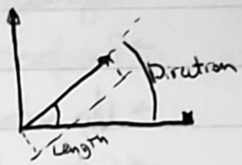
Introduction to Mechanics.

Vectors

- Vector = value + direction



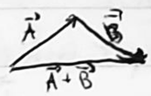
- Notation: Vector $= \vec{A}$

  Magnitude $= |\vec{A}|$

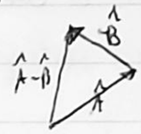
  Unit Vector $= \hat{A}$

  $\vec{A} = |\vec{A}|\hat{A}$

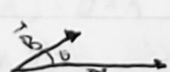
- Multiplication by a Scalar
  - $\vec{c} = b\vec{A}$ is a vector such that $|\vec{c}| = |b||\vec{A}|$, with $\vec{c}$ along $\vec{A}$ for $b>0$ and $\vec{c}$ opposite direction to $\hat{A}$ for $b<0$
  - $\nearrow \vec{A}$ $\quad \nearrow \vec{C},\ b>0$ $\quad \swarrow \vec{C},\ b<0$
- Addition of Vector
  - Geometrically: Tip to tail



  - Subtraction $\to \hat{A} - \hat{B} = \hat{A} + (-1)\hat{B}$

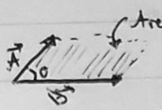


- Properties of ① and ②
  - Commutative $\Rightarrow \vec{A}+\vec{B} = \vec{B}+\vec{A}$
  - Associative $\Rightarrow (\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$
  - Distributive $\Rightarrow c\vec{A}+c\vec{B} = c(\vec{A}+\vec{B})$
- Here we use Euclidean vectors to def. properties. In math, these prop. define objects called vectors.
  - $\exists \vec{\emptyset}$, such that $\vec{A}+\vec{\emptyset} = \vec{A}$, $\forall \vec{A}$
  - $\vec{A} + (-\vec{A}) = \vec{\emptyset}$, $\forall \vec{A}$
  - $1\hat{A} = \hat{A}$, $\forall \hat{A}$
- Vectors so defined form a mathematic vector space
- Other examples: space of n-tuples $(x_1, x_2 \dots x_n)$, space of $(m \times n)$ matrices
- Multiplication of 2 Vectors.
  - Dot-Product $\Rightarrow \underbrace{\vec{A}}_{v} \cdot \underbrace{\vec{B}}_{v} = \underbrace{|\vec{A}||\vec{B}|\cos\theta}_{\text{Scalar}}$



$$\vec{A}\cdot\vec{B} = \emptyset \Rightarrow \begin{cases} \vec{A} = \emptyset \\ \vec{B} = \emptyset \\ \cos\theta = \emptyset,\ \vec{A} \perp \vec{B} \end{cases}$$

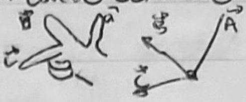
- Cross Product → $\underbrace{\vec{A}}_{\downarrow} \times \underbrace{\vec{B}}_{\downarrow} = \underbrace{\vec{C}}_{\text{vector}}$



→ magnitude of $\vec{C} = |\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$.

→ 2 vectors define a plane, which can be described by its normal

→ Up or down? Right Hand Rule



→ Cartesian $\vec{A} = \hat{x}$, $\vec{B} = \hat{y}$, $\vec{C} = \hat{z}$

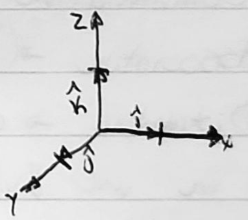
- Components of a Vector
  - the geometrical operations we have introduced are independent of the coordinate system: we impose the coordinate system physics should not depend on this.
  -  $\vec{A} = (A_x, A_y)$
  - Length remains the same for all reference frames. $|\vec{A}| = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$, $\forall$ xyz coordinate system
  - Vector Notation is compact Ex. $\vec{A} = \vec{B} \rightarrow \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases}$

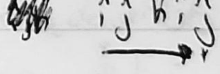
    $- c\vec{A} = (cA_x, cA_y)$.

- Base Vectors
  - We write A_x (and not $\vec{A}_x$) because there is an implicit direction, "along $\hat{x}$".
  - Define: $\hat{i}$ = unit vector, along x axis

    $\hat{j}$ = " " " y

    $\hat{k}$ = " " " z.





  - By Definition: $\hat{i}\cdot\hat{i} = 1 = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k}$

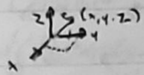
    $\hat{i}\cdot\hat{j} = 0 = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k}$

    $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

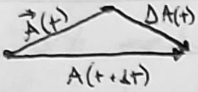
  - Base vectors reconcile geometrical and analytical (formal) points of view: $\vec{A} = (A_x, A_y, A_z) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
  - Ex. $\vec{A}\cdot\hat{i} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot\hat{i} = A_x$
  - Ex. $\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

    $= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$
  - Ex. $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

Position Vector

- Cartesian Coordinate System 
- Each point in space has a position vector associated with it → $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- $\vec{r}$ depends on the choice of coordinate system
- The same point in space will have different position vectors in different coordinate systems

Derivative of a Vector

- 

- Time Evolution: $A(t) \to A(t+\Delta t)$
- $\Delta A = \vec{A}(t+\Delta t) - \vec{A}(t)$
- $\frac{dA}{dt} = \lim_{t\to 0} \frac{\Delta A}{\Delta t}$ = rate of change of $\vec{A}$. $\quad \Delta\vec{A} = \left(\frac{dA}{dt}\right)\Delta t$
- There is a change in both magnitude and direction

Vector Calculus

- $\frac{d(c\vec{A})}{dt} = \frac{dc}{dt}\vec{A} + c\frac{d\vec{A}}{dt}$
- $\frac{d(\vec{A}\cdot\vec{B})}{dt} = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$
- $\frac{d(\vec{A}\times\vec{B})}{dt} = \frac{\Delta\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$
- where $\vec{A} = \vec{A}(t)$, $\vec{B} = \vec{B}(t)$, $c = c(t)$

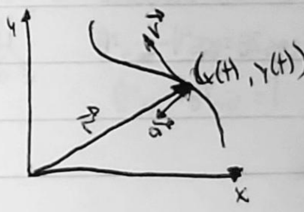
The properties follow from the vector space being a linear space

Elementary Kinematics

- In 1D → position: $x = x(t)$

  velocity: $v = \frac{dx(t)}{dt} = \lim_{t\to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$

  acceleration: $a = \frac{dv(t)}{dt}$



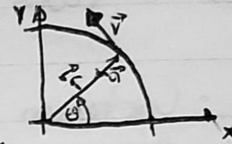
- In several dimensions → position: $\vec{r}(t) = (x(t), y(t))$

  velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt}$

  acceleration: $\vec{a}(t) = \frac{dv}{dt}$
- Formally: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

  $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

- Uniform Circular Motion



$x = r\cos\omega t$
$y = r\sin\omega t$

  - $\vec r = r(\cos\omega t\,\hat i + \sin\omega t\,\hat j)$
  - $|\vec r| = (r^2\cos^2\omega t + r^2\sin^2\omega t)^{\frac12} = (r^2)^{\frac12} = r$
  - $\vec v = \frac{dr}{dt} = \frac{d(r\cos\omega t\,\hat i + r\sin\omega t\,\hat j)}{dt} = r\omega(-\sin\omega t\,\hat i + \cos\omega t\,\hat j) = \vec r\omega$
  - $\vec a = \frac{dv}{dt} = r\omega^2(-\cos\omega t\,\hat i - \sin\omega t\,\hat j) = -\vec r\,\omega^2$
    - acceleration is centripetal
  - Straight forward: $\vec r \to \vec v \to \vec a$
  - What about this way? $\longleftarrow$

Solving Kinematic Equations

- Later dynamics will give us forces and thus, the acceleration. How do we get the details of motion from that?
- Kinematics = Differential Equations; thus integrating the equations of motion
- $\vec a = \frac{dv}{dt} \to \int_{t_1}^{t_2} dv = \int_{t_1}^{t_2} \vec a\,dt$.

  $v_2(t) - v_1(t) = \int_{t_1}^{t_2} \vec a\,dt$.
- Recall: $\int d\vec v = \int \vec a\,dt \to \vec v = \int a\,dt + C$
  - constant of integration = initial condition
- DE + IC = Solution
- $v_2(t) = v_1(t) + \int_{t_1}^{t_2} \vec a\,dt$.
- For position integrate again
- $r_2(t) = v_1(t) + \int_{t_1}^{t_2} v(t)\,dt$
- Ex Uniform Acceleration $\vec a$ = const

  $v = \vec v_0 + \vec a t$ $\quad v_0 = v(t=0)$ $\quad \int \vec a\,dt = \vec a t$

  $\vec r = \vec r_0 + \vec v_0 t + \frac12 \vec a t^2$ $\quad r_0 = r(t=0)$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$1 = \sin^2\theta + \cos^2\theta$.

- Ex Motion in a gravitational field.

  $v_0 = (v_{0x}, v_{0y}, v_{0z})$ at $t=0$

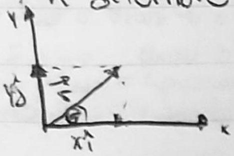
  $$\Rightarrow \begin{cases} x = x_0 + v_{0x}t \\ y = y_0 + v_{0y}t \\ z = z_0 + v_{0z}t - \frac12 g t^2 \end{cases}$$
- Confine to zx-plane: $x_0 = y_0 = z_0 = 0$ and $v_{0y} = 0$

  $$\begin{cases} x = v_{0x}t \\ z = v_{0z}t - \frac12 g t^2 \end{cases} \Rightarrow z = \frac{v_{0z}}{v_{0x}}x - \frac{g}{2v_{0x}^2}x^2$$

Range: $x = \frac{v_0^2}{2g}\sin 2\theta = x_0$

Polar Coordinates

- An alternate way of specifying a point on a plane:



cartesian: (x, y)

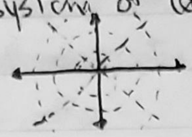
polar: (r, θ)

Conversions: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$

- Amounts to another system of coordinates

Unit vectors $\hat{i}\ \hat{j}$ $\quad \hat{i} \to \vec{e}_x$

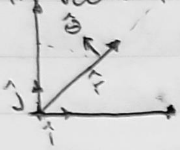
Also: $\hat{j} \to \vec{e}_y$



Unit vectors $\hat{r}\ \hat{\theta}$

Also: $\hat{r} \to \hat{e}_r$, $\hat{\theta} \to \hat{e}_\theta$

- In vector form:



$\left.\begin{array}{l} \hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta \\ \hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta \end{array}\right\}$ rotation by θ

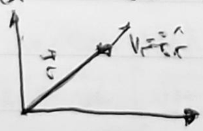
$\vec{r} = x\hat{i} + y\hat{j} = \underset{\uparrow \text{ Scalar}}{r}\,\hat{r}$ ← note $\hat{\theta}$ is implicit as $\hat{r} = \hat{r}(\theta)$

- This is called 'plane polar' or cylindrical coordinates.
- $\vec{r} = x\hat{i} + y\hat{j}$

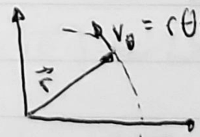
$\vec{v} = \frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j}$

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

- Now obtain in cylindrical
  - Uniform: $\dot{\theta} = \omega$ = constant angular velocity.
  - Circle: r = constant
  - $\vec{r} = x\hat{i} + y\hat{j} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$ } Cartesian.
  - $\vec{v} = \frac{dr}{dt} = r\omega[-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}] = r\omega\hat{\theta}$.
  - $\vec{r} = r\hat{r} \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} =$
  - $\frac{d\hat{r}}{dt} = \omega\hat{\theta} = \dot{\theta}\hat{\theta}$
  - $\vec{r} = r\hat{r}(\theta)$
  - $\vec{v} = \frac{d(r\hat{r})}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
  -



Radial Component



tangential Component

} Cylindrical

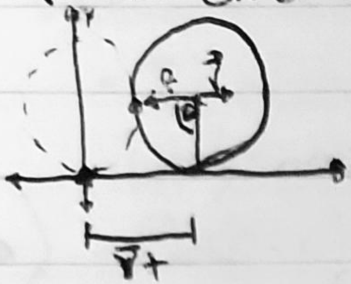
  - $\vec{a} = \frac{dv}{dt} = \frac{d[-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}]}{dt}\omega r = -\omega^2 r\hat{r} = -\omega^2\vec{r}$ } Cartesian
  - $\vec{a} = \frac{dv}{dt} = r\frac{d}{dt}(\dot{\theta}\hat{\theta}) = r[\ddot{\theta}\hat{\theta} + \dot{\theta}\frac{d\hat{\theta}}{dt}] = r\dot{\theta}\frac{d\hat{\theta}}{dt} = \omega r\frac{d\hat{\theta}}{dt} = -\omega^2 r\hat{r}$ } Cylindrical
  - $\vec{a} = \frac{dv}{dt} = \frac{d}{dt}[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}] = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$ } Cylindrical
  - $\vec{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)\hat{r}}_{\text{centripetal}} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}_{\text{coriolis}}$

$\vec{r} = r\hat{r}$

$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\vec{a} = (\quad)\hat{r} + (\quad)\hat{\theta}$

• Pebble on a Rolling Wheel.



