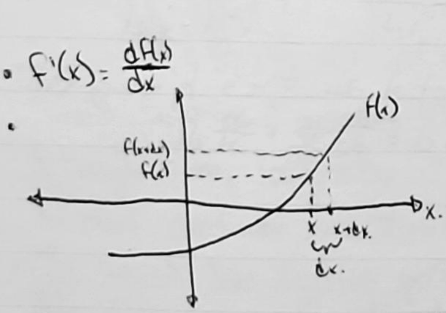
" ", denotes rough idea but not true in rigor.

Chain Rule

- $y = e^{(\sin(\cos e^{x^3}))}$
  $y' = e^{(\sin(\cos e^{x^3}))} \cdot \cos(\cos e^{x^3}) \cdot (-\sin e^{x^3}) \cdot e^{x^3} \cdot 3x^2$
- $y = \varphi(\psi(x^3)^5)$
  $y' = \varphi'(\psi(x^3)^5) \cdot 5(\psi(x^3))^4 \cdot \psi'(x^3) \cdot 3x^2$
- $y = \sin(\varphi(x^3 + \psi(x^2)))$
  $y' = \cos(\varphi(x^3 + \psi(x^2))) \cdot \varphi'(x^3 + \psi(x^2)) \cdot (3x^2 + \psi'(x^2) \cdot 2x)$
- $y = f(x^2, \sin x)$ where $f(u,w)$ - is an arbitrary Function in 2 variables
  $\frac{\partial y}{\partial x} =$ "$\frac{\partial f(x^2, \sin x)}{\partial (x^2)} 2x + \frac{\partial f(x^2, \sin x)}{\partial (\sin x)} \cdot \cos x$" ← wrong notations
  → better to write in:
  $\frac{\partial y}{\partial x} = \left.\frac{\partial f(u,w)}{\partial u}\right|_{u=x^2, w=\sin x} 2x + \left.\frac{\partial f(u,w)}{\partial w}\right|_{u=x^2, w=\sin x} \cdot \cos x.$
  → even better notation (maple)
  $\frac{\partial y}{\partial x} = D_1(f)(x^2, \sin x) \cdot 2x + D_2(f)(x^2, \sin x) \cdot \cos x.$ the subscript on the D_n represents the derivative with respect to the n^{th} argument.
- $g = f(\varphi(x^2 y, y\sin(q(x))), \psi(xy - \sin x))$
  $\frac{\partial g}{\partial x} = D_1(f)(\varphi(x^2y, y\sin(q(x))), \psi(xy-\sin x)) \cdot (D_1(\varphi)(x^2y, y\sin(q(x))) \cdot 2xy + D_2(\varphi)(x^2y, y\sin(q(x)))(y\cos(q(x)))q'(x))$
  $+ D_2(f)(\varphi(x^2y, y\sin q(x)), \psi(xy - \sin x)) \cdot \psi'(xy - \sin(x)) \cdot (y - \cos x)$

Differential of a Function

- $f'(x) = \frac{df(x)}{dx}$
- 

  dx is infinitesimally small
  "dx is not equal to zero but $(dx)^2 = 0$" ↔ "$dx \neq 0$ but $(dx)^2 = 0$"
  $f(x+dx) = f(x) + df(x).$
  $df = f'(x)dx$
- $f(x,y,z) \rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ - differential of $f(x,y,z)$
  $f(x+dx, y+dy, z+dz) = f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dx +$ "higher order terms"
  ↖ Taylor expansion.
- $f(x_1 \ldots x_n) = \frac{\partial f}{\partial x_1}dx_1 + \ldots \frac{\partial f}{\partial x_n}d_n$
- $f'(x) = \frac{df}{dx} \Rightarrow f'(x)dx = df.$

- x, y, z are variables such that $x^3 + y^3 + z^3 + xyz = 1$.
- What about dx, dy, dz?
  - $(3x^2 + yz)dx + (3y^2 + xz)dy + (3z^2 + xy)dz = 0$.
- Consider x, y as independent variables, then z depends on x, y.
- Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.
  - $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.
  - $dz = -\frac{(3x^2+yz)dx}{(3z^2+yx)} - \frac{(3y^2+xz)dy}{(3z^2+xy)}$, so
  - $\frac{\partial z}{\partial x} = \frac{-(3x^2+yz)}{(3z^2+yx)}$, $\frac{\partial z}{\partial y} = -\frac{(3y^2+xz)}{(3z^2+xy)}$
- Let x, y, z, t be variables such that $xyzt = 3$, & $x^2+y^2+z^2+t^2 = 5$. Let z, t be independent variables.
  - $yzt\,dx + xzt\,dy + xyt\,dz + xyz\,dt = 0$

    $2x\,dx + 2y\,dy + 2z\,dz + 2t\,dt = 0 = x\,dx + y\,dy + z\,dz + t\,dt$.
- z, t are independent so dz, dt are independent and dx, dy are dependent on dz, dt.
  - $$\begin{array}{r|l} y & yzt\,dx + xzt\,dy + xyt\,dz + xyz\,dt = 0 \\ xzt & x\,dx + y\,dy + z\,dz + t\,dt = 0. \end{array}$$
  - $(zty^2 - x^2zt)dx = -(xy^2t - xz^2t)dz - (xy^2z - xzt^2)dt$

    $dx = -\frac{xy^2t - xz^2t}{zty^2 - x^2zt}dz - \frac{xy^2z + xzt^2}{zty^2 - x^2zt}dt$.

    $dy = -\frac{x^2t - y^2t}{x^2zt - y^2zt}dz - \frac{x^2yz - yzt^2}{x^2zt - y^2zt}dt$
  - Now you can find partial derivatives by seeing that

    $dx = \frac{\partial x}{\partial z}dz + \frac{\partial x}{\partial t}dt$, & $dy = \frac{\partial y}{\partial z}dz + \frac{\partial y}{\partial t}dt$.

Calculus of Variations.

Functionals vs. Functions.

- function - $f(x) = x^2$
  - $\to \boxed{x \to x^2} \to$
  - a relation between a set that associates to every element of a first set exactly one element of the second.
- Functional - $S(f(x)) = -f(1)^2$
  - $\to \boxed{} \to$
  - $-x^2 - x + 3 \to \boxed{S} \to -(1^2 - 1 + 3)^2 = -9.$
  - $1 - \sin(\pi x) \to \boxed{S} \to -(1 - \sin(\pi))^2 = -1.$
  - A functional is a real-valued function on a vector space, usually of functions
  - $S(f) = \int_0^1 -f'(x)^2\,dx$, $S(x^2 - x) = \int_0^1 -(2x-1)^2\,dx = \dots = 0$
  - functionals are mapping from a set of functions to the real numbers
- An example of a functional we will consider will look as such.
  $S(f) = \int_a^b L(f(x), f'(x))\,dx.$
- We want to find minima, maxima, and critical points of S.
- Recall: Maxima, Minima, Critical points of a function
  - $y = f(x, y, z)$; If x, y, z is a min or max of f, then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0^{(*)}$ at this points.
  - Solution of a system $(*)$ is called a critical point of f
- Although it may look like it, the functions are not composite, to avoid confusion lose the argument of x; $S(f) = \int_0^2 (f'^2 - f^2)\,dx$.
- The search for critical points, being by thinking of finding some $\frac{\delta S}{\delta f} \to$ this is a functional derivative.

$\frac{\delta S}{\delta f}$, δ denotes functional derivative

- $S(f + \delta f)$, where δf is a "small" function.

$$S(f + \delta f) = \int_0^2 \left((f + \delta f)'^2 + (f + \delta f)^2\right) dx.$$

$$= \int_0^2 (f'^2 + 2\delta f' f' + \delta f'^2) - (f^2 + 2f\delta f + \delta f^2)\,dx.$$, Assume δf is infinitesimal.

$$= \underbrace{\int_0^2 (f'^2 - f^2)\,dx}_{S(f)} + \underbrace{\int_0^2 (2f'\delta f' - 2f\delta f)\,dx}_{\delta S(f)}$$

- $\delta S(f) = \int_0^2 (2f'\delta f' - 2f\delta f)\,dx$

IBP →

$\int u v' dx = uv - \int u' v dx$.

For a discrete integral $\int_a^b u v' dx = uv\Big|_a^b - \int_a^b u' v dx$.

sometimes uv is considered the boundary term.

$\frac{\partial \varphi}{\partial x} = \varphi_x$

$\frac{\partial \varphi}{\partial y} = \varphi_y$

$\frac{\partial^3 \varphi}{\partial x^2 \partial y} = \varphi_{xxy}$.

Assume ~~...~~ computation. →

- $\frac{\delta S}{\delta f}$ - ?
- $\delta S(f) = \int_0^2 (2f'\delta f' - 2f\,\delta f)dx = 2f'\delta f\Big|_0^2 + \int_0^2 (-2f''\delta f - 2f\delta f)dx$.
  - → Here. $u = 2f'$ & $v = \delta f$
  - $= 2f'\delta f\Big|_0^2 + \int_0^2 (-2f'' - 2f)\,\delta f\,dx$.
- $\frac{\delta S}{\delta f} = -2f'' - 2f$.
- $S(f) = \int_a^b L(f, f')dx$. in ex $L(f, f') = f'^2 - f^2$
  - → Here: L is not a functional but a regular general expression in the two variables f & f'
- $S(f + \delta f) = \int_a^b L(f + \delta f, f' + \delta f')dx$.

$= \int_a^b \left(L(f, f') + \frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'}\delta f'\right)dx$.

$S(f) + \int_a^b \left(\frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'}\delta f'\right)dx =$

$S(f) + \frac{\partial L}{\partial f'}\delta f\Big|_a^b + \int_a^b \left(\frac{\partial L}{\partial f}\delta f - \frac{d}{dx}\left(\frac{\partial L}{\partial f'}\right)\delta f'\right)dx$. , $u = \frac{\partial L}{\partial f'}$, $v = \delta f$.

- $\frac{\delta S}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dx}\left(\frac{\partial L}{\partial f'}\right)$
- If asked to find functional derivation, do not use this formula. Should derive the formula itself (go through the steps)
- This is the simplest case, the functionals can end up looking as such $S(f, g, h) = \int_a^b L(f, g, h, f', g', h', f'', g'', h'' \ldots)$.
- Ex $S(\varphi(x, y)) = S(\varphi)$.

$S(\varphi(x, y)) = \iint_D \left[\varphi^3 - \left(\frac{\partial \varphi}{\partial x}\right)^3 - \left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}\right]dx\,dy$

- Ex. $S(f) = \int_1^2 (f''^2 - f'^3)dx$.

$S(f + \delta f) = S(f) + \int_1^2 (2f''\delta f'' - 3f'^2\delta f')dx$.

$= S(f) + \int_1^2 (-2f'''\delta f' + (3f'^2)'\delta f)\,dx$.

$= S(f) + \int_1^2 (-2f^{(4)}\delta f + (6f'f'')\delta f$

$= S(f) + \int_1^2 (-2f^{(6)} + 6f'f'')\,\delta f\,dx$.

$\frac{\delta S}{\delta f} = -2f^{(6)} + 6f''f'$

$S(f) = \int_1^2 L(f, f', f'')dx$.

