Complex Variables

- Number Systems: -1×10^6 BC 1, 2, 3, 4, 5... } with natural numbers
  ↳ Solving linear equations }
  - " with fractional numbers $\frac{1}{n}$
  - inclusion of "Ø" as a number (~1000 A.D)
  - negative numbers
  - rational numbers $\frac{p}{q}$, p, q integers, now any equation $ax+b=0$ could be solved.
- Idea: Generalize (extend) number systems so a given class of problems can be solved inside the number system
- The introduction of i was highly successful
  - solves many practical problems
  - computations that look hard with real numbers become easy with complex numbers
- Next step in generalization: William Rowan Hamilton (Oct 16 1843)
  - Quaterions: $a+bi+cj+dk$, where i, j, k satisfy $i^2=j^2=k^2=-1$ and have non commutative property $ij=-ji$
  - Quaterions allow to describe a single rotation around some axis, Now performing 2 rotations, one after the other, around a different axis are always equivalent to a single rotation around another axis. Quaternions have the remarkable property that if R_1 describes the first rotation, R_2 the second one then $R_3 = R_1 R_2$ describes the 3rd equivalent rotation

Manipulation of Complex Numbers

Addition → • $(2+3i)+(4+5i) = (2+4)+(3i-5i) = 6-2i$

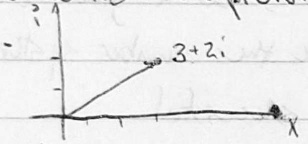
Multiplication → • $(2-i)(3+2i) = 6+4i-3i-2i^2 = 8+i$

Division → • $\frac{(1+i)}{(3+i)} = \frac{(1+i)(3-i)}{(3+i)(3-i)} = \frac{3-i+3i+1}{9-3i+3i+1} = \frac{4+2i}{10} = \frac{2}{5}+\frac{1}{5}i$

• Standard form of complex numbers: $a+bi$

- ∴ closed under addition, multiplication, & divisions
- ∴ complex "field"

• Geometric Interpretation

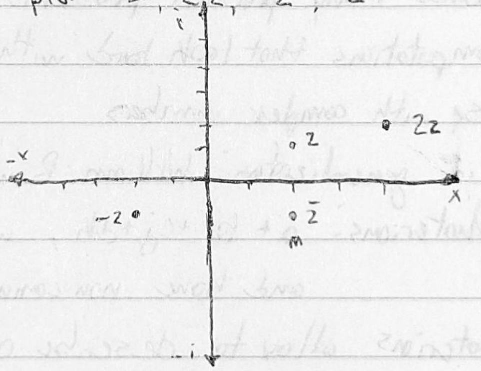


- Given $z = 2+i$, plot z, $2z$, $-z$, $\bar{z}$

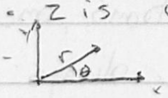
$2z = 4+2i$

$-z = -2-i$

$\bar{z} = 2-i$



- ∴ z is a 2D vector



$r = \sqrt{x^2+y^2}$

- cartesian: $x+iy = z(x,y)$
- polar: $x = r\cos\theta$, $y = r\sin\theta$

$x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

- Before we go any further with $\cos\theta + i\sin\theta$ we introduce the exponential function for complex arguments $e^{i\theta} = \cos\theta + i\sin\theta$ which will make computations much more convenient

• Miracle #1 - $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\cos\theta_2\sin\theta_1 + i^2\sin\theta_1\sin\theta_2$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

is shortened to

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i\theta_1 + i\theta_2} = e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

- Miracle #2 - $\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} = \frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

$$= \frac{\cos\theta_1 \cdot \cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin_2\theta}{\cos^2\theta_2 + i\sin\theta_2\cos\theta_2 - i\sin\theta_2\cos\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{(\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

By using the rule of how to divide exponential functions

$$\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} = \frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i\theta_1 - i\theta_2} = e^{i(\theta_1 - \theta_2)} = \cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

- Other properties can be verified e.g. taylor expansion

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + \frac{i\theta}{1} - \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!}$$

$$= \left[1 - \frac{(i\theta)^2}{2!} + \dots\right] + i\left[\theta - \frac{(i\theta)^3}{3!} + \dots\right]$$

$$= \cos\theta + i\sin\theta$$

- Also $y = e^{i\theta}$ satisfies the ODE

(*) $\frac{dy}{d\theta} = iy$ with $y(0) = 1$

How about $\cos\theta + i\sin\theta$?

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i[\cos\theta + i\sin\theta]$$

and $(\cos\theta + i\sin\theta)\big|_{\theta=0} = 1 + i \cdot 0 = 1$

So $e^{i\theta}$ and $\cos\theta + \sin\theta$ satisfy the same linear ODE with the same initial conditions. A uniqueness theorem says they are the same functions!

- Let us multiply complex numbers using the first miracle & interpret the result geometrically
  - $z = r(\cos\theta + i\sin\theta)$

    $w = s(\cos\phi + i\sin\phi)$

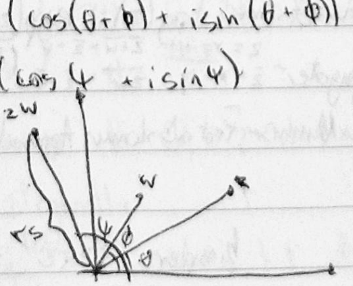
Consider $z \cdot w$

- $z \cdot w = rs\,([\cos\theta\cos\phi - \sin\theta\sin\phi] + i[\cos\theta\sin\phi + \sin\theta\cos\phi])$

$$= rs\,(\cos(\theta + \phi) + i\sin(\theta + \phi))$$

$$= a\,(\cos\psi + i\sin\psi)$$

- Geometrically:



• Definition: - $\arg(z)$ are called all values θ such that $z = r(\cos\theta + i\sin\theta)$

- Is $\arg(z)$ a function? It's not a normal function, it is a multivalue function. We'd like to have unique values so we define.

Some books say $[0, 2\pi]$ $\mathrm{Arg}(z)$

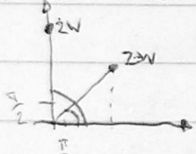
• Definition: - $\mathrm{Arg}(z)$ to be the value of $\arg(z)$ for which $-\pi < \mathrm{Arg}(z) < \pi$. $\mathrm{Arg}(z)$ is the principle value of $\arg(z)$

• Bad News: Later you will have to define your principle value of argument functions to execute applications (to avoid discontinuity at $-\pi, \pi$)

• Can we write 1st miracle: ~~...~~, $r(\cos\theta + i\sin\theta) \cdot s(\cos\phi + i\sin\phi)$
$= rs(\cos(\theta+\phi) + i\sin(\theta+\phi))$ in terms of $\mathrm{Arg}(z)$, $\mathrm{Arg}(w)$, like
$\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$

Reminder: $z = x + iy = r(\cos\theta + i\sin\theta) \Rightarrow r = \sqrt{x^2+y^2}$

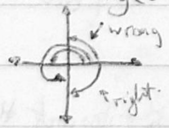
• Ex ① $z = w = 1 + i = \sqrt{1^2+1^2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $\qquad \mathrm{Arg}(z) = \frac{\pi}{4} = \mathrm{Arg}(w)$
$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



$zw = (1+i)^2 = 2i$
$= \sqrt{2}^2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2i$

So, $\mathrm{Arg}(zw) = \frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \mathrm{Arg}(z) + \mathrm{Arg}(w)$

• Ex ② $z = -1$, $w = i$
$\mathrm{Arg}(zw) = \mathrm{Arg}(-i) = -\frac{\pi}{2} \neq \frac{3}{2}\pi = \pi + \frac{\pi}{2} = \mathrm{Arg}(-1) + \mathrm{Arg}(i)$



• Nothing comes free: multival → single val, but addition property destroyed

• What about $\arg(zw) \stackrel{?}{=} \arg(z) + \arg(w)$

We have
$$\left.\begin{aligned} \arg(z) &= \theta + 2n\pi \\ \arg(w) &= \phi + 2m\pi \\ \arg(zw) &= \theta + \phi + 2l\pi \end{aligned}\right\} \quad \{\theta + \phi + 2l\pi\} = \{\theta + 2n\pi\} + \{\phi + 2m\pi\} = \{\theta + \phi + 2(n+m)\pi\}$$

• The definition of other functions $\log(z)$, $\exp(z)$, $\sin(z)$ is built from $\arg(z)$

• Other operations: - Complex Conjugate: $z = x + iy$, $\bar{z} = x - iy$ $\qquad \overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\cdot\bar{w}$, $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

- Powers of Complex Numbers: First and slower approach: $z = r(\cos\theta + i\sin\theta)$
$z^n =$ ~~$r^n(\cos\theta + i\sin\theta)^n$~~ $\to r^n(\cos(n\theta) + i\sin(n\theta))$

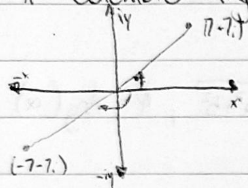
Simpler: $z = re^{i\theta} \to z^n = r^n e^{in\theta}$

• To understand its roots it is sufficient to understand $z^n=1$. Previously $z^{\frac{1}{2}}$ had only one value, now it has $z\,(\pm\sqrt{\ })$. Before looking closer into roots and power let us find out how to find $Arg(z)$.

• To determine $Arg(z)$ where $z=x+iy$

- Sketch z, easily recognizable $Arg(z)$, if not calculate $\arctan(\frac{x}{y})$ & then sketch to get $Arg(z)$

• Ex Calculate $Arg(7+7i)$



graphically: $Arg(7+7i)=\frac{\pi}{4}$

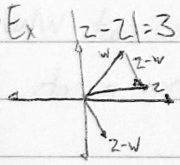
Calculate $Arg(-7-7i)$: $\arctan(\frac{x}{y})=\arctan(\frac{-7}{-7})=\frac{\pi}{4}$

$\therefore$, $Arg(z)=\frac{-3\pi}{4}$

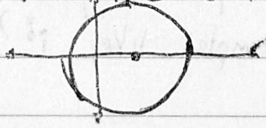
• Sketching sets of points in the complex plane

- often easily recognizable without computation
- if not, substitute $z=x+iy$ and solve the algebraic condition to get $y=y(x)$

• Ex $|z-2|=3$



So $z-w$ is the distance between z & w.



