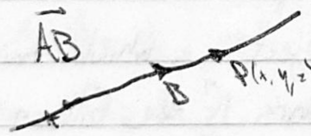
Vector Geometry

- Scalar - real number
- Vector - a quantity with magnitude and direction.
- Representation of Vectors - (in $\mathbb{R}^2 / \mathbb{R}^3$).
  - Arrows - $\vec{a} \neq \vec{b}$
  - Coordinates - $(0,0)$, $(1,1)$ $\vec{n} = (1,1)$. - In general in $\mathbb{R}^2$ $\vec{a} = (a_1, a_2)$.
  - $\vec{a} = \vec{b}$ $\iff$ $a_1 = b_1, a_2 = b_2$
- Geometry of Vectors - Sum of $\vec{F}_1$ & $\vec{F}_2$ $\rightarrow$ $\vec{F}_1 + \vec{F}_2 = (f_{1a} + f_{1b}, f_{2a} + f_{2b})$.
  - $\rightarrow$ Parallelogram Method
  - $\rightarrow$ Tip to Tail Addition.
- Scalar Multiple - $\vec{a}$, $2\vec{a}$, $3\vec{a}$, $\frac{1}{2}\vec{a}$, $-\vec{a}$
  - $\vec{a} \parallel \vec{b}$ - parallel. $\iff$ $\vec{a} = t\vec{b}$
- Vector Subtraction $\vec{b} - \vec{a} = \vec{b} + (-\vec{a})$.
- Ex.1 Find $\vec{w}$ starting at $\vec{a} = (-2,0,1)$ & ends at $\vec{b} = (4,5,2)$.

  $\vec{w} = \vec{b} - \vec{a} = (4,5,2) - (-2,0,1) = (6,5,1)$
- Ex.2



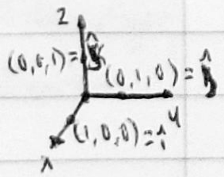
$\vec{AB} = (6,5,1)$, since $\vec{AB} \parallel \vec{AP} \iff \vec{AB} = t\vec{AP}$

$\vec{AP} = (x+2, y, z-1)$

$(x+2, y, z-1) = t(6,5,1) \Rightarrow$ $x = 6t-2$, $y = 5t$, $z = 1t+1$

- Dot Product - $\vec{a} \cdot \vec{b}$ = scalar

  $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.
  - $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$.



- Cross Product - $\vec{a} \times \vec{b}$ = vector
  - $\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
  - $\vec{a} \times \vec{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$
  - Produces perpendicular vector.

Darts before 10: ||||

Master These Concepts

- Determinants (Computation)
- Solving Linear Systems.

Euclidean n-space.

- $\mathbb{R}^n = \{(a_1, a_2, \ldots a_n) \mid a_1, a_2, \ldots a_n\}$ (n-tuple)
- Define - $\vec{a} = (a_1, a_2, \ldots, a_n)$, $\vec{b} = (b_1, b_2, \ldots, b_n)$
  1. $\vec{a} = \vec{b} \iff a_1 = b_1, a_2 = b_2, \ldots a_n = b_n$
  2. $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n)$.
  3. $K\vec{a} = (Ka_1, Ka_2, \ldots Ka_n)$
  4. $\vec{0} = (0, 0, 0)$
  5. $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Online Def. 4.1 Vector Space.

$\mathbb{R}(+) \to$ general V.S.
$\vec{0} = (0, 0 \ldots 0)$
$\vec{v} = (a_1, a_2, \ldots a_n)$
$-\vec{v} = (-a_1, -a_2, \ldots -a_n)$.

- V - Vector Space
- $\vec{u}, \vec{v}, \vec{w}$ - vectors
- Scalars - number (Real #) < Real Vector Space / Complex Vector Space
- Let V be an arbitrary nonempty set of objects on which two operations are defined: addition and multiplication by scalars. If the following axioms are satisfied by all objects $\vec{u}, \vec{v}, \vec{w}$ in V and all scalars K, m, then we call V a vector space

1. If objects $\vec{u}$ and $\vec{v}$ are in V, then $\vec{u} + \vec{v}$ is in V
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. There is an object $\vec{0}$ in V, called a zero vector for V, such that $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$
4. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
5. For each $\vec{u}$ in V, there is an object $-\vec{u}$ in V, called a negative of u, such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$
6. If K is any scalar and $\vec{u}$ is any object in V, then $K\vec{u}$ is in V
7. $K(\vec{u} + \vec{v}) = K\vec{u} + K\vec{v}$
8. $(K + m)\vec{u} = K\vec{u} + m\vec{u}$
9. $K(m\vec{u}) = (Km)\vec{u}$
10. $1\vec{u} = \vec{u}$

Examples of Vector Spaces

- ① The set $V = \{\vec{0}\}$ define ① $\vec{0} + \vec{0} = \vec{0}$, 10 properties in VS hold so
  ② $k\vec{0} = \vec{0}$ $V = \{\vec{0}\}$ is a vector space
- ② Let $V = \mathbb{R}^n$ with standard operations of addition and scalar multiplication
  $V = \mathbb{R}^n \Rightarrow$ Vector Space
- ③ Show that the set of V of all 2x3 matrices with real entries is a vector space if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication

A_1: $u = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $u+v = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \in V \Rightarrow A_1$ holds

$v = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

A_2 $u + v = v + u \Rightarrow A_2$ holds

A_3 $u + (v + w) = (u + v) + w \Rightarrow A_3$ holds

A_4 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ zero vector A_4 holds

A_5 $-u = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$ - negative of u $\Rightarrow A_5$ holds

A_6 $ku = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix} \in V \Rightarrow A_6$ holds.

A_{789} similarly A_7, A_8, A_9 holds

A_{10} $1 \cdot u = u \Rightarrow A_{10}$ holds

Thus $V = M_{2\times3}$ is a vector space.

Generalized, $V = M_{n\times m} \Rightarrow$ vector space.

① Vector Space
② Matrix Space
③ Function

- Ex ④ Let V be the set of all real value function defined $[a,b]$. If $f = f(x)$, $g = g(x)$ are real functions and k is a real number. Define $f+g$ and kf to $f+g(x) = f(x) + g(x)$. Then V is a vector space.
  $kf(x) = k(f(x))$

  A_{1-10} hold

Check axioms on assignment.

- ⑤ Let $V = \mathbb{R}^3$ and define addition and scalar multiplication as follows. If $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$ then $\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$, $k\vec{u} = (ku_1, ku_2, 0) \in V$.
  A_{1-9} hold but A_{10} doesn't. V is not a vector space.
  A_{10} $1 \cdot u = u \Rightarrow 1 \cdot (u_1, u_2, u_3) = (u_1, u_2, 0) \neq (u_1, u_2, u_3)$

Properties of Vectors in a Vector Space.

- Theorem 4.11 - Let V be a vector space, $\vec{u}$ in V and K a real number.

  Then (a) $0 \cdot \vec{u} = \vec{0}$ (b) $K \cdot \vec{0} = \vec{0}$ (c) $(-1)\vec{u} = -\vec{u}$

  (a) If $K\vec{u} = \vec{0}$, then $K = 0$ or $\vec{u} = \vec{0}$

  (b) $K\vec{0} + K\vec{0} = K(\vec{0} + \vec{0}) = K\vec{0}$ proof by A_7.

  $$(-K\vec{0} + K\vec{0}) + K\vec{0} = K\vec{0} - K\vec{0}$$

  $$\vec{0} + K\vec{0} = \vec{0}$$

  $$K\vec{0} = \vec{0}$$

- Theorem 4.12 - Let V be a vector space, $\vec{u}$ in V and K a real number.

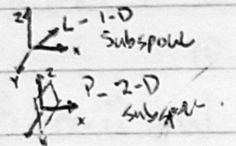
  If $\vec{u} + \vec{v} = \vec{u} + \vec{w}$, then $\vec{v} = \vec{w}$.

  - Corollary 1 - There is exactly one zero vector in this vector space
  - Corollary 2 - There is exactly one negative vector of vector u
  - Suppose O_1 & O_2 are zero vectors

  $$O_1 = O_1 + O_2 = O_2 \quad \therefore \text{ only 1 unique zero vector.}$$

4.2 Subspaces

• In $\mathbb{R}^3$ -



(nonempty)

• Definition - 4.2.1 - A subset $W \neq \emptyset$ of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined in V

$0^{W \neq \emptyset}$

• Theorem 4.2.1 - If W is a subset of one or more vectors from a vector space V, then W is a subspace of V, if the following hold

a) If $\vec{u}, \vec{v}$ are vectors in W, then $\vec{u}+\vec{v} \in W$ (addition closed in W)

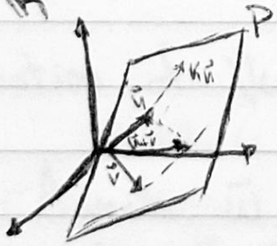
b) If $\vec{u} \in W$, K is a real number, then $K\vec{u} \in W$ (scalar multiplication in V)

Since $W \leq V$ A 2,3,7,8,9,10 are automatically satisfied.

• Proof - If W is a subspace of V, then A1 & A6 hold. Thus a) & b) hold. If a) & b) hold then we have A1 & A6. Since $W \neq \emptyset$ $\exists\, \vec{u} \in W$. $\vec{0} = 0 \cdot \vec{u} \in W$ by b). A4 holds. $-\vec{u} = (-1)\cdot\vec{u} \in W$, A5 holds. $\therefore$, W is a vector space (subspace).

$\exists \to$ There exists

• Ex ① Any plane W through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$



$\vec{u}+\vec{v} \in P$

$\vec{u},\vec{v} \in P$ $\quad \vec{u}+\vec{v} \in P \Rightarrow$ a) holds

$K\vec{u} \in P \Rightarrow$ b) holds

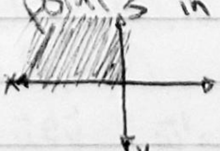
$\therefore P$ is a subspace

$(x, y, z) \in P$

$(x', y', z') \in P \quad \vec{u}+\vec{v} = (x+x', y+y', z+z') \in P$

$K\vec{u} = (Kx, Ky, Kz) \quad \in P$

• Ex ② Let W be the set of all points in $\mathbb{R}^2$ that lie in the 2nd quadrant, that is



$W = \{(x,y) \mid x \leq 0, y \geq 0\}$

Then W is not a subspace of $\mathbb{R}^2$. Note that addition is closed. Multiplication by scalar is not.

• Ex ③ If V is a nonzero vector space, V has at least 2 subspaces.

$\{\vec{0}\}$ V ← trivial.

$\mathbb{R}^3$ $\{\vec{0}\}$ $\mathbb{R}^3$

• Ex ④ Show that $W = \left\{ \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \middle| a_{12}, a_{21} \text{ are real numbers} \right\} \subseteq M_{2\times2}$ is a subspace of $M_{2\times2}$. ∵ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$, ∴ $W \neq \emptyset$. Let $A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$.

$A + B = \begin{pmatrix} 0 & a_{12}+b_{12} \\ a_{21}+b_{21} & 0 \end{pmatrix} \in W$ so a) holds

$kA = \begin{pmatrix} 0 & ka_{12} \\ ka_{21} & 0 \end{pmatrix} \in W$ so b) holds.

By the subspace test, W is a subspace

• Ex ⑤ Let P_k consist of all polynomials of degree $\leq k$.

$P_k = \{ p_k(x) \}$, where $p_k(x)$ is a polynomial of degree k.

If $f(x) + p(x)$ & their degree $\leq k$, their sum will be $\leq k$.

• Ex ⑥ Solution Space of a homogeneous system

System: $A\underline{x} = b$ $\quad A_{m\times n}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_m \end{pmatrix}$

$A\underline{x} = 0$ (*) → homogeneous system, all solutions of (*) form a subspace of $\mathbb{R}^n$

474.61
$+\ 2.569$
477.179

1ai - 4
ii - 4
bi - 1.854×10^{-4}
ii - 1×10^{7}

2a - 477.18
b - −0.2095
c - 7.711×10^{8}

3a - 75.9
b - 3.5×10^{3}

4a i - 3.1177073
ii - 0.19540
iii - 0.6747183
iv - 1.4821

b i 12794
c - i 0.13183496
- ii 791

5a - 3m
b - 0.3km
c - 3000mm
d = ~~[illegible]~~ 0.000457Kg
e - 0.457g
f - 127ml
g = 0.127L

6a - 46800 m^2
b - 30.1935 m^2
c - 301935 cm^2
d - 3.01935×10^{7} mm^2

7 - 10.8mL → 11mL

8a - 4.312821
b - ~~[illegible]~~955 cm^3
c - 3.61 Kg.

