Intro

- 2 Main Areas of Calculus
  - Differential
  - Integral.

Dimensionality of Space

- $\mathbb{R}$ - set of real numbers $x \in \mathbb{R}$

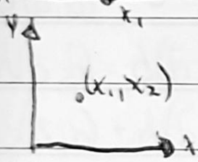
  $\mathbb{R}^2$ - set of pairs $(x_1, x_2) \in \mathbb{R}^2$

  $\mathbb{R}^3$ - set of triples $(x_1, x_2, x_3) \in \mathbb{R}^3$

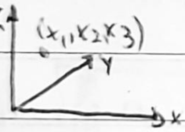
  $\mathbb{R}^n$ - set of n $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$

- 1D - 

$\mathbb{R}$

  2D - 

$\mathbb{R}^2$

  3D - 

$\mathbb{R}^3$

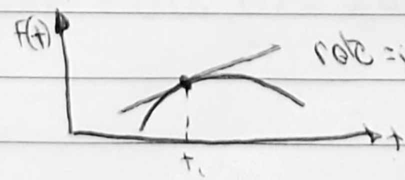
- Example: $\mathbb{R}^n \Rightarrow$ $n = 3$ Classical Physics

  $n = 4$ Relativity

  $n = 1, 2$ Quantum Effects

  $n \geq 10$ String Theory.

Basic Methods of Calculus

- ① Linearization 

rate = inst. rate of change.

② Complex Shapes Can Be Approximated by A Large Number of Simpler Shapes.

Calculus In Higher Dimensions

- $f: \mathbb{R} \to \mathbb{R}$ (functions of a variable)

  $f: \mathbb{R}^2 \to \mathbb{R}$ (functions of 2 variables)

  $f: \mathbb{R}^3 \to \mathbb{R}$ (functions of 3 variables)

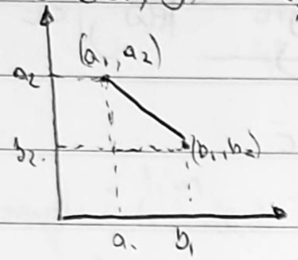
- Functions of 2 variables are described by

  ① Formula e.g $f(x,y) = 4x^2 + y^2$

  ② Graph - Set of points in 3D $(x, y, f(x,y))$, where $(x,y) \in$ domain of f.

  ③ Contour Plot - Choose $C \in \mathbb{R}$. Find all (x,y) for which $f(x,y) = C$. They typically will form a line. Now this is done for many C values

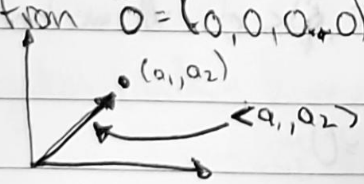
Limits

- Metric and norm in $\mathbb{R}^n$
  - $\bar{x} = (x_1, x_2 \ldots x_n)$ - a point in $\mathbb{R}^n$
  - $x_1, x_2, x_3 \ldots x_n \in \mathbb{R}^n$
- Euclidean Metric (distance)
  - $d(\bar{a}, \bar{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$
  - 

    $\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$

- Satisfies:
  1. $d(\bar{a}, b) = 0 \qquad a = b$
  2. $d(\bar{a}, \bar{b}) = d(\bar{b}, \bar{a})$
  3. $d(\bar{a}, b) + d(\bar{b}, \bar{c}) \geq d(\bar{a}, \bar{c})$
- Norm
  - $\|\bar{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2}$
  - Point $\bar{a} = (a_1, a_2, a_3 \ldots a_n)$ can be represented by a vector from $\bar{0} = (0, 0, 0 \ldots 0)$ to $\bar{a}$.

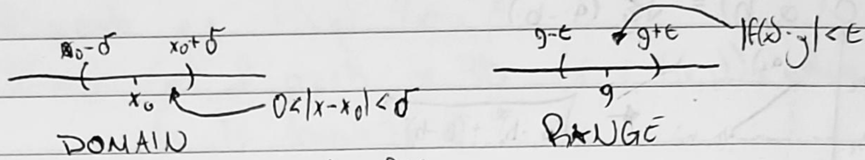
$\boxed{\bar{0} = \text{Origin}}$

  - 

  - Vectors are typically represented with angled brackets (< >)
  - Length of Vector $= d(\bar{a}, \bar{0}) = \sqrt{\sum_{i=1}^{n} \bar{a}_i^2}$

Limits

- Definition in $\mathbb{R}$: - g is the limit of $f(x)$ as $x \to x_0$ if
  - to any $\epsilon > 0$ corresponds some $\delta > 0$ such that for every x satisfying $0 < |x - x_0| < \delta$, we have $|f(x) - g| < \epsilon$

- 

$$g = \lim_{x \to x_0} f(x)$$

- How does this translate into higher dimensions?
  - $\lim_{(x,y)\to(x_0,y_0)}$ or $f(x,y) = g$.

- General form - Let $f: S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$
  - $\bar{a} \in S$, and let f be defined in some spherical neighbourhood of $\bar{a}$.
  - We say that $\lim_{\bar{x}\to\bar{a}} f(\bar{x}) = g$ if for every $\epsilon < 0$ corresponds some $\delta > 0$ such that for every $\bar{x}$ satisfying $0 < d(\bar{x}, \bar{a}) < \delta$ we have $|f(\bar{x}) - g| < \epsilon$

- Example: Find $\lim_{(x,y)\to(x_0,y_0)} \frac{2x^2y}{x^2+y^2}$
  - Try $x = y$ $\lim_{x\to 0} \frac{2x^3}{x^2+x^2} = \lim_{x\to 0} \frac{2}{2}x = 0$.
  - Take $\epsilon > 0$
  - How to choose $\delta > 0$ such that $0 < d((x,y),(0,0)) < \delta$ implies $|f(\bar{x}) - g| < \epsilon$
  - $0 < \sqrt{x^2+y^2} < \delta \Rightarrow \frac{2x^2|y|}{x^2+y} < \epsilon$
  - Note that $|x| \le (x^2+y^2)^{\frac{1}{2}}$ and $|y| \le (x^2+y^2)^{\frac{1}{2}}$
  - Therefore, $x^2|y| \le (x^2+y^2)^{\frac{3}{2}}$

$$\frac{2x^2|y|}{x^2+y^2} \le \frac{2(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2} = 2(x^2+y^2)^{\frac{1}{2}}.$$

  - Result: If $0 < (x^2+y^2)^{\frac{1}{2}} < \delta$ then $\left|\frac{2x^2y}{x^2+y^2}\right| < \epsilon$ for $\delta = \frac{1}{2}\epsilon$,
  - Observation: ~~Hard~~ to show a limit exists, Easy ~~hard~~ to show one doesn't

- Example: Show limit doesn't exist
  - $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$
  - Assume $y = ax$, $a \in \mathbb{R}$
  - $= f(x, ax) = \frac{x\,ax}{x^2+(ax)^2} = \frac{ax^2}{x^2+a^2x^2} = \frac{ax^2}{x^2(1+a^2)} = \frac{a}{1+a^2}$
  - $= \lim_{x\to 0} \frac{a}{1+a^2} \Rightarrow$ Does Not Exist. because the limit varies with a. If it existed the limit would be independent of a.

- Example: $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$
  - $y = ax$, $a \in \mathbb{R}$
  - ① $= f(x, ax) = \frac{x\,a^2x^2}{x^2+x^4a^4} = \frac{x^3a^2}{x^2(1+x^2a^4)} \Rightarrow \lim_{x\to 0} f(x, ax) = 0.$
  - ② $= f(x, x^2) = \frac{x\,x^4}{x^2+x^8} = \frac{x^5}{x^2+x^8} \Rightarrow \lim_{x\to 0} f(x, x^2) = 0$
  - ③ $f(y^2, y) = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$ $\lim_{y\to 0} f(y^2, y) = \frac{1}{2}$.
  - Limit does not exist.

- Integrated limits - have nothing to do with limits of multivariable calculus
  - $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) \neq \lim_{x,y\to 0} f(x,y)$.

Continuity

- A Function of 2 variables (x,y) is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
- Note: Function must be defined on (a,b) to be Continuous at (a,b)

Partial Derivatives

- $$\left.\begin{aligned} f_x(a,b) &= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \\ f_y(a,b) &= \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \end{aligned}\right\}$$ / Partial Derivatives with respect to x and y
- Notation: $f_1(a,b) = f_x(a,b)$.
- Partial Derivates can also be denoted by $\frac{\partial f}{\partial x} = f_x(x,y)$ or $\frac{\partial f}{\partial y} = f_y(x,y)$.

Computing Partial Derivates

- For $f_x(x,y)$, treat y as a constant $(f'(y)=0)$, vice versa
- Ex.
  - $f_x(x,y) = yx^2 + y \Rightarrow y2x$
  - $f(x,y) = e^{xy^2}$
    $f_y(x,y) = 2yx\, e^{xy^2}$.
  - $f(x,y) = e^{xy^2}$
    $f_x(x,y) = e^{xy^2} \cdot y^2$.
- Change variable if needed.

Higher Derivates

- $f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$
  $f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ ← second partial derivation
  $f_{xy}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
  $f_{yx}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ ← Mixed Second Partial Derivatives.
- Notation: $f_{xx}(x,y)$ can be written as $\frac{\partial^2 f}{\partial x^2}$

  $f_{xy}(x,y)$ can be written as $\frac{\partial^2 f}{\partial x\, \partial y}$

Example

- $f(x,y) = xy^2 + xe^x$

$\frac{\partial f}{\partial x} = y^2 + e^x + xe^x$ $\qquad \frac{\partial f}{\partial y} = 2xy$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial (2xy)}{\partial x} = 2y$

$\frac{\partial^2 f}{\partial x^2} =$ [redacted] $+ e^x + e^x + xe^x$ $\qquad \frac{\partial^2 f}{\partial y^2} = 2x.$ $\qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial (y^2 + e^x + xe^x)}{\partial x} = 2y.$

Clairaut Theorem

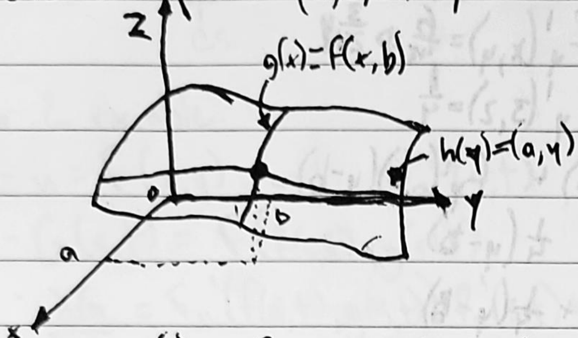
- Let f be defined in some neighbourhood (a,b), and let f_{xy}, f_{yx} be continuous in the neighbourhood. Then,
$f_{xy}(a,b) = f_{yx}(a,b)$

PDE's

- Partial differentiation equations are equations involving partial derivatives where unknowns are functions.
- $\left.\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \right\}$ [illegible] Equation Laplace's

14.4 Geometric Interpretation of Partials (tangent planes, linear approx.).

- function $f(x,y)$ represented as a set of point (x,y,z), $z=f(x,y)$
- Consider point (a,b,c), $c=f(a,b)$



- Note: - $g'(a)$ = Slope of tangent line at $x=a$
  - $h'(b)$ = Slope of tangent line at $y=b$
  - Those 2 tangent lines define a plane, the Tangent plane to the surface $z=f(x,y)$ at point (a,b,c).
- General equation of a plane passing through (a,b,c), $z-c = A(x-a) + B(y-b)$
- A & B to be found.