Here $|z-2|=3 \Rightarrow$

• Ex If $z=x+iy$, then we call the real part of z: $Re(z)=x$
and " " " " imaginary part of z: $Im(z)=y$

• Roots & Powers - $z=\alpha^{\frac{1}{n}}$ means all values of z such that $z^n=\alpha$

• Calculate: $1^{\frac{1}{3}}$ (by clumsy method)

$z=1^{\frac{1}{3}} \Rightarrow z^3=1 \Rightarrow z=r(\cos\theta+i\sin\theta) \Rightarrow z^3=r^3(\cos 3\theta+i\sin 3\theta)=1$

$|z^3|=|r^3|\cdot|\cos 3\theta+i\sin 3\theta|=1$ $\qquad |\cos 3\theta+i\sin 3\theta|=\sqrt{\cos^2\theta+\sin^2\theta}=1$

$|r^3|=1$, $r=1=|z|$

So $\cos(3\theta)+i\sin(3\theta)=1+i0$

$\left.\begin{array}{l}\cos 3\theta=1\\ \sin 3\theta=0\end{array}\right\}$ $3\theta=0+2k\pi$ $\quad k=\pm1,\pm2,\pm3\ldots$

$\theta=\frac{2k\pi}{3}$

3 solutions: $k=0,1,2$

- more elegantly

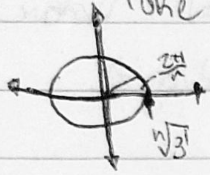
$z^3=r^3e^{3i\theta}=1$, take $|\ |$ of both sides $r^3=1$

$\therefore e^{3i\theta}=1=e^{2k\pi i}$

$\theta=\frac{2k\pi}{3}$, $k=0,\pm1,\pm2\ldots$

• Ex $3^{\frac{1}{n}}$ $\therefore z^n = 3 = r^n e^{ni\theta}$

Take $|x| \Rightarrow 3 = r^n \therefore r = 3^{\frac{1}{n}}$



$e^{n\theta i} = 1 = e^{2\pi k i}$ $\quad \theta = \frac{2\pi k}{n}$ $\quad k = 0, 1, 2 \ldots n-1$

$z = 3^{\frac{1}{n}} e^{\frac{2\pi k}{n} i}$

• Ex More generally $\alpha = a + ib \in \mathbb{C}$, but n still real integer

$z^n = \alpha$, $z = re^{\theta i}$

$\therefore r^n e^{n\theta i} = \alpha$

By writing $\alpha = a + ib = |\alpha| e^{i\phi}$ with $|\alpha| = \sqrt{a^2 + b^2}$, $\phi = \text{Arg}(\alpha)$

$r^n e^{n\theta i} = |\alpha| e^{i(\phi + 2\pi k)}$

$\theta = \frac{\phi}{n} + \frac{2\pi k}{n}$

so $z = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\phi}{n} + \frac{2\pi k}{n}\right)}$, $k = 0, 1, 2 \cdots n-1$

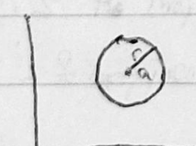
• Restricted for real numbers : $x^3 = 1$ has only one solution. Every complex polynomial has at least one root. When counting multiplicities, every n^{th} degree polynomial has n roots

• What if exponent is complex like, i^i?

— Soon

• Planar Sets

$|z - a| < r$

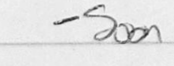


Open set.

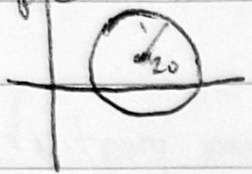
• Open set – around every point one can make a small circle lying in the set.

• Boundary Points – every circle around a boundary point has at least one point outside the set and at least one point inside the set.

•  – Set is not connected.

•  – Set is connected.

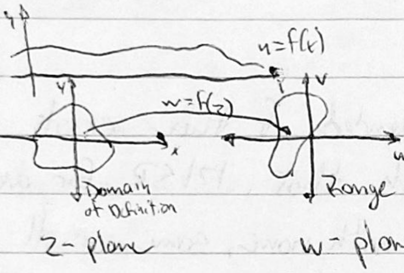
• Example: Punctured Disk $\{z : 0 < |z - z_0| < \rho\}$



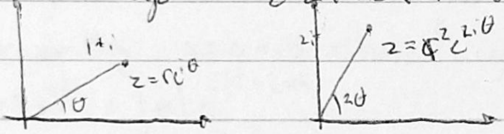
• A set of points is bounded if there exists a ~~positive~~ positive number R such that $|z| < R$ for every z in S. A region is a domain together with none, some, or all boundary points.

Chapter 2: Analytic Functions

- Real Function
- Complex function

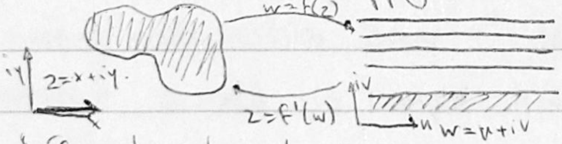


- Ex Find image of $\{z : Re(z) \geq 0, Im(z) \geq 0\}$ under $w = f(z) = z^2$



eg. $(1+i)^2 = 1+2i-1 = 2i$ $\therefore$, $\frac{1}{4}$ plane is mapped to $\frac{1}{2}$ plane.

- There is an intimate connection between analytic ~~functions~~ complex functions and the Laplace $\nabla^2 u = 0$ (more later)
- With a conformal mapping $w = f(z)$, that maps



diff.cult shaped z-domains into simple shaped w-domains we can solve the Laplace equation in the z-domain by solving it in the w-domain and transforming back the solution with $z = f^{-1}(w)$

- For the following introduction of Differentiations everything from real numbers carries over and works even better.
- Sequences:

$z_n \to z$

- Def: $\lim_{n \to \infty} z_n = z$ if for each $\varepsilon > 0$, $\exists$ N: $n > N \Rightarrow |z_n - z| < \varepsilon$
- Def: $\lim_{z \to a} f(z) = l$ iff for each $\varepsilon > 0$, $\exists$ $\delta > 0$, $0 < |z-a| < \delta \Rightarrow |f(z) - l| < \varepsilon$
- Note: In reals $|z-a| < \delta$ is an interval, in complex it becomes a disk.
  - Theorems on limits, like $\lim_{z \to z_0} [f(z) \cdot g(z)] = [\lim_{z \to z_0} f(z)] \cdot [\lim_{z \to z_0} g(z)]$ carrying over from the reals.

Most general complex
Function $f(x,y) = u(x,y) + i v(x,y)$.

Analyticity.

- Arbitrary maps $u(x,y)$, $v(x,y)$ too general to provide ground for nice theory
- Better: put in structure by restricting to functions $f(z)$
- How to recognize that

$x^2 - y^2 + i2xy$ is analytic $(= z^2)$

$x^2 - y^2 + i3xy$ " not " (needs $\bar{z}$)

$e^x(\cos y + i\sin y)$ " " $(= e^z)$?

- Substituting $x = \frac{(z+\bar{z})}{2}$, $y = \frac{(z-\bar{z})}{2i}$ and checking independence of $\bar{z}$ needs simplification which may be too difficult
- Answer: Analyticity := Differentiability (to be) on an open set Ω.

Differentiating Complex Value Functions

- $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ if the limit exists
- $f(z) = z^2$

$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)^2 - f(z)}{\Delta z} = \frac{2z\Delta z + \Delta z^2}{\Delta z} = 2z.$

- Note: In world of reals one can approach a point from 2 sides: Left or right, in complex case we have infinitely many angles & all must give $f'(z)$
- All the usual rules hold
  - $(f+g)' = f' + g'$ $(cf)' = cf'$ $(fg)' = f'g + fg'$

$\left(\frac{f}{g}\right) = \frac{f'}{g} - \frac{fg'}{g^2}$ $f(g(z))' = g'(z) \cdot f'(g(z))$

- How to do $\frac{d}{dz}$ if $f(z)$ is given as $f(z) = u(x,y) + iv(x,y)$
  - $f(z) = x^2 - y^2 + 2ixy$, find $f'(z)$

use $z = x + iy$ $x = \frac{z+\bar{z}}{2}$

$\bar{z} = x - iy$ $y = \frac{z-\bar{z}}{2i}$

$\therefore f(z) = \frac{(z+\bar{z})^2}{4} - \frac{(z-\bar{z})}{4(-1)} + \frac{2i(z^2-\bar{z}^2)}{4i}$

$= \dots = z^2$

$\therefore f'(z) = 2z = 2x - 2iy.$

• Ex $f(z) = x^2-y^2-2ixy$, find $f'(z)$

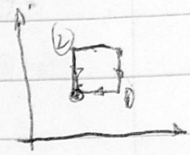
$$= \frac{z^2}{4} + \frac{2z\bar z}{4} + \frac{\bar z^2}{4} + \frac{z^2}{4} - \frac{2z\bar z}{4} + \frac{\bar z^2}{4} - \frac{z^2}{2} + \frac{\bar z^2}{2}$$

$$= \bar z^2$$

$$f'(z) = 2\bar z \cdot \frac{d\bar z}{dz} = ?$$

• Use limit definition with $F(z,\bar z) = \bar z$:

$F' = \lim_{z\to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = \dots = \lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$ when $\Delta z = \Delta x + i\Delta y$



Path ①: $\lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1$

Path ②: $\lim_{\Delta y\to 0} \frac{-i\Delta y}{i\Delta y} = -1$

• ∴ $\frac{d\bar z}{dz}$ is undefined & $\frac{dF(z,\bar z)}{dz}$ is undefined if F depends on $\bar z$

∴ need ~~only~~ fast test to determine if F is differentiable

i.e. $\frac{\partial F}{\partial \bar z} = 0$

• Let us set $f = F(z,\bar z) = u(x,y) + iv(x,y)$

• What are conditions for differentiability on $u(x,y)$, $v(x,y)$?

$x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$

$$\therefore\ 0 = \frac{\partial F}{\partial \bar z} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \bar z} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial \bar z}$$

$$= \frac{\partial(u+iv)}{\partial x}\cdot\frac{1}{2} + \frac{\partial(u+iv)}{\partial y}\cdot\frac{(-1)}{2i}$$

$$= \frac{1}{2}u_x + \frac{1}{2}iv_x + \frac{1}{2}iu_y - \frac{1}{2}v_y = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

• We obtain the so called Cauchy-Riemann equations as necessary conditions for differentiability: $u_x = v_y$, $u_y = -v_x$

• Applied to example

$\underbrace{x^2-y^2}_{u} + i\underbrace{2xy}_{v}$ check CR: $u_x = 2x$, $v_y = 2x$ ✓

$u_y = -2y$, $-v_x = -2y$ ✓

$\underbrace{x^2-y^2}_{u} + i\underbrace{(-2xy)}_{v}$ check C.R. $u_x = 2x$, $v_y = -2x$ ✗

$u_y = -2y$, $-v_x = 2y$ ✗ } CR fails except when $x=y=0$

• Definition - A complex value function $f(z)$ is called analytic on an open set G if it has a derivative at every point of G.

• Necessary and sufficient condition for analyticity

- In an open set G: $f = f(z) = u(x,y) + iv(x,y)$ is defined
  - u_x, u_y, v_x, v_y exists in G and are continuous at z_0
  - CR satisfies z_0

then f is differentiable in z_0

- If - u_x, u_y, v_x, v_y exists in G and are continuous in G
  - CR are satisfied in G

  then $f(z)$ is analytic in G.

