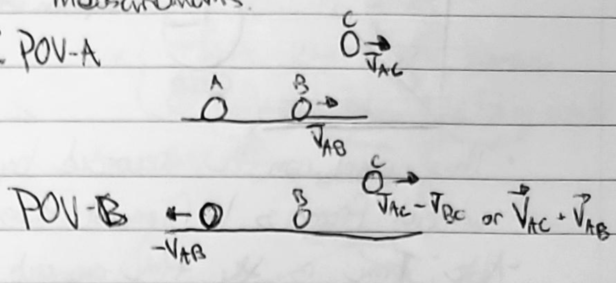
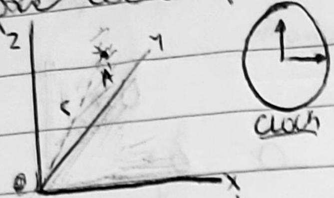
Einstein's Special Theory of Relativity

- Review - Classical Relativity: Deals with how observers in different reference frames compare results of measurements.
  - Relative Motion: POV-A



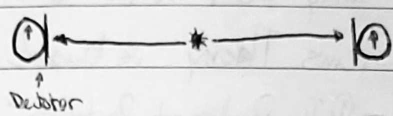
- When dealing with special relativity, inertial reference frames are only dealt with when velocity is constant (no acceleration).
- Einstein's Theory is based on 2 postulates:
  - ① The Relativity Postulate: The laws of physics are the same for all observers in all inertial reference frames (does not say that measure value are the same).
  - ② The Speed of Light Postulate: The speed of light is the same for all observers for all inertial reference frames. Any particle with mass travels at a speed less than c. ($c = 2.998 \times 10^8 \frac{m}{s}$) Classical Physics fall apart ($K = \frac{1}{2}mv^2$) because it states that more energy = more speed ($\infty E = \infty v$) Need an equation that show v getting asymptotically close to v. The postulate also states that the speed of light in a moving vessel is the same as a source at rest.

- Proof of Speed of Light Postulates
  - "light source" = pi meson (pion)
  - Pion decays into 2 gamma rays ($\pi^0 \to \gamma + \gamma$)
  - Pion moves a speeds close to c.
  - If classical physics were correct, then light velocity = $c + c \approx 2c$
  - But experiments show it only travels c.

- In relativity, space and time are entangled (4-D (x, y, z, T))
- An event is something to which we can assign three space coordinates and one time coordinate.
- 

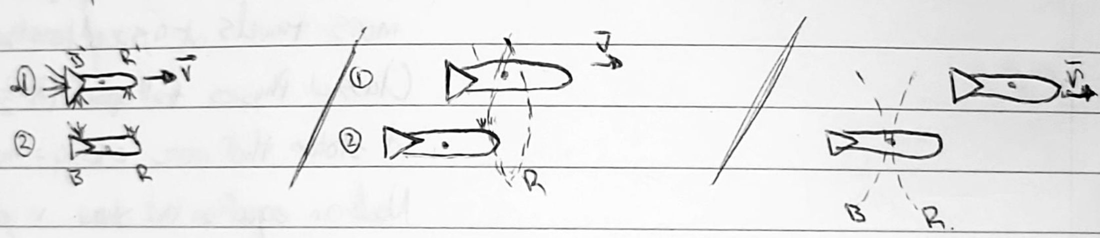
Explosion at point A. $x = 2$ m $z = 1$m

$y = 3$ m $t = 3:00$ pm

- This event can be recorded by different observes, in different interial frames. In General, these measurements will differ.
- Note: Time is the time measured on location of event. Observer at the origin detects light at the $t = \frac{s}{c}$ on a clock in-sync with the one on site.
- Process in Synchronizing Clocks: 

Light source placed equidistant between clock on site and one off site. Clock with start in sync when light hits detector

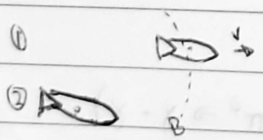
Relativity of Simultaneity.



- Meteor hits
- Red/blue light given off in respective areas

- Ship 1 detects red light first.
- Ship 2 " both light at the same time.

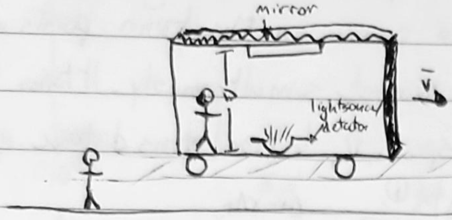
- Ship 2 measure that it is half way between strikes and concluded the events happened simultaneously



- The blue light eventually reaches ship 1 and concludes the event were not simultaneous.

- Simultaneity is not an absolute concept, it is a relativity one.

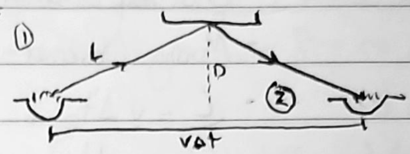
The Relativity of Time.

- Observers who move relative to each other generally will find different results for the time interval between events.
- 

In car $\begin{cases} \text{① Pulse leaves light source} \\ \text{② Reflected Pulse detected by detector} \end{cases}$

$$\Delta t_0 = \frac{2D}{c}$$ ← distance light travelled; ← speed of light.

Out of Car.



$$\Delta t = \frac{2L}{c} \rightarrow \frac{2\sqrt{D^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c} \rightarrow \Delta t = \frac{2}{c}\sqrt{\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2$$

$$(c\Delta t)^2 = (c\Delta t_0)^2 + (v\Delta t)^2$$

$$(c^2 - v^2)\Delta t^2 = c^2 \Delta t_0^2$$

$$\Delta t = \sqrt{\frac{c^2 \Delta t_0^2}{c^2 - v^2}} \rightarrow \frac{c\Delta t_0}{\sqrt{c^2 - v^2}} \rightarrow \Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Time Dilation.

* Must happen for events at ~~events~~ the same location
It is the proper time.

Some Short Forms

- The Speed Parameter $\beta = \frac{v}{c}$
- The Lorentz Factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Example

- Search light blinks on and off. Ground interval measures 3.25×10^{-3} s; ONboard officer measures 4.5×10^{-5} s

$\Delta t_0 = 4.5 \times 10^{-5}$

$\Delta t = 3.25 \times 10^{-3}$

① Officer measures proper time.

② $v =$

$$\Delta t = \Delta t_0 \gamma$$

$$\frac{1}{\gamma} = \frac{\Delta t_0}{\Delta t}$$

$$\sqrt{1 - \beta^2} = \frac{\Delta t_0}{\Delta t}$$

$$1 - \beta^2 = \sqrt{\frac{\Delta t_0}{\Delta t}}$$

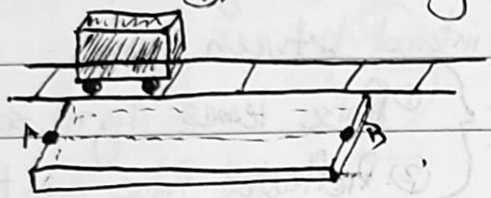
$$\beta = \sqrt{\frac{\Delta t_0}{\Delta t}^2 - 1}$$

$$\beta = \sqrt{\frac{4.5 \times 10^{-5}}{3.25 \times 10^{-3}}^2 - 1}$$

$$\beta = 0.99904$$

The Relativity of Length.

- Considering measuring length of a train platform



- For a person on the ground $L_0 = B - A$
- For a person in the train points need to be measured simultaneously. It takes time, t, at speed, $\vec{v}$, to travel this distance.

- Stationary Observer - $L_0 = v\Delta t$. (1)
- Moving Observer - $L = v\Delta t_0$ (2)

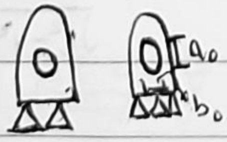
$\Delta t = \gamma \Delta t_0$

$\therefore$ - $\frac{L}{L_0} = \frac{v\Delta t_0}{v\Delta t} \rightarrow \frac{L}{L_0} = \frac{\Delta t_0}{\Delta t} \rightarrow \frac{L}{L_0} = \frac{1}{\gamma} \rightarrow L = \frac{L_0}{\gamma}$ } Length Contraction.

- L_0 is the proper length (rest length).

Example

- 2 spaceship are marked by a circle other by ell. [illegible]
- Major axis is 1.5 x minor axis
- How fast to be considered a circle?



$a_0 = 1.5 b_0$.

Note: Lengths only along direction of travel are measured.

$L_0 = \gamma L \rightarrow a_0 = \gamma b_0$

$\frac{a_0}{b_0} = \gamma \rightarrow 1.5 = \gamma$

$1.5 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

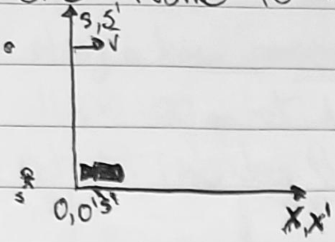
$1.5^2 = \frac{1}{1 - \frac{v^2}{c^2}}$

$\frac{v}{c} = \sqrt{1 - \frac{1}{1.5^2}}$

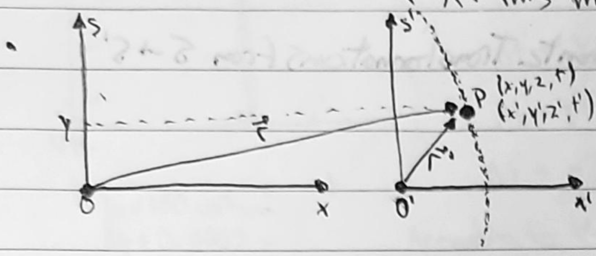
$\frac{v}{c} = 0.745$.

Lorentz Transformation

- Set of equations that relate space/time coords of one frame to those of another moving at relative speed $\vec{v}$.

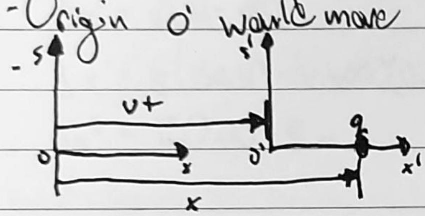


- $t = t' = 0$, origins coincide.
- Frame of ship described by (x', y', z', t') at speed v
- " of observer " " (x, y, z, t).
- At this moment a pulse of light is emitted, spherically at O, O'



- Distance to point P - $r = ct$ according to s ; $r' = ct'$ according to s' ①
- Since r & r' differ, t & t' differ.
- $y, z / y', z'$ are the same coordinates.
- $\underset{①}{x = ct}$, $\underset{②}{x' = ct'}$

- Consider a case where $v \ll c$.
  - Origin O' would move distance $O' = vt$ in time instead t.



  - point $q \rightarrow x' = x - vt$.
  - That is called a Galilean Transformation
  - A more general eqn for high speeds is $x' = G(x - vt)$ ③, where G a function to be determined and has the property $G \rightarrow 1$ and $v \rightarrow 0$.

- Einstein postulates states laws of physics same in both frames, $\therefore$ should be able to write the inverse transform
  - $x = G(x' + vt')$ ④ → Note: change sign of v in accordance to frame (s').
- Sub ① + ② into ③ + ④
  - $ct' = G(ct - vt)$ ③' $\xRightarrow{\text{solve for } t'}$ $t' = Gt\left(1 - \frac{v}{c}\right)$

    $ct = G(ct' + vt)$ ④' $\xRightarrow{\text{sub. } 3' \text{ into } 4'}$ $G^2 = \dfrac{c}{(c+v)\left(1 - \frac{v}{c}\right)} \Rightarrow G^2 = \dfrac{1}{1 - \frac{v^2}{c^2}}$

    $G = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

    $G = \gamma$

- Thus: $\left.\begin{array}{l} x = \gamma(x' + vt') \\ x' = \gamma(x - vt) \end{array}\right\}$ Lorentz Transformation.