$S(f + \delta f) = S(f) + \int_0^b \left(\frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'}\delta f' + \frac{\partial L}{\partial f''}\delta f''\right)dx$

$\frac{\delta S}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial f''}$

• $S(\varphi) = \iint_D (\varphi_t^2 - \varphi_x^2 - \varphi^2)\,dx\,dt$

- $S(\varphi+\delta\varphi) = S(\varphi) + \int (2\varphi_t \delta\varphi_t - 2\varphi_x \delta\varphi_x - 2\varphi\delta\varphi)\,dx\,dt$

$= S(\varphi) + \int (-(2\varphi_t)_t \delta\varphi + (2\varphi_x)_x \delta\varphi - 2\varphi\,\delta\varphi)\,dx\,dt.$

$= S(\varphi) + \int (-2\varphi_{tt} + 2\varphi_{xx} - 2)\,\delta\varphi\,dx\,dt$

- $\frac{\partial S}{\partial\varphi} = -2\varphi_{tt} + 2\varphi_{xx} - 2\varphi.$

• $S(\varphi) = \int (\varphi_{xt}^3 - \sin\varphi)\,dx\,dt$

- $S(\varphi+\delta\varphi) = S(\varphi) + \int (3\varphi_{xt}^2\,\delta\varphi_{xt} - \cos\varphi\,\delta\varphi)\,dx\,dt.$

$= S(\varphi) + \int (-(3\varphi_{xt}^2)_t\,\delta\varphi_x - \cos\varphi\,\delta\varphi)\,dx\,dt$

$= S(\varphi) + \int (-6\varphi_{xt}\varphi_{xtt}\,\delta\varphi_x - \cos\varphi\,\delta\varphi)\,dx\,dt$

$= S(\varphi) + \int ((6\varphi_{xt}\varphi_{xtt})_x\,\delta\varphi - \cos\varphi\,\delta\varphi)\,dx\,dt.$

$= S(\varphi) + \int (6\varphi_{xxt}\varphi_{xtt}\,\delta\varphi + 6\varphi_{xt}\varphi_{xxtt} - \cos\varphi)\,\delta\varphi\,dx\,dt.$

- $\frac{\partial S}{\partial\varphi} = 6\varphi_{xxt}\varphi_{xtt} + 6\varphi_{xt}\varphi_{xxtt} - \cos\varphi$

• $f(x_1, x_2, \ldots x_N)$

$df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\,dx_i \quad , \; i = 1,2\ldots N \quad , \; \Sigma \to \int$

$x_i \Rightarrow x(i).$

$df = \int_{i=1}^{N} \frac{\delta f}{\delta x(i)}\,\delta x(i).$

• $S(f,g) = \int_0^1 (f'^2 - f'g' + g'^2 - e^{fg})\,dt.$

- ~~$S(f+\delta f, g)$~~

$S(f, g+\delta g) = \int_0^1 ((-f' + 2g')\,\delta g' - e^{fg} f\,\delta g)\,dt + S(f,g)$

$= S(f,g) + \int_0^1 (-(-f'' + 2g'')\,\delta g - e^{fg} f\,\delta g)\,dt$

$= S(f,g) + \int_0^1 (f'' - 2g'' - fe^{fg})\,\delta g\,dt.$

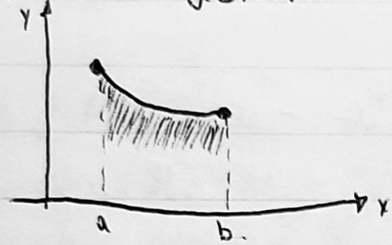
- $\frac{\partial S}{\partial g} = f'' - 2g'' - fe^{fg}$

• $S(f,g) =$

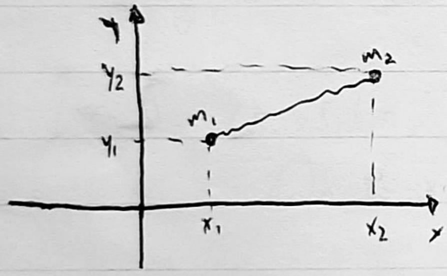
Maximizing & Minimizing Applications.

-  - length is ~~b~~ given
  want to ~~max~~ maximize area A.

- Min Area = 0 
- Max Area = a circle.

- 
  - ball rolls down the curve by gravity.
  - want to minize curve length to time is small as possible.

  - $t = \int_a^b \sqrt{\frac{1+y'^2}{2gy}}\,dx$.
  - $\frac{\delta T}{\delta y} = 0 \rightarrow$ to solve this case which is an ODE for $y(x)$.

- 
  - two particles ~~att~~ rotated to each other by some force.
  - want to know the movement of these particles

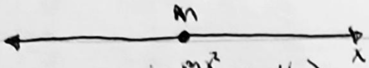
  - $L(x_1, y_1, ~~x_2~~, y_2, x_1', y_1', x_2', y_2') \leftarrow$ Lagrangian of this ~~...~~ system.
  - $S = \int_{t_1}^{t_2} L(x_1, y_1, x_2, y_2, x_1', y_1', x_2', y_2')\,dt \leftarrow$ action.
  - to find critical points of the motion (usually minizing). $\frac{\delta S}{\delta x_1} = 0$, $\frac{\delta S}{\delta y_1} = 0$, $\frac{\delta S}{\delta x_2} = 0$, $\frac{\delta S}{\delta y_2} = 0$

- - Assume a partile in free space with no forces applied to it, the Lagrangian is

(x, y, z) - position
$(\dot{x}, \dot{y}, \dot{z})$ - velocity

  - $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$
  - $S = \int \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,dt$.
  - $\frac{\delta S}{\delta x} = 0$, $\frac{\delta S}{\delta y} = 0$, $\frac{\delta S}{\delta z} = 0$ - equation of motion.
  - $\frac{\delta S}{\delta x} = -mx''$ similar for the others.

- Suppose a particle that can only move across a line



- $L = \frac{m\dot{x}^2}{2} - V(x)$, where $V(x)$ is the given potential energy function.
- $S(x) = \int_{t_1}^{t_2} \left(\frac{m\dot{x}^2}{2} - V(x)\right) dt$
- $S(x + \delta x) = \int_{t_1}^{t_2} \left(\frac{m(\dot{x} + \delta\dot{x})^2}{2} - V(x + \delta x)\right) dt$

  $= S(x) + \int_{t_1}^{t_2} (m\dot{x}\delta\dot{x} - \dot{V}(x)\delta x) dt$

  $= S(x) + \int_{t_1}^{t_2} (-m\ddot{x} - \dot{V}(x))\delta x \, dt$
- $\frac{\delta S}{\delta x} = -m\ddot{x} - \dot{V}(x) = 0$
- $m\ddot{x} = \underbrace{-\dot{V}(x)}$ ← this is Newtons second law.

  ↳ which is force.

- For m particles $(x_1, y_1, z_1) \ldots (x_m, y_m, z_m)$
  - $L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \ldots + \frac{m_m}{2}(\dot{x}_m^2 + \dot{y}_m^2 + \dot{z}_m^2) - V(x_1, y_1, z_1, \ldots x_m, y_m, z_m)$
  - $\frac{\delta S}{\delta x_1} = 0 \ldots \frac{\delta S}{\delta z_m} = 0 \iff 2^{nd}$ Newton Law.
- In special relativity, we have the Lagrangian
  - $L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c}}$
  - $S(x) = \int_{t_1}^{t_2} L dt$
  - $\frac{\delta S}{\delta x} = 0$
- $q_1 \ldots q_n$ - coordinates; n - number of degrees of freedom
  - Function of time, t
- $L(q_1 \ldots q_n, \dot{q}_1 \ldots \dot{q}_n)$ - Lagrangian
  - $S = \int_{t_1}^{t_2} L dt$ ← action
- Equation of motion is
  - $\frac{\delta S}{\delta q_1} = 0, \ldots, \frac{\delta S}{\delta q_n} = 0$; the functional derivatives.
- Ex. $x_1, y_1, z_1, x_2, y_2, z_2$ - coordinates.

  $L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - \frac{Gm_1m_2}{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}}$

  $\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0, \ldots, \frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0$ - eqn of motion

  $\frac{\partial L}{\partial x_1} + \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1}$

  $\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}$

  $\frac{\partial}{\partial t}(m_1\dot{x}_1) = -\frac{\partial}{\partial x_1}\left(\frac{Gm_1m_2}{\ldots}\right)$ } force

  $\underbrace{\frac{\partial}{\partial t}(m_1\dot{y}_1)}_{\text{momenta}} = \frac{\partial}{\partial y_1}\left(\frac{Gm_1m_2}{\ldots}\right)$

• $\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}}_{\text{momentum corresponding to coordinate } q_i} = \underbrace{\frac{\partial L}{\partial q_i}}_{\text{corresponding "Force" to } q_i}$ - Equation of motion, $i = 1, \ldots, n$

• Energy Derivation for 1 degree of freedom

- $\frac{dL}{dt}$, $L(q, \dot q)$ → chain rule

• $\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot q}\frac{d\dot q}{dt}$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\dot q_i + \frac{\partial L}{\partial \dot q_i}\frac{d\dot q_i}{dt}$$

$$= \frac{d}{dt}(\dot q)$$

$$= \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\dot q_i\right)} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q - L\right)$$

- $E = \frac{\partial L}{\partial \dot q}\dot q - L$ is called energy.

• $\frac{dL}{dt} = \sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_{i=1}^{n}\frac{\partial L}{\partial \dot q_i}\frac{d\dot q_i}{dt}$

$$= \sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot q_i}\frac{d\dot q_i}{dt}\right)$$

$$= \frac{d}{dt}\sum_{i=1}^{n}\frac{\partial L}{\partial \dot q_i}\dot q_i$$

$$0 = \frac{d}{dt}\left(\sum_{i=1}^{n}\frac{\partial L}{\partial \dot q_i}\dot q_i - L\right) \Rightarrow E = \sum_{i=1}^{n}\frac{\partial L}{\partial \dot q_i}\dot q_i - L$$

$$0 = \frac{\partial E}{\partial t}$$

• Ex. $L = \frac{m\dot q^2}{2} - U(q)$

$$\frac{\partial L}{\partial \dot q} = m\dot q$$

$$E = m\dot q \cdot \dot q - \left(\frac{m\dot q^2}{2} - U(q)\right)$$

$$= \frac{m\dot q^2}{2} + U(q)$$

However if $m(t)$ E is not constant $\frac{\partial E}{\partial t} \neq 0$

- $L(t, q, \dot{q})$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$$
$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$$
$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = -\frac{\partial L}{\partial t}.$$
$$E = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$$
$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} \quad \text{"Energy flow"}$$

- From Ex where m is a fxn of t.

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$
$$= -\frac{\dot{m}(t)}{2}\dot{q}^2$$

- $E = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$; if $\frac{\partial L}{\partial t} = 0$, then

$\frac{dE}{dt} = 0 \iff E = \text{constant} \iff E$ is conserved.

- $L = L(t, q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$

$\frac{\partial L}{\partial t} = 0 \iff L = L(\cancel{t}, q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$.

- Ex. $L = \frac{m\dot{q}^2}{2} - \lambda q^2$, then $\frac{\partial L}{\partial t} = 0 \iff m, \lambda$ do not depend on t.

$\iff \frac{dm}{dt} = \frac{d\lambda}{dt} = 0$.

- $L(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$

$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}_i}} = \frac{\partial L}{\partial q_i}$, $i = 1 \dots n$.

↳ momentum corresponding to q_i

If $\frac{\partial L}{\partial q_i} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ or $\frac{\partial L}{\partial \dot{q}_i} = \text{constant}$. (conservation of momentum).

- Ex. $L = \dot{q}_1^2 + 3\dot{q}_2^2 - \sin((q_1 - q_2)^2)$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} = -\cos((q_1 - q_2)^2) \cdot 2(q_1 - q_2).$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial q_2} = +\cos((q_1 - q_2)^2) \cdot 2(q_1 - q_2).$$
$$\frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial q_2} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial \dot{q}_2}\right) = 0 \quad \text{since}$$

↳ $\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial \dot{q}_2} = 2\dot{q}_1 + 6\dot{q}_2 = \text{constant}$ - conservation law.

- If $L(\dot{q}_1, \dot{q}_2, q_1, q_2) = \underbrace{L(\dot{q}_1, \dot{q}_2, q_1 + \epsilon, q_2 + \epsilon)}_{(*)}$, then $(\text{let } \epsilon^2 \to 0)$.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial \dot{q}_2}\right) = 0$ or $\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial \dot{q}_2} = \text{constant}$

- $L = \underbrace{L + \frac{\partial L}{\partial q_1}\epsilon + \frac{\partial L}{\partial q_2}\epsilon}_{\text{Taylor expansion of } (*) \text{ by } \epsilon} + \underbrace{\dots}_{\epsilon^2 \to 0}$

$\cancel{L} = \cancel{L} + \frac{\partial L}{\partial q_1}\epsilon + \frac{\partial L}{\partial q_2}\epsilon \Rightarrow 0 = \frac{\partial L}{\partial q_1}\epsilon + \frac{\partial L}{\partial q_2}\epsilon$.

$0 = \frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial q_2}$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \frac{\partial L}{\partial q_i} \quad ; i=1,2$$

$$\frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial q_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1} + \frac{\partial L}{\partial \dot q_2}\right) = 0 \iff \frac{\partial L}{\partial \dot q_1} + \frac{\partial L}{\partial \dot q_2} = \text{constant.}$$

- $L = \dot q_1^2 + \dot q_2^2 + \dot q_3^2 - V(q_1 + q_2, q_2 - 2q_3)$

$L = L(\dot q_1, \dot q_2, \dot q_3, q_1 + \epsilon, q_2 - \epsilon, q_3 - \frac{\epsilon}{2})$

$L = L + \frac{\partial L}{\partial q_1}\epsilon - \frac{\partial L}{\partial q_2}\epsilon - \frac{1}{2}\frac{\partial L}{\partial q_3}\epsilon$

$0 = \frac{\partial L}{\partial q_1} - \frac{\partial L}{\partial q_2} - \frac{1}{2}\frac{\partial L}{\partial q_3}$

$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial \dot q_2} - \frac{1}{2}\frac{\partial L}{\partial \dot q_3}\right)$

$\text{constant} = \frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial \dot q_2} - \frac{1}{2}\frac{\partial L}{\partial \dot q_3}$

$2\dot q_1 - 2\dot q_2 - \dot q_3 = \text{constant.}$

- $L = \dot q_1^2 + \dot q_2^2 + \dot q_3^2 - U(q_1 + 2q_2)$
- $L = L(\dot q_1, \dot q_2, \dot q_3, q_1 + \epsilon, q_2 - \frac{1}{2}\epsilon, q_3)$ ①

$L = L(\dot q_1, \dot q_2, \dot q_3, q_1, q_2, q_3 + \epsilon)$ ② $\Big\}$ Conservation.

