Literary Review: *Solitons in nuclear time-dependent density functional theory*

Matthew Olsson
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Preliminary Information

This literary review will go through and detail the writings in Yoritaka Iwata's 2004 paper *Solitons in nuclear time-dependent density functional theory*. In this paper, the practicality and presence in sub-atomic many-body nucleon systems. Sub-atomic many-body nucleon systems refer to the building blocks of an atom (protons, neutrons, even down to elementary particles) and the multiple nucleons that can combine together to create one object. This is done by closely and carefully monitoring the relation of a solitons mass dependence and energy to the many-body nucleons nuclear time-dependent density functional formalism. This paper looks at a specific portion of the density functional theory in which the nuclear structure of many-bodied systems are examined by way of quantum mechanical computer modelling.

The actuality of real nuclear solitons comes from the application of density functional theory (nuclear time-dependent). This paper defines a soliton of having the following properties: 'non linearity, dispersive property, ... quantum effect with the fermi statistics, many-body effect leading to the collectivity' (Iwata 2004) . The two common properties of non linearity refers to the relation between the independent and dependent variables, and the dispersive properties refers to the propagation variables of the wave. The two uncommon proprieties refer to the statistical mechanics of the nuclear system using quantum principles and the random materialization of ordered movement in the many-body system. Here, the nuclear time-dependent density functional formalism (will be referred to as NTDDFF) will be applied consistently to the many-body systems.

To garner a sense of scale for these solitons, the common size and energy measurements of be used. These are the femto-meter for length measurements (1fm $=10^{-15}$m) and the mega-electron volt for energy measurements (1MeV$=1.60218 \times 10^{-13}$J).

Considering the solitons properties, a nucleic system with these properties can be thought that they are in a fluid like state. This fluid state is called the perfect fluidic state and contains many special associations namely; the conserved number of vortexes, the phenomenon that is the low-energy collisions of heavy weighted atoms, and the preservation of the nuclear systems information. This creates an opportunity for a nuclear engineer to use these properties for other applications.

This relates to the course work by first and foremost being about the physical application of a soliton in nature and directly by using the two-soliton solution of the KdV equation.

Mathematical Background

The paper begins by introducing a few key formulas starting with the basic equation of a linear wave.

$$u(x,t) = Ae^{i(kx-\omega t+\alpha)} \tag{1}$$

The A value is our amplitude, k is the wave number (frequency), x is the one-dimensional spatial direction, ω is angular frequency, t is time, and finally α is the phase shift. In the case of multi-dimensions waves are considered to be plane waves. Their value will stay constant across the perpendicular plane. Equation [1] can quickly be applied to some ideas and the first is the advection equation.

$$\partial_t u + c\partial_x u = 0 \tag{2}$$

The c value denotes a real constant which is the propagation speed. This is classified as a first order linear hyperbolic equation, and ends up having the linear dispersion of $\omega = ck$. The solution came by way of the d'Alembert's formula, which is a tool used to solve one-dimensional partial differential equations (PDEs). This can handle higher order PDEs and can consider equations that are of modern physical interest.

Namely, this is the Klein-Gordon equation details about quantum scalars or pseudoscalar fields which are fields of particles that no associated spin value and looks as such:

$$\partial_t^2 u - c^2\partial_x^2 u + \left(\frac{mc^2}{\hbar}\right)^2 u = 0 \tag{3}$$

This is of importance as the solutions of this equation can be compared with the solution of the previous equation ([2]). The solution of the Klien-Gordon equation is $\omega = c^2(k^2 + \frac{m^2c^2}{\hbar^2})$, and is asymptotically equal to the advection equation solution of $\omega = ck$. It is worth noting that for when the mass of the spinless particle is zero, the solution drops to $\omega = \pm ck$.

However, the focus can be returned to particles with non-relativistic properties and the Schrödinger equation is also introduced.

$$i\partial_t u + c\partial_x^2 u = 0 \tag{4}$$

Here, the constant is defined as being $c = -\frac{\hbar}{2m}$. The Schrödinger equation does not fall into the $\omega = ck$ case, rather having this solution instead $\omega = ck^2$. Cases, like this, that do not satisfy the $\omega = ck$ are called dispersive waves. Thus Schrödingers equation is an example of dispersive wave equations.

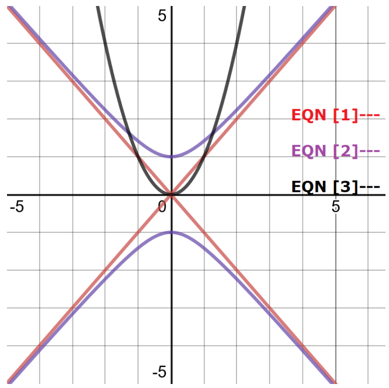

Figure 1: Plot showing the different dispersion relations on the (k, ω) axes

The focus next is on the the nonlinear dispersive waves. In the interest of length, only the Korteweg-de Vries (KdV) equations will be discussed. To begin, the properties that make a soliton are re-introduced. Since however, the paper deals with the idea that there many-nulceon systems (many solitons) they will be tweaked to include the idea of interactions at its core. Solitons satisfy the properties that: the wave moves at constant velocity and maintains its momentum and shape after collisions, and the wave could experiences a shift in phase and time delay during the collisions. The equation that has the information about the soliton solution is contained in the KdV equation. The equation follows as such:

$$\partial_t u + \alpha u \partial_x u + \partial_x^3 u = 0 \tag{5}$$

In this equation α, is a real constant and the second term (αu) influences the waves propagation speed. This demonstrates that the speed depends on the state of the wave itself. Further considering the idea that the soliton is a plane wave, the solution (for small amplitudes) is approximately $\omega = ck - k^3$. Nevertheless, the KdV equations reaches an exact solution of:

$$u = \frac{3c}{\alpha}\text{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct)\right] \tag{6}$$

Here c, is the speed of the wave. This equation turns out to be of the same form as the d'Alembert's solution. As synchronous as this maybe, this is only the case for the one-soliton solution. The two-soliton solution is much larger. The next equation will display the solution and the sequential equation will contain the two-soliton solutions individually for large t.

$$u = \frac{72}{\alpha}\frac{3 + 4\text{cosh}(2x - 8t) + \text{4x-64t}}{(3\text{cosh}(x - 28t) + \text{cosh}(3x - 36t))^2} \tag{7}$$

$$u = \frac{12\kappa_i}{\alpha}\text{sech}^2[\kappa_i(x - 4\kappa_i^2 t) + \delta_i] \tag{8}$$

i=1,2, κ_1=1, κ_2=2, and δ_i are constants. Since the two-soliton solution exists, the theoretical work behind it can be confirmed as true. Moreover, this solution of two-solitons can be extended to N-soliton solutions.

Physical Phenomena and Theory

To transition to the application of solitons to the nuclear time-dependent density functional theory, the environment of the many-nucleon system will be observed. The scale of the atoms being observed is put to subjection. The atoms range from being heavy elements (lead, uranium) and to light (helium, hydrogen). For the purpose of the paper, the soliton motion is wanted to be researched at the size of the nucleus (10^{-15}m to 10^{-13}m). The energy levels corresponding to this scale are around several 10s of MeV per each nucleon. A powerful feature that is used in this paper is the idea that the many-nucleons have finite-bodies and not like the many-electron system (infinite-bodies). This feature allows for the many-nucleon system to be self-bound which means that the wave is localized to it. With this feature, the properties of a soliton can be applied to the nucleon. The self-bound nucleus alone satisfies that it moves with a constant velocity and does not changing with shape. This can then be furthered to the many-nucleon system and applied to the collision case of solitons.

- a nucleus propagates with constant velocity and maintain its shape and momentum after collisions
- as a result of a collision, the nucleus may experiences a phase shift or time delay

This application of the soliton to the many-nucleon system is done to study and be worked into the nuclear time-dependent density functional theory (NTDDFT). The NTDDFT is a theoretical model in physics that observes the dynamics of nuclear collisions, the degrees of freedom for the collisions, the time dependence of these interactions, its non-linearity, and the dispersive properties all coincidentally. The solutions of the theory show that there is unitary time evolution, and that the dispersive property is non-relativistic. Unitary time is a quantum state in which the evolution of time is represented by an operator.

All of this cohesion between the disperion relations, soliton equations, and nuclear theory is extremely important. Sub-atomic theories that deal with time-dependence do not that the framework that has been built up to this point. The advantages and type of framework that has been built are that the theoretical ideas are much simpler/easier to understand and that the theoretical parts of the TDDFT can actually be calculated itself. The TDDFT theory is typically calculated in three-dimensional space and the self-bound nuclei have localized, stationary solutions. The tools that allow for this are the non-linearity, dispersive properties, and the unitary time-dependence. The Experimental portion studies the physical properties of helium and oxygen. They monitored propagation with time, nulceon count vs. size, energy levels of the nucleons, and more. The models that used the solitons equations were used to predict the information to be found. The theoretical, and calculable ideas that are presented in TDDFT has been shown experimentally in the paper. The existence of solitons in the nulceon scale gives light to the fact that they exist in transparency and in fluidity. This allows and brings vision into the legitimacy of different physics and different scales. In this paper, quantum phenomena (untarity) can be seen working with non-relativistic disperive properties. Non-linearity is in tandum with soliton propagation at the core of these physical ideas. Perhaps the interactions between these ideas can be expanded upon and applied to new areas.

Closing Remarks

This paper deatils and demonstrates some quite interesting and complex physical ideas. It delves into the very being of the building blocks of matter and challenges one look deeper. Although the math behind a lot of the concepts is at the upper echelon of the subject, the essential ideas can be perused with ease. Such a paper allows for an expert, a hobbyist, or the humbled undergraduate to have a view into the inner-workings of the atom.

Possible places to follow up that deal with this topic are seemingly endless. One of great interest would be the application of solitons to particle physics (quarks, gluons, etc.), a deeper looking into the quantum world of TDDFT, and if solitons could be applied to systems of relativistic speeds.