- To get time transformation
  - Sub ⑥ into ⑤:

$$x = \gamma[\gamma(x - vt) + vt']$$

$$t' = \frac{1}{v}\left[\frac{x}{\gamma} - \gamma(x - vt)\right]$$

$$t' = \gamma\left[\frac{x}{v\gamma^2} - \frac{x - vt}{v}\right]$$

$$t' = \gamma\left[\frac{x}{v}\left(1 - \frac{v^2}{c^2}\right) - \frac{x}{v} + t\right]$$

$$t' = \gamma\left[\cancel{\frac{x}{v}} - \frac{xv}{c^2} - \cancel{\frac{x}{v}} + 1\right]$$

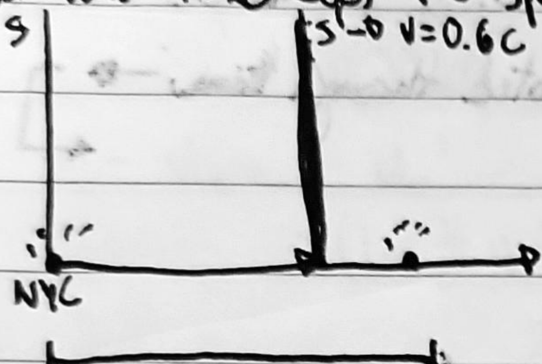
$$t' = \gamma\left[t - \frac{x}{v}\frac{v^2}{c^2}\right]$$

- Complete set of Lorentz Transformations from $S \to S'$
  - $x' = \gamma(x - vt)$
  - $y' = y$
  - $z' = z$
  - $t' = \gamma\left[t - \frac{x}{v}\left(\frac{v}{c}\right)^2\right]$
- For the inverse $S' \to S$, change • to •' and $v \to -v$
  - $x = \gamma(x' + vt')$
  - $y = y'$
  - $z = z'$
  - $t = \gamma\left[t' + \frac{x'}{v}\left(\frac{v}{c}\right)^2\right]$
- To analyze differences between coords of a pair of events simply substitute difference into Lorentz eqns
  - $\Delta x = \gamma[\Delta x' + v\Delta t']$

Example

- 2 Explosions occur. In NYC one goes of at 11:00am exactly. Second is 150 K away at exactly 11:00:00.003 am. Spaceship travels along a line passing through these two locations, Its clock also reads 11:00:00 am at NYC expl. Travels at 0.600c from NYC to 2nd loc.

A) at what time does the spaceship say the 2nd expl. takes place -



$\Delta x = 150 \times 10^3$ m.
$\Delta t = 0.0003$ s

$$\Delta t' = \gamma\left[\Delta t - \frac{\Delta x}{v}\left(\frac{v^2}{c^2}\right)\right]$$

$$\Delta t' = \gamma\left[0.003 - \frac{15 \times 10^5}{\cancel{0.6c}} \frac{0.6^{\cancel{2}}}{c^2}\right]$$

$$\Delta t' = \gamma\left[0.003 - \frac{150 \times 10^3 (.6)}{c}\right]$$

$$\Delta t' = -2.5 \times 10^{-7} \text{ s}$$

According to the spaceship 2nd explosion occured before the one in NYC.

b) How far apart are the 2 event (spaceship observer)?

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

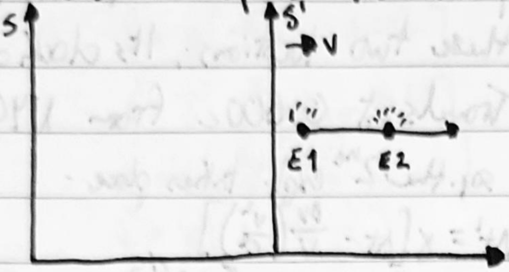
$$\Delta x' = \gamma(150 \times 10^3 - (0.600c)(0.003))$$

$$\Delta x' = 120 \times 10^3 \text{ m.}$$

Distance looks contracted.

The Relativity of Velocities.

- Assume 2 observers in S & S' are watch a particle that travels at a speed that we want to compare.



- Particle sends out 2 pulses
- Each observer measure space/time coord. of E1 & E2. Their measurements related through

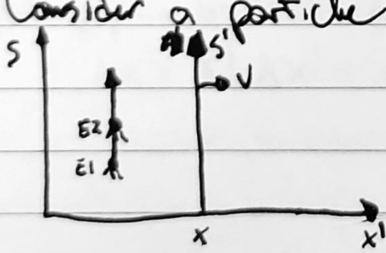
$$\Delta x = \gamma(\Delta x' + v\Delta t')$$
$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

- Speed is denoted by u

$$u_x = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v\Delta t')}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)} = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c^2}\frac{\Delta x'}{\Delta t'}}$$

$$u_x = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}$$ ← x component of velocity transfer from $S' \to S$.

- Consider a particle that travels along y-axis in frame S



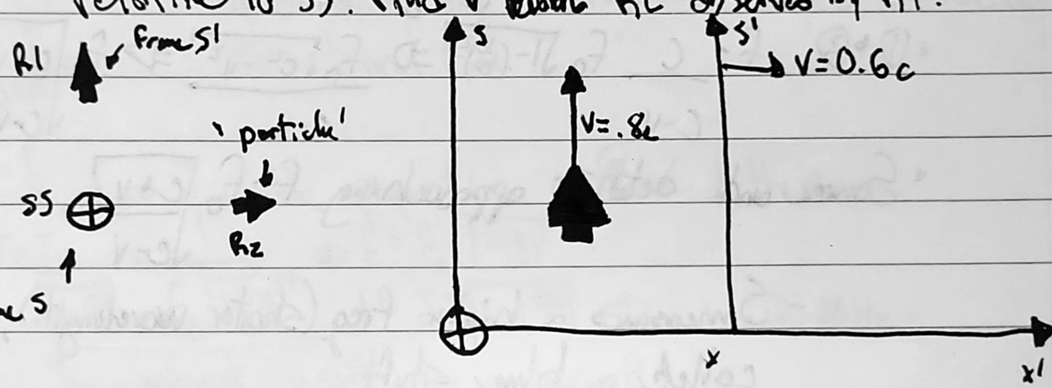
S looks like

S' look like

$$\Delta y = \Delta y'$$
$$\Delta t = \gamma\left(\Delta t' + v\frac{\Delta x'}{c^2}\right)$$
$$u_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma\left(\Delta t' + v\frac{\Delta x'}{c^2}\right)}$$
$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$ ← y component of velocity transfer from $S' \to S$

- Similarly, we would find a particle moving in the z-axis, in frame S: $U_z = \dfrac{U_z'}{\gamma(1+\frac{v}{c^2}U_x')}$
- Complete Velocity Transformations from $S' \to S$
  - $U_x = \dfrac{U_x' + V}{1+\frac{V}{c^2}U_x'}$
  - $U_y = \dfrac{U_y'}{\gamma(1+\frac{V}{c^2}U_x')}$
  - $U_z = \dfrac{U_z'}{\gamma(1+\frac{V}{c^2}U_x')}$.

- Example: 2 Rockets leave. $R1 = v_1 = 0.6c$ relative to SS, $R2 = v_2 = .8c$ relative to SS. Find v of R2 observed by R1.



$U_x' = \dfrac{U_x - V}{1-\frac{V}{c^2}U_x}$ $\quad U_x = 0$, NO SPEED IN x FROM R2 (PARTICLE).

$U_x' = -V$ $\quad u' = \sqrt{u_x'^2 + u_y'^2}$

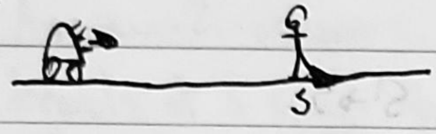
$U_x' = -0.6c$ $\quad u' = 0.88c$.

$U_y' = \dfrac{U_y}{\gamma(1-\frac{V}{c^2}\cancel{U_x})}$

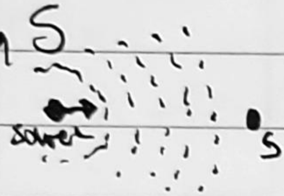
$U_y' = \dfrac{U_y}{\gamma}$

$U_y' = \dfrac{0.8c}{\gamma} \Rightarrow U_y' = 0.64c$

The Doppler Effect for Light.

- 

 - In the rest frame $f_0 = \frac{1}{T_0}$
 - We want to measure freq at S
- Let T be the time interval between wave crests measured by S



Note: Since the source is moving the emission of wave crests occur at different locations in space.

- During interval T, crests ahead move distance cT and source move at a distance vT. $\Rightarrow \lambda = T(c-v)$
- Freq S measures ① $- f = \frac{c}{\lambda} = \frac{c}{(c-v)T}$ (T_0 is the measured in the rest frame and is the proper time)

 ② $- T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{1}{f_0\sqrt{1-(\frac{v}{c})^2}}$
- ① + ② $f = \frac{c}{c-v} f_0\sqrt{1-(\frac{v}{c})^2} \Rightarrow \frac{f_0\sqrt{c^2-v^2}}{c-v} \Rightarrow f_0\sqrt{\frac{c+v}{c-v}}$
- Source and detector approaching $f = f_0\sqrt{\frac{c+v}{c-v}}$
 - S measures a higher freq (shorter wavelength), this is called a blue shift.
- Source and detector separating $f = f_0\sqrt{\frac{c-v}{c+v}}$
 - S measure a lower freq (longer wavelength), this is called a red shift.
- Can be written as:

 Approach $\rightarrow f = f_0\sqrt{\frac{1+\beta}{1-\beta}}$ Separate $\rightarrow f = f_0\sqrt{\frac{1-\beta}{1+\beta}}$

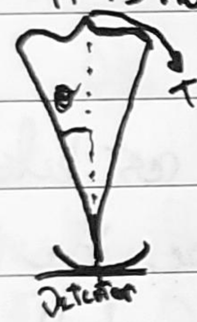
Example

- Spaceship moves at 0.92c relative to us and emits $f_0 = 3.50$ MHz according to spaceship. Freq approach?

App. $f = f_0\sqrt{\dfrac{c+.92c}{c-.92c}} = 3.5\text{ MHz}\sqrt{\dfrac{1.92}{0.08}} = 17.15\text{ MHz}$

Away. $f = f_0\sqrt{\dfrac{c-.92c}{c+.92c}} = 3.5\sqrt{\dfrac{0.08}{1.92}} = 0.71\text{ MHz}$

- What is the freq measured as it passes directly overhead.
  - $\Rightarrow$ It is not f_0. There is a transverse doppler effect.



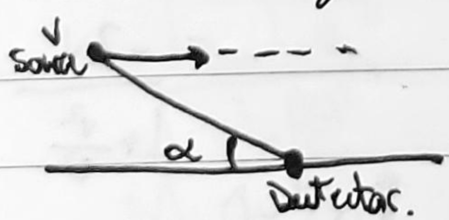
- Consider wave crest emitted just before and after the perpendicular. They will travel same distance and have the same travel time.

$f = \frac{1}{T}$, where T is the proper time in frame of the source

$T = \dfrac{T_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \dfrac{1}{f_0\sqrt{1-\left(\frac{v}{c}\right)^2}} \Rightarrow f = f_0\sqrt{1-\left(\frac{v}{c}\right)^2}$

$f = 3.5\sqrt{1-\left(\frac{.92c}{c}\right)^2} \Rightarrow f = 1.37\text{ MHz}$.

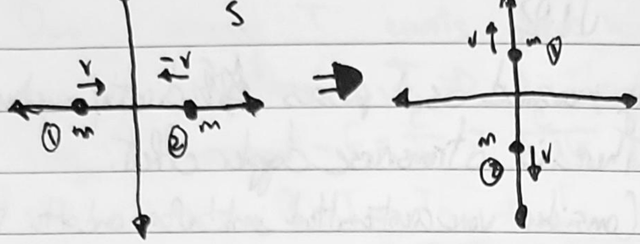
- Note: The general result for an angle θ



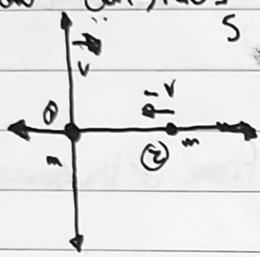
$$f = \frac{f_0}{\gamma}\,\frac{1}{1-\frac{v}{c}\cos\alpha}$$

Relativistic Dynamics.

- Consider dynamical quantities such as force, momentum, and energy.
- We know $K = \frac{1}{2}mv^2$ fall apart at high speeds but what about $\bar{p} = m\bar{v}$?
- Consider 2 particles moving with equal and opposite velocities in frame S.



- Now consider a frame where ① is at rest before the collision



  - Here we would use the Lorentz transformations to find the x & y components of velocity (initial & final)
  - U_{2bx}' = Speed of particle 2 before collision

$$U_{2bx}' = \frac{U_{2x} - v}{1 - \frac{v}{c^2}U_{2x}} = \frac{-v-v}{1-\frac{v}{c^2}\cdot -v} = \frac{-2v}{1+\frac{v^2}{c^2}}$$

Before Collision.

