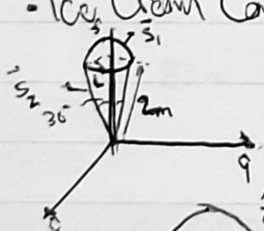
$= \iint z^3\, s\,ds\,d\phi$ $\quad A_s\, d\vec{a} = 0$.

$z=5 \quad = \int_0^5\int_0^{\frac{\pi}{2}} z^3\, s\,ds\,d\phi$

$= 125\int_0^5 s\,ds \int_0^{\frac{\pi}{2}} d\phi$

$= 125\pi$.

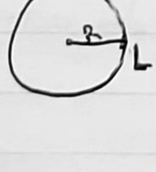
- Ice Cream Cone.



$\vec{A}(r,\theta,\phi) = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta}$

$\oint_L \vec{A}\cdot d\vec{l}$

$\hat{z}$ into page.



$R = 2\sin 30$. $\quad d\vec{l} = r\sin\theta\,d\phi\,\hat{\phi}$

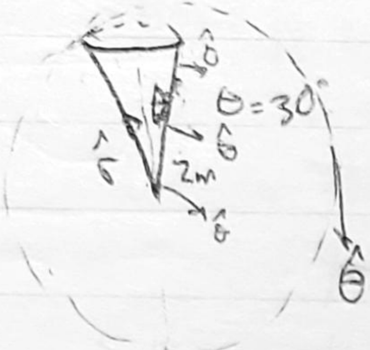
$\vec{A}\cdot d\vec{l} = 0$ because $\hat{\phi}\cdot\hat{r} = 0$

$\hat{\phi}\cdot\hat{\theta} = 0$.

so $\oint_L \vec{A}\cdot d\vec{l} = 0$.

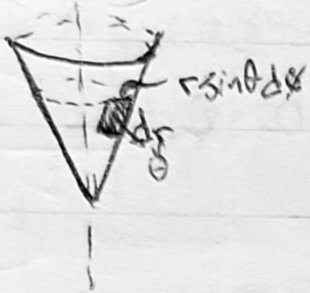
Calc

Calculate $\iint \vec{A}\cdot d\vec{a}$ on cone part.



$$d\vec{a} = r\sin\theta\, dr\, d\phi\, \hat{\theta}$$

$$\iint \vec{A}\cdot d\vec{a} =$$



$$= \iint (r\sin\theta\,\hat{r} + r\sin\theta\,\hat{\theta})\cdot r\sin\theta\, dr\, d\phi\,\hat{\theta}$$

$$= \iint r^2 \sin^2\theta\, dr\, d\phi$$

$$= \frac{1}{4}\int_0^2 r^2 dr \int_0^{2\pi} d\phi$$

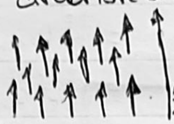
$$= \frac{1}{4}\left(\frac{8}{3}\right)(2\pi)$$

$$= \frac{16}{12}\pi = \frac{4}{3}\pi.$$

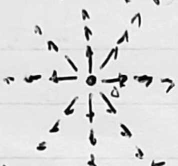
Vector Functions

- $\vec{V}(x,y,z)$ is complicated.
- $\vec{V} = f(x,y,z)\hat{x} + g(x,y,z)\hat{y} + h(x,y,z)\hat{z}$
- $\vec{V} = V_x(x,y,z)\hat{x} + V_y(x,y,z)\hat{y} + V_z(x,y,z)\hat{z}$
- Sampling Diagram - length of vector $\propto |\vec{V}|$

  direction of vector is direction of $\vec{V}$.

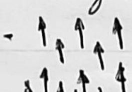
  -  $\hat{z}$ $\quad \vec{V} = 0\hat{x} + 0\hat{y} + C\hat{z}$ where C = some constant.

  - $\vec{E}$ produced by point charge.

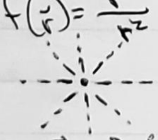


- Divergence - $\vec{\nabla}\cdot\vec{V} = \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right]\cdot\left[V_x\hat{x} + V_y\hat{y} + V_z\hat{z}\right]$

  $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial y}$ (scalar).

  - Divergence is a local value and varies from point to point.
  -  $\vec{V} = C\hat{y}$, $\vec{\nabla}\cdot\vec{V} = 0$.
  -  $\vec{V}' = C' y\hat{y}$, $\vec{\nabla}\cdot\vec{V}' = C'$

- Think of Derivatives ~~as~~ as measurements.
  - $\frac{\partial V_x}{\partial x} \simeq \frac{\Delta V_x}{\Delta x}$ ← difference in x component between points / ← difference in x direction.

- Griffiths 1.1.6.
  - $\vec{f}(\vec{r}) = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$
  -  $r = \sqrt{x^2+y^2+z^2}$

Div = 0, cause measurement is taking place at origin

Flux of a vector function

$\iint \vec{V}\cdot d\vec{a}$

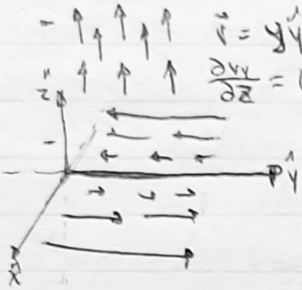
  - $\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$
  - $\frac{\partial}{\partial z}\left[\frac{z}{r^3}\right] = \frac{1}{r^3} + z\left[\frac{-3}{r^4}\left(\frac{\partial r}{\partial z}\right)\right]$

    $\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{z}$ . . . .

    . . . . . .

  - By symmetry $\frac{\partial}{\partial z}\left[\frac{z}{r^3}\right] \Rightarrow \frac{\partial}{\partial x}\left[\frac{x}{r^3}\right]$

  $= \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = \frac{3}{r^3} - \frac{3r^2}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0$.

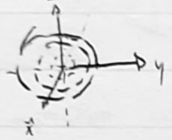
- Curl - measured at point in space
  - $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$
  - Consider the difference between a divergence term and curl term
    - Div term $\frac{\partial v_z}{\partial z}$, "changes" in the same direction
    - Curl term $\frac{\partial v_x}{\partial y}$; "changes" in the perpendicular direction.
  - 

    $\vec{v} = y\hat{y}$

    $\frac{\partial v_y}{\partial z} = 0$, $\frac{\partial v_z}{\partial x} = 0$ curl $\vec{v} = 0$

    sampled on x, y plane

    $\vec{v} = x\hat{y}$

    $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \hat{z}$
  - 

    sampled on x, y plane.

    $\vec{v} = -y\hat{x} + x\hat{y}$

    Curl $\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = 2\hat{z}$ $\qquad \partial_x = \frac{\partial}{\partial x}$

- ∇^2 scalar operator

  $\vec{\nabla}^2$ vector operator.
- Consider 1.26(d)
  - $\vec{v} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$
  - $\nabla^2\vec{v} = \underbrace{\nabla^2 v_x\hat{x}}_{\frac{\partial^2 x^2}{\partial x^2} + \frac{\partial^2 x^2}{\partial y^2} + \frac{\partial^2 x^2}{\partial z^2} = 2.} + \nabla^2 v_y\hat{y} + \nabla^2 v_z\hat{z}$ $\qquad \nabla^2 v_y = 6x$ $\qquad \nabla^2 v_z = 0.$

    $= 2\hat{x} + 6x\hat{y} + 0\hat{z}.$

Cartesian Tensors.

- Important - $\vec{\nabla}\times\vec{\nabla}f = 0$
  - $\vec{\nabla}\cdot(\vec{\nabla}\times f) = 0$.
- $\vec{\nabla}\times\vec{\nabla}f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \Rightarrow [\vec{\nabla}\times\vec{\nabla}f]_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 0$. by symmetry. $\vec{\nabla}\times\vec{\nabla}f = 0$
- $\vec{\nabla}\cdot(\vec{\nabla}\times f) = \frac{\partial}{\partial x}\left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]$

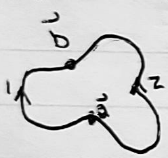
  $= 0$

  by symmetry, $\vec{\nabla}\cdot(\vec{\nabla}\times f) = 0$.

Vector Integral Calculus.

- Gradient Theorem - If I calculate the path integral from $r = a$, to $r = b$, it is just the value of the vector fxn evaluated at the two positions
  - $\int_a^b \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$
    1D, 0D
  - Essentially Fundamental Theorem of Calculus
- Stokes Theorem - If I calculate the curl of some function and evaluate that at a surface integral, we achieve a closed line integral around surface s.
  - $\iint (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$
    2D, 1D
  - Note: ∞ # of ~~[illegible]~~ open surfaces with same boundary
- Divergence Theorem - $\iiint_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oiint_S \vec{v} \cdot d\vec{s}$
  3D, 2D.
  - S "wraps" V.
- Gradient Theorem is implied by Stoke's theorem.
- $\iint_S [\vec{\nabla} \times \vec{\nabla} f] \cdot d\vec{a} = \oint_P \vec{\nabla} f \, d\vec{l}$

$$\underset{\text{because } \vec{\nabla} \times \vec{\nabla} f = 0.}{\overset{\shortparallel}{0}}$$



Then we see that for any closed loop $\oint_P \vec{\nabla} f \cdot d\vec{l} = 0$.

$$\oint \vec{\nabla} f \cdot d\vec{l} = \int_a^b \vec{\nabla} f \cdot d\vec{l} + \int_b^a \vec{\nabla} f \, d\vec{l} = 0$$

$$\therefore {}_1\int_a^b \vec{\nabla} f \cdot d\vec{l} = \int_a^b \vec{\nabla} f \, d\vec{l}$$

Flipped limits of int to aid (-).

- Application of Divergence Theorem.
  - Begin with Maxwell Equation.
  - $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
  - $\iiint (\vec{\nabla} \cdot E)\, d\tau = \frac{1}{\epsilon_0} \iiint_V \rho \, d\tau$

    $\oiint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \iiint dq = \frac{Q_{enc}}{\epsilon_0}$ charge inside surface s.
  - Charge on the surface is connected to charge inside volume.
  - Another $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ magnetic flux. $\frac{d \Phi_m}{dt}$

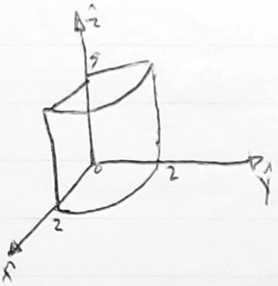
    $\iint_S \vec{\nabla} \times \vec{E} \, da = -\frac{\partial}{\partial t} \iint_D \vec{B} \cdot d\vec{a}$

    $\underbrace{\oint \vec{E} \, d\vec{l}}_{\text{Definition of EMF}} =$

Griffiths

1.43 $\vec{v} = s(2+\sin^2\phi)\hat{s} + s\sin\phi\cos\phi\,\hat{\phi} + 3z\hat{z}$

Show divergence is true.



$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}[s v_s] + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$
$$= \frac{1}{s}\frac{\partial}{\partial s}[s^2(2+\sin^2\phi)] + \frac{1}{s}\frac{\partial}{\partial\phi}[s\sin\phi\cos\phi] + \frac{\partial}{\partial z}(3z)$$
$$= 2(2+\sin^2\phi) + \frac{s}{s}[-\sin^2\phi + \cos^2\phi] + 3$$
$$= 4 + 2\sin^2\phi + \sin^2\phi - \cos^2\phi + 3$$
$$= 7 + \sin^2\phi + \cos^2\phi$$
$$= 8$$

$$\iiint_V \vec{\nabla}\cdot\vec{v}\,d\tau = 8\iiint s\,ds\,d\phi\,dz$$
$$= 8\int s\,ds\int d\phi\int dz$$
$$= 8\left[\frac{s^2}{2}\right]_0^2\left[\phi\right]_0^{\frac{\pi}{2}}\left[z\right]_0^5$$
$$= 8[2]\left[\frac{\pi}{2}\right][5]$$
$$= 40\pi$$

$$\oint_S \vec{v}\cdot d\vec{a} = \int_{\substack{(1)\\ \text{Top}}} + \int_{\substack{(2)\\ \text{Bottom}}} + \int_{\substack{(3)\\ XZ}} + \int_{\substack{(4)\\ YZ}} + \int_{\substack{(5)\\ \text{Curved}}}$$

$$= 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi + 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)\,d\phi\int_0^5 dz$$
$$= 15\pi + 25\pi$$
$$= 40\pi$$

Top: $d\vec{a} = s\,ds\,d\phi\,\hat{z}$ $(z=5)$

$\vec{v}\cdot d\vec{a} = \vec{v}_z\cdot da = (3z)(s\,ds\,d\phi)$

$= 15\,s\,ds\,d\phi$

Bottom: $d\vec{a} = (s\,d\phi\,ds)(-\hat{z})$ $(z=0)$

$\vec{v}\cdot da = v_z\cdot da = (3z)(s\,ds\,d\phi)$

$= 0$

XZ: $d\vec{a} \parallel -\hat{y} = -\hat{\phi}$ on x axis $(\phi=0)$

$\vec{v}\cdot d\vec{a} = v_\phi\,da_\phi$

$= -2(s\sin\phi\cos\phi)(d\phi\,dz)$

$= 0$

YZ: $\vec{v}\cdot d\vec{a} = 0$

Curved: $d\vec{a} = s\,d\phi\,dz\,\hat{s}$ $(s=2)$

$\vec{v}\cdot d\vec{a} = v_s\cdot da$

$= 2(2+\sin^2\phi)s\,d\phi\,dz$

$= 4(2+\sin^2\phi)$

$\vec{E}$ of ring

- 

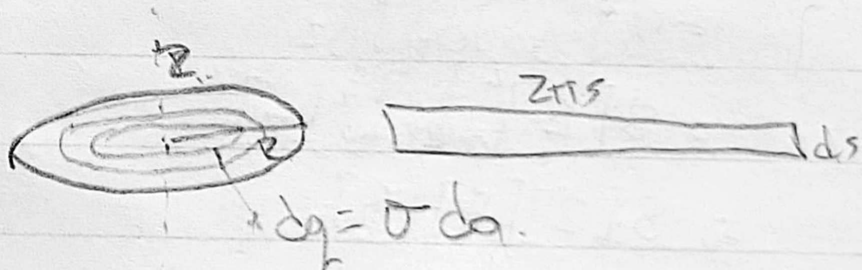
$\int d\vec{E} \parallel \hat{z}$ by symmetry.

$$\vec{E} = \hat{z}\int dE_z$$
$$= \hat{z}\int dE\cos\theta$$
$$= \hat{z}\int \frac{dq}{4\pi\epsilon_0}\left[\frac{z}{(r^2+z^2)^{\frac{3}{2}}}\right]$$
$$= \hat{z}\left[\frac{1}{4\pi\epsilon_0}\right]\left[\frac{z}{(r^2+z^2)^{\frac{3}{2}}}\right]\int dq$$
$$= \frac{z\,Q_{TOT}}{4\pi\epsilon_0(r^2+z^2)^{\frac{3}{2}}}\hat{z}$$

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0(r^2+z^2)}, \quad dq =$$
$$\cos\theta = \frac{z}{(r^2+z^2)^{\frac{1}{2}}}$$

$\vec{E}$ of Disk.