A: • Line $y=b$ plane tangent to $z=g(x)$ curve at $x=a \Rightarrow z = c + g'(a)(x-a)$ where $c=f(a,b)$
- $z-c = g'(a)(x-a)$ compare to $z-c = A(x-a) + \cancel{B(y-b)} = 0$.
- $g'(a) = A \Rightarrow \left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$

B: • Line $x=a$ plane tangent to $z=h(y)$ curve at $y=b \Rightarrow z = c + h'(b)(y=b)$
$z-c = h'(b)(y-b) \Rightarrow z-c = \cancel{A(x-a)} + B(y-b)$
- $h'(y) = B \Rightarrow \left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$

- Tangent Plane to $z=f(x,y)$ at (a,b)
  - $z - f(a,b) = f_x'(a,b)(x-a) + f_y'(a,b)(y-b)$

Example:

- Find eqn of tangent plane at $(3,2)$ for $f(x,y)=\frac{1}{24}x^2+\frac{3}{48}y^2$

  $a=3, b=2, c=\frac{5}{8} \Rightarrow \frac{1}{24}(3^2)+\frac{3}{48}(2^2)$

- $f_x'(x,y)=\frac{1}{12}x$ $\quad\quad$ $f_y'(x,y)=\frac{6}{48} \Rightarrow \frac{3}{24}y$

  $f_x'(3,2)=\frac{1}{4}$ $\quad\quad$ $f_y'(3,2)=\frac{1}{4}$

- $z-f(a,b)=f_x'(a,b)(x-a)+f_y'(a,b)(y-b)$

  $z-\frac{5}{8}=\frac{1}{4}(x-a)+\frac{1}{4}(y-b)$

  $z=\frac{5}{8}+\frac{1}{4}(x-a)+\frac{1}{4}(y-b)$

  $z=\frac{5}{8}+\frac{1}{4}(x-3)+\frac{1}{4}(y-2)$

- The tangent plane approximates the original surface near tangent point (a,b)
- Close to $x=a$, $y=b$
  - $f(x,y) \approx \underbrace{f(a,b)+f_x'(a,b)(x-a)+f_y'(a,b)(y-b)}_{L(x,y)=\text{linear approximation}}$
  - If f is differentiable.

Example:

- $f(x,y)=xe^{xy}$ at $(1,0)$

  $a=1, b=0, c=1$

  $f_x'(x,y)=e^{xy}+xye^{xy}$ $\quad\quad$ $f_y'(x,y)=x^2e^{xy}$

  $f_x'(1,0)=1$ $\quad\quad$ $f_y'(1,0)=1$

- $L(x,y)=1+1(x-1)+1(y)$

  $L(x,y)=1+(x-1)+y$

  $L(x,y)=x+y$

- $xe^{xy} \approx x+y$ near $(1,0)$

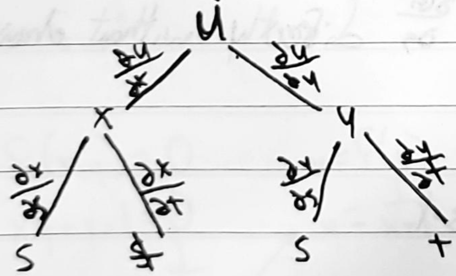
The Chain Rule

- For 1 variable:
  - $u = f(x)$ and $x = f(s)$ so $u = f(f(s))$ and $u = G(s)$
  - $G'(s) = \frac{dG}{ds} \Rightarrow f'(f(s)) \cdot f'(s)$

- For 2 variable:
  - $u = f(x,y)$ and $x = f(s,t)$, $y = g(s,t)$
  - $G(s,t) = f(f(s,t), g(s,t))$
  - $\frac{\partial G}{\partial s} = f_x'(f(s,t), g(s,t)) \cdot f_s'(s,t) + f_y'(f(s,t), g(s,t)) \cdot g_s'(s,t)$
  - $\frac{\partial G}{\partial s} = \frac{\partial f}{\partial x}\Big|_{\substack{x=f(s,t)\\ y=g(s,t)}} \cdot \frac{\partial f}{\partial s} + \frac{\partial f}{\partial y}\Big|_{\substack{x=f(s,t)\\ y=g(s,t)}} \cdot \frac{\partial g}{\partial s}$
  - $\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$

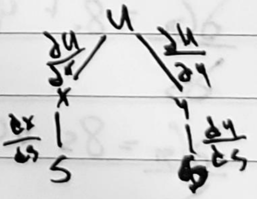
All mean the same

- Pictorial Representation



Another Case:

$u = f(x,y)$ $x = f(s)$ $y = g(s)$



$$\frac{du}{ds} = \frac{\partial u}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial u}{\partial y} \cdot \frac{dy}{ds}$$

Example

- $f(x,y) = x^2 + y^2$, $x = 2s + t$, $y = s - t$
- $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2x)(2) + (2y)(1)$$
$$\frac{\partial f}{\partial s} = 4x + 2y \rightarrow = 4(2s+t) + 2(s-t) \quad = 10s + 2t.$$
$$= 8s + 4t + 2s - 2t$$

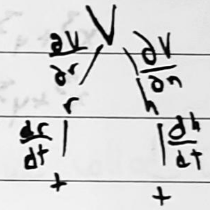
- $f(x,y) = x^2 + y$, $x = 2s + t$, $y = s - t$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2x)(2) + (1) \cdot (1)$$
$$= 4x + 1$$
$$= 4(2s + t) + 1$$
$$= 8s + 4t + 1$$

- Note: We could differentiate $\frac{\partial G}{\partial s}$ directly without chain rule
$$G(s,t) = f(f(s,t), g(s,t))$$
$$G(s,t) = (2s + t)^2 + s - t$$
$$\frac{\partial G}{\partial s} = 4s^2 + 4st + t^2 + s - t$$
$$\frac{\partial G}{\partial s} = 8s + 4t + 0 + 1 - 0$$
$$\frac{\partial G}{\partial s} = 8s + 4t + 1$$

Example

- The radius of a cylinder is decreasing at $1.2 \frac{cm}{s}$. Its height increases at $3 \frac{cm}{s}$. At what rate does volume change? (initial $r = 80cm$ $h = 150cm$)
- $V = \pi r^2 h$, $h(t)$, $r(t)$ $\qquad r = 80 cm$

  $h'(t) = 3 \frac{cm}{s}$ $r'(t) = -1.2 \frac{cm}{s}$ $\qquad h = 150 cm$



- $\frac{dV}{dt} = ?$ $\quad V(t) = \pi r^2(t) h(t)$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

$$= 2\pi h r(-1.2) + \pi r^2 (3)$$

$$= \cancel{2\pi(150)}(-1.2) + \pi(80)^2(3)$$

$2\pi(80)(150)$

$$= -\cancel{9}90477.9 + 60.318.6$$

$$\frac{\partial V}{\partial t} = -30159.3 \frac{cm^3}{s}$$

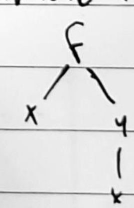
Implicit functions

- $F(x,y) = 0$ $\qquad y = f(x)$ $\qquad$ find: $\frac{dy}{dx} = ?$, differentiate $F(x,y)$.

  $xy + y - 1 = 0$ $\qquad y = \frac{1}{x+1}$

  $y = \frac{1}{x+1}$

$$\frac{d(F(x,y))}{dx} = 0.$$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\partial F}{dx} \cdot \frac{\partial y}{\partial F} = \frac{-F_x}{F_y}$$

Example.

$y = f(x)$, $x^3 + y^3 = 6xy$. find $\frac{dy}{dx} = ?$

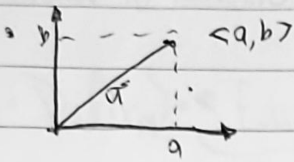
$$x^3 + y^3 - 6xy = 0$$
$$x^3 + y^3 - 6xy = f(x, y)$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{(-3x^2 + 6y)}{3y^2 - 6x}$$

$$= \frac{(-3x^2 - 6y)}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x}$$

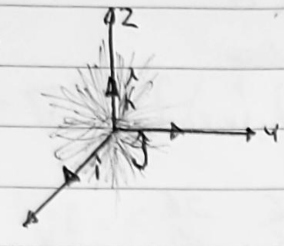
14.6 Directional Derivatives and Gradient



Unit Vectors: $\hat{i} = \langle 1,0,0 \rangle$

$\hat{j} = \langle 0,1,0 \rangle$

$\hat{k} = \langle 0,0,1 \rangle$

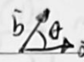


$\bar{u} = \langle a,b \rangle = a\hat{i} + b\hat{j}$

$|\bar{u}| = \sqrt{a^2 + b^2 + c^2}$, if $|\bar{u}| = 1$, we called $\bar{u}$ a unit vector

- Note: To make a vector a unit vector, $\frac{\bar{u}}{|\bar{u}|}$
- Scalar product of $\bar{a} = \langle a_1, a_2, a_3 \rangle$ $\quad \bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cos\theta$ aka dot product.

  $\bar{b} = \langle b_1, b_2, b_3 \rangle$



$\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

- If $|\bar{a}| \neq 0$ and $|\bar{b}| \neq 0$ and $\bar{a} \cdot \bar{b} = 0$, then $\bar{a} \perp \bar{b}$

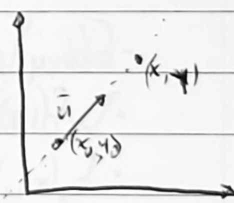
Recall

- $f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$, rate of change of f when y is constant.

Directional Derivatives

- Let $\bar{u}$ = a given unit vector

Let $\bar{u} = \langle a,b \rangle$, $x = x_0 + ah \quad h \in \mathbb{R}$



$y = y_0 + bh$

$z = f(x,y) = f(x_0 + ah, y_0 + bh)$

$\Delta z = z - z_0$, where $z_0 = f(x_0, y_0)$

$\Delta z = f(x_0 + ah, y_0 + bh) - f(x_0, y_0)$

$\lim_{h \to 0} \frac{\Delta z}{h} = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$

$D_{\bar{u}} = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$

- $D_{\bar{u}} f(x_0, y_0)$ - rate of change of f in the direction of $\bar{u}$ evaluated at (x_0, y_0).
- If $\bar{u} = \langle 1, 0 \rangle = \hat{i} \Rightarrow D_{\bar{u}} f(x,y) = f_x$

  If $\bar{u} = \langle 0, 1 \rangle = \hat{j} \Rightarrow D_{\bar{u}} f(x,y) = f_y$.

Theorem

- If $\bar{u} = \langle a, b \rangle$ is a unit vector,

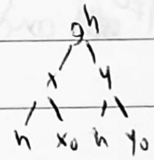
  $D_{\bar{u}} f(x,y) = a f_x(x,y) + b f_y(x,y)$.