Noether's Theorem

- $L = 5\dot q_1^2 + \dot q_2^2 + 3\dot q_3 - U(q_1 - 3q_2 + 11q_3)$
- $L(q_1, q_2, q_3, \dot q_1, \dot q_2, \dot q_3) = L(q_1 + \epsilon, q_2 + \epsilon, q_3)$

$q_1 \to q_1 + \epsilon_1$

$q_2 \to q_2 + \epsilon_2$

$q_3 \to q_3 + \epsilon_3$

$q_1 - 3q_2 + 11q_3 \to q_1 - 3q_2 + 11q_3 + \overset{0}{\overset{\shortparallel}{(\epsilon_1 - 3\epsilon_2 + 11\epsilon_3)}}$

- 1) $\epsilon_1 = 3\epsilon, \quad \epsilon_2 = \epsilon, \quad \epsilon_3 = 0$

2) $\epsilon_1 = 11\epsilon, \quad \epsilon_2 = 0, \quad \epsilon_3 = -\epsilon$

- $L(q_1, q_2, q_3, \dot q_1, \dot q_2, \dot q_3) = L(q_1 + 11\epsilon, q_2, q_3 - \epsilon)$

$L = L + \left(\frac{\partial L}{\partial q_1}\cdot 3 + \frac{\partial L}{\partial q_2}\right)\epsilon$

$3\frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial q_2} = 3\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_2}\right) = \frac{d}{dt}\left(3\frac{\partial L}{\partial \dot q_1} + \frac{\partial L}{\partial \dot q_2}\right) = 0.$

$3\frac{\partial L}{\partial \dot q_1} + \frac{\partial L}{\partial \dot q_2} = \text{constant.}$

$3\cdot 10\dot q_1 + 2\dot q_2 = \text{constant.}$

- $L = \dot q_1^2 + \dot q_2^2 - \sin(q_1^2 + q_2^2)$

$L(q_1, q_2, \dot q_1, \dot q_2) = L(\cos\alpha\, q_1 + \sin\alpha\, q_2, -\sin\alpha\, q_1 + \cos\alpha\, q_2, \cos\alpha\, \dot q_1 + \sin\alpha\, \dot q_2,$

$-\sin\alpha\, \dot q_1 + \cos\alpha\, \dot q_2)$

$$\left\{ q_1^2 + q_2^2 \qquad \begin{array}{l} q_1 \to \cos\alpha \cdot q_1 + \sin\alpha \cdot q_2 \\ q_2 \to -\sin\alpha \cdot q_1 + \cos\alpha \cdot q_2 \end{array} \right\}$$

$$\downarrow$$

$$q_1 \to \text{''} \quad \text{''} = q_1 + \alpha q_2$$
$$q_2 \to \text{''} \quad \text{''} = q_2 - \alpha q_1$$

- $\left\{ \begin{array}{ll} \sin\alpha = \alpha - \frac{\alpha^3}{3!} + \ldots\ldots = \alpha & \alpha^2 = 0 \\ \cos\alpha = 1 - \frac{\alpha^2}{2!} + \ldots\ldots = 1 & \end{array} \right\}$
- $\left. \begin{array}{l} q_1 \to q_1 + \alpha q_2 \\ q_2 \to q_2 - \alpha q_1 \end{array} \right\}$ is infinitesimal symmetry of L.
- $q_1^2 + q_2^2 = (q_1 + \alpha q_2)^2 + (q_2 - \alpha q_1)^2$
  $= q_1^2 + \cancel{2\alpha q_1 q_2} + \cancel{\alpha^2 q_2^2} + q_2^2 - \cancel{2\alpha q_1 q_2} + \cancel{\alpha^2 q_1^2}$
  $= q_1^2 + q_2^2$
- $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = L(q_1 + \epsilon q_2, q_2 - \epsilon q_1, \dot{q}_1 + \epsilon \dot{q}_2, \dot{q}_2 - \epsilon \dot{q}_1)$ if $\epsilon^2 = 0$ $(\epsilon = \alpha)$
  $L = L + \left(\frac{\partial L}{\partial q_1} \cdot q_2 - \frac{\partial L}{\partial q_2} \cdot q_1 + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_2 - \frac{\partial L}{\partial \dot{q}_2} \cdot \dot{q}_1\right)\epsilon$
- $\frac{\partial L}{\partial q_1} q_2 - \frac{\partial L}{\partial q_2} q_1 + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_2 - \frac{\partial L}{\partial \dot{q}_2} \dot{q}_1 = 0$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} q_2 - \frac{\partial L}{\partial \dot{q}_2} q_1\right) + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_2 - \frac{\partial L}{\partial \dot{q}_2} \dot{q}_1 = 0$
  $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} q_2 - \frac{\partial L}{\partial \dot{q}_2} q_1\right) = 0, \quad \frac{\partial L}{\partial \dot{q}_1} q_2 - \frac{\partial L}{\partial \dot{q}_2} q_1 = \text{constant}$
  $2\dot{q}_1 q_2 - 2\dot{q}_2 q_1 = \text{constant}.$
- $L = \dot{q}_1^2 \dot{q}_2^2 - e^{q_1 q_2}$
  $q_1 \to f q_1 = (1+\epsilon) q_1 \qquad \{ f = (1+\epsilon), \quad \epsilon^2 = 0 \}$
  $q_2 \to f^{-1} q_2 = (1+\epsilon)^{-1} q_2 \qquad \{ f^{-1} = \frac{1}{1+\epsilon}, \quad 1 - \epsilon + \epsilon^2 - \epsilon^3 + \ldots \}$
  $q_1 \to = q_1 + \epsilon q_1$
  $q_2 \to = q_2 - \epsilon q_2$
- $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = L(q_1 + \epsilon q_1, q_2 - \epsilon q_2, \dot{q}_1 + \epsilon \dot{q}_1, \dot{q}_2 - \epsilon \dot{q}_2)$
  $L = L + \left(\frac{\partial L}{\partial q_1} q_1 - \frac{\partial L}{\partial q_2} q_2 + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 - \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2\right)\epsilon$
  $\frac{\partial L}{\partial q_1} q_1 - \frac{\partial L}{\partial q_2} q_2 + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 - \frac{\partial L}{\partial \dot{q}_2} = 0$
  $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} q_1 - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} q_2 + \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 - \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 = 0$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} q_1 - \frac{\partial L}{\partial \dot{q}_2} q_2\right) = 0 \to \frac{\partial L}{\partial \dot{q}_1} q_1 - \frac{\partial L}{\partial \dot{q}_2} = \text{constant}.$

• $L(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n)$

Let $q_1 \to q_1 + \epsilon \delta q_1$ be an infinitesimal symmetry of L.

$\vdots\; q_n \to q_n + \epsilon \delta q_n$

$L(q_1 + \epsilon \delta q_1, \dots q_n + \epsilon \delta q_n, \dot{q}_1 + \epsilon \delta \dot{q}_1, \dots \dot{q}_n + \epsilon \delta \dot{q}_n)$ up to ϵ $(\epsilon^2 = \epsilon^2 = 0)$

• Recall: $f(x_1 + h_1, \dots, x_n + h_n) = f(x_1, \dots x_n) + \frac{\partial f(x_1 \dots x_n)}{\partial x_1} h_1 + \frac{\partial f(x_1 \dots x_n)}{\partial x_2} h_2 + \dots \frac{\partial f(x_1 \dots x_n)}{\partial x_n} h_n$

- Taylor expansion.

• $L = L + \frac{\partial L}{\partial q_1} \epsilon \delta q_1 + \dots \frac{\partial L}{\partial q_n} \epsilon \delta q_n + \dots \frac{\partial L}{\partial \dot{q}_1} \epsilon \delta \dot{q}_1 \dots + \frac{\partial L}{\partial \dot{q}_n} \epsilon \delta \dot{q}_n$

$\frac{\partial L}{\partial q_1} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1}, \dots, \frac{\partial L}{\partial q_n} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n}$

$S = \int_a^b L \, dt.$

$\frac{\delta S}{\delta q_1} = 0$

$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \delta q_1 + \dots + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) \delta q_n + \frac{\partial L}{\partial \dot{q}_1} \delta \dot{q}_1 + \dots \frac{\partial L}{\partial \dot{q}_n} \delta \dot{q}_n$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \cdot \delta q_1 + \frac{\partial L}{\partial \dot{q}_1} \cdot \delta \dot{q}_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \delta q_1\right)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \delta q_1 + \frac{\partial L}{\partial \dot{q}_2} \delta q_2 + \dots \frac{\partial L}{\partial \dot{q}_n} \delta q_n\right) = 0$

$\frac{\partial L}{\partial \dot{q}_1} \delta q_1 + \frac{\partial L}{\partial \dot{q}_2} + \dots \frac{\partial L}{\partial \dot{q}_n} = \text{constant}.$

Differential Forms (k-forms, $k = 0, 1, 2, \dots$).

- $x_1, \dots, x_n$ - cartesian coordinates.
- 0-forms $\equiv$ functions in $x_1, x_2, \dots, x_n$

  1-forms are expressions like $f_1(x_1 \dots x_n)dx_1 + \dots + f_n(x_1 \dots x_n)dx_n$
- Ex. 1-form in x, y, z.

  $xy\,dx - \sin(x^2 - z^2)dy$.

  $e^{x-z+y^2}dx - \sin(\tan(xyz^2))dy + 2xy^6z^{10}dz$.
- Ex 1-form in w

  $e^{w^3}dw$, $w^3 dw$, $2dw$, dw, ...
- Ex 1-form in x, y, z.

  $f(x,y,z)$

  $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$
- 2-forms are expressions of a form
  - $\sum_{1 \le i,j \le n} f_{ij}(x_1 \dots x_n)\,dx_i \wedge dx_j$ $\quad \Lambda$ or $\wedge$ is called wedge product.
  - $dx \wedge dy = -dy \wedge dx$ for arbitrary variable x, y.
- Wedge Product (Exterior Product).
  - Main property $\to$ $dx \wedge dy = -dy \wedge dx$
  - In particular, let $x = y$, then $dx \wedge dx = -dx \wedge dx$

    $\Rightarrow dx \wedge dx = 0$
  - $\sum_{1 \le i,j \le n} f_{ij}(x_1 \dots x_n)\,dx \wedge dy = \sum_{1 \le i < j \le n} f_{ij}(x_1 \dots x_n)\,dx \wedge dy$.
- Ex. Variable u, v

  $f(u,v)\,\cancel{du \wedge du} + g(u,v)\,du \wedge dv + h(u,v)\,dv \wedge du + \cancel{p(u,v)\,dv \wedge dv}$

  Since $du \wedge du = 0 = dv \wedge dv$, & $du \wedge dv = -dv \wedge du$.

  $g(u,v)\,du \wedge dv - h(u,v)\,du \wedge dv$.

  $\underbrace{(g - h)}_{q(u,v)}\,du \wedge dv$

  $q(u,v)\,du \wedge dv$.
- 3-forms are expressions of a form:

  $\sum_{1 \le i < j < k \le n} f_{ijk}(x_1 \dots x_n)\,dx_i \wedge dx_j \wedge dx_k$.

- Ex. 3-form in 2 variables
  $f(u,v)\, du \wedge dv \wedge du = 0$ ∴ no 3-form in 2 variables
  $du \wedge dv \wedge du = - du \wedge du \wedge dv = 0$

- 1-variable → u
  - 1 ↳ 0-forms → $f(u)$
  - 1 ↳ 1-forms → $f(u)\, du$
  - 1 ↳ 2-forms → $f(u)\, du \wedge du = 0$
  - ↳ 3....-forms → 0
- 2-variables → u, v
  - 1 - 0-forms → $f(u,v)$
  - 2 - 1-forms → $f(u,v)\, du + g(u,v)\, dv$
  - 1 - 2-forms → $f(u,v)\, du \wedge dv$
  - - 3.....-forms → 0.