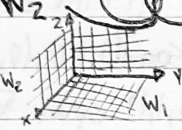
Review

- W is a subspace if ① $\vec{u}+\vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$
  ② $K\vec{u} \in W$ where $\vec{u} \in W$, K is a real number.
- $\vec{u} \in W$, $\vec{0} = 0 \cdot \vec{u} \in W$

Building Subspaces From Given Subspaces

- $W_1 \cap W_2$

- $V = \mathbb{R}^3$



$W_1 \cap W_2$ - x-axis

- Theorem 4.2.2 - If $W_1, W_2, \ldots W_r$ are subspace of V, then $\bigcap_{i=1}^{r} W_i$ is also a subspace of V

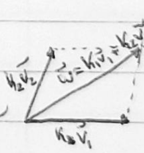
Proving (+) & (K) is closed → ~~Proof.~~ $\cdot\ \vec{u} \in \bigcap_{i=1}^{r} W_i$ if $\vec{u} \in W_i$ for all i

- Proof - Since W_i is a subspace, $W_i \ni \vec{0}$ for every i $(1 \le i \le r)$ so $\vec{0} \in \bigcap_{i=1}^{r} W_i \neq \emptyset$. $\forall \vec{u}, \vec{v} \in \bigcap_{i=1}^{r} W_i \rightarrow \vec{u}, \vec{v} \in W_i\ \forall i$ (since W_i is a subspace) $\vec{u}+\vec{v} \in W_i$ and $K\vec{u} \in W_i$ (for all i). So $\vec{u}+\vec{v} \in \bigcap_{i=1}^{r} W_i$, $K\vec{u} \in \bigcap_{i=1}^{r} W_i$. Thus, $\bigcap_{i=1}^{r} W_i$ is a subspace

$\forall$ - for all

Linear Combination of Vectors

- Definition - A vector $\vec{w}$ is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ if it can be written in the form $\vec{w} = K_1\vec{v}_1 + K_2\vec{v}_2 + \ldots K_r\vec{v}_r$, where $K_1, K_2 \ldots K_r$ are vectors

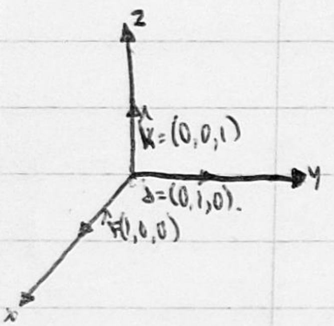


- Example in $\mathbb{R}^2$ - $(3,6) = (3,0) + (0,6) = 3(1,0) + 6(0,1)$
- Ex. Consider vector $\vec{u} = (1,2,-1)$ & $\vec{v} = (6,4,2)$ in $\mathbb{R}^3$. Show that $\vec{w} = (9,2,7)$ is a linear combination of $\vec{u}$ & $\vec{v}$, & $\vec{w}' = (4,-1,8)$ is not.
  - Assume that $\vec{w} = K_1\vec{u} + K_2\vec{v}$ for some scalars K_1, K_2. That is $(9,2,7) = K_1(1,2,-1) + K_2(6,4,2)$
  $(9,2,7) = (K_1 + 6K_2,\ 2K_1 + 4K_2,\ -K_1 + 2K_2)$

$$\begin{aligned} K_1 + 6K_2 &= 9 \\ 2K_1 + 4K_2 &= 2 \\ -K_1 + 2K_2 &= 7 \end{aligned} \quad \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$K_1 = -3$ Thus $\vec{w} = -3\vec{u} + 2\vec{v}$
$K_2 = 2$

Spanning Set.



- Definition - If $\vec{v}_1, \vec{v}_2 \dots \vec{v}_r$ are vectors in a vector space V and if every vector in V is a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_r$, then we say that $\vec{v}_1, \vec{v}_2 \dots \vec{v}_r$, span V. Denotably, $V = \text{span}\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_r\}$, $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_r\}$ - spanning set of V.
- Example in $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$ in $\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

  $(a,b) = a(1,0) + b(0,1)$ $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

  $\mathbb{R}^2 = \text{span}\{(1,0),(0,1)\}$ $\mathbb{R}^3 = \text{span}\{(1,0,0),(0,1,0),(0,0,1)\} = \text{span}\{\hat{i},\hat{j},\hat{k}\}$
- Ex. Determine if $\vec{v}_1 = (1,0,0)$, $\vec{v}_2 = (1,1,0)$, $\vec{v}_3 = (1,1,1)$ span $\mathbb{R}^3$

$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$ Let $\vec{v} = (a,b,c)$ be any vector in $\mathbb{R}^3$. Assume that

$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$

$(a,b,c) = k_1(1,0,0) + k_2(1,1,0) + k_3(1,1,1)$

$a = k_1 + k_2 + k_3$
$b = k_2 + k_3$
$c = k_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$, A has inverse $A\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \to A^{-1}A\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span in $\mathbb{R}^3$

$V \ni \vec{v}_1, \vec{v}_2$ $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0) \in \mathbb{R}^3$

$k_1\vec{v}_1 + k_2\vec{v}_2$ $k_1\hat{i} + k_2\hat{j} = (k_1,0,0) + (0,k_2,0) = (k_1,k_2,0) \subsetneq \mathbb{R}^3$

x, y - plane $\to$ subspace

$\text{Span}\{\hat{i}\,\hat{j}\} = \{(k_1,k_2,0) \mid k_1,k_2 \in \mathbb{R}\}$

in general $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$. $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 \mid k \in \mathbb{R}\} \subseteq V$

- Theorem 4.2.3 - If $S = \{v_1 \dots v_s\}$ is a subset of V, then
  - (a) - $W = \text{span}\{S\} = \{k_1\vec{v}_1 + \dots k_s\vec{v}_s\}$ is a subspace
  - (b) - W is the smallest subspace of V that contains S

① If $\vec{w} = k_1\vec{u}_1 + k_2\vec{u}_2 + \dots k_n\vec{u}_n$, then $\vec{w}$ is a linear combination of $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n$

② $\underset{\text{spanning set}}{\text{Span}\{\vec{u}_1, \dots \vec{u}_n\}} = \{\text{all } k_1\vec{u}_1 + \dots k_n\vec{u}_n\} = \underset{\text{subspace}}{W}$.

• Ex Spanning set for $P_n(x) = \{ a_0 + a_1x + a_2x^2 + \dots a_nx^n \}$

$a_0 + a_1x + a_2x^2 \dots + a_nx^n$

$V \ni \vec{v}_1, \vec{v}_2 = a_0(1) + a_1(x) + a_2(x)^2 + a_n(x^n)$

$-$ l.c. $1, x, x^2 \dots x^n$

$\Rightarrow P_n(x) = \text{span}\{1, x, x^2 \dots x^n\}$

4.3 Linear Independence

- Ex ① Consider $\vec{a}=(2,5,0)$, $\vec{b}=(1,0,0)$, and $\vec{c}=(0,1,0)$. Then $\vec{a}=2(1,0,0)+5(0,1,0)=2\vec{b}+5\vec{c}$. So there is a relation among $\vec{a},\vec{b},\vec{c}$.

  $\vec{a}-2\vec{b}-5\vec{c}=0 \quad (k_1\vec{a}+k_2\vec{b}+k_3\vec{c}=\vec{0})$

  linear independent

- ② Consider $\vec{b}=(1,0,0)$ & $\vec{c}=(0,1,0)$

  $k_1\vec{b}+k_2\vec{c}=0 \iff$

  $(k_1,k_2,0)=(0,0,0) \iff k_1=k_2=0$, only solution is 0(zero) solution

  $\{\vec{b},\vec{c}\}\to$ linearly independent.

- Definition - If $S=\{\vec{u}_1,\vec{u}_2,\dots\vec{u}_r\}$ is a non-empty set of vectors is a vector subspace V, then the vector equation $k_1\vec{u}_1+k_2\vec{v}_2+\dots k_r\vec{v}_r=\vec{0}$ has at least one solution, namely $k_1=0, k_2=0, \dots k_r=0$. If this is the only solution, then S is called linearly independent. If there are other solutions, then S is called linearly dependent.

- ③ The polynomials $p_1=1-x$, $p_2=5+3x-2x^2$, & $p_3=1+3x-x^2$ form a linear dependent set in $P_2(x)$.

  Consider $k_1p_1+k_2p_2+k_3p_3=0$

  $k_1(1-x)+k_2(5+3x-2x^2)+k_3(1+3x-x^2)=0$

  $(k_1+5k_2+k_3)+(-k_1+3k_2+3k_3)x+(-2k_2-k_3)x^2=0$

  $\Rightarrow k_1+5k_2+k_3=0$

  $-k_1+3k_2+3k_3=0$

  $0-2k_2-k_3=0$

  $$\begin{pmatrix} 1 & 5 & 1 & 0 \\ -1 & 3 & 3 & 0 \\ 0 & -2 & -1 & 0 \end{pmatrix}$$

  ANS: $3p_1-p_2+2p_3=0$

- Theorem 4.3.1 - A set S with 2 or more vectors is
  - ⓐ linearly dependent if atleast one of vectors in S is expressible as a linear combination of others.
  - ⓑ linearly independent if no vector in S is expressible as a linear combination of others.

$d = \frac{m}{V}$

$\lambda = \frac{m}{dwh}$

$\lambda =$

9.6
+5.4
15.0
+6
21.0 cal

$\hat{i}, \hat{j}, \hat{k}$ form the standard basis set.

- Proof ⓐ - Assume $S = \{\vec{u}_1, \vec{u}_2, \dots \vec{u}_r\}$ $(r \geq 2)$ is linearly dependent. Thus there are scalars $k_1, k_2, \dots k_r$ not all zero. Such that

  $k_1\vec{u}_1 + k_2\vec{u}_2 + \dots k_r\vec{u}_r = \vec{0}$

  Assume $k_1\vec{u}_1 \neq 0$, $k_1\vec{u}_1 = -k_2\vec{u}_2 \dots k_r\vec{u}_r$

  $\vec{u}_1 = \left(-\frac{k_2}{k_1}\right)\vec{u}_2 - \dots - \frac{k_r}{k_1}\vec{u}_r \rightarrow$ linear combination $\vec{u}_2 \dots \vec{u}_r$

  Assume $\vec{u}_1$ is a linear combination of other

  $\vec{u}_1 = -k_2\vec{u}_2 + \dots k_r\vec{v}_r$

  $1 \cdot \vec{u}_1 + k_2\vec{u}_2 + \dots k_r\vec{u}_r = 0$ since $1 \neq 0, \dots \vec{u}_1, \vec{u}_2, \vec{u}_r$ are linearly dependent

- ⓑ Vectors $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$ in $\mathbb{R}^3$ form a linearly independent set.

  Consider $k_1\hat{i} + k_2\hat{j} + k_3\hat{k} = \vec{0}$

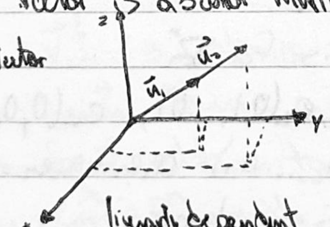
  $(k_1, k_2, k_3) = (0,0,0)$ $k_1 = 0 = k_2 = k_3$

- In General for $\mathbb{R}^n$, $e_1 = (1,0 \dots 0)$, $e_2 = (0,1 \dots 0)$, $e_n = (0,0, \dots 1)$ form a set in $\mathbb{R}^n$

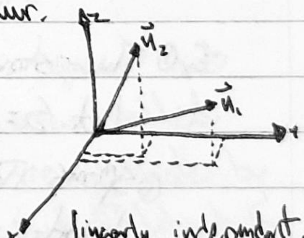
- Theorem 4.3.2 - ⓐ A set S that contains a zero vector is linearly dependent

  ⓑ A set with exactly two vectors is linear if neither vector is a scalar multiple of the other.

- Geometrically - 2 Vector

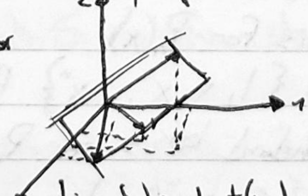


linearly dependent



linearly independent.

- 3 Vector



linearly dependent (coplanar)

linearly independent (not coplanar).

$Q = cm\Delta T$

$\frac{Q}{cm} = \Delta T$

- How many vectors can we find such that these vectors are linearly independent?

  $\mathbb{R}^2$, $\vec{a} = (1,0)$, $\vec{b} = (0,1)$

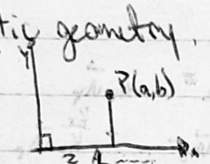
  $\mathbb{R}^3$, $\hat{i}$, $\hat{j}$, $\hat{k}$

  ⋮

- Theorem 4.3.3 - Let $S = \{\vec{u}_1, \vec{u}_2, \dots \vec{u}_r\}$ in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

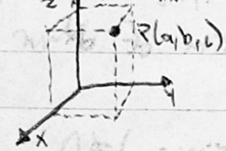
4.4 Coordinates and Bases

- In analytic geometry, we use rectangular systems
  - $\mathbb{R}^2$

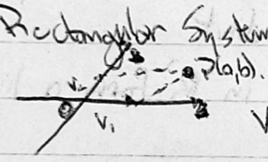


one to one correspondence

  - $\mathbb{R}^3$



- Non-Rectangular System
  -



$V_1 \not\parallel V_2$

  - In algebra, $\mathbb{R}^2$ $\overrightarrow{OP} = a\vec{v}_1 + b\vec{v}_2$

- Definition - (Basis for vector space) If V is a vector space ($V \neq \{0\}$) and $S = \{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$ is a set of vectors on V, S is called a basis for V if the following hold:
  - a) S is linearly independent
  - b) S spans V

- Ex ① The standard basis for $\mathbb{R}^n$
  Let $S = \{e_1 = (1, 0 \ldots 0), e_2 (0, 1, \ldots 0), \ldots e_n (0, 0, \ldots 1)\}$, then S ① spans $\mathbb{R}^n$. As $(a_1, a_2 \ldots a_n) = a_1 e_1 + a_2 e_2 \ldots a_n e_n$ and S is ② linearly independent. S is a basis.