- Proof: $g(h) = f(x_0 + ah, y_0 + bh)$

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\bar{u}} f(x_0, y_0)$$

- By the chain Rule

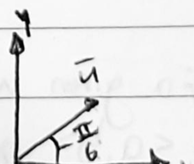


  - $g'(h) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h}$

    $= f_x a + f_y b$

  - Then if $g'(0)$, $h = 0$, $x = x_0$, $y = y_0$
  - $g'(0) = a f_x(x,y) + b f_y(x,y) = D_{\bar{u}}$ $\square$

Example



- $f(x,y) = x^3 - 3xy + 4y^2$
- $D_{\bar{u}} f(1,2) = 2$
- $f_x(x,y) = 3x^2 - 3y$ $\quad \bar{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle$ $\quad \bar{u} = \langle \cos\theta, \sin\theta \rangle$

  $f_y(x,y) = -3x + 8y$ $\quad \bar{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

- $D_u F(1,2) = (3x^2 - 3y)\Big|_{\substack{x=1 \\ y=2}} \left(\frac{\sqrt{3}}{2}\right) + (-3x + 8y)\Big|_{\substack{x=1 \\ y=2}} \left(\frac{1}{2}\right)$

  $D_u f(1,2) = \frac{13 - 3\sqrt{3}}{2}$

Gradient

- $D_{\bar{u}} f(x,y) = \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\nabla f \text{ or grad } f \text{ gradient of } f} \cdot \underbrace{\langle a, b \rangle}_{\bar{u} \text{ unit vector}}$

∇ is read "DEL" or "~~[illegible]~~nabla"

- $\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y}$

$\nabla f = \langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \rangle$

Example

- $f(x,y) = \sin x + e^{xy}$
- $\nabla f = \langle f_x(x,y), f_y(x,y) \rangle$

$= \langle \cos x + ye^{x}, xe^{xy} \rangle$

- $\nabla f(0,1) = \langle \cos(0) + 1e^{0}, 0e^{01} \rangle$

$= \langle 1+1, 0 \rangle$

$= \langle 2, 0 \rangle$

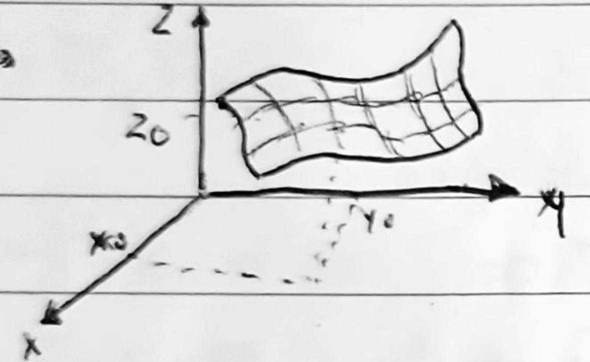
Maximization of Directional Derivative

- Let f be differentiable function $f(x,y)$. The max value of $D_{\bar{u}} f(x,y)$ is $|\nabla f(x,y)|$ and it occurs for $\bar{u} = \frac{\nabla f}{|\nabla f|}$
- Proof: $D_{\bar{u}} f(x,y) = \nabla f \cdot \bar{u} = |\nabla f| \cdot |\bar{u}| \cdot \cos\theta$

$= |\nabla f| \cdot \cos\theta$ * because $\bar{u}$ is a unit vector.

$\therefore$, maximum value when $\cos\theta = 1$, $\theta = 0$

- $\max D_{\bar{u}} f(x,y) = |\nabla f|$, $\bar{u}$ and ∇f in the same direction.

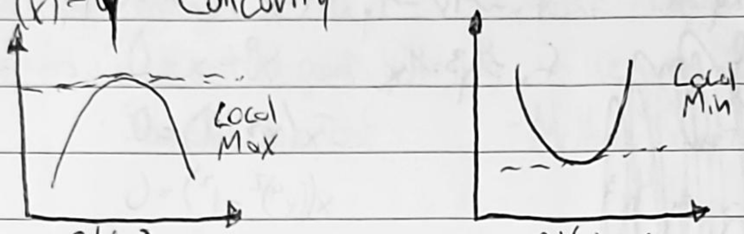


If $z = f(x,y)$ to be elevation, the $\nabla f(x,y)$ is the steepest direction

14.7 Maxima and Minima

- $f'(x) = 0$ - Maxima/Minima

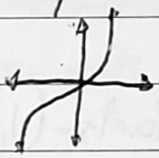
  $f''(x)$ - Concavity

- 

  Extrema.

  $f'(x) = 0$ $\quad$ $f'(x) = 0$

  $f''(x) < 0$ $\quad$ $f'(x) > 0$

- If $f''(x) = 0$, could be an inflution point. 

Theorem

- If f has a local min/max at (a,b), then $\begin{cases} f_x(a,b) = 0 \\ f_y(a,b) = 0 \end{cases}$ or $\nabla f(a,b) = \langle 0, 0 \rangle$
- Points where $\nabla f(x,y) = 0$ are called critical points (or stationary or stable).
- How to find critical points? Solve system $f_x = 0, f_y = 0$
- Ex, $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

  $\begin{cases} f_x = 2x - 2 \\ f_y = 2y - 6 \end{cases}$ $\quad$ $x = 1, \ y = 3$

Second Derivate Test

- $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$
- If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum
- If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum
- If $D < 0$, the $f(a,b)$ does not have a maximum or minimum.

Example

- $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y$ $\quad (x^3)^3 - x = 0$

$f_y = 4y^3 - 4x$ $\quad x^9 - x = 0$

$x(x^8 - 1) = 0$

$x((x^4)^2 - 1^2) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x((x^2)^2 - 1^2)(x^4 + 1) = 0$

$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

Critical Points - $(-1,-1), (0,0), (1,1)$.

$x\underset{<0}{(x-1)}\underset{>0}{(x+1)}\underset{>0}{(x^2+1)}\underset{>0}{(x^4+1)} = 0$. $\quad x = 0, -1$ or 1

$x = 0, y = 0$ or $x = 1, y = 1$ or $x = -1, y = -1$

- Determine the nature of these points.

$f_{xx} = 12x^2 \quad f_{xy} = -4$

$f_{yy} = 12y^2 \quad f_{yx} = -4$

- $D = 12x^2 12y^2 - 16$

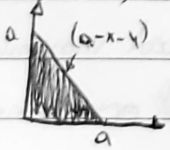
$\left.\begin{array}{ll} D(-1,-1) = 144 - 16 & > 0 \\ f_{xx}(-1,-1) = 12 & > 0 \end{array}\right\}$ Min at $(-1,-1)$

$D(0,0) = -16 \quad < 0$ Saddle at $(0,0)$

$\left.\begin{array}{ll} D(1,1) = 144 - 16 & > 0 \\ f_{xx}(1,1) = & > 0 \end{array}\right\}$ Min at $(1,1)$

- Break up a positive number $a > 0$ into a sum of 3 (+) #'s so that their product is maximize.

- Numbers $x, y, a - x - y$ $\quad f(x,y) = xy(a - x - y)$



$f_x = ay - 2xy - y^2 = y(a - 2x - y)$ solve system

$f_y = ax - x^2 - 2xy = x(a - x - 2y)$ of equations

$y = \frac{a}{3}, \; x = \frac{a}{3}$

$f_{xx} = \frac{-2a}{3}$

$f_{yx} = -\frac{a}{3}$

$f_{yy} = \frac{-2a}{3}$

$D = \left(\frac{-2a}{3}\right)\left(\frac{-2a}{3}\right) - \left(\frac{-a}{3}\right)^2$

$= \frac{4a^2}{9} - \frac{a^2}{9}$

$= \frac{3a^2}{9} = \frac{a^2}{3} > 0$

$\left.\begin{array}{l} D > 0 \\ f_{xx} < 0 \end{array}\right\}$ local max.

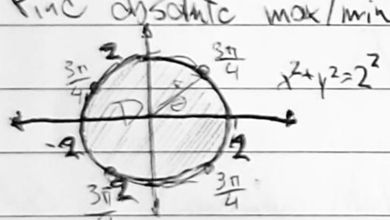
$\frac{a}{3} \cdot \frac{a}{3} \cdot \frac{a}{3} = \frac{a^3}{27}$ = ANS.

Types of Optimization

- Local
  - Find local min/max of $f(x,y)$
  - Methods: Find critical points ($f_x=0, f_y=0$)
    - Second Derivative Test
- Absolute
  - Find largest/smallest value of $f(x,y)$ in region D.
  - Method: find critical points in D, evaluate $f(x,y)$ at crit. points.
    - Check values at the boundary of D (use parimeterization).
    - Choose points where f is largest/smallest

Example

- Find absolute max/min of $f(x,y) = 2xy$ on a disk ~~D~~ $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 4\}$



$f_x = 2y \qquad 2y = 0 \qquad y = 0 \qquad f(0,0) = 0$

$f_y = 2x \qquad 2x = 0 \qquad x = 0$ } Critical Points

$x = 2\cos\theta \qquad \theta \in [-\pi, \pi) \qquad f(x,y) = f(2\cos\theta, 2\sin\theta)$

$y = 2\sin\theta \qquad\qquad = 2 \cdot 2\cos\theta \cdot 2\sin\theta$

$= 8\cos\theta\sin\theta$

$g(\theta) = 8\sin\theta\cos\theta$

$g'(\theta) = -8\sin^2\theta + 8\cos^2\theta$

$g'(0) = -8\sin^2\theta + 8\cos^2\theta$

$\sin^2\theta = \cos^2\theta$

$\left(\frac{\sin\theta}{\cos\theta}\right)^2 = 1$

$\tan\theta = 1$

$\theta = \tan^{-1}(1)$

$\theta = \pm\frac{3\pi}{4}$ or $\frac{\pi}{4}$

$\theta = \pm\frac{3\pi}{4}$ ← Suspect Points

$\theta = \pm\frac{\pi}{4}$

$f(\frac{\pi}{4}, \frac{\pi}{4}) = f(\sqrt{2}, \sqrt{2}) = 4$ MAX

$f(-\sqrt{2}, -\sqrt{2}) = -4$ MIN

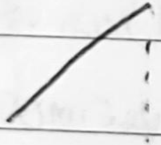
$f(-\sqrt{2}, \sqrt{2}) = -4$ MIN

$f(-\sqrt{2}, \sqrt{2}) = 4$ MAX

} Parimeterization

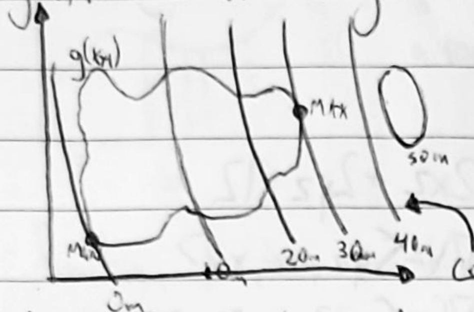
∴, the maximums are 4, minimums are -4.

Perimetorization

- Circle of radius, R, and centered at (a,b) has perimetorization
  - $\begin{cases} x = a + R\cos\theta \\ y = b + R\sin\theta \end{cases}$  $\theta = [0, 2\pi)$ or $[-\pi, \pi]$
- For a polygon, perimeterize each side
  -   $y = ax + b$ $\begin{cases} x = t \\ y = at + b \end{cases}$ $t \in [x_1, x_2]$

Constrainted Optimization

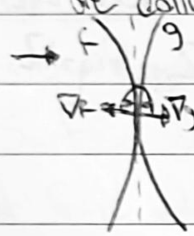
- Problem: find extreme values of $f(x,y)$ subject to $g(x,y)=K$, where $g(x,y)$ and K are given



At extremium, $g(x,y)=K$ is tangent to $f(x,y)$'s contour line

- Note → for $f(x,y)$, vector $\nabla f(a,b)$ is normal to contour lines at (a,b)

→ Two ~~curves~~ curves are ~~[illegible]~~ tangent at (a,b) if their gradients are colinear



$\nabla f = \lambda \nabla g$
constant.

Method of Lagrange Multipliers

- To find extremum of $f(x,y)$ subject to $g(x,y)=K$

1) Solve the system $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=K \end{cases}$ or explicitly $\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = K \end{cases}$ unknowns - x, y, λ.

Lagrange Multipliers.

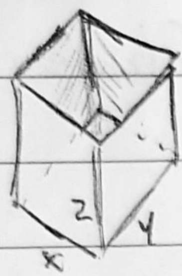
We obtain one or more solutions

2.) Evaluate $f(x,y)$ at all solutions (x,y)

- Largest = Max, Smallest = Min.