- Any 2-forms in 3 variables can be written as a linear combination of the 3 wedge products

3-variables → u, v, w
- 1 - 0-forms → $f(u,v,w)$
- 3 - 1-forms → $f(u,v,w)\, du + g\, dv + h\, dw$
- 3 - 2-forms → $f\, du \wedge dv + g\, du \wedge dw + h\, dv \wedge dw$
- 1 - 3-forms → $f\, du \wedge dv \wedge dw$
- - 4....-forms → 0.

4-variables → x, y, z, t.
- 1 - 0 → ~~dx,dy~~, $f(x,y,z,t)$
- 4 - 1 → dx, dy, dz, dt.
- 6 - 2 → $dx \wedge dy,\ dx \wedge dz,\ dx \wedge dt,\ dy \wedge dz,\ dy \wedge dt,\ dz \wedge dt$
- 4 - 3 → $dx \wedge dy \wedge dz,\ dx \wedge dy \wedge dt,\ dy \wedge dz \wedge dt,\ dx \wedge dz \wedge dt$
- 1 - 4 → $dx \wedge dy \wedge dz \wedge dt$.
- - 5... → 0

- For n variables, there are n forms.
- The number of terms in each form also follow Pascal's triangle.
- Ex $x\, dx \wedge dy \wedge dz - y\, dz \wedge dx \wedge dy = (x-y)\, dx \wedge dy \wedge dz$

Wedge Product of Differential Forms

- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$
  $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$; α, β, γ are differential forms.
  $\alpha \wedge (f\beta) = f\, \alpha \wedge \beta$; f is a function, α & β are " "
  $(f\alpha) \wedge \beta = f(\alpha \wedge \beta)$
- $\alpha = x\, dx - xy\, dy + dz$
  $\beta = y\, dx + z\, dy + x\, dz$.
  $\alpha \wedge \beta = (x\, dx - xy\, dy + dz) \wedge (y\, dx + z\, dy + x\, dz)$.
  $= xy\, \cancel{dx \wedge dx} + xz\, dx \wedge dy + x^2\, dx \wedge dz + (-xy^2)\, dy \wedge dx + (-xyz)\, \cancel{dy \wedge dy}$
  $+ (-x^2y)\, dy \wedge dz + y\, dz \wedge dx + (z)\, dz \wedge dy + x\, \cancel{dz \wedge dz}$.
  $= (xz + xy^2)\, dx \wedge dy + (x^2 - y)\, dx \wedge dz + (-x^2y - z)\, dy \wedge dz$.

- $\gamma = x\,dx\wedge dy - z\,dy\wedge dz + xyz\,dx\wedge dz$

$\alpha\wedge\gamma = (x\,dx - xy\,dy + dz)\wedge(x\,dx\wedge dy - z\,dy\wedge dz + xyz\,dx\wedge dz)$

$= zx\,dx\wedge dy\wedge dz - x^2y^2z\,dy\wedge dx\wedge dz + x\,dz\wedge dx\wedge dy$

$= (xz + x^2y^2z + x)\,dx\wedge dy\wedge dz.$

- If α is in k-forms ; β is in l-forms then $\alpha\wedge\beta$ is in $k+l$-forms.

Exterior Differential

- If f is a function (0-form), the $df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i$ is a 1-form.
- If ω is a k-form, then there exists $d\omega$ which is a $(k+1)$-form.
- Ex. $\alpha = xy\,dx - z\,dy + xyz\,dz$

$d\alpha = d(xy)\wedge dx - d(z)\wedge dy + d(xyz)\wedge dz$

$= (x\,dy + y\,dx)\wedge dx - dz\wedge dy + (yz\,dx + xy\,dz + xz\,dy)\wedge dz$

$= x\,dy\wedge dx - dz\wedge dy + yz\,dx\wedge dz + xz\,dy\wedge dz$

$= -x\,dx\wedge dy + (xz+1)\,dy\wedge dz + yz\,dx\wedge dz.$

- Ex. $\beta = xyz\,dx\wedge dy - x^2yz^3\,dx\wedge dz + x^3y^2z^9\,dy\wedge dz.$

$d\beta = d(xyz)\wedge dx\wedge dy - d(x^2yz^3)\wedge dx\wedge dz + d(x^3y^2z^9)\wedge dy\wedge dz$

$= xy\,dz\wedge dx\wedge dy - x^2z^3\,dy\wedge dx\wedge dz + 3x^2y^2z^9\,dx\wedge dy\wedge dz.$

$= (xy + x^2z^3 + 3x^2y^2z^9)\,dx\wedge dy\wedge dz.$

- $\mathbb{R}^3(x,y,z)$

$d\big($ 0-forms $\to f(x,y,z) \to f.$

$d\big($ 1-forms $\to f\,dx + g\,dy + h\,dz \sim (f,g,h)$ ← denoted by this vector.

$d\big($ 2-forms $\to f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy \sim (f,g,h).$

3-forms $\to f\,dx\wedge dy\wedge dz \sim f.$

- $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \sim \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \Leftrightarrow \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
- $(f,g,h) \sim f\,dx + g\,dy + h\,dz$
- $d(f\,dx + g\,dy + h\,dz) = \left(\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial z}dz\right)\wedge dy + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)\wedge dz.$

$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy\wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)dz\wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy$

$\simeq \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$

- $\operatorname{curl}(f,g,h).$

- $(f,g,h) \sim f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy.$
- $d((f,g,h)) = df\wedge dy\wedge dz + dg\wedge dz\wedge dx + dh\wedge dx\wedge dy$

  $= \frac{\partial f}{\partial x}\,dx\wedge dy\wedge dz + \frac{\partial g}{\partial y}\,dy\wedge dz\wedge dx + \frac{\partial h}{\partial z}\,dz\wedge dx\wedge dy.$

  $= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx\wedge dy\wedge dz \sim \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$
- $\operatorname{div}(f,g,h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

Properties of d and $\wedge$

- ① $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$

② Let α be in k-form and β be in m-form.

- $\deg \alpha = k$, $\alpha\wedge\beta = \pm\beta\wedge\alpha$

  $\deg \beta = m$ $\quad = (-1)^{km}\,\beta\wedge\alpha$

$dx\wedge dy = -dy\wedge dx$

for $m=k=1$

- Ex $k=1$.

  $\alpha = dx$, $\beta = dy_1\wedge\cdots\wedge dy_m$

  $\alpha\wedge\beta = dx\wedge dy_1\cdots\wedge dy_m$

  $(-1)^m\beta\wedge\alpha = dy_1\wedge\cdots dy_m\wedge dx.$

③ $d^2 = 0$ or $\alpha \to d\alpha \to d(d\alpha) = 0.$

④ $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$, where α is of k-form

- $f(x,y,z) = \alpha$

  $d\alpha = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

  $d(d\alpha) = d\left(\frac{\partial f}{\partial x}\right)\wedge dx + d\left(\frac{\partial f}{\partial y}\right)\wedge dy + d\left(\frac{\partial f}{\partial z}\right)\wedge dz$

  $= \left(\frac{\partial^2 f}{\partial y\partial x}dy + \frac{\partial^2 f}{\partial z\partial x}dz\right)\wedge dx + \left(\frac{\partial^2 f}{\partial x\partial y}dx + \frac{\partial^2 f}{\partial z\partial y}dz\right)\wedge dy + \left(\frac{\partial^2 f}{\partial x\partial z}dx + \frac{\partial^2 f}{\partial y\partial z}dy\right)\wedge dz.$

$dx\wedge dy + dy\wedge dx = 0$

- $f(x_1, \dots, x_n)$

  $df = \sum_{1\le i\le n}\frac{\partial f}{\partial x_i}dx_i$

  $d^2 f = \sum_{1\le i,j\le n}\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i = 0$

  $= \sum_{1\le i<j\le n}\underbrace{\left(\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\wedge dx_i + \frac{\partial^2 f}{\partial x_i\partial x_j}dx_i\wedge dx_j\right)}_{=0}$

- $\alpha = f\,dx_1\wedge\cdots\wedge dx_k$, $\beta = g\,dy_1\wedge\cdots\wedge dy_m$

Substitution into a Differential Form

- $\alpha = x^2 dx$, $\quad x = e^{y^2}$

  $= (e^{y^2})^2 d(e^{y^2})$

  $= e^{2y^2} \cdot e^{y^2} 2y\,dy$

  $= 2ye^{3y^2} dy$

- $\beta = x\,dy - y\,dx \qquad x = u - v, \quad y = u^2 + v^2$

  $= (u-v)d(u^2+v^2) - (u^2+v^2)d(u-v)$

  $= (u-v)(2u\,du + 2v\,dv) - (u^2+v^2)(du - dv)$

  $= (2u^2 - 2uv - u^2 - v^2)du + (2uv - 2v^2 + u^2 + v^2)dv$

- $\gamma = xy\,dx \wedge dy \qquad x = uv, \quad y = u^2 + v^2$

  $= uv(u^2+v^2)d(uv) \wedge d(u^2+v^2)$

  $= uv(u^2+v^2)(v\,du + u\,dv) \wedge (2u\,du + 2v\,dv)$

  $= (u^3v + uv^3)(v\,du + u\,dv) \wedge (2u\,du + 2v\,dv)$

  $= (u^3v + uv^3)(2v^2 - 2u^2)du\,dv$

- $\alpha = x\,dx \qquad x = u^2 - v^2 + uv$

  $\tilde{\alpha} = (u^2 - v^2 + uv)d(u^2 - v^2 + uv)$

  $= (u^2 - v^2 + uv)(2u\,du - 2v\,dv + u\,dv + v\,du)$

  $= (u^2 - v^2 + uv)(2u + v)du + (u^2 - v^2 + uv)(-2v + u)dv$

- $\beta = xy\,dx \wedge dy$, $\quad x = t^2$, $y = 1 - t$. two form in one variable $= 0$

  $\tilde{\beta} = 0$.

Integration of Differential Forms

- ① Integral of a n-form (in n variables) over n-dimensional "cube"
  - $\alpha = f(x_1, \dots, x_n)\, dx_1 \wedge \dots dx_n$
  - Cube denoted by Γ
  - $\Gamma = \left\{ (x_1, \dots, x_n) ;\ \begin{array}{l} a_1 \le x_1 \le b_1 \\ a_2 \le x_2 \le b_2 \\ \vdots \\ a_n \le x_n \le b_n \end{array} \right\}$
  - $\int_\Gamma \alpha = \int_{\substack{a_1 \le x \le b_1 \\ \vdots \\ a_n \le x \le b_1}} f dx_1 \wedge \dots dx_n = \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(x_1, \dots x_n)\, dx_1 \dots dx_n.$

- Consider $\alpha = x^2 y\, dx \wedge dy = -x^2 y\, dx dy$
  $\int_\Gamma \alpha = \int\int x^2 y\, dx \wedge dy = \int_{a_2}^{b_2}\int_{a_1}^{b_1} x^2 y\, dx\, dy.$
  $\int_\Gamma \alpha = -\int x^2 y\, dy \wedge dx = -\int\int x^2 y\, dy\, dx$

Orientation in $\mathbb{R}^n$

- $\mathbb{R}^1$ - , two orientations.
- $\mathbb{R}^2$ -

- $\int_\Gamma \alpha$, when α is a K-form in $x_1, \dots x_n$ $(0 \le k \le n)$
  Γ is a k-dimension "surface" given by
  $\Gamma = \left\{ (x_1, \dots x_n) ;\ \begin{array}{l} x_1 = g_1(u_1, \dots u_k) \quad 0 \le u_1 \dots u_k \le 0 \\ x_2 = g_2(u_1 \dots u_k) \\ x_n = g_n(u_1, \dots u_k) \end{array} \right\}$
  $\alpha = \sum_{1 \le i_1 \dots \le i_k \le n} f_{i_1 \dots i_k}(x_1, \dots x_n)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}.$

k-dimensional in ~~hypersurface~~ n-dimensional space

$\Gamma \subseteq \mathbb{R}^n$
$\dim \Gamma = k.$

- Eg. $n=2$, $k=1$
  $\alpha = x dx - y dy$
  $\Gamma = \{ (x, y) ;\ x = t^3, y = 1 - t^2, -1 \le t \le 1 \} \subset \mathbb{R}^2$

$$\int_\Gamma \alpha = \int_\Gamma (x\,dx - y\,dy) = \int_{-1}^{1} [t^3\, d(t^3) - (1-t^2)\, d(1+t^2)] = \int_{-1}^{1} [(3t^5)dt - (-2t+2t^3)dt]$$

$$= \left[3\tfrac{1}{6}t^6\right]_{-1}^{1} - \left[-t^2 + \tfrac{2}{4}t^4\right]_{-1}^{1} = 0$$