ω = constant , $\theta = -\omega t$ ← means clockwise rotation

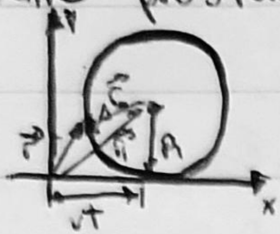
$vt = R\theta$

$$\begin{cases} x = R\theta - R\sin\theta \\ y = R(1-\sin\theta) \end{cases}$$

$$\begin{cases} \dot{x} = R\dot{\theta} - R\cos\theta\,\dot{\theta} = v - v\cos\theta = \omega R - \omega R\cos\omega t \\ \dot{y} = R\sin\theta\,\dot{\theta} = v\sin\theta = -\omega R\sin\omega t \end{cases}$$

$$\begin{cases} \ddot{x} = v\omega\sin\theta = -v\omega\sin\omega t = -\omega^2 R\sin\omega t \\ \ddot{y} = v\omega\cos\theta = v\omega\cos\omega t - \omega^2 R\cos\omega t \end{cases}$$

• Same problem but with a moving reference frame



$\vec{r} = \vec{r}_1 + \vec{r}_2$, where the center moves to the right with a constant velocity, v, at height R:

$\vec{r}_1 = vt\hat{i} + R\hat{j} = R\omega t\hat{i} + R\hat{j}$

The pebble rotates about the center:

$\vec{r}_2 = -R\sin\omega t\,\hat{i} + R\cos\omega t\,\hat{j}$

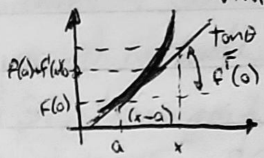
→ $\vec{V} = \vec{V}_1 + \vec{V}_2 = (R\omega - R\omega\cos\omega t)\hat{i} - (R$

Approximation Methods

Taylor Series

- $f(x) = f(a) + f'(a)\frac{x-a}{1!} + f''(a)\frac{(x-a)^2}{2!} \dots + f^{n}(x)\frac{(x-a)^n}{n!} + \underbrace{R_n(x)}_{n}$

  nth order residual



- Calculus: conditions for $R_n(x) \to \emptyset$
- Basic Idea: Successive Approximations
- For a 'smooth' function, knowing

  $f(a), f'(a), f''(a) \dots$

  allows us to predict $f(x)$ for $x \neq a$. The closer x is to a the better the prediction; fewer terms to achieve the same precision or higher precision with the same number of terms
- $\frac{df}{dx} = \lim_{x \to 0} \frac{\Delta F}{\Delta x}$ ← Definition of derivative.
- Special Case: Binomial Expansion $f(x) = (1+x)^n$ near $x = 0$
  - $f(x) = (1+x)^n|_{x=0} = 1$
  - $f'(x) = n(1+x)^{n-1}|_{x=0} = n$

  $\Rightarrow (1+x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$
  - Valid (converges) for $|x| < 1$
- Running Average Filter

```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1   → 16 × 1/16 + 4/16 + 6/16 + 4/16 + 1/16
 1 5 10 10 5 1
```

$\dots x_{i-2} + x_{i-1} + x_i + x_{i+1} + x_{i+2} \dots$

$\downarrow$

$y_i = \frac{1}{16}x_{i-2} + \frac{1}{4}x_{i-1} + \frac{3}{8}x_i + \frac{1}{4}x_{i+1} + \frac{1}{16}x_{i+2}$

- Ex $n = \frac{1}{2}$ i.e. $\sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \underline{O(x^3)}$

  "order of x^3"

What can we say about $[\sqrt{L+1} - \sqrt{L}]$, for $L >> 1$

$\sqrt{L+1} - \sqrt{L} = \sqrt{L}\left[\sqrt{1+\frac{1}{L}} - 1\right]$

$= \sqrt{L}\left[1 + \frac{1}{2}\frac{1}{L} - \frac{1}{8}\frac{1}{L} + O(\frac{1}{L})\right]$

$\dots$

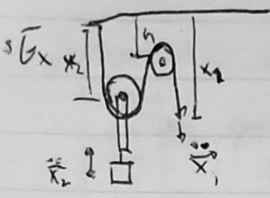
$= \frac{1}{2L}$

- Other expansions
  - $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$
  - $\sin x = \cancel{x}\ \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$
  - $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \dots$

Netons Laws

- 1 - Isolated bodies moves at constant velocity with respect to inertial systems. (Postulates the existence of special frames of reference [coordinate systems] called "inertial frames"
- 2 - $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt}$, where m is the mass of the body (defined in terms of the N2L, $m' = m\frac{a}{a'}$), $\vec{a}$ is the acceleration (in any inertial frame!), $\vec{F}$ is the total force acting on it, $\sum_i \vec{F}_i$. When $\vec{a} \neq 0$, the body is not isolated. Define "mass" by characteristics of a body as $\vec{F} \to N\vec{F}$ (N springs in parallel) $\vec{a} = N\vec{a}$. $\vec{F} \parallel \vec{a}$
- 3 - Forces always appear in pairs of equal and oppositely directed forces. Postulates that all forces are due to interactions between different objects. If acceleration of a body is due to an outside force, then somewhere in the universe there must be an equal and opposite force acting on another body.
- Non-Inertial Reference Frames - $\vec{F}_{apparent} = \vec{F}_{TRUE} + \vec{F}_{FICTITOUS}$
- $= \vec{F}_{TRUE} - M\ddot{\vec{R}}$
- Limits of Applicability - Relativity - wrong operational definition of mass; different concept of space & time
  - QM - different nature of measurement itself
  - Newtonian vs. Aristotelian Mechanics.
- Units - SI: meter, kg, second, 1 Newton ($1\ kg\frac{m}{s^2}$)
- Procedure for solving newtonian mechanics
  1. Divide the system into subsystems, which may be treated as a point mass
  2. Draw a force diagram for each point mass. Draw only forces on masses.
  3. Choose a coordinates system. Express the forces acting on masses in component form; write down the equations of motion for each mass
  4. Write down relations between forces on the basis of N3L
  5. Write down constants for the path
  6. Identify unknown/known variables; ensure there enough equations to solve.

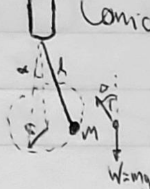
S Ex



$$l = x_2 + \pi R_2 + (x_2 - h) + \pi R_1 + (x_1 - h)$$

$$\frac{d^2(l)}{dt^2} = 0 \Rightarrow 0 = \ddot{x}_2 + \ddot{x}_2 + \ddot{x}_1 \Rightarrow \ddot{x}_2 = \frac{1}{2}\ddot{x}_1$$

- Ex Conical Pendulum



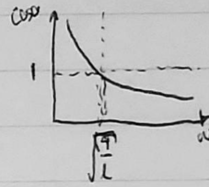
Along y: $y = \text{const} \Rightarrow \ddot{y} = 0$ $\qquad T\cos\alpha - W = 0$ *

Along x: $a_r = -\omega^2 r$ $\qquad -T\sin\alpha = -M\omega^2 r = -M\omega^2(l\sin\alpha)$

$$T = M\omega^2 l \quad **$$

Combine (*) and (**) $\quad Ml\omega^2\cos\alpha = Mg \Rightarrow \cos\alpha = \frac{g}{l\omega^2}$

$\max(\cos\alpha) = 1$ when $\omega = \sqrt{\frac{g}{l}}$



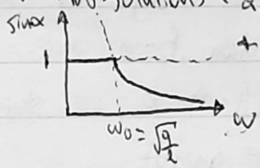
① $\omega = \sqrt{\frac{g}{l}}$, $\cos\alpha = 1$, $\sin\alpha = 0$; bob hangs vertically

② In getting (**), we divide both sides by $\sin\alpha$ - not allowed when $\sin\alpha = 0$. If $\sin\alpha = 0$, then (**) $T\sin\alpha = Ml\omega^2\sin\alpha$ is valid for all ω values

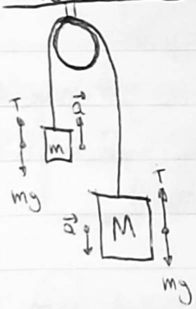
a)- $\omega < \sqrt{\frac{g}{l}}$: $\sin\alpha = 0$, $\alpha = 0$ is the only solution

b)- $\omega > \sqrt{\frac{g}{l}}$: two solutions $\begin{cases} \alpha = 0, \text{ unstable} \\ \alpha = \arccos\left(\frac{g}{l\omega^2}\right), \text{ stable} \end{cases}$

+ happens a critical frequency $\left(\omega = \sqrt{\frac{g}{l}}\right)$



- Ex The Atwood Machine



N2L: $T - mg = m\ddot{x}_m$

$T - Mg = M\ddot{x}_M$

Constraint: $\ddot{x}_m = -\ddot{x}_M = a$

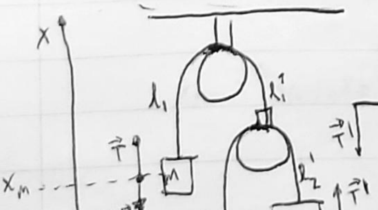
Eliminate T: $m\ddot{x}_m + mg = M\ddot{x}_M + Mg$

$ma + mg = -Ma + Mg$

$(m+M)a = (M-m)g$

$$a = \frac{(M-m)g}{m+M}$$

Tension: $T = \frac{2Mmg}{M+m}$



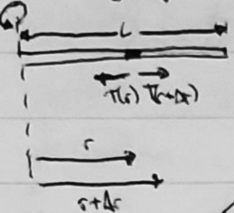
N2L: $\vec{T} - mg = m\ddot{x}_m$, $\quad T' - Mg = M\ddot{x}_M$, $\quad T - 2T' = 0$, massless pulley

Constraint: $x_m + l_1 + l_1' + \frac{l_2 + l_2' + x_M}{2} = \text{Const}$

$\frac{d^2}{dt^2}$: $\ddot{x}_m = -\frac{1}{2}\ddot{x}_M$

EFTS: $\ddot{x}_m = \frac{2M - m}{4M + m}g$

• Ex Twirling Rope.



For the small section of the rope, Δr:

$$T(r+\Delta r) - T(r) = -(\Delta m) r\omega^2$$

with $\Delta m = \frac{\Delta r}{L} M$

Taking the limit of $\Delta r \to 0$:

$$\lim_{\Delta r \to 0} \frac{T(r+\Delta r) - T(r)}{\Delta r} = \frac{dT}{dr} = -\frac{mr\omega^2}{L}$$

Integrate: $\int_{T_0}^{T(r)} dT = -\int_0^r \frac{M\omega^2}{L} r\,dr$

$$T(r) - T_0 = -\frac{M\omega r^2}{L2} \Rightarrow T(r) = T_0 - \frac{M\omega r^2}{2L}$$

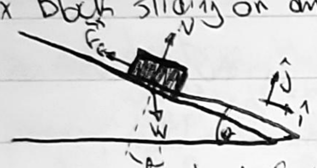
Because: $T_0 = T(r=0) = ?$ But, the other end is free

$T(r=L) = 0$

$$\Rightarrow T(L) = 0 = T_0 - \frac{M\omega^2 r^2}{2L}\Big|_{r=L} = T_0 - \frac{M\omega^2 L^2}{2L} \Rightarrow T_0 = \frac{M\omega^2 L^2}{2L}$$

finally $\Rightarrow T(r) = \frac{M\omega^2 L^2}{2L} - \frac{M\omega^2 r^2}{2L} = \frac{M\omega^2}{2L}(L^2 - r^2)$

• Ex Block sliding on an Incline.



Along x: $m\ddot{x} = W\sin\theta - f_f$

Along y: $m\ddot{y} = N - W\cos\theta = 0$ (*)

a) No friction: $f_f = 0 \Rightarrow \ddot{x} = \frac{W}{m}\sin\theta = \frac{mg}{m}\sin\theta$.

Assuming $V_0 = \dot{x}(t=0) = 0$, i.e. starting from rest

$v = \cancel{v_0} + at = g\sin\theta t$

$x = x_0 + \cancel{v_0 t}^{\,0} + \frac{1}{2}at^2 = x_0 + \frac{1}{2}g\sin\theta t^2$

b.) with friction: $0 < f_f \le f_f^{(max)} = \mu N$.

The block starts to slide, i.e. $\ddot{x}$ is just above 0 when:

$m\ddot{x} = W\sin(\theta_{max}) - \mu N = 0 \Rightarrow W\sin(\theta_{max}) = \mu N$, but from $W\cos(\theta_{max}) = N$

$$\Rightarrow \begin{cases} W\sin(\theta_{max}) = \mu N \\ W\cos(\theta_{max}) = N \end{cases} \Rightarrow \mu = \tan\theta_{max}$$

• Ex Viscosity: a moving body sets of movement in the fluid

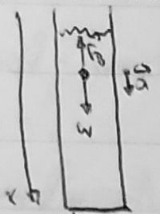
Reaction force $\vec{f}_v = -c\vec{v}$ Note $\vec{f} \uparrow\downarrow \vec{v}$

N2L: $m\ddot{x} = -c\dot{x}$ or $m\dot{v} = -cv$

Math Detour: $\dot{x} + ax = 0$ try $x = x_0 e^{-at} \Rightarrow \dot{x} = x_0(-a)e^{-at} = -ax$

a) $\Rightarrow$

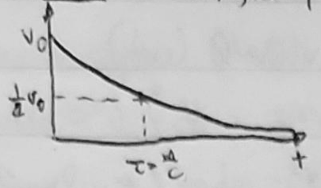
a) Terminal Velocity of a body falling in a fluid



$m\vec{a} = m\vec{g} - c\vec{v}$

$\vec{a} = \varnothing \leftarrow$ constant velocity $\rightarrow \vec{v} = \vec{v}_{TERMINAL} = \frac{m}{c} g$

No gravity: $v(t) = v_0 e^{-\frac{t}{\tau}}$, $\tau = \frac{m}{c}$



$\frac{1}{e} \simeq 0.37$

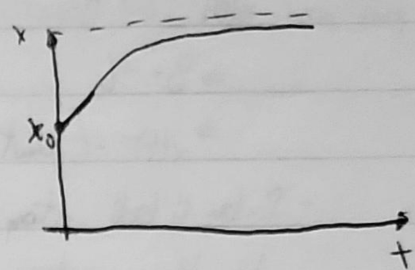
when $t = \tau = \frac{m}{c}$, $e^{-\frac{t}{\tau}} = e^{-1} \simeq 0.37$

$v(t) = 0.37 v_0$

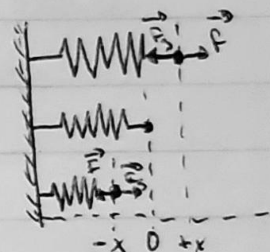
Second Integration: $x(t) = x_0 + \int_0^t v(t')\,dt'$

$= x_0 + v_0\tau(1 - e^{-\frac{t}{\tau}})$

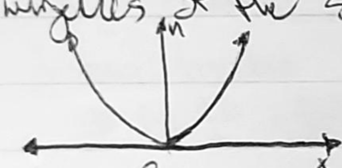
$= x_0 + \frac{v_0 m}{c}(1 - e^{-\frac{t}{\tau}})$



Linear Restoring Force

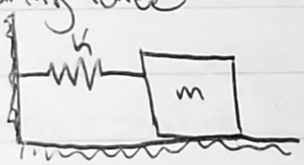


- $F_s = -kx$, $\leftarrow$ Hooke's Law
- Origin in the empirical nature of Hooke's Law
- $\vec{F}_s$ is a restoring force, always opposes the applied $\vec{F}$
- Energetics of the spring:



$U \propto x^2$

- Restoring force $\longleftrightarrow$ stable equilibrium
- Ex①



$m\ddot{x} = -kx$

$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$

Denote $\omega = \sqrt{\frac{k}{m}}$

$\Big\} \; \ddot{x} + \omega^2 x = 0$

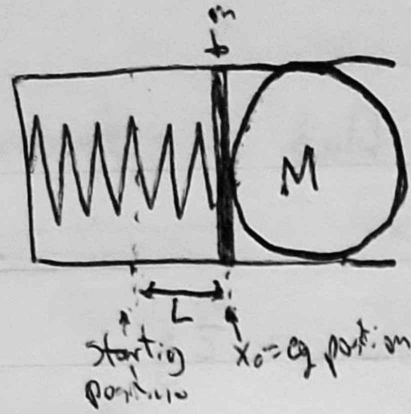
Solution: $x = A\sin\omega t + B\cos\omega t \rightarrow$ the meaning of ω: Angular frequency

IC: Determine A&B from IC, as usual.

$x = A\sin\omega t + B\cos\omega t \rightarrow \dot{x} = A\omega\cos\omega t - B\omega\sin\omega t$

$\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t = -\omega^2 x$

• Ex. ②



Q: What position and velocity do m & M part company?

