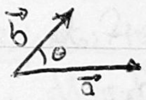
Vector Calculus

- Study of vector (and scalar) fields.
- $\vec{a} = \langle a_1, a_2, a_3 \rangle$ ← vector notation.
- $\vec{a} = \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3$
- Sum of Vectors
  - $\vec{a} + \vec{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$
- Multiplication by Constant.
  - $c\vec{a} = \langle a_1c, a_2c, a_3c \rangle$
- Properties
  - $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
  - $\vec{a} + 0 = \vec{a}$, $\langle 0,0,0 \rangle$
  - $(cd)\vec{a} = c(d\vec{a})$
- Multiplication of Vect
  - Dot Product → $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

    $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$



→ $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$

perpendicular (orthogonal)

  - Vector Product → $\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1a_2 - a_2b_1 \rangle$

→ $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

↳ $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} \times \vec{b} = 0$

→ $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ & $\vec{b}$

→ $\hat{i} \times \hat{j} = \hat{k}$, $\hat{k} \times \hat{i} = \hat{j}$, $\hat{j} \times \hat{k} = \hat{i}$

→ $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{i} \times \hat{k} = -\hat{j}$, $\hat{k} \times \hat{j} = -\hat{i}$

→ $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

  - Interesting Properties → $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

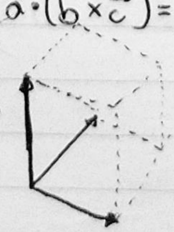
→ $\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

dot product

Multiplication by scalar

→ $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ← triple vector product identity.

\- →



$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

Fields

- Scalar field is a function, $f: \mathbb{R}^n \to \mathbb{R}$
  - $\mathbb{R}^n$: vector argument; $\mathbb{R}$: scalar value.
- Vector field is a function $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}$
  - $\mathbb{R}^n$: vector argument; $\mathbb{R}$: vector value

usually $n = 2$ or 3

- Arguments of scalar and vector fields are interpreted as position in space
- Example: $f(x,y,z) = x^2y + y^2z$, Temperature $= T(x,y,z)$ ← scalar field

  $F(x,y,z) = z\hat{i} + (x^2+y)\hat{j} + xyz\hat{k}$, gravitational force ← Vector field
- Vector field is a function that assigns a vector to each point in space.
- $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$
- Scalar field is a function that assigns a scalar to each point in space
- $f(x,y) = xyz + x^2y$

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- Gradient field is a special type of field
- $\vec{F}(x,y,z) = \nabla f(x,y,z) \Rightarrow \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

  $= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

  $= f_x\hat{i} + f_y\hat{i} + f_z\hat{k}$
- Ex. $\nabla f(x,y,z) = \nabla(xyz + x^2y) = \langle yz + 2xy,\ xz + x^2,\ xy + x^2y \rangle$

  $= (yz + 2xy)\hat{i} + (xz + x^2)\hat{j} + (xy + x^2y)\hat{k}$

  at (x,y,z), ∇f indicates direction of most rapid increase of f
- Some vector fields are gradients of scalar fields
- If there exists scalar field, f, such that $\vec{F} = \nabla f$, then $\vec{F}$ is conservative.

Calculus of Vectors Fields

- Vector calc covers integration and differentiation.
- In that order as well

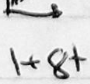
Line Integrals

- Two Types: ① With respect to arc length (line integral of scalar fields)
  ② With respect to coordinates (line integral of vector field)
- Let C be a curve in $2D^{\text{or } 3D}$ space $A_{t_1} \overset{C}{\frown} B_{t_2}$
  - Define LI with respect to arc length of scalar f over C.   LI = Line Integral
    $\int_C f(x,y,z)\,ds \overset{def}{=} \int_{t_1}^{t_2} f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$ where $x(t), y(t), z(t)$ is a parameterization of C
  - Ex find $\int_C xy\,ds$ where C is quarter of a circle

Parameterization: $x(t) = \cos t$, $y(t) = \sin t$ $t \in [0, \frac{\pi}{2}]$ $t_1 = 0$, $t_2 = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} xy\,ds = \int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} \cos t \sin t\,dt$

$= \int_0^{2\pi} \cos t \sin t\,dt = \left[\frac{1}{2}\sin^2 t\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$