- $\sum \vec{E}_{rings}$



$dq = \sigma\, da$

$$\vec{E}_{disk} = \int \frac{dq\, z}{4\pi\epsilon_0(z^2+s^2)^{\frac{3}{2}}}$$
$$= \int \frac{\sigma 2\pi s\, ds\, z}{4\pi\epsilon_0(z^2+s^2)^{\frac{3}{2}}}\hat{z}$$
$$= \frac{\sigma z}{2\epsilon_0}\int \frac{s}{(z^2+s^2)^{\frac{3}{2}}}ds\ \hat{z}$$
$$= \frac{\sigma z}{2\epsilon_0}\left[\frac{-1}{\sqrt{z^2+s^2}}\right]_0^R \hat{z}$$
$$= \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{z}$$

- $\lim_{R\to\infty} E_{disk} = \frac{\sigma}{2\epsilon_0}\hat{z}$

Dirac Delta Function.

- 1D - 3 Properties.
  - ① $\delta(x) = 0$, anywhere except $x=0$.
  - ② $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$
  - ③ $\hat{O} = \int_{-\infty}^{\infty} \delta(x)\,dx$

    $\hat{O}\,F(x) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$.
- 1.44(c) $\int_2^6 (3x^2-2x-1)\,\delta(x-3)\,dx = 3(3^2) - 2(3) - 1 = 20$.

  but $\int_{-6}^{2} (3x^2-2x-1)\,\delta(x-3)\,dx = 0$
- One possible representation
  - $\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} \le x \le \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$

  $\delta(x) = \lim_{n\to\infty} \delta_n(x)$
  - Property ① - True
    - ② - $\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)\,dx = 3$ integrals $= \int_{-\infty}^{-\frac{1}{2n}} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \int_{\frac{1}{2n}}^{\infty}$

      $\lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x)\,dx = \lim_{n\to\infty} n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} dx = \lim_{n\to\infty} n\left[\frac{1}{n}\right] = 1$
    - ③ - $\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = \lim_{n\to\infty} n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x)\,dx$ ← assume $\int f(x)\,dx = F(x)$

      $= \lim_{n\to\infty} n\left[F\left(\tfrac{1}{2n}\right) - F\left(-\tfrac{1}{2n}\right)\right] \Rightarrow F'(x)\big|_{=0} = f(0)$.
  - Satisfies all 3 properties of the DDF.

3D Dirac Delta Function

- Recall $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$, except at $r=0$ where it is undefined.
- $\iiint_{\text{sphere}} \vec{\nabla}\cdot\frac{\hat{r}}{r^2}\,d\tau = \oiint \frac{\hat{r}}{r^2}\cdot d\vec{a}$ $\quad d\vec{a} \parallel \hat{r}$, r^2 is constant on sphere

  $= \frac{1}{R^2}\oiint d\vec{a} = \frac{1}{R^2}\left[4\pi R^2\right] = 4\pi$.

  Finally, $\frac{1}{4\pi}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \delta^3(\vec{r})$
- 3 Properties.
  - ① $\delta^3(\vec{r}-\vec{0}) = 0$, except at $\vec{r}=\vec{0}$
  - ② $\iiint_{\text{all space}} \delta^3(\vec{r}-\vec{a})\,d\tau = 1$
  - ③ $\iiint$

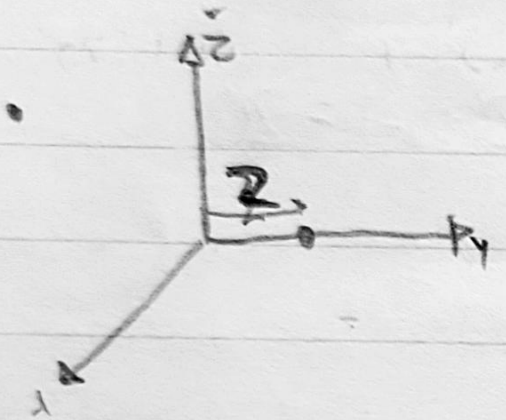
- 1.47 (b) - point charge

$\iiint \rho(\vec{r})\,d\tau = q\iiint \delta^3(\vec{r}-\vec{R})\,d\tau = 1$

(c) - shell of charge

$Q = \int \rho\,d\tau = \int A\,\delta(r-R)\,\underbrace{4\pi r^2\,dr}_{d\tau} = A4\pi R^2$

no ϕ or θ dependence



cartesian - $\rho(\vec{r}) = q\,\delta(x)\,\delta(y-2)\,\delta(z)$

spherical - $\rho(\vec{r}) = q\,\delta(r-2)\,\delta(\theta-\frac{\pi}{2})\,\delta(\phi-\frac{\pi}{2})$

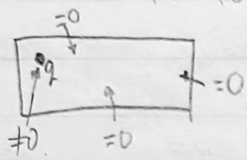
- 1.48 (d) - $\iiint_{V} \vec{r}\cdot(\vec{d}-\vec{r})\,\delta^3(\vec{e}-\vec{r})\,d\tau$, $\vec{e}=(3,2,1)$

V = sphere of radius 1.5 centered at (2,2,2) , $\vec{d}=(1,2,3)$

- sphere $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$
- $\vec{e}=x$, $x_1=(2,2,2)$, $2=R$
- $R^2=2.25$
- $\vec{e}$ is in sphere
- $\vec{e}\cdot(\vec{d}-\vec{e})$

Divergence of an Electric Field & Gauss's Law

Most Important Idea

- $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ one of Maxwell's Equations. which is in differential form
  - ↳ refers to point in space
- 

Here $\vec{\nabla}\cdot E = 0$ everywhere but at the point charge ("blows up")

- Gauss's Law is the "integral form" of $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$
  - integrate both sides $\iiint_V \vec{\nabla}\cdot\vec{E}\,d\tau = \iiint_V \frac{\rho}{\epsilon_0}\,d\tau$
  - Now, there's a region V, no longer point charge.
  - $\oiint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\iiint_V \underbrace{\rho\, d\tau}_{\frac{C}{m^3}\cdot m^3} = \frac{1}{\epsilon_0}\iiint_V dq = \frac{Q_{ENC}}{\epsilon_0}$

- Consider a point charge.
  - Experiment shows $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$
  - Apply $\vec{\nabla}$ to $\vec{E}$ $\quad \vec{\nabla}\cdot E = \vec{\nabla}\cdot\left[\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right]$

$$= \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\left[\frac{\hat{r}}{r^2}\right] \Rightarrow \vec{\nabla}\cdot\left[\frac{\hat{r}}{r^2}\right] = 4\pi\delta^3(r)$$

$$= \frac{q}{4\pi\epsilon_0}4\pi\delta^3(r) = \frac{q\delta^3(r)}{\epsilon_0} \underset{\rho = q\delta^3(r)}{} = \frac{\rho}{\epsilon_0}.$$

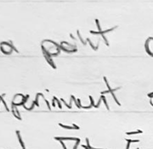
- Problem 2.9 - Given $\vec{E} = kr^3\hat{r}$. (a) Find $\rho(r)$ everywhere

(b) Find Q_{TOT} inside sphere radius R center (0,0,0).

(a) In spherical coordinates $\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial(r^2E_r)}{\partial r} + \ldots\ldots$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2kr^3) = \frac{\rho}{\epsilon_0}$$

$$\frac{5r^4k}{r^2} = \frac{\rho}{\epsilon_0} \Rightarrow 5r^2\epsilon_0 k = \rho$$

(b) 

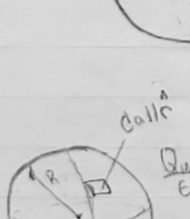
$dq = \rho\,d\tau \quad Q_{INSIDE} = \iiint_{sphere}\rho\,d\tau.$

$$\int_0^R\int_0^\pi\int_0^{2\pi}(5r^2\epsilon_0 k)\,r^2\sin\theta\,dr\,d\phi\,d\theta$$

$$= (4\pi)(5\epsilon_0 k)\int_0^R r^2\,dr$$

$$Q_{TOT} = \int_0^R(5k\epsilon_0 r^2)(4\pi r^2)\,dr$$

$$= 20\pi k\epsilon_0\left[\frac{r^5}{5}\right]_0^R = 20\pi k\epsilon_0\frac{R^5}{5} = 4\pi k\epsilon_0 R^5$$

2nd way → 

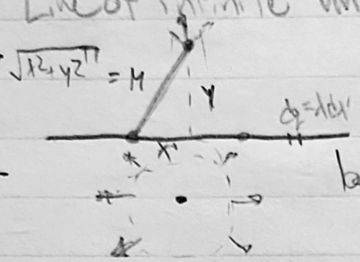
$\frac{Q_{IN}}{\epsilon_0} = \oiint_{sphere}\vec{E}\cdot d\vec{a}, \quad \vec{E} = kr^3\hat{r}$

$\Rightarrow E\,da, \quad |\vec{E}| = kR^3 \Rightarrow \oiint \vec{E}\,d\vec{a} = |\vec{E}|\oiint da = (kR^3)(4\pi R^2) = \frac{Q_{TOT}}{\epsilon_0}$

$4\pi k\epsilon_0 R^5 = Q_{TOT}$

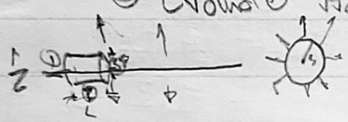
- Using $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{IN}}{\epsilon_0}$ in situations of high symmetry.
- ① Use intuition to find direction of $\hat{E}$.
  ② Choose S (only 1 S will be useful)
  ③ Apply $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}$
  depending on region your interested in.

- Problem - Line of infinite uniform charge density (λ).



Rotate 90°.
looks like point charge.

- Hard way $\vec{E} = \int \frac{dq}{4\pi\epsilon_0}\hat{\mathscr{r}} = \int_{-\infty}^{\infty} \frac{\lambda dx'}{4\pi\epsilon_0[x'^2+y^2]}\cdot\frac{y}{\sqrt{x'^2+y^2}}$
- Easy way (Gauss's Law).
  ① determine $\vec{E}$ direction
  ② Find S. → $\vec{E} \parallel d\vec{a} \to |\vec{E}|da$, $E \perp da \to 0$.
  ③ Evaluate $\oint_S \vec{E}\,d\vec{a} = \frac{Q_{IN}}{\epsilon_0}$



$\vec{E} \parallel \hat{s}$

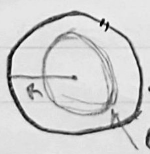
$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{IN}}{\epsilon_0}$

$\oint \vec{E}\cdot da = \cancel{\iint_{①}}^{0} + \iint_{②} + \cancel{\iint_{③}}^{0} = |\vec{E}| \iint d\vec{a} = \frac{\lambda L}{\epsilon_0}$

$|\vec{E}|\, 2\pi s L = \frac{L\lambda}{\epsilon_0} \Rightarrow |\vec{E}| = \frac{\lambda}{2\pi s\epsilon_0}$

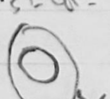
$\sigma = \frac{\text{charge}}{\text{area}}$.

- Problem 2.11 - Uniform shell of sphere



$E_{out} \parallel \hat{r}$
← not S for integration

- $\oint_S E\,da = \frac{Q_{IN}}{\epsilon_0} = \frac{0}{\epsilon_0}$. $|\vec{E}| = 0$.

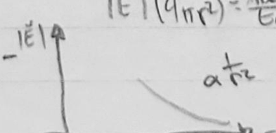


$r > R$.

$\oint \vec{E}\,d\vec{a} = \frac{Q_{IN}}{\epsilon_0}$

$|\vec{E}| \oint d\vec{a} = \cancel{|\vec{E}|(4\pi r^2\sigma)} = |\vec{E}| \oint d\vec{a} = \frac{4\pi R^2\sigma}{\epsilon_0}$

$|\vec{E}|(4\pi r^2) = \frac{4\pi R^2\sigma}{\epsilon_0} \Rightarrow \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{r} = |\vec{E}|$



Calculating $\vec{E}$

① • Given $\rho(\vec{r}) \to$ produces $\vec{E}$

$$\vec{E} = \sum \frac{q_i}{4\pi\epsilon_0 \mathscr{r}_i^2} \hat{\mathscr{r}}_i = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2} \hat{\mathscr{r}} \qquad dq = \lambda dx' \text{ or } \sigma da' \text{ or } \rho d\tau$$

② • $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \to \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{in}}{\epsilon_0}$.