- Until that moment, $x(t) = A\sin\omega t + B\cos\omega t$

where $\omega = \sqrt{\frac{k}{m+M}}$

- Differentiate: $v(t) = \dot{x}(t) = \omega A\cos\omega t - \omega B\sin\omega t$

- Loaded gun: at $t = 0$, $x(0) = -L$, $v(0) = 0$ (rest)

- From these IC determine A & B:

$x(0) = -L = A\sin 0 + B\cos 0$ $\quad\quad$ $v(0) = \omega A\cos 0 + \omega B\sin 0 = 0$

$\Rightarrow B = -L$ $\quad\quad$ $\Rightarrow A = 0$

$\Rightarrow x(t) = -L\cos\omega t$ $\quad\quad$ $\Rightarrow v(t) = \omega L\sin\omega t$

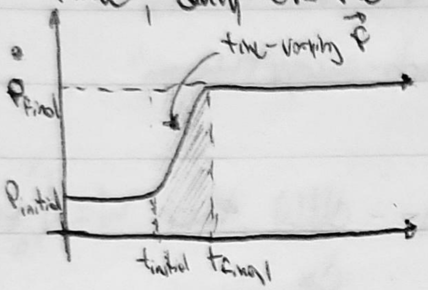
- Piston & ball stay in touch while accelerating; part company when the piston begins to decelerate and the ball continues with constant velocity. The parting point $v = v_{MAX}$

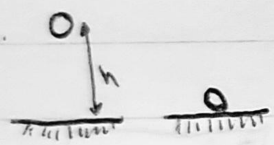
- But: $\max(\sin\omega t) = 1$, when $\omega t_{MAX} = \frac{\pi}{2}$

- $x(t_{MAX}) = -L\cos\frac{\pi}{2}$ $\quad\quad$ $v(t_{MAX}) = \omega L\sin(\frac{\pi}{2}) = \omega L$

$= 0$ $\quad\quad$ $v(t_{MAX}) = \sqrt{\frac{k}{m+M}}\, L$

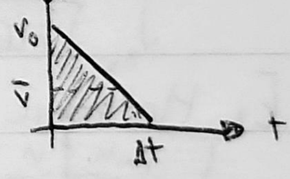
Momentum

- $\vec{p} = m\vec{v}$
- N2L: $\vec{F} = m\vec{a} \rightarrow \vec{F} = \frac{\partial \vec{p}}{\partial t}$ ← more fundamental / Easier to generalize to complex systems
- more universal than m & $\vec{v}$ separately
- Impulse $= \int_0^t \vec{F}\,dt = \vec{p}(t) - \vec{p}(0)$
- change in momentum is independent of how the force may vary in time, only on the total impulse, $\int \vec{F}\,dt$
- 

- Ex. 

Object dropped from height, h, (initially at rest) hits ground with velocity, $v = \sqrt{2gh}$. Suppose it takes distance, s and time Δt to stop it.

- $F\Delta t = mv_0$ or $F = \frac{mv_0}{\Delta t}$ = force of deceleration
- For F = constant, the average velocity is $\bar{v} = \frac{1}{2}v_0$ and the distance travelled $s = \frac{1}{2}v_0\Delta t$



$$\Rightarrow F = \frac{mv_0}{t} = \frac{mv_0}{2s/v_0} = \frac{mv_0^2}{2s} = \frac{m2gh}{2s} = mg\frac{h}{s}$$

or $F = ma$ where $a = \frac{h}{s}g$

- For $h = 2m$, $s = 1cm \rightarrow a = 200g$! $F = 200 \cdot$ weight! "Pulling g's"
- Ankle $\simeq 5cm^2 \Rightarrow \frac{\text{force}}{\text{unit area}} \simeq 2\times10^4 \frac{N}{cm^2} \triangleq$ Compressive strength of human bones
- Moral: Violence of a short collision; safety in extending it.

Work and Kinetic Energy

- Momentum Conservation - integrate N2L w.r.t time, t
- $\vec{f} = \frac{d\vec{p}}{dt} \Rightarrow \vec{f} = \emptyset \Rightarrow \vec{p} = \text{constant}$
- Energy Conservation - integrate w.r.t. position, x
- $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

  ↳ $f = f(x) = mv\frac{dv}{dx} = \frac{1}{2}m\frac{d(v^2)}{dx} = \frac{dK}{dx}$, where $K = \frac{1}{2}mv^2$ = Kinetic energy

  ↳ $K - K_0 = \underbrace{\int_{x_0}^{x} f(x)\,dx}_{=\text{work}=W}$

  | ① Often know $f = f(x)$ not $f = f(t)$
  | ② Known as "work-energy theorem"

  ↳ integrate form of N2L
- If we could define $U(x)$ such that $f(x) = -\frac{dU(x)}{dx}$, then $W = \int_{x_0}^{x} f(x)\,dx =$

$= -\int_{x_0}^{x} dU = -U(x) + U(x_0) = K - K_0$

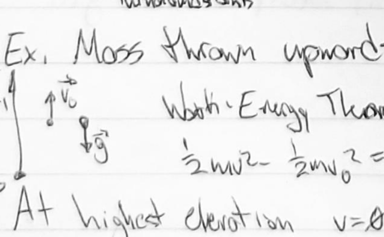
- $K_0 + U(x_0) = K + U(x) = E = \text{constant}$.
- Formal Solution: $\frac{1}{2}mv^2 = K \Rightarrow v = \frac{dx}{dt} = \pm\sqrt{\frac{K}{m}} = \sqrt{\frac{2}{m}[E - U(x)]}$

$$\Rightarrow \frac{dx}{\pm\sqrt{\frac{2}{m}[E - U(x)]}} = dt$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}[E - U(x)]}} = t - t_0$$

| Note: $U \leq E$ for a physical
| solution to exist

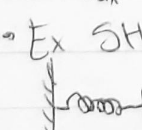
- Ex Harmonic Potential - $U(x) = \frac{1}{2}kx^2$



E = constant

Motion confined to $U(x) \leq E$

- Ex. Mass thrown upwards



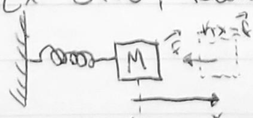
Work-Energy Theorem

$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z_1} f\,dz = -mgz'\Big|_{z_0}^{z_1} = -mg(z - z_0)$

At highest elevation $v = \emptyset \Rightarrow \emptyset - \frac{1}{2}mv_0^2 = -mg(z_1 - z_0)$

$z_{max} = z_0 + \frac{v^2}{2g}$

- Ex SHO, reconsidered



$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -k\int_{x_0}^{x} x'\,dx' = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$

I.C. $v_0 = \emptyset$ at some x_0 = equilibrium

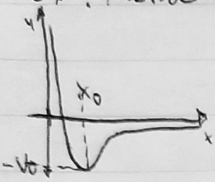
$v^2 = \frac{k}{m}x^2 - \frac{k}{m}x_0^2$

$\omega = \sqrt{\frac{k}{m}}$

$\frac{dx}{dt} = v = \sqrt{\frac{k}{m}}\sqrt{x_0^2 - x}$

Harmonic Motion

- Motivation - All potentials look harmonic near their minima
- $V(x) = V(x_0) + \frac{x-x_0}{1!}V'(x_0) + \frac{(x-x_0)^2}{2!}V''(x_0) + \dots$ ← Taylor series expansion
- Minimum @ x_0 → $V'(x_0) = 0$
- Ex. Morse Potential. $V(x) = U_0\left[1 - e^{-\frac{(x-x_0)}{\delta}}\right]^2 - U_0$



describes the interaction between two hydrogen atoms in a H_2 molecule looks harmonic for $|x - x_0| \lesssim 0.1\delta$

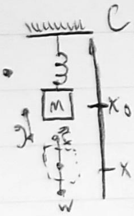
- SHO General Solution

  $-m\ddot{x} + kx = 0 \Rightarrow x = A\sin(\omega t + \varphi_0)$ where $\omega_0 = \sqrt{\frac{k}{m}}$ and φ_0 is initial phase determined by I.C.

- Most General I.C.: $x_0 = 0$, $v_0 = 0$
  - $x(0) = A\sin\varphi = x_0$ $\dot{x}(0) = \omega_0 A\cos\varphi_0 = v_0$
  - $\tan\varphi_0 = \dfrac{\frac{x_0}{A}}{\frac{v_0}{\omega_0 A}} = \dfrac{x_0\omega_0}{v_0}$ $A^2 = A^2\sin^2\varphi_0 + A^2\cos^2\varphi_0 = x_0^2 + \frac{v_0^2}{\omega_0^2}$

- Other equivalent form:
  - $x = A\sin(\omega_0 t + \varphi_0) = \underbrace{A\sin\varphi_0}_{C}\cos\omega_0 t + \underbrace{A\cos\varphi_0}_{D}\sin\omega_0 t = C\cos\omega_0 t + D\sin\omega_0 t$

  C, D determined by I.C.



- SHO in the presence of a constant force

  N2L: $m\ddot{x} = +f - W = -k(x - x_0) - mg$ where equilibrium is now at $0 = -k(x - x_0) - mg \Rightarrow x = x_0 - \frac{mg}{k}$

- Change Variables: $y = x - \left(x_0 - \frac{mg}{k}\right) \Rightarrow \ddot{y} = \ddot{x}$

  $\Rightarrow m\ddot{x} = m\ddot{y} = -k\left(y - \frac{mg}{k}\right) - mg = -ky$ or $m\ddot{y} = -ky$

  ⇒ same SHO around the new equilibrium $x_0 - \frac{mg}{k}$

- Damped H.O - General Solution

  $-m\ddot{x} + \underbrace{c\dot{x}}_{\text{Damping}} + kx = 0 \rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

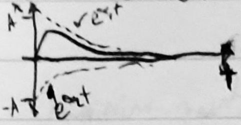
  with $\gamma = \frac{c}{2m}$, $\omega_0^2 = \frac{k}{m}$

- The most general solution: $x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$
- The sign of $(\gamma^2 - \omega_0^2)$ determines the nature of the solution

  ① $\overbrace{\gamma^2 - \omega_0^2 > 0}^{\text{overdamping}} \Rightarrow \alpha_{1,2}$ are both real, -ve, $|\alpha_2| > |\alpha_1|$

  ⇒ Solution is the sum of 2 ~~damping~~ decaying damping exponentials, one faster than the other

• Ex. I.C. $x_0=0 \Rightarrow x(t=0)=A_1+A_2=0 \Rightarrow A_1=-A_2=A$



$\Rightarrow x = A(e^{\alpha_1 t}-e^{\alpha_2 t})$

$\Rightarrow A(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t}) = v_0$

$\Rightarrow A = \frac{v_0}{\alpha_1-\alpha_2}$

• $\gamma^2-\omega_0^2<0 \Rightarrow \alpha_{1,2} = -\gamma \pm i\sqrt{\omega_0^2-\gamma^2}$

- $\omega_d=\sqrt{\omega_0^2-\gamma^2}=\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}$
- Quite frequently, $\gamma^2 \ll \omega_0^2$ and $\omega_d \simeq \omega_0$
- $x(t)=C_1e^{-(\gamma+i\omega_d)t}+C_2e^{-(\gamma+\omega_d)t}$

$= e^{-\gamma t}(C_1e^{-i\omega_d t}+C_2e^{i\omega_d t})$

- For x(t) to represent a physical solution, $x(t)=x^*(t)$

$\Rightarrow C_1e^{-i\omega_d t}+C_2e^{i\omega_d t} = C_1^*e^{i\omega_d t}+C_2^*e^{-i\omega_d t}$

$\Rightarrow x(t)=e^{-\gamma t}|d|[e^{-i(\omega_d t+\varphi_0)}+e^{i(\omega_d t+\varphi_0)}]$ $\qquad e^{i\alpha}+e^{-i\alpha}=2\cos\alpha$

$=e^{-\gamma t}|d|\,2\cos(\omega_d t+\varphi_0)$

$\Rightarrow$ Let $A=2|d| \Rightarrow x(t)=\underset{\text{amplitude}}{A}\,\underset{\text{decay}}{e^{-\gamma t}}\,\underset{\text{oscillation}}{\cos(\omega_d t+\varphi_0)}$, where $\omega_d=\sqrt{\omega_0^2-\gamma^2}$

• $\omega_d=\sqrt{\omega_0^2-\gamma^2}=\frac{1}{2}\omega_0 \Rightarrow \gamma=\sqrt{\frac{3}{4}}\,\omega_0$

- 2 successive maxima are $T_d=\frac{2\pi}{\omega_d}$ apart, with $\sin(\omega_d t+\varphi_0)=1$ for both

$\Rightarrow \frac{x(t_{max})}{x(t_{max}+T_d)} = \frac{e^{-\gamma t_{max}}}{e^{-\gamma(t_{max}+T_d)}} = \frac{1}{e^{-\gamma T_d}}$

$\Rightarrow \gamma T_d = \gamma\frac{2\pi}{\omega_d} = \sqrt{\frac{3}{4}}\,\omega_d = \frac{2\pi}{\frac{1}{2}\omega_0} = 2\sqrt{3}\,\pi \simeq 10.88$

$\Rightarrow e^{-10.88} = 0.00002 = 20$ ppm

- Very highly damped → next peak is 20 ppm

• Quality Factor, $Q = 2\pi\,\frac{\text{Energy Stored in Osc}}{\text{Energy lost per period}}$

- Recall $\dot{E} = \frac{dE}{dt} = -c\dot{x}^2$
- Energy lost in a cycle: $\Delta E = \int_0^{T_d}\dot{E}\,dt = \int_0^{\frac{2\pi}{\omega_d}} -c\dot{x}^2\,dt$

where $\dot{x}(t) = Ae^{-\gamma t}[-\gamma\sin(\omega_d t+\varphi_0)+\omega_d\cos(\omega_d t-\varphi_0)]$

- Let $\theta=\omega_d t+\varphi_0 \leftrightarrow dt=\frac{1}{\omega_d}d\theta$
- $\Delta E=\frac{1}{\omega_d}\int_0^{2\pi}-c\dot{x}^2\,d\theta \ldots$

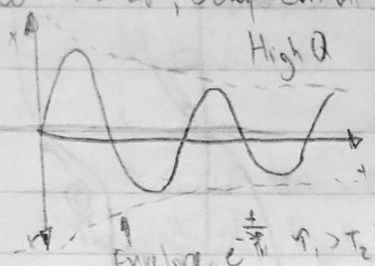
$\Rightarrow \Delta E = -\frac{cA^2}{\omega_d}e^{-2\gamma t}\int_0^{2\pi}[\gamma^2\sin^2\theta-2\gamma\omega_d\sin\theta\cos\theta+\omega_d^2\cos^2\theta]\,d\theta$

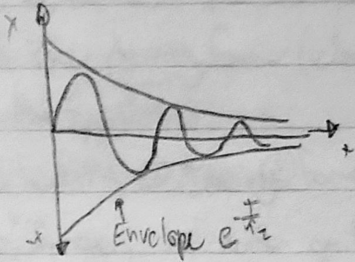
$\Rightarrow \Delta E = -\frac{cA^2}{\omega_d}e^{-2\gamma t}\,\pi[\gamma^2+\omega_0^2] = -2m\gamma A^2\omega_0^2e^{-2\gamma t}\left(\frac{\pi}{\omega_d}\right)$

$\Rightarrow$ Energy Stored $= E(t) = \frac{1}{2}m\omega_0^2A^2e^{-2\gamma t}$

$\Rightarrow Q = 2\pi\frac{E}{\Delta E} = 2\pi\,\frac{\frac{1}{2}}{2\gamma\left(\frac{\pi}{\omega_d}\right)} = \frac{\omega_d}{2\gamma}$

- Let $\tau = \frac{1}{2\gamma}$, decay constant





$\left.\begin{array}{l}\tau_1 > \tau_2 \\ Q_1 > Q_2\end{array}\right\} \rightarrow$ High Q $\rightarrow$ long lived oscillation

- Forced H.O.
  - $m\ddot{x} + c\dot{x} + Kx = F_0 \cos\omega_D t$ driving frequency
  - Algebra gets simpler if we solve in the complex domain:

    $m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$ & $Re\{x(t)\}$ is the physically relevant part of the solution
  - Expect an oscillatory solution; available frequencies are