- ② The standard basis for $P_n(x)$
  We can prove $S = \{1, x, x^2, \ldots x^n\}$ forms a basis for $P_n(x)$. $S \rightarrow$ the standard basis for $P_n(x)$
  $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = $ linear combination of $1, x, x^2 \ldots x^n$
  ~~Thus~~ S spans $P_n(x)$

- ③ $\mathbb{R}^3 \rightarrow S = \{\hat{i}, \hat{j}, \hat{k}\}$ standard basis
  Show that $S = \{\vec{v}_1 = (1,0,0), \vec{v}_2 = (1,1,0), \vec{v}_3 = (1,1,1)\}$ is a basis in $\mathbb{R}^3$
  Let $k_1 v_1 + k_2 v_2 + k_3 v_3 = (0,0,0)$
  $k_1(1,0,0) + k_2(1,1,0) + k_3(1,1,1) = (0,0,0)$
  $k_1 = 0 = k_2 = k_3$
  Thus S is a basis.

• ④ Standard basis for M_{mn}

M_{23} $S=\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}_{M_1}\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}_{M_2}\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}_{M_3}\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}_{M_4}\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}_{M_5}\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}_{M_6}\right\}$ – standard basis for M_{23}

$K_1M_1+K_2M_2 \ldots +K_6M_6=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$

$M=\begin{pmatrix}M_{11}&M_{12}&M_{13}\\m_{21}&m_{22}&m_{23}\end{pmatrix}=m_{11}M_1+m_{12}M_2+m_{13}M_3+m_{21}M_4+m_{22}M_5+m_{23}M_6$

• Theorem 4.4.1 – If $S=\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_n\}$ is a basis for a vector space V, then every vector $\vec{v}$ in V, can be expressed in the form $\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2+\ldots c_n\vec{v}_n$ in exactly one way.

• Proof – ① Since S is a basis, S spans V $\forall \vec{v}\in V$

$V=c_1\vec{v}_1+c_2\vec{v}_2+\ldots c_n\vec{v}_n$

② Assume that $\vec{v}=a_1\vec{v}_1+a_2\vec{v}_2 \ldots a_n\vec{v}_n$

②–① $\vec{0}=(c_1-a_1)\vec{v}_1\pm(c_2-a_2)\vec{v}_2 \ldots (c_n-a_n)\vec{v}_n$

Since S is linearly independent, we have $c_1-a_1=c_2-a_2 \ldots =c_n-a_n=0$

so $c_1=a_1$, $c_2=a_2 \ldots c_n=a_n$.

Coordinates Relative to A Basis

• If $S=\{\vec{v}_1,\vec{v}_2\ldots\vec{v}_n\}$ is a basis for vector space V and $\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2+\ldots c_n\vec{v}_n$, then $c_1,c_2,\ldots c_n$ are called the coordinates of $\vec{v}$ relative to S and we denote by $(\vec{v})_S=(c_1,c_2\ldots c_n)\in\mathbb{R}^n$ → the coordinate vector of $\vec{v}$ relative to S

• Ex ① If $V=\mathbb{R}^n$ and S is the standard basis for $\mathbb{R}^n$ then $(\vec{v})_S=\vec{v}$

• ② a) Find the coordinate vector $\vec{v}=(5,-1,9)$ relative to the basis $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ ($\vec{v}_1=(1,2,1)$, $\vec{v}_2=(2,9,0)$, $\vec{v}_3=(3,3,4)$)

b) Find the vector in $\mathbb{R}^3$ whose coordinate vector relative to S is $(\vec{u})_S=(-1,3,2)$

b → $\vec{u}=(-1)\vec{v}_1+(3)\vec{v}_2+2\vec{v}_3=(11,31,7)$

a → Let $\vec{v}=k_1\vec{v}_1+k_2\vec{v}_2+k_3\vec{v}_3$

$(5,-1,9)=k_1(1,2,1)+k_2(2,9,0)+k_3(3,3,4)$

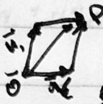
$k_1+2k_2+3k_3=5$ $\quad k_1=1$ $\quad$ Thus $(\vec{v})_S=(1,-1,2)$.

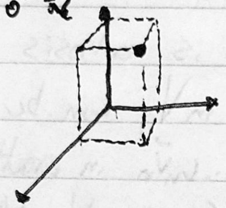
$2k_1+9k_2+3k_3=-1$ $\quad k_2=-1$

$k_1+0+4k_3=9$ $\quad k_3=2$

4.5 Dimension

- 1-D (line) 

$\vec{0}p = k\vec{u}$ Span$\{\vec{u}\}$
- 2-D (plane) 

$\vec{0}p = k_1\vec{u}_1 + k_2\vec{v}$ Span$\{\vec{u}, \vec{v}\}$
- 3-D 
Span$\{\vec{u}, \vec{v}, \vec{w}\}$

- If $S \ni \vec{0}$, S is linearly dependent
  1. $S = \{\vec{u}\}$ is linearly independent iff $\vec{u} \neq \vec{0}$
  2. $S = \{\vec{u}_1, \vec{u}_2\}$ is linearly independent iff $\vec{u}_1 \neq k_2\vec{u}_2$, $\vec{u}_2 \neq k_1\vec{u}_1$

Finite on Infinite Dimensional Vector Space

- Definition - A non-zero space V is called "finite dimensional" if it contains a basis with finitely many vectors. If no such basis, S, exists, V is called "infinite dimensional". In addition, $\{\vec{0}\}$ is called 0-dimensional.
- Ex) R^n: P_n M_{mn} are finite dimensional.
  P = {all real polynomials} is infinite dimensional
  $1, x, x^2, \dots x^n, x^{n+1} \dots$ infinite basis
- Theorem 4.5.1 & 4.5.2 -
  a. If $S = \{u_1, u_2, \dots u_n\}$ is a basis for V, then every set with less than n vectors from V does not span V.
  b. If $S = \{u_1, u_2, \dots u_n\}$ is a basis for V, then every set with more than n vectors from V is linearly dependent.
  c. Any two bases for a "finite dimensional" vector space have the $\underbrace{\text{same number}}_{\text{Dimension of V.}}$ of vectors

Dimensions of Vector Space

- Definition - The dimension of a finite dimensional vector space, $V \neq \{\vec{0}\}$ is defined to be the number of vectors in a basis for V. In addition, if $V = \{\vec{0}\}$, the dimension of $\{\vec{0}\}$ is 0. Denoted by dim{V}.

- Ex ① $\dim\{\mathbb{R}^n\} = n$
  $\dim\{P_n(x)\} = n+1$
  $\dim\{M_{22}\} = 4 \Rightarrow \dim\{M_{mn}\} = m \cdot n$

- Theorem 4.5.4 - ⓐ If $\dim V = n$ and $S = \{u_1, u_2, \ldots u_n\}$ is a set of n, linearly independent vectors of V, then S is a basis for V.
  ⓑ If $\dim V = n$ & $S = \{u_1, u_2, \ldots u_2\}$ spans V, then S is a basis.

- ② $\dim \mathbb{R}^2 = 2$, $S = \{(1,0), (0,1)\}$ is linearly independent. By theorem 4.5.4, S is a basis for $\mathbb{R}^2$

- Theorem 4.5.3 - (Plus/Minus Theorem) Let S be a non-empty set of vectors in V.
  ⓐ If S is a linearly independent set and if $\vec{v}$ is a vector outside of span{S}, then the set $S \cup \{\vec{v}\}$ is still linearly independent. $\cup$: union
  ⓑ If $\vec{v}$ is a vector in S that can be written as a linear combination of others and $S - \{\vec{v}\}$ denotes the set obtained by removing $\vec{v}$ from S then $\text{Span}\{S\} = \text{Span}\{S - \{\vec{v}\}\}$.

- ③ Show that $\vec{p}_1 = 1 - x^2$, $\vec{p}_2 = 2 - x^2$, & $\vec{p}_3 = x^3$ are linearly independent.
  Sol 1: $a_1\vec{p}_1 + a_2\vec{p}_2 + a_3\vec{p}_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0 \Rightarrow$ L.I.
  Sol 2: $\vec{p}_1, \vec{p}_2$ are L.I. as they are not scalar multiples.
  Since $\vec{p}_3 \notin \text{span}\{\vec{p}_1, \vec{p}_2\}$, by Thm 4.5.3, $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ are L.I.

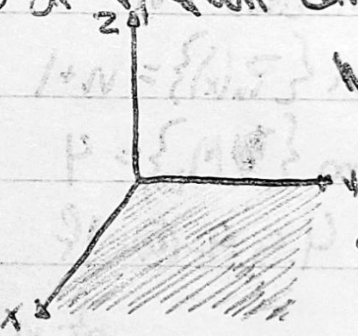
- Theorem 9.5.5 - Let $\dim(V) = n$

  (a) If $S = \{v_1, v_2 \ldots v_r\}$ is a linearly independent set of vectors in V and if $r < n$, then S can be enlarged to a basis on V.

  (b) If S spans V but is not a basis then S can be reduced to a basis of V

  (c) If W is a subspace of V then $\dim(W) \le \dim(V)$

- (d) $\dim(W) = \dim(V)$ iff $W = V$



$W = \text{span}\{\hat{i}\, \hat{j}\}$

$\dim W = 2 < 3 = \dim \mathbb{R}^3$

$\dim(\mathbb{R}^3) = 3$

$\hat{i}, \hat{j}$ are linearly independent

$\hat{k}, \hat{u} = (1,1,0)$ " " "

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = 0$

4.7 Row Space, Column Space, and Null Space of a Matrix

- ① Understanding the solution space of $A\vec{x}=0$ in A
- ② A computational algorithm for finding a basis for a vector space
- A basis for vector space V, $S=\{u_1, u_2 \ldots u_r\}$
  - ① $Span(S)=V$
  - ② S is linearly independent
- Ex ① Let $A=\begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^3$, $\vec{c}_1=\binom{2}{3}$, $\vec{c}_2=\binom{1}{-1}$, $\vec{c}_3=\binom{0}{4} \in \mathbb{R}^2$

  $\vec{r}_1=(2,1,0)$, $\vec{r}_2=(3,-1,4)$

  $Span\{\vec{r}_1, \vec{r}_2\}=\{k_1\vec{r}_1+k_2\vec{r}_2\}$ ← Subspace of $\mathbb{R}^3$

  ← Row space of A.

  $Span\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ ← Column Space of A

  $A\vec{x}=0$ (*) ← Solution space of (*)

  ← Null space of A.

- In general $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{matrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{matrix} \in \mathbb{R}^n$

  $\vec{c}_1 \quad \vec{c}_2 \quad \cdots \quad \vec{c}_n \in \mathbb{R}^n$

- $Span\{\vec{r}_1, \vec{r}_2, \ldots \vec{r}_m\}$ - Row Space of A - $R(A)$

  $Span\{\vec{c}_1, \vec{c}_2, \ldots \vec{c}_n\}$ - Column Space of A - $C(A)$

  The solution space of $A\vec{x}=0$ - Null space of A - $N(A)$

- Theorem 4.7.1 & 4.7.2 - (a) Elementary row operations do not change the row space on the null space of a matrix

  $\underset{R(A)}{A} \rightarrow \underset{R(E)}{E}$ (Echelon Form)

  (b) The non-zero row vectors is an echelon form form a basis for $R(E)=R(A)$.

- ② Find a basis for nullspace of A ($A\vec{x}=0$)

$$A=\begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix} \xrightarrow{r_2-2r_1} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & 5 & 8 \end{bmatrix} \xrightarrow[\substack{r_1+r_2 \\ (-1)r_1}]{r_3+r_2} \begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}=E$$

$x_1+4x_3-4x_4=0$

$x_2-5x_3+8x_4=0$ ⇒ $\cancel{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}+x_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}+x_3\begin{pmatrix} 4 \\ -5 \\ 0 \\ 0 \end{pmatrix}}$

- ③ Find a basis for the space spanned by the vectors $\vec{v}_1=(1,1,-1,4)$, $\vec{v}_2=(2,1,3,0)$, & $\vec{v}_3=(0,1,-5,8)$

Let $V=\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

Step 1: Form a matrix A with $\vec{v}_1,\vec{v}_2,\vec{v}_3$ as row vectors

$$A=\begin{bmatrix}1&1&-1&4\\2&1&3&0\\0&1&-5&8\end{bmatrix}$$ Thus $R(A)=V$.

Step 2: Reduce A to echelon form

$$A\to E=\begin{bmatrix}1&0&4&-4\\0&1&-5&8\\0&0&0&0\end{bmatrix}$$

Step 3: $\vec{w}_1=(1\ 0\ 4\ -4)$ $\vec{w}_2=(0\ 1\ -5\ 8)$ form a basis for $R(A)$.

Remark:
① $\dim R(A)=2=$
$=4-2$
# of columns − dim of null space.

- ④ Find a basis for $C(A)$

Step 1: Let $B=A^T=\begin{bmatrix}1&2&0\\1&1&1\\-1&3&5\\4&0&8\end{bmatrix}$ your row vectors of are column vectors of A

$C(A)\cong R(B)$

$$\begin{bmatrix}1&2&0\\1&1&1\\-1&3&5\\4&0&8\end{bmatrix}\begin{matrix}R_2-R_1\\R_3+R_1\\R_4-4R_1\end{matrix}\begin{bmatrix}1&2&0\\0&-1&1\\0&5&5\\0&8&8\end{bmatrix}\sim\begin{bmatrix}1&2&0\\0&1&-1\\0&0&0\\0&0&0\end{bmatrix}$$ Then $\alpha_1=(1,2,0)$ & $\alpha_2=(0,-1,1)$ form a basis for $R(B)$

Then $\beta_1=\alpha_1^T=\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\beta_2=\alpha_2^T=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ form a basis for $C(A)$

$\dim(C(A))=\dim(R(A))$

- $A\xrightarrow{\text{row operation}}B$ B is row equivalent to A. Row operations do not change $(R(A))$ but they change the $C(A)$

$$A=\begin{array}{c}\begin{matrix}c_1&c_2\end{matrix}\\\begin{bmatrix}1&0\\0&-1\\1&0\end{bmatrix}\end{array}\xrightarrow{R_3-R_1}\begin{bmatrix}1&0\\0&-1\\0&0\end{bmatrix}$$ $R(A)=R(E)$, $C(A)\neq C(E)$

However, the relationship of linear independence & l.d. among columns do not change.

$$A=\begin{array}{c}\begin{matrix}c_1&c_2&c_3\end{matrix}\\\begin{bmatrix}1&0&0\\0&1&2\\0&0&0\end{bmatrix}\end{array}\sim\begin{array}{c}\begin{matrix}c_1&c_2&c_3\end{matrix}\\\begin{bmatrix}1&0&0\\0&1&2\\0&0&0\end{bmatrix}\end{array}$$

- Theorem 4.7.3 - If A & B are row-equivalent matrices then
  (a) A given set of column vectors of A is l.i. iff the corresponding column vectors of B are l.i.
  (b) A given set of column vectors of A forms a basis for $C(A)$ iff the corresponding column vectors of B form a basis for $C(B)$.