  - If f represents density, $\int_C f\,ds$ represents mass.
- Let C be a curve in 2D/3D space $A_{t_1} \overset{C}{\frown} B_{t_2}$ oriented
  - Define LI with respect to coordinates of vector field $\vec{F}$ over C.
    $\vec{F}(xyz) = P(xyz)\hat{i} + Q(xyz)\hat{j} + R(xyz)\hat{k}$
  - $\int_C \vec{F}(xyz)\,d\vec{r} = \int_C P(xyz)dx + Q(xyz)dy + R(xyz)\hat{k}\,dz$. $d\vec{r} = \langle dx, dy, dz \rangle$
  - $\int_{t_1}^{t_2} P(xyz)\,x'(t)dt + Q(xyz)\,y'(t)dt + R(xyz)\,z'(t)dt$ $\vec{F} = \langle P, Q, R \rangle$
  - $\int_{t_1}^{t_2} [P(xyz)x'(t) + Q(xyz)y'(t) + R(xyz)z'(t)]\,dt$ $\vec{F} \cdot d\vec{r} = Pdx + Qdy + Rdz$
  - Ex $F(x,y) = y\hat{i} + x\hat{j}$  $A = (1,2)$ $B = (9,8)$. $\frac{dx}{dt} = x'(t) \Rightarrow dx = x'(t)dt$

Parameterization: $x(t) = 1 + 8t$ $y(t) = 2 + 6t$ $t \in [0,1]$

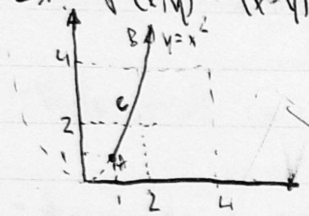
$\int_C \vec{F} \cdot d\vec{r} \Rightarrow \int_0^1 P(x(t),y(t))x'(t)dt + Q(x(t),y(t))y'(t)dt = \int_0^1 y(t)x'(t)dt + x(t)y'(t)dt$

$= \int_0^1 (2+6t)(8)dt + (1+8t)(6)\,dt = \int_0^1 (16+48t)dt + (6+48t)dt = \ldots 70$

- $\int_C \vec{F}\,d\vec{r}$ represents work done by force field, $\vec{F}$, along curve C.

Properties of Line Integrals

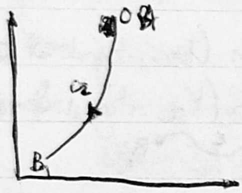
- Ex. $\vec{F}(x,y) = (x-y)\hat{i} + (x+y)\hat{j}$. Compute $\int_C \vec{F}\, dr$ for all



Parameterization. $x(t) = t$ $\quad t_1 = 1$ $\quad \int_C \vec{F} \cdot dr = \ldots \frac{35}{3}$

$y(t) = t^2$ $\quad t_2 = 2.$

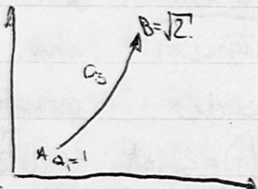
$x'(t) = 1$ $\quad y'(t) = 2t$



Parameterization: $x(t) = 2-t$ $\quad t_1 = 0$ $\quad \int \vec{F}\, dr = \frac{39}{3}$

$y(t) = 2 - t^2$ $\quad t_2 = 1$

$x'(t) = -1$ $\quad y'(t) = -4t + 2t$



Parameterization: $x(t) = t^2$ $\quad t_1 = 1$ $\quad \int_C \vec{F}\, dr = 39.$

$y(t) = t^4$ $\quad t_2 = \sqrt{2}$

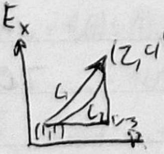
$x'(t) = 2t$ $\quad y'(t) = 4t^3$

- $\int_C \vec{F}\, dr$ depends on orientation of C, but not on details of parameterization.
- Let $-C$ be C but in the opposite direction
  - $\int_C \vec{F}\, d\vec{r} = -\int_{-C} \vec{F}\, d\vec{r}$.
- Remark: $\int_C f(x,y,z)\, ds = \int_{-C} f(x,y,z)\, ds$.
- If C is closed $A = B$, we use notation $\oint_C \vec{F}\, d\vec{r}$ $\quad A=B$.
- Sometimes orientation is indicated $\oint_C \vec{F}\, d\vec{r}$. or $\oint_C \vec{F}\, d\vec{r}$

  clockwise $\qquad$ Counterclockwise.

Path Dependence and Independence

- Ex $\quad C_1$: $\vec{F}(x,y) = y\hat{i} + 2x\hat{j}$ $\quad \int_{C_1} = \frac{23}{2}$.



$x(t) = 1+t$ $\quad t \in [0,1]$

$y(t) = 1 + 3t$

C_2: $x(t) = t$ $\quad t \in [1,2]$. $\quad \int_{C_2} = \frac{35}{3}$

$y(t) = t^2$

C_{3_1}: $x(t) = t$ $\quad t \in [1,2]$ $\quad \int_3 = \int_{31} + \int_{32} = 13.$

$y(t) = 1$

C_{32}: $x(t) = 2$ $\quad t \in [1,4]$

$y(t) = t$

- Q: When is $\int_C \vec{F} \cdot d\vec{r}$ independent of path joining A & B?
- A: (Yes for conservative forces) Fundamental theorem of ~~calculus~~ line integrals
- Let $f: \mathbb{R}^3 \to \mathbb{R}$ (scalar field) have continuous partial derivatives at each point of curve C. Let C be parameterized by $\vec{r}(t) = (x(t), y(t), z(t))$ for $a \le t \le b$. Then $\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$
- Recall Fundamental theorem of calculus.
  - $\int_a^b g(t)\,dt = [G(t)]_a^b = G(b) - G(a)$
- Implications: Let $\vec{F} = \vec{\nabla} f$, $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)).$$

no dependence on shape of C, only on position of end points of C.