③ • Given $\rho(\vec{r})$, calculate $V(\vec{r})$ "scalar potential"

$$\vec{E} = -\vec{\nabla} V(\vec{r})$$

Where does potential come from?

• $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ 2nd Maxwell Equation.

- Electrostatics $\to \frac{\partial}{\partial t} = 0$, so $\vec{\nabla} \times \vec{E} = 0$

• Also, identity $\vec{\nabla} \times (\vec{\nabla} f) = 0$, ALWAYS TRUE.

• Conclude from this that $\vec{E} = \vec{\nabla} f$

$$\vec{E} = -\vec{\nabla} V$$

How does one calculate $V(\vec{r})$?

• ① if $\vec{E}$ is known, $\vec{E} = -\vec{\nabla} V$ (goes from $V \to \vec{E}$).

$$\therefore -\int_R^{\vec{r}} \vec{E} \cdot d\vec{l} = \int_R^{\vec{r}} \vec{\nabla}(V) \cdot d\vec{l} \quad , R = \text{reference point } (V=0 \text{ here})$$

$$= V(\vec{r}) - V(R)$$

② if $\rho(\vec{r})$ is known, Assume $V(r) = 0$ @ ∞.

- For a pt charge, $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

$$V(r) = -\int_\infty^r \vec{E} \cdot d\vec{l}$$

$$\vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr. \qquad \hat{r} \cdot d\vec{l} = dr.$$

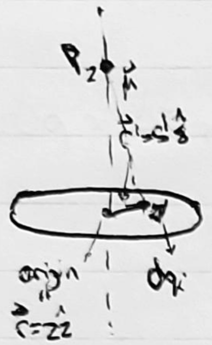
$$V(r) = -\int_\infty^r \frac{q}{4\pi\epsilon_0} \frac{dr}{r^2} \Rightarrow V = \frac{q}{4\pi\epsilon_0 r}$$

- By superposition, $V = \sum_i \frac{q_i}{4\pi\epsilon_0 \mathscr{r}_i}$ or $V = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}}$

• Problem 2.25 (c)

- Uniform disk of total charge Q, radius R. Whats V(z)?

- $V = \int \frac{dq}{4\pi\epsilon_0 r}$



$dq = \sigma da = \sigma\, ds\, s\, d\phi$

$V(z) = \iint \frac{\sigma\, ds\, s\, d\phi}{4\pi\epsilon_0 \sqrt{z^2 + s'^2}} = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^R \frac{s'\,ds'}{\sqrt{z^2+s'^2}}$

$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+s^2}\right]_0^R = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2} - z\right]$

Symmetry $\Rightarrow \vec{E} \parallel \hat{z}$, $E_z = -\frac{\partial V}{\partial z}$

$\vec{E} = \frac{\sigma}{2\epsilon_0}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]\hat{z}$

• Problem 2.28

- Find $V(\vec{r})$ inside sphere of uniform charge density.

Must be a function of r only. For simplicity, put P on z axis.

$V = \int \frac{dq}{4\pi\epsilon_0 r}$, $dq = \rho\, d\tau' = \rho r'^2 dr' d\theta d\phi$

$r = \sqrt{z^2 + r'^2 - 2zr'\cos\theta'}$

$V(z) = \frac{\rho}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_0^R r'^2 dr \int_0^\pi \frac{\sin\theta' d\theta}{\sqrt{z^2 + r'^2 - 2zr'\cos\theta}}$

$= \frac{\rho}{4\pi\epsilon_0}(2\pi) \int_0^R r'^2 dr' \left[\frac{1}{zr'}\sqrt{z^2 + r'^2 - 2zr'\cos\theta'}\right]_0^\pi \Rightarrow \frac{1}{zr'}\left[\sqrt{z^2+r'^2+2zr'\cos\theta} - \sqrt{z^2+r'^2-2\ldots}\right.$

$\frac{1}{zr'}\left[|z+r'| - |z-r'|\right]$

if $r' < z$, $= 2r'/zr'$

if $r' > z$, $= \frac{2z}{zr'} = \frac{2}{r'}$

$z = r \Rightarrow$

$= \frac{\rho}{\epsilon_0}\left[\int_0^z \frac{r'^2}{z} dr' + \int_z^R \frac{r'^2 dr'}{r'}\right]$

$= \frac{Q}{8\pi\epsilon_0 R}\left[3 - \frac{r^2}{R^2}\right]$

• Laplace/Poisson Eq.

- maxwell → $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

- sub $\vec{E} = -\nabla V$ Poisson Laplace

$\nabla^2 V = -\frac{\rho}{\epsilon_0}$ if $\rho = 0$, $\nabla^2 V = 0$.

- Pt charge $\Rightarrow \int \vec{E} \cdot d\vec{l} = \int_a^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} dr = \frac{q}{4\pi\epsilon_0 b} - \frac{q}{4\pi\epsilon_0 a}$ for $a = b \Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$.

& $\iint (\nabla \times \vec{E}) \cdot d\vec{a}$ MUST BE 0

Potential is Potential Energy.

- $\vec{F} = -\vec{\nabla} U$ ← Potential Energy.

$$\int_a^b \vec{\nabla} U \cdot d\vec{l} = -\int_a^b \vec{F} \cdot d\vec{l}$$

ΔV.

$$U(\vec{b}) - U(\vec{a}) = -\int q \vec{E} \cdot d\vec{l} = q\left[-\int_a^b \vec{E} \cdot d\vec{l}\right] =$$

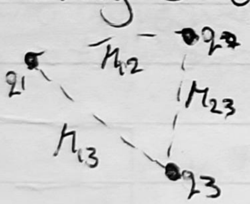
$$U(\vec{b}) - U(\vec{a}) = q\left(-V(\vec{b}) - V(\vec{a})\right).$$

- $V = \dfrac{\text{Potential Energy}}{\text{charge}} = \dfrac{J}{C} = \text{volt},$

- Point Charge - $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, r = distance from charge.

- Potential Energy - thought expt
  - $+q$ ⟷ $+q$ held stationary.
  - When released the fly apart, from PE to KE.
  - $+q_1$ ⟵ r ⟶ $+q_2$ Bring from infinity to r. HOLD
  - $U = W = \int_{\infty}^{r} \vec{F} \cdot d\vec{l} = \int_{\infty}^{r} -\frac{kq_1q_2}{r^2}\,dr. = \frac{kq_1q_2}{r}$

- Bring 3 charges together



$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2q_3}{r_{23}}$$

3 charges, $\frac{3!}{2!1!} = 3$ interactions

- Bring N charges.

$$U = \sum_i\sum_j \frac{q_iq_j}{4\pi\epsilon_0 r_{ij}}, \quad i \neq j, \text{ no double counting}$$

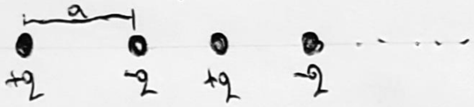
$$U = \frac{1}{2}\sum_i\sum_j \frac{q_iq_j}{4\pi\epsilon_0 r_{ij}}$$

- 2.31 (b)

$\frac{4!}{2!2!} = 6$. $U = 4\left(\frac{q^2}{4\pi\epsilon_0 a}\right) - 2\left(\frac{q^2}{4\pi\epsilon_0\sqrt{2}a}\right).$

=

· 2.33. Work per particle to assemble infinite chain of alternating



$U = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} = q_2 \underbrace{\left[\frac{q_1}{4\pi\epsilon_0 r_{12}}\right]}_{V \text{ produced by } 1.}$

$U_{\text{per particle}} = -qV$ produced by rest.

$= -q\, 2\left[\frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon_0 2a} + \frac{q}{4\pi\epsilon_0 3a} \dots\right]$

$= -2q\left[\frac{q}{4\pi\epsilon_0 a}\right]\underbrace{\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots\right]}$ Recall $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4!} \dots$

$= \frac{-2q^2}{4\pi\epsilon_0 a}\ln 2$ 2. $\to$ $\ln(2)$.

Potential Maps & $\vec{E}$

- Map $V = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i} = \iiint \frac{\rho\, d\tau}{4\pi\epsilon_0 r}$
  $\vec{E} = -\vec{\nabla} V$.
- Equipotential Surface.
  - Potential is the same.
- $W(\text{to assemble}) = U(\text{P.E stored}) = \frac{1}{2}\sum_i \sum_j \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$
  $= \frac{1}{2}\sum_i q_i V(\vec{r}_i)$ produced by all other charges.

$-v_f$ $\overset{\longleftarrow d \longrightarrow}{e^- \quad e^-}$ v_f electrons are released. what are velocities after a long time?

Cons of energy. $\frac{e^2}{4\pi\epsilon_0 d} = (2)\frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{e^2}{4\pi\epsilon_0 d}}$

$4e^-$ Held at 4 corners of square $\frac{?}{?}$
What are $\vec{v}$ after long time?
$U_{\text{initial}} = U_f + KE_f$
$4\left(\frac{e^2}{4\pi\epsilon_0 a}\right) + 2\left(\frac{e^2}{4\pi\epsilon_0 \sqrt{2}a}\right) = 4\left(\frac{1}{2}mv^2\right)$

|— d —|

• q • $-2q$

m $5m$

Cons of $\vec{p}$

$$0 = mv\hat{x} + 5m\left(\frac{-v}{5}\right)\hat{x}$$

Cons of $\vec{E}$

$$U_i = K_f + U_f$$

$$\frac{(-2q)(q)}{4\pi\epsilon_0 d} = \frac{1}{2}mv^2 + \frac{1}{2}(5m)\left(\frac{v}{5}\right)^2 + \frac{(-2q)(q)}{4\pi\epsilon_0 \frac{d}{2}}$$

• Discrete → Continuous.

$$W = \frac{1}{2}\sum_i q_i \underbrace{V(\vec{r}_i)}_{q_i \text{ does not contribute}} \quad \Big\} \text{ Discrete.}$$

$$W = \frac{1}{2}\int dq\, V \Rightarrow \frac{1}{2}\iiint \rho(\vec{r}) V(\vec{r})\, d\tau \quad \Big\} \text{ Continuous.}$$

• Consider a pt charge.

Using $W = \frac{1}{2}\sum_i q_i V(\vec{r}_i)$

$W = 0$ b.c. $V(\vec{r}_i) = 0$.

Using $W = \frac{1}{2}\iiint \rho V(\vec{r})\, d\tau$

$\rho = q\delta^3(\vec{r})$

$V = \frac{q}{4\pi\epsilon_0 r}$

$$W = \frac{1}{2}\int q\delta(r)\frac{q}{4\pi\epsilon_0 r}4\pi r^2\, dr = \frac{q^2}{2\epsilon_0}\int_0^\infty r\, dr \to \infty$$

• Infinite Classical self energy.

• Most Important Eqn.

$W = \frac{1}{2}\iiint \rho V(\vec{r})\, d\tau$ use $\vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} + f\vec{\nabla}\cdot\vec{A}$ (1)

$W = \frac{\epsilon_0}{2}\iiint_V |E|^2\, d\tau + \oint_{S} V\vec{E}\cdot d\vec{a}$ (2)

boundary of V.

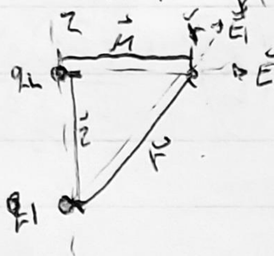
$\lim_{V \to \text{all space}} W = \frac{\epsilon_0}{2}\iiint_{\text{all space}} |E|^2\, d\tau$ (3).

· Problem 2.37.

$$W = \frac{\epsilon_0}{2}\iiint |E|^2 d\tau.$$

$$|\vec{E}| = E$$

Consider 2 pt charge



$$\vec{E}_1 = \frac{q_1}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}_2 = \frac{q_2}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

$$W = \frac{\epsilon_0}{2}\iiint \underbrace{(\vec{E}_1 + \vec{E}_2)\cdot(\vec{E}_1 + \vec{E}_2)}_{(\vec{E}_{TOT})^2} d\tau.$$

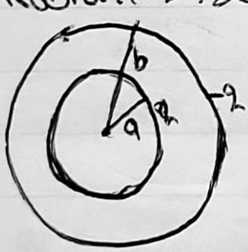
self energy of q_1 — self energy of q_2.

$$W = \cancel{\frac{\epsilon_0}{2}\iiint E_1^2 d\tau}, \cancel{\frac{\epsilon_0}{2}\iiint \vec{E}_2 d\tau} + \underbrace{2\frac{\epsilon_0}{2}\iiint \vec{E}_1\cdot\vec{E}_2 d\tau}_{\text{interaction energy}}$$

$$= \epsilon_0 \iiint \vec{E}_1 \cdot \vec{E}_2 d\tau$$

Answer must $= \frac{q_1 q_2}{4\pi\epsilon_0 \mathscr{r}}$.