•5. $A=\begin{bmatrix}1&1&-1&4\\2&1&3&0\\0&1&-5&8\end{bmatrix}$ is $E=\begin{array}{cccc} c_1' & c_2' & c_3' & c_4' \\ \end{array}\begin{bmatrix}1&0&4&-4\\0&1&-5&8\\0&0&0&0\end{bmatrix}$

c_1', c_2' are l.i. $\quad c_3' = 4c_1' - 5c_2'$

$c_4' = -4c_1' + 4c_2'$

Step 2: c_1', c_2' form a basis for $C(E)$

Step 3: By Thm 4.7.3 $c_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$ $c_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ form a basis for $C(A)$.

Since $c_3' = 4c_1' + (-5)c_2'$, $c_4' = -4c_1' + 8c_2'$, $c_3 = 4c_1 + (-5)c_2$, $c_4 = -4c_1 + 8c_2$

•Theorem 4.7.4: If a matrix E is in row echelon form then the column vectors that contain the leading 1's form a basis $C(E)$

$\begin{bmatrix}1&0&-4&4\\0&1&*&4\\0&0&0&0\end{bmatrix}$

•6 Find a basis for $R(A)$ where $A=\begin{bmatrix}1&1&-1&4\\2&1&3&0\\0&1&-5&8\end{bmatrix}$ consisting entirely of row vectors of A. Express the row vectors not in the basis as linearly independent of basis vectors.

$S=\{v_1,v_2,v_3\}$ Find a subset of S that forms a basis for $\text{Span}\{v_1,v_2,v_3\}$.

Solution Step 1 Let $B = A^T = \begin{bmatrix}1&2&0\\1&1&1\\-1&3&-5\\4&0&8\end{bmatrix}$

Step 2: Reduce A^T to row echelon form

$E=\begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\\0&0&0\end{bmatrix}$

Step 3: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ form a basis for $C(E)$.

Step 4: $\begin{bmatrix}1\\1\\-1\\4\end{bmatrix}\begin{bmatrix}2\\1\\3\\0\end{bmatrix}$ form a basis for $C(A^T)$.

Step 5: $[1\ 1\ -1\ 4]$, $[2\ 1\ 3\ 0]$ form a basis $R(A)$.

4.7 Rank and Nullity

- $\dim(R(A)) = \dim(C(A))$, $\dim(N(A))$.
- Theorem 4.8.1 - If A is any matrix, then the row space and the column space of A have the same dimension ($\dim(R(A)) = \dim(C(A)) \rightarrow \text{rank}(A)$).
- Proof: Let E be the reduced echelon form of A. It follows from Theorem 4.7.1 that $\dim(R(A))$ = # of non-zero rows on E = # of leading 1's. By Theorem 4.7.8 $\dim(C(A))$ = # of leading 1's. Thus $\dim(R(A)) = \dim(C(A))$
- Definition - Rank & Nullity
  - $\dim(R(A)) = \dim(C(A)) \rightarrow \text{Rank}(A)$.
  - Nullity of A is the $\dim(N(A))$.
- Ex① Find rank & nullity of the matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

Step 1: $A \rightarrow E = \begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus $\dim(R(A)) = 2$, $\text{Rank}(A) = 2$.

Consider $A\underset{\sim}{x} = 0$, $E\underset{\sim}{x} = 0$.

$x_1 + 4x_3 - 4x_4 = 0 \qquad x_3 = s \qquad x_1 = -4s + 4t$

$x_2 - 5x_3 + 8x_4 = 0. \qquad x_4 = t. \qquad x = 5s - 8t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 4 \\ 5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ -8 \\ 0 \\ 1 \end{pmatrix}$$ Thus $\dim(N(A)) = 2$.

$\text{rank}(A) + \text{Nullity}(A) = 2 + 2.$ = # of leading 1's + # of non-leading vectors

- Dimension Theorem for Matrices (Theorem 4.8.2).
  - If A is a matrix with n columns then $\text{rank}(A) + \text{Nullity}(A) = N$.
- Proof: Since A has n columns, the homogeneous linear systems $A\underset{\sim}{x} = 0$ has n unknowns (variables). These fall into two categories the leading variables + non-leading variables. # of leading variables + # of non leading variables $= n$.

· Proof (cont.): $A \to E$, $\text{rank}(A) + \text{nullity}(A) = n$.

(# of leading variables + # of "free" or non-leading ~~variables~~ variables).

5 Eigenvalues & Eigenvectors

- Ex① Consider $A=\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}$ & $\underline{x}=\begin{bmatrix}-1\\ 1\end{bmatrix}\neq 0$. What is the relationship between $A\underline{x}$ & $\underline{x}$?

$$A\underline{x}=\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}4-6\\ -3+5\end{bmatrix}=\begin{bmatrix}-2\\ 2\end{bmatrix}=2\begin{bmatrix}-1\\ 1\end{bmatrix}=2\underline{x}$$

$A\underline{x}=2\underline{x}$ - 2 is the eigenvalue of A.
- $\underline{x}$ is the eigenvector corresponding to 2.

- More Generally: $A\underline{x}=\lambda\underline{x}$, $\lambda=$ scalar, $\underline{x}\neq 0$
- Definition - If A is an $n\times n$ matrix, then a non-vector $\underline{x}$ in $\mathbb{R}^n$ is called an eigenvector of A if $A\underline{x}=\lambda\underline{x}$ for some scalar λ, the scalar λ is said to be the eigenvalue of A.
- Q - How to find eigen values?

$$A\underline{x}=\lambda\underline{x} \Rightarrow A\underline{x}=0=\lambda\underline{x} \Rightarrow 0=\lambda I\underline{x}-A\underline{x}$$
$$0=(\lambda I-A)\underline{x}\,,\ (\underline{x}\neq 0)\ ①$$

λ is an eigenvalue iff ① has a non-zero solution
iff $|\lambda I-A|=0$.

- Theorem 5.1.1 - If A is an $n\times n$ matrix then λ is an eigenvalue of A iff. it satisfies the equation
  $|\lambda I-A|=0$ ⊛
  - ⊛ - is the characteristic equation.
  - Left side ⊛ - polynomial of λ - characteristic polynomial
  - $|\lambda I-A|=\lambda^n+a_1\lambda^{n-1}+\dots\, a_n$

- Ex Find eigen values of $A=\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}$

$$\lambda I-A=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}=\begin{bmatrix}\lambda+4 & +6\\ -3 & \lambda-5\end{bmatrix}$$

$$|\lambda I-A|=(\lambda+4)(\lambda-5)-(6)(-3)$$
$$=\lambda^2-\lambda-20+18=\lambda^2-\lambda-2=0 \Rightarrow (\lambda-2)(\lambda+1)=0$$

Eigenvalues are $\lambda=2,\ -1$

• ③ Find eigenvalues of $A=\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 18 \\ -4 & -4 & -11 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-5 & -8 & -16 \\ -4 & \lambda-1 & -18 \\ 4 & 4 & \lambda+11 \end{vmatrix} \underset{R_2+R_3}{=} \begin{vmatrix} \lambda-5 & -8 & -16 \\ 0 & \lambda+3 & \lambda+3 \\ 4 & 4 & \lambda+11 \end{vmatrix} = (\lambda+3)\begin{vmatrix} \lambda-5 & -8 & -16 \\ 0 & 1 & 1 \\ 4 & 4 & \lambda+11 \end{vmatrix}$$

$$\underset{C_3-C_2}{=} (\lambda+3)\begin{vmatrix} \lambda-5 & -8 & -8 \\ 0 & 1 & 0 \\ 4 & 4 & \lambda+7 \end{vmatrix} = (\lambda+3)\big((\lambda-5)(\lambda+7)+32\big) = (\lambda+3)(\lambda+3)(\lambda-1)$$

$$= (\lambda+3)^2(\lambda-1) = 0$$

∴, eigenvalues $\lambda = -3, 1$

• Theorem 5.1.2 - If A is an $n \times n$ matrix & λ is a real number, then the following are equivalent

- ⓐ λ is ~~equival~~ an eigenvalue of A
- ⓑ The system of equation $(\lambda I - A)\vec{x} = 0$ has non-zero solutions
- ⓒ There is a non-zero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$
- ⓓ λ is a solution of $|\lambda I - A| = 0$.

• Q_2 - How to find eigenvector?

$(\lambda I - A)\vec{x} = 0$ (**) - Eigen vectors are non-zero solutions of (**)

solution space - Eigen vectors $\neq 0$.

nullspace $(\lambda I - A)$ - $N(\lambda I - A)$ - Eigen space to λ - E_λ

• ④ Find the eigenvector of $A=\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ bases of eigen spaces of A.

Step ① Find eigenvalues of A

They are 2 & -1

Step ② $(\lambda I - A)\vec{x} = 0 \Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \lambda+4 & 6 \\ -3 & \lambda-5 \end{bmatrix}$ $\qquad E_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

If $\lambda = 2 \Rightarrow \begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 + x_2 = 0$, $x_1 = -x_2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A basis in E_2 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

If $\lambda = -1 \Rightarrow \begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $x_1 + 2x_2 = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis in E_{-1} is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$E_{-1} = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

• ⑤ Find eigenvectors of $A=\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$

Step ① Eigenvalues $= -3$ & 1

Step ② $|\lambda I - A| = \begin{vmatrix} \lambda-5 & -8 & -16 \\ -4 & \lambda-1 & -8 \\ 4 & 4 & \lambda+11 \end{vmatrix}$

$\lambda = -3 \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + x_2 + 2x_3 = 0$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1 \Rightarrow \begin{bmatrix} -4 & -8 & -16 \\ -4 & 0 & -8 \\ 4 & 0 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

⑥ $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & 0 & 0 \\ -a_{21} & \lambda - a_{22} & 0 \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix}$

$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})$

Eigen values are a_{11}, a_{22}, a_{33}

• ⑦ Let λ be an eigenvalue A an $\underline{x}$ be a corresponding eigenvector ($A\underline{x} = \lambda \underline{x}$, $\underline{x} \neq 0$). Show,

① λ^2 is an eigen ~~vector~~ value of A^2

② $2 + 2\lambda^2$ is " " " $A + 2A^2$

Given $A\underline{x} = \lambda \underline{x}$

$A^2 \underline{x} = A(A\underline{x}) = A\lambda \underline{x} = \lambda A \underline{x} = \lambda \lambda \underline{x} = \lambda^2 \underline{x}$

Thus λ^2 is eigen value of A^2.

5.1 Eigenvalues & Eigenvectors

- $A\bar{x} = \lambda\bar{x}$ $\quad (\bar{x} \neq 0)$ $\qquad$ $A = n \times n$

eigenvalue $\quad$ corresponding eigenvector

- ① $|\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \dots a_n = p(\lambda)$

$p(\lambda) = 0$; $\lambda_1, \lambda_2, \dots \lambda_n$

- ② $(\lambda I - A)\bar{x} = 0$ (*)

E_λ - solution space of (*) - Eigenspace of $A \to \lambda$

- A has eigenvalue λ

A^2 " " " λ^2

$A + 2A^2$ " " " $\lambda + 2\lambda^2$

5.2 Diagonalization

- Similar Matrices $\quad A\underline{\bar{x}}=0$

$A \sim = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad A \to R \quad R\underline{\bar{x}}=0.$

- Definition: Two $n\times n$ matrices A & B are called similar (denoted $A\sim B$) if there exists an invertible matrix P such that, $B = P^{-1}AP$

- Ex ① Let $A=\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$, $P=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, & $D=\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ Show that $A=P^{-1}DP \quad (D\sim A)$

$|P| = 1-2=-1 \qquad P^{-1} = \frac{1}{(-1)}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$P^{-1}DP = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$

$\therefore A\sim D$

- $P^{-1}DP = A \qquad$ Take $Q = P^{-1}$

$D = PAP^{-1}$

$D = Q^{-1}AQ \Rightarrow A\sim D$

- The Diagonalization Problem - Given a $n\times n$-matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix? $A\sim D$? (D: diagonal)

- Definition - A $n\times n$ matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix D.

- Theorem 5.2.1 - If A is an $n\times n$ matrix, the the following are equivalent: ① A is diagonalizable

② A has n linearly independent eigenvectors.