- If C is closed and $\vec{F}$ is conservative, then $\oint_C \vec{F} \cdot d\vec{r} = 0$.
- If $\vec{F}$ is conservative, $\vec{F} = \vec{\nabla} f$, then f is called potential of $\vec{F}$.
- Given $\vec{F}$, how do we know if $\vec{F}$ is conservative?
- If it is, how do we find potential $\vec{F}$?

Fundamental Theorem of Line Integrals.

- If C is parameterized by $\vec{r}(t) = (x(t), y(t), z(t))$
  - $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, where $f(x, y, z)$ is a scalar field.
- Proof: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, $d\vec{r} = (dx, dy, dz)$

$$\nabla f \cdot d\vec{r} = \int_C \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \int_a^b \frac{\partial f}{\partial x} x'(t)\,dt + \frac{\partial f}{\partial y} y'(t)\,dt + \frac{\partial f}{\partial z} z'(t)\,dt$$

$$= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt$$

- Define $G(t) = f(x(t), y(t), z(t))$

  $G'(t)$ = expression in square brackets
- $\int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt = \int_a^b G'(t)\,dt$
- By fundamental theorem of calculus.
  - $\int_a^b G'(t)\,dt = G(b) - G(a)$ so $f(\vec{r}(b)) - f(\vec{r}(a))$ ✓

- This implies independence of paths for line integrals of ∇f
- Theorem: Let $\vec{F}$ be conservative, i.e. $\vec{F} = \nabla f$ for some f. If the above is true in a region, D, then $\int_C \vec{F}\, d\vec{r}$ is path independent for any curve C in region D.
- Existence of potential $\Rightarrow$ path independence.
- One can also show $\Leftarrow$ holds
- The Practice: Existence of potential is equivalent to path independence.
- Terminology: $\vec{F} = \nabla f$

  $\vec{F}$ conservative

  $\vec{F}$ admit's potential

  $\}$ The same meaning.
- If $\vec{F} = \nabla f$, f is called potential of $\vec{F}$

① How to determine if $\vec{F}$ is conservative? ② How to find potential?

A① • Theorem: Let P & Q be continuous functions $\underbrace{\text{of 2 variables } (x,y)}$ and continuously differentiable in the region $R = [a,b] \times [b,c]$

- Then $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative in region R, if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ for any $(x,y) \in R$

A② • Explicit computation of

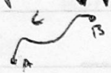
- Example: $\vec{F} = \underbrace{(6xy - y^3)}_{P}\hat{i} + \underbrace{(4y + 3x^2 - 3xy^2)}_{Q}\hat{j}$

  - $\frac{\partial P}{\partial y} = 6x - 3y^2$ $\qquad \frac{\partial Q}{\partial x} = 6x - 3y^2$
  - $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ is conservative, there exists potential $f(x,y)$ such that $\vec{F} = \nabla f$

- Finding Potential: - $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ $\quad \vec{F} = (P, Q)$
  - $\therefore \frac{\partial f}{\partial x} = P$ & $\frac{\partial f}{\partial y} = Q$
  - $\begin{cases} \frac{\partial f}{\partial x} = 6xy - y^3 \\ \frac{\partial f}{\partial y} = 4y + 3x^2 - 3xy^2 \end{cases}$ ← $f(x,y)$ is unknown.
  - $\int \partial f = \int 6xy - y^3\, \partial x \Rightarrow f(x,y) = 3x^2y - xy^3 + \underbrace{C(y)}$ arbitrary function of y.
  - Substitute to 2nd equation.
  - ~~[illegible]~~
  - $\frac{\partial f}{\partial y} = 3x^2 - x3y^2 + c'(y)$ ← this must equal
  - $\cancel{3x^2} - \cancel{3xy^2} + c'(y) = \cancel{3x^2} - \cancel{3xy^2} + 4y$
  - $c'(y) = 4y \Rightarrow c(y) = \int 4y\, dy = 2y^2 + c$ constant *.

Potential $= f(x,y) = 3x^2y - xy^3 + 2y^2 + c$

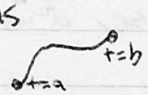
- Note: Potential is not unique (unique up to a constant)

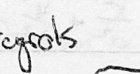
- Remark: Why Conservative?
  - In physics, potential is usually $V = -f$ (Potential Energy)
  - Suppose $\vec{F}$ be force field with potential f, $\vec{F} = \nabla f$ or $\vec{F} = -\nabla V$
  -  $\int_C \vec{F}\,d\vec{r}$ = work done by $\vec{F}$, denoted by W.