- $U_{1bx}' = 0$ $\quad p_{xb}' = m(u_{2xb}' + u_{1xb}') = \frac{-2mv}{1+\frac{v^2}{c^2}}$ $\quad p_{yb}' = 0$

After Collision

- Particle 1

$$U_{1xa}' = \frac{U_{1xa} - v}{1-\frac{v}{c^2}U_{1xa}} = -v$$

$$U_{1ya}' = \frac{U_{1ya}}{\gamma(1-\frac{v}{c^2}U_{1xa})} = \frac{v}{\gamma}$$

Particle 2

$$U_{2xa}' = \frac{U_{2xa} - v}{1-\frac{v}{c^2}U_{2xa}} = -v$$

$$U_{2ya}' = \frac{U_{2ya}}{\gamma(1-\frac{v}{c^2}U_{2xa})} = \frac{-v}{\gamma}$$

$p_{xa}' = -2mv$

$p_{ya}' = 0$

$P_{before} \neq P_{after}$, need to revise formula.

in S p is conserved
in S' p in not conserved.

· To satisfy Einstein's postulates that the laws of physics are same for both frames (conservation of $\bar{p}$), we need to revise our definition of p.

· $p = mv = m\frac{\Delta x}{\Delta t}$

· To find a relativistic expression, the Δt with Δt_0 (proper time).

$p = m\frac{\Delta x}{\Delta t_0}$, where Δx is the distance covered by the moving particle as viewed by S or the displacement of frame S viewed by S', and Δt_0 is the proper time measured in the frame of the moving particle.

· $p = m\frac{\Delta x}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\gamma = mv\gamma \Rightarrow p = mv\gamma$

$\bar{p} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$

Note: This transformation is between the particle and lab frame.

· Check with the past example.

From S: $P_{tot} = 0$ ✓

From S': $P_{yB}' = P_{yA}' \Rightarrow P_{tot\,y}' = 0$. } y-component

$v_2 = u_{2xb}' = \frac{-2}{1+\frac{v^2}{c^2}}$ so $P_{xb}' = 0 + \frac{m u_{2xb}'}{\sqrt{1-\frac{v^2}{c^2}}}$

$P_{xA}' =$

$v_A = \sqrt{u_{xA}'^2 + u_y'^2}$

$= \sqrt{-v^2 + (\frac{v}{\gamma})^2}$

$= \sqrt{1 + \frac{1}{\gamma^2}}$

$= \sqrt{1 + (1 - \frac{v^2}{c^2})}$

$= \sqrt{2 - \frac{v^2}{c^2}}$

$P_{xA} = \frac{m u_{xA}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m u_{xA}}{\sqrt{1-\frac{v^2}{c^2}}}$

$= \frac{-mv + -mv}{\sqrt{1-\frac{v^2}{c^2}}}$

$= \frac{-2mv}{\sqrt{1-\frac{1}{c^2}(v^2(2-\frac{v^2}{c^2}))}}$

$= \frac{-2mv}{\sqrt{(1-\frac{v^2}{c^2})^2}} = \frac{-2mv}{1-\frac{v^2}{c^2}}$

$P_{xb}' = 0 + \frac{m u_{2xb}'}{\sqrt{1-\frac{v_2^2}{c^2}}}$

$= \frac{\frac{-2mv}{1+\frac{v^2}{c^2}}}{\sqrt{1-\frac{1}{c^2}\frac{4v^2}{1+\frac{v^2}{c^2}}}}$

$= \frac{\frac{-2mv}{1+\frac{v^2}{c^2}}}{\sqrt{\frac{(1+\frac{v^2}{c^2})^2 - \frac{4v^2}{c^2}}{(1+\frac{v^2}{c^2})^2}}}$

$=$

$= \frac{-2mv}{\sqrt{1-\frac{2v^2}{c^2}+\frac{v^4}{c^4}}}$

$= \frac{-2mv}{\sqrt{(1-\frac{v^2}{c^2})^2}}$

$= \frac{-2mv}{1-\frac{v^2}{c^2}}$

$P_{xb}' = P_{xA}'$

So momentum is conserved here.

Example

- What is the % difference between the Newtonian & relativistic values for the momentum of a meteorite travelling at $72\frac{km}{s}$

$P_R = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ $\quad P_N = mv$

$\% \text{ diff} = \left(\frac{P_R - P_N}{P_R}\right) \times 100\%$

$= \left(1 - \frac{P_N}{P_R}\right) \times 100\%$

$= \left(1 - \frac{\cancel{mv}}{\cancel{mv}}\sqrt{1-\frac{v^2}{c^2}}\right) \cdot 100\%$

$= \left(1 - \sqrt{1-\frac{v^2}{c^2}}\right) \cdot 100\%$ ← Use binomial expansion $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \ldots$

$= \left(1 - \left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right) \cdot 100\%$

$= 1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right)$

$= 1 - \left[1 - \frac{v^2}{2c^2}\right]$

$= \frac{v^2}{2c^2} \rightarrow \frac{1}{2}\left(\frac{72 \times 10^3}{2.998 \times 10^8}\right)^2 \rightarrow 2.88 \times 10^{-6}\ \%$

- Recall $F = ma = m\frac{\Delta v}{\Delta t} = \frac{dp}{dt}$.

The Relativity of Force

- $f = \frac{dp}{dt} \Rightarrow m \times \frac{dv}{dt} \Rightarrow \frac{d}{dt}\left[\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right]$

- Consider a case where $\vec{f}$ & $\vec{V}$ are both along the x-axis

$$f = m\frac{d}{dt}\left[\frac{v_x}{\sqrt{1-\frac{v_x^2}{c^2}}}\right]$$

$$= m\frac{d}{dt}\left[v_x\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$= m\left[\left[\frac{dv_x}{dt}\right]\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{1}{2}} + \frac{d}{dt}v_x\cancel{\tfrac{1}{2}}\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-\cancel{2}v_x}{c}\right)\right]$$

$$= m\left[a\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{1}{2}} + a\frac{v_x^2}{c^2}\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{3}{2}}\right]$$

$$= ma\left[\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{1}{2}} + \frac{v_x^2}{c^2}\left(1-\frac{v_x^2}{c^2}\right)^{-\frac{3}{2}}\right]$$

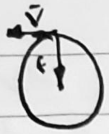
$$= ma\left(\frac{1}{\sqrt{1+\frac{v_x^2}{c^2}}} + \frac{v_x^2}{c^2}\frac{1}{\sqrt{\left(1-\frac{v_x^2}{c^2}\right)^3}}\right) \ldots \quad F_x = \frac{ma}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}}$$

$$\cancel{= \frac{ma}{\sqrt{1+\frac{v_x^2}{c^2}}}\left(1+\frac{v_x^2}{c^2}\frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}}\right)}$$

$$a = \frac{F}{m}\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}$$ Note: as $v \to c$, $a \to 0$.

$$\cancel{= \frac{ma}{\sqrt{1+\frac{v_x^2}{c^2}}}\left(1+\frac{1}{\frac{v_x^2}{c^2}-1}\right)}$$

$$\cancel{= \frac{ma}{\sqrt{1+\frac{v_x^2}{c^2}}}}$$

- Consider a case with circular motion.
- 

$$f = \frac{d}{dt}\left[\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right] \quad v = \text{constant.}$$

$$f = \frac{dv}{dt}\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$f = \frac{ma}{\sqrt{1-\frac{v^2}{c^2}}}$$

Relativistic Work

- Recall: $W = \int F\,dx$, and work is equal to the kinetic energy gained
- Find kinetic energy gained by moving particle initially at rest from x_1 to x_2 where it will have velocity v.
- $K = W = \int_{x_1}^{x_2} \frac{ma}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}}\,dx$. since $a = \frac{dv}{dt}$ $a\,dx = \frac{dv}{dt}dx$ and $\frac{dx}{dt} = v_x$ so $dv_x\,v_x$

$$= \int_0^v \frac{m v_x}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}}\,dv \Rightarrow m\int_0^v \frac{v_x}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}}\,dv$$

$$= m\int_0^v \frac{v_x}{\left(\frac{1}{c^2}\left(c^2-v_x^2\right)\right)^{\frac{3}{2}}}\,dv \Rightarrow mc^3\int \frac{v_x}{\left(c^2-v_x^2\right)^{\frac{3}{2}}}\,dv_x \Rightarrow \text{integration by substitution} \ldots$$

$$K = mc^3\left[\frac{1}{\sqrt{c^2-v_x^2}}\right]_0^v$$

$$K = \frac{mc^3}{\sqrt{c^2-v^2}} - \frac{mc^3}{\sqrt{c^2}}$$

$$= \frac{mc^3}{\sqrt{c^2-v^2}} - \frac{mc^3}{c}$$

$$= m\cancel{c^3}\left[\frac{1}{\left(c^2-v^2\right)^{\frac{1}{2}}} - \frac{1}{c}\right]$$

$$= mc^2\left[\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} - 1\right]$$

$$= mc^2\left[\gamma - 1\right]$$

} Kinetic energy is comprised of a Total energy $E_T = \gamma mc^2$ and subtracted by an energy that is present even when at rest, rest energy $E_R = mc^2$

↑ depends on v.

• Total Energy = K + Rest Energy

$\gamma mc^2 = K + mc^2$

Relationship Between Total Energy and Momentum

• $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow p^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}}$ ①

$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow E^2 = \frac{(mc^2)^2}{1-\frac{v^2}{c^2}}$ ②

• Sub ① to ②:

$1-\frac{v^2}{c^2} = \frac{(mc^2)^2}{E^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{(mc^2)^2}{E^2}$

$\rightarrow v^2 = c^2\left[1-\frac{(mc^2)^2}{E^2}\right]$

$p^2 = \frac{m^2v^2E^2}{(mc^2)^2}$

$p^2 = \frac{m^2E^2}{(mc^2)^2} c^2\left[1-\frac{(mc^2)^2}{E^2}\right]$

$p^2 = \frac{E^2}{c^2}\left[1-\frac{(mc^2)^2}{E^2}\right]$

$p^2c^2 = E^2\left[1-\frac{(mc^2)^2}{E^2}\right]$

$p^2c^2 = E^2 - (mc^2)^2$

Common Units.

E = MeV - Mega Electron Volts

$p = \frac{MeV}{c}$ - " " " per speed of light

Example

• A cosmic ray ~~photon~~ proton moves at speed where moon-earth travel time is 1.5 s. $d_{earth\,moon} = 3.8\times10^5$ km. a) At what fraction of c is it moving? b) Rest Energy in MeV? c) Kinetic E? d) Total E? e) p?

a) $v = \frac{d}{t}$

$= \frac{3.8\times10^8}{1.5} \cdot \frac{1}{c}$

$v = 0.845c$

b) $mc^2 = E_R$

$E_R = (1.6727\times10^{-27})\cdot c^2$

$E_R = 1.5033\times10^{-10}\,J \cdot \frac{1eV}{1.6022\times10^{-19}J}$

$E_R = 9.383\times10^8\,eV$

$E_R = 938.3$ MeV

c) $K = \gamma mc^2 - mc^2$

$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$

K = 816.3 MeV

d) $E_T = 1754.6$ MeV

e) $pc = \sqrt{E^2-(mc^2)^2}$

$p = \frac{1482.6}{c}$

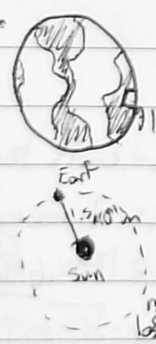
$p = 1482.6 \frac{MeV}{c}$

Conservation of Mass/Energy

- Mass and energy are not independent
- Therefore, they cannot be separately conserved
- Principle of Conservation of mass:
  - In an isolated system of particles the relativistic total energy remains constant
  - $E_{before} = \sum_{j=1}^{k} E_j = \sum_{j=1}^{k} (K_j + m_j c^2) = \sum_{j=1}^{k} \gamma_j m_j c^2$
  - $E_{After} = \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} (K_i + m_i c^2) = \sum_{i=1}^{n} \gamma_i m_i c^2$ } Note number of particles doesn't have to be the same
  - $E_{Before} = E_{After}$
  - Mass can be created or destroyed but when this happens, an equivalent amount of energy simultaneously vanishes or comes into being and vice versa
  - Mass and energy are different aspects of the same thing.

Example:

- Solar energy reaches earth at a rate of 1.4 kW per square meter of surface ⊥ to the sun. By how much does the mass of the sun decrease per second to account for this energy loss? The mean radius of earths orbit is 1.5×10^{11} m.