- Procedure for Diagonalizing a Matrix A.
  - ① Find n l.i. eigenvectors of A. $\quad \underset{\bar{p}_1}{\lambda_1}\ \underset{\bar{p}_2}{\lambda_2} \cdots \underset{\bar{p}_n}{\lambda_n}$
  - ② Let $P = (\bar{p}_1, \bar{p}_2 \cdots \bar{p}_n)$.
  - ③ $P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, where λ_1 is the eigenvalue corresponding to $\bar{p}_1$

- ② a) show that $A=\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}$ is diagonalizable
  b) find a matrix P that diagonalizes A ($P^{-1}AP=D$)
- ③ Diagonalize $A=\begin{bmatrix}5 & 8 & 16\\ 4 & 1 & 8\\ -4 & -4 & -11\end{bmatrix}$

- Properties & Eigen vectors
- Theorem 5.2.2 - If $v_1, v_2, \ldots v_k$ are eigenvectors on matrix A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$, then $v_1, v_2, \ldots v_k$ are linearly independent.
- Theorem 5.2.3 - If a $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable
- Similarity Invariants
  - Note that if $P^{-1}AP=B$ $(A \sim B)$, then $\det(B)=\det(P^{-1}AP)$
    $=\det(P^{-1})\det(A)\det(P)$
    $=\det(A)\det(P^{-1})\det(P)$
    $=\det(A)$
  - In General, any property that is shared by all similar matrices is called a similarity invariant
  - $|\lambda I - A|$
    $|\lambda I - B| = |\lambda I - P^{-1}AP| = |\lambda P^{-1} I P - P^{-1}AP| = |P^{-1}(\lambda I - A)P| = |P^{-1}||\lambda I - A||P|$
    $= |P^{-1}||P||(\lambda I - A)| = |\lambda I - A|$

6 Inner Product Spaces

6.1 Inner Products

$\vec{u} = (u_1, u_2 \dots u_n)$
$\vec{v} = <v_1, v_2 \dots v_n>$

- $\mathbb{R}^n$ - dot product - $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 \dots\dots u_nv_n$ (scalar).
- $(\mathbb{R}^n, \cdot)$ - Euclidean Inner Product Space.
- $(V, < >)$ - General Inner Product.
- Definition - An inner product on a real space V is a function that associates a real # $<\vec{u}, \vec{v}>$ with each pair of vectors $\vec{u}$ and $\vec{v}$ in V in such way that the following axioms hold ($\vec{u}, \vec{v}, \vec{w}$ in V, k-scalar)
  - ① $<\vec{u}, \vec{v}> = <\vec{v}, \vec{u}>$ - symmetric axiom
  - ② $<\vec{u}+v, \vec{w}> = <\vec{u}, \vec{w}> + <\vec{v}, \vec{w}>$ additivity axiom
  - ③ $<k\vec{u}, \vec{w}> = k<\vec{u}, \vec{w}>$ Homogeneity Axiom
  - ④ $<\vec{v}, \vec{v}> \geq 0$ & $<\vec{v}, \vec{v}> = 0$ iff $\vec{v} = 0$ Positivity Axiom
- Ex ① If $\vec{u} = (u_1, u_2 \dots u_n)$, $\vec{v} = (v_1, v_2, \dots v_n) \in \mathbb{R}^n$, then dot product →
  $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots u_nv_n$ is an inner product
  $(\mathbb{R}^n \cdot)$ - Euclidean Inner Product Space Standard Inner Product

  Weighted inner product

  Let $w_1, w_2, \dots w_n$ be positive real numbers which we call weights, then the following formula defines an inner product in $\mathbb{R}^n$

  $<\vec{u}, \vec{v}> = w_1u_1v_1 + w_2u_2v_2 + \dots\dots w_nu_nv_n$ - Weight Euclidean Inner Product.
- Ex ② Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$. Show that $<\vec{u}, \vec{v}> = 2u_1v_1 + 3u_2v_2$ satisfies the 4 axioms.

  ① $<u, v> = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = <v, u>$

  ② If $\vec{w} = <w_1, w_2>$, then $\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2)$

  $<\vec{u}+\vec{v}, \vec{w}> = 2(u_1+v_1)w_1 + 3(u_2+v_2)w_2 = (2u_1w_1 + 3u_2w_2) + (2v_1w_1 + 3v_2w_2) =$
  $<\vec{u}, \vec{w}> + <\vec{v}, \vec{w}>$

  ③ $<k\vec{u}, \vec{v}> = 2ku_1w_1 + 3ku_2w_2 = k(2u_1w_1 + 3u_2w_2) = k(\vec{u}, \vec{v})$

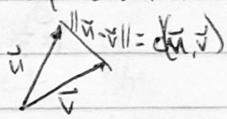
  ④ $<\vec{v}, \vec{v}> = 2v_1v_1 + 3v_2v_2 = 2v_1^2 + 3v_2^2 \geq 0$

  $<\vec{v}, \vec{v}> = 0 \iff 2v_1^2 + 3v_2^2 = 0 \iff v_1 = v_2 = 0 \iff \vec{v} = \vec{0}$

• Length & Distance in an Inner Product Space

$(\mathbb{R}^n, \cdot)$ - $\|\vec{u}\| = (\vec{u} \cdot \vec{u}) = \sqrt{\langle u u \rangle}$, $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

• Definition 2 - If V is an inner product space, then norm (a length) of a vector u in V is denoted by $\|\vec{u}\|$ and is defined by $\|\vec{u}\| = \langle \vec{u}, \vec{v} \rangle^{\frac{1}{2}}$. The distance between two vectors $\vec{u}$ & $\vec{v}$ is denoted by $d(\vec{u}, \vec{v})$ and is defined by $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

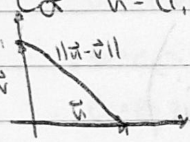


• Remark - It is important to keep in mind that norm & distance depend on the inner product being used. If the inner product is changed, then the norm & distance also change.

• Ex ③ $\mathbb{R}^n$ $\vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle$ $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \dots u_n^2}$

$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots (u_n - v_n)^2}$

• Ex ④ Let $\vec{u} = (1, 0)$ & $\vec{v} = (0, 1)$ in $\mathbb{R}^2$ with Euclidean Inner Product



We have $\|\vec{u}\| = \sqrt{1^2 + 0^2} = 1$, $\|\vec{v}\| = \sqrt{0^2 + 1^2} = 1$

$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$.

However if we changed to weighted inner product

$\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$

$\|u\| = \sqrt{2u_1^2 + 3u_2^2} = \sqrt{2(1^2)} = \sqrt{2}$, $\|v\| = \sqrt{2(0^2) + 3(1^2)} = \sqrt{3}$.

$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(2u_1 - 3v_1)^2 + (3u_2 - 3v_2)^2} = \sqrt{\sqrt{2}^2 + \sqrt{3}^2} = \sqrt{5}$.

• Theorem 6.11 - (Algebraic Properties) If $\vec{u}, \vec{v}, \vec{w} \in V$ and k is a scalar cont*

a) $\langle \vec{0}, \vec{u} \rangle = \langle \vec{u}, \vec{0} \rangle = 0$

b) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

c) $\langle \vec{u}, k\vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

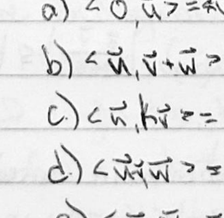
d) $\langle \vec{u} - \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle$

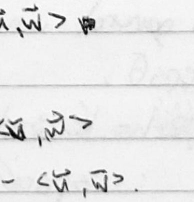
e) $\langle \vec{u}, \vec{v} - \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{w} \rangle$.

• Proof c) - By A1 $\langle \vec{u}, k\vec{v} \rangle = \langle k\vec{v}, \vec{u} \rangle = k\langle \vec{v}, \vec{u} \rangle$ by A3.

• Unit Spheres & Circles

$\|\vec{u}\| = 1$



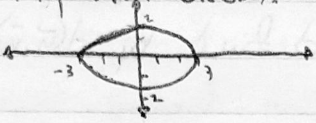


• Ex ⑤ a) Sketch the unit circle in x-y plane using the Euclidean Inner Product

b) Sketch " " " in x-y plane use $\langle \vec{u}, \vec{v} \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$

If $\vec{u} = (x, y)$, then $\|\vec{u}\| = \langle u, u \rangle^{\frac{1}{2}} = \sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2}$. Thus $\|\vec{u}\| = 1$ gives $1 = \frac{1}{9}x^2 + \frac{1}{4}y^2$ (unit circle).

$1 = \frac{1}{3^2}x^2 + \frac{1}{2^2}y^2$



• Inner Product Generated by Matrices

- Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let A be an $(n \times n)$ invertible matrix.

$\langle \vec{u}, \vec{v} \rangle = A\vec{u} \cdot A\vec{v}$ (2) — inner product generated by A.

$A = I$

$\vec{u} \cdot \vec{v} = (u_1 \dots u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v^T u \Rightarrow A\vec{u} \cdot A\vec{v} = (Av)^T (Au) = v^T A^T A u$

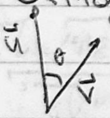
$\langle \vec{u}, \vec{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \dots w_nu_nv_n$

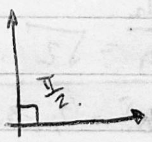
$W = \begin{pmatrix} \sqrt{w_1} & & \\ & \sqrt{w_2} & \\ & & \ddots & \\ & & & \sqrt{w_n} \end{pmatrix}$ $\quad W^T \cdot W = \begin{pmatrix} w_1 & & \\ & w_2 & \\ & & \ddots & \\ & & & w_n \end{pmatrix}$

$\langle u, v \rangle = v^T W^T W u = w_1u_1v_1 + w_2u_2v_2 \dots w_nu_nv_n$.

• Angle & Orthogonality

- $\mathbb{R}^2$





- In $\mathbb{R}^2, \mathbb{R}^3$, $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$1 \geq |\cos\theta| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

- If in a general inner product space, $\left|\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right| \leq 1$, we can define $\cos\theta$.

• Theorem 6.2.1 - If $\vec{u}$ & $\vec{v}$ are vectors in a real inner product space, then $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$

• Proof (Outline) - If $\vec{u} = \vec{0}$, ① holds

- If $\vec{u} \neq \vec{0}$, let $a = \langle \vec{u}, \vec{u} \rangle$, $b = 2\langle \vec{u}, \vec{v} \rangle$, $c = \langle \vec{v}, \vec{v} \rangle$, & $t \in \mathbb{R}$

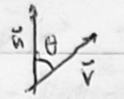
- $0 \leq \langle t\vec{u} + \vec{v}, t\vec{u} + \vec{v} \rangle = t\langle \vec{u}, t\vec{u} + \vec{v} \rangle + \langle \vec{u}, t\vec{u} + \vec{v} \rangle$

$t^2 \langle \vec{u}, \vec{u} \rangle + t\langle \vec{u}, \vec{v} \rangle + t\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle = at^2 + bt + c \geq 0$

- $b^2 - 4ac \leq 0$

⋮

- $|\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\| \|\vec{v}\|$

- Angle between vectors
  -  $0 \le \theta \le \pi$
  - We define $\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\|\vec{v}\|}$
- Ex ② $\mathbb{R}^4$ - Euclidean Space, Find the $\cos\theta$ of the angle between vectors $\vec{u} = (1,2,3,4)$ & $\vec{v} = (1,0,1,0)$

  $\|\vec{u}\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$, $\|\vec{v}\| = \sqrt{1^2+0^2+1^2+0^2} = \sqrt{2}$

  $\langle \vec{u}, \vec{v} \rangle = 1\cdot 1 + 2\cdot 0 + 3\cdot 1 + 4\cdot 0 = 4$

  $\cos\theta = \dfrac{4}{\sqrt{30}\sqrt{2}} = \dfrac{2}{\sqrt{15}}$
- Orthogonality
  - In $\mathbb{R}^2$ & $\mathbb{R}^3$ $\vec{u} \perp \vec{v}$ iff $\theta = \frac{\pi}{2}$, iff $\cos\theta = 0$, $\vec{u}\cdot\vec{v} = 0$
  - Define - Two vectors $\vec{u}$ & $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$
  - Ex ③ In $\mathbb{R}^n$ with $\langle \vec{u}, \vec{v} \rangle = \vec{u}\cdot\vec{v}$, the standard basis $e_1 = (1,0 \dots 0)$, $e_2 = (0,1 \dots 0)$.

    $e_n = (0,0 \dots 1)$

    $e_1 \cdot e_2 = 1\cdot 0 + 0 \cdot 1 + \dots = 0$

    $e_i \cdot e_j = (0, \dots \underset{i}{1} \dots 0) \cdot (0 \dots \underset{j}{1} \dots 0)_{i \ne j} = 0$. $e_i \perp e_j$

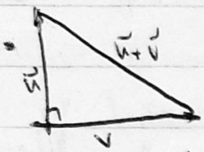
    $\|e_i\| = \sqrt{1} = 1$
  - Ex ④ M_{22} Orthogonality in M_{22}. If M_{22} has the inner product defined by $\langle \vec{u}, \vec{v} \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$, where $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ then the matrix S $u = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ are orthogonal since $\langle u, v \rangle = \langle \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \rangle = 0$

    $(1\ 0\ 0\ 2) \cdot (0\ 2\ 3\ 0) = 0$.
- Properties of Length & Distance.
  - Theorem 6.22 - Let $\vec{u}, \vec{v}$ be vectors in an ~~vector~~ inner product space V & k is a scalar. then (properties of Length).
    - L_1 - $\|u\| \ge 0$, L_2 - $\|u\| = 0$ iff $u = 0$, L_3 - $\|ku\| = |k|\|u\|$, L_4 - $\|u+v\| \le \|u\| + \|v\|$

L_4 = Triangle Inequality.