Example

- Build a box (no lid) of max volume using $12 m^2$ of cardboard

- 

$$V(x,y,z) = xyz$$
$$S(x,y,z) = xy + 2xz + 2yz = 12$$
$$\begin{cases} \nabla V = \lambda \nabla S \\ S(x,y,z) = 12 \end{cases} \quad \begin{aligned} \nabla V &= \langle yz, xz, xy \rangle \\ \nabla S &= \langle y+2z, x+2z, 2x+2y \rangle \end{aligned}$$

- $$\begin{cases} yz = \lambda(2z+y) & \cdot x \\ xz = \lambda(2z+x) & \cdot y \\ xy = \lambda(2x+2y) & \cdot z \\ 2xz+2yz+xy = 12 \end{cases} \quad \begin{cases} xyz = \lambda(2zx+xy) \\ xyz = \lambda(2yz+xy) \\ xyz = \lambda(2xz+2yz) \\ 2xz+2yz+xy = 12 \end{cases} \Rightarrow \begin{aligned} x &= 2 \\ y &= 2 \\ z &= 1 \end{aligned}$$

Remark 1

- In Lagrange Method, we assume $\nabla g \neq 0$ on $g(x,y) = k$
- Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$ under this assumption

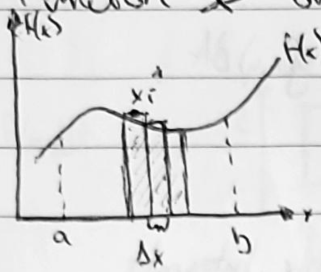
Remark 2

- For problem with constraints

•

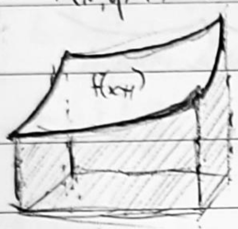
Multiple Integrals

- For $f(x)$ - function of one variable



- $\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$, where x_i^* is a representative point of a subinterval

- For $f(x,y)$ - function of two variables
  - $z = f(x,y)$ is a surface so we talk about the volume under the surface.



- Consider Region $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$
- S - Surface above R, given by $z = f(x,y)$
- Goal: find volume above R and under S.
- Method: Divide R into smaller rectangles
  - $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $\quad i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.
- Choose representative point (x_i^*, y_j^*) in each R_{ij}
- Construct rectangular box above each R_{ij} of height $f(x_i^*, y_j^*)$
- Add volumes of all of them and volume of R under S
  $= \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$, $\quad \Delta A = \Delta x \, \Delta y$
- $\iint_R f(x,y)\,dA = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$

- Define Double Integral - If $f(x,y) \ge 0$ for $(x,y) \in R$,
  $\iint_R f(x,y)\,dA$ represents the volume under S above R.

Properties of Double Integrals

- $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
- $\iint_R c\cdot f(x,y)\,dA = c\iint_R f(x,y)\,dA$.

Practical Method for Computing Double Integrals

- Fubini's Theorem: M1 → $\iint_R f(x,y)\,dA = \underbrace{\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx}_{\text{iterated integral}} = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$

Example

- for $R=[0,2]\times[1,2]$, compute $\iint_R (x-3y^2)\,dA$
- $\int_0^2\int_1^2 (x-3y^2)\,dy\,dx$

$= \int_0^2 [yx-y^3]_1^2\,dx$

$= \int_0^2 [2x-8-x+1]\,dx$

$= \int_0^2 [x-7]\,dx$

$= \left[\frac{x^2}{2}-7x\right]_0^2$ ~~dx~~

$= 2-14$

$= -12$

Order of Integration (Sometimes) matters!

- $\iint_{[1,2]\times[0,\pi]} y\sin(xy)\,dA = \int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy$ ★

$= \int_0^\pi [-\cos(xy)]_1^2\,dy$

$= \int_0^\pi (-\cos 2y+\cos y)\,dy$

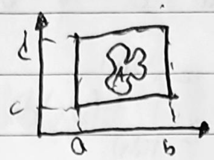
$= \left[-\frac{1}{2}\sin 2y+\sin y\right]_0^\pi$ ~~dy~~

$= 0$

Double Integrals Over General Regions

General Region

-  $\iint_A f(x,y)\,dA$

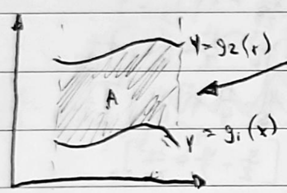


$A \subset [a,b]\times[c,d]$

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } f(x,y) = A \\ 0 & \text{if } \notin A \end{cases}$$

- $\iint_A f(x,y)\,dA \overset{def}{=} \iint_{[a,b]\times[c,d]} \tilde{f}(x,y)\,dA.$

Special Cases - Practical Applications

- 

Type I Region or vertically simple.

- ~~$A = \{(x,y)\in\mathbb{R}^2 : a \le x \le b,\ g_1(x) \le g_2(x)\}$~~ $A = \{(x,y)\in\mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

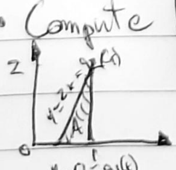
Method of Computing Integrals Over Type I

- $\iint_A f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$

- Remark: Variable bounds must be inside.

Example

- Compute $\iint_A f(x,y)\,dA$ over A.



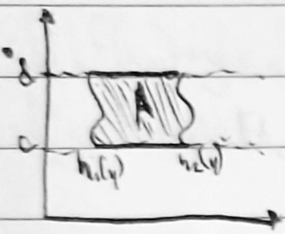
$f(x,y) = x+y$

$0 \le x \le 1$

$0 \le y \le 2x$

$$\iint_A (x+y)\,dA = \int_0^1\int_0^{2x}(x+y)\,dy\,dx$$
$$= \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_0^{2x} dx$$
$$= \int_0^1 \left[2x^2 + \tfrac{1}{2}(2x)^2\right] dx$$
$$= \int_0^1 [2x^2 + 2x^2]\,dx = \int_0^1 4x^2\,dx$$
$$= \left[\tfrac{4}{3}x^3\right]_0^1$$
$$= \frac{4}{3}$$

~~Also~~ Type II Region - Horizontal Simple

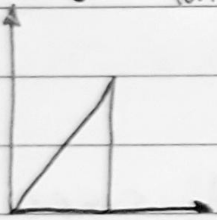


$A = \{(x,y) : \mathbb{R}^2 : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$\iint_A f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Example

- Same as last but II



$0 \le y \le 2$

$h_1(y) \le x \le h_2(y)$

$\frac{y}{2} \le x \le 1$

$\iint_A (x+y)\,dA = \int_0^2 \int_{\frac{y}{2}}^{1} (x+y)\,dx\,dy$

$= \int_0^2 \left[\frac{x^2}{2} + xy\right]_{\frac{y}{2}}^{1} dy$

$= \int_0^2 \left[\frac{1}{2} + y - \frac{y^2}{8} - \frac{y^2}{2}\right] dy$

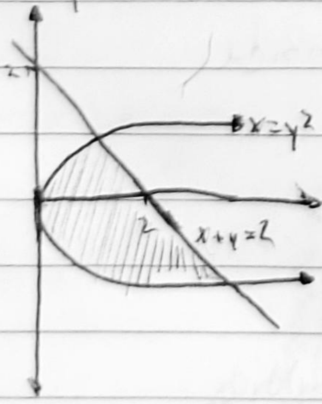
$= \left[\frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24}\right]_0^2$

$= 1 + 2 - \frac{40}{24}$

$= \frac{4}{3}$

Example

- Evaluate $\iint_A (6x + 2y^2)\,dA$, where R is bounded by $x = y^2$ and $x + y = 2$



$\left.\begin{array}{l} x+y=2 \\ x=y^2 \end{array}\right\}$ solutions $=$ $(1,1)$ & $(4,-2)$

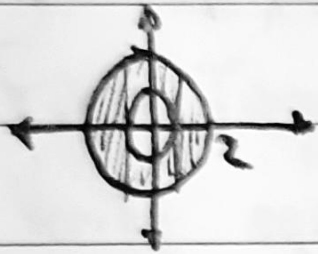
$\left.\begin{array}{l} -2 \le y \le 1 \\ y^2 \le x \le 2-y \end{array}\right\}$ Type II $\int_{-2}^{1}\int_{y^2}^{2-y} (6x + 2y^2)\,dx\,dy$

$\vdots$

$= \frac{99}{2}$

Example:

- $\iint_A f(x,y)\,dA = 2$, where A is a ring (annulus)

- 

$\iint_A f(x,y)\,dA = \iint_{A_1} + \iint_{A_2} + \iint_{A_3} + \iint_{A_4}$

Change of Variable

- $z = f(x,y)$ $\quad x = g(u,v)$, $y = h(u,v)$

$$\frac{\partial f}{\partial x} + (x-y)\frac{\partial f}{\partial y} = 0$$

$u = G(x,y)$

$v = H(x,y)$

$x = u+v \qquad u = \frac{1}{2}(x+y)$

$y = u-1 \qquad v = \frac{1}{2}(x-y)$

- $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{1}{2} + \frac{\partial f}{\partial v}\cdot\frac{1}{2}$$

$\therefore \cdots$

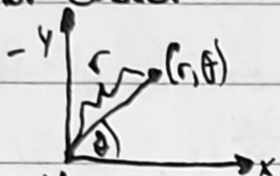
- $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u}\frac{1}{2} + \frac{\partial f}{\partial v}\cdot\frac{1}{2}.$$

Methods of Computing $\iint_A f(x,y)\,dA$.

Note: Polar coordinates are defined on a polar graph that looks like

- We change variables to exploit symmetry of A
- Polar coordinates:

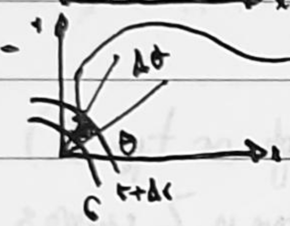


$r = \sqrt{x^2+y^2}$ $\quad x = r\cos\theta$

$\tan\theta = \left(\frac{y}{x}\right)$ $\quad y = r\sin\theta$

$\tan\theta = \left(\frac{1}{1}\right)$ $\quad r = \sqrt{1+1}$ $\quad \left(\sqrt{2}, \frac{\pi}{2}\right)$

$= \frac{\pi}{4}$ $\quad r = \sqrt{2}$



Polar Rectangle $A = \Delta A$.

$\Delta A = \Delta r \cdot a \rightarrow$ where $a = 2\pi r \cdot \frac{\Delta\theta}{2\pi}$

$\Delta A = r\,\Delta r\,\Delta\theta$ $\quad a = r\Delta\theta$

- $\iint_A f(x,y)\,dA \Rightarrow \iint_{D'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

- Example
  - Compute $\iint_D y\,dA$, where D is

- 1.) Describe D in polar for $(r,\theta) \in D$, $0 \le r \le b$, $0 \le \theta \le \pi$

$$\iint_D y\,dA = \int_0^{\pi}\int_a^b r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi}\int_a^b r^2 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\tfrac{1}{3}r^3\sin\theta\right]_a^b d\theta = \int_0^{\pi} \tfrac{1}{3}[b^3\sin\theta - a^3\sin\theta]\,d\theta = \tfrac{1}{3}(b^3-a^3)\int_0^{\pi}\sin\theta\,d\theta$$

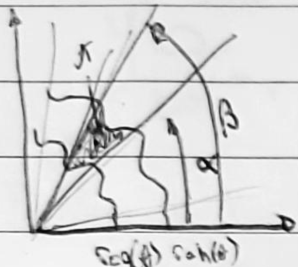
$$= \tfrac{1}{3}(b^3-a^3)[-\cos\theta]_0^{\pi} = \tfrac{1}{3}(b^3-a^3)[-\cos\pi - -\cos 0] = \tfrac{1}{3}(b^3-a^3)(2)$$

$$= \tfrac{2}{3}(b^3-a^3)$$

Polar Coordinates & Applications

- $\iint_D f(x,y)\,dA = \iint_{D'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

  ↑ Express in Polar coordinates ↑ Jacobian

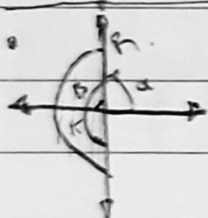
- 

  $\iint_A f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$ $A=\{(r,\theta)\in\mathbb{R}^2 : g(\theta)\le r\le h(\theta),\ \alpha\le\theta\le\beta\}$

Example

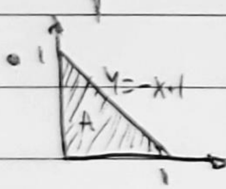
- 

  $R_1 \le r \le R_2$
  $0 \le \theta \le 2\pi$
  $\int_0^{2\pi}\int_{R_1}^{R_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

- 

  $0 \le r \le R$
  $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$
  $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