- Eg. $n=3$, $k=2$.

$$\alpha = x\,dx\wedge dy - y\,dx\wedge dz + z\,dy\wedge dz$$

$$\Gamma = \{(x,y,z);\ x = u-v,\ y = u^2+v^2,\ z = uv,\ \begin{matrix} 0\le u\le 1 \\ -1\le v\le 0\end{matrix}\}$$

$$\int_\Gamma \alpha = \int_\Gamma (x\,dx\wedge dy - y\,dx\wedge dz + z\,dy\wedge dz).$$

$$= \int_{\substack{0\le u\le 1\\ -1\le v\le 0}} \left[(u-v)\,d(u-v)\wedge d(u^2+v^2) - (u^2+v^2)\,d(u-v)\wedge d(uv) + (uv)\,d(u^2+v^2)\wedge d(uv)\right]$$

$$= \int \Big[(u-v)(du-dv)\wedge(2u\,du + 2v\,dv) - (u^2+v^2)(du-dv)\wedge(v\,du + u\,dv) +$$
$$(uv)(2u\,du + 2v\,dv)\wedge(v\,du + u\,dv)\Big]$$

$$= \int_{-1}^{0}\int_{0}^{1} \left[(u-v)(2v+2u) - (u^2+v^2)(u+v) + uv(2u^2-2v^2)\right] du\,dv$$

Degenerate case $\Rightarrow$

- Eg $n=3$, $k=0$

$$\alpha = x - yz.$$

$$\Gamma = \{(x,y,z);\ x=2,\ y=-1,\ z=3\} = \{(2,-1,3)\}$$

$$\int_\Gamma \alpha = \int_{(2,-1,3)} x - yz = (2) - (-1)(3) = 5$$

- $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\wedge dy$$

$$= \int_{\mathbb{R}^2} e^{-(r\cos\varphi)^2 - (r\sin\varphi)^2}\,d(r\cos\varphi)\wedge d(r\sin\varphi)$$

$$= \int_{\mathbb{R}^2} e^{-r^2}(-r\sin\varphi\,d\varphi + \cos\varphi\,dr)\wedge(r\cos\varphi\,d\varphi \wedge \sin\varphi\,dr)$$

$$= \int_{\mathbb{R}^2} e^{-r^2}(r\cos^2\varphi + r\sin^2\varphi)\,dr\,d\varphi = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\varphi$$

$$= \int_0^\infty e^{-r^2} r\,dr \int_0^{2\pi} d\varphi = \int_0^\infty e^{-r^2}\,d\left(\tfrac{1}{2}r^2\right)[2\pi] = \pi\int_0^\infty e^{-t}\,dt = \pi\left(-e^{-t}\right)\Big|_0^\infty$$

$$= \pi$$

$$I^2 = \pi \Rightarrow I = \sqrt{\pi}$$

- $V_3 \Rightarrow \{(x,y,z);\ x^2+y^2+z^2 \le r^2\}$, Volume of circle proof.

$A_r \Rightarrow \{(x,y,z);\ x^2+y^2+z^2 = r^2\}$. , Area of circle.

$V_n(r) \Rightarrow \{(x_1,\dots x_n);\ x_1^2 + \dots x_n^2 \le r^2\}$, Volume of n-dimensional "sphere"

$A_n(r) \Rightarrow \{(x_1,\dots x_n);\ x_1^2 + \dots x_n^2 = r^2\}$, Area of n-dimensional "sphere"

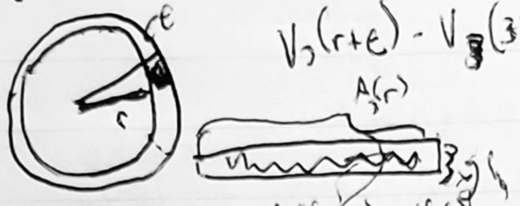
Kilroy

$\Rightarrow$

• $V_3(r) = u_3 r^3 \Rightarrow V_2(r) = \pi r^2 \Rightarrow A_2 = 2\pi r$

$V_n(r) = u_n r^n \Rightarrow 3u_3 r^2$

$A_n(r) = V_n'(r)$

• 

$V_3(r+\epsilon) - V_3(r) \simeq A_3 \epsilon$

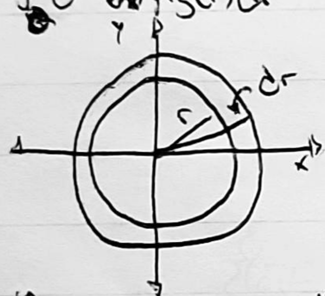
$A_3(r) = \lim_{\epsilon \to 0} \frac{V_3(r+\epsilon) - V_3(r)}{\epsilon} = V_3'(r)$

• $\int_{\mathbb{R}^3} e^{-x^2-y^2-z^2}\, dx\, dy = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy \int_{\mathbb{R}} e^{-z^2} dz$

$= \sqrt{\pi}\ \sqrt{\pi}\ \sqrt{\pi}$

$= \pi^{\frac{3}{2}}$

$\int_0^\infty e^{-r^2} A_3(r)\, dr$



$\int_0^\infty e^{-r^2} A_2(r)\, dr$

• $\int_0^\infty e^{-r^2} A_3(r)\, dr = \int_0^\infty e^{-r^2} 3u_3 r^2\, dr = 3u_3 \int_0^\infty e^{-r^2} r^2\, dr = \pi^{\frac{3}{2}}$

$\Rightarrow \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-r^2} dr = re^{-r^2}\Big|_0^\infty - \int_0^\infty r\, d(e^{-r^2}) = -\int_0^\infty r(-2re^{-r^2})\, dr$

$\int_0^\infty r^2 e^{-r^2} dr = \frac{\sqrt{\pi}}{4}$

$3u_3 \cdot \frac{1}{4} \pi^{\frac{1}{2}} u_3 = \pi^{\frac{3}{2}}$

$\frac{3}{4} u_3 = \pi$

$u_3 = \frac{4}{3}\pi \Rightarrow V_3 = \frac{4}{3}\pi r^3,\ A_3 = 4\pi r^2.$

Another Method →

• $\int_0^\infty e^{-tx^2} dx = \int_0^\infty e^{-(\sqrt{t}x)^2}\, d(x\sqrt{t}) \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{t}} \int_0^\infty e^{-(\sqrt{t}x)^2}\, d(x\sqrt{t})$

$= \frac{1}{\sqrt{t}} \int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2\sqrt{t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$

$\int_0^\infty (-x^2 e^{-tx^2})\, dx = \frac{1}{2}\sqrt{\pi} \cdot \left(-\frac{1}{2} t^{-\frac{3}{2}}\right) = -\frac{1}{4}\sqrt{\pi}\, t^{-\frac{3}{2}}$

$t=1 \quad \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{4}\sqrt{\pi}$

Stokes' Theorem (or Generalized Stokes Theorem)

$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

$\beta = \alpha \Rightarrow p = q$

$\alpha \wedge \alpha = (-1)^{p^2} \alpha \wedge \alpha$.

If p is odd $\Rightarrow \alpha \wedge \alpha$ 0

If p is even $\Rightarrow \alpha \wedge \alpha = \alpha \wedge \alpha$ "empty statement"

- $\int_\Gamma d\alpha = \int_{\partial\Gamma} \alpha$ — α is a k-form in $x_1, \dots x_n$; Γ is a $k+1$-dimensional subspace in $\mathbb{R}^n$
- $\partial\Gamma$ is (in a sense) a boundary of Γ
- $\int_a^b f'(x)dx = f(b) - f(a)$ can be written as →→→→→→

$\int_{[a,b]} d(f(x)) = \int_{\partial[a,b]} f(x)$

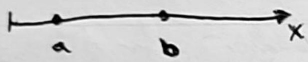
$\alpha = f(x)$ - 0-form

$d\alpha = d(f(x)) = f'(x)dx$.

$\Gamma = \{x;\ a \le x \le b\}$

- $\partial[a,b] = \{a,b\}$ $\Rightarrow \int_{\partial[a,b]} f(x) = \int_{\{a,b\}_{-,+}} f(x) = -f(a) + f(b)$.

↑ true up to sign (orientation should be chosen).



- $\Gamma = \{(x_1 \dots x_n);\ x_1 = g_1(u_1 \dots u_k), \dots x_n = g_n(u_1 \dots u_k);\ 0 \le u_1 \dots \le u_k \le 1\} \subseteq \mathbb{R}^n$

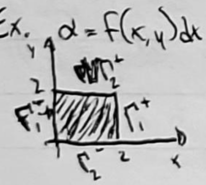
$\partial\Gamma$ as a set $\Gamma_1^+ \cup \dots \cup \Gamma_k^+ \cup \Gamma_1^- \dots \cup \Gamma_k^-$ ($\cup$ = union, $\subseteq$ subset).

$\Gamma_i^+ = \Gamma \cap \{u_i = 1\}$

$\Gamma_i^- = \Gamma \cap \{u_i = 0\}$.

$\partial\Gamma = \sum_{1 \le i \le k} (-1)^{i-1}(\Gamma_i^+ - \Gamma_i^-) = \Gamma_1^+ - \Gamma_1^- + \Gamma_2^+ - \Gamma_2^- + \Gamma_3^+ - \Gamma_3^- + \dots$

- $\int_\Gamma d\alpha = \sum_{i=1} (-1)^{i-1} \int_{\Gamma_i^+} \alpha - (-1)^{i-1} \int_{\Gamma_i^-} \alpha$
- Ex. $\alpha = f(x,y)dx + g(x,y)dy$; $\Gamma = \{(x,y);\ 0 \le x, y \le 2\}$



$\Gamma_1^+ = \Gamma \cap \{x = 2\} \equiv \Gamma|_{x=2}$

$\Gamma_1^- = \Gamma \cap \{x = 0\} \equiv \Gamma|_{x=0}$

$\Gamma_2^+ = \Gamma \cap \dots \equiv \Gamma|_{y=2}$

$\Gamma_2^- = \Gamma \cap \dots \equiv \Gamma|_{y=0}$

$\int_\Gamma d\alpha = \int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_0^2\int_0^2 \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$

- when $x = a$ the $f(x,y)$ term drops out because $d(a) = 0$.

$\int_{\partial\Gamma} \alpha = \int_{\Gamma_1^+} \alpha - \int_{\Gamma_1^-} \alpha - \int_{\Gamma_2^+} \alpha + \int_{\Gamma_2^-} \alpha = \int_{\substack{x=2 \\ 0 \le y \le 2}} \alpha - \int_{\substack{x=0 \\ 0 \le y \le 2}} \alpha - \int_{\substack{y=2 \\ 0 \le x \le 2}} \alpha + \int_{\substack{y=0 \\ 0 \le x \le 2}} \alpha$

$= \int_0^2\int_0^2 \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \int_0^2 g(2,y)dy - \int_0^2 g(0,y)dy - \int_0^2 f(x,2)dx + \int_0 f(x,2)dx$

$= \int_0^2 g(x,y)\Big|_{x=0}^{x=2} dy - \int_0^2 f(x,y)\Big|_{x=0}^{x=2} dx$.

$=$

- $\int_\Gamma d\alpha = \int_{\Gamma_1^+} \alpha - \int_{\Gamma_1^-} \alpha - \int_{\Gamma_2^+} \alpha + \int_{\Gamma_2^-} \alpha + \int_{\Gamma_3^+} \alpha \dots = \sum_{i=1}^{k+1} (-1)^{i-1} \left(\int_{\Gamma_i^+} \alpha - \int_{\Gamma_i^-} \alpha\right)$

- $u_1, \dots u_{k+1}$ are coordinates in Γ
- $\Gamma_i^+ = \Gamma|_{u_i =}$ its max

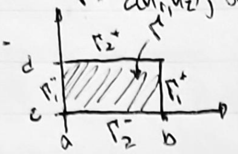
$\Gamma_i^- = \Gamma|_{u_i =}$ its min.