- Let us take a closer look at the intricate relationship between differentiability, CR relations, & analyticity.
- Ex Take $f = z\bar{z} = x^2 + y^2$. What do the CR say?

  $u_x = 2x$, $v_y = 0$

  $u_y = 2y$, $v_x = 0$

  $\therefore$ for $x \neq 0$ or $y \neq 0$ the CR rel. are not satisfied $\therefore$ no differentiability.

  What about $x = y = 0$? In an open neighbourhood of $z = 0$ we have:
  - $f = z\bar{z}$ is defined
  - 1st derivates of u, v exist and are continuous
  - C.R. is satisfied in $z = 0$

  $\therefore$ $f = z\bar{z}$ is differentiable in $z = 0$ but in no other neighbouring point, i.e. there is no open neighbourhood of $z = 0$ in which f is differentiable

  $\therefore$ f is not analytic anywhere because because it would have to be differentiable in an open neighbourhood but it is differentiable only in a single point

  The property we are interested in most is the more restrictive property of analyticity (differentiability in an open neighbourhood), not differentiability in a single point

- What equation does v satisfy if C.R. hold?

  $$\left.\begin{aligned} u_x = v_y &\rightarrow \partial_y u_x = \partial_y v_y \\ u_y = -v_x &\rightarrow \partial_x u_y = -\partial_x v_x \end{aligned}\right\} \quad 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \text{ or } v_{xx} + v_{yy} = 0$$

  similarly $u_{xx} + u_{yy} = 0$

  These are the Laplace equations in 2 dimensions.

- Ex. Determine whether $u = x^2$ can be the real part of an analytic function $f = u + iv$

  $\Rightarrow$ Do not try to solve C.R. for v, just test the necessary condition $u_{xx} + u_{yy} = 0$

  Here $u = x^2$ gives $u_{xx} + u_{yy} = 0 \neq 2 + 0 \neq 0$

  $\therefore$ u cannot be the real part of an analytic function

• Ex Given $u = y^3 - 3x^2y$

Find v such that $f = u + iv$ is analytic

Do simple necessary test first.

$$u_{xx} + u_{yy} = (-6xy)_x + (3y^2 - 3x^2)_y = -6y + 6y = 0 \quad \checkmark$$

Now solve C.R. to find v.

$$v_y = u_x = -6xy \qquad ①$$

$$v_x = -u_y = -3y^2 + 3x^2 \qquad ②$$

Integrate ①:

$$v = -\frac{6}{2}xy^2 + g(x)$$

Substitute into ②:

$$v_x = -3y^2 + g'(x) = -3y^2 + 3x^2$$

$$\therefore g'(x) = 3x^2$$

$$g(x) = x^3 + C$$

$$\therefore v = -3xy^2 + x^3 + C$$

$$\therefore f = u + iv = y^3 - 3x^2y + i(-3xy^2 + x^3 + C)$$

• Definition: A real-valued function $\phi(x,y)$ is said to be harmonic in a domain D if all its 2^{nd} order partial derivates are continuous in D and if at each point in D, ϕ satisfies $\phi_{xx} + \phi_{yy} = 0$.

• For any analytic function $f(z) = u(x,y) + iv(x,y)$ the real and imag. parts both satisfy the Laplace equation. It's shown later; u, v have continuous partial derivatives of all orders, so u, v are harmonic, $f(z)$ defines a family of curves $u(x,y) = \text{const.}$ and family of curves $v(x,y) = \text{const}$, called 'level curves' or 'isotherms'

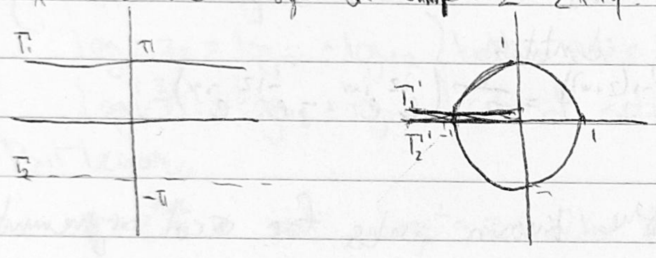
Geometric Property of Level Curves

• Orthogonality $u = \text{const} \perp v = \text{const}$ because $\vec{\nabla}u \perp \vec{\nabla}v$

as $(\vec{\nabla}u)\cdot(\vec{\nabla}v) = \begin{pmatrix} u_x \\ u_y \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = u_x v_x + u_y v_y \underset{C.R.}{=} v_y v_x - v_x v_y = 0.$

Elementary Functions

- Recall: $\arg(z) = \operatorname{Arg}(z) + 2\pi k$.
  $e^{z} = e^{x+iy} = e^{x}(\cos x + i\sin y)$.
  $e^{z} = 1$ iff $z = i2\pi k$
  $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + i2\pi k \Rightarrow e^{z+i2\pi k} = e^{z} \Rightarrow e^{z}$ is periodic
- Ex Find the image of stripe $z = \{x+iy : -\pi \le y \le \pi\}$ under $w = f(z) = e^{z}$



z-plane w-plane

Consider images of boundaries T_1, T_2:
T_1: $z = x + i\pi \rightarrow w = e^{z} = e^{x}e^{i\pi}$
T_2: $z = x - i\pi \rightarrow w = e^{z} = e^{x}e^{-i\pi}$
(also $x = 0$, $-\pi < y \le \pi \Rightarrow w = e^{0}e^{iy}$ = unit circle)

- So z stripe maps to w plane minus origin (later will look at inverse mapping $f^{-1}(w) = \ln z$
- Other functions:
  - We know $e^{i\theta} = \cos\theta + i\sin\theta$ and consequently
    $e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$.
  - All justifications like $e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} \cdots\cdots = \cos(\theta) + i\sin\theta$
  - Carry over to complex numbers: $e^{iz} = \cos z + i\sin z$
    $e^{-iz} = \cos z - i\sin z$
  - Addition, Subtraction - $\sin z = \dfrac{e^{iz} - e^{-iz}}{2}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
- Having the same relations ↑ ↗ as for real arguments we get same differentiation rules & same trigonometric relations for $\sin(z_1 + z_2)$ ...
- The extension of applicability of a real function to complex values is called "analytic function" (a.c.).
- By a.c. further $\cosh = \frac{1}{2}(e^{z} + e^{-z})$, $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$

- So $\cosh(iz) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

$\cos(iz) = \frac{1}{2}(e^{iiz} + e^{-iiz}) = \frac{1}{2}(e^{-z} + e^{z}) = \cosh z$

$\sinh(iz) = \frac{1}{2}(e^{iz} - e^{-iz}) = i\sin z$

$\sin(iz) = \frac{1}{2i}(e^{iiz} - e^{-iiz}) = \frac{-1}{2i}(-e^{-z} + e^{z}) = i\sinh z$

- ∴ Complex variables unmask relationships between elementary function. Previously difficult relations are easy. $e^{z} \cdot e^{w} = e^{z+w}$ gives more complicated trig identities

∴ $\sin(z+w) = \frac{1}{2i}\left(e^{i(z+w)} - e^{-i(z+w)}\right) = \frac{1}{2i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right)$

$= \ldots\ldots$

- Also for $\cos(z+w)$ the known rules for real arguments apply
- $\sinh(z+w) = \frac{1}{i}\sin(iz+iw) = \ldots$
- Also for $\cosh(z+w)$ the known rules for real arguments apply
- The logarithmic function - e^{z} is a single valued but not one to one (each stripe $(2n-1)\pi < y \leq (2n-1)\pi \to$ whole complex plane $/\{0\}$)

Therefore the inverse $\log z$ will be multivalued

$w = \log z$ iff $z = e^{w}$

iff $z = e^{u+iv}$

iff $z = e^{u}e^{iv}$

So definition of log: $u = \mathrm{Log}|z|$ L indicating the single valued real logarithm

$v = \arg(z)$

$= \mathrm{Arg}(z) + 2k\pi$

∴ v is multivalued

- If $z \neq 0$, then $\log z := \mathrm{Log}|z| + i\arg z$

$= \mathrm{Log}|z| + i\,\mathrm{Arg}\,z + i2k\pi$, $k = 0, 1, 2 \ldots$

- Ex $\log(1) = \mathrm{Log}|1| + i\,\mathrm{Arg}(1) + i2k\pi$

$= 0 + i0 + i2k\pi$, $k = \pm 0, 1, 2 \ldots$

- Ex $\log(1+i) = \mathrm{Log}|1+i| + i\arg(1+i)$

$= \mathrm{Log}\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$

$= \frac{1}{2}\mathrm{Log}\,2 + i\left(\frac{\pi}{4} + 2k\pi\right)$, $k = \pm 0, 1, 2 \ldots$

- Eg $\log i = \mathrm{Log}|i| + i(\mathrm{Arg}(i) + 2k\pi)$
$= \mathrm{Log}\,1 + i(\frac{\pi}{2} + 2k\pi) = i\pi(2k + \frac{1}{2}),\ k = \pm 0, 1, 2$
- Nice properties of multivalued arg:

$\arg z_1 z_2 = \arg z_1 + \arg z_2$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

carry over to log:

$$\left.\begin{array}{l}\log z_1 z_2 = \log z_1 + \log z_2 \\ \log\left(\frac{z_1}{z_2}\right) = \log z_1 + \log z_2\end{array}\right\}\text{both sides represent sets of solution}$$

But beware:

$z = e^{\log z}$ but: $\log e^{z} = z + i2k\pi,\ k = \pm 0, 1, 2 \ldots$

although log is multivalued, all values of $\log z$ are mapped by exp function to z.

General Power Definition.

- $z^{\alpha} = (e^{\log z})^{\alpha} := e^{\alpha \log z} = e^{\alpha(\mathrm{Log}|z| + i\arg z)}$
- Ex: $i^i = e^{i \log i}$
$= e^{i(\mathrm{Log}|i| + i(\mathrm{Arg}\, i + 2k\pi))}$
$= e^{-(\frac{\pi}{2} + 2\pi k)}$, $k = \pm 0, 1, 2 \ldots$, infinitely many real values.
- How many values have z^n and $z^{\frac{1}{n}}$ (n integer)?
- Ex. $z^2 = e^{2(\log|z| + i(\mathrm{Arg} z + 2k\pi))}$
$= e^{2\log|z| + i2\mathrm{Arg} z + i4k\pi}$
$= e^{2\log|z| + i2\mathrm{Arg} z}$ since $e^{i4k\pi} = 1$

$\therefore$ unique value.