1m² has $1.4 \times 10^3 \frac{J}{s}$ from sun.

Earth

1.5×10^{11} m

Sun

Total power $= 1.4 \times 10^3 \frac{J}{s} \cdot 4\pi (1.5 \times 10^{11})^2$

$= 3.96 \times 10^{26} \frac{J}{s}$

mass lost per second → $\Delta m = \frac{\Delta E}{c^2}$

$\Delta m = \frac{3.96 \times 10^{26}}{c^2}$

$\Delta m = 4.4 \times 10^9 \frac{kg}{s}$

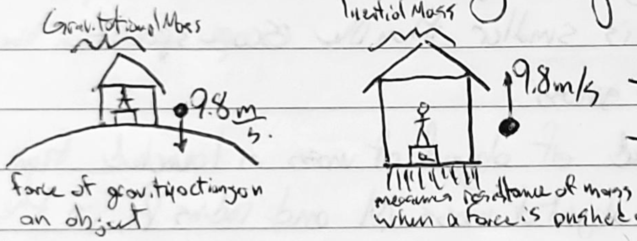
A 2×10^{30} kg sun would shine for ___ at this rate

$\frac{2 \times 10^{30} \text{ kg}}{4.4 \times 10^9 \frac{\text{kg}}{s}} = 4.54 \times 10^{20} \text{ s} \cdot \frac{1 \text{ y}}{60 \cdot 60 \cdot 24 \cdot 363 \text{ s}}$

$= 1.4 \times 10^{13}$ years.

General Relativity

- ~~Special~~ Special Relativity expresses the physical equivalence of inertial reference frames
- For non-inertial (accelerating) frames Einstein formulated the Equivalence Principle of General Relativity:
  - "Provided observations are in a small enough region of spacetime, it is not possible by experiment to distinguish between an ~~accelerated~~ accelerating frame and an inertial frame in a suitably chosen gravitational potential"
  - In an enclosed room, you cannot tell if you are at rest on earth or accelerating through interstellar space at $9.8 \frac{m}{s^2}$.
  -

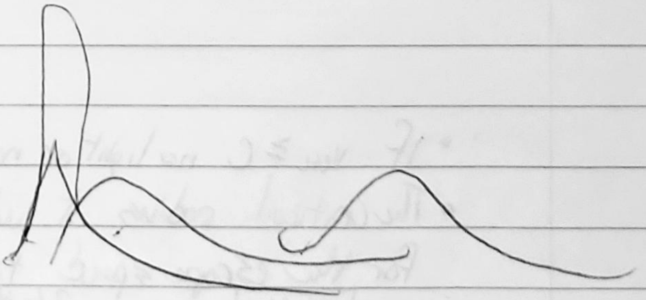


- Scale would read same weight
- Objects would fall at the same rate.

- Equivalence implies that the gravitational mass, m_g, that appears in $F = -\frac{Gm_1m_{2g}}{r^2}$ and the inertial mass, m_i, that appears in $F = m_i a$ must be equal.
- $-\frac{Gm_{1g}m_{2g}}{r^2} = \cancel{m_i} a$

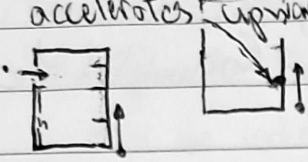
$$-\frac{GM}{r^2} = a$$

$$9.8\frac{m}{s^2} = a.$$

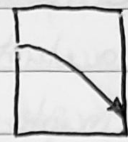


$$\frac{1eV}{1.602 \times 10^{-19}} = 1$$

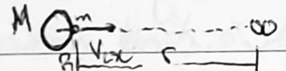
- A second consequence is that light "falls" under the influence of gravity
- Consider beam of light projected across a room that accelerates upward.



so it looks like the light went like this



- By the equivalence principle a room placed in a gravitational potential with $g=a$ would follow same path and be interpreted as being attracted by the gravitational field.
- This causes problems in astronomy (i.e. gravitational lensing)
- Another consequence is the existence of blackholes
- Blackhole - An object of large mass localize in a small space such that light is smaller than the escape speed from the surface of the mass.
- The escape speed of object of mass m launched from the surface of a massive object of mass M and radius R, is the minimum speed it must have in order to leave the vicinity of the object forever.



- $E = K + U = \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right)$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

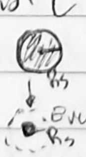
$$E_r = \frac{1}{2}mv_{esc}^2 - \frac{GMm}{R}$$

$$v_{esc}^2 = \frac{2GM}{R}$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

- If $v_{esc} \geq c$ no light or matter can ever escape from the object (blackhole)
- The critical radius at which mass M must be compressed for the escape speed to equal c is called the Schwarzchild Radius.
- $c > \sqrt{\frac{2GM}{R_s}} \Rightarrow R_s = \frac{2GM}{c^2}$

Compress common



- Event horizon - Any event occuring with in here is invisible to the outside observer.

Example

- Calculate the Schwarzchild radius for the sun
- $M_s = 1.99 \times 10^{30}$ kg $\qquad R_s = \frac{2GM_s}{c^2}$ $\qquad R_{sun} = 6.96 \times 10^5$ Km

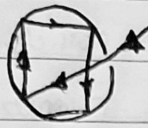
$$= \frac{2(6.67 \times 10^{-11})(1.999 \times 10^{30})}{(2.998 \times 10^8)^2}$$

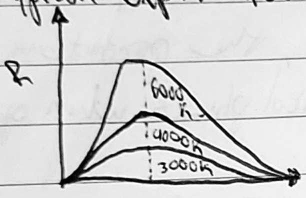
$$R_s = 2.95 \times 10^3 \text{ m} \approx 3 \text{ Km}$$

- Real blackholes need mass greater than $3 \cdot M_s$.

Quantum Mechanics

- From 1890 → 1925 a series of experiments were carried out whose results could not be interpreted within classical theories of mechanics, thermodynamics, and EM.
- The eventually led to the development of Quantum Mechanics.
- Ex.- Discovery of particle-like properties of light and wave-like properties of matter.
  - Discovery of x-rays & radioactivity.
  - the emission of sharp spectral lines by atoms in gas discharges.
- Just as the predictions of special relativity reduce to those of classical physics for $v \ll c$, the predictions of quantum theory reduce to those of classical physics when applied to macroscopic objects.
- How does Quantum Mechanics differ from classical mechanics?
  - Classical physics contends that matter and energy are infinitely divisible, while in quantum theory, physical quantities come in discrete values.
  - will examine several phenomena that force us to accept the idea that energy is quantized.
- Thermal Radiation -
  - A hot object will emit thermal radiation
  - red → orange → yellow → "white" hot
  - Predominant wavelength emitted becomes shorter as object is heated
  - Thermal radiation actually consists of a continuous spectrum and is emitted by all objects regardless of temperature.
  - Amount of radiation coming from object depends on not only on its temp. but also on such factors as the ability of surface to reflect radiation incident on it
  - To study Thermal radiation need an object that is a perfect absorber of em-radiation (reflects none) called a "black body"

- Small hole in a hollow cavity is a good approximation.
-  Incident radiation undergoes multiple reflections in interior of cavity eventually being absorbed
- very little such radiation will re-emerge from the hole
- The blackbody radiation that does emerge from the hole is representative of the radiation emitted by the walls of cavity at some particular temp
- The blackbody is characterized by its radiance $\left(\frac{\text{power}}{\text{area}}\right)$ per wavelength interval
- Typical experimental results:

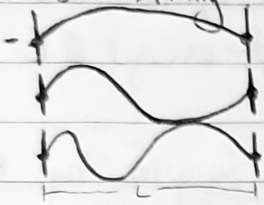


• Area under curve grows rapidly with temp
• $\lambda_{max} \propto \frac{1}{T}$, $P \propto T^4$
• Stefan-Boltzman Law: $P = \sigma A T^4$ Area of radiating surface.
• $\sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^2}$ } Stefan-Boltzman Constant.

• Also Weins displacement law: $\lambda_{max} \cdot T = 2.898 \times 10^{-3}$ m K
• Distributions of wavelengths depends only on T of cavity not on material of which it is made.

- Microscopically thermal radiation is associated with the ~~the~~ thermal motions (osc's) of atoms and molecules that make up the cavity walls
- 1800's attempted to use EM & Thermodynamics to try and explain results
- based on osc'ing charges radiating em-radiation.
- Assumed cavity filled with EM standing waves and that each wave contributes one energy KT to the radiation in the cavity. ($K = 1.381 \times 10^{-23} \frac{J}{K}$)
- Result comes from law of equipartition of energy which states that for any classical system containing a large # of entities of the same kind that the energy power per degree of freedom is $\frac{1}{2}kT$

- Here entities are 'standing waves' which, have DOF in electric field amplitude
- Exhibit SHM → which yields another DOF in the potential energy.
  - ie. pendulum, spring.
- # of standing waves can be obtained through analogy with waves on a string:



- Note there is no limit to how high the freq. can be.

etc. - $\lambda = \frac{2L}{n}$ $n = 1, 2, 3 \ldots$

- # of possible standing waves between λ_1 & λ_2 is:
  - $n_2 - n_1 = 2L\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$
- In small interval from λ to $\lambda + d\lambda$ # of standing waves is:
  - $N(\lambda)d\lambda = 2L\left(\frac{1}{\lambda} - \frac{1}{\lambda + d\lambda}\right)$
  - $N(\lambda)d\lambda = \frac{2L}{\lambda^2} d\lambda$
- Extending to 3D:
  - $N(\lambda)d\lambda = \frac{8\pi V}{\lambda^4} d\lambda$
  - and $\frac{N(\lambda)}{V} = \frac{8\pi}{\lambda^4}$
- Radiance relates to energy density through $R(\lambda) = \frac{c}{4} u(\lambda)$
- $u(\lambda) =$ (# of standing waves per unit volume)(Energy per standing wave)
  $= \frac{8\pi}{\lambda^4} kT$
- Thus $R(\lambda) = \frac{c}{4} \frac{8\pi}{\lambda^4} kT = \frac{c 2\pi kT}{\lambda^4}$ } Rayleigh-Jeans Formula.
- But this formula fails to agree with experimental results.
- & presents a serious problem for classical physics.

- In 1900 Max Planck came up with theory that could explain experimental results
- He realized he needed to make $R \to 0$ as $\lambda \to 0$ (as $f \to \infty$)
- In classical phys., freq can have any amount of energy
- Planck proposed that oscing atoms can only emit or absorb radiation in discrete amounts (quanta) with one integer multiples of a basic amount of energy ε
- $E = n\varepsilon$, $n = 1, 2, 3 \ldots$
- Proposed energy of the quanta is given by $\varepsilon = hf$ — prop. constant
- Since energy of quanta is prop to f, and no individual standing wave can have energy greater than kT, no standing wave can exist whose energy quanta is greater than kT.
- This limited the high freq. (low λ) radiant intensity and solved the UV-catastrophe
- Planck obtained: $R(\lambda) = \frac{c}{4}\frac{8\pi}{\lambda^4}\left(\frac{hc}{\lambda}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)$
- h = Planck constant $= 6.626 \times 10^{-34}\ \frac{J}{s}$

Example

- $T = 2.735K$. What's λ_{max}? Freq? Total Power on Earth?