· Problem 2.38. Two spherical shells



① $\frac{\epsilon_0}{2}\iiint (\vec{E}_{TOT})^2 d\tau$

② $W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d\tau$

$E_{TOT} \neq 0$, when $\underline{a \le r \le b}$.

$$\vec{E}_{TOT} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$W = \frac{\epsilon_0}{2}\iiint |E_{TOT}|^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int_a^b \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\left[\int_a^b \frac{1}{r^2} dr\right]$$

$$= \frac{1}{2}\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$W = W_1 + W_2 + \epsilon_0 \iiint \vec{E}_1 \cdot \vec{E}_2 d\tau.$$

$$= \frac{\epsilon_0}{2}\int_a^\infty \frac{q^2}{(4\pi\epsilon_0)^2} 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_b^\infty \frac{q^2}{(4\pi\epsilon_0 r^2)^2} 4\pi r^2 dr.$$

$$+ \frac{\epsilon_0}{2}\int_b \frac{-q^2}{(4\pi\epsilon_0 r^2)^2} 4\pi r^2 dr.$$

1 = inner r = a

2 = outer r = b.

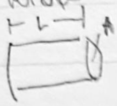
$\vec{E}_1 = 0$, $r < a$

$= \frac{+q}{4\pi\epsilon_0 r}\hat{r}$, $r \ge a$

$\vec{E}_2 = \frac{-q}{4\pi\epsilon_0 r}\hat{r}$, $r \ge b$

$W =$

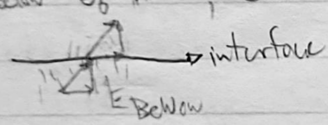
Insulators vs. Conductors, Metals

- Metals have free $\bar{e}$
- Metals obey ohm's law $\Delta V = IR \Rightarrow I = \frac{\Delta V}{R}$ effect, cause, "response function"
-  $R = \rho \frac{L}{A}$ resistivity (material property)

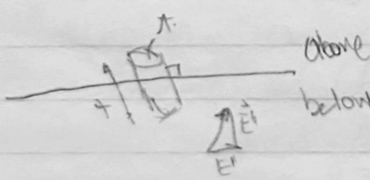
- $\sigma = \frac{1}{\rho}$ * conductivity $\Rightarrow R = \frac{L}{\sigma A} \Rightarrow I = \sigma \frac{\Delta V}{L} A$ or $\frac{I}{A} = \frac{\sigma \Delta V}{L} \Rightarrow \vec{J} = \sigma \vec{E}$
  effect, cause

- Typical Metal: Sodium (Na)
  Solid Neon (Ne)
  Ground state atom config
  $Na (z=11): 1s^2 2s^2 2p^6 3s^1$ ← free e^-
  $Ne (z=10): 1s^2 2s^2 2p^6$

Boundary Conditions

- $E_{above} - E_{below} = \frac{\sigma}{\epsilon_0} \hat{n}$, $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$ } same as above b/c $E = -\nabla V$
- Proof: 

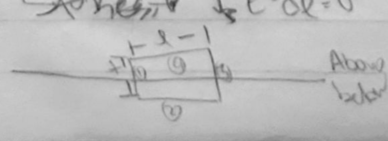
- $\vec{\nabla} \cdot E = \frac{\rho}{\epsilon_0}$ or $\vec{\nabla} \times \vec{E} = 0$
- $\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{inside}}{\epsilon_0}$



On ends $\vec{E} \cdot d\vec{a} = E^{\perp} da$
On curved $\vec{E} \cdot d\vec{a} = E^{\parallel} \cancel{da}$
Consider $\lim_{t \to 0} \oint_s \vec{E} \cdot d\vec{a}$
curved $\oint \vec{E} \cdot d\vec{a}$

$= (E^{\perp}_{above} - E^{\perp}_{below}) A$
RHS $\lim_{t \to 0} = \frac{Q_{inside}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$

- Now Consider $\vec{\nabla} \times \vec{E} = 0$
  $\oint_s \vec{\nabla} \times \vec{E} \cdot da = 0$
  Stokes ⇒ $\oint \vec{E} \cdot d\vec{l} = 0$



(1) & (3) $\vec{E} \cdot d\vec{l} = E^{\perp} dl$
(2) & (4) $\vec{E} \cdot d\vec{l} = E^{\parallel} dl$

$$\lim_{t\to 0} \oint = \oint_2 + \oint_4$$

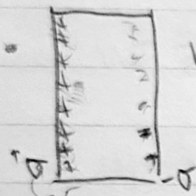
$$= \vec{E}^{\parallel}_{above}\, l - \vec{E}^{\parallel}_{below}\, l$$

$$E^{\parallel}_{above} = E^{\parallel}_{below}$$

Combine both

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

Thought Experiment

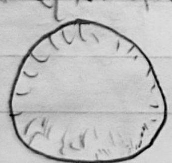


- neutral piece of metal
  - periodic array of ion cores, $[1s^2 2s^2 2p^6]$ in Na
  - as gas of free e⁻

initial current - lasts ~ ns

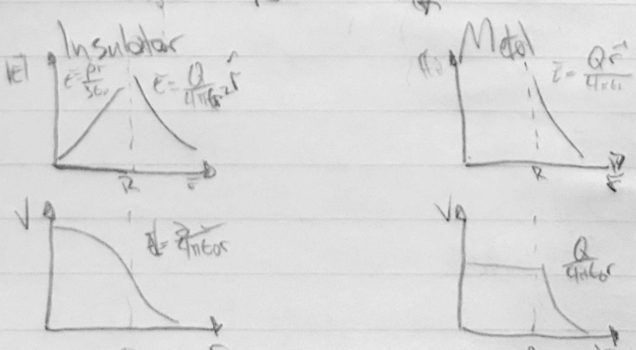
eventually stop $I \to 0$ $\vec{E}_{inside} = 0$

- Dump 10^6 $\bar{e}$ on metal spheres, they will be free (in 3d)
  they will, by mutual repulsion go to surfaces.



- Surface of Metal
  $E = \frac{\sigma}{\epsilon_0}\hat{n}$ all surface

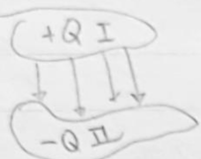
- Comparison: charged metal plate vs uniform charge insulating sphere
  metal: $\sigma = \frac{Q_{TOT}}{4\pi}$
  insulator: $\rho = \frac{3}{4}\frac{Q_{TOT}}{\pi R^3}$



· Problem 2.39.

$\vec{E}_a = \frac{q_a}{4\pi r^2}\hat{r}$
$\vec{E}_b = \frac{q_b}{4\pi r^2}\hat{r}$

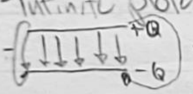
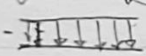
Capacitor

· 

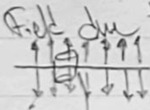
$\Delta V = -\int_I^{II} \vec{E}\, d\vec{l}$

$C \equiv \frac{Q}{\Delta V}$

· Uses: ① Store energy.
② Rectification AC → DC voltage
③ Radio tuner. $\omega_{res} = \frac{1}{\sqrt{LC}}$
④ DRAM memory.

· Parallel Plate Capacitor
- Infinite plate approximation.
-  ← reality.
-  $\vec{E} = 0$ ← approximation.

✱ - Field due to one plate

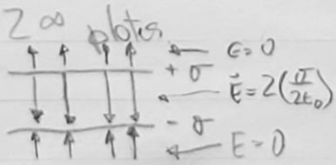


$\sigma = \frac{Q}{A}$ $\oint_S \vec{E}\, d\vec{a} = \frac{Q_{in}}{\epsilon_0}$

$|\vec{E}|$ same | $\vec{E}\cdot d\vec{a} = |\vec{E}|\cdot d\vec{a}$ } ends
$\vec{E}\cdot d\vec{a} = 0$ } curved part

$2|\vec{E}|A = \frac{Q_{in}}{\epsilon_0}$
$|\vec{E}| = \frac{Q}{A}\frac{1}{2\epsilon_0}$
$\vec{E} = \pm\frac{\sigma}{2\epsilon_0}$

- 2 ∞ plates

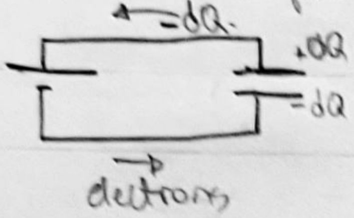


- Calculate, C, for capacitor

$|\Delta V| = \int \vec{E}\cdot d\vec{l}$ → $Q = \sigma A$
Area A
d separation
$= \int \vec{E}\, dl$
$= \frac{\sigma}{\epsilon_0} d$

$C \equiv \frac{Q}{|\Delta V|} = \frac{\sigma A}{\frac{\sigma d}{\epsilon_0}} = \frac{A\epsilon_0}{d}$

Energy Stored in Capacitor

- Start neutral plates 0 energy, $\Delta V = 0$
- 

$\Delta U = (dQ)V$ voltage between plates

$U_{TOT} = \int \Delta U$

$= \int_0^{Q_F} V\,dQ \quad = \frac{Q_F^2}{2C} = \frac{1}{2} C \Delta V^2$

$= \frac{A\epsilon_0}{d} \frac{|E|^2 d^2}{2}$ volume

$= \frac{\epsilon_0 |E|^2}{2} Ad$ energy per volume

Same as $U = \iiint_{\text{all space}} \frac{\epsilon_0 E^2}{2} d\tau$, $\vec{E} = 0$ except for between plates

Chapter 3.

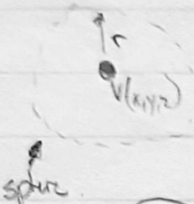
- Laplace's Equation
  - Start with $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

    $\vec{\nabla} \times \vec{E} = 0 \Rightarrow$ implies $\vec{E} = -\vec{\nabla} V$

    substitute $\nabla^2 V = \frac{-\rho}{\epsilon_0}$ ← poissons equation

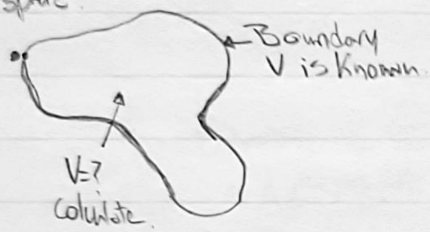
    if $\rho = 0$ then $\nabla^2 V = 0$ ← Laplace

    ↑ partial differential equation.

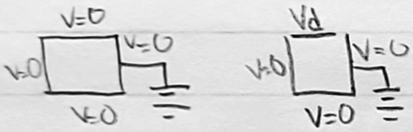


① then $V(x,y,z)$ has no local max or minima.

② $V(x_0, y_0, z_0) = \frac{1}{4\pi R^2} \oint V \, d\vec{a}$



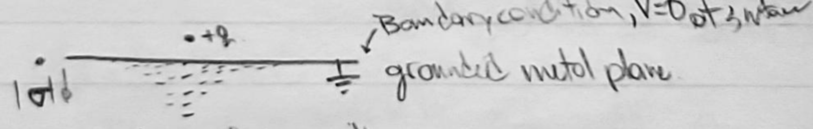
- 2-D Examples



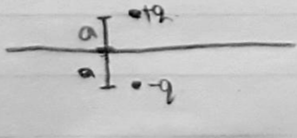
$V(x,y,z)$ is different inside because the boundary conditions determine $V(x,y,z)$

Method of images.

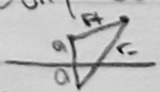
- Artificially recreating boundary conditions.
- Because of uniqueness theorem



Replace this with



The only solution is

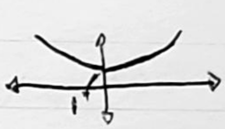
$V =$ 

$= \frac{q}{4\pi\epsilon_0 r_+} - \frac{q}{4\pi\epsilon_0 r_-} = 0$

Review

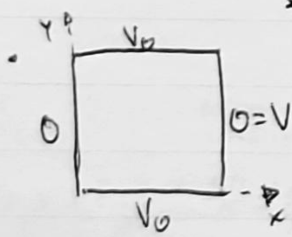
- $\frac{d^2f}{dx^2} = +k^2 f$ | Solution $Ae^{kx} + Be^{-kx}$
- $\frac{d^2f}{dx^2} = -k^2 f$ | $Ae^{ikx} + Be^{-ikx}$

- Note: $Ae^{ikx} + Be^{-ikx} = A[\cos(kx) + i\sin(kx)] + B[\cos(kx) + i\sin(kx)]$.
$= C\cos kx + D i \sin kx$.
Oscillating solutions; multiple zeros.

Alternating $\rightarrow$ • Solutions for $\frac{d^2f}{dy^2} = +k^2 f$.
- Hyperbolic functions - $\sinh(x) = (e^{x} - e^{-x})/2$,  , ODD FXN, Not oscillating
$\frac{d}{dx}(\sinh(x)) = \cosh(x)$
- $\cosh(x) = (e^{x} + e^{-x})/2$,  , EVEN FXN, Not oscillating.
$\frac{d}{dx}(\cosh(x)) = +\sinh(x)$

- Two Solutions to $\frac{d^2f}{dx^2} = +k^2 f$.
  - $Ae^{kx} + Be^{-kx}$
  $C\cosh kx + D\sinh kx$ } are equivalent

2D Problem



$\nabla^2 V = 0$
Assume $V(x,y) = V(x)V(y)$
Soln - ① Separation
② Substitution
③ Division
④ Application of Boundary Conditions

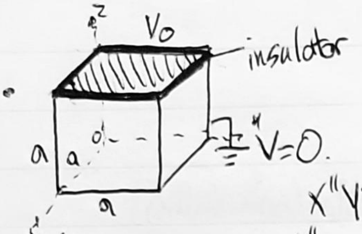
$\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right] XY = 0$

$X''Y + XY'' = 0 \overset{\text{divide by } XY}{\Rightarrow} \frac{X''Y}{XY} + \frac{XY''}{XY} = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$.

$f(x) + g(y) = 0$
$-k^2 + k^2 = 0$.

CHOOSE x to BE ~~ZERO~~ TWO ZEROS, OSCILLATING ZEROS

3D Problem (3.16)



$\nabla^2 V = 0$.