Circle radius R, centered at (a,b) has equation $(x-a)^2+(y-b)^2=R^2$

- 

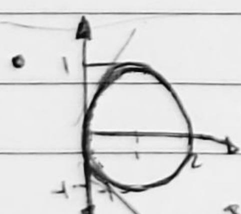
  $\iint_A f(x,y)\,dA = \int_0^1\int_0^{-x+1} f(x,y)\,dy\,dx = \iint_{A'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

  $\rightarrow \int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{\sin\theta+\cos\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

  $0\le\theta\le\frac{\pi}{2}.$

  $y=-x+1 \rightarrow r\sin\theta = -r\cos\theta + 1$ $0 \le r \le \frac{1}{\sin\theta+\cos\theta}$

  $r = \frac{1}{\sin\theta+\cos\theta}$

- 

  $0 \le r \le 2\cos\theta.$ $(x-1)^2+y^2=1$ $(r\cos\theta)^2 + r(\sin\theta)^2 - 2r\cos\theta = 0.$

  $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $x^2-2x+1+y^2-1=0$ $r^2(\cos^2\theta+\sin^2\theta) = 2r\cos\theta$

  $x^2+y^2-2x=0$ $r = 2\cos\theta.$

  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

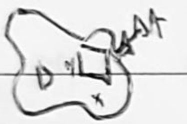
Applications of Double Integrals

- 1) Area of D → Area $= \iint_D dA$

- 2) Volume above region D in xy plane and surface $f(x,y) = z$.

$$V = \iint_D f(x,y)\,dA.$$

- 3) Mass of lamina with density $\rho(x,y)$



$\Delta m \approx \rho(x,y)\Delta A.$ Total Mass → $m = \iint_D \rho(x,y)\,dA.$

$\rho(x,y) = \frac{\Delta m}{\Delta A}$ $\left[\frac{kg}{m^2}\right]$

- 4.) Center of mass of region D with mass density $\rho(x,y)$

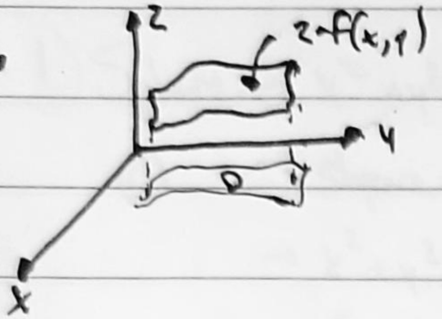
Center of mass at point (x_{cm}, y_{cm}) $x_{cm} = \frac{1}{m}\iint_D x\rho(x,y)\,dA.$ also $x_{cm} = \frac{1}{A}\iint_D x\,dA$

$y_{cm} = \frac{1}{m}\iint_D y\rho(x,y)\,dA.$ also $y_{cm} = \frac{1}{A}\iint_D y\,dA.$

- 5.) Average value of $f(x,y)$ over D

$$\bar{f} = \frac{\iint_D f(x,y)\,dA}{\iint_D dA} = \frac{\iint_D f(x,y)\,dA}{A}.$$

15.5 Surface Area

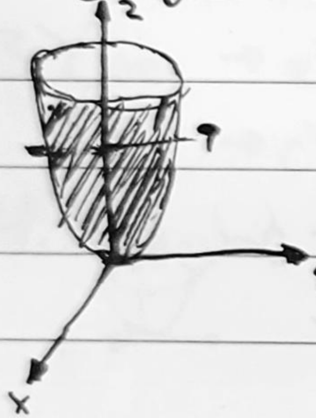


Area of surface $z=f(x,y)$ above region D:

$$A=\iint_D \sqrt{1+f_x^2+f_y^2}\,dA$$

Example

• Find area of $z=x^2+y^2$ under the plane $z=9$.



$$A=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$A=\iint_D \sqrt{1+(2x)^2+(2y)^2}\,dA$$

$A=\iint_D \sqrt{1+4(x^2+y^2)}\,dA$ → Polar Form → $\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$.

$A=\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$ $\quad u=4r^2+1$ → $du=8r\,dr$ $\quad r\,dr=\frac{1}{8}du$

$A=\int_0^{2\pi}\int_1^{37} \sqrt{u}\,\frac{1}{8}\,du\,d\theta$. $\quad r=0 \to u=1;\ r=3 \to u=37$

$A=\left[\int_1^{37}\sqrt{u}\,\frac{1}{8}\,du\right]\int_0^{2\pi} d\theta$

$A=\frac{\pi}{6}(37\sqrt{37}-1)$

Surfaces in 3D

- 1.) Sphere → $x^2+y^2+z^2=R^2$

→ sphere with radius R centered at (0,0,0)

→ $x^2+y^2+z^2 \le R^2$

→ Solid ball of radius R.

→ $(x-a)^2+(y-b)^2+(z-c)^2=R$ centered at (a,b,c).

2.) Cylinder → $x^2+y^2=R^2$ centered at (0,0)

→ $(x-a)^2+(y-b)^2=R^2$ centered at (a,b)

3.) Cone → $z^2=x^2+y^2$ centered at (0,0,0) and $\alpha=\frac{\pi}{4}$

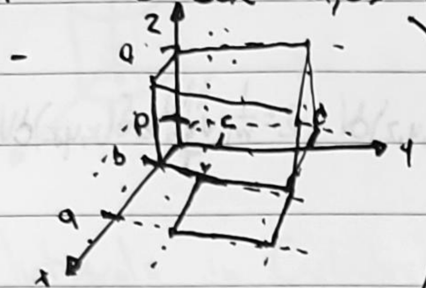
→ $c^2(z-h)^2=(x-a)^2+(y-b)^2$ vertex at (a,b,h) & $\tan\alpha=c$

4.) Paraboloid → $z=x^2+y^2$ vertex at (0,0,0)

→ $z-c=(x-a)^2+(y-b)^2$ vertex at (a,b,c)

16.5 Triple Integrals

- $\iiint_T f(x,y,z)\,dV$ → Element of Volume.
  - T: solid in 3D
- Let T be box $[a,b]\times[c,d]\times[p,q]$



Just a fuckin cube where the subintervals divide into $[a,b]\to\Delta x$, $[c,d]\to\Delta y$, & $[e,f]\to\Delta z$.

- This partition T into small blocks of volume $\Delta V=\Delta x\Delta y\Delta z$.
- Let x_i^*, y_i^*, z_i^* be representative of a point in ΔV.
- $\iiint_T f(x,y,z)\,dV = \lim_{\Delta V\to 0}\sum f(x_i^*,y_i^*,z_i^*)\Delta V$.

Fubini's Theorem

- If $T=[a,b]\times[c,d]\times[p,q]$, then $\iiint_T f(x,y,z)\,dV=\int_p^q\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$
- This is a iterated integral.

Example

- $T=\{(x,y,z): -1\le x\le 1,\ 2\le y\le 3,\ 0\le z\le 1\}$, $f(x,y,z)=xy+yz$. Find $\iiint$
- $\int_0^1\int_2^3\int_{-1}^1 (xy+yz)\,dx\,dy\,dz=\int_{-1}^1\int_2^3\int_0^1 xy+yz\,dz\,dy\,dx=\int_{-1}^1\int_2^3\left[xyz+\frac{1}{2}yz^2\right]_0^1 dy\,dx=\int_{-1}^1\int_2^3 xy+\frac{1}{2}y\,dy\,dx$

$=\int_{-1}^1\left[\frac{1}{2}xy^2+\frac{1}{4}y^2\right]_2^3 dx=\int_{-1}^1\left[\frac{9}{2}x+\frac{9}{4}-2x-1\right]dx=\int_{-1}^1\frac{5}{2}x+\frac{5}{4}\,dx=\left[\frac{5}{4}x+\frac{5}{4}x\right]_{-1}^1$

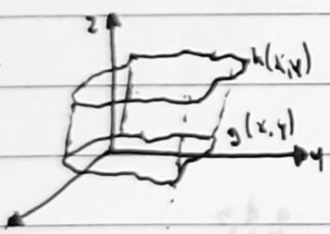
$=\frac{5}{2}$

Applications

- If $\delta(x,y,z)$ has a mass density $\left(\frac{kg}{m^3}\right)$ then mass of T
  $m = \iiint_T \delta(x,y,z)\,dV$
- Centroid of Mass of T, $(\bar{x},\bar{y},\bar{z})$
  → $\bar{x} = \frac{1}{m}\iiint_T x\,\delta(x,y,z)\,dV$, $\bar{y} = \frac{1}{m}\iiint_T y\,\delta(x,y,z)\,dV$, $\bar{z} = \frac{1}{m}\iiint_T z\,\delta(x,y,z)\,dV$
- Average Volume of f over T
  → $\bar{f} = \frac{1}{V}\iiint_T f(x,y,z)\,dV$
- Three main applications of triple integrals

δ = density
if $\delta(x,y,z) = 1$,
$(\bar{x},\bar{y},\bar{z})$ is centroid.

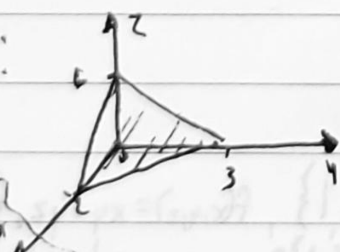
Triple Integral Continuation

- 

- T is a solid between surfaces $z = g(x,y)$ & $z = h(x,y)$
- This is a z simple figure

- $T = \{(x,y,z) \in \mathbb{R}^3 : g(x,y) \le z \le h(x,y),\ (x,y) \in D\}$
- $\iiint_T f(x,y,z)\,dV = \iint_D \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\,dA.$

- Example:
  $A = (2,0,0)$
  $B = (0,3,0)$
  $C = (0,0,6)$



Find mass of pyramid assuming density is $\delta(x,y,z) = z$

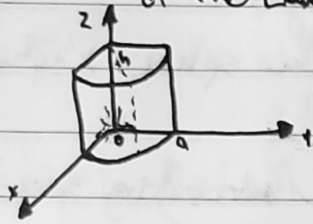
$g(x,y) = 0$

$h(x,y) = z = ax + by + c$ → sub & solve → $6 - 3x - 2y$

$\iiint_T \delta(x,y,z)\,dV = m = \iint_D \int_0^{6-3x-2y} \delta(x,y,z)\,dz\,dA = \iint_D \int_0^{6-3x-2y} z\,dz\,dA = \iint_D \left[\tfrac{1}{2}z^2\right]_0^{6-3x-2y} dA$

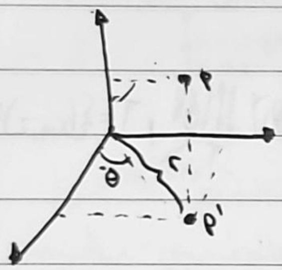
$= \frac{1}{2}\iint_D (6-3x-2y)^2\,dA = \frac{1}{2}\int_0^2\int_0^{-\frac{3}{2}x+3} (6-3x-2y)^2\,dy\,dx \ldots = 9$

- Example: Find the mass of a portion of a solid cylinder $x^2+y^2=a^2$ which lies in between planes $z=0$ & $z=h$ and in the first octant of the coord system, assuming density $= \delta(x,y,z) = x$



$$m=\iiint_T x\,dV = \iint_D \int_0^h x\,dz\,dA = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h x\,dz\,dy\,dx = \dots \frac{1}{3}a^3h$$

Triple Integrals in Cylindrical Coordinates.



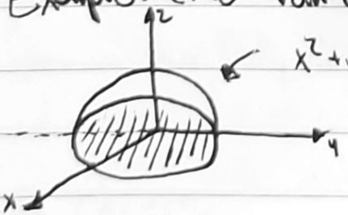
- Cartesian $\rightarrow$ Cylindrical
  $(x,y,z) \rightarrow (r,\theta,z)$
- $x = r\cos\theta$ $\quad z=z$
  $y = r\sin\theta$

- z = constant plane
  θ = constant plane
  r = constant cylinder

Triple Integrals

- Let T be z-simple $T=\{(x,y,z): g(x,y) \le z \le h(x,y), (x,y \in D)\}$
- $I = \iiint_T f(x,y,z)\,dV = \iint_D \int_{g(r\cos\theta,r\sin\theta)}^{h(r\cos\theta,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta.$
- Note: $dV = r\,dz\,dr\,d\theta$
  $\Delta V \approx r\,\Delta z\,\Delta r\,\Delta\theta.$

- Example: Find volume of a sphere with radius r.