$\therefore$ z^n has unique value as $e^{i2\pi k m} = 1$

$z^{\frac{1}{q}}$ has q values as there are q different values $e^{i\frac{2k\pi}{q}}$, $k = 0, 1, \ldots q-1$

$\therefore$ $z^{\frac{m}{q}} = (z^{\frac{1}{q}})^m$ has q values

$\therefore$ z^{α}, α transcendental real (not rational) $\Rightarrow \infty$ many values

$\therefore$ z^{w}, w has imag part $\Rightarrow \infty$ many values

- Analogous to Arg we will call the single-valued complex logarithm Log:

$\mathrm{Log}\, z := \mathrm{Log}|z| + i\,\mathrm{Arg} z$

single-valued complex logarithm; single-valued real logarithm; single valued argument function.

• Log z is principle value of log z

$\text{Log}(i) = \text{Log}|i| + i\,\text{Arg}\,i = \text{Log}\,1 + i\frac{\pi}{2} = i\frac{\pi}{2}$

principle value

• Similarly principal value of power defined through $z^{\alpha} = e^{\alpha \text{Log}(z)}$

• Ex. p.v. of $i^i = e^{i\text{Log}(i)} = e^{ii\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$

• Single valuedness has its price:

$\text{Arg}(zw) \neq \text{Arg}(z) + \text{Arg}(w)$

$\text{Log}(zw) \neq \text{Log}(z) + \text{Log}(w)$

where both identities hold for arg and log

Differentiation of $\log z = w$

• $z = e^w$ ∴ $\frac{1}{\frac{dw}{dz}} = \frac{dz}{dw} = e^w = z$

∴ $\frac{dw}{dz} = \frac{d\log z}{dz} = \frac{1}{z}$

• Also Log z is analytic (differentiable in open domain) in the whole complex plane expect on non-positive real axis

orbits
↓
• $z_0, f(z_1), f(f(z_1)), \dots$

i) Show that if $f(z)$ is analytic in a neighbourhood of $z = \zeta$

ii) $f(\zeta) = \zeta$

iii) If $|f'(\zeta)| < 1$

then there is a disk around ζ with the property that all orbits launched from inside the disk remain confined to the disk and converge to ζ

Solution:

$\lim_{z \to \zeta} \left| \frac{f(z) - f(\zeta)}{z - \zeta} \right| = |f'(\zeta)| < 1$ and $f(\zeta) = \zeta$, we can pick a number ρ lying between $|f'(\zeta)|$ and 1 such that $|f(z) - f(\zeta)| = |f(z) - \zeta| \le \rho |z - \zeta|$. Then we have

$|z_n - \zeta| \le \rho |z_{n-1} - \zeta| \le \dots \le \rho^n |z_0 - \zeta|$

$\Rightarrow 0$ for $n \to \infty$

$f(\zeta) = \zeta \Rightarrow \zeta$ is a fixed point if meeting previous conditions $\Rightarrow$ attractor; otherwise $\Rightarrow$ repellor.

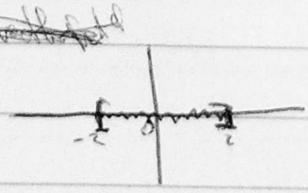
• Ex $f(z) = z^2$

$\zeta^2 = \zeta = 0$ $\Rightarrow$ is an attractor

$\zeta^2 = \zeta = 1$ $\Rightarrow$ is repellor

• Julia set – The filled Julia set for a polynomial function, $f(z)$ is defined to be the set of points that launch bounded orbits through iteration of f. The Julia set is the boundary of the filled Julia set.

• $f(z) = z^2 - z$



How does our definition of the derivative of a complex valued analytic function $f(z) = u(x,y) + iv(x,y)$ compare to the derivative of a 2-D vector function, $\vec{F}(x,y) = (Re(z), Im(z))^T = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$?

In other words, does this commutativity diagram hold?

$$\begin{array}{ccc} f(z) \in \mathbb{C} & \xrightarrow[\text{differential}]{\text{total}} & df = \frac{df}{dz}\, dz \\ \Big\updownarrow{\scriptstyle \text{2-D form}} & & \\ \vec{F}\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R} & \xrightarrow[\text{differential}]{\text{total}} & d\vec{F} = \nabla \vec{F} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} \overset{?}{=} \frac{df}{dz}\, dz \end{array}$$

① $\downarrow \longrightarrow$ We have $\vec{F}(x,y) = \vec{F}(x_0 + \Delta x, y_0 + \Delta y) = \vec{F}(x_0,y_0) + \nabla\vec{F}(x_0,y_0) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \ldots$

$$= \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \ldots$$

$$= \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} u_x \Delta x + u_y \Delta y \\ v_x \Delta x + v_y \Delta y \end{pmatrix}\Bigg|_{\substack{x = x_0 \\ y = y_0}} + \ldots$$

② $\longrightarrow\downarrow$ With $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$ we get $\partial z = \frac{\partial x}{\partial z}\partial x + \frac{\partial y}{\partial z}\partial y = \frac{1}{2}\partial x + \frac{1}{2i}\partial y$

and $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = \frac{1}{2}u_x + \frac{1}{2i}u_y + i\left(\frac{1}{2}v_x + \frac{1}{2i}v_y\right) = \frac{1}{2}\left(u_x + v_y + i(v_x - u_y)\right)$

So $\frac{\partial f}{\partial z}\Delta z = \frac{1}{2}[\qquad] \cdot (\Delta x + i\Delta y)$

$$= \frac{1}{2}\left[(u_x + v_y)\Delta x - (v_x - u_y)\Delta y + i\{(v_x - u_y)\Delta x + (u_x + v_y)\Delta y\}\right]$$

In general it doesn't work. But let's find where it does.

OR $u_x = v_y$, $u_y = -v_x$. Because of the Cauchy-Riemann conditions we see that the property holds for only complex analytic functions.

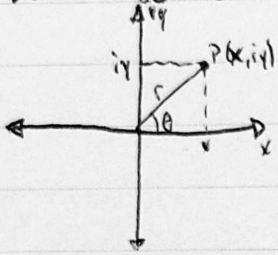
Classes of Numbers

- Natural - $\mathbb{N}$ - $1, 2, 3 \ldots$
- - $\mathbb{Z}$: $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$
- Rational - $\mathbb{Q}$ - $\left\{ \frac{a}{b},\ a, b \in \mathbb{Z},\ b \neq 0 \right\}$
- Real - $\mathbb{R}$ - $\left\{ ax^2 \pm b,\ a, b \in \mathbb{R} \right\}$
- Imaginary - $\mathbb{C}$ - If $z \in \mathbb{C}$, $z = a + bi$; $a, b \in \mathbb{R}$
- For every polynomial $P(x) \in \mathbb{C}[x]$ of degree n, $P(x) = 0$ has a solution in $\mathbb{C}$. It has exactly n solutions in $\mathbb{C}$
- Ex. $2x^3 + 3x^2 + (1 + 2i)x - 3 = 0$ has 3 solutions (may not be distinct)
- A complex number $z \in \mathbb{C}$, can be written in the form
  - $z = a + ib$; $a, b \in \mathbb{R}$
  - $i^2 = -1$.

Polar Form of Complex Numbers

$z = x + iy$
$= r\cos\theta + i r\sin\theta$

- Let P be a point in the complex plane, corresponding to the complex number (x, y) or $x + iy$.



$z = x + iy = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$. This is polar form. Also we see that $z = \sqrt{x^2+y^2}$ is the modulus of z. θ is the argument of z (sometimes called the amplitude of z).

-

De Moivre's Formula

- Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Know the $\cos(\alpha+\beta)$ & $\sin(\alpha+\beta)$ identity.

$$z_1 z_2 = (r_1(\cos\theta_1 + i\sin\theta_1)) \cdot (r_2(\cos\theta_2 + i\sin\theta_2))$$
$$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)]$$
$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

- Shows us the modulus is multiplied and the arguments are added

- $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(-\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$
$$= \frac{r_1}{r_2}\cos(\alpha+\beta) + i\sin(\alpha+\beta)$$
$$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

- The converse is true for division.
- $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Roots of Complex Numbers

- If $w^n = z$

- Ex. find w such that $w^3 = 1+i$

$z = r(\cos\theta + i\sin\theta)$

$w = \rho(\cos\alpha + i\sin\alpha)$

$w^n = \rho^n(\cos(n\alpha) + i\sin(n\alpha)) \Rightarrow \rho^n = r \Rightarrow \rho = r^{\frac{1}{n}}$

$\Rightarrow n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta + 2k\pi}{n}$, $k = 0, 1, 2 \ldots n-1$

$w = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$, $k = 0, 1, 2 \ldots n-1$

- Ex. find cubic roots of $z = 1+i$

$w^3 = z$

$z = r(\cos\theta + i\sin\theta)$

$1+i =$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$w = \cancel{2^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)} = 2^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$ $k = 0, 1, 2$

$w_0 = 2^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$

$w_1 = 2^{\frac{1}{6}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{\pi}\right)\right)$

$w_2 = 2^{\frac{1}{6}}\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)$

- Ex find the cubic roots of unity.

$w^3 = 1$.

$z^n = 1$, $z = x + iy$, $z^n = (x+iy)^n = 1$

$z = r(\cos\theta + i\sin\theta)$, $z^n = r^n(\cos n\theta + i\sin n\theta)$

$$\begin{cases} r^n = 1 \\ n\theta = 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

then $\begin{cases} r = 1 \\ \theta = \frac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots n-1 \end{cases}$

$z_k = 1 \cdot \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$, $k = 0, 1, 2 \ldots n-1$

They are on the unit circle

For $n = 3$, $z_1 = 1 \cdot (\cos(0) + i\sin(0)) = 0$

$z_2 = 1 \cdot \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = \ldots$

$z_3 = 1 \cdot \left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = \ldots$

Fundamental Theorem of Algebra.

$p(z) = (z - z_1) \cdot Q(z)$
If $p(z) = 0$, then
$\begin{cases} z = z_1 \\ Q(z) = 0. \end{cases}$

- Let $p(z) \in \mathbb{C}[z]$, a polynomial of degree n with coefficients in $\mathbb{C}$. Then $p(z) = 0$ has at least one solution in $\mathbb{C}$. This is equivalent to saying $p(z) = 0$ has exactly n solution in $\mathbb{C}$. $z^n - 1 = 0$, $p(z) = z^n - 1$. If $p(z)$ is a polynomial of degree, n, with real coefficients of course $p(z) = 0$ has with exactly n solutions.

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_0$, $a_i \in \mathbb{R}$ $i = 0, 1, 2 \dots n$.

If $p(z_1) = 0$

$a_n z_1^n + a_{n-1} z_1^{n-1} + \dots a_1 z_1 + a_0 = 0$

$a_n (\bar{z}_1^n) + a_{n-1} (\bar{z}_1^{n-1}) + \dots a_1 (\bar{z}_1) + a_0 = 0$.

Then $\bar{z}_1$ is also a root of $p(z)$.

- Theorem - If $p(z)$ is a polynomial of degree, n, with real coefficients, then the complex roots are in pairs (if z_1 is a root, the $\bar{z}_1$ is also a root). If 'r' is the number of real roots of $p(z)$ and 's' is the number of totally complex roots then $n = r + 2s$.