• $k=0$ $\quad \Gamma=\{a\le u\le b\}$

- 

$\Gamma_1^+=\{a\}$ $\quad \Gamma_1^-=\{b\}$

• $k=1$ $\quad \Gamma=\{(u_1,u_2);\ a\le u_1\le b,\ c\le u_2\le d\}$

- 

$\Gamma_1^+=\Gamma|_{u_1=b}$ $\quad \Gamma_1^-=\Gamma|_{u_1=a}$

$\Gamma_2^+=\Gamma|_{u_2=d}$ $\quad \Gamma_2^-=\Gamma|_{u_2=c}$

• $\alpha = f(x,y,z)\,dx\wedge dy + g(x,y,z)\,dx\wedge dz + h(x,y,z)\,dy\wedge dz$

$\Gamma=\{(x,y,z);\ 0\le x,y,z\le 1\}$

$\int_\Gamma d\alpha = \int_{\Gamma_1^+}\alpha - \int_{\Gamma_1^-}\alpha - \int_{\Gamma_2^+}\alpha + \int_{\Gamma_2^-}\alpha + \int_{\Gamma_3^+}\alpha - \int_{\Gamma_3^-}\alpha$

$d\alpha = \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) dx\wedge dy\wedge dz$

$\int_\Gamma d\alpha = \int_0^1\int_0^1\int_0^1 \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) dx\,dy\,dz$, $\quad \Gamma_1^+=\Gamma\cap\{x=1\}$, $\Gamma_1^-=\Gamma\cap\{x=0\}$ ...

$\Downarrow$

$= \int_0^1\int_0^1 h(1,y,z)\,dydz - \int_0^1\int_0^1 h(0,y,z)\,dydz - \int_0^1\int_0^1 g(x,1,z)\,dxdz + \int_0^1\int_0^1 g(x,0,z)\,dxdz$
$+ \int_0^1\int_0^1 f(x,y,1)\,dxdy - \int_0^1\int_0^1 f(x,y,0)\,dxdy.$

$= \int_0^1\int_0^1 f(x,y,z)\Big|_{z=0}^{z=1} dxdy - \int_0^1\int_0^1 g(x,y,z)\Big|_{y=0}^{y=1} dxdz + \int_0^1\int_0^1 h(x,y,z)\Big|_{x=0}^{x=1} dydz.$

Determinants

• $A=\begin{pmatrix}a&b\\c&d\end{pmatrix} \Rightarrow \det A = ad-bc$

• $\omega = dx\wedge dy \Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\hat x\\\hat y\end{pmatrix}$

$x=a\hat x+b\hat y \qquad dx=a\,d\hat x+b\,d\hat y$

$y=c\hat x+b\hat y \qquad dy = c\,d\hat x + d\,d\hat y$

• $\omega = dx\wedge dy = (a\,d\hat x + b\,d\hat y)\wedge(c\,d\hat x+d\,d\hat y) = ad\,d\hat x\wedge d\hat y + bc\,d\hat y\wedge d\hat x$

$= \underbrace{(ad-bc)}_{\text{determinant of } A}\, d\hat x\wedge d\hat y$.

• $A=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&\ddots&\vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix}$, $\det A = ?$

• $\omega = dx_1\wedge\dots\wedge dx_n = (a_{11}dy_1+\dots a_{1n}dy_n)\wedge\dots\wedge(a_{n1}dy_1+\dots a_{nn}dy_n)$

$\tilde x = A\tilde y$

$\tilde x = \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, \tilde y = \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \Rightarrow \begin{cases} x_1 = a_{11}y_1 + \dots a_{1n}y_n \\ \vdots \\ x_n = a_{n1}y_1+\dots a_{nn}y_n\end{cases}$

$= \det A\; dy_1\wedge\dots\wedge dy_n$

- Property of Determinants
- $\det(AB) = \det(A)\det(B)$
- $\vec{x} = A\vec{y}$, $\vec{y} = B\vec{z}$

$\det A \, dy_1 \wedge \ldots \wedge dy_n = \det A \cdot \det B \, dz_1 \wedge \ldots \wedge dz_n$

$\vec{x} = AB\vec{z}$

$w = \det(AB) \, dz_1 \wedge \ldots \wedge dz_n$

$\det(AB) = \det(A)\det(B)$

Pfaffian

- A matrix A is skew-symmetric if $A^t = -A$.
  " " " " symmetric if $A^t = A$.
- 2×2 Skew-Symmetric
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

    $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \Rightarrow$ $a = -a$ $\quad$ $b = -c$ $\quad$ $d = -d$
  - $\det\begin{pmatrix} a & b \\ -b & 0 \end{pmatrix} = b^2$
- $\det(A^t) = \det(-A)$

  $\det(A) = (-1)^n \det(A)$
- If n is odd, $\det A = -\det(A) \Rightarrow 0$
  - $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$
- if n is even, $\det A = (\mathrm{Pf} A)$.
  - $\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ & & & 0 \end{pmatrix} = A$

  $= \det(A) = (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})^2 \leftarrow$ Pfaffian

Examples

$\cdot\ S(f,g) = \int\!\!\int (f_{xy}^2 + g_{xy}^2 - \sin(f^2 g^3))\,dx\,dy.$

$\frac{\delta S}{\delta f} = ?$

$f_{xy} \to f_{xy} + \delta f_{xy}$

$f \to f + \delta f$

$S(f+\delta f, g) = S(f,g) + \int (2 f_{xy}\,\delta f_{xy} - \cos(f^2 g^3)\cdot 2fg^3\,\delta f)\,dx\,dy.$

$= S(f,g) + \int (2 (f_{xy})_{xy}\,\delta f - \cos(f^2 g^3) 2fg^3\,\delta f)\,dx\,dy.$

$F(x_1 + dx_1, x_2 + dx_2, \ldots) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots$

$= S(f,g) = \int [2(f_{xxyy}) - \cos(f^2 g^3) 2fg^3]\,\delta f\,dx\,dy$

$\int F H_u\,du = -\int F_u H\,du$ - Our IBP

$\frac{\delta S}{\delta f} = 2f_{xxyy} - \cos(f^2 g^3) 2fg^3$

$\cdot\ L = 2q_1'^2 + 3q_2'^2 + q_3'^2 + 5q_4'^2 - \sin(q_1 - 2q_3) - \cos(q_2 + 2q_4)$

$\frac{d}{dt}\frac{\partial L}{\partial q_i'} = \frac{\partial L}{\partial q_i}, \quad i = 1, \ldots 4.$

$\frac{d}{dt}(4q_1') = -\cos(q_1 - 2q_3) \Rightarrow 4q_1'' + \cos(q_1 - 2q_3) = 0$

$\frac{d}{dt}(6q_2') = \sin(q_2 + 2q_4) \Rightarrow 6q_2'' - \sin(q_2 + 2q_4) = 0$

$\frac{d}{dt}(2q_3') = 2\cos(q_1 - 2q_3) \Rightarrow 2q_3'' - 2\cos(q_1 - 2q_3) = 0$

$\frac{d}{dt}(10q_4') = 2\sin(q_2 + 2q_4) \Rightarrow 10q_4'' - 2\sin(q_2 + 2q_4) = 0$

equations of motion.

$p_1 = 4q_1', \quad p_2 = 6q_2', \quad p_3 = 2q_3', \quad p_4 = 10q_4'$

$E = \sum_{i=1}^{4} \frac{\partial L}{\partial q_i'} q_i' - L = \sum p_i q_i' - L.$

Hamiltonian $\Rightarrow$

$E = 2q_1'^2 + 3q_2'^2 + q_3'^2 + 5q_4'^2 + \sin(q_1 - 2q_3) + \cos(q_2 + 2q_4).$

$\frac{dE}{dt} = 4q_1'q_1'' + 6q_2'q_2'' + 2q_3'q_3'' + 10q_4'q_4'' + \cos(q_1 - 2q_3)(q_1' - 2q_3') - \sin(q_2 + 2q_4)(q_2' + 2q_4')$

$\ldots\ldots = 0$ energy is conserved

$2p_1' + p_3' = 0, \quad 2p_1 + p_3 = C_1$

$2p_2' - p_4' = 0, \quad 2p_2 - p_4 = C_2$

Pfaffian

- $A^t = -A$, if $A = (a_{ij})$; $i,j = 1, \ldots, n$ then $a_{ij} = -a_{ji}$ & $a_{ii} = 0$
- $n = 2$, $A = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}$ $\quad n = 3$, $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$
- $\det A = \begin{cases} = 0, \text{ if } n = 1, 3, 5, 7, \ldots \\ (\mathrm{Pf}A)^2, \text{ if } n = 2, 4, 6, 8 \ldots \end{cases}$

  where $\mathrm{Pf}A$ is a polynomial in a_{ij}
- $n = 2$, $\det A = \begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = 0 - (-a_{12})(a_{12}) = (a_{12})^2 \Rightarrow \mathrm{Pf}A = a_{12}$

  $n = 4$, $\det A = \begin{vmatrix} 0 & \cdots & \cdots \\ -0 & \cdots & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & -0 \end{vmatrix} = (a_{12}a_{34} + \ldots)^2 \Rightarrow \mathrm{Pf}A = a_{12}a_{34} + \ldots$
- $w = \sum_{1 \le i \le j \le n} a_{ij}\, dx_i \wedge dx_j \leftarrow$ 2-form.

  $w \wedge w = \ldots \leftarrow$ 4-form.

  $w \wedge w \wedge w = \ldots \leftarrow$ 6-form

  $w \wedge w \wedge w \wedge w = \ldots \leftarrow$ 8-form

  $\underbrace{w \wedge w \wedge w \wedge w \ldots \wedge w}_{m} = \ldots \leftarrow$ 2m-form

  $2m = n \Rightarrow m = \frac{n}{2}$

  $\underbrace{w \wedge w \wedge w \wedge \ldots \wedge w}_{\frac{n}{2} \text{ terms}} = \ldots \leftarrow$ n-form

  $\uparrow \mathrm{Pf}(A)(\frac{n}{2})!$

Vector Fields.

- A vector field in $x_1, \dots x_n$ is:
  $V = f_1(x_1 \dots x_n)\frac{\partial}{\partial x_1} + \dots + f_n(x_1 \dots x_n)\frac{\partial}{\partial x_n}$
- Ex. $xy^2 e^{x+y}\frac{\partial}{\partial x} - \sin(y^3 x)\frac{\partial}{\partial y} = \alpha$, $e^{-u^2}\frac{\partial}{\partial u} = \beta$
  $u\frac{\partial}{\partial u} - v\frac{\partial}{\partial u} = \gamma$, $uvw\frac{\partial}{\partial w} = \mu$.
- $V(g) = f_1\frac{\partial g}{\partial x_1} + \dots + f_n\frac{\partial g}{\partial x_n}$
- Ex $\alpha(x^3y^2) = xy^2 e^{x+y}\frac{\partial}{\partial x}(x^3 y) - \sin(xy^3)\frac{\partial}{\partial y}(x^3 y^2)$
  $= xy^2 e^{x+y} 3x^2 y - \sin(xy^3)\, x^3\, 2y$
- $V(gh) = V(g)\cdot h + V(h)\, g$

Change of Variables

- $V = x^3\frac{\partial}{\partial x}$ $\quad x = u^3$
  Write V in terms of u
- Old change of variables
  - $w = x^3 dx \Rightarrow w = u^2$; $d(u^3) = 3u^2 du$;
    $w = (u^3)^3 \cdot 3u^2 = 3u^{11} du$.
- Need to write $\frac{\partial}{\partial x}$ in terms of $\frac{\partial}{\partial u}$
  - $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial g}{\partial u}$
  - $\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial x}$
  - $x = u^3 \Rightarrow \frac{\partial x}{\partial x} = \frac{\partial u}{\partial x} \Rightarrow 1 = 3u^2\frac{\partial u}{\partial x}$
  - $\frac{\partial}{\partial x} = \frac{1}{3u^2}\frac{\partial}{\partial u}$.
- $V = \frac{1}{3u^2}(3^3)^3\frac{\partial}{\partial u} = \frac{u^7}{3}\frac{\partial}{\partial u}$
- $\mu = x^2\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ $\quad x = u^2 + v^2 \Rightarrow \begin{cases} 1 = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \\ 0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \end{cases}$ Solve for $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$
  $= (u^2+v^2)^2\frac{\partial}{\partial x} - uv\frac{\partial}{\partial y}$ $\quad y = uv$
  $\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$ $\quad \begin{cases} 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \\ 1 = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} \end{cases}$ solve for $\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}$.
  $\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$

• $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $\qquad x = r\sin\varphi\ ,\ y = r\cos\varphi$

$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$

$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}$

$\begin{cases} 1 = r\cos\varphi\frac{\partial\varphi}{\partial x} + \sin\varphi\frac{\partial r}{\partial x} \Rightarrow \cos\varphi = r(\cos^2\varphi + \sin^2\varphi)\frac{\partial\varphi}{\partial x} \Rightarrow \frac{\partial\varphi}{\partial x} = \frac{\cos\varphi}{r} \\ 0 = -r\sin\varphi\frac{\partial\varphi}{\partial x} + \cos\varphi\frac{\partial r}{\partial x} \Rightarrow \sin\varphi = \frac{\partial r}{\partial x} \end{cases}$