① Assume $V(x,y,z) = X(x)Y(y)Z(z)$

② Substitute $\Rightarrow \nabla^2 V = 0$

$$X''YZ + XY''Z + XYZ'' = 0$$

③ Divide $\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

④ $\frac{X''}{X} = -\alpha^2$, $\frac{Y''}{Y} = -\beta^2$, $\frac{Z''}{Z} = +\alpha^2 + \beta^2$

Boundary Conditions
α { ① $V(x=0) = 0$, ② $V(x=a) = 0$ }
β: ③ $V(y=0) = 0$, ④ $V(y=a) = 0$
⑤ $V(z=0) = 0$
⑥ $V(z=a) = V_0$.

①&② $\rightarrow X(0) = 0$, $X(a) = 0$

③&④ $\rightarrow Y(0) = 0$, $Y(a) = 0$

⑤&⑥ $\rightarrow Z(0) = 0$, $Z(a) = \Sigma$

Find α, $X'' = -\alpha^2 X$.

$$X = A\cos(\alpha x) + B\sin(\alpha x)$$

$$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0.$$

$$X(a) = 0 = B\sin(\alpha a) \Rightarrow \alpha a = m\pi \Rightarrow \alpha = \frac{m\pi}{a} \quad m = 1,2,3,\dots$$

Find β $Y'' = -\beta^2 Y$

$$Y = A\cos(\beta x) + B\sin(\beta x)$$

$$Y(0) = 0 = A(1) + B(0) \Rightarrow A = 0.$$

$$Y(a) = 0 = B\sin(\beta a) \Rightarrow \beta a = n\pi \Rightarrow \beta = \frac{n\pi}{a}, \quad n = 1,2,3,4\dots$$

$$\frac{Z''}{Z} = +\alpha^2 + \beta^2 = \gamma^2$$

$$\gamma^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2}$$

$$Z = Ae^{\gamma z} + Be^{-\gamma z}$$

$$Z(0) = 0 = A + B \Rightarrow A = -B$$

$$\cancel{Z(a) = 0 = Ae^{\gamma a} + Be^{-\gamma a}}$$

$$Z = C\sinh(\gamma z) \Rightarrow \gamma = \gamma_{mn} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{a^2}} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

- At this point, we have an infinite set of solutions

$$V(x,y,z) = \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh(\gamma_{mn} z), \quad \gamma_{mn} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$\therefore V(x,y,z) = \sum_m \sum_n \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sinh(\gamma_{mn} z)C_{mn}.$$

- Apply B.C. ⑥ and orthogonality to find C_{mn}, when $z = a$, $V = V_0$

- $V_0 = \sum_m \sum_n \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh(\gamma_{mn} a) C_{mn}$

Kronecker delta.

- Recall $\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m'\pi x}{a}\right) dx = \delta_{mm'} \frac{a}{2}$ $\qquad \delta_{mm'} = 1$ if $m = m'$, 0 otherwise
- multiply by $\sin\left(\frac{m'\pi x}{a}\right)$ & $\int_0^a dx$.
  & $\sin\left(\frac{n'\pi y}{a}\right)$ & $\int_0^a dy$.

$$V \int_0^a \sin\left(\frac{m'\pi x}{a}\right) dx \int_0^a \sin\left(\frac{n'\pi y}{a}\right) dy = \sum_m \sum_n C_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m'\pi x}{a}\right) dx \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy \; \sinh(\gamma_{mn} a)$$

- Because of orthogonality, only one of infinite number of terms survives: $m = m'$, $n = n'$
- RHS $\Rightarrow C_{mn} \left(\frac{a}{2}\right)\left(\frac{a}{2}\right) \sinh(\gamma_{mn} a)$.

LHS $\Rightarrow$ Consider $\int_0^a \sin\left(\frac{m\pi x}{a}\right) dx.$, $u = \frac{m\pi x}{a}$. $dx = \frac{a}{m\pi} du$.

$= \frac{a}{m\pi} \int_0^{m\pi} \sin u \, du$, if m is even, $= 0$

if m is odd $\frac{a}{m\pi} \int_0^{\pi} \sin(u) du = \frac{2a}{m\pi}$.

LHS $= V_0 \left[\frac{2a}{m\pi}\right]\left[\frac{2a}{n\pi}\right]$, if m, n are odd, 0 otherwise.

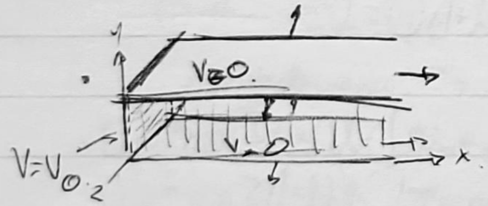
RHS $= C_{mn} \left(\frac{a}{2}\right)\left(\frac{a}{2}\right) \sinh(\gamma_{mn} a)$

$$C_{mn} = \frac{16 V_0}{\pi^2 m n} \frac{1}{\sinh(\gamma_{mn} a)}.$$

- Final Solution:

$$V(x,y,z) = \frac{16 V_0}{\pi^2} \sum_m \sum_n \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \frac{\sinh(\gamma_{mn} z)}{\sinh(\gamma_{mn} a)}, \quad \gamma_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$$

Problem 3.14 (Infinite Slot).



$$V(x,y) = \frac{4V_0}{\pi} \sum_n \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a}.$$

- Q: What is σ (surface charge density) on the side where $V = V_0$?

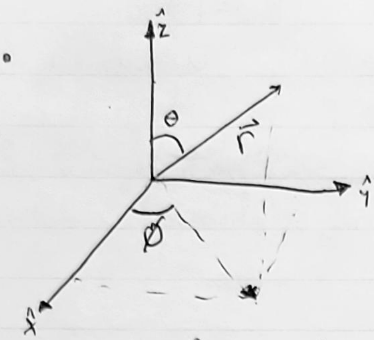
A: Its metal, $\vec{E} \perp$ surface $\therefore$, $\sigma = \epsilon_0 \left(\frac{\partial V}{\partial x}\right)$.

$|\vec{E}| = \frac{\sigma}{\epsilon_0}$

Arfken - Mathematical Method for Physicists

Boas - Mathematical Methods for Physical Sciences.

Separation of Variables in Spherical Coordinates.



Legendre's D.E.

$(1-x^2)\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial x}{\partial x} + l(l+1) = 0.$

series solution.

→ gives Legendre polynomial, $P_l(x)$

$l=0,\ P_0(x) = 0$

$l=1,\ P_1(x) = x$

$l=2,\ P_2(x) = \frac{3x^2-1}{2}$ ODD FXN

$l=3,\ P_3(x) = \frac{5x^3-3x}{2}$

$l=4,\ P_4(x) = \frac{35x^4 - x^2 -}{2}$ EVEN FXN.

Useful Properties

- Orthogonality ①

$\int_{-1}^{1} P_l(x) P_{l'}(x)dx = \left[\frac{2}{2l+1}\right]\delta_{ll'}$

- Rodrigues Formula. ②

$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$

- Separation of $\nabla^2 V(r,\theta,\phi) = 0$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

- By azimuthal symmetry $\frac{\partial^2 V}{\partial\phi^2}$ term $= 0$.

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] = 0.$$

- Assume $V(r,\theta) = R(r)\Theta(\theta)$

$$= \frac{\Theta}{r^2}\frac{\partial}{\partial r}(r^2 R') + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\,\Theta'] = 0.$$

- Multiply by $\frac{r^2}{R\Theta}$

$$= \frac{1}{R}\frac{\partial}{\partial r}(r R') + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\,\Theta'] = 0.$$

- Set the terms equal to a constant.

$$\frac{1}{R}\frac{\partial}{\partial r}(r R') = l(l+1) \quad , \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\,\Theta'] = -l(l+1).$$

- 2 ODEs to solve

① $r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$

② $\frac{1}{\sin\theta}\left[\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] + l(l+1)\Theta = 0$

- Look for solutions of $R = \sum_{m=0}^{\infty} a_m r^m$ and substitute.

① $r^2\frac{d^2}{dr^2}\left[\sum_{m=0}^{\infty} a_m r^m\right] + 2r\frac{d}{dr}\left[\sum_{m=0}^{\infty} a_m r^m\right] - l(l+1)\sum_{m=0}^{\infty} a_m r^m = 0.$

$\Rightarrow r^2\sum_m m(m-1)a_m r^{r-2} + 2r\sum_m m a_m r^{m-1} - l(l+1)\sum_m a_m r^m = 0$

$\Rightarrow \sum_m [m(m-1) + 2m - l(l+1)]a_m r^m = 0.$

$\Rightarrow \sum_m [m(m+1) - l(l+1)]a_m r^m = 0$, either $m=l$ or $m=l+1)$

$\Rightarrow R = A_l r^l + \frac{B}{r^{l+1}}$

- Demonstrate that ② is Legendre's Equation.
  - $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0.$
  - same as $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)y = 0.$
  - Change variables, $x = \cos\theta$, and use chain rule.

$$\frac{d\theta}{dx}\frac{d}{d\theta}\left[(1-\cos^2\theta)\frac{dy}{d\theta}\frac{d\theta}{dx}\right] + l(l+1)y = 0.$$

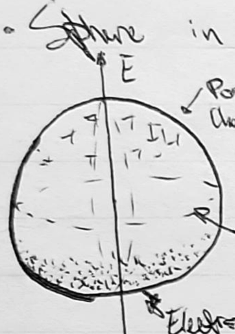
- if $x = \cos\theta$, $\frac{dx}{d\theta} = -\sin\theta$, & $\frac{d\theta}{dx} = -\frac{1}{\sin\theta}$, also $(1-\cos^2\theta) = \sin^2\theta$
- $-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta\left[-\frac{1}{\sin\theta}\right]\frac{dy}{d\theta}\right] + l(l+1)\,y = 0$
= $\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dy}{d\theta}\right] + l(l+1)\,y = 0$
- This is equation 2.

- Putting it all together
  - $V(r,\theta) = R(r)\,\Theta(\theta) = \sum_l \left[A_l r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta)$.

    # of solns

Example 3.8.

- ~~Sphere~~ Sphere in constant $\vec{E} = \underbrace{E_0 \hat{z}}$
  Creates azimuthal symmetry.



$\vec{F} = q\vec{E}$, separates charge.

$\vec{E}=0$, ∴ V = constant because $\vec{E} = -\vec{\nabla} V$.

Assume $V=0$.

- We know general solution.
  $V(r,\theta) = \sum_l \left[A_l r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta)$
- Need boundary conditions.
  1. $V(r=R) = 0$.
  2. $V(r \gg R) = -E_0 z$, since $\vec{E} = E_0\hat{z} = -\vec{\nabla} V$
     → In spherical → $= -E_0 r\cos\theta$.
- Apply Boundary Conditions to general solution.
  1. - Bc $V(r=R) = 0$
     - General $\sum_l \underbrace{\left[A_l R^l + \frac{B_l}{R^{l+1}}\right]}_{\text{must be zero,}} P_l(\cos\theta) = 0$ ↑ is not zero everywhere.
     - $B_l = -A_l R^{2l+1}$
  2. - General $\sum_l \left[A_l r^l - \frac{A_l R^{2l+1}}{r^{l+1}}\right] P_l(\cos\theta)$. when $r \gg R$ second term goes to zero.
     - $\sum_l A_l r^l P_l(\cos\theta) = \underbrace{-E_0 r\cos\theta}_{\text{only } l=1 \text{ term involved}}$.
     = ∴ $A_1 = -E_0$.
- Final solution.
  $V(r,\theta) = E_0 r\cos\theta - E_0 \frac{R^3}{r^2}\cos\theta$.

Beginning of 3.19.

- Assume $V = k\cos 3\theta$ on surface of a sphere, radius R, Find $V(r,\theta)$ everywhere $r<R$, $r>R$.
- Inside, $r<R$.
  - We know; $V(r,\theta) = \sum\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta)$
  - B.C. ① at $r=R$, $V = k\cos 3\theta$.
    - ② at $r=0$, Not infinite. it is average V on sphere.
    - $\therefore$ $B_\ell = 0$ inside.

For $\cos 3\theta$
$= \mathrm{Re}(e^{i3\theta})$
$= \mathrm{Re}[(\cos\theta + i\sin\theta)^3]$
$= 4\cos^3\theta - 3\cos\theta.$ } tells us about gen sol.

$P_1 = \cos\theta$
$P_3 = \frac{5\cos^3\theta - 3\cos\theta}{2}.$

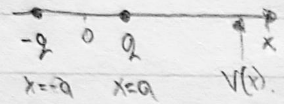
Hence only P_1, P_3 contribute.

Multipole Expansion of the Scalar Potential

- Eg. Use binomial approximation get same result as multipole.