$x^2+y^2+z^2=R^2$

$z=\sqrt{R^2-x^2-y^2}$

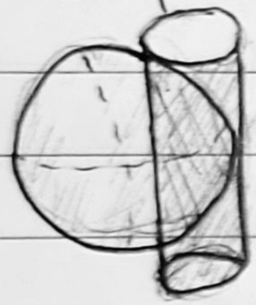
$$V=\iiint_T dV = \iint_D \int_0^{\sqrt{R^2-x^2-y^2}} dz\,dA$$

$$= \int_0^{2\pi}\int_0^R \int_0^{\sqrt{R^2-r^2}} dz\,r\,dr\,d\theta \cdot 2 = 2\int_0^{2\pi}\int_0^R \sqrt{R^2-r^2}\,r\,dr\,d\theta$$

$$=2\int_0^{2\pi} d\theta \int_0^R \sqrt{R^2-r^2}\,r\,dr = -4\pi\int_{R^2}^{0} \sqrt{u}\,\frac{1}{2}\,du = 4\pi\int_0^{R^2}\sqrt{u}\,\frac{1}{2}\,du = 2\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{R^2}$$

$$=2\pi\frac{2}{3}R^3 = \frac{4}{3}\pi r^3$$

Vivani's Solid

- Example of application of double/triple integrals
- Find volume of a solid interior to both the sphere, $x^2+y^2+z^2=4$ and cylinder $(x-1)^2+y^2=1$
-  -Find the volume of the intersecting sphere and cylinder.

- Two ways to approach the problem: - $V=\iint f(x,y)\,dA$ (1)
  - $V = \iiint_T dV$, $T=\{(x,y,z): 0\le z\le f(x,y), (x,y)\in D\}$ (2)

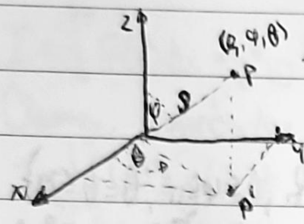
- Method 1:
  - $V = 2\iint_D \sqrt{4-x^2-y^2}\,dA = 2\int_0^2\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \sqrt{4-x^2-y^2}\,dy\,dx =$

- Method 2:
  - $V = 2\iiint dV = 2\int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$ ... $V = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta$
  - Use cylindrical coordinates
    - $x = r\cos\theta$, $y = r\sin\theta$, $z = z$
    - $0 \le z \le \sqrt{4-r^2}$, $0 \le r \le 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
  - $V = 4\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\sqrt{4-r^2}\,r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}}\left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right]_0^{2\cos\theta} d\theta$

$= \frac{32}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)\,d\theta = \frac{32}{3}\left[\int_0^{\frac{\pi}{2}}d\theta - \int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta\right] = \frac{32}{3}\cdot\frac{\pi}{2} - \frac{32}{3}\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta$

$= \frac{32}{3}\frac{\pi}{2} - \frac{32}{3}\cdot\frac{2}{3} \approx 9.64405$

Spherical Coordinates.

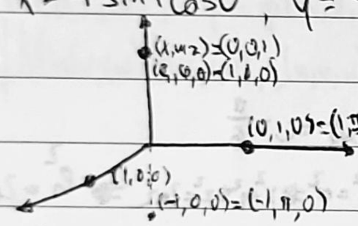


- ρ is distance from the origin
- $\rho = \sqrt{x^2+y^2+z^2}$
- φ is the angle between P and positive z-axis
- θ is the angle from x-axis to P (azimuth angle)

ρ is rho

- $\rho \in [0,\infty)$, $\underbrace{\varphi \in [0,\pi],\ \theta \in [0,2\pi)}_{\text{measured in rads.}}$

- Conversions to Cartesian Plane
  - $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho\cos\varphi$
  -



- Consider paraboloid $z = x^2+y^2$
  - $z = x^2+y^2 \Rightarrow \rho\cos\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta$

$$\cos\varphi = \rho\sin^2\varphi(\cos^2\theta+\sin^2\theta)$$
$$\cos\varphi = \rho\sin^2\varphi$$
$$\frac{\cos\varphi}{\sin^2\varphi} = \rho$$

Integration

- $\iiint_T f(x,y,z)\,dV = ? \Rightarrow \iiint_{T'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\underbrace{\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}_{\text{in place of } dV.}$
- One can show that, $\Delta V = \underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\,\Delta\rho\,\Delta\varphi\,\Delta\theta$

$$dV = \rho^2\sin\varphi\,\Delta\rho\,\Delta\varphi\,\Delta\theta$$

- $T' = \begin{cases} 0 \le \rho \le R \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$

HintsPr.9

$\delta(x,y,z)$ - density of mass

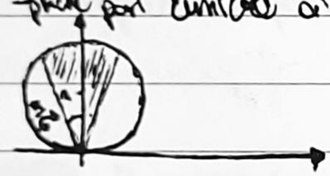
$\iiint_T \delta(x,y,z)\,dV$ - Total mass

Spherical Coordinates

- Volume of sphere of radius R → $V=\iiint_T dV$
- Sphere in spherical coordinates → $T=\{(\rho,\varphi,\theta): 0\le\rho\le R,\ 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi\}$
- $V=\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \Rightarrow \int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi \int_0^{R} \rho^2\, d\rho \Rightarrow [\theta]_0^{2\pi}[-\cos\varphi]_0^{\pi}[\tfrac13\rho^3]_0^{R}$

$\Rightarrow 2\pi\cdot 2\cdot\frac13 R^3 \Rightarrow \frac43\pi R^3$

Find the Volume of Ice-Cream Cone



- Sphere part centered at (0,0,a)
  - φ = constant $= c = \frac{\pi}{6}$
  - $x^2+y^2+(z-a)^2=a^2 \Rightarrow x^2+y^2+z^2=2az \Rightarrow \rho^2 = 2a\rho\cos\varphi \Rightarrow \rho=2a\cos\theta$
  - $0\le\theta\le 2\pi;\ 0\le\varphi\le\frac{\pi}{6};\ 0\le\rho\le 2a\cos\theta$
- $V=\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2a\cos\theta}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/6}\left[\frac13\rho^3\sin\varphi\right]_0^{2a\cos\theta} d\varphi\, d\theta = \frac13\int_0^{2\pi}\int_0^{\pi/6}(2a)^3\cos^3\varphi\sin\varphi\,d\varphi\,d\theta$

$=\frac{8a^3}{3}\int_0^{2\pi}\int_0^{\pi/6}\cos^3\varphi\sin\varphi\,d\varphi\,d\theta = \frac{8a^3}{3}\int_0^{2\pi}d\theta\int_0^{\pi/6}\cos^3\varphi\sin\varphi\,d\varphi = \frac{16\pi a^3}{3}\int_0^{\pi/6}\cos^3\varphi\sin\varphi\,d\varphi$

$=\frac{16\pi a^3}{3}\left[-\frac14\cos^4\varphi\right]_0^{\pi/6} = \cancel{\frac{16\pi a^3}{3}} = \frac{7\pi a^3}{12}$

Find the centroid of the cone

- $\bar{x} = \dfrac{\iiint_T x\, dV}{V}$, $\bar{y} = \dfrac{\iiint_T y\, dV}{V}$, $\bar{z} = \dfrac{\iiint_T z\, dV}{V}$
- By symmetry $\bar{x}=\bar{y}=0$
- $\bar{z}=\frac{1}{V}\iiint z\,dV = \frac{12}{7\pi a^3}\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2a\cos\varphi} \underbrace{\rho\cos\varphi}_{z}\ \overbrace{\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}^{dV} = \frac{12}{7\pi a^3}\int_0^{2\pi}\int_0^{\pi/6}\left[\frac14\rho^4\right]_0^{2a\cos\varphi}\cos\varphi\sin\varphi\,d\varphi\,d\theta$

$=\frac{12}{7\pi a^3}\int_0^{2\pi}\int_0^{\pi/6}\frac{16a^4}{4}\cos^4\varphi\sin\varphi\cos\varphi\,d\varphi\,d\theta = \frac{48a^4}{7\pi a^3}\int_0^{2\pi}d\theta\int_0^{\pi/6}\cos^5\varphi\sin\varphi\,d\varphi = \frac{48\cdot 2\pi a^4}{7\pi a^3}\left[-\frac16\cos^6\varphi\right]_0^{\pi/6}$

$=\frac{96a}{7}\left[-\frac16\cos^6\varphi\right]_0^{\pi/6} = \frac{37a}{28}$

Change of Variables in Integrals

- $I = \int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$
- Change coordinates from x to t, $x = g(t)$, $t = g^{-1}(x)$ $\frac{dx}{dt} = g'(t) \Rightarrow dx = g'(t)\,dt$
- $I = \int_{g^{-1}(b)} f(g(t))\, g'(t)\,dt$

  $= \int_{g^{-1}(a)} f(g(t))\,g'(t)\,dt \leftarrow$ Region of integration at new coordinate

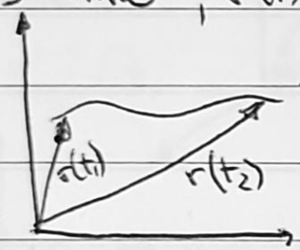
- $I = \iiint_T f(x,y,z)\,dV$
- Change coordinates from (x,y,z) to (u,v,w)
- $x = h(u,v,w)$, $y = p(u,v,w)$, $z = q(u,v,w)$
- $I = \iiint_T f(h(u,v,w), p(u,v,w), q(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

General Jacobian in 3-D:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

13 Vector Functions.

- $\langle a,b,c \rangle$ - is a vector in 3D space
- $\langle a,b,c \rangle \in \mathbb{R}^3$
- Vector functions : $f: \mathbb{R} \to \mathbb{R}^3$
- functions of many variables : $f: \mathbb{R}^3 \to \mathbb{R}$
- Vector Function : $\vec{r}(t) = \langle t^2, \sin t, e^{-t} \rangle$ (Example)
- In general, vector function has the form
  - $\vec{r}(t) = \langle f(t), h(t), g(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ ($\hat{i}, \hat{j}, \hat{k}$ are unit vectors)
- If t is time, $\vec{r}(t)$ represents position in space ($\vec{r}$ - position vector



$t_2 > t_1$ - Vector Function, $\vec{r}(t)$ represents a curve in 2D/3D space parameterized by time.

- Define: $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

  $\vec{r}''(t) = \langle f''(t), g'(t), h'(t) \rangle$

  $\int \vec{r}(t)dt = \langle \int f(t)dt, \int g(t)dt, \int h(t)dt \rangle$

- $\vec{r}(t)$ represents space curves.

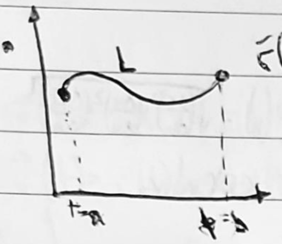
  1.) Straight Line = $y = mx+b$, $x = t$, $y(t) = mx(t)+b$, $\Rightarrow \vec{r}(t) = \langle t, at+b \rangle$

  2.) Circle = $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$, $t \in [0, 2\pi)$

  $= \langle a\cos(\omega t), a\sin(\omega t) \rangle$, where ω = constant angular velocity.