    $\underline{\omega_0} = \sqrt{\frac{k}{m}}$, $\underline{\omega_d} = \sqrt{\omega_0^2 - \gamma^2}$ with $\gamma = \frac{c}{2m}$, or $\underline{\omega}$
  - Steady state Solution $\Rightarrow x(t) = Ae^{i(\omega t + \varphi)}$
  - $-m\omega^2 A + i\omega c A + kA = F_0 e^{i\varphi}$

    Re: $A(k - m\omega^2) = F_0 \cos\varphi$

    Im: $Ac\omega = F_0 \sin\varphi$

    $\tan\varphi = \frac{c\omega}{k - m\omega^2}$

    $F_0^2 \sin^2\varphi + F_0 \cos^2\varphi = A^2(k - m\omega^2)^2 + A^2c^2\omega^2$

    $A = F_0[(k - m\omega^2)^2 + c^2\omega^2]^{-\frac{1}{2}}$

    Using $\omega_0^2 = \frac{k}{m}$ and $\gamma = \frac{c}{2m}$, rewrite as

    $\tan\varphi = \frac{2m\gamma\omega}{K - m\omega^2} = \frac{2\gamma\omega}{\frac{k}{m} - \omega^2} = \frac{2\gamma\omega}{\omega_0^2 - \omega}$

    $A = \frac{\frac{F_0}{m}}{\left[\left(\frac{k - m\omega^2}{m}\right)^2 + \left(\frac{c}{m}\right)^2\omega^2\right]^{\frac{1}{2}}} = \frac{\frac{F_0}{m}}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{\frac{1}{2}}}$

    ① $\omega \to \omega_0 \Rightarrow \tan\varphi \to \infty \Rightarrow \varphi = \frac{\pi}{2}$

    For a weakly damped case, $\gamma \to 0$, $A(\omega) \to \infty$ as $\omega \to \omega_0$

    Actual Max(A):

    $\frac{dA}{d\omega} = 0 \overset{EFTS}{\Rightarrow} \omega_r^2 = \omega_0^2 - 2\gamma^2$ ω_r = resonant frequency

    Resonance disappears when $2\gamma^2 \geq \omega_0^2$:

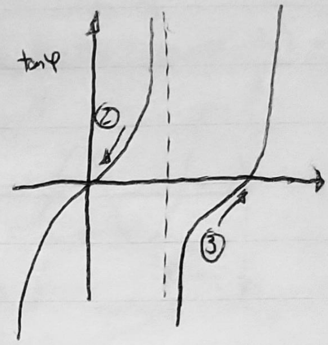
    $A|_{2\gamma^2 = \omega_0^2} = \frac{\frac{F}{m}}{(\omega_0^2 + \omega)^{\frac{1}{2}}}$

2 $\omega \to \emptyset \Rightarrow A = \dfrac{F_0/m}{\omega_0^2} = \dfrac{F_0}{k}$.

$\tan\varphi = +\emptyset \to \varphi = \emptyset$.

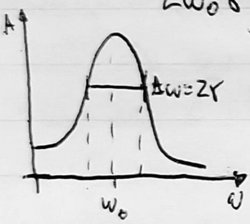
3 $\omega \to \infty \Rightarrow A \to 0$

$\tan\varphi = -0 \to \varphi = \pi$



Sharpness of Resonance and Q

- Weak damping, $\gamma \ll \omega_0$; near $\omega \simeq \omega_0$:
  - $\omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) \simeq 2\omega_0(\omega_0-\omega)$ and $\gamma\omega \simeq \gamma\omega_0$
- $A(\omega) \simeq \dfrac{F_0/m}{[4\omega_0^2(\omega_0-\omega)^2 + 4\gamma\omega_0]^{\frac{1}{2}}} = \dfrac{F_0/m}{2\omega_0} \dfrac{1}{[[\omega_0^2-\omega^2] + \gamma^2]^{\frac{1}{2}}}$
- max $A(\omega) \simeq \dfrac{F_0/m}{2\omega_0\gamma} = A_{max}$, where $\omega = \omega_0$ | and, when $(\omega_0-\omega)^2 = \gamma^2$ $A(\omega) = \dfrac{A_{max}}{\sqrt{2}}$

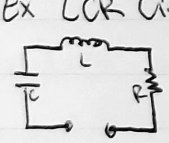


$\omega_0 \pm \gamma \to \gamma$ is a measure of the width of the resonance curve

Recall: $Q = \dfrac{\omega_d}{2\gamma} \simeq \dfrac{\omega_0}{2\gamma}$ (ω_d = weak damping)

$\Rightarrow \Delta\omega = 2\gamma \simeq \dfrac{\omega_0}{Q}$ or $\dfrac{\Delta\omega}{\omega_0} \simeq \dfrac{1}{Q}$ ← operational definition of Q

- Ex LCR Circuit.



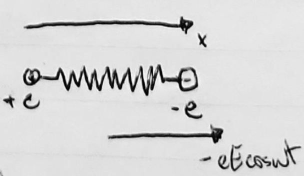
$L\dfrac{\partial^2 N}{\partial t} + R\dfrac{dN}{dt} + \dfrac{1}{C}V = V_0 e^{i\omega t}$

$x \to N$, charge — $k \to \frac{1}{C}$, capacitance

$\dot{x} \to i$, current — $c \to R$, resistance $\Rightarrow$ resonance near $\omega_0 = \sqrt{\frac{k}{m}}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

$m \to L$, inductance — $F \to V$, voltage

- Ex dielectric constant of an insullator
  - Lorentz model of an insulator:
  - Applied external electric field $\vec{E}$



$\to$ Force $\vec{F} = -e\vec{E}$

$\to m_e\ddot{x} + c\dot{x} + kx = -eE\cos\omega t$

$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = -\dfrac{eE_0}{m_e} e^{i\omega t}$

$\Rightarrow x_0 = -\dfrac{eE}{m_e}[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2]^{-\frac{1}{2}}$, $\tan\varphi = \dfrac{2\gamma\omega}{\omega_0^2-\omega^2}$

$\Rightarrow$ ~~Real~~ take the Re(x) and find physical solution

$$\Rightarrow x = \frac{-eE}{m_e}\frac{\omega_0^2-\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cos\omega t - \frac{eE}{m_e}\frac{2\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\sin\omega t$$

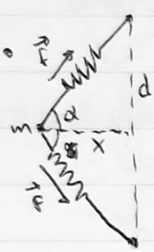
$\Rightarrow$ $\cos\omega t$ term: in-phase with the driving force (elastic or dispersive part of displacement)

$\Rightarrow$ $\sin\omega t$ term: out of phase with the driving force $\rightarrow$ absorption

- The induced dipole moment, $p_x = -ex$, is never quite in phase with the applied electric field, $E_x = E_0\cos\omega t$. For n electrons per unit volume, polarization $P_x = \chi E_x$ with susceptibility

$$\Rightarrow \chi(\omega) = \frac{ne^2}{m_e}\frac{\omega_0^2-\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{\frac{1}{2}}} - i\frac{ne^2}{m_e}\frac{2\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{\frac{1}{2}}}$$

A Note on Anharmonicity



- As in the lab, two springs (of natural length, l_0) connected as shown, will produce a force $f \propto x^3$

$\rightarrow f = -k(\sqrt{d^2+x^2}-l_0)$, $\cos\alpha = \frac{x}{\sqrt{d^2+x^2}}$

- $m\ddot{x} = -2k(\sqrt{d^2+x^2}-l_0)\frac{x}{\sqrt{d^2+x^2}}$
- $m\ddot{x} = -2kx + 2kx\frac{l_0}{d}\frac{1}{\sqrt{1+(\frac{x}{d})^2}}$
- Near center, $\frac{x}{d} \ll 1 \Rightarrow (1+z)^{-\frac{1}{2}} \simeq (1 - \frac{1}{2}z \ldots$ for $z \ll 1)$
- $m\ddot{x} = -kx(2-2\frac{l_0}{d}) - \frac{kx^3 l_0}{d^3}$

- In fact for $d \simeq l_0$, the first term vanishes and this is purely anharmonic, $F \propto x^3$, oscillator. However in general:
  - $\ddot{x} + \omega_0^2 x = -\beta x^3$ with $\omega_0^2 = \frac{k}{m}(2-2\frac{l_0}{d})$, $\beta = \frac{k l_0}{m d^2}$
  - use perturbation theory approach (

    $\rightarrow x = x^{(1)} + x^{(2)} + \ldots$ where $x^{(1)} = A\cos\omega t$

    $\rightarrow$ and $\omega = \omega_0 \pm \Delta\omega$ is the exact value (unknown).
  - Can rewrite as

    $\rightarrow \frac{\omega_0^2}{\omega^2}\ddot{x} + \ddot{x} + \omega_0^2 x = -\beta x^3 + \frac{\omega_0^2}{\omega^2}\ddot{x}$

    $\rightarrow \frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\beta x^3 - (1-\frac{\omega_0^2}{\omega^2})\ddot{x}$

- But $x^{(1)} = A\cos\omega t \rightarrow \ddot{x} = -A\omega^2\cos\omega t \Rightarrow \frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = \emptyset$
  - with $x = x^{(1)} + x^{(2)}$, 1st order terms vanish on the LH side
  - $\underbrace{\frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)} + \omega_0^2 x^{(2)}}_{\text{free system term}} = -\beta(x^{(1)})^3 - \underbrace{(1-\frac{\omega_0^2}{\omega^2})(-\omega^2 A\cos\omega t)}_{\text{driven @ resonance } \omega}$

- Runaway Resonance! non-physical i.e. two RHS terms must cancel

$\rightarrow -\beta(x^{(1)})^3 - (1-\frac{\omega_0^2}{\omega^2})(-\omega^2 A\cos\omega t) = \emptyset$

$\rightarrow -\beta A^3\cos^3\omega t - (1-\frac{\omega_0^2}{\omega^2})(-\omega^2 A\cos\omega t) = \emptyset$

$\rightarrow -\beta A^3 \frac{3}{4}\cos\omega t - (1-\frac{\omega_0^2}{\omega^2})(-\omega^2 A\cos\omega t) + \ldots \cos 3\omega t = 0$

$\rightarrow -\frac{3}{4}\beta A^2 - (\omega_0^2 - \omega^2) = \emptyset$

Use: $\cos 3x = 4\cos^3 x + 3\cos x \rightarrow$

- and since $\omega = \omega_0 \pm \Delta\omega$, $\omega^2 - \omega_0^2 = 2\omega_0\Delta\omega + \Delta\omega^2 \approx 2\omega_0\Delta\omega$

$\rightarrow \Delta\omega = \frac{3\beta A^2}{8\omega_0}$

- The observed resonance frequency will have a [weak] amplitude dependence

- $\omega = \omega_0 + \frac{3\beta}{8\omega_0}A^2$ Note the absence of the linear term A.

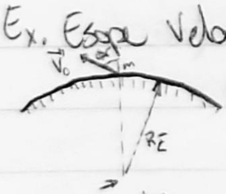
- This is only an approximation as higher order terms and in the fourier expansions which we truncated to a single term $\cos(n\omega t)$, $n=1$, will contribute

- For a more complete description of $\omega(A)$, need a numerical solution

Work and Energy in 3D

- A generalization - Define work done by a force acting on a particle that moves along a specific path from $\vec{r}_a$ to $\vec{r}_b$ as:
  - $W_{a\to b} = \oint_a^b \vec{F}\cdot d\vec{r}$
- $\vec{F}\cdot d\vec{r} = F(\vec{r})\,dr\cos\theta$, $\forall \vec{r}$ along the path
- Since $F = m\frac{d\vec{v}}{dt}$ where $\vec{v} = \frac{d\vec{r}}{dt}$
  - $\Rightarrow W_{a\to b} = m\oint_a^b \frac{d\vec{v}}{dt}\,d\vec{r} = m\oint_a d\vec{v}\,\vec{v} = m\oint_a m\vec{v}\,d\vec{v} = \frac{1}{2}mv^2\Big|_a^b$
  - $W_{a\to b} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$
- Conservative forces - forces such that $\oint_a^b \vec{F}\cdot d\vec{r}$ is independent of path (only depends on endpoints).
- Ex. Radial forces depend only on $r = |\vec{r}|$

  - For simplicity, consider 2D case: $\hat{r}, \hat{\theta}$ (result remains valid in 3D)
  - Choose the origin of force as the origin of the coordinate system
  - At $\vec{r} = r\hat{r}$, $\vec{F}(\vec{r}) = f(r)\hat{r}$ while $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$
  - $\vec{F}(\vec{r})\cdot d\vec{r} = f(r)dr + 0$
  - $W_{a\to b} = \oint_a^b \vec{F}\cdot d\vec{r} = \int_{r_a}^{r_b} f(r)dr$ = independent of path.
- ~~Only such forces~~ Only for such conservative forces can define a scalar potential, $U(r)$ $W_{a\to b} = \int_{r_a}^{r_b} f(r)dr = U_a - U_b$
- Note the sign convention: $\int_{r_a}^{r_b} \Rightarrow U_a - U_b$ $\left(U \propto -\frac{d}{dr}\right.$
- Force conservative forces, $W_{a\to b} = K_b - K_a = U_a - U_b$ lead to
  - $K_a + U_a = K_b + U_b$.
- Ex. Escape Velocity



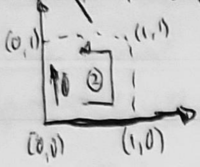
  - $\vec{F} = -G\frac{Mm}{r^2}\hat{r} = -mg\frac{R_E^2}{r^2}\hat{r}$
  - $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$
  - $\vec{F}\cdot d\vec{r} = -mg\frac{R_E^2}{r^2}\hat{r}\cdot(dr\,\hat{r} + r\,d\theta\,\hat{\theta})$
  - Again, the component along $\hat{\theta}$ contributes 0 $\to$ gravity as a conservative force
  - $\vec{F}\cdot d\vec{r} = -mg\frac{R_E^2}{r^2}dr$
  - $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mg\frac{R_E^2}{r}\Big|_{R_E}^{r}$ $\Rightarrow$ $v_0 \geq \sqrt{2gR_E} \simeq 11\,\frac{km}{s}$
  - Note: v_0 is independent of direction.

- Ex. Inverse square force
  - Eg. gravity, Coulomb
  - $U_b - U_a = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r} = -\int_{r_a}^{r_b} f(r)\,dr = -\int_{r_a}^{r_b} \frac{A}{r^2}\,dr$

  $\Rightarrow U(r) = \frac{A}{r} + \text{const}, \quad f(r) = \frac{A}{r^2}$

  - Const = ? A ~~popular~~ popular convention: $U(\infty) = 0$ but in general, only differences in $U(r)$ have physical actuality

- Ex. path-dependent force



$\vec{F} = A(xy\hat{i} + y^2\hat{j})$

① $d\vec{r} = dy\,\hat{j} \rightarrow \vec{F}\cdot d\vec{r} = Ay^2\,dy$ and

$\int_{①} \vec{F}\cdot d\vec{r} = A\int_{0,0}^{0,1} y^2\,dy = \frac{A}{3}$

② $\oint_{②} \vec{F}\cdot d\vec{r} = \int_{0,0}^{1,0} + \int_{1,0}^{1,1} + \int_{1,1}^{0,1}$

$y = 0 \Rightarrow F = 0 \Rightarrow \int = 0$

$d\vec{r} = dy\,\hat{j} \Rightarrow \vec{F}\cdot d\vec{r} = Ay^2\,dy$, $A\int_0^1 y^2\,dy = \frac{A}{3}$

$d\vec{r} = dx\,\hat{i} \Rightarrow \int_1^0 1\cdot x\,dx = -\frac{A}{2}$

$\Rightarrow \oint_{②} \vec{F}\cdot d\vec{r} = 0 + \frac{A}{3} - \frac{A}{2} = -\frac{A}{6} \neq \frac{A}{3} \neq \oint_{①} \vec{F}\cdot d\vec{r}$.