$\begin{cases} \frac{\partial}{\partial x} = \sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial\varphi} \\ \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial y} \end{cases}$

$\begin{cases} 0 = r\cos\varphi\frac{\partial\varphi}{\partial y} + r\sin\frac{\partial\varphi}{\partial y} \Rightarrow -\sin\varphi = \frac{\partial\varphi}{\partial x} \\ 1 = -r\sin\varphi\frac{\partial\varphi}{\partial y} + \cos\varphi\frac{\partial\varphi}{\partial y} \Rightarrow \cos\varphi = \frac{\partial r}{\partial y} \end{cases}$

$\frac{\partial}{\partial x} = \sin\frac{\partial}{\partial x} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial x}$

$\frac{\partial}{\partial y} = \cos\varphi\frac{\partial}{\partial y} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial y}$

$V = r\sin\varphi\left(\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial\varphi}\right) + r\cos\varphi\left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right)$

$= (r\sin^2\varphi + r\cos^2\varphi)\frac{\partial}{\partial r} + \left(r\sin\varphi\cdot\frac{1}{r}\cos\varphi - r\cos\varphi\frac{1}{r}\sin\varphi\right)\frac{\partial}{\partial\varphi}$

$= r\frac{\partial}{\partial r}$

• $\mu = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}$ $\qquad x = u^3v^2\ ,\ y = u^2v^3$

$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$

$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$

$1 = 3u^2v^2\frac{\partial u}{\partial x} + 2u^3v\frac{\partial v}{\partial x} \quad \Big| \quad 3v$

$0 = 2uv^3\frac{\partial u}{\partial x} + 3u^2v^2\frac{\partial v}{\partial x} \quad \Big| \quad -2u$

$3v = (9u^2v^3 - 4u^2v^3)\frac{\partial u}{\partial x}\ ;\ \frac{\partial u}{\partial x} = \frac{3}{5}\frac{1}{u^2v^2}$

$2v = (4u^3v^2 - 9u^3v^2)\frac{\partial v}{\partial x}\ ;\ \frac{\partial v}{\partial x} = -\frac{2}{5}\frac{1}{u^3v}$

Differential Operators

- $V = \sum_{i_1 \ldots i_n \geq 0}^{N} f_{i_1 \ldots i_n}(x_1, \ldots x_n) \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$
- Ex. $x^2 - 1 + (x^3 + x)\frac{\partial}{\partial x} - e^x\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x}\right)^3$

  $xy - xy^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \sin(x-y)\left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial}{\partial x}\frac{\partial}{\partial y} - e^{xy}\left(\frac{\partial}{\partial y}\right)^3$

  $xy\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \left(\frac{\partial}{\partial z}\right)^2$

  $3 - xy^2 z \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z} - x\left(\frac{\partial}{\partial y}\right)^4$
- $V = \ldots \equiv \sum f_{i_1 \ldots i_n}(x_1, \ldots x_n) \frac{\partial^{i_1 + \ldots i_n}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$
- Ex $x^2 - 1 + (x^3 + x)\frac{\partial}{\partial x} - e^x\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial x}\right)^3 \equiv x^2 - 1 + (x^3 + x)\frac{\partial}{\partial x} - e^x\left(\frac{\partial^2}{\partial x^2}\right) + \frac{\partial^3}{\partial x^3}$

  $\frac{\partial}{\partial x}\frac{\partial}{\partial y} \equiv \frac{\partial^2}{\partial x \partial y}$

Product of Differential Operators

- $V = x^2 \frac{d}{dx}$, $\mu = x^5 \left(\frac{d}{dx}\right)^3$
- V, μ are differential operators such that $(v\mu)(f) = v(\mu(f))$ for arbitrary f.
- $v(\mu(f)) = x^2 \frac{d}{dx}\left(x^5\left(\frac{d}{dx}\right)^3(f)\right) = x^2 \frac{d}{dx}\left(x^5 \frac{d^3 f}{dx^3}\right)$

  $= x^2\left(5x^4 \frac{d^3 f}{dx^3} + x^5 \frac{d^4 f}{dx^4}\right)$

  $= 5x^6 \frac{d^3 f}{dx^3} + x^7 \frac{d^4 f}{dx^4}$

  $v\mu = 5x^6 \frac{d^3}{dx^3} + x^7 \frac{d^4}{dx^4}$
- $\mu(v(f)) = x^5 \frac{d^3}{dx^3}\left(x^2 \frac{df}{dx}\right) = x^5 \frac{d^2}{dx^2}\left(2x\frac{df}{dx} + x^2 \frac{d^2 f}{dx^2}\right)$

  $= x^5 \frac{d}{dx}\left(2\frac{df}{dx} + 2x\frac{d^2 f}{dx^2} + 2x\frac{d^2 f}{dx^2} + x^2 \frac{d^3 f}{dx^3}\right)$

  $= x^5\left(2\frac{d^2 f}{dx^2} + 4\frac{d^2 f}{dx^2} + 4x\frac{d^3 f}{dx^3} + 2x\frac{d^3 f}{dx^3} + x^2\frac{d^4 f}{dx^4}\right)$

  $= 2x^5 \frac{d^2 f}{dx^2} + 4x^5 \frac{d^2 f}{dx^2} + 6x^6 \frac{d^3 f}{dx^3} + x^7 \frac{d^4 f}{dx^4}$

  $= 6x^5 \frac{d^2 f}{dx^2} + 6x^6 \frac{d^3 f}{dx^3} + x^7 \frac{d^4 f}{dx^4}$
- $\mu v = 6x^5 \frac{d^2}{dx^2} + 6x^6 \frac{d^3}{dx^3} + x^7 \frac{d^4}{dx^4}$
- Note: $v\mu \neq \mu v$
- $v = x - \frac{d}{dx}$, $\mu = 1 + x^2 \frac{d^2}{dx^2}$

  $v(\mu(f)) = \left(x - \frac{d}{dx}\right)\left(f + x^2 \frac{d^2 f}{dx^2}\right)$

  $= xf + x^3 \frac{d^2 f}{dx^2} - \frac{df}{dx} - 2x\frac{d^2 f}{dx^2} - x^2 \frac{d^3 f}{dx^3}$

  $= xf - \frac{df}{dx} + (x^3 - 2x)\frac{d^2 f}{dx^2} - x^2 \frac{d^3 f}{dx^3}$

  $v\mu = x - \frac{d}{dx} + (x^3 - 2x)\frac{d^2}{dx^2} - x^2 \frac{d^3}{dx^3}$

- $\alpha = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, $\beta = x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial x\partial y}$

$$\alpha(\beta(f)) = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2}{\partial x\partial y}\right)$$

$$= 2x^2\frac{\partial^2 f}{\partial x^2} + x^3\frac{\partial^3 f}{\partial x^3} + xy^2\frac{\partial^3 f}{\partial x^2\partial y} - x^2 y\frac{\partial^3 f}{\partial x^2\partial y} - 2y^2\frac{\partial^2}{\partial x\partial y} - y^3\frac{\partial^3}{\partial x\partial y^2}$$

$$= 2x^2\frac{\partial^2 f}{\partial x^2} + x^3\frac{\partial^3 f}{\partial x^3} + (xy^2 - x^2y)\frac{\partial^3 f}{\partial x^2\partial y} - 2y^2\frac{\partial^2 f}{\partial x\partial y} - y^3\frac{\partial^3 f}{\partial x\partial y^2}$$

$$\alpha\beta = 2x^2\frac{\partial^2}{\partial x^2} + x^3\frac{\partial^3}{\partial x^3} + (xy^2 - x^2y)\frac{\partial^3}{\partial x^2\partial y} - 2y^2\frac{\partial^2}{\partial x\partial y} - y^3\frac{\partial^3}{\partial x\partial y^2}$$

- Computations can be done without f, it was just used for beginning exercises.
- Rule: $\frac{d}{dx}x = x\frac{d}{dx} + 1$.
- Ex. $x^2\frac{d^2}{dx^2}\cdot x\frac{d}{dx} = x^2\frac{d}{dx}\frac{d}{dx}x\frac{d}{dx} = x^2\frac{d}{dx}\left(x\frac{d}{dx}+1\right)\frac{d}{dx}$

$$= x^2\frac{d}{dx}x\frac{d^2}{dx^2} + x^2\frac{d^2}{dx^2} = x^2\left(x\frac{d}{dx}+1\right)\frac{d^2}{dx^2} + x^2\frac{d^2}{dx^2}$$

$$= x^3\frac{d^3}{dx^3} + 2x^2\frac{d^2}{dx^2}$$

- Extension of Rule: $\frac{d}{dx}g(x) = g(x)\frac{d}{dx} + g'(x)$

Change of Variables in Differential Operators

- $V = x - x\frac{d}{dx} + \frac{d^2}{dx^2}$ $\qquad x = \sin y \Rightarrow 1 = \cos y\frac{dy}{dx}$; $\frac{dy}{dx} = \frac{1}{\cos y}$.

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \frac{1}{\cos y}\frac{d}{dy}$$

$=$

$$= \sin y - \sin y\left(\frac{1}{\cos y}\frac{d}{dy}\right) + \left(\frac{1}{\cos y}\frac{d}{dy}\right)^2$$

$$= \sin y - \tan y\frac{d}{dy} + \left(\frac{1}{\cos y}\frac{d}{dy}\right)\left(\frac{1}{\cos y}\frac{d}{dy}\right)$$

$$= \sin y - \tan y\frac{d}{dy} + \frac{1}{\cos y}\left(\frac{1}{\cos y}\frac{d^2}{dy^2} + \frac{\sin y}{\cos^2 y}\frac{d}{dy}\right)$$

$$= \sin y - \left(\tan y \cancel{\frac{d}{dy}} - \frac{\sin y}{\cos^3 y}\right)\frac{d}{dy} + \frac{1}{\cos^2 y}\frac{d^2}{dy^2}$$

- $\frac{\partial^3}{\partial x^2\partial y}$ $\qquad x = $ "u, v" , $y = $ "u, v"

$$= \underbrace{\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)^2}_{\frac{\partial^2}{\partial x^2}} \underbrace{\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)}_{\frac{\partial}{\partial y}}$$

Commutator of Vector Fields.

- $\upsilon = x^3 \frac{d}{dx}$ $\quad \mu = \sin x \frac{\partial}{dx}$

$\upsilon\mu = x^3 \frac{d}{dx}\left(\sin \frac{d}{dx}\right) = x^3\left(\cos x \frac{d}{dx} + \sin x \frac{d^2}{dx^2}\right) = x^3 \cos x \frac{d}{dx} + x^3 \sin x \frac{d^2}{dx^2}$

$\mu\upsilon = \sin x \frac{d}{dx}\left(x^3 \frac{d}{dx}\right) = \sin x\left(3x^2 \frac{d}{dx} + x^3 \frac{d^2}{dx^2}\right) = 3x^2 \sin x \frac{d}{dx} + x^3 \sin x \frac{d^2}{dx^2}$

$\upsilon\mu - \mu\upsilon = x^3 \cos x \frac{d}{dx} + x^3 \sin x \frac{d^2}{dx^2} - 3x^2 \sin \frac{d}{dx} - x^3 \sin x \frac{d^2}{dx^2}$

$= \left(x^3 \cos x - 3x^2 \sin x\right)\frac{d}{dx}$

$[\upsilon,\mu] \overset{def}{=} \upsilon\mu - \mu\upsilon = (x^3 \cos x - 3x^2 \sin x)\frac{d}{dx}$ ← this is commutator.

- Definition - Let A, B be "linear operator" then $(AB)C = A(BC)$ but $(AB) \neq (BA)$ in general. Commutator $[A,B] \overset{def}{=} AB - BA$.
- Properties of Commutator.
  - ① $[A,B] = -[B,A]$
  - ② $[A,BC] = [A,B]C + B[A,C]$
  - ③ $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ ← Jacobi identity.
- If υ and μ are vector fields, then $[\upsilon,\mu]$ is also a vector field
  - Proof - $\upsilon = f\frac{d}{dx}$, $\mu = g\frac{d}{dx}$

$[\upsilon,\mu] = f\frac{d}{dx}\left(g\frac{d}{dx}\right) - g\frac{d}{dx}\left(f\frac{d}{dx}\right) = f\left(g' + g\frac{d}{dx}\right)\frac{d}{dx} - g\left(f' + f\frac{d}{dx}\right)\frac{d}{dx}$

$= fg'\frac{d}{dx} + fg\frac{d^2}{dx^2} - gf'\frac{d}{dx} - fg\frac{d^2}{dx^2} = (fg' - gf')\frac{d}{dx}$

- Ex. $\alpha = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $\beta = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$[\alpha,\beta] = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$

$= xy\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} - x^2\frac{\partial^2}{\partial x\partial y} + y\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial^2}{\partial y^2} - \cancel{y\frac{\partial}{\partial x}} - xy\frac{\partial^2}{\partial x^2} -$

$- y^2\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial x\partial y} + x\frac{\partial}{\partial y} + xy\frac{\partial^2}{\partial x\partial y}$

$= 0$

Vector Fields and 1-forms

- $\alpha = x^3\frac{d}{dx}$, $\omega = x^4 dx$, $x = \sin y$, $\frac{1}{\cos y} = \frac{dy}{dx}$

$\alpha = (\sin y)^3 \frac{1}{\cos y}\frac{\partial}{\partial y}$, $\omega = (\sin y)^4 \cos y\, dy$

$(\alpha,\omega) \overset{def}{=} x^3 x^4 = x^7$ - in terms of x.