In 2D, any curve $y = f(x)$ can be written as vector function $\vec{r}(t) = \langle t, f(t) \rangle$

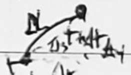
13.3 Length of a Curve



$\vec{r}(t) = \langle f(t), g(t) \rangle$

- Arc length of $\vec{r}(t)$ from $t \in [a,b]$ is $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$
  - $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

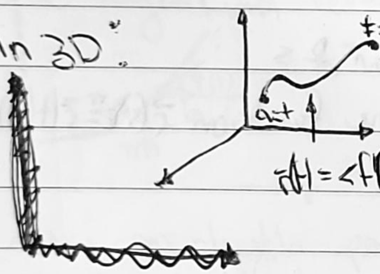
Note: For small Δt



There is no radical curve

$\Delta L \approx \Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$

- In 3D:



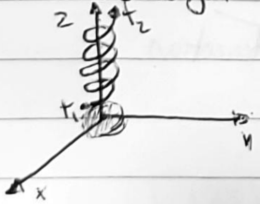
- $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt.$

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle.$ $L = \int_a^b |\vec{r}\,'(t)|\, dt.$

$= \int_a^b |\vec{r}\,'(t)|\, dt.$

Example

- Consider cylindrical helix
- $\vec{r}(t) = \langle a\cos t, a\sin t, kt \rangle$, a, k are positive constants.
- find arc length for $t \in [t_1, t_2]$



- $L = \int_a^b ds$, where $ds = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$

$= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + k^2}\, dt$

$= \sqrt{a(\sin t^2 + \cos t)^2 + k^2}\, dt$

$= \sqrt{a^2 + k^2}\, dt$

$L = \int_{t_1}^{t_2} ds$

$L = \int_{t_1}^{t_2} \sqrt{a^2 + k^2}\, dt$

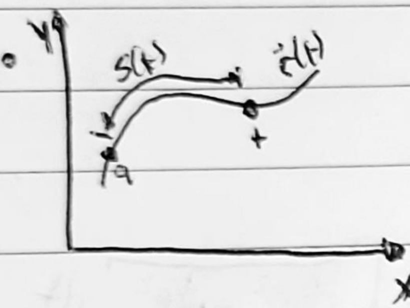
$L = \sqrt{a^2 + k^2} \int_{t_1}^{t_2} dt$

$L = \sqrt{a^2 + k^2}\,(t_2 - t_1)$

Note:

- Parameterization is not unique
- $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $t \in [0, 2\pi)$

Making Parameterization Unique

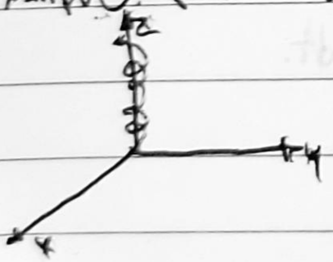
- 

- $s(t) = \int_a^t |r'(u)|\,du$ $\quad r'(u) = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}$
- By fundamental theorem of calculus
- $\frac{ds}{dt} = |r'(t)| \Rightarrow ds = |r'(t)|\,dt$

$*$

- Using $(*)$, we can reparameterize $\vec{r}(t)$ as follows
  - 1.) Write $\frac{ds}{dt} = |r'(t)|$
  - 2.) Solve $(*)$ for t as a function of s
  - 3.) Reparameterize $\vec{r}$ to become function $\vec{r}(s) = \vec{r}(t(s))$

- Example: $\vec{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$

- 

starting @ $t=0$

$r'(t) = \langle -3\sin 3t, 3\cos 3t, 4 \rangle$

$|r'(t)| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} \Rightarrow t=0$

$= \sqrt{16+9} = \sqrt{25} = 5.$

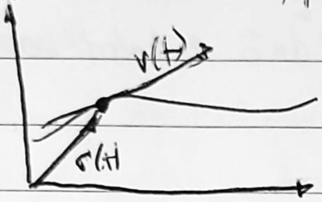
- $\frac{ds}{dt} = 5 \Rightarrow s(t) = \int_0^t |r'(u)|\,du = \int_0^t 5\,du = 5[u]_0^t = 5t.$

- $s = 5t \Rightarrow t = \frac{s}{5} \quad \Rightarrow \underbrace{\vec{r}(s) = \langle \cos \tfrac{3s}{5}, \sin \tfrac{3s}{5}, \tfrac{4s}{5} \rangle}_{\text{arc length parameterization.}}$

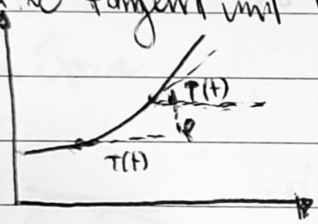
Remark

- If $\vec{r}(t)$ is arc length parameterization then $\int_0^a |\vec{r}\,'(s)|\,ds = a$

Curvature

- $\vec{r}(t) = \langle x(t), y(t)\rangle$ ← position vector
- $\vec{v}(t) = r'(t) = \langle x'(t), y'(t)\rangle$ ← velocity vector
- 

- Define tangent unit vector.



  - $T(t) = \dfrac{\vec{v}(t)}{|v(t)|}$
  - Angle φ depends on time

- We use arc-length parameterization
  - Def: $K = \left|\dfrac{d\varphi}{ds}\right|$ ← curvature
  - Better formula: $\vec{T} = \hat{i}\cos\varphi + \hat{j}\sin\varphi$

$$\frac{d\vec{T}}{ds} = \frac{dT}{d\varphi}\frac{d\varphi}{ds} = (-\hat{i}\sin\varphi + \hat{j}\cos\varphi)\frac{d\varphi}{ds}$$

$$\frac{d\vec{T}}{ds} = \left|\vec{b}\frac{d\varphi}{ds}\right| = \left|\frac{d\varphi}{ds}\right| = K = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{dT}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{1}{\frac{ds}{dt}}\right|\cdot\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{v}\left|\frac{dT}{dt}\right|$$

  - $K = \frac{1}{v}\left|\frac{dT}{dt}\right|$, where $T(t) = \dfrac{r'(t)}{|r'(t)|}$, $v = |r'(t)|$

- Example: Find curvature of $x^2+y^2=a^2$

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\vec{v}(t) = r'(t) = \langle -a\sin t, a\cos t\rangle$

$v = |\vec{v}(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$T(t) = \dfrac{v(t)}{v} = \dfrac{\langle a\sin t, a\cos t\rangle}{a} = \langle \sin t, \cos t\rangle$

$dT/dt = \langle -\cos t, \sin t\rangle$

$\left|\frac{dT}{dt}\right| = \sqrt{(\cos t)^2 + (\sin t)^2} = 1$

$K = \dfrac{1}{a}$

Curvature Summary

• $K = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{r}'(t)|}\left|\frac{d\vec{T}}{dt}\right|$

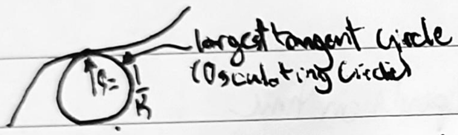
• $\vec{T}(t) = \frac{\vec{r}'(t)}{|r'(t)|}$

• $x^2 + y^2 = a^2 \Rightarrow x = a\cos t \, , y = a\sin t$

• We found $K = \frac{1}{a}$

• Radius of Curvature $= \rho = \frac{1}{K}$ (for circle, $\rho = a$)

• In General,



• Note: Straight line and circle have constant curvature

Example

• Curvature of helix for $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$

$$\vec{v}(t) = r'(t) = \langle -a\omega\sin\omega t, a\omega\cos\omega t, b\rangle$$

$$|\vec{v}(t)| = |r'(t)| = \sqrt{a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t + b^2} = \sqrt{\omega^2 a^2 + b^2}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|r'(t)|} = \left\langle \frac{-a\omega\sin\omega t}{\sqrt{\omega^2 a^2 + b^2}}, \frac{a\omega\cos\omega t}{\sqrt{\omega^2 a^2 + b^2}}, \frac{b}{\sqrt{\omega^2 a^2 + b^2}} \right\rangle$$

$$\frac{dT}{dt} = \left\langle \frac{-a\omega^2\cos\omega t}{\sqrt{\omega^2 a^2 + b^2}}, \frac{-a\omega^2\sin\omega t}{\sqrt{\omega^2 a^2 + b^2}}, 0 \right\rangle$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\frac{\omega^4 a^2}{\omega^2 a^2 + b^2}} = \frac{\omega^2 a}{\sqrt{a^2\omega^2 + b^2}} = \frac{\omega^2 a}{\sqrt{a^2\omega^2 + b^2}}$$

$$K = \frac{1}{|r'(t)|} \cdot \left|\frac{dT}{dt}\right| = \frac{1}{\sqrt{a^2\omega^2 + b}} \cdot \frac{\omega^2 a}{\sqrt{a^2\omega^2 + b}} = \frac{\omega^2 a}{a^2\omega^2 + b}$$

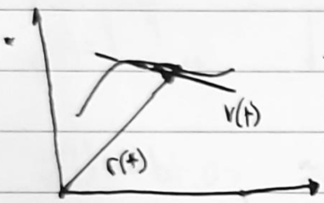
$$K = \frac{\omega^2 a}{a^2\omega^2 + b^2}$$

Alternative Formula

- $K = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ ← cross product

- Cross Product → $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

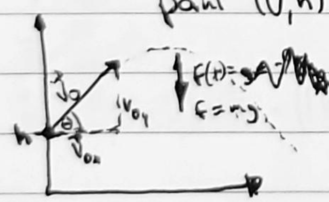
Motion in Space

- 

  - $r(t)$ = position
  - $v(t) = r'(t)$ = velocity
  - $a(t) = v'(t) = r''(t)$ = acceleration.

- Newtons Second Law
  - $\vec{F}(t) = m\vec{a}(t)$
- $\vec{F}(t)$, m: Find the trajectory if force is known
- Given function $\vec{F}(t)$, find $r(t)$ such that $\vec{F}(t) = m\dfrac{d^2\vec{r}(t)}{dt^2}$

- Example: In 2D space, find trajectory of a projectile starting at $t=0$, at point $(0,h)$ with velocity $\langle v_{0x}, v_{0y} \rangle = \vec{v}_0$, and $\vec{F}(t) = \langle 0, -mg \rangle$



$\langle 0, -mg \rangle = m \langle x''(t), y''(t) \rangle$

$\langle 0, -g \rangle = \langle x''(t), y''(t) \rangle$

$0 = x''(t)$; $-g = y''(t)$

$\int 0\,dt = \int x''(t)\,dt$ $\quad \int -g\,dt = \int y''(t)\,dt$

$c = x'(t)$ $\quad -gt + c = y'(t)$ $\quad y'(0) = v_{0y}$

$c = x'(t) = v_{0x}$ $\quad \int -gt + v_{0y}\,dt = \int y'(t)\,dt$

$x'(t) = v_{0x}$ $\quad -\frac{1}{2}gt^2 + v_{0y}t + c = y(t)$ $\quad h = c$

$\int x'(t)\,dx$ $\quad -\frac{1}{2}gt^2 + v_{0y}t + h = y(t)$

$x(t) = v_{0x}t + c$

$x(t) = v_{0x}t$

Summary

$\langle 0, -mg \rangle = m \langle x''(t), y''(t) \rangle$

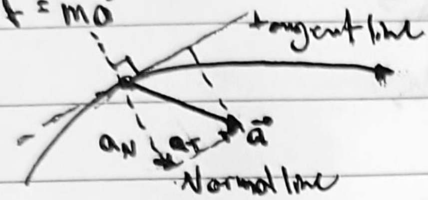
$\langle x(0), y(0) \rangle = \langle 0, h \rangle$

$\langle x'(0), y'(0) \rangle = \langle v_{0x}, v_{0y} \rangle$

$x(t) = v_{0x}t$

$y(t) = h + v_{0y}t - \frac{1}{2}gt^2$

Motion In Space Continued

- $\vec{r}(t)$, $\vec{v}(t) = \vec{r}'(t)$, $\vec{a}(t) = \vec{r}''(t) = \vec{v}'(t)$
- $\vec{F} = m\vec{a}$
- 

a_T - Tangential Component (increasing speed)

a_N - Normal Component (Curving Trajectory)

- if $a_N = 0 \Rightarrow$ straight trajectory

- $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ ← Tangent unit vector

- $|\vec{T}(t)|^2 = 1$ $\qquad \vec{a} = \langle a_1\ a_2\ a_3 \rangle$

$\vec{T}(t) \cdot \vec{T}(t) = 1$ $\qquad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\dfrac{d}{dt}[\vec{T}(t) \cdot \vec{T}(t)] = \dfrac{d(1)}{dt}$ $\qquad |\vec{a}|^2 = \underbrace{a_1^2 + a_2^2 + a_3^2}_{\vec{a} \cdot \vec{a}}$ ← Dot Product $\qquad |\vec{a}|^2 = \vec{a} \cdot \vec{a}$

$T'(t)T(t) + T'(t)T(t) = 0$

$2T(t)T'(t) = 0$

$T(t)T'(t) = 0$ ← If the dot product of 2 vectors $= 0$, then they are perpendicular!