- And this is a non-conservative force.
- In general, $\vec{F} = \sum \vec{F}_i = \vec{F}_{\text{cons}} + \vec{F}_{\text{noncons}}$ and

  $\rightarrow W_{a\to b}^{(\text{total})} = -(U_b + U_a) + W_{a\to b}^{(\text{noncons})}$ (Recalling $W_{a\to b}^{(\text{total})} = K_b - K_a$)

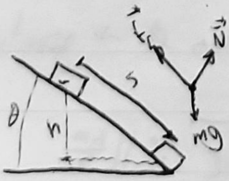
  $\rightarrow K_b - K_a = -(U_b + U_a) + W_{a\to b}^{(\text{noncons})}$

  $\rightarrow E_b - E_a = W_{a\to b}^{(\text{noncons})}$

- Thus in general: $W_{a\to b} = \oint_a^b \vec{F}\,d\vec{r}$ but needs to know the path before the calculation may proceed, for particular cases, where the calculation gets simple:

  $\rightarrow \oint \vec{F}\,d\vec{r}$ = independent of path or $\oint \vec{F}\,d\vec{r} = 0$.

- Ex



① at h: $U_a = mgh \rightarrow K_a = 0 \qquad E_a = mgh$

② at 0: $U_b = 0 \rightarrow K_b = \frac{1}{2}mv^2 \qquad E_b = \frac{1}{2}mv^2$

- force of friction $\Rightarrow$ NON CONS.
- $W_{a\to b} = \int (\sum \vec{F}_i) d\vec{r} = \int \vec{W} dr + \int \vec{N} d\vec{r} + \int \vec{f}_f d\vec{r}$

① $\int \vec{W} d\vec{r} = -(U_b - U_a)$

② $\int \vec{N} d\vec{r} = \emptyset$ since $d\vec{r} \perp \vec{N}$

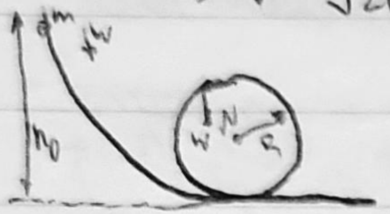
③ $\int \vec{f}_f d\vec{r} = -f s = -\mu M g \cos\theta\, s = -\mu M g \cos\theta \frac{h}{\sin\theta} = -\mu M g h \cot\theta$.

- C.o.E $E_b - E_a = W_{a\to b}^{(NONCONS)}$

$\Rightarrow \frac{1}{2}mv^2 - mgh = -\mu M g h \cot\theta$.

$\Rightarrow v = \sqrt{2(1-\mu \cot\theta}$

• Ex



Roller Coaster

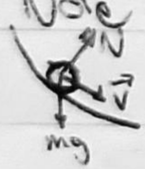
(a) initially, $E_a = mgh_0$

(b) $E_b = mg(2R) + \frac{1}{2}mv_b^2$

- But: $-N - mg = -f_{centripetal} = -\frac{mv^2}{R}$.
- If the rider is to experience "normal" weight at the top of the track: $N = mg \Rightarrow 2mg = \frac{mv_b^2}{R}$ or $mv_b^2 = 2mgR$

$\Rightarrow E_b = mg(2R) + \frac{1}{2}(2mgR) = 3mgR$

- C.o.E: $E_a = mgh_0 = E_b = 3mgR \Rightarrow \boxed{h_0 = 3R}$
- Note that $\forall t$ $\vec{N} \perp \vec{v}$ and $\int_a^b \vec{N} \cdot d\vec{r} = \int_0^{t_b} \vec{N} \cdot \vec{v} dt = \emptyset$



which is why we write $E_a = E_b$.

Momentum of a System of Particles

- Def'n: $\vec{p} = m\vec{v}$ N2L: $F = ma \Rightarrow \frac{dp}{dt}$.
- This is easier to generalize to complex systems, and is, in fact, more functional, as in $\Delta p \Delta x \geq \hbar$
- A system of particles

masses $m_1, m_2, m_3 \ldots$

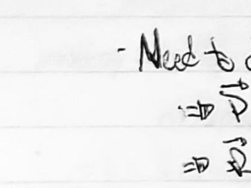
velocity v_1, v_2, v_3

- Total momentum of a system: $\vec{P} = \sum \vec{p}_j = \sum_j \vec{m}_j \vec{v}_j$
- forces on individual masses:
  $\vec{f}_j^{(int)} + \vec{f}_j^{(ext)} = \frac{dp_j}{dt}$, added up: $\sum_j f_j^{(int)} + \sum_j f_j^{(ext)} = \sum \frac{dp_j}{dt}$
- N3L: $\sum_j \vec{f}_j^{(int)} = \emptyset$!
- Total External force: $\vec{F}_{ext} = \sum_j \vec{f}_j^{(ext)}$
- Σ and $\frac{d}{dt}$ are linear operations: $\sum \frac{d}{dt} = \frac{d}{dt}(\Sigma)$
  $\Rightarrow \vec{F}_{ext} = \frac{d}{dt} \sum \vec{P}_j = \frac{dP}{dt}$
- and in particular: $\vec{F}_{ext} = \emptyset \Rightarrow \vec{P} = \text{constant}$
- Center of Mass: Particle: $F = \frac{dp}{dt}$ or $\vec{f} = m\vec{a} = m\ddot{\vec{r}}$
  System: $\vec{f} = \frac{d\vec{P}}{dt}$ or $\vec{F} = M\ddot{\vec{R}}$
- Need to define $\vec{R}$:
  $\Rightarrow \vec{P} = \sum_j m_j \dot{\vec{r}}_j \Rightarrow M\ddot{\vec{R}} = f = \frac{dP}{dt} = \sum_j m_j \ddot{\vec{r}}_j$
  $\Rightarrow \vec{R} = \frac{1}{M} \sum_j m_j \vec{r}_j = \frac{\sum_j m_j \vec{r}_j}{\sum_j m_j}$

  system behaves under $\vec{f}_{ext}$ as if all the mass is concentrated at this point

  $\Rightarrow$ Note: Translations only (cannot rotate a point)
- Practical Importance: in the absence of any external force on the system, the center of mass moves with constant $\vec{v}_{c.m.}$, $\vec{F}_{ext} = \emptyset \Rightarrow \dot{\vec{R}} = \text{const}$ since $P = \sum m \dot{\vec{r}} = \sum m \frac{dR}{dt} = MV_{cm}$
- Extended Bodies: $\Sigma \to \int$
  - $R_{CM} = \frac{1}{M} \int \vec{r}\, dm = \frac{1}{M} \int_V \vec{r} \rho(r)\, dV$

- Ex. Spring gun recoil



V_0 = muzzle speed

Initial momentum $= \emptyset =$ final momentum

gun: $V_x = V_{recoil}$

marble: $V_x = V_0 \cos\theta - V_{recoil}$

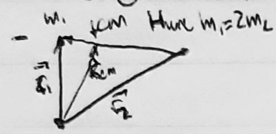
$\emptyset = m(V_0 \cos\theta - V_{recoil}) - MV_{recoil}$

$V_{recoil} = \frac{m V_0 \cos\theta}{M + m}$

Where's y component? Balanced in internal force but external makes us think otherwise.

- Note 1: can ascribe $\vec{p}$ to radiation, when $m = \emptyset$ | P is more fundamental than m, v,
- Note 2: Relativistic mass depends on v

- Center of Mass Coordinates



by defn: $R_{cm} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$
$\vec{r} = \vec{r}_1 - \vec{r}_2$ $\Big\}$ (*)

Solving (*) for $\vec{r}_1$ & $\vec{r}_2$ gives:

$\Rightarrow \vec{r}_1 = R_{cm} + \dfrac{m_2}{m_1+m_2}\vec{r}$
$\Rightarrow \vec{r}_2 = R_{cm} - \dfrac{m_1}{m_1+m_2}\vec{r}$ $\Big\}$ (**)

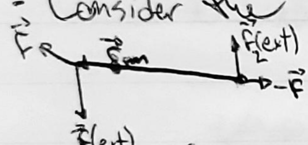
- In the reference frame associated with the CM:

$\Rightarrow r_1' = \dfrac{m_2}{m_1+m_2}\vec{r} = \dfrac{m_2}{m_1+m_2}(r_1 - r_2) = \dfrac{m_2}{m_1+m_2}(\vec{r}_1{}' - \vec{r}_2{}')$

$\Rightarrow r_2' = \dfrac{m_1}{-m_1+m_2}(\vec{r}_1{}' - \vec{r}_2{}')$

Δr is the same in any reference frame
$\vec{r}_1 - \vec{r}_2 = \vec{r}_1{}' - \vec{r}_2{}'$

- Consider the forces



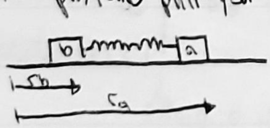
$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_1(ext) + \vec{f} \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2(ext) - \vec{f} \end{cases}$$

$$\Rightarrow \begin{cases} (m_1+m_2)\ddot{R}_{cm} = \vec{F}_1^{(ext)} + \vec{F}_2^{(ext)} \\ \mu\ddot{\vec{r}} = \vec{f} + \dfrac{\mu}{m_1}\vec{F}_1^{(ext)} - \dfrac{\mu}{m_2}\vec{F}_2^{(ext)} \end{cases}$$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2}$ = reduced mass

- Note: a single 2-particle problem became two one particle problems

- Ex. "push-me pull-you" IC: $v_b(\emptyset) = \emptyset$, $v_a(\emptyset) = v_0$, $m_b = m_a = m$



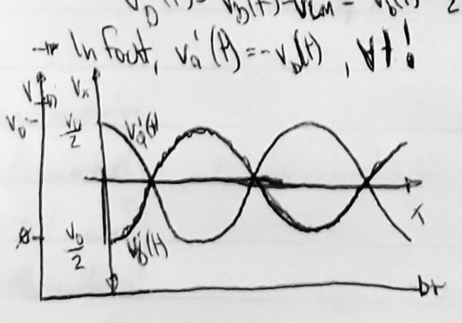
C.o. Momentum (no friction):
$(m+m)v_{cm} = m v_a(\emptyset) + m \cancel{v_b(\emptyset)}^{\,0}$
$v_{cm} = \frac{1}{2}v_0$

$\Rightarrow$ No friction: CM moves uniformly with v_{cm}
$\Rightarrow$ Change to CM Coordinates:

$v_a'(t) = v_a(t) - v_{cm} = v_a(t) - \frac{1}{2}v_0\big|_{t=\emptyset} = \frac{1}{2}v_0$

$v_b'(t) = v_b(t) - v_{cm} = v_b(t) - \frac{1}{2}v_0\big|_{t=\emptyset} = -\frac{1}{2}v_0$

$\Rightarrow$ In fact, $v_a'(t) = -v_b'(t)$, $\forall t$!



- Rocket Motion
  - System = Rocket + gas
  - Gas is emitted with $\vec{u}$ relative to rocket
Time	Rocket Mass	Rocket Velocity	Momentum
t	M	$\vec{v}$	$M\vec{v}$
$t+dt$	$M-dM$	$\vec{v}+d\vec{v}$	$\underbrace{(M-dM)(\vec{v}+d\vec{v})}_{\text{rocket}} + \underbrace{dM(\vec{v}-\vec{u})}_{\text{gas}}$
  - Co. Momentum
    - $\rightarrow$ $\cancel{Mv} = dM(\vec{v}-\vec{u}) + (M-dM)(\vec{v}+d\vec{v})$
    - $\rightarrow$ $0 = -dM\vec{u} + Mdv - \cancel{dMdv}$

      $dM\vec{u} = MdV$

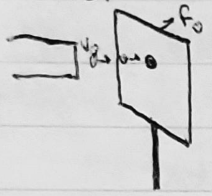
      $dv = u\frac{dM}{M}$
    - $\rightarrow$ Integrate $\rightarrow$ $\vec{v} = \vec{u}\ln M + \text{const}$
    - $\rightarrow$ IC says at rest in free space. $v=0 = \vec{u}\ln M_0 + \text{constant}$ where M_0 = initial mass of rocket + gas it carries (as fuel)
    - $\rightarrow$ $\text{const} = -u\ln M_0 \Rightarrow \vec{v} = \vec{u}\ln\frac{M}{M_0} = -u\ln\frac{M_0}{M}$
    - $\rightarrow$ Note: Independent of how the mass is released only on $|\vec{u}|$ & $\frac{M_0}{M}$
  - Alternatively the r.o.c of momentum:
    - $\rightarrow$ $\frac{dP}{dt} = \frac{dM(v-u) + (M+dM)(v+dv) - mv}{dt} = \frac{-dM\vec{u} + Md\vec{v}}{dt}$
    - $\rightarrow$ thence: $\vec{F}_{ext} = M\frac{dv}{dt} - \vec{u}\frac{dM}{dt}$ $\leftarrow$ the fundamental rocket equation
  - Ex Rocket in gravitational field

    $\hat{k}$, $\vec{F} = mg$, $\vec{u}$

    $F = mg \Rightarrow \frac{dv}{dt} = \frac{u}{m}\frac{dM}{dt} + g$ I.C.: $\vec{v}(0) = 0$ start at rest
    - $\rightarrow$ Integrate: $v(t) = -u\ln\left(\frac{M_0}{M}\right) + gt$ $M = M_0$ $\leftarrow$ rocket + fuel
    - $\rightarrow$ vertical-takeoff: $\vec{g} = -g\hat{k}$, $\vec{v} = v\hat{k}$; $\vec{u} = -u\hat{k}$
    - $\rightarrow$ $v(t) = u\ln\frac{M_0}{M} - gt$
- Momentum transport
  - Impulse $= \int F\,dt = \Delta p$
  - So far, 1 - short time $\rightarrow$ large force, when deliver a given Δp

    2 - Δp "delivered" to the rocket by the exhaust @ constant $\vec{u}$, changing mass $\Rightarrow$ need a force, $F = (\vec{v}-\vec{u})\frac{dM}{dt}$
  -

• Consider a stream of particle hitting paddle:



$\bar{f} = f_{AVG} = ?$

$(\Delta p)_{i,collision} = -\Delta p_{i,drop} = -\left[\cancel{mv_{FIN}}^{\,0} - m v_{init}^{\,=v_0}\right]$

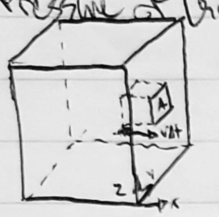
$= + m v_0$ ← impulse on paddle

$\Delta p_{total} = \Delta p_{i,collision} \cdot$ # of collisions per time $\cdot$ time

$\bar{f} = \frac{\Delta p_{total}}{time} = m v_0 \cdot \underline{(\text{rate of collision})} \propto v_0^2$

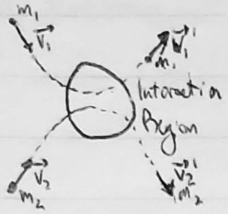
$\propto v_0$

• Pressure of Gas:



- n atoms per unit volume in random 3D motion
- 1) a single elastic collision with a wall elastic
  - → $\vec{V}_{FIN} = -\vec{V}_{INI} = -\vec{v}$
  - → $(\Delta p_{ATOM}) = -2m\vec{V}$ & $(\Delta p_{wall}) = +2mV$
- 2) principle superposition of motion; same in all directions
  - → Say in x, $(\Delta p)_{wall_x} = 2m\vec{V}_x$
  - → Since it's in 1D, we can drop vector notation, $\Delta p_{wall_x} = 2mv_x$
- 3) random motion; $\frac{1}{2}$ of atoms moves towards A, $\frac{1}{2}$ away from A.
  - → On average: # of atoms striking $A = \frac{1}{2} n A (v_x \Delta t)$
- 4) momentum change due to collisions
  - → $\frac{dp_x}{dt} (\frac{1}{2} n A v_x)(2mv_x) = mnAv_x^2$
  - → But force (due to collisions) per unit area is the pressure
    - $P_x = \frac{F_x}{A} = \frac{1}{A}\frac{dp_x}{dt} = mnv_x^2$
- For a distribution of velocities, $P_x = mn\overline{v_x^2}$
- 3D random → all directions are equivalent, on average
  - → $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2}$
    - $P = \frac{1}{3}nm\overline{v^2}$ since $P_x = P_y = P_z = P$ on average
- Kinetic theory of gases: temperature is a measure of kinetic energy
  - → $\frac{1}{2}mv_x^2 = \frac{1}{2}kT$ → $P = nkT = \frac{N}{V}kT$
  - → Or $PV = NkT$
  - → And we recover the ideal gas law from a simple mechanical argument
    - within a fudge factor of $\frac{2}{3}$.