Euler Formula.

- By assuming that the infinite series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$ of elementary calculus holds for when $x = i\theta$, then we can arrive at the result. $e^{i\theta} = \cos\theta + i\sin\theta$.
- This is called Euler's Formula for complex numbers.
- $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$
- Then $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos\theta + i\sin\theta)$.

$(e^{i\theta})^n = e^{in\theta}$ and we have De Moivre's Formula.

Dot and Cross Product.

- The dot product (scalar product) ~~is defined~~ of z_1 & z_2 is defined as the real number $z_1 z_2 = x_1 x_2 + y_1 y_2$
$= |z_1||z_2| \cos\theta$, where θ is the angle between z_1 & z_2 which lies between 0 and π.

- Ex. Let $z_1 = 1+i$, $z_2 = 2i$

$z_1 z_2 = 1 \cdot 0 + 1 \cdot 2 = 2$

Do in $|z_1||z_2|\cos\theta$.

- The cross product of z_1 and z_2 is defined as the vector $z_1 \times z_2 = (0, 0, x_1 y_2 - x_2 y_1)$ in perpendicular to the complex plane having modulus $|z_1 \times z_2| = |x_1 y_2 - x_2 y_1| = |z_1| \cdot |z_2| \sin\theta$.
- Ex. Let $z_1 = 1+i$, $z_2 = 2i$

$z_1 \times z_2 = (0, 0, 2 \cdot 1 - 1 \cdot 0) =$
$= (0, 0, 2)$

- Theorem. Let z_1 and z_2 be non-zero, then

(1) A necessary and sufficient condition that z_1 & z_2 be perpendicular is that $z_1 \cdot z_2 = 0$

(2) ~~z_1 & z_2 are parallel if and only if~~ The modulus of the projection of z_1 & z_2 is $\frac{|z_1 z_2|}{|z_2|}$

(3) z_1 & z_2 are parallel iff $|z_1 \times z_2| = 0$.

(4) The area of the parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

Complex Function

- A complex function is a function defined from $\mathbb{C}$ to $\mathbb{C}$
- In real analysis, $f: \mathbb{R} \to \mathbb{R}$
  - Ex $f: \mathbb{R} \to \mathbb{R}$

    $x \to \frac{\sqrt{x^2-1}}{x}$ $\quad x^2-1 \geq 0$, $x \neq 0$
- A complex function has its domain of definition, D_f,
  - Ex $f: \mathbb{C} \to \mathbb{C}$, $D_f = \mathbb{C} \setminus \{0\}$.

    $z \to \frac{1}{z^2}$
- The concept of a limit is as in real analysis central to the development of our subject. Given a complex function, f, and complex numbers l and c, we say that
  - $\lim_{z \to c} f(z) = l$
  - if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - l| < \varepsilon$ for all z such that $0 < |z-c| < \delta$
- Formally, this definition is the same for real analysis as is complex function.
- Remark: The two-dimensional nature of the definition does not introduce complications. For real numbers, there are only two directions along which a sequence of points can approach a real number. However in the complex plane there are infinitely many path towards a point and the existence of limit requires that the limit exist and be the same for every possible path.
- Ex $f: \mathbb{C} \to \mathbb{C}$

  $f(z) = ?$, $z = x+iy$

  $f(z) = z^2+1 = x^2 - y^2 + 2xyi + 1. = (x^2-y^2+1) + 2ixy$

  $= u(x,y) + iv(x,y)$
- A complex function, f, can be written as $f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real functions.
- Ex. Let $f(x+iy) = u(x,y) + iv(x,y)$, where $v(x,y) = 0$ for all x, y & $u(x,y) = \frac{xy}{x^2+y^2}$ $(u(x,y) \neq (0,0))$, $\lim_{z \to 0} f(z) = \to$

  $\lim_{x \to 0} f(x+i0) = 0$ $\quad$ $z = re^{i\theta}$, $\theta = \frac{\pi}{4}$ $\quad$ $\lim_{r \to 0} f(re^{i\theta}) = \frac{1}{2}$.. for all values of r, then

  $\lim_{y \to 0} f(0+iy) = 0$ $\quad$ $= r(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$ $\quad$ the limit does not exist

  $f(z) = \frac{r^2\cos\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos\theta\sin\theta$.

• Remark: In that example, we obtained different limiting values when 0, was being approached from different angles by a straight line paths. It might be thought that the $\lim_{z\to 0} f(z)$ would exist if all straight line paths led the same result. But this is not true.

• Ex. Let $f(x+iy) = u(x,y) + iv(x,y)$, where $v(x,y)=0$ for all x, y and $u(x,y) = \frac{xy^2}{x^2+y^4}$, $((x,y)\neq(0,0))$. ① Prove that $\lim_{r\to 0} f(re^{i\theta})$ exists. ② Prove that $\lim_{z\to 0} f(z)$ does not exist.

①
$$f(re^{i\theta}) = \frac{r\cos\theta\,(r\sin\theta)^2}{r^2\cos^2\theta + r^4\sin^4\theta} = \frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta}$$

$$|f(re^{i\theta})| = \left|\frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta}\right|$$

$$\leq \left|\frac{r^3}{r^2\cos^2\theta}\right| = \left|\frac{r}{\cos^2\theta}\right| < \varepsilon$$, when r is very small whenever $r < \varepsilon\cos^2\theta$.

We proved that when θ in fixed θ, is $f(re^{i\theta}) = 0$

For every $\varepsilon > 0$, there exists $\delta > 0$, such that $|r| < \delta$ implies $|f(re^{i\theta})| < \varepsilon$. We proved that whenever $|r| < \varepsilon\cos^2\theta$ then $|f(re^{i\theta})| < \varepsilon$ then $\lim_{r\to 0} f(re^{i\theta}) = 0$.

② Let $z = y^2 + iy$. If $y \to 0$, then $z \to 0$.

$$\lim_{y\to 0} f(y^2+iy) = \lim_{y\to 0} \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$ Thus the $\lim_{z\to 0} f(z)$ does not exist.

• Ex. Prove that the function $f: z \to |z|^2$ is continuous at every point c.

$$f'(c) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{|z+h|^2 - |z|^2}{h}$$

$$= \frac{|x+iy+h| - |x+iy|^2}{h}$$

$$= \frac{(x+iy+h)^2 - (x+iy)^2}{h}$$

$$= \frac{\cancel{x^2} - \cancel{y^2} + \cancel{2ixy} + 2xh + 2iyh - \cancel{x^2} - \cancel{2ixy} + \cancel{y^2}}{h}$$

$$= 2x + 2iy.$$

$$f(z) = \begin{cases} z, & \text{if } z \geq 0 \\ -z, & \text{if } z < 0 \end{cases}$$, let $z = x+iy$.

$(x+iy+h)(x+iy+h)$

$x^2 + iyx + xh + iyx - y^2 + iyh + xh + ihy + h^2$

$x^2 - y^2 + 2iyx + 2xh + 2iyh$

$(x+iy)(x+iy)$

$x^2 + 2ixy - y^2$

• Remark - The 'standard' calculus of limits familiar with real functions, applies also to complex functions, and the proofs are formally identical.

• If $\lim_{z \to c} f(z) = l_1$ and $\lim_{z \to c} g(z) = l_2$, then, $Kf(z)$, $f(z) \pm g(z)$, $f(z) \cdot g(z)$, and (provided $l_2 \neq 0$) $\frac{f(z)}{g(z)}$. Have limits respectively Kl_1, $l_1 + l_2$, $l_1 l_2$, and $\frac{l_1}{l_2}$. Also, the continuity of f and g at c implies the continuity of Kf, $f \pm g$, $f \cdot g$, and (when $g(c) = 0$) $\frac{f}{g}$.

The O and o Notations.

• Big O, small o

• Let f and Φ, be complex functions. Suppose that Φ is in some way "better known" than f.

• $f(z) = O(\Phi(z))$ as $z \to \infty$ meaning there is a positive constant K such that $|f(z)| \le K|\Phi(z)|$ for all z sufficiently large $|z|$.

• Ex. $z^3 + 30 = O(z^3)$, $\quad z^3 = O(z^3+30)$

$|z^3 + 30| \le 2|z^3|$, when $|z|$ is very large

$|z^3 + 30| \le K|z^3|$, for $|z|$ very large

$|z^3 + 30| \le |z^3| + |30| \le |z^3| + |z^3| = 2|z^3|$

$z^3 = O(z^3 + 30)$

If $f(z) = O(\Phi(z))$, it does not imply that $\Phi(z) = O(f(z))$.

Let $f(z) = 1$, $\Phi(z) = z$, $|f(z)| \le 1 \cdot |z|$, $f(z) = 1 = O(z)$

But $z \neq O(1)$

$|z| \not\le K \cdot 1$

• $f(z) = O(\varphi(z))$ as $z \to 0$ means that there is a constant $K > 0$, such that $|f(z)| \le K|\varphi(z)|$ for all sufficiently small $|z|$.

• $f(z) = o(\varphi(z))$ as $z \to \infty$ means $\lim_{|z| \to \infty} \left|\frac{f(z)}{g(z)}\right| = 0$.

• $f(z) = o(\varphi(z))$ as $z \to 0$ means $\lim_{z \to 0} \frac{f(z)}{g(z)} = 0$.

• Let, $w^5 = 1$

$w = re^{i\theta}$ to solve. What about $w^5 = z$, where w is a function of z? $z = re^{i\theta} = re^{i(\theta + 2k\pi)}$

$w^5 = re^{i(\theta + 2k\pi)}$

$w = r^{\frac{1}{5}} e^{i\left(\frac{\theta + 2k\pi}{5}\right)}$, $k = 0, 1, 2, 3, 4.$

∴ w can be seen as a collection of 5 single-valued functions. W is a multiple-valued function. w as ~~[illegible]~~ 5 single-valued functions is called branches of the multiple-value function by properly restricting θ.

• For example: $w = r^{\frac{1}{5}}\left(\cos\frac{\theta}{5} + i\sin\frac{\theta}{5}\right)$

The first interval $0 \leq \theta < 2\pi$ is called the principal range of θ and this corresponds to the principal ~~[illegible]~~ branch of the multiple-valued function.