- Triangle Inequality - $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

$$\|\vec{u}+\vec{v}\|^2 = \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle u,v\rangle + \langle v,v\rangle = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$
$$\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$
$$\|u+v\| \le \|u\| + \|v\|$$

- 

$\|\vec{u}+\vec{v}\|^2 = \|u\|^2 + \|v\|^2$

$\vec{u} \perp \vec{v}$ $\langle \vec{u}, \vec{v}\rangle = 0$

- Theorem 6.2.3 - (Generalized Theorem of Pythagoras). If $\vec{u}$ & $\vec{v}$ are orthogonal, then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, $\langle u,v\rangle = 0$.

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\underbrace{\langle u,v\rangle}_{0} + \langle v,v\rangle$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

- Pythagoras - If $\vec{u}$ & $\vec{v}$ are orthogonal ($\langle \vec{u},\vec{v}\rangle = 0$) then
  $\|\vec{u}+v\|^2 = \|u\|^2 + \|v\|^2$
  - $\langle u,v\rangle = \vec{u}\cdot\vec{v}$ - Euclidean
  - $\hat{i}, \hat{j}, \hat{k} \Rightarrow \|\hat{i}\| = \|\hat{j}\| = \|\hat{k}\| = 1$. orthonormal basis
  - ① Orthonormal Set (basis)
    ② How to find it?
  - Orthogonal & Orthonormal sets $S = \{u_1, u_2 \ldots u_n\} \le V$

- Definition - A set of vectors in an inner product space V is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal. An orthogonal set is an orthogonal set with unit vectors.

  $\langle v_i, v_j\rangle = 0$ $i \ne j$
  $\langle v_i, v_i\rangle = 1$

- Ex① Let $u_1 = (1,0,0)$, $u_2 = (0,1,1)$, $u_3 = (0,1,-1)$

$\langle u_1, u_2\rangle = 1\cdot0 + 0\cdot1 + 0\cdot1 = 0 = \langle u_1, u_3\rangle$

$\langle u_2, u_3\rangle = 0\cdot0 + 1\cdot1 + 1\cdot(-1) = 0$

(u_1, u_2, u_3) orthogonal set

$\|u_1\| = 1$, $\|u_2\| = \sqrt{2} \ne 1$, $\|u_3\| = \sqrt{2} \ne 1$

Normalizing Process

- If $\vec{v} \neq 0$, $\frac{v}{\|v\|}$ - unit vector with the same direction as $\vec{v}$
- $\left\|\frac{v}{\|v\|}\right\| = \frac{1}{\|v\|}$, $\|v\|=1$

- $S = \{u_1, u_2 \ldots u_n\}$ - Basis

$(u)_S$, $u = k_1u_1 + k_2u_2 + k_3u_3$

(k_1, k_2, k_3).

Coordinates relative to orthonormal basis

- Theorem 6.3.1 - If $S = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_n\}$ is an orthonormal basis in an inner product space V and $u \in V$ then

$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \ldots \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$

- That is $(\vec{u})_S = (\langle \vec{u}, v_1 \rangle, \langle \vec{u}, \vec{v}_2 \rangle, \ldots \langle \vec{u}, \vec{v}_n \rangle)$. Given that $\langle v_i, v_j \rangle = 0$ if $i \neq j$, $\langle v_i, v_j \rangle = 1$ if $i = j$
- $\vec{u} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots k_n\vec{v}_n$
- $\langle \vec{u}, \vec{v}_i \rangle = \langle k_1v_1 + k_2v_2 \ldots k_n\vec{v}_n, \vec{v}_i \rangle = k_1\langle \vec{v}_1, v_i \rangle + k_2\langle \vec{v}_2, v_i \rangle \ldots k_n\langle \vec{v}_n, \vec{v}_i \rangle$

$= 0 + 0 + 0 + k_i\langle \vec{v}_i, v_i \rangle + 0 + 0 \ldots$

$= k_i \cdot 1 = k_i$.

- Ex ③ Let $v_1 = (1,0,0)$, $v_2 = (0, -\frac{4}{5}, \frac{3}{5})$, $v_3 = (0, \frac{3}{5}, \frac{4}{5})$.

Then $S = \{v_1, v_2, v_3\}$ - orthonormal basis for $\mathbb{R}^3$ express $u = (1,2,3)$ as a l.c. of S and find $(u)_S$.

Solution $\langle u, v_1 \rangle = \langle (1,2,3), (1,0,0) \rangle = 1$

$\langle u, v_2 \rangle = \langle (1,2,3), (0, -\frac{4}{5}, \frac{3}{5}) \rangle = \frac{1}{5}$.

$\langle u, v_3 \rangle = \langle (1,2,3), (0, \frac{3}{5}, \frac{4}{5}) \rangle = \frac{18}{5}$

Then $u = 1\vec{v}_1 + \frac{1}{5}\vec{v}_2 + \frac{18}{5}\vec{v}_3$

Hence $(u)_S = (1, \frac{1}{5}, \frac{18}{5})$

- Theorem 6.3.7 - If S is an orthonormal basis for an n-dimensional inner product space V, and if $(u)_S = (u_1, u_2, \ldots u_n)$ and $(V)_S = (v_1, v_2, \ldots v_n) \in \mathbb{R}^n$

Then (a) $\|u\| = \sqrt{u_1^2 + u_2^2 + \ldots u_n^2}$

(b) $d(u,v) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \ldots (u_n-v_n)^2}$

- Q: If S is an orthogonal set of non-zero vectors is S l.i?

  A: Yes.

- Theorem 6.3.3 - If $S = \{v_1, v_2, \dots v_n\}$ is an orthogonal set of nonzero vectors, then S is linearly independent

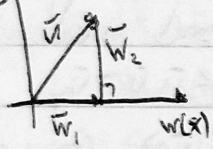
- Proof: Consider $k_1 v_1 + k_2 v_2 + \dots k_n v_n = \vec{0}$

  $\langle k_1 v_1 + k_2 v_2 + \dots k_n v_n, v_i \rangle = \langle 0, v_i \rangle = \vec{0}$

  $k_1 \underbrace{\langle v_1, v_i \rangle}_{= 0} + \dots k_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0}$ so $k_i = 0$

  $\therefore$ $(v_1 v_2 \dots v_n)$ are linearly independent.

- Orthogonal Projection

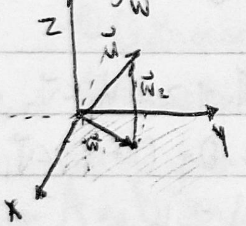
  - 

    $\vec{u} = \vec{w}_1 + \vec{w}_2$

    $w_1 \in W$, $w_2 \perp W$

    $w_2 = \vec{u} - w_1$

  - $\vec{w}_1 = \text{Proj}_W u$

  - 

    $\vec{u} = \vec{w}_1 + \vec{w}_2$

    $w_1 \in W$, $w_2 \perp W$

    $\vec{w}_2 = \vec{u} - \vec{w}_1 = \vec{u} - \text{Proj}_W u \perp W$

- Theorem 6.3.4 - If W is finite-dimensional subspace of an inner product space V, then every $\vec{u} \in V$ can be expressed in exactly one way as $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1 \in W$, & $\vec{w}_2 \perp W$. $\vec{w}_1$ is called the orthogonal projection of u on W, denoted by $w_1 = \text{Proj}_W(u)$

  $\vec{w}_2 = u - \text{proj}_W(u)$

- Theorem 6.3.5 - Let W be a finite dimensional subspace of V

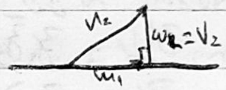
  (a) If $\{v_1 v_2 \dots v_r\}$ is an orthonormal basis for W and $\vec{u} \in V$ then $\text{Proj}_W(u) = \langle u, v_1 \rangle v_1 + \langle u v_2 \rangle v_2 + \dots \langle u v_r \rangle v_r$

  (b) If $\{v_1 v_2 \dots v_n\}$ is an orthogonal basis for W

  $$\text{Proj}_W u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \dots \frac{\langle u, v_r \rangle}{\|v_r\|^2} v_r$$

- Theorem 6.3.6 - Every non-zero finite-dimensional inner product space has an orthonormal basis

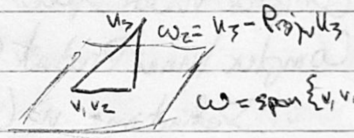
- Proof - Let V be a non-zero finite-dimensional inner product space and let $\{u_1, u_2, \dots u_n\}$ be any basis for V. It suffices to construct $\{u_1, u_2 \dots u_n\}$ into an orthogonal basis "Gram-Schmidt process"
  - $\{u_1, u_2 \dots u_n\} \to \{v_1, v_2 \dots v_n\}$ orthogonal
  - Step 1: Let $v_1 = u_1$



  - Step 2: Illustrate as diagram

$w_2 = u_2 - w_1 = u_2 - \mathrm{Proj}_W u_2$

$$v_2 = \bar{u}_2 - \mathrm{Proj}_W u_2$$
$$= \bar{u}_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$



  - Step 3: $v_3 = u_3 - \mathrm{Proj}_W u_3$
$$= u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$u_1 \perp u_2$, $v_3 \perp v_1$, $v_3 \perp v_2$

  - Step 4: Continuing in this way we will obtain after n-steps an orthogonal basis

- Ex 0 Consider $\mathbb{R}^3$ with Euclidean inner-product. Applying Gram-Schmidt Process to transfer the basis $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, and $u_3 = (0,0,1)$ into an orthogonal basis, then normalize the ortagonal basis to obtain an ~~orthogonal~~ orthonormal

$\frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$

$\bar{v} \neq 0$, $v/\|v\|$ = unit vector

Step 1: $v_1 = u_1 = (1,1,1)$

$\langle u_2, v_1 \rangle = 2$

$\|v_1\|^2 = 3$

Step 2: $v_2 = u_2 - \mathrm{Proj}_W u_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = u_2 - \frac{2}{3}(1,1,1)$
$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$\langle u_3, v_1 \rangle = 1$

$\langle u_3, v_2 \rangle = \frac{1}{3}$

$\|v_1\|^2 = 3$

$\|v_2\|^2 = \frac{2}{3}$

Step 3: $v_3 = u_3 - \mathrm{Proj}_W u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$
$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$
$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$\frac{1}{2} \cdot \frac{2}{3}$

Let $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1)$

$w_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$w_3 = \frac{v_3}{\|v_3\|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

- Assume matrix A is called orthogonal if $A^T = A^{-1}$ or equivalently $AA^T = AA^{-1} = I$
- Ex ① If $A = \begin{bmatrix} 1 & -i \\ 1+i & 1-i \end{bmatrix}$, $B = \begin{bmatrix} i & 1-2i \\ 2-4i & 3 \end{bmatrix}$ find AB

  $AB = \begin{bmatrix} -4-i & 1-8i \\ -3-5i & 6-4i \end{bmatrix}$
- Complex Conjugate: $z = a+bi$, then $\bar{z} = a^2+b^2 =$
- $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$
- Division: $\dfrac{z_1}{z_2 \neq 0} = \dfrac{z_1\bar{z}_2}{z_2\bar{z}_2} = \dfrac{z_1\bar{z}_2}{|z_2|^2}$
- Complex Vector Spaces: $\mathbb{R}^n$ $\mathbb{C}^n$
- Complex Inner Product Spaces, Euclidean Product Spaces.
  - Definition - $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, \ldots v_n)$ are in $\mathbb{C}^n$, then the complex Euclidean inner product denotedly $u \cdot v$ ($\langle u, v\rangle$) is given by $u \cdot v = u_1\bar{v}_1 + u_2\bar{v}_2 + \ldots u_n\bar{v}_n$
- Ex ② $u = (1, 1+i, 1-i)$ $v = (1, 1-i, i)$

  $u \cdot v = 1 \cdot \bar{1} + (1+i)\overline{(1-i)} + (1-i)(\bar{i}) = 1 + (1+i)(1+i) + (1-i)(-i) = i$
- Ex ③ Weighted Inner Product in $\mathbb{C}^2$

  Let $u = (u_1, u_2)$, $v = (v_1, v_2)$ $\in \mathbb{C}^2$

  $\langle u, v\rangle = 3u_1\bar{v}_1 + 4u_2\bar{v}_2$

  Let $u = (2i, -i)$, $v = (-i, 3i)$.

  $\langle u, v\rangle = 3(2i)(\bar{-i}) + 4(-i)(\bar{3i}) = 6(i)(i) + 12(-i)(-i) = -18$

4.6 Change of Basis

- $V \cdot \mathbb{R}^2$ $\quad B = \{\overset{u_1}{(1,0)}, \overset{u_2}{(0,1)}\}$ $\quad B' = \{\overset{u_1'}{(1,0)}, \overset{u_2'}{(1,1)}\}$

  old basis $\qquad\qquad$ new basis

- Given $u \in \mathbb{R}^2$ $\quad (u)_B = P(u)_{B'}$
- Coordinate Matrices. - If $S = \{u_1, u_2 \dots u_n\}$ is a a basis o vector space V, then each $\bar{v} \in V$ can be expressed uniquely as a linear combination of basis vectors say, $\bar{v} = k_1 v_1 + k_2 v_2 + \dots k_n v_n = (v_1 v_2 \dots v_n)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$
  - $(\bar{v})_S = (k_1 k_2 \dots k_n)$ (row vector)
  - $[\bar{v}]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$ (coordinate matrix of $\bar{v}$ relative to S).

- Ex ① Let $\bar{v}_1 = (1,0,0)$, $\bar{v}_2 = (0, -\frac{4}{5}, \frac{3}{5})$, $\bar{v}_3 = (0, \frac{3}{5}, -\frac{4}{5})$.