  $(1+x)^n$, $x \ll 1$

  Taylor $(1+x)^n \simeq 1 + nx + \frac{n(n-1)x^2}{2!} + \dots$



$V = \sum \frac{q_i}{4\pi\epsilon_0 r_i}$

$= \frac{q}{4\pi\epsilon_0(x-a)} - \frac{q}{4\pi\epsilon_0(x+a)} = \frac{q}{4\pi\epsilon_0 x}\left[\frac{1}{(1-\frac{a}{x})} - \frac{1}{(1+\frac{a}{x})}\right]$

limit as $x \gg a$, then $\frac{a}{x}$ is small

$\frac{1}{(1-\frac{a}{x})} = (1-\frac{a}{x})^{-1} \simeq 1 + \frac{a}{x}$

$\frac{1}{(1+\frac{a}{x})} \simeq 1 - \frac{a}{x}$

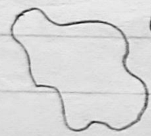
$V(x) \simeq \frac{q}{4\pi\epsilon_0 x}\left[(1+\frac{a}{x}) - (1-\frac{a}{x})\right] \simeq \frac{2qa}{4\pi\epsilon_0 x^2}$

- Compare to multipole expansion

  $V(\vec{r}) \simeq \frac{Q_{TOT}}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} + \dots$

  $Q_{TOT} = \text{Monopole} = q + -q = 0$

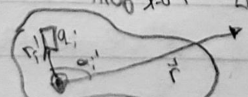
  $\vec{p} = \sum q_i \vec{r}_i' = q(a\hat{x}) + (-q)(-a\hat{x}) = 2qa\hat{x}$

- 
  complicated charge distribution $\Big\}$ Characteristics
  ① Q_{TOT}
  ② $\vec{p} = \sum q_i \vec{r}_i'$

$\hat{r} = \hat{x}$

- Multipole Result $= \frac{0}{4\pi\epsilon_0 x} + \frac{2qa\hat{x}\cdot\hat{x}}{4\pi\epsilon_0 x^2}$, same as binomial
- ③ Third characteristic is quadrupole (3x3) tensor.
- Calculating $\vec{p}$
  - $\vec{p} = \sum q_i \vec{r}_i'$ Discrete charge

    $\vec{p} = \iiint \vec{r}' \rho(\vec{r}') d\tau'$ Continuous Charge

    $\frac{C}{m^3}$ $m^3 \Rightarrow dq$ essentially
- Griffiths - Spherical Coordinates
  - $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\left[\sum q_i r_i'^{\,l} P_l(\cos\alpha)\right]$
  - 

  - One can show that this is identical to

    $V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} + \underbrace{\frac{1}{4\pi\epsilon_0 r^3}\hat{r}Q\hat{r}}_{\text{Quadrupole}} + \dots$

    row column

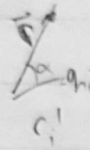
• DEMO that spherical & cartesian expressions are the same.

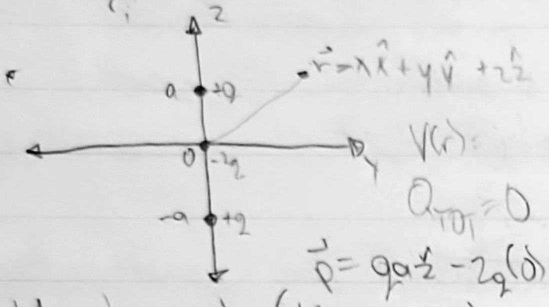
$\ell=0$ $\frac{1}{4\pi\epsilon_0}\frac{1}{r^{0+1}}\sum_i q_i (r_i')^0 \underbrace{P_0(\cos\alpha_i')}_{=1} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_i q_i = \frac{Q_{TOT}}{4\pi\epsilon_0 r}$ ← monopole

$P_1(\cos\alpha_i') = \cos\alpha_i'$

$\ell=1$ $\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\sum_i q_i (r_i')^1 P_1(\cos\alpha_i') = \frac{1}{4\pi\epsilon_0 r^2}\sum q_i (r_i')^2 \cos\alpha_i'$, $\cos\alpha_i' = \frac{\vec r \cdot \vec r'}{|\vec r|}$

$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\underbrace{\left[\sum q_i \vec r_i'\right]}_{\vec p} \cdot \left(\frac{\vec r}{r}\right)$





$V(\vec r) =$

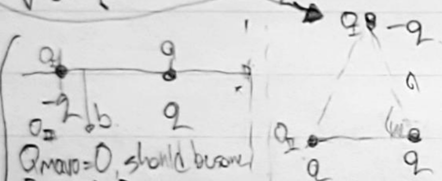
$Q_{TOT} = 0$

$\vec p = qa\hat z - 2q(0) + q(-a\hat z) = 0$

• Use binomial (keep 3 terms)

$V \propto \frac{1}{r^3}$

• Ex



$Q_{mono} = 0$, should be same for $\vec p = 0$

Use the diagrams to demonstrate the $\vec p$ is independent of origin only if monopole moment $= 0$

O_I

$\vec p = -q(0) + q(a\hat x) \Rightarrow \vec p_I = qa\hat x$

O_{II}

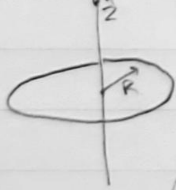
$\vec p = -q(b\hat y) + q(a\hat x + b\hat y) \Rightarrow \vec p_{II} = qa\hat x$

$\cos 30 = \frac{\sqrt3}{2}$

O_I: $\vec p_I = (-q(0) + q(-a\sin 30\hat x - a\cos 30\hat y) + q(a\sin 30\hat x - a\cos 30\hat y)) = -2qa\frac{\sqrt3}{2}\hat y$

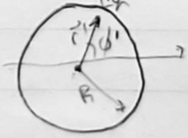
O_{II}: $\vec p_{II} = -q[a\cos 60\hat x + a\sin 60\hat y] + q(0) + q(a\hat x) \neq \vec p_I$

• Problem 3.28 Uniform ring of charge.



$Q_{mono} = Q_{TOT} = \lambda[2\pi R]$

$\vec p = 0$ by symmetry.



$\vec p = \int dq\, \vec r'$

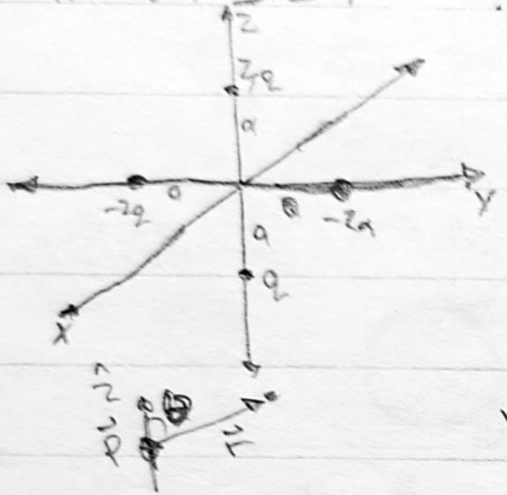
$dq = \lambda d\ell = \lambda R\, d\phi'$ $\vec r' = R\cos\phi'\hat x + R\sin\phi'\hat y$

$\vec p = \int_0^{2\pi}[\lambda R\, d\phi'][R\cos\phi'\hat x + R\sin\phi'\hat y]$

$= \lambda R^2\left[\hat x\int_0^{2\pi}\cos\phi' d\phi \overset{0}{} + \hat y\int_0^{2\pi}\sin\phi' d\phi\right] \overset{0}{}$

$= 0$

• Problem 3.29. Find $V(\vec{r})$ in spherical coords



$Q_{TOT} = 0$

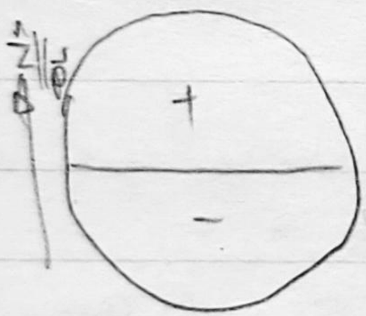
$\therefore V(\vec{r}) \cong V_{dip} = \frac{1}{4\pi\epsilon_0 r^2} \vec{p}\cdot\hat{r}$

$\vec{p} = -2q(a\hat{x}) + (-2q)(-a\hat{x}) + 3q(a\hat{z}) + q(-a\hat{z})$

$= 2qa\hat{z}$

$V \cong \frac{1}{4\pi\epsilon_0} \frac{2qa\hat{z}\cdot\hat{r}}{r^2} = \frac{2qa\cos\theta}{4\pi\epsilon_0 r^2}$

• Problem 3.27 Sphere (non-uniform)



$\rho(\vec{r}) = \frac{kR}{r^2}(R-2r)\sin\theta$

Intuition: $Q_{MONO} = 0$

$\vec{p} = \iiint (x'\hat{x} + y'\hat{y} + z'\hat{z})\rho\, d\tau$

$= \iiint z'\rho\, d\tau\ \hat{z} = \hat{z}\iiint \cancel{r\cos\theta}^{\,0}\, \rho\, r^2 \sin\theta\, d\theta\, dr\, d\phi$

$\vec{p} = 0$

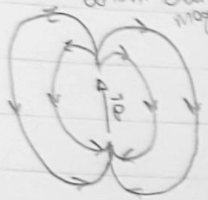
Approximate $\vec{E}$ Produced by Arbitrary Charge Distribution

-  $\rho(\vec{r})$ produces $V(\vec{r})$

  $V(\vec{r}) \simeq V_{mono}(\vec{r}) + V_{dip}(\vec{r}) + V_{quad}(\vec{r}) + \dots$

  $\simeq \frac{Q_{TOT}}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} + \dots$

The electric dipole looks like bar magnet



- By definition, $\vec{E} = -\vec{\nabla} V(\vec{r})$

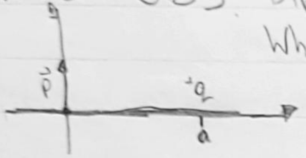
  $\vec{E}_{mono} = -\nabla \frac{Q_{TOT}}{4\pi\epsilon_0 r} = \frac{Q_{TOT}}{4\pi\epsilon_0 r^2}$

  $\vec{E}_{dip} = -\vec{\nabla}\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = -\vec{\nabla}\left[\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right]$ choose $\hat{z}$ such that $\vec{p} \parallel \hat{z}$

  recall $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

  $\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$

- Problem 3.33 - Dipole $p\hat{z}$ at origin, charge q at $(a,0,0)$. What is the force on q?



$\vec{F} = q\vec{E}_{dip}$, $r = a$ @ q, $\theta = 90°$ @ q

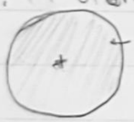
$\hat{r} = \hat{x}$, $\hat{\theta} = -\hat{z}$

$\vec{F} = \frac{\vec{p}}{4\pi\epsilon_0 a^3}\left[2\cos(90)\hat{x} + \sin(90)(-\hat{z})\right] q$

$= -\frac{\vec{p}\,q}{4\pi\epsilon_0 a^3}\hat{z}$

Insulators in Presence of $\vec{E}$

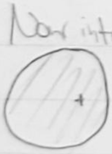
- In contrast to metals, $\vec{E} \neq 0$ inside. Inside insulators we can have permanent dipoles. (H_2O molecule) or "introduced" dipoles
- Eg. H-atom



spherical symmetric charge cloud $\quad \vec{p} = 0$

$\vec{p} \equiv \int \vec{r}\,dq = \iiint \vec{r}'\rho(\vec{r}')\,d\tau' = 0$

Now introduce $\vec{E}$



$\rightarrow E_{x1}$ $\quad$ → proton $\quad$ ← electron

$\longrightarrow \vec{p}$ (induced by field)

$\vec{p}_{induced} = \alpha\vec{E}$, α called ~~polarizability~~ polarizability

$\vec{p}_{induced} \ll$ permanent $\vec{p}$

Debye = 1D

$1D = 3.34 \times 10^{-30}$ Cm

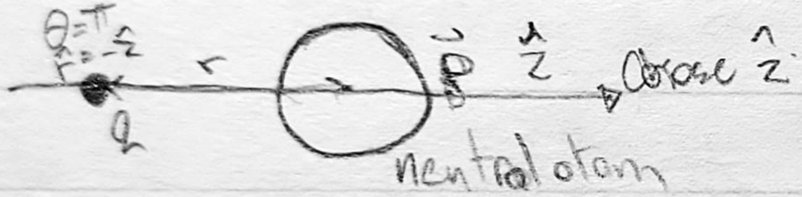
	p
HF	$= 1.73D$
H_2O	$= 1.83D$
HCl	$= 1.04D$

• Coordinate free form of E_{dipole}

$$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right], \text{ true always.}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right], \text{ only true when } \vec{p} \parallel \hat{z}$$

• Problem 4.4 - What is the force from atom and q?



$$\vec{p} = \frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}\hat{z}$$

$$\vec{E}_{\substack{induced\\dipole}} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\overset{-1}{\cos(\pi)}(-\hat{z}) + \overset{0}{\sin(\pi)}\hat{\theta}\right]$$

$$= \frac{p}{2\pi\epsilon_0 r^3}\hat{z}$$

$$\vec{F}_{on\,Q} = q\left[\frac{\alpha q}{4\pi\epsilon_0 r^2}\right]\left[\frac{1}{2\pi\epsilon_0 r^3}\right]\hat{z}$$

$$= \frac{\alpha q^2}{8\pi^2\epsilon_0^2 r^5}\hat{z}$$

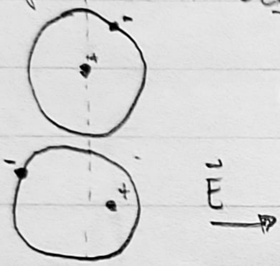
Review.

- Induced vs. Permanent Dipoles
- $\vec{p} = \iiint \vec{r}\,' \rho(\vec{r}) d^3\vec{r}\,'$
- N_2: $\vec{p} = 0$, H_2O: $\vec{p}\downarrow$ $\delta-$ $\delta+$

} Permanent Dipoles.

- $\vec{p} = 0$ if $\vec{E}_{EXTERNAL} = 0$.
  $\vec{p} \approx \alpha\vec{E}$ if $\vec{E}_{ext} \neq 0$

} induced.

- 

} "stretching" of an atom.

$q\vec{E}_{INT}$ ⟵ ⟶ $q\vec{E}_{EXT}$

Response of Dipoles to $\vec{E}_{EXT}$.