• Let the principal branch be. $w = r^{\frac{1}{5}}\left(\cos\frac{\theta}{5} + i\sin\frac{\theta}{5}\right)$, $0 \leq \theta < 2\pi$

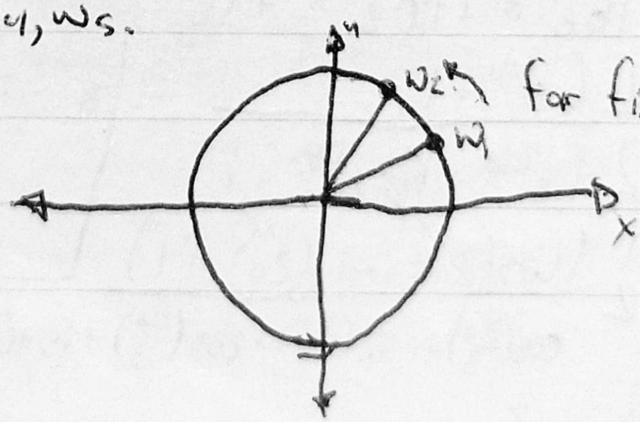
If $w_1 = r^{\frac{1}{5}} e^{i\frac{\theta}{5}}$ w_2, w_3, w_4, w_5.

$w_2 = r^{\frac{1}{5}} e^{i\left(\frac{\theta + 2\pi}{5}\right)}$

$w_3 = r^{\frac{1}{5}} e^{i\left(\frac{\theta + 4\pi}{5}\right)}$

$w_4 = r^{\frac{1}{5}} e^{i\left(\frac{\theta + 6\pi}{5}\right)}$

$w_5 = r^{\frac{1}{5}} e^{i\left(\frac{\theta + 8\pi}{5}\right)}$



for fixed θ.

• If we start at the point ~~z~~ $z=z_1$ in the plane and make one complete cycle counterclockwise around the origin, we see that the value of w ~~...~~ on returning to z_1 is $w_1 e^{i\frac{2\pi}{5}}$. $z \to z_1$, $w_1 = r_1^{\frac{1}{5}} e^{\frac{i\theta}{5}}$, $w_2 = r^{\frac{1}{5}} e^{i\left(\frac{\theta+2\pi}{5}\right)} = w_1 e^{i\frac{2\pi}{5}}$
As θ increases from θ to $\theta_1 + 2\pi$, which is what happens when one complete cycle is made. After 5 complete cycles the value of w is $w_1 e^{\frac{10\pi i}{5}}$. So that the original value of w is obtained after five revolutions about the origin. Because different values of $f(z)$ are obtained by successively encircling $z=0$, we call $z=0$ a branch point. We can restrict ourselves to a particular single-valued function, ~~...~~ ~~...~~ usually the principal branch by insuring not more than one complete cycle about the branch point. In the case of the principal ~~...~~ range $0 \le \theta < 2\pi$, we can construct ~~...~~ a cut, called a branch cut or branch line on the positive real axis. The purpose being that we do not allow ourself to cross this cut. If we do cross the cut, another branch of the function is obtained.

• Remark: If another interval for θ is chosen, the branch line or cut is taken to be some other line in the z plane eminating from the branch point.

• Suppose that $z = e^{w}$, where $z = r(\cos\theta + i\sin\theta)$. Lets define $\ln z = \ln(r(\cos + i\sin\theta))$ $(r \neq 0)$.

$$z = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta = re^{i\theta}$$

$$w = u + iv$$

$$z = e^{u+iv} = e^{u} \cdot e^{iv} = re^{i\theta}.$$

Then, $r = e^{u}$, $e^{iv} = e^{i\theta}$ $\nearrow$ $\begin{cases} r = e^{u} \\ v = \theta + 2k\pi, \ k = 0, \pm 1, \pm 2, \ldots \end{cases}$

$z = e^{w}$, $u + iv$.

$$\ln z = w \Rightarrow w = \ln r + i(\theta + 2k\pi), \ k \in \mathbb{Z}$$
$$= u + iv.$$

An equivalent way of saying that is to write $\ln z = \ln r + i\theta$, θ can assume infinitely many values that differ by 2θ.

• It follows that $\ln z$ is an infinitely many-valued function of z with infinitely many branches. The particular branch of $\ln z$ which is real when z is real and positive is called the principal branch.

• Note: The function $\ln(z-a)$ has $z = a$ as the branch point.

• Ex. $\ln(1-i)$

- If f is a complex function we say f is continuous if $\lim_{z \to c} f(z) = f(c)$
- $\lim_{z \to c} f(z) = l$. If for every $\epsilon > 0$, there exists $\delta > 0$, such that whenever $|z-c| < \delta$, we have $|f(z) - l| < \epsilon$.

} This is used in the $f: z \to \mathbb{R}^R$ solution.

Differentiability.

- f is said to be differentiable at a point c in $\mathbb{C}$ if $\lim_{z \to c} \frac{f(z)-f(c)}{z-c}$ exists. This is called the derivative of f at c, $f'(c)$.
- Like real functions, polynomials are differentiable at every point in the plane, and a rational function $f(z) = \frac{p(z)}{g(z)}$ are differentiable at every point except where $g(z) = 0$.
- Ex. $f(z) = z^2$ $\quad \mathbb{C} \to \mathbb{C}$

$$\lim_{z \to c} f(z) \frac{f(z)-f(c)}{z-c} = \lim_{z \to c} \frac{z^2 - c^2}{z-c} = \lim_{z \to c} \frac{(z-c)(z+c)}{z-c} = \lim_{z \to c} (z+c) = 2c$$

- Let $z = x+iy$, $f(z) = u(x,y) + iv(x,y)$.

$\lim_{z \to c} \frac{f(z)-f(c)}{z-c}$ $\quad c = a+ib$, $z \to a+ib$, $x+iy \to a+ib$

$$= \lim_{z \to c} \frac{f(x+ib) - f(a+ib)}{(x+ib) - (a+ib)} = \lim_{z \to c} \frac{(u(x,b) + iv(x,b)) - (u(a,b) + iv(a,b))}{x-a}$$

$$\Rightarrow \frac{u(x,b) - u(a,b)}{x-a} + i \frac{v(x,b) - v(a,b)}{x-a}.$$

$$= \lim_{x \to a} \frac{u(x,b) - u(a,b)}{x-a} + \lim_{z \to c} i \frac{v(x,b) - v(a,b)}{x-a}$$

- If f is differentiable at $c = a+ib$, then the existence of $f'(c)$ implies the existence of $\lim_{z \to c} \frac{u(x,b)-u(a,b)}{x-a}$ & $\lim_{z \to c} \frac{v(x,b)-v(a,b)}{x-a}$. That is the existence at the point (a,b) of the partial derivatives $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ at (a,b) $\left[f'(c) = \frac{\partial u}{\partial x}\Big|_{(a,b)} + i \frac{\partial v}{\partial x}\Big|_{(a,b)} \right]$. If we keep x fixed at a we will have $\frac{f(a+iy) - f(a+ib)}{(a+iy) - (a+ib)} = \frac{u(a,y) - u(a,b)}{i(y-b)} + \frac{i v(a,y) - v(a,b)}{i(y-b)} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

$\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = i \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$. Then, if f is differentiable at $z = a+ib$ we have:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

These are the Cauchy-Riemann equations.

- If f is differentiable at $z = a+ib$ then the Cauchy-Riemann equations are satisfied for $x = a$, and $y = b$.

• A function f, is continuous at $z=z_0$ if $\lim_{z\to z_0} f(z)=f(z_0)$. For every $\epsilon>0$, there exists $\delta>0$, such that $|z-z_0|<\delta$, then $|f(z)-f(z_0)|<\epsilon$.

• Compute the limits of

① $\lim_{z\to\infty}\frac{4+z^2}{(z-1)^2} \Rightarrow \cancel{\frac{z^2+4}{z^2-2z+1}\ \frac{(z+2i)(z-2i)}{(z-1)^2} \Rightarrow \lim_{z\to\infty}\frac{4+z}{(z-1)^2}} = 4.$

$\lim_{z\to\infty}\frac{\frac{1}{z^2}}{\frac{1}{z^2}}\frac{4+z^2}{z^2-2z+1} = \frac{\frac{4}{z^2}+1}{1-\frac{2}{z}+\frac{1}{z^2}} = 1$

Class soln

$\lim_{z\to\infty}\frac{4+z^2}{(z-1)^2} = \lim_{z\to\infty}\frac{z^2}{z^2}\frac{\frac{4}{z^2}+1}{(1-\frac{1}{z})^2} \Rightarrow$ let $w=\frac{1}{z}$ & the limits change accordingly.

$\lim_{w\to 0}\frac{(2w)^2+1}{(1-w)^2} = \lim_{w\to 0}\left(\frac{(2\cdot 0)^2+1}{(1-0)^2}\right) = 1.$

• f is differentiable at $z=z_0$, if $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists and we put $f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

• We proved the following theorem:

- Theorem: Let $f:\mathbb{C}\to\mathbb{C}$, $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$. If f is differentiable at $z_0=x_0+iy_0$ then

$$\begin{cases} \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \\ \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} \end{cases} \text{. (Cauchy-Riemann).}$$

• For every $\epsilon>0$, there exists $\delta>0$, such that $0<|z-z_0|<\delta$, then $\left|\frac{f(z)-f(z_0)}{z-z_0}-c\right|<\epsilon$. This is equivalent to saying that $\left|\frac{f(z)-f(z_0)-c(z-z_0)}{z-z_0}\right|<\epsilon$, which implies that $f(z)\approx f(z_0)+c(z-z_0)$. Whenever z is in some neighbourhood of z_0.

• Theorem: Let $f(z)=u(x,y)+iv(x,y)$ and $z_0=x_0+iy_0\in\Omega$. Suppose that u_x, u_y, v_x, v_y exists in a neighbourhood of z_0 and are continuous at (x_0,y_0), and suppose CR-equations are satisfied. Then f is differentiable at $z=z_0=x_0+iy_0$.

Convergence Tests

Ratio
Integral
Root

- There exists an r such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$
  - If $r<1$, convergent. If $r>1$, divergent. If $r=1$, inconclusive.
- Root Test
  - Let $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$; where limsup is the limit superior
  - If $r<1$, series is convergent.
  - If $r>1$, series is divergent
  - If $r=1$, the test is inconclusive.
- Integral Test
  - Let $f:[1,\infty)\to\mathbb{R}_+$ be a non-negative monotonically decreasing function such that $f(n)=a_n$
  - If $\int_1^\infty f(x)dx = \lim_{t\to\infty}\int_1^t f(x)dx < \alpha$ then the series converges
  - If the integral diverges, then the series does as well.
- Radius of Convergence
  - ~~a power series~~ the radius of largest disk in which the series converges.