  $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ - basis for $\mathbb{R}^3$ if $v = (1,2,3)$.

  $v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \langle v, v_3 \rangle v_3$

  $= 1 v_1 + \frac{1}{5} v_2 + \frac{14}{5} v_3 \Rightarrow (v)_S = (1, \frac{1}{5}, \frac{14}{5})$

  $[v]_S = \begin{bmatrix} 1 \\ \frac{1}{5} \\ \frac{14}{5} \end{bmatrix}$

- Main Problem - (Change of Basis Problem) If we change the basis for a vector space from some old basis B to some new basis B', how is the old coordinate matrix $[v]_B$ of a vector $\bar{v}$ related to the new coordinate matrix $[v]_{B'}$, $[v]_B = P[v]_{B'}$
  - $\mathbb{R}^2$ $\quad B \to B'$ $\quad B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$

    $[v]_B \quad [v]_{B'}$

    Suppose that $[u_1']_B = \begin{bmatrix} a \\ b \end{bmatrix}$ and $[u_2']_B = \begin{bmatrix} c \\ d \end{bmatrix}$

    $u_1' = au_1 + bu_2 = (u_1\ u_2)\begin{bmatrix} a \\ b \end{bmatrix}$ $\quad u_2' = cu_1 + du_2 = (u_1, u_2)\begin{bmatrix} c \\ d \end{bmatrix}$

    $v = k_1 u_1 + k_2 u_2 \Rightarrow k_1' u_1' + k_2' u_2'$

    $[v]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ $\quad [v]_{B'} = \begin{bmatrix} k_1' \\ k_2' \end{bmatrix}$

    $(u_1'\ u_2') = (u_1\ u_2)\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

    $(u_1\ u_2)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = v = (u_1'\ u_2')\begin{bmatrix} k_1' \\ k_2' \end{bmatrix} \Rightarrow = (u_1, u_2)\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1' \\ k_2' \end{bmatrix}$

    $[v]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1' \\ k_2' \end{bmatrix}$, $\quad [v]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}[v]_{B'}$

    $P = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \left[[u_1']_B\ [u_2']_B \right]$ $\quad [v]_B = P[v]_{B'}$

· Solution - If we change basis for V from $B = \{u_1, u_2 \dots u_n\}$ to $B' = \{u_1', u_2' \dots u_n'\}$, then $\forall v \in V$, $[v]_B = P[v]_{B'}$ where $P = [[u_1']_B, [u_2']_B \dots [u_n']_B]$ the columns of P are coordinate matrices of the new basis vectors relative to the old basis. P is called a transition matrix from B' to B ($P_{B \to B'}$)

· Ex@ $B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$ in $\mathbb{R}^2$

$u_1 = (1,0)$, $u_2 = (0,1)$, $u_1' = (1,1)$, $u_2' = (2,1)$

...

· Theorem 4.6.1 - If P is the transition matrix from a basis B' to B, then (a) P is invertible

(b) P^{-1} is the transition matrix from B to B'

· Change of Orthonormal Basis

- V, $\langle u, v \rangle$ B B' orthonormal basis

· Theorem 7.1.5 - If P is the transition matrix from one orthonormal basis in B to another B' then

$P^{-1} = P^T$

P - orthogonal matrix ($PP^T) = I$

li - linear independent
ld - " dependent
lc - " combination
lt - " transformation

8 Linear Transformations

8.1 Intro to Linear Transformations

- $V \xrightarrow{F} W$

  $v \longrightarrow F(v) = w$
- If V, W are 2 vector spaces and F is a function from V to W, $F: V \to W$ $F(v) = w$

  $w = F(v) \to$ image of v

  $v \to$ preimage of w

  $V \to$ domain of F

  $W \to$ codomain of image space F

  $F(x, y) = (x-y, x+y, 5x)$ is a fxn from $\mathbb{R}^2$ to $\mathbb{R}^3$

  $v = (x, y) \in \mathbb{R}^2$

  $w = F(v) = (x-y, x+y, 5x)$
- Definition - If F $V \to W$ is a function, then F is called a linear transformation if
  - (a) $F(u+v) = F(u) + F(v)$ $\forall u, v \in V$
  - (b) $F(ku) = k F(u)$ $\forall u \in V$, $k =$ scalar.
  - If $W = V$, linear transformation = linear operator.

• Kernel & Range $\quad T: V \to W$

• $Ker(T) = \{v \mid T(v) = \vec{0}\}$

$R(T) = \{w \in W \mid \exists\, v \in V,\ s.t\ T(v) = w\} = \{T(v) \mid v \in V\}$

• Ex ① Let $T: V \to W$ be the transformation $T(v) = \vec{0}$

$Ker(T) = V \qquad R(T) = \{\vec{0}\} \leq W$

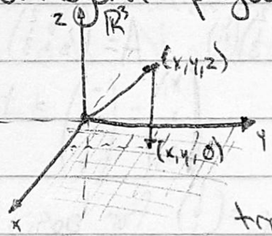
• ② Let $I: V \to W$ be the identity transformation

$I(v) = v$

$Ker(T) = \{0\} \leq V, \quad R(T) = V$

• ③ Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection on the xy plane

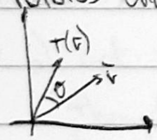
$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$Ker(T) = \{(0,0,z) \mid z \in \mathbb{R}^3\}$

$R(T)$ = the xy plane

• ④ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each vector in the xy plane through angle $\theta \neq 0$.



$Ker(T) = \{0\}$

$R(T) = \mathbb{R}^2$

• T is a one to one if $a \neq b \in V$

$\Longrightarrow T(a) \neq T(b) \ (\Longleftrightarrow$ if $T(a) = T(b)$, then $a = b)$

T is onto if $R(T) = W$

• Theorem 8.1.3 - If $T: V \to W$ is a l.t, then

a) $Ker(T)$ is a subspace of V

b) Range of T is a subspace of W

• Proof: $Ker(T) \neq \emptyset \quad \forall\ v_1, v_2 \in Ker(T) \quad T(v_1) = T(v_2) = 0$

$T(v_1 + v_2) = \ldots = 0$

$T(kv_1) = \ldots = 0$

$R(T) \to T(0) = 0 \Rightarrow K(T) \neq \emptyset$

$\forall\ w_1, w_2 \in R(T)\ \exists\ v_1, v_2 \in V$ such that $T(v_1) = w_1$, $T(v_2) = w_2$. Let $v = v_1 + v_2 \in V$

$T(v) = T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$ Thus $w_1 + w_2 \in R(T)$

$T(kv_1) = kT(v_1) = kw_1 \in R(T)$

By subspace test $R(T)$ is a subspace of W.

⑤ Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by a matrix $A_{m \times n}$.

$T(\bar{x}) = A\bar{x}$.

$\ker(T) = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = 0\} = N(A)$

Range $= \{b \in \mathbb{R}^n \mid b = A\bar{x}$ on some $\bar{x} \in \mathbb{R}^n\} = C(A)$

$A = (c_1, c_2 \ldots c_n)$

$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$b = A\bar{x} = c_1 x_1 + c_2 x_n + \ldots c_n x_n$

• ⑥ Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the l.t defined by

$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ x_1 + x_3 \\ 2x_1 + x_2 + 3x_3 \end{pmatrix}$ Find kernel & range of T.

$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 + x_3 = 0$

$x_2 + x_3 = 0$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is the basis for Ker(T)

~~for~~ $c_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, c_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for basis for $C(A) = R(T)$

• Theorem 8.1.4 - If $T: \mathbb{R}^n \to \mathbb{R}^n$ is multiplication by A then

a) $\ker(T) = N(A)$

b) $R(T) = C(A)$

• Proof - $\dim(C(A)) = \dim(R(A)) = \text{rank}(A)$

$\dim(N(A)) = \text{nullity}(A)$

$\text{rank}(A) + \text{nullity}(A) = \#$ of columns in A.

• Definition - If $T: V \to W$ is a linear transformation, then $\dim(R(T))$ is called rank of T, denoted by rank(T) and $\dim(\ker(T))$ is called the nullity of T denoted by Nullity(T).

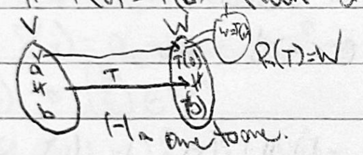
• Theorem 8.1.5 - Dimension theorem for linear transformations. If $T: V \to W$ is a l.t from a n-dimensional vector space V to a vector space W, then

$\text{rank}(T) + \text{nullity}(T) = n$ $(\dim(V))$ (dimension of domain).

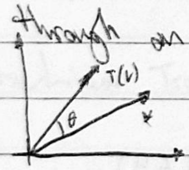
8.2 & 8.3 Isomorphisms

- ① One to one linear transformations
- ② Onto linear independence
- ③ Inverse linear transformations
- ④ Isomorphism

- One to One l.t. $T: V \to W$ $\quad u_1 \neq u_2$; $T(u_1) \neq T(u_2)$
- Definition - A linear transformation $T: V \to W$ is said to be ~~one~~ one to one if T maps distinct vectors in V to distinct vectors in W. That is $\forall\, a \neq b \in V$, $T(a) \neq T(b)$ or equivalently; If $T(a) = T(b)$, then $a = b$. $\forall\, a, b \in V$

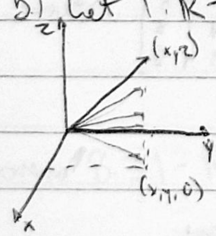


- Definition - If $T: V \to W$ and $R(T) = W$, then T is called onto
- Ex① a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the operator that rotates each vector through an angle θ. Then T is one to one. Is T onto?



~~[illegible]~~ $R(T) = \mathbb{R}^2$

b) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection on $\mathbb{R}^2$



T is not one to one as $(x,y,1)$ & $(x,y,0)$ both map to $(x,y,0)$.

- Theorem 8.2.1 - If $T: V \to W$ is a linear transformation, then the following statements are equivalent
  - a) T is one to one
  - b) $Ker(T) = \{0\}$
  - c) $Nullity(T) = 0$.

- Proof - $b \iff c$ is obvious; $a \Rightarrow b$, let $\forall\, v \in Ker(T)$ $\quad T(v) = 0 = T(0)$ Since T is 1-1, $\vec{v} = 0$ so $Ker(T) = 0$

$b \Rightarrow a$

Suppose $Ker(T) = \{0\}$, Let $u \neq v \in V$; $u - v \neq 0$ $\quad u - v \notin Ker(T)$

$T(u-v) \neq \vec{0}$; $T(u) - T(v) = T(u-v) \neq \vec{0}$. So $T(u) \neq T(v)$. Thus T is one to one

• ② $T: \mathbb{R}^5 \to \mathbb{R}^3$ is a multiplication by the matrix A. $T(\underline{x}) = A\underline{x}$

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 & 0 \\ 2 & 1 & 3 & 0 & -1 \\ 0 & 1 & -5 & 8 & 1 \end{bmatrix}$$

$Ker(T) = N(A)$

$Nullity(T) = Nullity(A)$

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 & 0 \\ 0 & -1 & 5 & -8 & -1 \\ 0 & 1 & -5 & 8 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 4 & 0 \\ 0 & -1 & 5 & -8 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rank(A) = 2$, $Nullity(A) = C(A) - Rank(A) = 5 - 2 = 3$

• Theorem 8.2.2 - If $T: V \to V$ then, TFSAE

a) T is one to one

b) $Ker(T) = \{0\}$

c) $Nullity(T) = 0$

d) $R(T) = V$ (T is onto)

• Proof - $a \Leftrightarrow b \Leftrightarrow c$, $c \Leftrightarrow d$ ~~so~~

$a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$.

- By dimension theorem, $Nullity(T) = \dim(V) - rank(T)$

$= \dim(V) - \dim(V) = 0$.

• Theorem 8.2.3 - If $T: \mathbb{R}^n \to \mathbb{R}^n$ is multiplication by an $n \times n$ matrix A then TFSAE

a) T is one to one

b) A is invertible

• ③ Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be multiplied by $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 0 & 4 & 8 \\ 3 & 9 & 1 & 8 \\ 1 & 1 & 4 & 8 \end{bmatrix}$. Determine is T is one to one.

$$|A| = \begin{vmatrix} 1 & 0 & 2 & 4 \\ 2 & 0 & 4 & 8 \\ 3 & 9 & 1 & 8 \\ 1 & 1 & 4 & 8 \end{vmatrix} \xrightarrow{R_2 - 2R_1} \begin{vmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 8 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

A is not invertible so T is not one to one.