$\underset{\text{Tangent Direction}}{T(t)} \perp \underset{\text{Normal Direction}}{T'(t)}$

- $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ ← Normal Unit Vector

- $\vec{N}(s) = \dfrac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} \Leftrightarrow K = \left|\frac{dT}{ds}\right|$

- $\vec{N}(s) = \dfrac{1}{K}\dfrac{dT}{ds}$

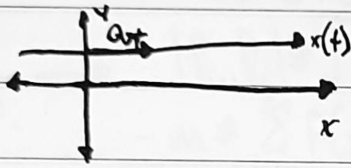
- $\vec a = \frac{d\vec v}{dt} = \frac{d(|\vec v|\vec T)}{dt} = \frac{d(|\vec v|)}{dt}\vec T + |\vec v|\frac{d\vec T}{dt} = \frac{d(|\vec v|)}{dt}\vec T + |\vec v|\overset{KN(s)}{\frac{dT}{ds}}\overset{v}{\frac{ds}{dt}}$

$= \vec a = \frac{dv}{dt}\vec T + Kv^2\vec N$

- $\vec a_T = \frac{dv}{dt}$, $a_N = Kv^2$

Example

- $x(t) = \frac{1}{2}bt^2$ ← constant
  $y(t) = 1$

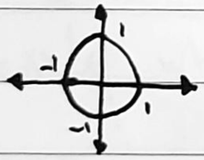


$\vec v(t) = \langle bt, 0\rangle$ $\quad v = |\vec v| = \sqrt{b^2t^2} = bt$

$\vec a(t) = \langle b, 0\rangle$ $\quad a_T = \frac{dv}{dt} = b$

$a_N = Kv^2 = 0$

- $x(t) = \cos(\omega t)$
  $y(t) = \sin(\omega t)$



$\vec v(t) = \langle -\omega\sin\omega t, \omega\cos\omega t\rangle$

$\vec v = |\vec v| = \sqrt{\omega^2\sin^2\omega t + \omega^2\cos^2\omega t} = \omega$

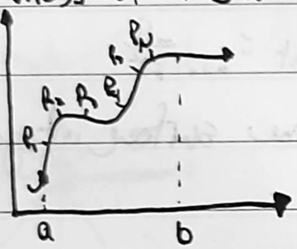
$\vec v = \omega$

$\vec a_T = \frac{dv}{dt} = 0$

$\vec a_N = Kv^2 = \frac{1}{r}\omega^2 = (1)(\omega^2) = \omega^2$

~~Types~~ Line Integrals

Types of Line Integrals

- 1) With respect to arc length
  2) With respect to coordinates
- Consider a smooth curve $\langle x(t), y(t), z(t)\rangle$. Let $f(x,y,z)$ be density of mass of the wire having the shape of C. ~~Mass~~ find the mass over $t \in [a,b]$
- 

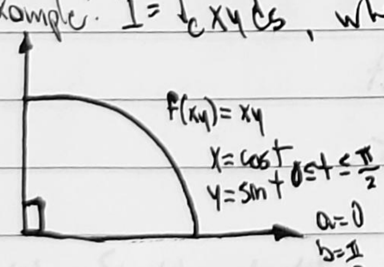
  - Points $P_0, P_1, P_2 \ldots P_N$ correspond to $a = t_0 < t_1 < t_2 \ldots \le t_N$
  - This divides $[a,b]$ into $[t_i, t_{i+1}]$ (assume $t_{i+1} - t_i = \frac{(b-a)}{n}$
  - $|P_{i+1} P_i| \approx \sqrt{x'(t_i^*)^2 + y'(t_i^*)^2 + z'(t_i^*)^2}\,\Delta t$, where $t_i^* \in [t_{i-1}, t_i]$, $\Delta t = t_i - t_{i-1}$
  - $m \approx \sum_{i=1}^{n} \underbrace{f(x(t_i^*), y(t_i^*), z(t_i^*))}_{\text{Density}} \underbrace{\Delta S_i^*}_{\text{Length}}$, $\Delta S_i^* = \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\,\Delta t$

  - $m = \underbrace{\int_C f(x,y,z)\,ds}_{\text{Line integral with respect to arc Length}} = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*), z(t_i^*))\,\Delta S_i^*$

Type 1.

- $\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}$

- Example: $I = \int_C xy\,ds$, where C is a quarter of a circle in the first quadrant.



$I = \int_C xy\,ds$

$= \cancel{\int_0^{\pi/2} xy}$

$= \int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{(\cos t)'^2 + (\sin t)'^2}\,dt$

$= \int_0^{\frac{\pi}{2}} \cos t \sin t = \left[\frac{1}{2}\sin^2 t\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$

- Applications
  - 1) Mass - $m = \int_C \delta(x,y,z)\,ds$, δ is the density function
  - 2) Center of Mass - $\bar{x} = \frac{1}{m}\int x\,\delta(x,y,z)\,ds$

    $\bar{y} = \frac{1}{m}\int y\,\delta(x,y,z)\,ds$

    $\bar{z} = \frac{1}{m}\int y\,\delta(x,y,z)\,ds$

• Example: Find x-coordinate of centre of mass of helix C, $x=3\cos t,\ y=3\sin t,\ z=4t,\ 0\le t\le \pi,\ \rho(x,y,z)=kz$

- $ds=\sqrt{(-3\sin t)^2+(3\cos t)^2+4^2}\,dt$
$=\sqrt{9+16}\,dt$
$=5dt$

$m=\int_C \rho(x,y,z)ds=\int_C kz\,ds=\int_C k(4t)5dt=\int_C k20t\,dt=20k\int_0 t\,dt$
$=20k\left[\frac{t^2}{2}\right]_0^{\pi}=10k\pi^2$

$\bar{x}=\frac{1}{m}\int_C x\,\rho(x,y,z)ds=\frac{1}{10k\pi^2}\int_0^{\pi}3\cos t\,k\cdot 4t\,5dt=\ldots\frac{-12}{\pi^2}$

• Note: Similarly to line integrals, we define surface integrals
- $\rho(x,y,z)\Rightarrow$ surface density (kg/m²)
- Let z be surface $f(x,y)$, $(x,y)\in D$
- Total Mass $\to m=\iint_D \rho(x,y,z)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dt$.
- Problem 7.

Type 2 Line Integrals

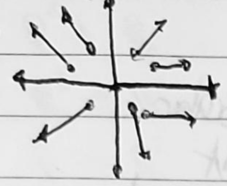
• With respect to coordinates.
- $\int_C f(x,y,z)dx=\int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$
- $\int_C f(x,y,z)dy=\int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$
- $\int_C f(x,y,z)dz=\int_a^b f(x(t),y(t),z(t))\,z'(t)dt$

• Example: $\int_C y\,dx$ along C given by $x(t)=t,\ y(t)=t^2,\ z(t)=t^3,\ 0\le t\le 1$
- $\int_C t^2\,dt \leftarrow y=t^2,\ x'=1$
$=\int_0^1 t^2\,dt$
$=\left[\frac{t^3}{3}\right]_0^1=\frac{1}{3}$
- Also for $\int_C x\,dy$.
- $\int_C t\,2t\,dt=\int_C 2t^2\,dt=\left[\frac{2}{3}t^3\right]_0^1=\frac{2}{3}$

Line Integrals and Vector Fields

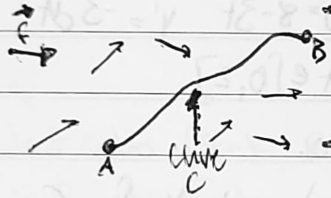
Vector Fields

- functions $\mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$ in 2D)
- $\vec{F}(x,y)$ - Vector field in 2D



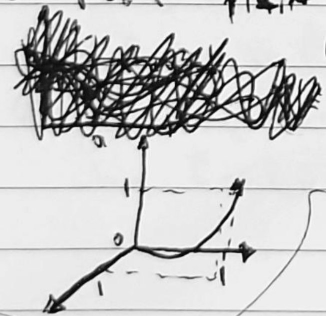
- Example - Gravity

- Algebraic description of vector fields
  - $\vec{F}(x,y) = (x^2+y^2)\,\hat{i} + xy\,\hat{j}$ or $\langle x^2+y^2, xy \rangle$
- Motivation:



- What is the work done by $\vec{F}$ while moving C from A to B?
- Answer: $W = \int_C \vec{F} \cdot d\vec{r}$ (line integral with respect to coordinates)
- $W = \int_C \vec{F} \cdot dr = \int_C P\,dx \pm Q\,dy \pm R\,dz$ where $\vec{F} = \langle P, Q, R \rangle$
  $= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$
- $dr = \langle dx, dy, dz \rangle$
- $\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy + R\,dz$

- Example: $\vec{F} = \langle y, z, x \rangle = y\hat{i} + z\hat{j} + x\hat{k}$



$$C: \begin{cases} x = t \\ y = t^2 & t \in [0,1] \\ z = t^3 \end{cases}$$

$x' = 1$

$y' = 2t$

$z' = 3t^2$

$W = \int_C \vec{F}\,dr$

$= \int_C y\,dx + z\,dy + x\,dz$

$= \int_0^1 t^2\,dt + t^3\,2t\,dt + t\,3t^2\,dt = \int_0^1 t^2\,dt + 2t^4\,dt + 3t^3\,dt$

$= \int_0^1 (t^2 + 3t^3 + 2t^4)\,dt$

$= \left[\frac{t^3}{3} + \frac{3t^4}{4} + \frac{2t^5}{5} \right]_0^1$

$= \frac{1}{3} + \frac{3}{4} + \frac{2}{5} = \frac{20}{60} + \frac{45}{60} + \frac{24}{60} = \frac{89}{60}$

• Note: Orientation of the curve is very important.

• Property

$$-\int_C P\,dx + Q\,dy + R\,dz = -\int_{-C} P\,dx + Q\,dy + R\,dz$$

curve in reverse direction

• Contrast

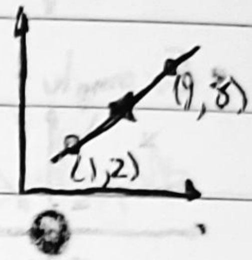
$$-\int_C f(x,y,z)\,ds = \int_{-C} f(x,y,z)\,ds$$ ← linear integral is scalar

• Example $\vec{F}(x,y) = \langle y, x \rangle$

$x(t) = 1+8t \quad x'(t) = 8\,dt$

$y(t) = 2+6t \quad x'(t) = 6\,dt$

$t \in [0,1]$



$= \int_C F\,dr$

$= \int_C y\,dx + x\,dy$

$= \int_C (2+6t)(8)\,dt + (1+8t)(6)\,dt \overset{\circ\circ\circ}{=} 70.$

$x(t) = 9-4t \quad x' = -4\,dt$

$y(t) = 8-3t \quad y' = -3\,dt$

$t \in [0,2]$

$= \int_C F\,dr$

$= \int_C y\,dx + x\,dy$

$= \int_C (8-3t)(-4)\,dt + (9-4t)(-3)\,dt \overset{\circ\circ\circ}{=} -70.$

• Sometimes, $\int_C \vec{F}\,d\vec{r}$ is independent of C, depends only on endpoints. These are conservative vector fields