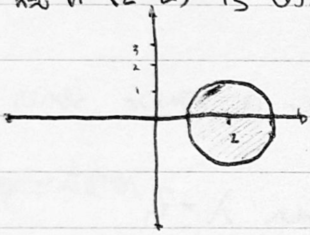
Power Series

- Power Series are functions of a complex variable, defined by $f(z)=\sum_{n=0}^{\infty} a_n z^n$, where (a_n) is a sequence of complex numbers. Or more generally by $g(z)=\sum_{n=0}^{\infty}(a_n)(z-z_0)^n$

$a_n(z-z_0)^n = z_n$

- Ex. $f(z)=\sum_{n=0}^{\infty} 2^n z^n$
  $= 1 + 2z + 4z^2 + 8z^3 + \dots$
- Ex $f(z)=\sum_{n=0}^{\infty} n!\,(z-1)^n$
- $\sum_{n=0}^{\infty} a_n$ is convergent iff $S_N = \sum_{n=0}^{N} a_n$, is convergent as a sequence.
- $\sum_{n=0}^{\infty} z_n$ is convergent iff $S_N=\sum_{n=0}^{N} z_n$. is convergent as a sequence of complex numbers
- If $\sum_{n=0}^{\infty}|z_n|$ is convergent, then we say that $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

- Theorem - Suppose that the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent for $z-a=d$. Then $\sum_{n=0}^{\infty} c_n(z-a)^n$ is absolutely convergent for all z such that $|z-a|<d$.
- Remark - A series that absolutely converges is convergent. The inverse is not true in general.
- Ex. $\sum_{n=0}^{\infty} \frac{1}{n^2}(z-2)^n$ is absolutely convergent for all z, such that $|z-2|<1$.



This theorem tells us that the series $\sum_{n=0}^{\infty} \frac{1}{n}(z-2)^n$ is absolutely convergent for all z inside the circle of center $z=a$ and radius d. But it is not telling us what is happening in the circle.

- Ex. $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

  Since $n! > 2^n$ for $n>3$, $\frac{n!}{2^n} > 1$.

  This means that $\sum_{n=0}^{\infty} \frac{n!}{2^n}$ diverges.

! Proof - We have $\sum_{n=0}^{\infty} c_n d^n$ is convergent as a series of complex numbers the sequence $(c_n d^n) \to 0$ as $n \to \infty$. Then there exists $h>0$ such that $|c_n d^n| < h$, for every $n \geq 0$. Let z such that $|z-a|<d \to \left|\frac{z-a}{d}\right|<1$. $\sum_{n=0}^{\infty}\left(\frac{z-a}{d}\right)^n$ is convergent because the geometric series $\sum_{n=0}^{\infty}\left(\frac{|z-a|}{d}\right)^n$ is convergent.

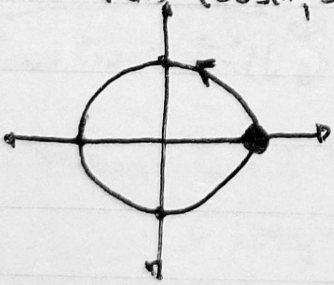
$\sum_{n=0}^{\infty} |c_n(z-a)^n| = \sum_{n=0}^{\infty} |c_n d^n| \left(\frac{|z-a|}{d}\right)^n \leq h\sum_{n=0}^{\infty}\left(\frac{|z-a|}{d}\right)^n$. We know that $\sum_{n=0}^{\infty} \frac{|z-a|^n}{d}$ converges, we are in $|z-a|<d$. Then the comparison test implies the $\sum_{n=0}^{\infty} c_n(z-a)^n$ is absolutely convergent for all of z such that $|z-a|<d$.

- Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ diverges for every $z \in \mathbb{C}$? No, at least for when $z=a$
- Theorem - Let $\sum_{n=0}^{\infty} c_n(z-a)^n$, then we have one of the three following possibilities.
  - ① The series converges everywhere
  - ② The series converges only for $z=a$
  - ③ There exists a positive real number R such that the series converges for all z, $|z-a|<R$, diverges for all z, $|z-a|>R$, and unknown on the circle. R is called the radius of convergence of the series.

- Theorem - Suppose that the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ has a radius of convergence R, then
  ① $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = R$
  ② $\lim_{n\to\infty} c_n^{-\frac{1}{n}} = R$
- Ex. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $c_n = \frac{1}{n!}$, $\Rightarrow \left|\frac{c_n}{c_{n+1}}\right| = \frac{(n+1)!}{n!} = n+1$.
  $\therefore \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \infty$.
- Theorem - Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence R.
  - If ① $\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lambda$, then $\lambda = R$.
  ② If $\lim_{n\to\infty} |c_n|^{-\frac{1}{n}} = \lambda$, then $\lambda = R$.
- Theorem - The power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ and $\sum_{n=0}^{\infty} c_n(z-a)^{n-1}$ have the same radius of convergence.
- Remark - If we differentiate a power series term by term, we do not need to change the radius of convergence.
- Theorem - The power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R \neq 0$, and let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then f is holomorphic within the open disc $V(a,R)$ and $f'(z) = \sum_{n=0}^{\infty} n c_n (z-a)^{n-1}$.

Parametric Representation

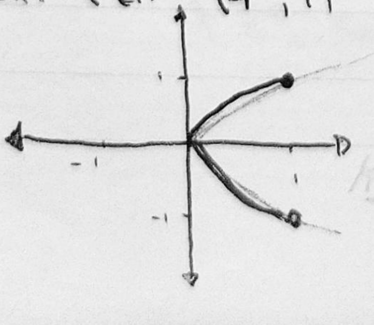
- We will define curves by means of parametric representation. That is a curve, or a path, C is defined as $C = \{(r_1(t), r_2(t)) : t \in [a,b]\}$ where $[a,b]$ is an interval, and r_1 and r_2 are two real continuous functions with domain $[a,b]$. This has some advantage over the standard approach $y = f(x)$, $C = \{(x, f(x)), x \in I\}$. ① There are problems when the curve becomes vertical, or crosses itself. ② The definition imposes an orientation, which is the direction of travel of a point on the curve as t increases from a to b. We shall find it useful to use vector notation and write $C = \{r(t) : t \in [a,b]\}$. $r(t) = (r_1(t), r_2(t))$.
- If $r(a) = r(b)$, we say that the curve C is closed.
- when the curve does not cross itself maybe when $r(a) = r(b)$, we says that the curve is simple. If $a \le t < t' \le b$ and $|t'-t| < |b-a|$ implies that $r(t) \neq r(t')$ & C is simple.
- Ex. $r(t) = (\cos(t), \sin(t))$ $t \in [0, 2\pi)$



Closed ✓

simple ✓

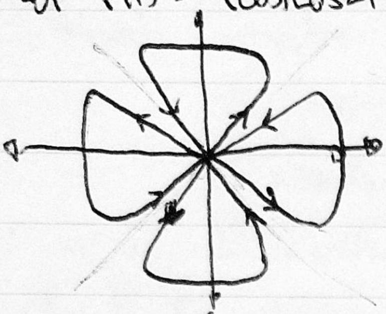
- Ex. $r(t) = (t^2, t)$ $t \in [-1, 1]$



$r(-1) = (1, -1)$, simple ✓

$r(1) = (1, 1)$ not closed ✓.

• Let $r(t) = (\cos t \cos 2t, \sin t \cos 2t)$ $\quad t \in [0, 2\pi)$



$r(0) = (\cos 2, 0)$ $\qquad$ simple ✗

$r(2\pi) = (\cos 2, 0)$ $\qquad$ closed ✓

$r(\pi) = (0, 1)$

$r(\frac{\pi}{4}) = r(\frac{3\pi}{4}) = r(\frac{5\pi}{4}) = \dots = 0$.

• Let $C = \{r(t) : t \in [a,b]\}$, where as $(r(t) = r_1(t), r_2(t))$. Suppose that r_1 and r_2 are differentiable and $r_1'(t)$ and $r_2'(t)$ are continuous on $[a,b]$, then the length, $\lambda(c)$, of c is given by

$\lambda(c) = \int_a^b \|r'(t)\| \, dt$

$= \int_a^b \sqrt{(r_1'(t))^2 + r_2'(t)^2} \, dt$

• Sometimes we write $\gamma(t) = \alpha(t) + i\beta(t)$

and can write it as $(\alpha(t), \beta(t))$, $t \in [a,b]$. $\|r(t)\|$ translates $|\gamma(t)|$ and $\gamma^* = \{\gamma(t), t \in [a,b]\}$ is the corresponding curve.

• Ex. Determine the length of the circumference of the circle $\{re^{it} : 0 \leq t \leq 2\pi\}$

$\gamma(t) = re^{it}$, $t \in [0, 2\pi)$.

$\|r'(t)\| = |\gamma'(t)| = |(r\cos t + ri\sin t)'| = |(-r\sin t, r\cos t)|$

$\lambda(c) = \int_0^{2\pi} \sqrt{(r^2\sin^2 t) + (r^2\cos^2 t)} \, dt = \int_0^{2\pi} r \, dt = r[t]_0^{2\pi} = 2\pi r$.

$\lambda(c) = \int_0^{2\pi} |\gamma'(t)| \, dt$.

$\gamma'(t) = rie^{it}$, $|\gamma'(t)| = r|i||e^{it}| = r$

$\lambda(c) = \int_0^{2\pi} r \, dt = 2\pi r$.

• Ex Determine the length of γ^*, where $\gamma(t) = t - ie^{it}$ on $t \in [0, 2\pi]$.

$\gamma(t) = t - i\cos t + \sin t = (t + \sin t, -i\cos t)$.

$\gamma^*(t) = \{(t + \sin t, -\cos t), t \in [0, 2\pi]\}$

$\lambda(c) = \int_0^{2\pi} \sqrt{(t + \sin t)^2 + (-\cos t)^2} \, dt = \int_0^{2\pi} \sqrt{t^2 + 2t\sin t + \sin^2 t + \cos^2 t} \, dt$.

Integration (Complex Integration)

- We aim to define the integral of a complex function along a curve in the complex plane. Let $\gamma:[a,b]\to\mathbb{C}$ be smooth and let f be a complex function whose domain includes γ^* ($\gamma^* = \{(\gamma_1(t),\gamma_2(t)), t\in[a,b]\}$). We define the integral $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. If $g:[a,b]\to\mathbb{C}$, then $g(t)=f(\gamma(t))\cdot\gamma'(t)$ $= u+iv$, and $\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$. We refer to $\int_\gamma f(z)dz$ as the integral of f along γ. If γ is a closed curve, we call it the integral around γ.