a) $\lambda_{max} T = 2.898 \times 10^{-3}$ mK

$\lambda_{max} = \frac{2.898 \times 10^{-3}}{2.735}$

$\lambda_{max} = 1.06 \times 10^{-3}$ m

b) $f = \frac{c}{\lambda}$

$= \frac{c}{1.06 \times 10^{-3}}$

$f = 2.83 \times 10^{11}$ Hz

c) $P = \sigma A T^4$

$P = \sigma (4\pi r^2) T^4$

$P = (5.67 \times 10^{-8})(4\pi(6.37 \times 10^6)^2)(2.735^4)$

$P = 1.62 \times 10^9$ W

Derivation of Weins Displacement Law

- $R(\lambda) = \frac{c}{4}\frac{8\pi}{\lambda^4}\left(\frac{hc}{\lambda}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)$

$$R(\lambda) = 2\pi c^2 h \left(\frac{1}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)$$

$$\frac{dR(\lambda)}{d\lambda} = 0 = 2\pi hc^2 \left[\frac{-5}{\lambda^6\left(e^{\frac{hc}{\lambda kT}}-1\right)} - \frac{e^{\frac{hc}{\lambda kT}}}{\lambda^5\left(e^{\frac{hc}{\lambda kT}}-1\right)^2}\frac{hc}{kT}\left(\frac{-1}{\lambda^2}\right)\right]$$

$$= \frac{2\pi hc^2}{\lambda^6\left(e^{\frac{hc}{\lambda kT}}-1\right)}\left[\underbrace{-5 + \frac{hc}{kT\lambda}\frac{e^{\frac{hc}{\lambda kT}}}{\left(e^{\frac{hc}{\lambda kT}}-1\right)}}_{\text{Set to zero}}\right]$$

$$5 = \frac{hc}{\lambda kT}\frac{e^{\frac{hc}{\lambda kT}}}{e^{\frac{hc}{\lambda kT}}-1}, \quad x=\frac{hc}{\lambda kT}$$

$$5 = x\frac{e^x}{e^x-1} = \frac{xe^x}{e^x-1}, \quad x=4.966$$

$$\frac{hc}{\lambda kT} = x$$

$$\frac{hc}{xk} = \lambda_{max}T$$

$$2.898\times10^{-3} = \lambda_{max}T$$

Derivation of Stefan-Boltzman Law

- $\frac{P}{A} = \int^{\infty} R(\lambda)\,d\lambda$

$$\frac{P}{A} = \int_0^{\infty} 2\pi hc^2\left(\frac{1}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)d\lambda, \quad x=\frac{hc}{\lambda kT}, \quad \lambda=\frac{hc}{kTx}, \quad \frac{d\lambda}{dx}=\frac{-hc}{\lambda T x^2}$$

$$\frac{P}{A} = 2\pi hc^2\int_{\infty}^{0}\left(\frac{kTx}{hc}\right)^5\left(\frac{1}{e^x-1}\right)\left(\frac{-hc}{\lambda T x^2}\right)dx$$

$$\frac{P}{A} = 2\pi hc^2\left(\frac{kT}{hc}\right)^4\left(\frac{\pi^4}{15}\right) \Rightarrow \frac{2}{15}\pi^5\frac{k^4\,T^4}{h^3\,c^2} \Rightarrow 5.67\times10^{-8}\,T^4$$

Hypothesis

- Planck's hypothesis quantized the energies of vibrating molecules and showed that the could only emit/absorb radiation in discrete bundles
- Einstein postulated that the EM-radiation itself was quantized
- The was put to the test with the photoelectric effect
-

Albert in Wonderland Problem

- Einstein and Lorentz play tennis. They stand 20m apart with no net, ~~and~~ play with a ball of 0.0580 kg, and gravity in negligible.

a.) Lorentz serves at 80 $\frac{m}{s}$. What is the KE and momentum?

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ ⇒ Note: $v \ll c$ → use binomal expansion $(1+x)^n = 1 + \frac{nx}{1!} + \ldots$

$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{v^2}{2c^2}$

$KE = mc^2(\gamma - 1)$

$KE = mc^2\left(\cancel{1} + \frac{v^2}{2c^2} - \cancel{1}\right)$

$KE = \frac{mv^2}{2}$ ⇒ $\frac{(0.058)(80^2)}{2} \approx 186\ J$.

$p = mv\gamma$ neglect

$= mv\left(1+\frac{v^2}{2c^2}\right)$

$= (0.058)(80^2)$

$p = 4.64\ \frac{kg\,m}{s}$

b.) Einstein slams a return at $1.8\times10^8\ \frac{m}{s}$. What are the balls KE & p?

$KE = mc^2(\gamma - 1)$

$= mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$

$= mc^2\left(\frac{1}{\sqrt{1-\left(\frac{1.8\times10^8}{c}\right)^2}} - 1\right)$

$KE = 1.3059\times10^{15}\ J$

$p = mv\gamma$

$p = (0.058)(1.8\times10^8)\left(\frac{1}{\sqrt{1-\left(\frac{1.8\times10^8}{c}\right)^2}}\right)$

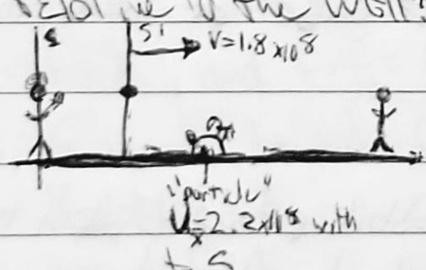
$p = 1.3055\times10^7\ \frac{kg\,m}{s}$

or

$(pc)^2 = E^2 - (mc^2)^2$ if its easier.

- Remember $1eV = 1.602\times10^{-19}\ J$.

c.) During Einsteins return a robot runs beside the court from Einstein to Lorentz at 2.2×10^8 relative to him. What is the speed of the robot relative to the wall?



$u_x' = \frac{u_x - V}{1 - \frac{u_x V}{c^2}}$ ← with respect to s'

$= \frac{2.2\times10^8 - 1.8\times10^8}{1 - \frac{(2.2\times10^8)(1.8\times10^8)}{c^2}}$

≈ 0.715

d.) The rabbit wears a LED that emits green $\lambda = 514$ nm light. What colour will i) Einstein see? ii) Lorentz?

i) $f = f_0\sqrt{\frac{1 \pm \beta}{1 \mp \beta}}$ $f = \frac{c}{\lambda}$ $\lambda = 1310$ nm

$f = f_0\sqrt{\frac{1-\beta}{1+\beta}}$ $f = \frac{c}{\lambda}$ Redshift

$f = f_0\sqrt{\frac{1-\frac{v}{c}}{1-\frac{v}{c}}}$

$f = \ldots$ $\lambda = \frac{c}{f}$

ii) ... $\lambda = 201$ nm Blue shift.

e) What does rabbit measure as length of court?

$L = \frac{L_0}{\gamma}$

$= 20\sqrt{1-\frac{v^2}{c^2}}$

$L = 13.59$ m

f) How long does the rabbit run according to players?

$t = \frac{d}{v} = 9.09 \times 10^{-8}$ s.

g) Rabbit carries a pocket watch to measure it, whats his time measurement?

$t = \frac{L}{v} = 6.177 \times 10^{-8}$ or $\Delta t_0 = \gamma \Delta t_0$ or lorentz transformation

h.) Assume & instant rabbit passes Einstein he hits the ball, what is the location of the ball when the ball reaches lorentz according to i) rabbit ii) Einstein.

i) $d = vt$

$d = (0.715 \times 10^8)(6.177 \times 10^{-8})$

$d = -4.42$ m

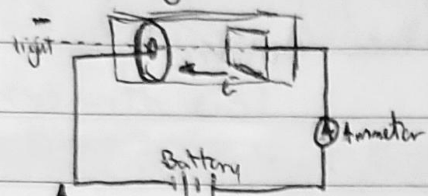
ii) $\Delta t = 9.09 \times 10^{-8}$ s

$v = 1.8 \times 10^8$ m/s

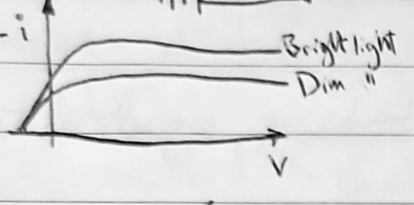
$d = 16.3636$ m

The Photoelectric Effect

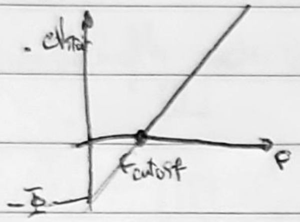
- e^- eject from metallic target upon illumination with light
- Current begins immediately



- making V increasingly neg., target eventually results in most energetic e^-'s not having enough energy to make it to collector & current stops.



- $K_{max} = eV_{stop}$; same for bright and dim lights.



- f_{cutoff} is the lowest frequency for which e^-'s eject

- Einstein resolved discrepancies with classical em.
- Proposed - $E=hf$ for photon.
- To just escape the target the e^- must have a minimum energy to overcome the binding energy of the e^- to the atomic nucleus of the atoms making up the target.
- The minimum E is called the work function
  - Φ notation
  - It is different for different metals.
- If energy of photons is less than Φ no e^-'s eject regardless of intensity
- $E = hf = \Phi$, and $f = f_0 = f_{cutoff}$ so $f_0 = \frac{\Phi}{h}$
- As f increase beyond f_0 e^-'s are ejected with max KE
  - $K_{max} = hf - \Phi$
  - $eV_{stop} = hf - \Phi$

Example

- Consider light shining on a photographic plate, The light will be recorded if it dissociates an AgBr molecule in the plate. The minimum E to do this is 10^{-19}. Evaluate λ_{cutoff}
- $E = hf_0$

$$\frac{E}{h} = \frac{c}{\lambda_0} \Rightarrow \lambda_0 = \frac{ch}{E} \Rightarrow 1.986 \times 10^{-6}\ m$$

- X-Ray $\rightarrow \lambda = 0.71Å$, e^- circle around at radius r, magnetic induction = B, $rB = 1.88 \times 10^{-4}\ Tm$
  - $K_{max} = \frac{1}{2}mv^2$ $\quad \frac{mv^2}{r} = qvB$

$$= \frac{m^2v^2}{2m} \qquad mv = qBr$$

$$= \frac{q^2B^2r^2}{2m} \qquad p = qBr$$

$$= 4.978976 \times 10^{-16}\ J \cdot \frac{1}{1.602 \times 10^{-19}} = 3.108\ KeV$$

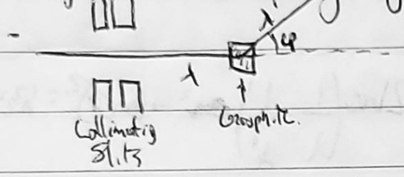
  - $K_{max} = hf - \Phi$

$$\Phi = \frac{hc}{\lambda} - K_{max}$$

$$\Phi = 2.29995 \times 10^{-15}\ J = 14.4\ KeV$$

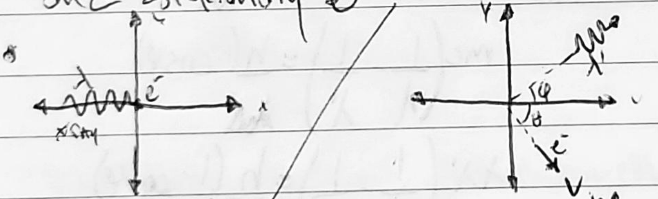
The Compton Effect.

- In 1916 Einstein proposed that not only energy but also momentum can be transferred by photons interacting with matter
- 1923, Compton did expt. to confirm this (studying interactions of x-rays with e^-'s).
- Comptons subjected graphite to x-rays.
  - Classically, free e^- would move in oscillatory motion under influence of waves,
- Compton measured intensity of the scattered x-rays as a function of wavelength and scattering angle



- The highest intensity of scattered x-ray at a wavelength longer than incident.

- He interpreted his results within the evolving quantum picture by assuming that the particle-like photons had collided with free-e^- in the target, losing energy to them and since $E = hf$ emerging with lower freq (higher λ)
- Treat as a relativistic elastic collision between incident photons and stationary e^-.