$(\alpha,\omega) = \sin^3 y \frac{1}{\cos y} \sin^4 y \cos y = \sin^7 y$ - in terms of y.

- Definition - Let $\alpha = f_1\frac{\partial}{\partial x_1} + \dots f_n\frac{\partial}{\partial x_n}$ & $\beta = g_1\frac{\partial}{\partial x_1} + \dots \frac{\partial}{\partial x_n}$ where α is a vector field & β is a 1-form, the $(\alpha,\beta) \overset{def}{=} f_1 g_1 + \dots f_n g_n$.

- (α, β) does not depend on coordinates.

$(\alpha, \beta) \xrightarrow{\text{compute}} (\alpha, \beta)_{\text{in } x_1 \dots x_n}$ change coordinates: $x_1 = \varphi_1(y_1 \dots y_n)$, $\vdots$, $x_n = \varphi_n(y_1 \dots y_n)$

→ now in $y_1 \dots y_n$

→ changed coordinates α, β into $y_1 \dots y_n$

Vector Fields and Infinitesimal Transformation.

- $\begin{cases} x_1 = \varphi_1(y_1, \dots y_n) \\ \vdots \\ x_n = \varphi_n(y_1, \dots y_n) \end{cases}$ a transformation from $x_1, \dots x_n$ to $y_1, \dots y_n$.
- Infinitesimal: $x_1 \simeq y_1, \dots, x_n \simeq y_n$

$$\begin{cases} x_1 = y_1 + \epsilon \varphi_1(y_1 \dots y_n), \text{ where } \epsilon \simeq 0 \ (\epsilon^2 = 0). \\ \vdots \\ x_n = y_n + \epsilon \varphi_n(y_1 \dots y_n) \end{cases}$$

- $x_1 = \hat{x}_1 + \epsilon \varphi_1(\hat{x}_1, \dots \hat{x}_n)$
  $\vdots$
  $x_n = \hat{x}_n + \epsilon \varphi_n(\hat{x}_1, \dots \hat{x}_n)$
- $f(x_1, \dots x_n) \rightarrow f(\hat{x}_1 + \epsilon\varphi_1, \dots \hat{x}_n + \epsilon \varphi_n)$

Taylor Expansion up to ϵ ($\epsilon^2 \simeq 0$).

$f(\hat{x}_1, \dots \hat{x}_n) + \frac{\partial f}{\partial x_1} \cdot \epsilon\varphi_1 + \dots + \frac{\partial f}{\partial x_n} \epsilon \varphi_n + \dots$

$= f + \epsilon\left(\varphi_1 \frac{\partial f}{\partial x_1} + \dots + \varphi_n \frac{\partial f}{\partial x_n} + \dots\right)$

Since $\hat{x}_n$ is multiplied by ϵ, $\hat{x}_n$ is indistinguishable from x_n.

- $\epsilon x_1 = \epsilon \hat{x}_1 + \cancel{\epsilon^2 \varphi_1} \Rightarrow \epsilon x_1 = \epsilon \hat{x}_1$
  $\vdots$
  $\epsilon x_n = \epsilon \hat{x}_n + \cancel{\epsilon^2 \varphi_n} \quad \epsilon x_n = \epsilon \hat{x}_n$
- $= f + \epsilon v(f)$ where $v = \varphi_1 \frac{\partial}{\partial x_1} + \dots \varphi_n \frac{\partial}{\partial x_n}$

$$v(f) = \lim_{\epsilon \to 0} \frac{f(x_1 + \epsilon\varphi_1, \dots x_n + \epsilon\varphi_n) - f(x_1, \dots x_n)}{\epsilon}$$

- Ex. $x^2 y \frac{\partial}{\partial x} - x e^{xy^2} \frac{\partial}{\partial y}$

$x = \hat{x} + \epsilon \hat{x}^2 \hat{y}$

$y = \hat{y} - \epsilon \hat{x} e^{\hat{x}\hat{y}^2}$

Lie Derivative

The infinitesimal change in v is applied to ψ.

- Notation: $L_v(\psi)$, where $v = \varphi_1 \frac{\partial}{\partial x_1} + \dots \varphi_n \frac{\partial}{\partial x_n}$ is a vector field and ψ is 'anything'.
- ① $v \to$ $x_1 = \hat{x}_1 + \epsilon \varphi_1$
  $\vdots$
  $x_n = \hat{x}_n + \epsilon \varphi_n$

② Make the infinitesimal change of coordinates to ψ.
Denote the result by $\hat{\hat{\psi}}$.

③ Write $\hat{\hat{\psi}} = \psi + \epsilon \underbrace{L_v(\psi)}_{\text{by definition.}}$ (Taylor expand $\hat{\psi}$).

- Ex. $L_{x^2 \frac{d}{dx}}(x^5 dx)$

① $x = \hat{x} + \epsilon \hat{x}^2$

② $x^5 dx \to (\hat{x} + \epsilon \hat{x}^2)^5 \, d(\hat{x} + \epsilon \hat{x}^2)$

$= x^5 + 5\epsilon x^4 \cdot x^2 (dx + \epsilon 2x dx)$

$= x^5 dx + \epsilon(x^5 2x \, dx + 5x^4 x^2 dx)$

③ $= x^5 dx + \epsilon(7x^6 dx)$

$L_{x^2 \frac{d}{dx}}(x^5 dx) = 7x^6 dx.$

Tensor Fields.

- $x_1, \dots x_n$; $dx_1 \dots dx_n$; $\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}$
- $\otimes$ - "all product" / "full product"
  - tensor product
- Λ - wedge product.
- Definition - (Tensor field) A tensor field of type (i,k) is an expression of the form

$$\sum_{1 \le i_1 \dots \le j_k \le n} f(x_1, \dots x_n)\, dx_{i_1} \otimes \dots \otimes dx_{i_i} \otimes \frac{\partial}{\partial x_{j_1}} \dots \otimes \frac{\partial}{\partial x_{j_k}}$$

- $dx_i \wedge dx_j = -dx_j \wedge dx_i$

  $dx_i \otimes dx_j \neq c\, dx_j \otimes dx_i$, iff $i \neq j$

  $dx_1 \otimes dx_2 \neq \pm dx_2 \otimes dx_1$, if x_1, x_2 are independent

  $\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \neq \pm \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}$, if (x, y) are independent.

- Ex $x^2 dx \otimes dx \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}$

  $\sin(x-1) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}$

  $\cos(x^2)\, dx \otimes dx$.

- Ex x, y

  $x^2 y\, dx \otimes dx - (x^3 y^3 + 1)\, dx \otimes dy + e^{x-y^2} dy \otimes dx + e^x dy \otimes dy$ (type 2,0)

  $y^3 e^x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - y \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} + xy^3 \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}$ (type 0,2)

  $xy\, dx \otimes \frac{\partial}{\partial x} - y\, dx \otimes \frac{\partial}{\partial y} + x^2 y\, dy \otimes \frac{\partial}{\partial x} - e^x dy \otimes \frac{\partial}{\partial y}$. (type 1,1)

- Assume: $(a+b) \otimes (c) = a \otimes c + b \otimes c$

  $a \otimes (b+c) = a \otimes b + a \otimes c$

Change of Variables

- $\alpha = x\, dx \otimes dx \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}$, $\sin y = x \Rightarrow dx = \cos y\, dy$, $\frac{\partial}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial}{\partial y}$, $\frac{\partial y}{\partial x} = \cos^{-1}(x)$.

  $= \sin y \cos y\, dy \otimes \cos y\, dy \otimes \left(\frac{1}{\cos y}\right) \frac{\partial}{\partial y} \otimes \left(\frac{1}{\cos y}\right) \frac{\partial}{\partial y} \otimes \left(\frac{1}{\cos y}\right) \frac{\partial}{\partial y}$

  $= \sin y \cos y \left(\frac{1}{\cos y}\right)\left(\frac{1}{\cos y}\right)\left(\frac{1}{\cos y}\right) \cos y\; dy \otimes dy \otimes \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}$.

  $= \frac{\sin y}{\cos y} dy \otimes dy \otimes \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}$.

- $\mu = v\, du \otimes \frac{\partial}{\partial v} + u\, dv \otimes \frac{\partial}{\partial u}$

$u = x^2 + y^2, \quad v = xy$ | $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}$

$du = 2x\,dx + 2y\,dy$ | $\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}$

$dv = y\,dx + x\,dy$.

$2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u} = 1.$ | $\Rightarrow \frac{\partial x}{\partial u} = \frac{x}{2(x^2-y^2)}, \quad \frac{\partial y}{\partial u} = \frac{y}{2(y^2-x^2)}$

$x \frac{\partial x}{\partial v} + y \frac{\partial y}{\partial v} = 1.$ | $\Rightarrow \frac{\partial x}{\partial v} = \frac{-y}{x^2-y^2}, \quad \frac{\partial y}{\partial v} = \frac{x}{x^2-y^2}$

$$\mu = xy(2x\,dx + 2y\,dy) \otimes \left(\frac{-y}{x^2-y^2}\frac{\partial}{\partial x} + \frac{x}{x^2-y^2}\frac{\partial}{\partial y}\right) + (x^2+y^2)(y\,dx + x\,dy) \otimes \left(\frac{x}{2}\frac{1}{x^2-y^2}\frac{\partial}{\partial x} - \frac{y}{2(x^2-y^2)}\frac{\partial}{\partial y}\right)$$

$$= \left(-xy \cdot 2x \cdot \frac{y}{x^2-y^2} + (x^2+y^2)y\frac{1}{2}\frac{x}{x^2-y^2}\right)dx \otimes \frac{\partial}{\partial x} + \left(xy\frac{x}{x^2-y^2} + (x^2+y^2)y\left(-\frac{y}{x^2-y^2}\right)\right)dx \otimes \frac{\partial}{\partial y} +$$

$$\left(xy\,2y\left(\frac{-y}{x^2-y^2}\right) + (x^2+y^2)\frac{x}{2}\frac{x}{x^2-y^2}\right)dy \otimes \frac{\partial}{\partial x} + \left(xy\,2y\frac{x}{x^2-y^2} + (x^2+y^2)x\left(-\frac{1}{2}\frac{y}{x^2-y^2}\right)\right)dy \otimes \frac{\partial}{\partial y}$$

E

Differential Forms as also Tensor Fields.

- $dx \wedge dy = dx \otimes dy - dy \otimes dx$ ①
  $\uparrow$ definition.
- Note: $dx \wedge dy = -dy \wedge dx$ where $\wedge$ is defined by ①
- Proof: $dx \otimes dy - dy \otimes dx = -(dy \otimes dx - dx \otimes dy)$
- Note: $dx \wedge dx = dx \otimes dx - dx \otimes dx = 0$.
- $dx \wedge dy \wedge dz = dx \otimes dy \otimes dz + dy \otimes dz \otimes dx + dz \otimes dx \otimes dy - dy \otimes dx \otimes dz$
  $- dz \otimes dy \otimes dx - dx \otimes dz \otimes dy.$
  $\uparrow$ definition.
- $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = -\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}$, in particular, $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} = 0$
  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}.$

"symmetric product"
- $dx\,dy \overset{\text{def}}{=} dx \otimes dy + dy \otimes dx.$
  $dx\,dy\,dz \overset{\text{def}}{=} dx \otimes dy \otimes dz + dy \otimes dz \otimes dx + dz \otimes dx \otimes dy + dy \otimes dx \otimes dz$
  $+ dz \otimes dy \otimes dx + dx \otimes dz \otimes dy.$
  $\frac{\partial}{\partial x}\frac{\partial}{\partial y} \overset{\text{def}}{=} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}$
- Ex. $x(dx)^2 - y(dx\,dy) + xy^2\,dy.$