- Ex $\int_\gamma z^m dz$, where $\gamma(t) = e^{it}$ $(0\le t\le 2\pi)$

$$\int_0^{2\pi} f(\gamma(t))\cdot\gamma'(t)dt = \int_0^{2\pi}(e^{it})^m(e^{it})'dt = \int_0^{2\pi} ie^{(m+1)it}dt$$

① $\int_0^{2\pi} i(\cos(m+1)t + i\sin(m+1)t)dt$

$= i\int_0^{2\pi}\cos(m+1)t\,dt - \int_0^{2\pi}\sin(m+1)t\,dt$

$= i(0) - (0)$

$= 0$

② If $m\neq -1$, then

$\int_0^{2\pi} ie^{(m+1)it}dt = \left(\frac{i}{(m+1)i}e^{(m+1)it}\right)\Big|_0^{2\pi}$

$= \left(e^{(m+1)2\pi i} - e^{(m+1)(0)i}\right)\frac{1}{m+1} = (1-1)\frac{1}{m+1}$

$= 0$

If $m=-1$, then

$\int_\gamma z^m dz = \int_0^{2\pi} i\,dt = 2\pi i$

$\int_\gamma z^m dz = \begin{cases} 0 & \text{if } m\neq -1 \\ 2\pi i & \text{if } m=-1\end{cases}$

- Theorem - Let $f:[a,b]\to\mathbb{C}$ be continuous, and let $F(x)=\int_a^x f(t)dt$, $\forall x\in[a,b]$, then $F'(x)=f(x)$. If $\Theta:[a,b]\to\mathbb{C}$ is any function such that $\Theta'=f$, then $\int_a^b f(t)dt = \Theta(b)-\Theta(a)$

- Proof - Write $f(t)=g(t)+ih(t)$

$\int_a^x f(t)dt = \int_a^x g(t)dt + i\int_a^x h(t)dt$

⋮

- Theorem - Let $\gamma:[a,b]\to\mathbb{C}$, be a piecewise smooth. Let F be a complex function defined on an open set containing γ^*, and suppose that $F'(z)$ exists and is continuous at each point of γ^*, then $\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a))$

Cauchy's Theorem

Contour – piecewise smooth, simply closed.

- Let γ^* determine by a piecewise smooth function, $\gamma: [a,b] \to \mathbb{C}$, be a contour, and let f be holomorphic in an open domain containing $I(\gamma) \cup \gamma^*$, then $\int_\gamma f(z)dz = 0$
- Theorem: $\gamma: [a,b] \to \mathbb{C}$, $F'(\gamma)$ exists and continuous on γ^*

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

- Ex $\int_{i}^{1-4i} 4z\,dz$.
- Ex. Evaluate $\int_c \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve c given by. a) $\gamma(t) = t^2 + it$

  b) The line from $z=0$ to $z=2i$ and ......

- Ex Cont.

a.) Doesnt depend on t.

$$\int_{3i}^{1-4i} 4z\,dz = 2z^2\Big|_{3i}^{1-4i} = 2(1-4i)^2 - 2(3i)^2$$
$$= 2(1-8i-16) + 18$$
$$= 2 - 16i - 32 + 18$$
$$= 12 - 16i$$

$$\begin{array}{r} 32 \\ -20 \\ \hline 12 \end{array}$$

b.) Line from $3i$ to $1-4i$,

- Ex. Cont. $\bar{z}$

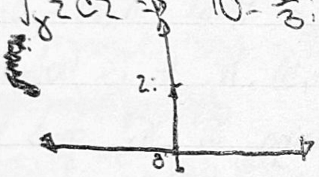
$\int_\gamma \bar{z}\,dz$, $x = t^2$, $y = t$. $t = [0,2]$.

$\int_\gamma \bar{z}\,dz = \int_0^2 f(\gamma(t))\gamma'(t)\,dt$ $\quad \bar{z} = x - iy = t^2 - it$

$$= \int_0^2 (t^2 - it)(2t + i)\,dt$$
$$= \int_0^2 2t^3 + it^2 - i2t^2 + t\,dt$$
$$= \int_0^2 2t^3 + t\,dt + i\int_0^2 -t^2\,dt$$
$$= \left[\tfrac{2}{4}t^4 + \tfrac{t^2}{2}\right]_0^2 + i\left[-\tfrac{t^3}{3}\right]_0^2$$
$$= \tfrac{16}{2} + 2 - i\,8/3$$
$$= 10 - \tfrac{8}{3}i$$

$$\frac{16}{3} - \frac{3}{3} = \frac{13}{3}$$

b) $\int_\gamma \bar{z}\,dz \Rightarrow 10 - \frac{8}{3}i$



$x = 0$, $y = t$. $t \in [0,2]$

$\gamma(t) = ti$

$$\int_\gamma \bar{z}\,dz = \int_0^2 (-ti)(i)\,dt$$
$$= \int_0^2 t\,dt = \left[\tfrac{t^2}{2}\right]_0^2 = 2.$$

- Theorem - Let f be a continuous function in a domain D, then the following are equivalent
  - ① f has an antiderivative in D
  - ② Every loop integral of f in D vanishes, i.e. $\oint_\gamma f(z)\,dz = 0$, for every closed contour γ.
  - ③ The contour integrals of f is independent of path in D.

- Question: What is the antiderivative of $\bar{z}$ in $\mathbb{C}$?

  This function has no antiderivative

- Definition - A domain, D, possessing the property that every loop in D, can be continuously deformed to a point is called a simply connected domain.
- Ex.

simply connected

not simply connected.

simply connected.

not simply connected

- Theorem:- If f is analytical in a simply connected domain and Γ is any loop in D, then $\int_\Gamma f(z)dz = 0$
- Proof - If we know that if γ_1 and γ_2 are obtained by deformation from one to another
  $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$

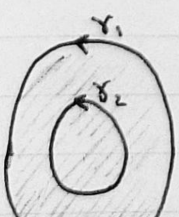
Deformation

- Theorem - Let γ_1 & γ_2 from $[a,b] \to \mathbb{C}$ be piecewise smooth curves such that $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$, $\gamma_1(t) \neq \gamma_2(t)$, $t \in (a,b)$. If f is holomorphic throughout an open set containing γ_1^*, γ_2^* and the region between them then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$. Let $\gamma = \gamma_1 \cup \gamma_2$ then f is holomorphic in D, containing $I(\gamma) \cup \gamma^*$. Then Cauchy's Theorem implies that $\oint_\gamma f(z)dz$, $\int_\gamma f(z)dz = 0$, where γ is the curve starting at A going through γ_1 to B and the from B going to A along $-\gamma_1$. $\int_{\gamma_2 \cup (-\gamma_1)} f(z)dz = \int_{\gamma_2} f(z)dz - \int_{\gamma_1} f(z)dz = 0$. Then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

$I(\gamma) \cup \gamma^*$

$\oint_\gamma f(z)dz = 0$

- Theorem - (The deformation theorem) Let γ_1 and γ_2 be contours with γ_2 lying wholly inside γ_1 and suppose that f is holomorphic in a domain containing the region between γ_1 and γ_2 then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.



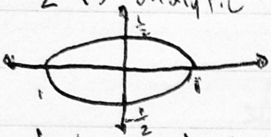
$\Rightarrow$

$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$

- Ex. $\int_\gamma \frac{1}{z}\,dz$, γ is the ellipse defined by $x^2+4y^2=1$.

  ① Parametrization ② If Γ is the unit circle, $\int_\Gamma \frac{1}{z}dz = 2\pi i$

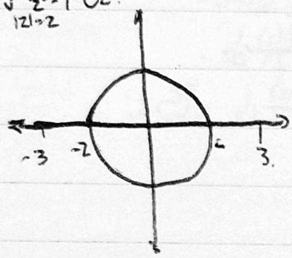
  $\frac{1}{z}$ is analytic on the plane with the origin deleted.



  inside the unit circle

  $$\int_\gamma \frac{1}{z}\,dz = \int_\Gamma \frac{1}{z}\,dz = 2\pi i$$

- Ex $\int_{|z|=2} \frac{e^z}{z^2-9}\,dz$.



  $\frac{e^z}{z^2-9}$ is holomorphic for ~~[illegible]~~ except where $z=\pm 3$. Since $|z|=2$ the $\int_{|z|} \frac{e^z}{z^2-9}dz = 0$.

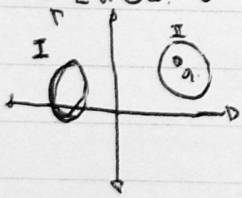
- Some Consequences of Cauchy's Theorem

  - Let $f(z)$ be analytic in a (simply connected) region R, then if a & z are two points in R, we know $\int_a^z f(z)dz$ is independent of the path in R, joining a & z.

- Ex. Evaluate $\oint_\Gamma \frac{1}{z-a}\,dz$, where Γ is a circle not passing through $z=0$, traversed once in the counterclockwise direction.

  ~~[illegible]~~.

  $\oint_\Gamma \frac{1}{z-a}\,dz = 0$.



  I) If the circle is outside $z=a$, the function is analytic on $\Gamma \cup I(\Gamma)$ then $\oint_\Gamma \frac{1}{z-a}dz = 0$

  II) If the circle covers $z=a$, the function $\frac{1}{z-a}$ is not holomorphic on a single point inside $(z=a)$ the $\oint_\Gamma \frac{1}{z-a}dz = 2\pi i$

- Theorem - (Cauchy's Integral Formula) Let Γ be a simple, closed, positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z is any point inside Γ, then

  $f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz$, For $f(z)=1$ example above $\int_\Gamma \frac{f(z)}{z-a}dz = \int_{C_r}\frac{f(z)}{z-a}dz$, C_r $|z-a|=R$

  $$\int_\Gamma \frac{f(z)}{z-a}dz = \int_{C_r}\frac{f(z)}{z-a}dz = \lim_{r\to 0}\int_{C_r}\frac{f(z)}{z-a}dz = \underbrace{\int_{C_r}\frac{f(z)-f(a)}{z-a}dz}_{0} + \underbrace{\int_{C_r}\frac{f(a)}{z-a}}_{f(a)\cdot 2\pi i}$$

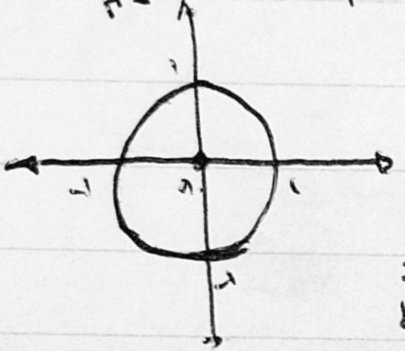
  $\int_\Gamma \frac{f(z)}{z-a}dz = 2\pi i f(z)$, $\int_{C_r}\frac{f(z)-f(a)}{z-a}dz \Rightarrow \left|\int_\Gamma f(z)dz\right| \le \max_{z\in\Gamma}|f(z)| \int_\Gamma |dz|$

  ..... $\lim_{r\to 0}(f(z)-f(a)) = 0$.

  $$f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz \text{ for } n=1,\ldots$$

- Theorem - If f is analytic in D, then all the consecutive derivatives exists ex f', f'', f''' are differentiable

- Ex. $\int_\Gamma \frac{e^{5z}}{z^3}\, dz$, $\Gamma : |z| = 1$.



$a = (0,0)$.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

$$f^{(n)}(a)\, 2\pi i \frac{1}{n!} = \int_\Gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$f^{(2)}(0)\, 2\pi i / 2! = \int_\Gamma \frac{f(z)}{(z-0)^{2+1}}\, dz$$

$$= \int_\Gamma \frac{f(z)}{z^3}\, dz.$$

$$\int_\Gamma \frac{f(z)}{z^3}\, dz = 25\pi i$$