$(pc)^2 = E^2 - \cancel{(mc^2)^2}$ NO MASS

$p = \frac{E}{c}$

$p = \frac{h}{\lambda}$

- $E_2 = E_1 \Rightarrow \frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \gamma mc^2 \Rightarrow \frac{h}{\lambda} + mc = \frac{h}{\lambda'} + (\gamma - 1)mc$
- $p_x = p_x' \Rightarrow \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\varphi + \gamma mv \cos\theta$

  $p_y = p_y' \Rightarrow 0 = \frac{h}{\lambda'}\sin\theta - \gamma mv \sin\theta$
- E_2 ① $\rightarrow \frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \gamma mc^2$

  ② $\rightarrow \frac{h}{c} = \frac{h}{\lambda'} + (\gamma - 1)mc^2$

  ③ $\rightarrow \gamma mv \cos\theta = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\varphi$

  ④ $\rightarrow \gamma mv \sin\theta = \frac{h}{\lambda'}\sin\varphi$

- We want to find the Compton Shift ($\Delta\lambda = \lambda' - \lambda$)
- Eliminate φ using ② & ③
  - Square ② $\to$ $\gamma^2 m^2 v^2 \cos^2\theta = \frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'}\cos\varphi + \frac{h^2}{\lambda'^2}\cos\varphi$ ③
  - Square ③ $\to$ $\gamma^2 m^2 v^2 \sin^2\theta = \frac{h^2}{\lambda'^2}\sin^2\varphi$ ④
  - Add ②+③: $\gamma^2 m^2 v^2 (\cos^2\theta + \sin^2\theta) = \frac{h^2}{\lambda} - \frac{2h^2}{\lambda\lambda'}\cos\varphi + \frac{h^2}{\lambda'^2}(\cos^2\theta + \sin\theta)$

$$\gamma^2 m^2 v^2 = \frac{h^2}{\lambda} - \frac{2h^2}{\lambda\lambda'}\cos\varphi + \frac{h^2}{\lambda'^2} = p^2 \quad ⑥$$

  - Sub in p^2 to: $E^2 = (pc)^2 + (mc^2)^2$

$$(\gamma mc^2)^2 = c^2(\gamma mv)^2 + (mc^2)^2$$

  - Sub in 1 & 6: $\left(\frac{hc}{\lambda} + mc^2 - \frac{hc}{\lambda'}\right)^2 = c^2\left(\frac{h^2}{\lambda^2} - \frac{2h^2}{\lambda\lambda'} + \frac{h^2}{\lambda'^2}\right) + (mc^2)^2$

$$\left(hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + mc^2\right)^2 =$$

$$h^2c^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)^2 + 2hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)mc^2 + \cancel{(mc^2)^2} = h^2c^2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2}\right) - \frac{2h^2c^2\cos\varphi}{\lambda\lambda'} + \cancel{(mc^2)^2}$$

$$h^2c^2\left(\cancel{\frac{1}{\lambda^2}} - \frac{2}{\lambda\lambda'} - \cancel{\frac{1}{\lambda'^2}}\right) + 2hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)mc^2 = h^2c^2\left(\cancel{\frac{1}{\lambda^2}} + \cancel{\frac{1}{\lambda'^2}}\right) - \frac{2h^2c^2}{\lambda\lambda'}\cos\varphi$$

  - Divide by $2hc^2$: $\frac{-2h^2c^2}{\lambda\lambda'} + 2hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)mc^2 = \frac{-2h^2c^2}{\lambda\lambda'}\cos\varphi$

$$\frac{-h}{\lambda\lambda'} + mc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{-h}{\lambda\lambda'}\cos\varphi$$

$$mc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h}{\lambda\lambda'}(1 - \cos\varphi)$$

$$\lambda\lambda'\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h}{mc}(1 - \cos\varphi)$$

$$\Delta\lambda = (\lambda' - \lambda) = \frac{h}{mc}(1 - \cos\varphi)$$

- $\frac{h}{mc}$ is the compton wavelength or λ_c
- Maximum shift will be, at $\varphi = 180°$ $\left((1 - \cos(180)) \to (1+1) \to \frac{2h}{mc}\right)$

Example

- Incident Photons have E = 10.39 keV and are compton scattered at 45°.

  - find E at the angle? $E = \frac{hc}{\lambda}$ → $\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$

  $\lambda = \frac{hc}{E} = 1.19345\times10^{-10}$ $\Delta\lambda = 7.10576\times10^{-13}$ m

$E = \frac{hc}{\lambda'}$ ← $\lambda' = \lambda + \Delta\lambda = 1.19345\times10^{-10} + 0.007106\times10^{-10}$ m

$E = 10.328$ keV $\lambda = 1.2005\times10^{-10}$ m

- find KE Given to scattered e^-?

$K = E - E' = 10.39 - 10.328 = 0.062$ keV

- find θ at which e^- is scattered

Divide (4)&(5)

$$\frac{\cos^2\theta}{\sin^2\theta} = \frac{\lambda'^2}{\lambda^2\sin^2\varphi} - \frac{2\lambda'^2\cos\varphi}{\lambda\lambda'\sin^2\varphi} + \frac{\lambda'^2\cos^2\varphi}{\lambda'^2\sin^2\theta}$$

$$\frac{1}{\tan^2\theta} = \frac{1}{\sin^2\varphi}\left[\frac{\lambda'^2}{\lambda^2} - \frac{2\lambda'}{\lambda}\cos\varphi + \cos^2\varphi\right]$$

$$= \frac{1}{\sin^2\varphi}\left[\frac{\lambda'}{\lambda} - \cos\varphi\right]^2$$

$$\tan\theta = \frac{\sin\varphi}{\left(\frac{\lambda'}{\lambda} - \cos\varphi\right)} \rightarrow \theta = \tan^{-1}\left[\frac{\sin\varphi}{\left(\frac{\lambda'}{\lambda} - \cos\varphi\right)}\right]$$

$\theta =$ 67.1°

- Blackbody radiation, photoelectron effect, & compton scattering expts established basis for treating photons.

Summary of Basic Properties of Photons

- ① Like an EM wave, photons travel at the speed of light.
- ② Photons have zero mass and zero rest energy
- ③ Photons carry enough E & p which are related to the freq. & wavelength of the em wave by: $E = hf = \frac{hc}{\lambda}$ & $p = \frac{h}{\lambda}$
- ④ Photons can be created or destroyed when radiation is emitted or absorbed
- ⑤ Photons can have particle like collisions with other particles.

Probability & Photons

- Probability that a photon is detected in particle expt is proportional to the Intensity
- Since I is $\propto |E|^2$ where E is electric field vector of light we say: probability to observe photons $\propto |E|^2$
- Accepting that light travels as a probability wave becomes more important when one considered a single photon version of double slit experiment
- An interference pattern still emerges but on a probability fringe
- This brought on the wave-particle duality
- de Broglie suggested there should be matter waves
  - $p = \frac{h}{\lambda}$
  - $\lambda = \frac{h}{p}$ ← de Broglie wavelength.
- There is no expt that can be setup to view wave like nature of macroscopic objects

Example

- A Beam of 77K cold neutrons, spacing = 0.4nm. Angle of 1st order Bragg peak?
(Note: Can be treated as an ideal gas [equilibrium temp / no change])

they have equipartition of energy → $\frac{1}{2}k_B T$

$K = \frac{3}{2}k_B T$ ← Kinetic energy

$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}}$

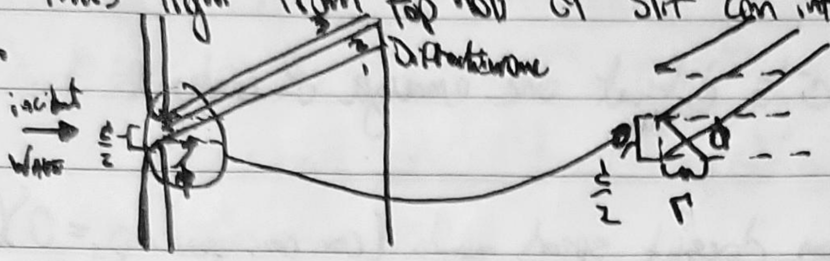
$n\lambda = d\sin\varphi$ ← Bragg Eqn

$\lambda = d\sin\varphi$

$\sin^{-1}\left(\frac{\lambda}{d}\right) = \varphi$ → $\sin^{-1}\left(\frac{\lambda}{d}\right) = \varphi = \sin^{-1}\left(\frac{h}{\sqrt{2mK}\,d}\right) = \varphi = \sin^{-1}\left(\frac{h}{\sqrt{2m\frac{3}{2}k_B T}\,d}\right)$

$= \sin^{-1}\left(\frac{6.626\times10^{-34}}{\sqrt{3(1.97\times10^{-23})77(1.38\times10^{-23})}\,0.7\times10^{-9}}\right) = 49.37° \approx 49°$

Single Slit Diffraction

- Diffraction also occurs from a single slit or aperature
- Consider light principle incident on slit of width d
- According to Huygen's Principle each portion of slit acts as a secondary source of waves
- Thus light from top half of slit can interfere with that bottom half.
- 

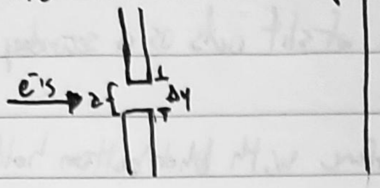
$$r = \frac{d}{2}\sin\phi$$

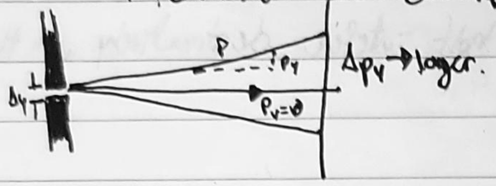
- If $r = \frac{\lambda}{2}$ waves from upper half interfere destructively with those of the lower half
- $\sin\phi_{min} = \frac{\lambda}{d}$

Direct Observation of Electron Diffraction

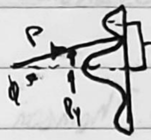
- G. Thomson passed an $\bar{e}$ beam through a small aperature & observed $\bar{e}$ diffraction directly
- 

Single Slit Linear

- Assume horizontally propagating e^- beam passes through linear aperture
- We know vertical component of e^- to within aperture dimension
- 

- Due to wave properties e^-'s diffract and emerge at angle ϕ
- Angle ϕ is unpredictable
- If $d >> \lambda$, diffracted beam doesn't spread much (can assume $p_y = 0$ (y momentum $= 0$))
- 

- The more you know about position the less you know about momentum &, vice versa.
- Quantitatively: ① $\Delta y \simeq d$ First Min $\rightarrow \sin\phi = \frac{\lambda}{d}$



$$p_{y\,1st\,Min} \rightarrow p\sin\phi = \frac{p\lambda}{d}$$

$$\Delta p_y \geq \frac{p\lambda}{d} = \frac{h}{\lambda}\cdot\frac{\lambda}{d} = \frac{h}{d} \quad ②$$

Eqn ①·② $\Delta y\,\Delta p_y \geq d\cdot\frac{h}{d} \Rightarrow \Delta y\,\Delta p_y \geq h$ ← Uncertainty Principle

- Heisenbergs Uncertainty Principle
  - $\Delta x\,\Delta p \geq \frac{\hbar}{2}$, where $\hbar = \frac{h}{2\pi}$
  - The result of $\frac{\hbar}{2}$ is rarely ever obtained.

Example

- e^- moves in x with $v = 1\times10^6$. The speed is measured within 1%.

a.) What precision can we measure its position?

$p_x = mv = (9.11\times10^{-31})(1\times10^6) = 9.11\times10^{-25}$

$\Delta p_x = 0.01 p_x = 9.11\times10^{-27}$

$\Delta x\,\Delta p_x \geq \frac{\hbar}{2} \Rightarrow \Delta x \geq \frac{\hbar}{2\Delta p_x} \Rightarrow \Delta x \geq 5.79\times10^{-9}\,m$

b.) What can be said about the p_y?

- If e^- has a $p_y = 0$, then $\Delta y \geq \infty$

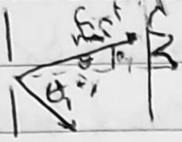
- Since we know we can easily locate the position of a baseball as it crosses a point (x,y,z directions) we need to consider that our calculation assumes there is only momentum in the x direction
- Suppose a tiny component in y-direction
- Assume it is 10^{-10} of x-component velocity & our errors in its measurement is 1%.
- The $\Delta p_y = 10^{-10} \Delta p_x \Rightarrow 6.1625\times10^{-12}$
- $\Delta y \geq \frac{\hbar}{2\Delta p_y} \Rightarrow \Delta y \geq 8.5\times10^{-24}\ m \approx 10^{-23}$
- h is so small that the limitations it impose are only visible on subatomic scale.
- Another form of the Uncertainty Principle involves E & t.
  - Consider a wave 
  - A counter counts how many crests per unit t
  - $f = \frac{N}{\Delta t}$, and the uncertainty in N is 1λ $(\Delta N \geq 1)$
  - $\Delta f = \frac{\Delta N}{\Delta t} \Rightarrow \Delta f \Delta t \geq \Delta N$

    $\Delta E \frac{\Delta t}{h} \geq \Delta N \Rightarrow \Delta E \Delta t \geq h$
  - Better Calculation $\rightarrow \Delta E \Delta t \geq \frac{\hbar}{2}$

Example

- The Strong force which holds protons and neutrons together in the nucleus which is believed to come from the exchange of particles in the nucleon (π-mesons $= 135 \frac{MeV}{c^2}$)
- $F = ma = \frac{m\,d(v)}{dt} = \frac{d(mv)}{dt} = \frac{d(p)}{dt}$
- How can a neutron/proton emit this and abide the cons. energy?
  - The answer is we can't measure ΔE accurate enough. $(\Delta E \geq \frac{\hbar}{\Delta t 2})$
- How long can a pion exist?
  - $\Delta t \geq \frac{\hbar}{\Delta E 2} \Rightarrow 2.438\times10^{-24}\ s.$
- Assume pions $v \approx c$, Estimate range of force?
  - $r \approx c\Delta t$

    $\approx 0.73\times10^{-15}\ m$

- Diffracted particle through slit.
-  For small θ $p=\sqrt{p_x^2+p_y^2} = p \approx p_x$

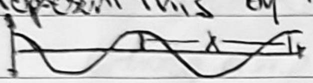
  $y \approx \frac{hD}{pd}$
- Photon transfers $\Delta p_y = 2p_{y,ph}$ to the e^-
  - $t = \frac{d}{v} = \frac{Dm}{p_x}$
  - Thus the y position by an amount $\Delta y = \Delta v_y t = \frac{\Delta p_y \Delta t}{m} = \frac{\Delta p_y l}{m p_x} = \frac{\Delta p_y D}{p_x}$
- If interference pattern is not to be significantly distorted, $\Delta y \approx y \Rightarrow \frac{\Delta p_y D}{p_x} = \frac{hD}{p_x d} \Rightarrow \Delta p_y = \frac{h}{d} \Rightarrow \Delta p_y d \ll h$
- Position can only be determined within 1λ $\Delta y_e \approx \lambda_{ph}$
- To determine which slit the photon came through must that $\Delta y_e \ll d$
- Thus if $\Delta p_y d \ll h$ then $\Delta p_y \Delta y_e$ certain must be smaller than h but this violates the uncertainty principle
- Particle and wave like properties cannot be observed at the same time
- This is Bohr's principle of complementarity.