Collisions between Masses



	Initial	Final
m_1:	$\vec{v}_1$	$\vec{v}_1'$
m_2:	$\vec{v}_2$	$\vec{v}_2'$

- C.o.M. $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$
- C.o.E. $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 + \Delta E$
  - → ΔE: energy converted into other forms by the collision (heat, sound, etc.)
- Ex. Head-On. $m_2 = 3m_1$, $\vec{v}_1 = -\vec{v}_2$

  $\vec{v}_1$ $\vec{v}_2$ m_1 m_2

  C.O.M into C.O.E

  $(m_1 - m_2)v = -m_1v_1' + m_2v_2'$ and $(m_1 + m_2)v^2 = m_1v_1'^2 + m_2v_2'^2$

  $m_1v_1' = m_2v_2' - (m_1 - m_2)v$

  ∴ $\frac{1}{m_1}(m_1v_1')^2 = (m_2v_2')^2 - 2m_2(m_1-m_2)vv_2' + (m_1-m_2)^2v^2$

  ∴ $\frac{m_2}{m_1}\left[m_1v_2'^2 + m_2v_2'^2 - 2(m_1-m_2)vv_2' + \frac{(m_1-m_2)^2}{m_2}v^2 - \frac{m_1(m_1+m_2)}{m_2}v^2\right] = 0$

  ∴ $\frac{v_2'}{v} = \frac{2(m_1-m_2) \pm \sqrt{4(m_1-m_2)^2 + 4(m_1+m_2)(m_2-3m_1)}}{2(m_1+m_2)}$

  In our case → $\frac{v_2'}{v} = \frac{-4m_1 \pm \sqrt{16m_1^2 + 0}}{8m_1}$ ↗ -1 → Reproduction of initial condition ↘ 0 → At rest after collision.

Collisions and CM Coordinates

- Center of mass has velocity $\vec{v}_{CM}$ from $m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)v_{cm}$

  ⇒ $\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$ = constant
- Relative to CM, velocities are $\vec{v}_{1c} = \vec{v}_1 - \vec{v}_{CM} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}(\vec{v}_1 - \vec{v}_2)$
- Notation: $\vec{v} = \vec{v}_1 - \vec{v}_2$ = relative velocity

  ⇒ $\vec{v}_{1c} = \frac{m_2}{m_1+m_2}\vec{v}$ and the momentum, in the CM frame

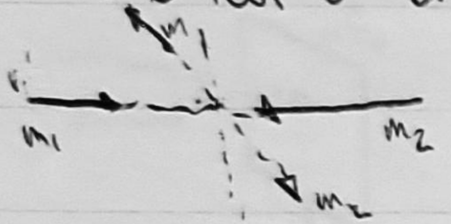
  $\vec{P}_{1c} = m_1\vec{v}_{1c} = \frac{m_1m_2}{m_1+m_2}\vec{v} = \mu\vec{v}$, where $\mu = \frac{m_1m_2}{m_1+m_2}$ is reduced mass
- for m_2: $v_{2c} = \vec{v}_2 - v_{cM} = \frac{-m_1}{m_1+m_2}\vec{v}$ and

  ⇒ $\vec{P}_{2c} = m_2\vec{v}_x = -\mu\vec{v}$
- In the CM reference frame, the C.OM. is trivially satisfied

  ⇒ $\vec{P}_{1c} - \vec{P}_{2c} = \mu\vec{v} - \mu\vec{v} = 0 = \vec{P}_{1c}' + \vec{P}_{2c}'$

  ∴ ⇒ $\left(m_1 + \frac{m_1^2}{m_2}\right)v_{1c}^2 = \left(m_1 + \frac{m_1^2}{m_2}\right)\vec{v}_{1c}'^2$

• The effect of an elastic collision is to rotate the vectors $\vec{v}_{1c}$ & $\vec{v}_{2c}$



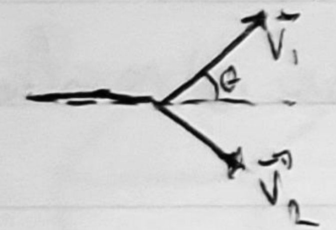
$$\frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2 = \frac{1}{2} \mu \vec{V}^2 = \frac{P^2}{2\mu}$$

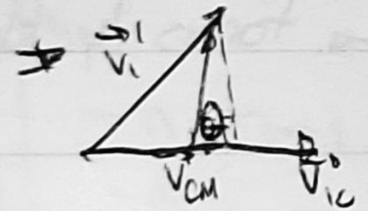
• Ex. PARTICLE AT REST

$m_1\ \vec{v}_1 \qquad m_2$

$$\vec{V}_{cm} = \frac{m_1}{m_1+m_2} V_1$$

$$\begin{cases} \vec{v}_{1c} = \vec{v}_1 - \vec{V}_{cm} = \dfrac{m_2}{m_1+m_2} v \\ \vec{v}_{2c} = -V_{cm} = -\dfrac{m_1}{m_1+m_2} \end{cases}$$





θ & θ_1 related

$$\tan\theta_1 = \frac{v_{1c}' \sin\theta}{\vec{V}_{cm} + v_{1c}' \cos\theta}$$

$$\left. \begin{aligned} V_{cm} &= \frac{m_1}{m_1+m_2} \\ V_{1c} &= \frac{m_2}{m_1 m_2} \end{aligned} \right\} \quad \frac{V_{cm}}{V_{1c}} = \frac{m_1}{m_2} \Rightarrow \tan\theta_1 = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta}$$

• Summary

- C.M. frame $\Rightarrow$ no restriction on scattering

Angular Momentum

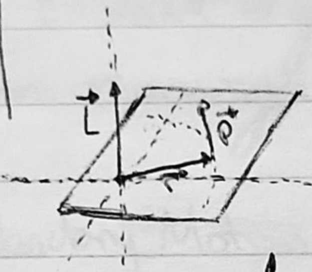
- A theorem of rigid body: any displacement of a rigid body can be decomposed into two independent motion; a translations of its C.O.M and a $\underbrace{\text{rotation}}_{\text{angular momentum}}$ about its C.O.M.

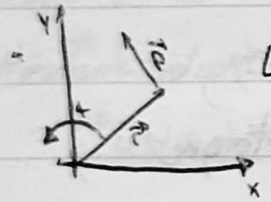
- Angular Momentum of a Particle
  - $\vec{L} = \vec{r} \times \vec{p}$ $\qquad [L] = \frac{kg\,m^2}{s}$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$$

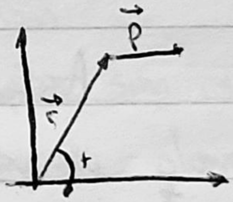
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - p_x y)\hat{k}$$

  - $|\vec{L}| = |\vec{r}||\vec{p}|\sin\theta$



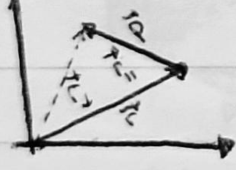


$L_z > 0$, up



$L_z < 0$, down

  - Geometrical Interpretation

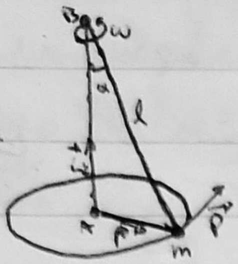


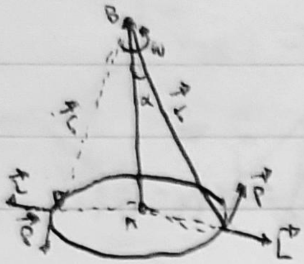
decompse.

  - Ex Trivial Case

$L = \vec{p} \times \vec{p} = 0$ for straight line motion

The Importance of the 3rd Dimension

-  - Steady circular motion

- 

Torque

- $\tau = \vec{r} \times \vec{F}$
- $|\tau| = |\vec{r}_\perp||\vec{F}| = |\vec{r}||\vec{F}_\perp|$
- 

As for, momentum → angular momentum
force → torque

- Features: - Depends on the origins we choose
  - Always $\perp \vec{F}$
- Ex 

Net $f = 0$ $\qquad$ $\vec{F} = -f\hat{j} \neq 0$

Net $\tau = 2Rf\hat{k} = 0$ $\qquad$ $\vec{\tau} = Rf\hat{k} \neq 0$

- Analogy: $f = \frac{\partial p}{\partial t} \Rightarrow \tau = \frac{\partial L}{\partial t}$ Torque is the time derivative of angular momentum
- Implication: Since $\vec{\tau} = \frac{\partial L}{\partial t}$, $\vec{\tau} = 0$, $\vec{L}$ = constant

Conservation of Angular Momentum

- For a system of particles: $\vec{\tau}_{TOTAL} = \sum_i \vec{r}_i \times \vec{F}_i$

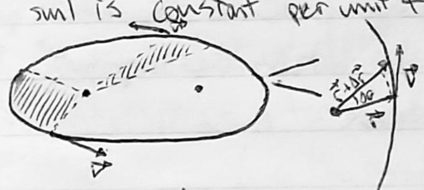
  $\vec{L}_{TOTAL} = \sum_i \vec{r}_i \times \vec{p}_i$

  $\Rightarrow \frac{\partial L_{TOTAL}}{\partial t} = \vec{\tau}_{TOTAL}$.

- The total torque due to internal forces is always zero (N3L + the same line of force for both particles in the interaction) $\Rightarrow$ a closed system (with no external forces applied) has no net torque
- $\vec{\tau}_{TOT} = 0$, $\vec{L} =$ constant $\leftarrow$ for a closed system.
- Radial Forces: The τ about origin $= 0$

$\Rightarrow$ angular momentum about origin is conserved.

- Kepler's Law of Planetary Motion
  - 1^{st} Law - planets move on ellipses, sun is one of the focii
  - 2^{nd} Law - the area swept by the planet's radius vector (from the sun) is constant per unit time.



Area Swept:

$\Delta A \simeq \frac{1}{2}(r+\Delta r)(r\Delta\theta)$

$\simeq \frac{1}{2} r^2 \Delta\theta + \cancel{\frac{1}{2} r \Delta r \Delta\theta}$

- Rate: $\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$
- In Polar: $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\quad \underbrace{\hat{r}\times\hat{r} = 0,\ \hat{r}\times\hat{\theta} = \hat{k}}$

  $\Rightarrow \vec{L} = \vec{r} \times m\vec{v} = r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = mr^2\dot{\theta}\hat{k}$

  $\Rightarrow \frac{\partial A}{\partial t} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2m}$, $L_z =$ const because of central force ($\vec{r} \times \vec{F} = 0$)

- Ex Bullet & Gravity

Gravitational Pull of a planet bends the trajectory, effective making it

b_{eff} = "Impact parameter"

C.O.AM: $m v_0 b_{eff} = m v_R R \Rightarrow b_{eff} = \frac{v_R}{v_0} R$

C.O.E: $\frac{1}{2} m v_0^2 = \frac{1}{2} m v_R^2 - G\frac{Mm}{R} \Rightarrow \frac{v_R^2}{v_0^2} = 1 - \frac{GMm/R}{m v_0^2/2} = 1 - \frac{U(R)}{K_0}$

$\Rightarrow b_{eff}^2 = \left(\frac{v_R}{v_0}\right)^2 R^2 = R^2\left[1 - \frac{U(R)}{K_0}\right]$

Cross Section of Sphere: $A = \pi r^2 \Rightarrow A_{eff} = A_{planet}\left[1 - \frac{U(R)}{K_0}\right]$

$\Rightarrow$ If $U(\infty) = 0$; $U(R) = -\frac{GMm}{R} < 0$

Note: If $\frac{1}{2} m v_0^2 = 0$, $A_{eff} \to \infty$.

- Bohr's Atom
  - Experimental fact: $\frac{1}{\lambda} = R_y\left(\frac{1}{\ell^2} - \frac{1}{n^2}\right)$, $\ell = 1, 2, \dots$, $n = \ell+1, \ell+2, \dots$
  - Bohr's Postulates - ① Atoms exist in stationary states a, b, ... w/ energies $E_a, E_b, \dots$
    - ② E.M. energy is emitted when atom undergoes a transition from a higher-E to a lower-E state. The emitted photon has the energy $hf = E_a - E_b$
    - ③ Angular Momentum is quantized, $n\hbar = \frac{nh}{2\pi}$
    - ④ Classical Mechanics describes orbital motion.
  - Eqn of Motion: $\frac{mv^2}{r} = \frac{e^2}{r^2}$ *

    Energy: $E = \frac{1}{2}mv^2 - \frac{e^2}{r}$

    $\Big\}\ E = -\frac{1}{2}\frac{e^2}{r}$ **

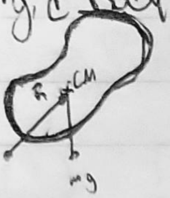
    A.M. Quantization: $mv_n r_n = n\hbar \Rightarrow m^2 v_n^2 r_n^2 = n\hbar$

    $\Rightarrow r_n^2 = \frac{n^2\hbar^2}{m^2 v_n^2} \overset{*}{=} \frac{n^2\hbar^2}{m^2}\frac{m r_n}{e^2} = \frac{n^2\hbar^2}{me^2}$

    $\overset{**}{\Rightarrow} E_n = -\frac{1}{2}\frac{e^2}{r_n} = -\frac{1}{2}\frac{me^4}{n^2\hbar^2}$

    $\Rightarrow hf = E_n - E_\ell$ and $\frac{1}{\lambda} = \frac{f}{c} = \frac{E_n - E_\ell}{hc} = \overbrace{\frac{2\pi^2}{c}\frac{me^4}{h^3}}^{R_y}\left(\frac{1}{\ell^2} - \frac{1}{n^2}\right)$
  - Remarkably, there are no adjustable parameters and the predicted value matches the experiment: $R_y = 109{,}700\,\frac{1}{cm}$
- Torque on a Conical Pendulum

• Rigid Body Considerations



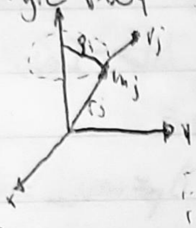
The total torque on a rigid body due to gravity.