Isomorphisms

- Definition - $T: V \to W$ is called an isomorphism ~~onto~~ if T is one to one & onto. In this case, we say V and W are isomorphism denoted by $V \cong W$. $(V \cong W)$
- Ex ① $P_{n-1} \cong \mathbb{R}^n$; ② $M_{22} \cong \mathbb{R}^4$
  $\dim(P_{n-1}) = n = \dim(\mathbb{R}^n)$ $\dim(M_{22}) = 4 = \dim(\mathbb{R}^4)$
- ① Define: $T: P_{n-1} \to \mathbb{R}^n$ by $[P(x)]_S \in \mathbb{R}^n$
  $S = \{1, x, \dots x^n\}$
  $P(x) = a_0 + a_1x + \dots a_nx^n \to \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$
  ② $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$
- Theorem 8.2.4 - If $\dim(V) = n$ then $V \cong \mathbb{R}^n$.
  $T: V \to \mathbb{R}^n$; $T(v) = [v]_S$. Let S be a basis in V $(S = \{v_1, \dots v_n\})$. T is one to one & onto & $V \cong \mathbb{R}^n$. Corollary, $V \cong W$ iff $\dim(V) = \dim(W)$.
- Inverse Linear Transformation - If $T: V \to W$ is one to one $(T: V \to R(T) \subseteq W)$
  We can define $T^{-1}: R(T) \to V$ given by $T^{-1}(w) = w$ (if $T(v) = w$). Then T^{-1} is a linear transformation called the inverse of T. $T^{-1} \circ T = I$
- Theorem 8.3.1 - If $T: \mathbb{R}^n \to \mathbb{R}^n$ is multiplied by an invertible matrix A, then T has an inverse and $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is multiplied by A^{-1}, ie if $T(\vec{x}) = A\vec{x}$ where A is invertible, then $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

Linear Transformations from $\mathbb{R}^n \to \mathbb{R}^n$ (4.9/4.10)

- If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then there exists an $n \times m$ matrix A such that $T(\bar{x}) = A\bar{x}$, $\bar{x} \in \mathbb{R}^n$. $[T] = A$

① Finding linear transformation from images & basis vectors

$S = \{v_1, v_2 \dots v_n\}$ – basis $T: V \to W$ for V; $T(v_1) \dots T(v_n)$

determine $T(u)$ completely; $\forall u \in V$ s $u = k_1v_1 + k_2v_2 + \dots k_nv_n$

Since T is a linear transformation

$$T(u) = T(k_1v_1 + k_2v_2 + \dots k_nv_n) = k_1T(v_1) + k_2T(v_2) \dots k_nT(v_n)$$

- Ex① Consider a basis $S = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$ and $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the l.t. such that

$T(v_1) = (1,0)$, $T(v_2) = (2,-1)$, $T(v_3) = (4,3)$

Find formula for $T(x_1, x_2, x_3)$ then compute $T(2,-3,5)$

Solution ① we first express (x_1, x_2, x_3) as a l.c. of $v_1 v_2 v_3$

$$k_1\begin{pmatrix}1\\1\\1\end{pmatrix} + k_2\begin{pmatrix}1\\1\\0\end{pmatrix} + k_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \to \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$k_1 + k_2 + k_3 = x_1$ $\quad k_1 = x_3$
$k_1 + k_2 = x_2$ $\quad k_2 = x_2 - x_3$
$k_1 = x_3$ $\quad k_3 = x_1 - x_2$

$$(x_1, x_2, x_3) = T(k_1v_1 + k_2v_2 + k_3v_3) = k_1T(v_1) + k_2T(v_2) + k_3T(v_3)$$
$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$
$$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)$$
$$T(2,-3,5) = (9, 23)$$

② Every l.t. $T: \mathbb{R}^n \to \mathbb{R}^n$ is a multiplication by matrix $A_{m \times n}$

$S = \{e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}\}$

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}4x_1 - 2x_2 - x_3\\3x_1 - 4x_2 + x_3\end{pmatrix} \quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-4\end{pmatrix}, T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$
$$= \begin{pmatrix}4 & -2 & -1\\3 & -4 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$A = \begin{bmatrix}4 & -2 & -1\\3 & -4 & 1\end{bmatrix} = [T(e_1), T(e_2), T(e_3)] \quad \text{Let } T(e_1) = \begin{bmatrix}a_{11}\\a_{21}\\ \vdots \\a_{m1}\end{bmatrix}, T(e_2) = \begin{bmatrix}a_{12}\\a_{22}\\ \vdots \\a_{m2}\end{bmatrix}, T(e_3) = \begin{bmatrix}a_{13}\\a_{23}\\ \vdots \\a_{m3}\end{bmatrix}$$

$$\text{Let } A = [T(e_1), T(e_2) \dots T(e_n)] = \begin{bmatrix}a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots a_{mn}\end{bmatrix}$$

$$T\begin{pmatrix}x_1\\x_2\\ \vdots \\x_n\end{pmatrix} = A\begin{pmatrix}x_1\\ \vdots \\x_n\end{pmatrix}$$

- Theorem 8.3.2: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a l.t. and if $S = \{e_1 \dots e_n\}$ is the standard basis for $\mathbb{R}^n$, then T is the multiplication by A ($T(\bar{x}) = A\bar{x}$) where $A = [T(e_1) \dots T(e_n)]$

  - A – standard matrix from T denoted by $[T] = A$ $\quad (T(\bar{x}) = A\bar{x} = [T]\bar{x})$

- Problem 1 - If $T_1: \mathbb{R}^n \to \mathbb{R}^k$, $T_2: \mathbb{R}^k \to \mathbb{R}^m$ then $T_1 \circ T_2: \mathbb{R}^n \to \mathbb{R}^m$

  $[T_2 \circ T_1] = [T_1] \cdot [T_2]$

- Theorem 8.3.3 - If $T_1: \mathbb{R}^n \to \mathbb{R}^k$ & $T_2: \mathbb{R}^k \to \mathbb{R}^n$ are l.t. then

  $[T_2 \circ T_1] = [T_2][T_1]$

- Problem 2 - If $T: \mathbb{R}^n \to \mathbb{R}^n$ is an one to one l.o. What the matrix for T^{-1}. $[T^{-1}] = [T]^{-1}$

- Ex (9, in 8.1)

- Theorem 8.3.4 - If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a l.t. then the following are true

  a) T is one to one

  b) $[T]$ is invertible.

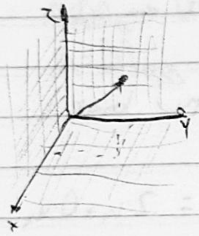
  Moreover if T is one to one we have $[T^{-1}] = [T]^{-1}$

8.4 Matrices of General Linear Transformations

• Recap: ① $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = [T][x]$ where $[x]$ = matrix transformation, $[T]$ = Standard matrix for T

② Coordinate Vector Matrix

$V, B = \{u_1, u_2 \dots u_n\} \ \forall u \in V$



$v = c_1u_1 + c_2u_2 + \dots + c_nu_n \quad (v)_S = (c_1, c_2 \dots c_n)$ ← coordinate vector of v

$(v)_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ ← coordinate vector for v

$T: V \to W \qquad B = \{u_1, u_2 \dots u_n\} \qquad B' = \{u_1', u_2', \dots u_n'\}$

$x \to T(x)$

$\underset{\mathbb{R}^n}{\vec{x}\,[x]_B} \xrightarrow{T_1} \underset{\mathbb{R}^m}{[T(x)]_{B'}} \qquad [T(\vec{x})]_{B'} = A[\vec{x}]_B = [T][\vec{x}]_B$ ②

T can be regarded as a matrix transformation

A - matrix for T with respect to B & B'

$A = [T]_{B' \cdot B}$ (B' – basis of codomain, B – Basis of domain)

$A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & & \ddots \\ & & & a_{mn} \end{bmatrix}$ (columns $c_1, c_2, \dots c_n$) $\quad u_1 \in B \quad u_1 = 1 \cdot u_1 + 0 \cdot u_2 - \dots 0 \cdot u_n$

$e_1 = [u_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [u_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2, [u_n]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_n$

$Ae_1 = A\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = c_1$

By ② $[T(u_1)]_{B'} = A[u_1]_B = Ae_1 = c_1, \quad c_2 = [T(u_2)]_{B'}, \quad c_n = [T(u_n)]_{B'}$

$A = ([T(u_1)]_{B'}, [T(u_2)]_{B'} \dots [T(u_n)]_{B'})$ ③

$[T]_{B' \cdot B} = ([T(u_1)]_{B'}, [T(u_2)]_{B'} \dots [T(u_n)]_{B'})$

$[T]_{B' \cdot B}[\vec{x}]_B = [T(\vec{x})]_{B'}$ ⑤

• Ex① Let $T: P_1 \to P_2$ be the l.t. defined by $T(p(x)) = xp(x)$. Find the matrix for T with respect to the standard bases.

$B = \{u_1, u_2\}$ and $B' = \{w_1, w_2, w_3\}$ where $u_1 = 1, u_2 = x$ & $w_1 = 1, w_2 = x, w_3 = x^2$

Solution: $T(u_1) = T(1) = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$; $[T(u_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T(u_2) = T(x) = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2$; $[T(u_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[T]_{B'B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

• Ex② Let $T: P_1 \to P_2$ be the same as in ex①. Verify formula ⑤ for any polynomial $p(x) = ax + b$. ⑤ $= [T]_{B' \cdot B}[\vec{x}]_B = [T(\vec{x})]_{B'}$

$[T]_{B'B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ a \end{bmatrix}$

$[ax + b]_B = \begin{bmatrix} b \\ a \end{bmatrix}$

$[T(ax + b)]_{B'} = ax^2 + bx]_B = [a \cdot x^2 + b \cdot x + 0 \cdot 1]_{B'} = \begin{bmatrix} 0 \\ b \\ a \end{bmatrix}$

Since L = R, formula ⑤ holds

Matrices for Linear Operators

- $T: V \to V$ $\quad [T]_{B'B}$ $\quad B = B'$ $\quad \{u_1, u_2 \ldots u_n\} = B$

$[T]_B = ([T(u_1)]_{B'}, [T(u_2)]_{B'} \ldots [T(u_n)]_{B'})$ (6)

$[T]_B [\bar{x}]_B = [T(x)]_B$ (7)

- Ex③ Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the l.o. be defined by
$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and let $B = \{u_1, u_2\}$ be the basis
where $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

ⓐ Find $[T]_B$: $T(u_1) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ -2+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2u_1 = 2 \cdot u_1 + 0 \cdot u_2$

$T(u_2) = T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ -2+8 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3u_2 = 0 \cdot u_1 + 3 \cdot u_2$

$[T(u_1)]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $[T(u_2)]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$[T]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

- Remark: ① Matrice of Identity $I: V \to V$

$I(\bar{x}) = \bar{x}$ $\quad B$-basis $\quad [I]_B = I$

② Theorem 8.4.1 - If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a l.t. and if B and B' are standard basis for $\mathbb{R}^n$ & $\mathbb{R}^m$ respectively then $[T]_{B'B} = [T]$

③ $T: V \to W$, $\bar{x} \to T(\bar{x})$; Find $T(\bar{x})$ in 3 steps

$\bar{x} \to [\bar{x}]_B \to [T]_{B'B}[\bar{x}]_B = [T(\bar{x})]_{B'} \to$ reconstruct $T(x)$

Recap

$$\begin{tikzcd} \bar{x} \arrow[r, "T"] \arrow[d] & T(x) \arrow[d] \\ \mathbb{R}^n \ni [x]_B \arrow[r] & [T(x)]_{B'} \in \mathbb{R}^m \end{tikzcd}$$

- $T: \underset{B}{V} \to \underset{B'}{W}$

$A = [T]_{B'B}$

In particular, if $V = W$, $B = B'$

$[T(x)]_B = [T]_{BB}[\bar{x}]_B$

$[T]_{B'B} = [[T(u_1)]_{B'} [T(u_2)]_{B'} \dots [T(u_n)]_{B'}]$

$[T]_B = [T]_{BB}$

$[T]_B = [[T(u_1)]_B \dots [T(u_n)]_B]$

8.5 Similarity

- $[T]_B$ depends on the basis B selected
- ① To choose a basis B that makes $[T]_B$ as simple as possible or the simplest form (diagonal matrix or triangle matrix) $\begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_n \end{pmatrix}$ $\begin{pmatrix} a_1 & a_2 & a_3 \\ & \ddots & a \\ & & a_n \end{pmatrix}$

② $[T]_B \overset{?}{=} [T]_{B'}$ $\quad [T]_{B'} = P^{-1}[T]_B P \quad P = P_{B' \to B}$

Recall change basis (4.6) $B = \{u_1, u_2 \dots u_n\}$, $B' = \{u_1', u_2' \dots u_n'\}$

$P_{B' \to B} = ([u_1']_B [u_2']_B \dots [u_n']_B)$ - transformation matrix from B' to B

$P_{B' \to B}[v]_{B'} = [v]_B$

- Ex① $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$B = \{e_1, e_2\} \quad [T]_B = [T] = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

$B' = \{u_1, u_2\}$ where $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad [T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$[T]_{B'} = P^{-1}[T]_B P \quad P = P_{B' \to B} = ([u_1]_B [u_2]_B) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\begin{tikzcd} {[x]_{B'}} \arrow[r, "{[T]_{B'}}"] \arrow[d, "P_{B' \to B}"'] & {[T(x)]_{B'} = [T]_{B'}[x]_{B'}} \\ {[x]_B} \arrow[r, "{[T]_B}", "T"'] & {[T(x)]_B = [T]_B[\bar{x}]_B} \arrow[u, "P^{-1}"'] \end{tikzcd}$$

$= [T]_B P [\bar{x}]_{B'}$

$[T]_{B'} = P^{-1}_{B' \to B}[T]_B P_{B' \to B}$

- Theorem 8.5.1 - Let $T: V \to V$ be a linear operator and let B & B' be basis for V. Then $[T]_{B'} = P^{-1}[T]_B P_{B' \to B}$ where $P = P_{B' \to B}$

- Definition - If A and B are squares matrices, we say B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$ denoted by $B \sim A$.

• Definition - A property with determinants

$-|B| = |P^{-1}AP| = |P^{-1}||A||P| = |A||P^{-1}||P|$

• Definition - Let $T: V \to V$ be a l.o. If there is a non-zero vector such that $T(\bar{x}) = \lambda \bar{x}$, the λ is the eigenvalue of T and $\bar{x}\ (\neq 0)$ is the eigenvector of T corresponding to λ.

$\lambda(\bar{x}) - T(\bar{x}) = 0$

$(\lambda I - T)(\bar{x}) = 0$