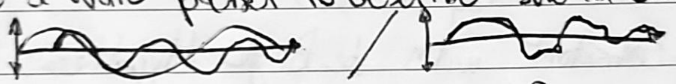
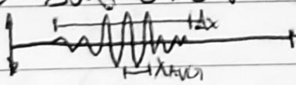
- One final question in the course:
  - What does the amplitude mean?
- Recall EM wave amplitudes meaning probability of finding photons
- Probability of finding photons $\propto |E|^2$
- Similarly probability to find said particle is proportion to the $|\text{Amplitude of the de Broglie wave}|^2$

Quantum Mechanics.

- Representing a matter wave
- de Broglie wave $\Rightarrow p = \frac{h}{\lambda}$
- Essentially assuming $\Delta p = 0$ (no uncertainty)
- According to Heisenberg $\Delta p \Delta x \geq \frac{\hbar}{2}$

  If $\Delta p = 0$, Δx has to be ∞.
- Can represent this by harmonic wave function



- Particle is completely unlocalized
- Classical particles are essentially completely localized
- Particle whose behavior must be described by quantum mechanics are approximately localized
- Use a wave packet to describe such a situation (b/c of size atoms etc.)
-  (Superposition)
- A particle described by this wave fxn is more likely to be found in some places than others.
- To describe approx. localized particles, superimpose large # of λ's such that they have considerable amplitude in a region of space and zero elsewhere
- 

→ Wave packet.
- Spread of λ's that make up wave packet is characteristic of uncertainty in momentum $\rightarrow \Delta p$
- Represent mathematically by.

$$\Psi(x) = \int_0^{\infty} A(\lambda) \cos \frac{2\pi}{\lambda} x \, d\lambda \text{ ①}$$

- If $A(\lambda)$ is a narrow distribution, $\Delta x \downarrow$
- If $A(\lambda)$ is a broader distribution, $\Delta x \uparrow$
- For convience:

$$\Psi(x) = \int_0^{\infty} A(k) \cos kx \, dk$$, where $k = \frac{2\pi}{\lambda}$ or the wave number.

$$\Psi(x) = \int_{\infty}^{0} \cancel{\tfrac{dk}{d\lambda}} A(k)\cos kx \, d\lambda \qquad \frac{dk}{d\lambda} = \frac{-2\pi}{\lambda^2}$$
$$= \int_{\infty}^{0} \frac{-2\pi}{\lambda^2} A(k)\cos kx \, dk$$
$$= \int_{0}^{\infty} A'(k) \cos kx \, dk$$

Schrodingers Approach to Quantics

- Systematic approach for finding wave-function of non-relative particle under the influence of a given potential.
- Energy is quantized for certain λ.
- Schrodingers Equation should be regarded as an eqn to agree with experiments ~~no~~ Can't be derived
- Properties the must be satisfied:
  - 1- Must conserve energy $\left(K = \frac{p^2}{2m}\right)$
  - 2- Must be consistent with de Brogle Hypothesis. $\left(K = \frac{\hbar k^2}{2\pi}\right)$
  - 3- Must be mathematically well behaved (continuous, singular, linear)
- Represent de Broglie wave

$$\Psi(x) = A\sin \frac{2\pi}{\lambda} x \, d_\lambda = A \sin kx$$

$$\frac{\partial^2 \Psi(x)}{\partial(k)} = A k \cos kx \qquad \frac{\partial^2 \Psi}{\partial^2 k^2} = \frac{-2mk}{\hbar^2} - \cancel{m}$$

$$\partial^2 \Psi(x)$$

$$\frac{\partial^2 \Psi}{dk} = \frac{\hbar^2}{2m}\frac{\partial \psi(x)}{dx^2}, \quad U\Psi(x) = E\Psi(x) \Big\} \text{One dimension time independent schrodinger eqn}$$

- In General would like to find the $\Psi(x)$ that satisfy the TISE for a $U(x)$, will also find solns occur for only certain energies E called eigenvalues.
- General Approach
  - 1- Write out TISE with appropriate $U(x)$ (If $U(x)$ is discontinuous, diff eqns for diff regions
  - 2- Find possible solutions ~~[illegible]~~ (Differential eqns).
  - 3- Apply boundary conditions to eliminate some solutions.
  - 4- If $U(x)$ is discontinuous apply continuity conditions on $\Psi(x)$ and $\frac{d\Psi}{dx}$
  - 5- Evaluate undetermined constants (A) by requiring proba finding particle somewhere in space 100

Probability density

$$P(x)dx = |\Psi(x)|^2 dx$$

$$\underbrace{\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1}_{\text{normalization condition}}$$

$\int_{x_1}^{x_2} |\Psi|^2 dx$ = the probability of finding a particle in $x_1 - x_2$.

Quantum Mechanics

- 1D → TISE ← $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi(x) = E\psi(x)$
- Probability to observe particles is proportional to $\psi^2(x)$
  - Define probability density: $P(x)dx = |\psi^2(x)|dx$
- Gives prob to observe particle in small interval dx at x
- Particle must be found somewhere:
  - $\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$
- That is the normalization condition.
- Probability of finding particle between x_1 and x_2 is
  - $P = \int_{x_1}^{x_2} |\psi(x)|^2 dx$
- No longer know absolute position of particle. thus can nolonger guarantee outcome of measurement of its position
- Instead determine the expected outcome or the average outcome of a number of measurements.
- Ex. Measure location of particle, Find position x_1, x_2, x_3 turns n_1, n_2, n_3 times etc.

$$x_{av} = \frac{n_1x_1 + n_2x_2 + n_3x_3 \ldots}{n_1 + n_2 + n_3} = \frac{\sum_i n_i x_i}{\sum n_i}$$

- Number of times n_i that one measures each x_i is proportional to probability of finding particle at that x_i
- Thus - $x_{av} = \frac{\int_{-\infty}^{\infty} P(x)x\,dx}{\underbrace{\int_{-\infty}^{\infty} P(x)\,dx}_{\to 1}} = \int_{-\infty}^{\infty} P(x)x\,dx = \int_{-\infty}^{\infty} |\psi(x)|^2 x\,dx$
- The average (or expected) value of any function of x is given by:
  - $[f(x)]_{av} = \underbrace{\langle f(x)\rangle}_{\text{Expected Value.}} = \int_{-\infty}^{\infty} P(x)f(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2 f(x)dx$
- 1D Particle in a Box / 1D infinite square well
  - ~~$\frac{\hbar^2}{2m}\frac{d^2}{dx^2} = E$~~
  - $\psi = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$

    $E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$

    $\lambda_n = \frac{2L}{n}$

- Confinement of a wave leads to quantization (Confinement Principle)
- Confined energy systems cannot exist in states with zero energy; they always have a minimum zero-point energy.
- Energies of free particles are not quantized.
- Ex, e^- in a 1D well. $L = 1\times10^{-10}$ m

a.) E_0 to E_1?

$$\Delta E = E_1 - E_0$$
$$= \frac{\hbar^2\pi^2}{2mL^2}(n_f^2 - n_i^2)$$
$$= \frac{3(6.626\times10^{-34})(\pi^2)}{2(9.11\times10^{-31})(1\times10^{-10})^2}$$
$$\Delta E = 113 \text{ eV}$$

c.) Find x_{av} when e^- is in n^{th} state

$$x_{av} = \int_{-\infty}^{\infty} |\psi|^2 x\,dx = \int_0^L |\psi|^2 x\,dx$$
$$= \int_0^L \frac{2}{L}\sin^2\frac{n\pi x}{L}\,x\,dx$$
$$= \frac{2}{L}\left[\frac{x^2}{4} - \frac{xL}{4\pi n}\sin\frac{2n\pi x}{L} - \frac{L^2}{8n^2\pi^2}\cos\frac{2\pi n x}{L}\right]_0^L$$
$$= \frac{2}{L}\left[\left(\frac{L^2}{4} - \frac{L^2}{4\pi n}\cancel{\sin 2\pi n} - \frac{L^2}{8n^2\pi^2}\cos\frac{2\pi n L}{L}\right) - \left(\frac{L^2}{4} - 0 - \frac{L^2}{8n^2\pi^2}\cos 0\right)\right]$$
$$= \frac{2}{L}\left[\frac{L^2}{4} - \cancel{\frac{L^2}{8\pi^2 n^2}} + \cancel{\frac{L^2}{8\pi^2 n^2}}\right] = \frac{L}{2}$$

$x_{av} = \frac{L}{2}$, and it is independent of state n.

b.) In the ground, prob. of e^- in $x_1 = 0.09\times10^{-10}$ m to $x_2 = 0.110\times10^{-10}$ m

0.09L 0.11L

$$P = \int_{.09L}^{.11L} |\psi|^2 dx = \int_{.09L}^{.11L} \left|\sqrt{\tfrac{2}{L}}\sin\tfrac{\pi x}{L}\right|^2 dx$$
$$= \int_{.09L}^{.11L} \frac{2}{L}\sin^2\frac{\pi x}{L}\,dx = \frac{2}{L}\left[\frac{x}{2} - \frac{L}{4\pi}\sin 2\frac{\pi x}{L}\right]_{.09L}^{.11L}$$
$$= \frac{2}{L}\left[\left(\frac{0.11L}{2} - \frac{L}{4\pi}\sin 2\frac{\pi(.11L)}{L}\right) - \left(\frac{0.09L}{2} - \frac{L}{4\pi}\sin 2\frac{\pi\, 0.09L}{L}\right)\right]$$
$$= \frac{2}{L}\left[\frac{0.11L - 0.09L}{2} + \frac{L}{4\pi}\sin\pi 0.18 - \frac{L}{4\pi}\sin\pi 0.22\right]$$
$$= 0.02 - \frac{1}{2\pi}(\sin\pi 0.22 - \sin\pi 0.18)$$
$$= 0.02 - \frac{1}{2\pi}(0.637 - 0.536)$$
$$= 0.02 - \frac{0.101}{2\pi} = 0.383\%$$

Calc MUST BE IN RADS

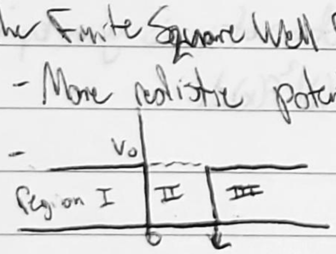
Another way of doing it, for small intervals $(x_2 - x_1)$

$$P(x)dx = |\psi|^2 dx$$
$$= \left|\sqrt{\tfrac{2}{L}}\sin\tfrac{\pi x}{L}\right|^2 \Delta x$$
$$= \frac{2}{L}\sin^2\frac{\pi x}{L}(0.11L - 0.09L)$$
$$= 2\sin^2\frac{\pi x}{L}(0.02)$$
$$= 0.04\sin^2\frac{\pi 0.1L}{L}$$
$$= 3.82\%$$

0.09L 0.11L

x = 0.1L

- The Finite Square Well Potential
  - More realistic potential



$$U(x) = V_0 \rightarrow x \le 0$$
$$= 0 \rightarrow 0 \le x \le L$$
$$= V_0 \rightarrow x \ge L$$

- In quantum mechanics it is only as $V_0 \to \infty$ that particle is absolutely trapped in region II. There is a finite prob. that particle is in region I & III ($\psi \ne 0$ for I & III)
- Outside well: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$, solution must be form $\psi = Ae^{\alpha x}$, $\frac{d\psi}{dx} = A\alpha e^{\alpha x}$, $\frac{d^2\psi}{dx^2} = \alpha^2\psi$

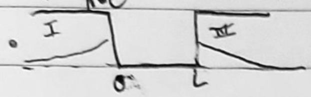
→ PTO

- $-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx} = (E-V_0)\Psi$

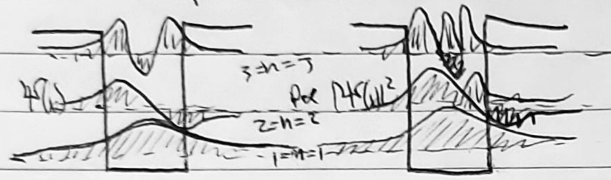
$-\frac{\hbar^2}{2m}\alpha^2\Psi = (E-V_0)\Psi$ ← $E<V_0$ so α is positive.