$\vec{\tau} = \sum_i \vec{r}_i \times (m_i \vec{g}) = \sum (m_i r_i) \times g = M \vec{R}_{CM} \times \vec{g} = R_{cm} \times (M\vec{g})$

Defn of center of mass

$\vec{\tau} = \vec{R}_{CM} \times W$

- In uniform field, torque on object can be treated as a point mass at C.M.
- To balance object ($\vec{\tau}_{net} = \emptyset$), place pivot at C.M.
- Angular Momentum Associated with a Fixed Axis' Rotation
  - A special case
  - Direction same, translation of the axis
  - Rigid Body → Internal Distances do not change



z - axis of motion

$\vec{L}_j = \vec{r}_j \times m \vec{v}_j$

Rigid object: $|\vec{v}_j| = \omega \rho_j$ } Only L matters.

$\rho_j = \perp$ distance to the z-axis $= \sqrt{x^2 + y^2}$; $|L_{z,j}| = \rho_j m_j v_j = m_j \rho_j^2 \omega$

and the total

$L_z = \sum_j L_{z,j} = \sum_j m_j \rho_j^2 \omega = I\omega$, * $I = \sum m_i \rho_i^2$ * Moment of inertia.

Continuous limit: $I = \int \rho^2 dm = \int (x^2 + y^2) dm$ + in ref frame

- Moment of Inertia - in terms of the dynamics of a pure rotation (i.e. the axis of rotation is at rest), fulfills the role similar to that of mass for the [point mass] translations.

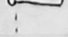
$KE = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \rho_i^2 \omega_i = \frac{1}{2} I \omega^2$

$\text{Torque} = \frac{\partial L}{\partial t} = \frac{\partial}{\partial t}(I\omega) = I\dot{\omega} = I\ddot{\theta}$

• Moments of Interia of Simple Uniform Objects

a.)  $I = \int \rho^2 dm = MR^2$

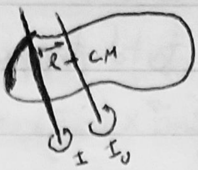
b.)  $I = \int \rho^2 dm = \frac{1}{2} MR^2$

c.)  $I = \int \rho^2 dm = \frac{1}{12} M\ell^2$ |  $I = \int \rho^2 dm = \frac{1}{3} m\ell^2$

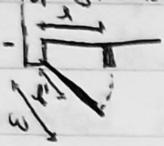
d.)  $I = \int \rho^2 dm = \frac{2}{5} MR^2$

- Parallel Axis Theorem
  - $I = I_0 + Ml^2$
  - $I_{center} = \frac{1}{12}ml^2$ - $I = I_0 + m\left(\frac{l}{2}\right)^2$

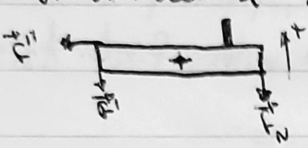
    $I_{end} = \frac{1}{3}ml^2$ - $\frac{1}{3} = \frac{1}{12} + \frac{1}{4}$

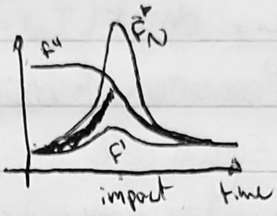


- Center of Percussion



Uniform Door $l' = \frac{1}{2}w$





① About Hinge: $I = \frac{1}{3}I\omega^2$

$\frac{\partial L}{\partial t} = \tau$ ∴ $\partial L = \tau\, dt \Rightarrow \Delta L = \int \tau\, dt$

Initial am: $I\omega_0$, final a.m. $= \emptyset \Rightarrow \Delta L = -I\omega_0$

During collision: $-\int f_d l\, dt = -I\omega_0$

② $\Delta \vec{p} = \int \vec{F}\, dt$ for the C.o.M., $\Delta p = mv_{cm} = ml'\omega_0$

$\Rightarrow -ml'\omega_0 = \int -(f' + f_d)\, dt$

CoM: $Ml'\omega_0 = \int (f' + f_d)\, dt$

CoAM: $I\omega_0 = l\int f_d\, dt$

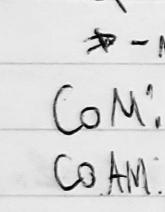
$\Rightarrow ml'\omega_0 - I\omega_0 \frac{1}{l} = \int f'\, dt$, or $\int f'\, dt = \left(Ml' - \frac{I}{l}\right)\omega_0$

A Clever choice: $l = \frac{I}{Ml'} \Rightarrow \int f'\, dt = \vec{f}'\Delta t = \emptyset$

For a uniform door of width w: $\left.\begin{aligned} I &= \frac{1}{3}mW^2 \\ l' &= \frac{1}{2}W \end{aligned}\right\}\ l = \frac{2}{3}W$

Note: works for all ω_0

- Baseball bat: need to hit the ball at the center of percussion



$\Delta(mv)_{ball} \approx -2Mv_0 = \int f\, dt$

Impulse on bat $\approx -\Delta(mv)_{ball} \approx 2MV_0 = -\int f\, dt$

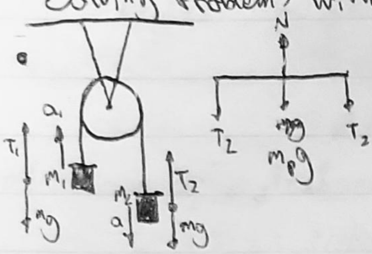
$\Delta L_{bat} \approx I\Delta\omega = -I\omega_0$

$\Rightarrow \int f\, dt \approx I\frac{\omega_0}{l} = Ml'\omega_0 + \int f_{hand}\, dt$

For $f_{hand} = \emptyset$, $\frac{I\omega_0}{l} = ml'\omega_0 \Rightarrow l \approx \frac{I}{Ml}$

$l' = \frac{L}{2}$, $I = \frac{ML^2}{3} \Rightarrow l \approx \frac{2}{3}L$

- Solving Problems with Torques



$$\begin{cases} m_1 g - T_1 = M_1 a \\ M_2 g - T_2 = m_2 a \\ N - T_1 - T_2 - m_p g = \emptyset \end{cases} \quad \text{Constraint } a_1 = a_2 = a$$

$\tau = T_1 R - T_2 R = I\alpha$ + Constraint: Doesn't slip $v = \omega R$, $a = \alpha R$

- Assume pulley is a disk: $I = \frac{1}{2} M_p R^2$

$\Rightarrow T_1 - T_2 = \frac{I\alpha}{R} = \frac{Ia}{R^2} = \frac{M_p a}{2}$

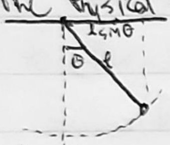
- Also: $m_2 g - m_1 g + T_1 - T_2 = -m_1 a - m_2 a$

$(m_2 - m_1)g + (T_1 - T_2) = -a(m_1 + m_2)$

$-(m_1 - m_2)g + \frac{M_p a}{g} = -a(m_1 + m_2)$

$a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{M_p}{2}}\right) g$

- The Physical Pendulum



Force along $\hat{\theta}$: $-mg\sin\theta$.

$\Rightarrow -mg\sin\theta = ml\ddot{\theta}$ (*)

Another point of view: a pure rotation

$I = ml^2$, $\alpha = \ddot{\theta}$, $\tau = -mgl\sin\theta$

$\Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta$

An approximation solution for small θ: $\sin\theta \approx \theta$; (*) becomes

$\Rightarrow l\ddot{\theta} + g\theta = \emptyset \Rightarrow$ harmonic motion

This solution involves $\sin\omega t$, $\cos\omega t$, $\omega = \sqrt{\frac{g}{l}}$, and $T = \frac{2\pi}{\omega} = 2\pi / \sqrt{\frac{g}{l}}$

Note how T is independent of amplitude of motion.

- Mass Pendulum

About Pivot P: I_P

" CM: I_{CM}

As in the case of simple pendulum

$+lMg\sin\theta = I_P\ddot{\theta} \xrightarrow{\text{small } \theta} \theta = A\sin\omega t + B\cos\omega t$

Parallel Axis Theorem: $I_P = I_{cm} + ml^2$

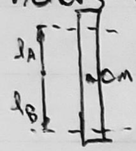
$\Rightarrow \omega = \sqrt{\frac{mlg}{I_{cm} + ml^2}} = \sqrt{\frac{gl}{\frac{I_{cm}}{m} + l^2}}$

I_{cm}/M has a simple physical meaning & $k = \sqrt{\frac{I_{cm}}{M}}$ = radius of gyration

In general, 'k' = "size" of the body

$\omega = \sqrt{\frac{gl}{k^2 + l^2}}$, (For simple pendulum, $k = \emptyset$).

• Katers Pendulum



$$T_A = 2\pi\left(\frac{K^2 + l_A^2}{g l_A}\right)^{\frac{1}{2}}$$ — Adjust T_A, T_B until $T_A = T_B = T$

$$T_B = 2\pi\left(\frac{K^2 + l_B^2}{g l_B}\right)^{\frac{1}{2}}$$

⇒ eliminate T: $K^2 = \frac{l_A l_B^2 - l_A^2 l_B}{l_B - l_A} = l_A l_B$

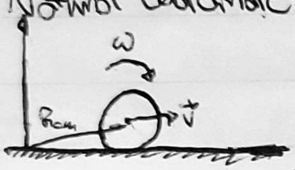
⇒ $T = 2\pi\left(\frac{l_A + l_B}{g}\right)^{\frac{1}{2}}$ ← only depends on $l_A + l_B$, so no need to know where CM is.

or $g = 4\pi^2 \frac{l_A + l_B}{T^2}$

• Motion with translation and rotation

- Principle of Decomposition - any motion can be described a translation of the CM and a rotation about the C.M.

- Natural Coordinates: C.M. Coordinates



e.g. Rolling Drum
- mass, M
- moment of inertia, I_0
- Position of CM, R_{cm}

- Without loss of generality, let the axis of rotation lie along z-axis ⇒ $L_z = I_0\omega + (\vec{R}_{CM} \times m\vec{v})_z$

- So far, R_{cm} is measured in an inertial reference frame; $\vec{v} = \dot{R}_{CM}$

- Proof: divide the body into small "particles" m_j, @ $\vec{r}_j$

$$\Rightarrow \vec{L} = \sum_j (\vec{r}_j \times M\dot{\vec{r}}_j) = \sum (\vec{R}_{cm} + \vec{r}_j') \times m_j (\dot{\vec{R}}_{cm} + \dot{\vec{r}}_j')$$

$$\vec{r} = \vec{R}_{cm} + \vec{r}_j'$$

$$\Rightarrow \vec{L} = \underset{①}{\vec{R} \times \sum_j m_j \dot{\vec{R}}} + \underset{②}{\sum_j m_j \vec{r}_j' \times \dot{\vec{R}}} + \underset{③}{\vec{R} \times \sum_j m_j \dot{\vec{r}}_j'} + \vec{R} \times \sum m_j \vec{r}_j' \times \dot{\vec{r}}_j'$$

② ⇒ $\sum_j m_j \vec{r}_j' = \sum m_j (\vec{r}_j - \vec{R}) = \sum m_j \vec{r}_j - (\sum m_j)\vec{R} = 0$.

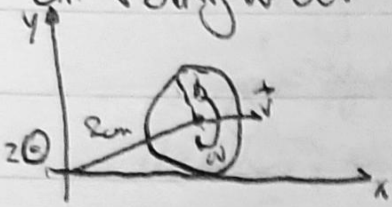
③ ⇒ Similarly, $\sum m_j \vec{r}_j' = 0$

⇒ $\frac{d}{dt}(\sum m_j \vec{r}_j') = 0$

① ⇒ $\vec{R} \times \sum m_j \dot{\vec{R}} = \vec{R} \times M\dot{\vec{R}} = \vec{R} \times m\vec{v}$

- finally, $\vec{L} = \underbrace{\vec{R} \times M\vec{v}}_{\text{a.m. due to the CM rotation}} + \underbrace{\sum_j \vec{r}_j' \times m_j \dot{\vec{r}}_j'}_{\text{a.m. due to motion about the CM.}}$

- Ex. Rolling wheel



$L_0 = I_0\omega = -\frac{1}{2}Mb^2\omega$

$(\vec R \times M_0\vec v)_z = -Mbv = -Mb(b\omega)$

$v = b\omega$. $\Rightarrow L_z = -\frac{1}{2}Mb^2\omega - Mb^2\omega = -\frac{3}{2}Mb^2\omega$

Similarly for Torque

$$\vec\tau = \sum_j \vec r_j \times \vec f_j = \sum_j (\vec r_j{}' \times \vec R)\times \vec f_j = \sum_j \vec r_j{}' \times \vec f_j + \vec R \times \vec F$$

where $F = \sum_j f_j$ = Total applied force

$\vec\tau_0 = \sum_j \vec r_j{}' \times \vec f_j$ = torque about CM.

$\vec R \times \vec F$ = torque due to total external force (applied @ CM)

For a fixed-axis rotation: $\tau_z = \tau_0 + (\vec R \times \vec F)_z$

Using $\tau_z = \frac{dL_z}{dt}$ where $L_z = I_0\omega + \frac{d}{dt}(\vec R \times m\vec v)_z$

$\Rightarrow I_z\dot\omega + \cancel{\dot{\vec R} \times m\vec v} + R \times M\dot{v} = I\alpha + (\vec R \times m\vec a)_z = I\alpha + (\vec R \times \vec F)_z$

$\Rightarrow \tau_0 = I_0\alpha$ (*)

Rotational motion about CM depends only on the torque about the CM; (*) remains valid even if the axis itself is accelerating.

- Summary
  - A pure rotation about an axis, no translation

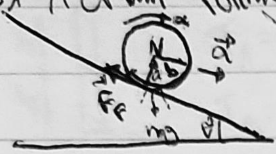
    $L = I\omega$ $\tau = I\alpha$ $K = \frac{1}{2}I\omega^2$

  - Rotation + Translation

    $L_z = I_0\omega + (\vec R \times m\vec v)_z$ $\tau_z = \tau_0 + (\vec R \times \vec F)_z$ $K = \frac{1}{2}I_0\omega^2 + \frac{1}{2}mv^2$

    where I_0, τ_0 are about CM.

• Ex A drum rolling down an incline



: Friction $= \vec{F}_f$; Normal force $= N$; weight $= mg$

acceleration $= a$; ang acceleration $= \alpha$; radius $= b$.

Translation: $mg\sin\theta - f_f = Ma$.

Rotation: $bf_f = I_0\alpha$

Solve: $f_f = \frac{I_0\alpha}{b}$, $I_0 = \frac{1}{2}Mb^2$, $\alpha = \frac{a}{b}$

$\Rightarrow mg\sin\theta - \frac{1}{2}ma = a$

$\Rightarrow a = \frac{2}{3}g\sin\theta$.

Choose origin on the plane

About A: $\tau_A = \tau_0 + (\vec{R}\times\vec{f}) = -bMg\sin\theta$.

$L_z = -I_0\omega + (\vec{R}\times m\vec{v})_z = -\frac{3}{2}mb^2\omega$

$\tau_z = \frac{dL_z}{dt} \Rightarrow -bmg\sin\theta = -\frac{3}{2}mb^2\dot{\omega}$

$a = \frac{2}{3}g\sin\theta$.

• Mod. Fixed - Work Energy

- Generalize to include rot. and trans energy.