- Torque. $\vec{F} = +q\vec{E}$, $+q$, $\vec{d}$, $-q$, $F = -q\vec{E}$, $\vec{E}\uparrow$

$\vec{p} = q\vec{d}$ $\quad \overbrace{\vec{F}_{NET} = 0 = q\vec{E} - q\vec{E}}^{\text{uniform field.}}$

$\vec{N} = \vec{r} \times \vec{F}$

$\vec{r} = \frac{\vec{d}}{2}, -\frac{\vec{d}}{2}$ $\quad \Sigma\vec{N}_i = \frac{\vec{d}}{2} \times q\vec{E} + \left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})$.

$= q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$

- Net force on $\vec{p}$ in non-uniform field

$-q$ —— $\vec{d}$ —— $+q$ ⟶ $\vec{E}(\vec{r} + \Delta\vec{r})$

⟶ $\vec{E}(\vec{r})$

$\vec{F}_{NET} = q[\vec{E}(\vec{r} + \Delta\vec{r}) - E(\vec{r})]$

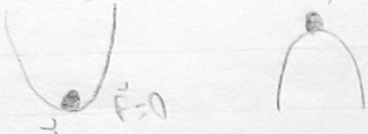
$= q\Delta E = q(\vec{d} \cdot \vec{\nabla}\vec{E}) = [\vec{p} \cdot \vec{\nabla}]\vec{E}$

Polarization, ($\vec{P} \neq \vec{p}_{dipole}$)

- $\vec{P} = \sum_i \vec{p}_i / V$, average dipole moment per volume

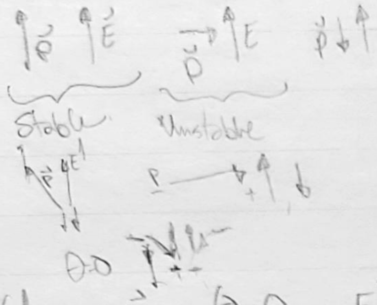
Torque in Terms of Energy

• Stable vs. Unstable equilibrium



$\vec F=0$ $\quad$ $F=0$

$\vec U_{min} = PE$ $\quad$ $U_{max}=P_E$



• $U=-\vec p\cdot\vec E$ $\quad$ $\theta=0,\ -pE$

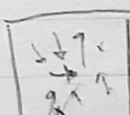
$\theta=\frac{\pi}{2},\ U=0$

$\theta=\frac{\pi}{2},\ U=0$

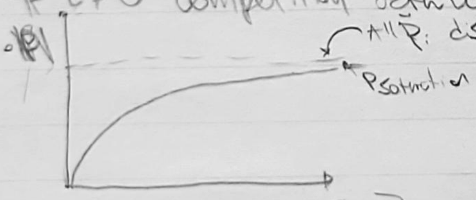
Curie's Law

• Material (liquid) composed of identical polar molecules of moment p.

• $\vec P$ if $\vec E_{EXT}=0$

• Snapshot in time  $\langle\vec P\rangle=0$, random thermal energy to disorder

• If $\vec E\neq 0$ competition between $\vec N=\vec p\times\vec E$ (order), k_BT (dis)



if N molecules

V volume

$P_{SAT}=\frac{N}{V}|\vec p|=\frac{Np}{V}$

if $\vec E_{EXT}\parallel\hat z$, then it can be shown that $\langle p_z\rangle = p\left[-\frac{1}{a}+\coth a\right]$, $a=\frac{p|\vec E|}{k_BT}$, ratio is competition between order & disorder.

For small a $\quad \coth \approx \frac{1}{a}+\frac{a}{3}-\frac{a^3}{45}$ $\qquad P=\frac{N}{V}\langle p_z\rangle=\frac{Np}{V}\left(\frac{p|\vec E|}{3k_BT}\right)=\left(\frac{Np^2}{3Vk_BT}\right)|\vec E|$

$\langle p_z\rangle\approx\frac{pa}{3}=\frac{p^2|\vec E|}{3k_BT}$ $\qquad \vec P=\chi\vec E$

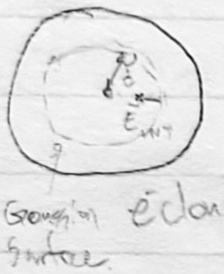
as T gets small, χ = susceptibility, χ gets large

Problem 4.2

- Given, that in an H-atom $\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$, Calculate α where $\vec{p} = \alpha \vec{E}$.

Equilibrium $q\vec{E}_{INT}$ $q\vec{E}_{EXT}$

Proton



$$\oint \vec{E}\, d\vec{a} = \frac{Q_{INT}}{\epsilon_0}, \quad |\vec{E}_{IN}| 4\pi d^2 = \iiint \frac{1}{\epsilon_0}\rho(\vec{r}')\, d\tau$$

$$= \frac{1}{\epsilon_0}\iiint \rho(\vec{r}')\, d\tau$$

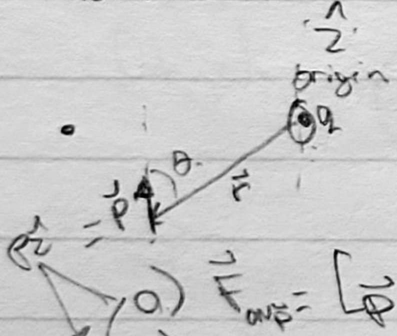
$$= \frac{q}{\epsilon_0 \pi a^3}\int_0^d e^{-\frac{2r'}{a}} \underbrace{4\pi r'^2 dr}_{d\tau}, \quad \frac{d}{a} \ll 1 \Rightarrow e^{-2r'/a} \simeq 1$$

for whole vol.

$$|\vec{E}_{in}| 4\pi d^2 = \frac{4q}{\epsilon_0 \pi a^3}\int_0^{d} dr'^2\, dr'$$

$$|\vec{E}_{N}| = \frac{1}{3\pi\epsilon_0 a^3}\underbrace{qd}_{p} \Rightarrow \underbrace{3\pi\epsilon_0 a^3}_{\alpha}|\vec{E}| = p.$$

Problem 9.9.



- Calculate (a) $\vec{F}$ on $\vec{p}$, and (b) $\vec{F}$ on q } MUST BE EQUAL & OPPOSITE

a) $\vec{F}_{on\,p} = [\vec{p}\cdot\vec{\nabla}]\vec{E}_q$ $\qquad E_q = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{q\vec{r}}{4\pi\epsilon_0 r^3}$

b) $\vec{F}_{on\,q} = q\vec{E}_{dip}$

$$\vec{F}_{on\,p} = p\frac{\partial}{\partial z}\left[\frac{q}{4\pi\epsilon_0}\frac{[x\hat{x}+y\hat{y}+z\hat{z}]}{[x^2+y^2+z^2]^{\frac{3}{2}}}\right]$$

$$= \frac{pq}{4\pi\epsilon_0}\left[-\frac{3}{2}\left(\frac{2z\,x\,\hat{x}}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) - \frac{3}{2}\frac{(2z)y\hat{y}}{r^5} - \frac{3}{2}\frac{(2z)z\hat{z}}{r^5} + \frac{\hat{z}}{r^3}\right]$$

$$= \frac{pq}{4\pi\epsilon_0}\left[-\frac{3z\vec{r}}{r^5} + \frac{\hat{z}}{r^3}\right]$$

Field Produced by Polarized Object.

- $\vec{P} = \frac{\sum \vec{p}_i}{V} \neq 0$
- Viewing material as a collection of dipoles.
- Single $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \vec{p} \cdot \hat{r}$
- $d\vec{p} = \vec{P}\, d\tau \Rightarrow V = \iiint \frac{1}{4\pi\epsilon_0 r^2} \vec{P} \cdot \hat{r}\, d\tau$
- Can replace set of dipoles by charges:

  $\sigma_b = \vec{P} \cdot \hat{n}$ [only at surface]

  $\rho_b = -\vec{\nabla} \cdot \vec{P}$

For non-zero $\vec{P}$ to exist either ① if E_{ext} must have set of permanent dipoles = Electret ② if no permanent dipoles, then $E_{ext} \neq 0$

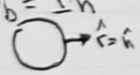
Problem 4.14

- Prove that bound charge = zero.
- Object -

$$Q^{B}_{TOT} = \oint \sigma_b\, da + \iiint \rho_b\, d\tau$$
$$= \oint \vec{P} \cdot \hat{n}\, da + \iiint -\nabla \cdot \vec{P}\, d\tau$$
$$= \oint \vec{P} \cdot \hat{n}\, da - \iiint \vec{P} \cdot \hat{n}\, da$$
$$= 0$$

Problem 4.10

- Sphere of radius, R, in which $\vec{P} = k\vec{r}$, non-uniform
- Find σ_b, ρ_b, & $\vec{E}$ everywhere
- A. $\sigma_b = \vec{P} \cdot \hat{n}$



  B. $\rho_B = -\nabla \cdot \vec{P} = -\vec{\nabla} \cdot [k(x\hat{x} + y\hat{y} + z\hat{z})] = -3k$ (uniform)

- E_{inside}

uniform $\rho_b = -3k$.

Use Gauss Law - $\vec{E} \parallel \hat{r}$ (symmetry).

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{inside}}{\epsilon_0} \qquad |\vec{E}| 4\pi r^2 = \frac{1}{\epsilon_0} \int_0^r$$

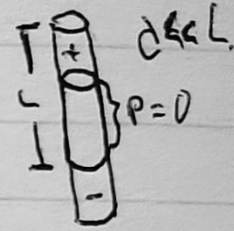
$$|\vec{E}| 4\pi r^2 = -3k(4\pi) \frac{r^3}{3\epsilon_0} \Rightarrow |\vec{E}_{in}| = -\frac{kr\hat{r}}{\epsilon_0} = -\frac{\vec{P}}{\epsilon_0}$$

$E_{OUTSIDE} = 0$

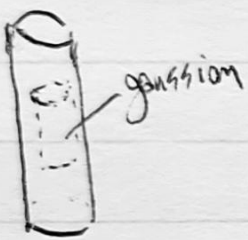
Problem 4.13

- Given cylinder, $\vec{P}$ uniform $\perp \hat{z}$ (not saying $\vec{P} \parallel \hat{s}$)

- Insight: Pretend $\vec{P} \parallel \hat{s}$ looks like dipoles with charged ends.
  - Equivalent to words two cylinders displaced by $\vec{d} = d\hat{z}$
- Return to $\vec{P} \perp \hat{z}$
- Use infinite cylinder approximation



- Uniform $\vec{P}$, only surface charges.



$\vec{E} \parallel \hat{s}$

$$\oint \vec{E} \cdot d\vec{a} = Q_{enc}/\epsilon_0$$

$$|E|(2\pi s L) = \frac{\pi s^2 L \rho}{\epsilon_0} \Rightarrow \vec{E} = \frac{\rho s}{2\epsilon_0}\hat{s} \quad \text{(inside)}$$

$$E_{TOT} = \vec{E}_{+cyc} + \vec{E}_{-cyc}$$

Electric Displacement ($\vec{D}$)

is not metallic free charges $\rho_b = -\vec{\nabla}\cdot\vec{P}$

- Divide total charge density into $\rho = \rho_f + \rho_b$
- Griffiths says ρ_f is any charge that isn't ρ_b
- Rather ρ_f is
- Consequences: for electrostatics

$$-\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = 0$$

$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho$$

$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho_f + \rho_b$$

$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho_f - \vec{\nabla}\cdot\vec{P}$$

$$\vec{\nabla}\cdot\underbrace{(\epsilon_0\vec{E} + \vec{P})}_{\text{Defines } \vec{D}} = \rho_f$$

Defines $\vec{D}$ ← what we have control over

- $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ → for point charge at origin $\rho = q\delta(\vec{r})$

$$\iiint \vec{\nabla}\cdot\vec{E}\,d\tau = \frac{1}{\epsilon_0}\iiint q\delta(\vec{r})$$

$$\oiint \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$|\vec{E}|4\pi r^2 = \frac{q}{\epsilon_0} \quad \leftarrow \text{Coulombs Law}$$

$$|\vec{E}| = \frac{q}{4\pi r^2\epsilon_0}$$

- For $\vec{D}$ $\quad \vec{\nabla}\cdot\vec{D} = \rho_f$

but $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\left[\epsilon_0\vec{E} + \vec{P}\right]$

$$\vec{\nabla}\times\vec{D} = \epsilon_0\overbrace{\vec{\nabla}\times\vec{E}}^{=0} + \vec{\nabla}\times\vec{P}$$

Two sources of $\vec{D}$

(i) ρ_f

(ii) $\vec{\nabla}\times\vec{P}$

- Recall: $\vec{\nabla}\cdot\vec{M}$ scalar $\qquad \frac{\partial M_y}{\partial y} \neq 0$

  - $\frac{\partial M_x}{\partial x}, \frac{\partial M_y}{\partial y}, \frac{\partial M_z}{\partial z}$

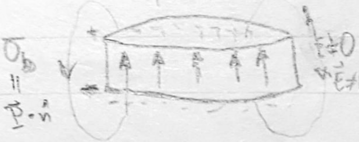
  $\vec{\nabla}\times\vec{M}$ contains

  - $\frac{\partial M_x}{\partial y}, \frac{\partial M_x}{\partial z}$, etc. $\qquad \vec{\nabla}\times\vec{M} \neq 0$

- Example where $\rho_f = 0$, but $\vec{D} \neq 0$ ← $\vec{P}$

• Ex. where $\rho_f = 0$ but $\vec{D} \neq 0$

Hockey puck with permanent $\vec{P}$

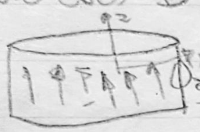


$\vec{P} = P_z \hat{z}$ inside

$\therefore \vec{D} = \epsilon_0 \vec{E} + 0$

$\vec{D} \neq 0$

Where does $\vec{D}$ come from?