$\alpha = \pm\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$ ← $\alpha \neq -$ b/c $\Psi \to \infty$ and $x \to \infty$

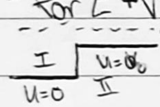
A&B aren't necessarily the same.

- $\Psi = A e^{+\sqrt{2m(V_0-E)}\frac{x}{\hbar^2}}$ ← REGION I $x<0$
- $\Psi = B e^{-\sqrt{2m(V_0-E)}\frac{x}{\hbar^2}}$ ← REGION III $x>L$
- 

- Region II, $U=0$. $\Psi(x) = A'\sin k_x + B'\cos k_x$
- wave fxns & prob densities look like.



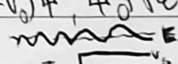
- Only a finite number of bound states, which is limited by the height of well.
- Each bound state has well defined energy Eigenvalue, E_n
- Particles with $E>V_0$ are not bound; energy not quantized
- We will see form of wave fxn when we consider Potential energy step.
- Consider particle of energy E in constant potential encountering another constant potential.
  - For $E > V_0$ → Region I → $-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \cancel{V_0}\Psi = E\Psi$ Note $K > K_0$



$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi$, Ψ of the form $\Psi = A\sin Kx + B\cos Kx$ $K = \frac{\sqrt{2mE}}{\hbar}$

→ Region II → $\frac{-\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = (E-V_0)\Psi$, Ψ of the form $= C\sin K_0 x + D\cos K_0 x$ $K_0 = \frac{\sqrt{2m|E-V_0|}}{\hbar}$

  - Recall $K = \frac{2\pi}{\lambda}$, thus we expect $\lambda_1 < \lambda_2$



  - Relationships between A, B, C, & D can be found by requiring $\Psi(x)$ & $\frac{d\Psi}{dx}$ is continuous at $x=0$.
  - First rewrite wave fxn as complex exponentials.

$\Psi(x) = A' e^{iKx} + B' e^{-iKx}$ $x<0$

$\Psi(x) = A'[\cos Kx + i\sin Kx] + B'[\cos Kx - i\sin Kx] = \underbrace{i(A'-B')}_{A}\sin Kx + \underbrace{(A'-B')}_{B}\cos Kx$

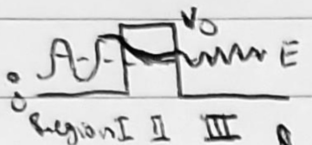
$\Psi(x) = i(C'-D')\sin K_0 x + (C'-D')\cos K_0 x$ $x>0$

  - Time Dependence → $\Psi(x,t) = \Psi(x)e^{-i\omega t}$ where ω comes from de Broglie relationship $E = hf$ & $E = \hbar\omega$

$E = \hbar\omega \Rightarrow \omega = \frac{E}{\hbar}$

- Interpret using uncertainty principle
- $\Delta E = V_0 - E + K$ particle can borrow energy
- $\Delta t \sim \frac{\hbar}{2\Delta E}$ for a time
- To penetrate a step, Δx, but is eventually reflected back

Potential Barrier



$k_1 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$ $\quad k_3 = k_1 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

λ are the same, amplitude changes

- The transmission coefficient is found to be:

$$T = \left[1 + \frac{\sinh^2(\alpha L)}{4\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}\right]^{-1}$$ where $\alpha = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

and $\sinh = \frac{e^x - e^{-x}}{2}$.

$$T = \left[\frac{\frac{4E}{V_0}\left(1-\frac{E}{V_0}\right) + \sinh^2(\alpha L)}{4\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}\right]^{-1}$$

$$= \frac{\frac{4E}{V}\left(1-\frac{E}{V_0}\right)}{\frac{4E}{V_0}\left(1-\frac{E}{V_0}\right) + \left(\frac{e^{\alpha L} - e^{-\alpha L}}{2}\right)^2}$$

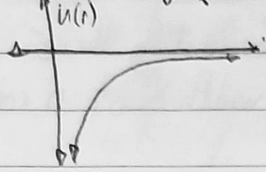
$$= \frac{4\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}{\frac{e^{2\alpha L}}{4}}$$ ← when $\alpha L \gg 1$, $e^{\alpha L} \gg e^{-\alpha L}$

$$T \simeq 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\alpha L}$$ ← in cases where $e^{-2\alpha L}$ is highly dominant

$T \propto e^{-2\alpha L}$

Quantum Mechanics of Hydrogen

- Simplest atom of 1e^- and 1p^+
- e^- is held by the coulomb potential
- $u(r) = \frac{-1}{4\pi\epsilon_0}\frac{e^2}{r}$ — permittivity of freespace $= 8.8842\times10^{-12}\frac{F}{m}$



$r \to \infty \quad u \to 0$

$r \to 0 \quad u \to -\infty$

- Want to find wave fxn and eigenvalues that define quantum states of H-atom, need to solve SE with $u(r)$ above.
- Need to use spherical coordinates (r, θ, φ).
- TISE: $-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] + u(r,\theta,\varphi)\psi = E\psi$
  - where $\psi = \psi(r,\theta,\varphi)$
- Look for solutions that are separable and can be factored as
  - $\psi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$
- Find that 3 quantum numbers label the solutions
  - $\psi_{nlm_l}(r,\theta,\varphi) = \underbrace{R_{nl}(r)}_{\text{Radial fxn}}\,\underbrace{\Theta_{lm_l}(\theta)}_{\text{Polar fxn}}\,\underbrace{\Phi_{m_l}(\varphi)}_{\text{Azimuthal fxn}}$

  n = principle quantum # $= 1, 2, 3 \ldots$
  L = angular momentum $= 0, 1, \ldots n-1$
  m_L = magnetic qn $= -L \ldots +L$
- Principle qn specifies the energy levels
  - $E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} = \frac{-13.6\,eV}{n^2}$
- sheet.
- Ground state potential TISE
  - $-\frac{\hbar}{2m}\left[\frac{1}{a_0^2}\psi_{100} + \frac{2}{r}\left(-\frac{1}{a_0}\psi_{100}\right)\right] - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\psi_{100} = E\psi_{100}$
  - rearrange
  - $\underbrace{-\frac{\hbar^2}{2m}\frac{1}{a_0^2}\psi_{100} + \frac{\hbar^2}{mra_0}\psi_{100} = E\psi_{100} + \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\psi_{100}}$ $\ldots = \frac{-13.6\,eV}{n^2}$

  equal equal

- As with 1-D wave fxns $\psi^* \psi = |\psi|^2$ gives the probability density of finding particle at location (r, θ, φ)
- Thus probability is: $|\psi_{nlm}|^2 \, dV$, where $dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$.
- $P = |R_{nl}(r)|^2 \, |\Theta_{lm_l}(\theta)|^2 \, |\Phi_{m_l}(\varphi)|^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$.

  $\Phi^* \Phi$

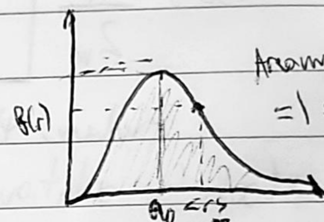
- Can find radial probability density $(P(r)dr)$ for locating the e^- somewhere within a thin spherical shell of width dr at a radius r by integrating over θ & φ.
- $P(r)dr = |R_{nl}(r)|^2 r^2 dr \int_0^{\pi} |\Theta_{lm_l}(\theta)|^2 \sin\theta \, d\theta \int_0^{2\pi} |\Phi_{m_l}(\varphi)|^2 d\varphi$
- Ex. Ground State

$$P(r)dr = \left|\frac{2}{a_0^{3/2}} e^{-\frac{r}{a_0}}\right|^2 r^2 dr \int_0^{\pi} \left(\frac{1}{\sqrt{2}}\right)^2 \sin\theta \, d\theta \int_0^{2\pi} \left|\frac{1}{\sqrt{2\pi}}\right|^2 d\varphi$$

$$= \frac{4 e^{-\frac{2r}{a_0}}}{a^3} \cdot \frac{1}{2} \int_0^{\pi} \sin\theta \, d\theta \, \frac{1}{2\pi} \int_0^{2\pi} d\varphi$$

$$= \frac{e^{-\frac{2r}{a_0}}}{\pi a^3} [\cos\theta]_0^{\pi} \, [\varphi]_0^{2\pi}$$

$$= \frac{e^{-\frac{2r}{a_0}}}{\pi a^3} (2)(2\pi) = \frac{4\pi e^{-\frac{2r}{a_0}}}{\pi a^3} = \frac{4 e^{-\frac{2r}{a_0}}}{a^3}$$



Area under curve $= 1 = \int_0^{\infty} P(r)dr$

$$\frac{dP(r)}{dr} = 0 \Rightarrow \frac{4}{a_0}\left[2re^{-\frac{2r}{a_0}} + r^2\left(-\frac{2}{a_0} e^{-\frac{2r}{a_0}}\right)\right] = 0$$

$$\frac{4}{a_0} 2re^{-\frac{2r}{a_0}}\left[1 - \frac{r}{a_0}\right] = 0 \Rightarrow 1 = \frac{r}{a_0} \Rightarrow r = a_0 \leftarrow \text{most probable}$$

- Ex find the expectation value of r

$$\langle r \rangle = \int \psi^* r \psi \, r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= \int R^* \Theta^* \Phi^* \, r R \Theta \Phi \, r^2 \sin\theta \, dr \, d\theta \, d\Phi$$

$$= \int |R|^2 r^3 dr \int |\Theta|^2 \sin\theta \, d\theta \int_0 |\Phi|^2 d\varphi$$

$$= \int r P(r) dr$$

$\langle r \rangle = \int P(r) dr.$

Ground State expectation value

$$\langle r \rangle = \int r \frac{4}{a_0} r^2 e^{-\frac{r}{a_0}} dr$$

$$= \frac{4}{a_0^3} \int r^3 e^{-\frac{r}{a_0}} dr = \frac{4}{a_0^3}\left[\frac{3!}{\left(\frac{2}{a_0}\right)^4}\right] = \ldots = \frac{3}{2} a_0$$

- Ex What is the probab of finding $\bar{e}$ in the ground state within a shell of thickness ~~dr~~ $0.01a_0$?
  - Shell is so thin that we can take the probability to be constant.
  - $P = P(a_0)\,\Delta r$

$$= \frac{4}{a_0^3} r^2 e^{-\frac{2a_0}{a_0}} \Delta r$$

$$= \frac{4}{a_0^3} a_0^2 e^{-\frac{2a_0}{a_0}} (0.01a_0)$$

$$= 0.04 e^{-2}$$

$$\approx 0.5\%$$

- Ex What is the prob of finding $\bar{e}$ in the ground state of hydrogen closer to the nucleus than the Bohr radius?

$$P = \int_0^{a_0} P(r)dr = \frac{4}{a_0^3}\int_0^{a_0} r^2 e^{-\frac{2r}{a_0}} dr \;\ldots \simeq 0.32 \simeq$$

- Ex a) What are the possible quantum numbers for the state $n=3$ in Hydrogen?

n	ℓ	m_ℓ
3	0	0
	1	-1, 0, 1
	2	-2, -1, 0, 1, 2

b.) What is the degeneracy of the $n=3$ level?
→ For H-atom, E only depends on n so all these states have same E, Thus the $n=3$ level is "9-fold ~~degenerate~~ degenerate"