- Trans.: C.M. $\vec{f} = m\ddot{\vec{R}} = m\dot{\vec{v}} = m\frac{dv}{dt}$

$f\cdot dR = \left(m\frac{dv}{dt}\right)\cdot(v\,dt) = M\,d\vec{v}\cdot\vec{v} = d\left(\frac{m}{2}v^2\right)$

$\int_{R_a}^{R_b} f\cdot dR = \frac{1}{2}Mv_b^2 - \frac{1}{2}mv_a^2$

- Rot: $\tau_0 = I_0\alpha = I_0\frac{d\omega}{dt}$

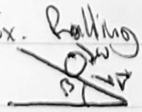
$d\theta = \omega t$

$\tau_0\,d\vec{\theta} = I_0\frac{d\omega}{dt}\omega dt = d(\omega^2 I_0)$

$\int_{\theta_a}^{\theta_b} \tau_0\,d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$

- Work-Energy Theorem: $K_b - K_a = W_{a\to b}$ where $K = \frac{1}{2}mv^2 + \frac{1}{2}I_0\omega^2$

• Ex. Rolling drum with energy



$\int_a^b f\,dr = \frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2$

$(mg\sin\theta - f_f)\,l = \frac{1}{2}mv^2$

$mgh = \frac{1}{2}mv^2 + f_f l$

$\int_{\theta_a}^{\theta_b} \tau\,d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$

$Fb\theta = \frac{1}{2}I_0\omega_0^2$

No slipping: $b\theta = l$, $b\omega = v$ $\Rightarrow fl = \frac{1}{2}I_0\omega^2 = \frac{1}{2}I_0\frac{v^2}{b^2}$

$\Rightarrow mgh = \frac{1}{2}\left(\frac{I_0}{b^2}+m\right)v^2 = \frac{1}{2}\left(\frac{m}{2}+m\right)v^2 = \frac{3}{4}mv^2$

$v = \sqrt{\frac{4}{3}gh}$

- Summary - angular momentum $\vec{L} = \vec{r} \times \vec{p}$
- torque $\vec{\tau} = \vec{r} \times \vec{F}$
- Moment of inertia $I = \int \rho^2 dm$
- Simple Rotation $L = I\omega$

$E = \frac{1}{2}mv^2 \Rightarrow E = \frac{1}{2}I\omega^2$

$F = ma \Rightarrow \tau = I\ddot{\theta} = I\alpha$

I plays the role of m

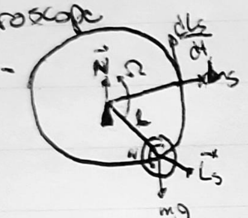
- Conservation of Angular Momentum $\vec{\tau} = \frac{d\vec{L}}{dt}$
- Combining Rotation and Translation
  - Chasles Theorem
  - $L_z = I_0\omega + (\vec{R} \times m\vec{v})_z$

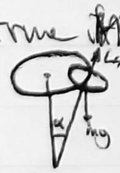
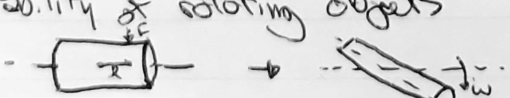
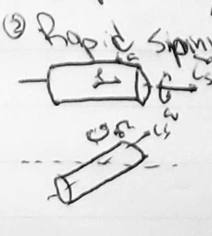
    $\tau_z = \tau_0 + (\vec{R} \times \vec{F})_z \leftarrow \tau_0 = I\alpha$

    $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
- Tools for solving problems
  - Conservation laws (a.m, m, E).
  - separation of rotation & translation
  - parallel axis theorem
  - calculations for definitions, eg. $I = \sum m_i \rho_i^2$
  - constraints

Advanced Topics

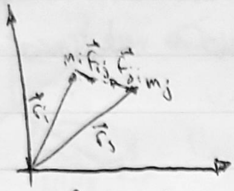
- More general treatment of the rigid body motion
- "Commutative"
  - an operation ⊛ is ~~commutative~~ means: $A ⊛ B = B ⊛ A$.
  - "Commutative" is a part of the definition of a vector
  - Fact: translations commute $\Rightarrow \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a vector
  - Fact: Rotations do not commute $\Rightarrow \vec{\theta} = \theta_x\hat{i} + \theta_y\hat{j} + \theta_z\hat{k}$ is not vector
  - But: Infinitesimal rotations do commute $\Rightarrow \omega = \frac{d\theta_x}{dt}\hat{i} + \frac{d\theta_y}{dt}\hat{j} + \frac{d\theta_z}{dt}\hat{k}$
    $= \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$

- Gyroscope
  - 
    L_s = spin angular momentum
    $\vec{L}_s = I_0\,\omega_s$
    ① L_s is a rotating vector
    $\left|\frac{dL_s}{dt}\right| = \Omega L_s$

    who did that
  - The direction of $\frac{dL_s}{dt}$ is tangential, in the horizontal plane
  - ② The torque of (mg) about ▲: $|\tau_{mg}| = lmg$, direction $\perp mg$, $\perp l$
    $\Rightarrow$ tangential
    $\Rightarrow \Omega L_s = lmg \Rightarrow \Omega = \frac{lmg}{I_0\omega_s}$
  - Ω bigger for smaller ω_s (until frictional torque becomes important)
  - true even if non-horizontal
    
    $\left|\frac{dL_s}{dt}\right| = \left|\frac{d(L_s)_h}{dt}\right| = \Omega L_s \sin\alpha$
    $\tau = (l\sin\alpha)mg \Rightarrow \Omega = \frac{lmg}{I_0\omega_s}$ still
- Stability of rotating objects
  - 
  - ① No spin - Torque $\tau = fl \Rightarrow \Delta L = I_0(\omega - \omega_0) = fl\Delta t \Rightarrow \omega = \frac{fl\Delta t}{I_0}$
  - ② Rapid spinning about long axis
    
    $\Omega = \frac{fl}{L_s} \Rightarrow \varphi = \Omega\Delta t = \frac{fl\Delta t}{L_s}$

Generalizing Angular Momentum

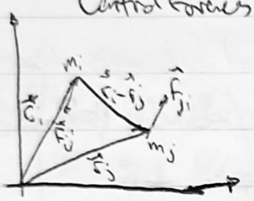
- Place of A.M. in mechanics
  - C.o.A.M is a separate independent, experimental fact; it does not follow from Newtons Laws
  - Ex



$$\left.\begin{aligned} N3L: -\vec{f}_{ij} &= \vec{f}_{ji} \\ \vec{\tau}_{ij} &= \vec{r}_i \times \vec{f}_{ij} \\ \vec{\tau}_{ji} &= \vec{r}_j \times \vec{f}_{ji} \end{aligned}\right\} \quad \vec{\tau}_{ij} + \vec{\tau}_{ji} = \vec{r}_i \times \vec{f}_{ij} - \vec{r}_j \times \vec{f}_{ij} = \emptyset$$

Central forces $\Rightarrow \sum_i \vec{\tau}_i = \emptyset \Rightarrow L$ is conserved.

  -



$N3L: -\vec{f}_{ij} = \vec{f}_{ji}$ but still $\sum \vec{\tau}_i = \emptyset$

$\Rightarrow$ C.o.AM does not follow from Newtons Laws.

- Tensor of inertia
  - Generalizing: $L = I\omega$
  - Simple rotation:  In general, $\vec{\omega} \not\parallel \vec{L}$

$\vec{\omega} = \frac{1}{I}\vec{L}$

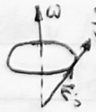
In the CM coordinates:

- $\vec{L} = \vec{R} \times M\vec{V} + \sum_j \vec{r}_j \times m_j \dot{\vec{r}}_j$
  $\vec{\tau} = \vec{R} \times \vec{F} + \sum_j \vec{r}_j \times \vec{f}_j$ $\qquad \tau = \frac{dL}{dt}$

$\Rightarrow \cancel{\vec{R} \times \vec{F}} + \sum \vec{r}_j \times \vec{f}_j = \frac{d}{dt}(\cancel{\vec{R} \times M\vec{V}}) + \frac{d}{dt}(\sum \vec{r}_i \times m_i \dot{\vec{r}}_i)$

① $\frac{d}{dt}(\vec{R} \times M\vec{V}) = \cancel{\vec{V} \times M\vec{V}} + \vec{R} \times M\frac{d\vec{V}}{dt} = \vec{R} \times M\vec{A} = \vec{R} \times \vec{F}$

②



$\dot{\vec{r}}_j = \vec{\omega} \times \vec{r}_j \leftarrow \vec{r}_j$ is a rotating vector

$\Rightarrow$ A.M. about CM

$\vec{L} = \sum \vec{r}_j \times m_j \dot{\vec{r}}_j = \sum_j \vec{r}_j \times m_j (\vec{\omega} \times \vec{r}_j)$

$\Rightarrow \sum_i \vec{r}_i \times \vec{f}_i = \frac{d}{dt}(\sum \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i))$

- In components: $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$
  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- $\vec{\omega} \times \vec{r} = (\omega_y z - \omega_z y)\hat{i} + (\omega_z x - \omega_x z)\hat{j} + (\omega_x y - \omega_y x)\hat{k}$
- $[\vec{r} \times (\vec{\omega} \times \vec{r})]_x = y(\vec{\omega} \times \vec{r})_z - z(\vec{\omega} \times \vec{r})_y$
  $= (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z$
- $L_x = \sum m_j (y_j^2 + z_j^2)\omega_x - \sum m_j x_j y_j \omega_y - \sum m_j x_j z_j \omega_z$
  $= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$
- $L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$
- $L_z = I_{zx}\omega_x + I_{zy}\omega_z + I_{zz}\omega_z$

Tensor of inertia

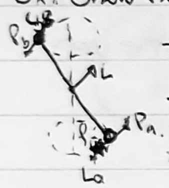
$$-\begin{pmatrix}L_x\\ L_y\\ L_z\end{pmatrix} = \vec{L} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \tilde{I}\vec{\omega}$$

- Tensor - higher order operator relating 2 vectors
- In general, angular momentum about each axis depends on angular velocity about all 3 axes ⇒ need tensor
- Details: $I_{xx} = \sum m_j(y_j^2 + z_j^2)$

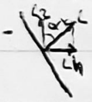
  $I_{xy} = \sum m_j x_j y_j = I_{yx}$

  I_{xz}

- Ex. Skew Rod.



- Each mass describes a circle of radius ; $\rho = l\cos\alpha$
- At any point, $\vec{r}$ is along the rod, $\perp \vec{\rho}$
- ⇒ $|\vec{L}| = |\vec{L}_a + \vec{L}_b| = |\vec{r}_a \times \vec{p}_a + \vec{r}_b \times \vec{p}_b| = 2m\omega l\cos\alpha$



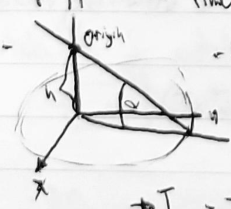
- $L_z = L\cos\alpha = \text{const}$

  $L_h = L\sin\alpha$, rotates about z.

⇒ $\frac{\partial L}{\partial t} = \frac{\partial L_h}{\partial t} = \vec{\tau}$ but $\frac{\partial L_h}{\partial t} = \omega L_h = \omega L\sin\alpha$

⇒ $\tau = \omega L\sin\alpha$ with the direction of $\vec{\tau}$ along $\frac{d\vec{L}}{dt}$ in the tangential direction, in the horizontal plane; it rotates with the rod. This torque is causing the direction L to change, while $|\vec{L}| = \text{constant}$.

- Note: need forces to provide the torque
  - time varying forces creates wear and vibrations



- the inertia tensor of a skew rod
  - $z = -h = -l\sin\alpha$ $\quad \left.\begin{array}{l} x = \rho\cos\omega t \\ y = \rho\sin\omega t\end{array}\right\} \rho = l\cos\alpha$

⇒ $I_{xx} = 2m(\rho^2\sin^2\omega t + h^2)$

⇒ $I_{xy} = I_{yx} = -2m\rho^2\sin\omega t\cos\omega t$

⇒ $I_{xz} = I_{zx} = 2m\rho h\cos\omega t$

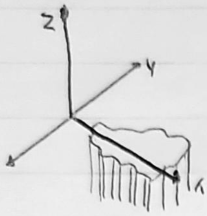
⇒ $\tilde{I} = 2m\begin{pmatrix} \rho^2\sin^2\omega t + h^2 & \dots & \dots \\ -\rho^2\sin\omega t\cos\omega t & \rho^2\cos^2\omega t + h^2 & \dots \\ \rho h\cos\omega t & \rho h\sin\omega t & \rho^2\end{pmatrix}$ ⇒ $\vec{L} = \tilde{I}\vec{\omega}$

EFTS: $\begin{cases} \tau_x = -2m\rho h\omega^2\sin\omega t \\ \tau_y = 2m\rho h\omega^2\cos\omega t \\ \tau_z = \frac{d}{dt}(\rho^2\omega) = \varnothing \end{cases}$

Principal Axes

- Symmetry simplifies: $\tilde{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$

- Eg.



$I_{xy} = -\sum m_j x_j y_j = \emptyset$

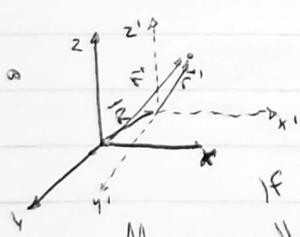
since for each (x_j, y_j) there is $(x_j, -y_j)$

But: not I_{xx}

- for a highly symmetric body aligning coordinates with axes of symmetry,
  - $\tilde{I} = \begin{pmatrix} I_{xx} & \emptyset & \emptyset \\ \emptyset & I_{yy} & \emptyset \\ \emptyset & \emptyset & I_{zz} \end{pmatrix}$

- Diagonalization of $\tilde{I}$ $\Rightarrow$ choice of coordinate ~~system~~ axes in which $\tilde{I}$ is diagonal | Possible for body of any shape
- Tensor of inertia w.r.t. principal axes is diagonal.
- In general - find principal axes of body (diagonalization)
  - Solve eulers equation - in p.a. system
  - Convert from the $(1,2,3)$ system back to the lab ~~frame~~ frame

Non-Inertial Reference frames:



$\vec{r} = \vec{R} + \vec{r}' \Rightarrow \dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}' \Rightarrow \ddot{\vec{r}} = \ddot{\vec{R}} + \ddot{\vec{r}}'$

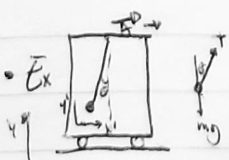
$\vec{v} = \vec{V} + \vec{v}' \Rightarrow \vec{a} = \vec{A} + \vec{a}'$

if $\vec{A} = \emptyset \Rightarrow \vec{a} = \vec{a}'$ and $F = ma = m\vec{a}' = \vec{F}'$

- More generally, $\vec{A} \neq \emptyset$; $F = m\vec{a} = m\vec{A} + m\vec{a}'$

$\Rightarrow \vec{F}' = m\vec{a}' = \underbrace{\vec{F}}_{\text{True force}} - \underbrace{m\vec{A}}_{\text{inertial force}}$

- Ex



y: $T\cos\theta - mg = \emptyset$

x: $T\sin\theta = mA$

$\Rightarrow A = g\tan\theta$.

noninertial observer $\vec{a}' = 0$

$\Rightarrow F = m\vec{a}' = 0 = T + mg - m\vec{A}$

$\Rightarrow F_x = T\sin\theta - mA = \emptyset \Rightarrow \tan\theta = \frac{A}{g}$, as before

$F_y = T\cos\theta - mg = \emptyset$.