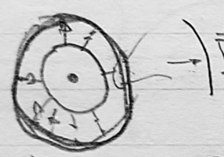


$\frac{\partial P_z}{\partial x} \neq 0$. $\therefore \vec{\nabla} \times \vec{P} \neq 0$

Comes from the boundary.

• 4.15 Thick shell (spherical)

$\vec{P} = \frac{k}{r} \hat{r}$, for $a < r < b$



$\vec{P} = 0$

$\frac{\partial P_r}{\partial r} \neq 0$ ~~but~~ not curl

$\frac{\partial P_r}{\partial \theta}, \frac{\partial P_r}{\partial \phi} \overset{\text{both}}{=} 0$.

• Find $\vec{E}$ ~~and~~ anywhere two ways

① Find $\vec{D}$, then $\vec{E}$

② Find σ_b, then calculate $\vec{E}$

• ~~Sources~~ Sources of ~~...~~ $\vec{D}$

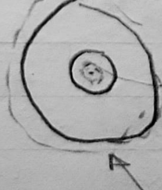
- $\rho_f = 0$, $\vec{\nabla} \times \vec{P} = 0$, $\vec{D} = 0 \to \epsilon_0 \vec{E} + \vec{P} = 0$, $\vec{E} = -\frac{\vec{P}}{\epsilon_0}$
- for $r < a$, $r > b$, $\vec{P} = 0$, $\vec{E} = 0$
- $a < r < b$, $\vec{E} = -\frac{1}{\epsilon_0} \vec{P} = -\frac{1}{\epsilon_0} \frac{k}{r^2} \hat{r}$

• Second Way

- $\vec{P} = \frac{k}{r^2} \hat{r}$, $\sigma_b = -\frac{k}{a}$ at $r = a$, $\sigma_b = \frac{k}{r}$ at $r = b$.
- $\rho_b = \vec{\nabla} \cdot \vec{P} = \frac{1}{r^2}\left[\frac{\partial}{\partial r} r^2 P_r\right] + 0 + 0 = -\frac{1}{r^2} \frac{\partial}{\partial r}(kr)$

$\rho_b = -\frac{k}{r^2}$

•



$\vec{E} \parallel \hat{r}$

GS

$r < a$, $Q_{inside} = 0$

$\vec{E} = 0$

$r > b$, $Q_{inside} = 0$

$\vec{E} = 0$

- $Q = 0$ (Proof)

$Q = \iint \sigma_b \, da + \iint \sigma_b \, da + \iiint \rho_b \, d\tau$

$= \left(-\frac{k}{a}\right)(4\pi a^2) + \left(-\left(\frac{k}{b}\right)\right)(4\pi b^2) + \int_a^b \left(-\frac{k}{r^2}\right) \underbrace{4\pi r^2}_{d\tau}$

$= -4\pi k a - 4\pi k b - 4\pi k[b - a]$

-

- (4.16) Carve out cavity of different shapes in $\vec{E}$, or uniformly polarized dielectrics.

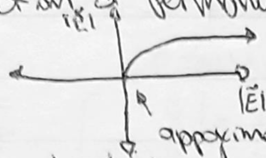
Insight: Super impose same shapes with reverse $\vec{P}$

~~LINEAR, HOMOGEN~~

LINEAR, ISOTROPIC, HOMOGENEOUS, DIELECTRICS.

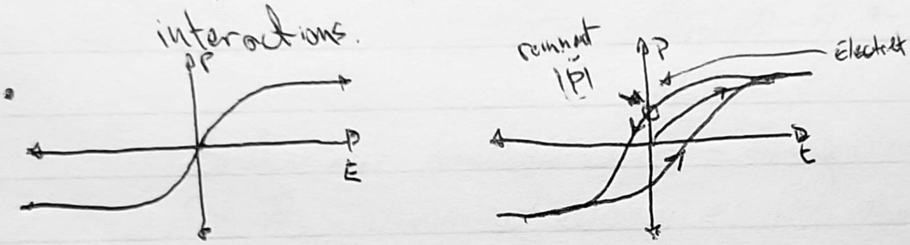
- Abbreviated to lih dielectrics.
- This is a response to a quantity of dipoles in an electric field.
- Consider a collection of permanent dipoles in an external field.

$$\vec{P} = \frac{\sum \vec{p}_i}{V}$$



approximately linear for small $|\vec{E}|$

- Effect of Interdipole Interactions.
  - an "ideal gas" cannot exhibit phase transition (cannot liquify).
  - Why not? Phase transitions are due to interparticle interactions.



no interdipole interactions

with interactions.

- $\vec{P}$ is a response ~~function~~.
- $\vec{P} = F(\vec{E})$. → linear response function → $\vec{P} = \chi_e \vec{E}$

susceptibility = response function.

- General Response Function.
  - $P_i = \underbrace{\sum_j \alpha_{ij} E_j}_{\text{linear response}}$ $\alpha_{ij} = 3\times3$ matrix, 2^{nd} rank tensor.

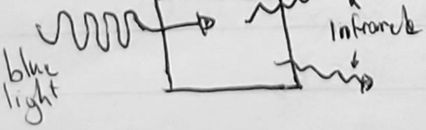
    P_i: ith component of vector

  - $P_i = \sum_j \alpha_{ij} E_j + \underbrace{\sum_j \sum_k \beta_{ijk} E_j E_k + \ldots}_{\text{Non-linear response, only important for large } |\vec{E}| \text{ (lasers)}}$

    Non-linear ~~resp~~ response, ~~only~~ only important for large $|\vec{E}|$ (lasers).

  - Eg. Nonlinear Effect, parametric down conversion

    certain crystals



(1 blue photon) $hf = 2hf'$ (2 infrared photons).

• Most general linear response $\Rightarrow \vec{P} = \epsilon_0 \chi_{m} \vec{E}$

$\uparrow$ Effect, $\uparrow$ response (3×3 matrix), $\uparrow$ cause.

$$\hookrightarrow \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

χ_{m} denotes matrix.

• Lets assume all $\chi_{ij} = 0$ except χ_{zx}.

- $\vec{P} = \epsilon_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \chi_{zx} & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \chi_{zx}\epsilon_0 E_x \end{bmatrix}$, $\vec{P} \parallel \hat{z}$ produced by $\vec{E} \parallel \hat{x}$

• Isotropic implies $\chi_{m} = \begin{bmatrix} \chi_e & 0 & 0 \\ 0 & \chi_e & 0 \\ 0 & 0 & \chi_e \end{bmatrix}$, all diagonals are the same, all off diagonal = 0

$\rightarrow \vec{P} = \chi_e \epsilon_0 \vec{E}$

$\uparrow$ scalar.

• Homogeneous - same material throughout.

• What $|\vec{E}|$ is needed for non-linear effects?

- roughly $|\vec{E}|$ inside H-atom, $E = \frac{ke^2}{r}$, $r \simeq 1\text{Å}$, $|\vec{E}| \sim 1 \frac{GV}{m}$

• $\vec{D}$ in lih materials. $(\vec{D} = \epsilon_0 \vec{E} + \vec{P})$

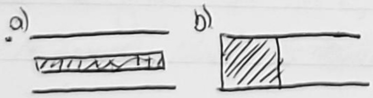
- in lih only, $\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$

$$= \epsilon_0 \underbrace{(1+\chi_e)}_{\text{Dielectric constant, } \epsilon_r} \vec{E}$$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

• Maxwell's Equation predicts light that moves with speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ & in an insulator $V = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_0}}$ } if $\epsilon_r < 1$ than $V > C$.

Problem 4.19.



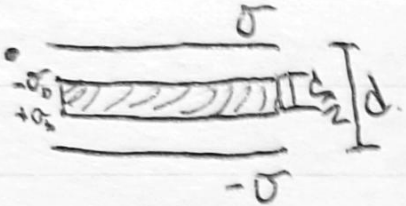
both cases, half of volume filled with same lih material (ϵ_r).

• Review: ① Metals are equipotentials. (doesnt mean charge density is uniform).

② $\vec{E} \perp$ metals, $|\vec{E}| = \frac{\sigma}{\epsilon_0}$

③ Definition $C \equiv \frac{Q}{\Delta V}$

④ Parallel Plate capacitor with vacuum $C = \frac{A\epsilon_0}{d}$.



Vac part: $|\vec{E}| = \frac{\sigma}{\epsilon_0}$

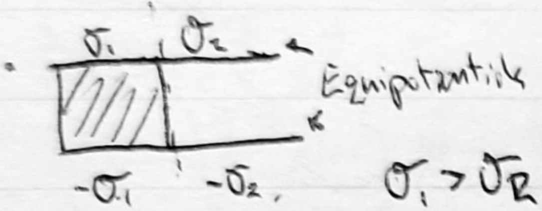
Lin part: $|\vec{E}| = \frac{\sigma}{\epsilon_r \epsilon_0}$

$\Delta V = E_{vac} \frac{d}{2} + E_{lin} \frac{d}{2}$

$= \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon_r \epsilon_0}\frac{d}{2}$

$Q = \sigma A$

$C_a = \frac{A\epsilon_0}{d}[\epsilon \ldots]$



$C_b = \frac{A\epsilon_0}{d}\left(\frac{\epsilon_r + 1}{2}\right)$

$\sigma_1 > \sigma_2$

Basic Ideas

Physics Ideas
1. $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$
   $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ In this course $-\frac{\partial\vec{B}}{\partial t} = 0$.
2. Superposition
3. $\iiint_V \vec{\nabla}\cdot\vec{F}\,d\tau = \oiint_S \vec{F}\cdot d\vec{a}$ ← $\vec{F}$ is some general field
4. Work $\equiv \int \vec{F}\cdot d\vec{l}$ ← $\vec{F}$ is force.
5. Total Energy $= U+K$ is constant if $W_{nc}=0$
6. $\vec{F}_c = -\nabla U$ or $\underbrace{\Delta U}_{U(b)-U(a)} = -\int_a^b \vec{F}\cdot d\vec{l}$

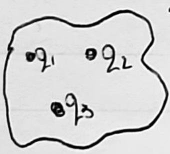
Consequences

- $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ ① start $\rightarrow$ $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \frac{q\delta^3(\vec{r})}{\epsilon_0}$

  pt. charge @ origin, $\rho = q\delta^3(\vec{r})$ $\iiint d\tau\,\vec{\nabla}\cdot\vec{E} = \iiint d\tau \frac{q\delta^3(\vec{r})}{\epsilon_0}$ ← integrate over a sphere.

  Symmetry says $\vec{E}\parallel\hat{r}$, $\vec{E}(r)$ only

  $\oiint_{sphere} \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} \Rightarrow |\vec{E}| \underbrace{\oiint da}_{4\pi r^2} = \frac{q}{\epsilon_0} \Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$

- 

  $\vec{E} = ?$, superposition implies that $\vec{E}_{TOT} = \sum_i \vec{E}_i$

  $\vec{E}_{TOT} = \int \frac{dq}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

  $dq = \lambda\,dl'$ ← 1D

  $dq = \sigma\,da'$ ← 2D

  $dq = \rho\,d\tau'$ ← 3D.

- High Symmetry Gauss's Law

  $-\oiint \vec{E}\cdot d\vec{a} = \frac{Q_{in}}{\epsilon_0}$

- $\vec{\nabla}\times\vec{E} = 0$ ① & $\vec{\nabla}\times\nabla f = 0$ ③ (always)

  implies

  $\vec{E} = -\vec{\nabla}V$, V = scalar potential.

  Same as $\Delta V = -\int\vec{E}\cdot d\vec{l}$ ③

  $V(b)-V(a) = -\int_a^b \vec{E}\cdot d\vec{l}$ ④, assume pt charge $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$ ⑤

  ⑥ & ④ imply, assume $V(\infty)=0$

  $V_{pt\,charge} = \frac{q}{4\pi\epsilon_0 r}$ scalar.

- Superposition implies

$$V_{TOT} = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i} \quad \text{or} \quad \int \frac{dq}{4\pi\epsilon_0 r_i}$$, dq same as $\vec{E}$

- $\Delta U = -\int \vec{F} \cdot d\vec{l}$ assume

$$\Delta U = -\int \frac{q_i q_j}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\vec{l} = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

$$U_{TOT} = \frac{1}{2} \sum_i \sum_j \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

$$U_{TOT} = \frac{1}{2} \iiint \rho(\vec{r}_i) V(\vec{r}) \, d\tau$$

$$U_{TOT} = \iiint_{\text{all space}} \frac{\epsilon_0}{2} |\vec{E}|^2 \, d\tau$$

Methods to find $\vec{E}$, V

$\vec{E}$ • Given $\rho(\vec{r})$ or set of charges

① - $\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2} \hat{r}$

② - or if symmetry $\oint_{GS} \vec{E} \cdot d\vec{a} = \frac{Q_{in}}{\epsilon_0}$, GS = Gaussian Surface.

③ - If V is known $\vec{E} = -\vec{\nabla} V$

V ① • From point charge or $\rho(\vec{r})$

- $V = \int \frac{dq}{4\pi\epsilon_0 r}$

②. From $\vec{E}$ then

- $\Delta V = -\int \vec{E} \cdot d\vec{l}$

③. if $\rho = 0$, and V known on boundaries, solve

- $\nabla^2 V = 0$. (uniqueness theorem) (method of images, fourier series)

④ • Multipole expansion.

Know V, $\vec{E}$ in and around insulators & metals

Know bound charge density, ρ_b, σ_b.