    $W = \int_C \vec{F}\,d\vec{r} \overset{FTCI}{=} f(\vec{r}(b)) - f(\vec{r}(a)) = -V(\vec{r}(b)) + V(\vec{r}(a)) = V(\vec{r}(a)) - V(\vec{r}(b))$
- Recall Netwons 2^{nd} Law
  - $F = ma$
  - $\vec{F}(\vec{r}(t)) = m\frac{d^2\vec{r}}{dt^2} = m\vec{v}'(t)$ where $v(t) = \frac{d\vec{r}}{dt} \Rightarrow dr = v(t)\,dt$.
  - $\int_a^b \vec{F}\,d\vec{r} = \int_a^b m\vec{v}'(t) \cdot v(t)\,dt = \int_a^b m\frac{d(\frac{1}{2}\vec{v}(t)\cdot v(t))}{dt}\,dt = m\left[\frac{1}{2}\vec{v}(t)\vec{v}(t)\right]_a^b$

    $\left[\frac{1}{2}mv(t)^2\right]_a^b = \frac{1}{2}m\vec{v}^2(b) - \frac{1}{2}m\vec{v}^2(a) = W = V(A) - V(B)$
  - $\frac{1}{2}m\vec{v}_B^2 - \frac{1}{2}m\vec{v}_A^2 = V(a) - V(b) \Rightarrow \frac{1}{2}m\vec{v}_A^2 + V(a) = \frac{1}{2}mv_b^2 + V(b)$
  - total energy is constant in conservative vector field.

Green's Theorem

- Fundamental Theorem of Line Integrals
  - $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ 
  - $\oint_C \vec{F}\cdot d\vec{r} = 0$ (because $\vec{r}(b) = \vec{r}(a)$ for closed curve) 
- What if $\vec{F}$ is not conservative?
  - $\oint_C \vec{F}\cdot d\vec{r} = ?$
- Theorem - Let D be closed (boundary belongs to it.) region in $\mathbb{R}^2$ with sectionally smooth boundaries. $P(x,y)$, $Q(x,y)$ - continuous functions with continuous partial derivatives. Let C be a boundary of D, counter-clockwise oriented. Then

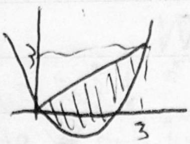
  $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$
  - Note: If $\oint_C \vec{F}\,d\vec{r}$ is conservative $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\oint \vec{F}\,d\vec{r} = 0$.
- Proof - Consider in special region D. 
  - $\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$
  - along C_2 $x(t) = b$, $x'(t) = 0$, $dx = x'(t)\,dt = 0\cdot dt$ so C_2 & $C_4 = 0$.
  - $\int_{C_1} P\,dx = \begin{cases} x(t) = t \\ y(t) = g_1(t) \\ t \in [a,b] \end{cases} = \int_a^b P(t, g_1(t))\,dt.$
  - $\int_{C_3} P\,dx = \begin{cases} x(t) = t \\ y = g_2(t) \\ t \in [b,a] \end{cases} = \int_b^a P(t, g_2(t))\,dt$
  - $\oint P\,dx = \int_a^b P(t, g_1(t))\,dt + \int_b^a P(t, g_2(t))\,dt = -\int_a^b [P(t, g_2(t)) - P(t, g_1(t))]\,dt$ $\Longrightarrow$

• $= -\int [P(x,g_2(x)) - P(x,g_1(x))]\,dx = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$

$= -\iint_D \frac{\partial P}{\partial y}\,dA$

• And similarly for $\iint_D \frac{\partial Q}{\partial x}\,dA$.

• Ex. find $\oint_C 3xy\,dx + 2x^2\,dy$



$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x - 3y = x$

$I = \iint_D x\,dA = \int_0^3 \int_{x^2-2x}^{x} x\,dy\,dx = \int_0^3 [xy]_{x^2-2x}^{x}\,dx = \int_0^3 [3x^2 - x^3]\,dx$

$= [x^3 - \frac{x^4}{4}]_0^3 = \frac{27}{4}$

• Ex. $F = (0,x) = 0\hat{i} + x\hat{j} = x\hat{j}$

$\oint_C \vec{F}\,d\vec{r} = \oint_C 0\,dx + x\,dy = \oint_C x\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \iint_D (1-0)\,dA = \iint_D dA = A$ (area of D)

$A = \oint_C x\,dy$

• Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$A = \oint_C x\,dy$

Parameterization! $x(t) = a\cos t$ $\quad t = [0, 2\pi)$

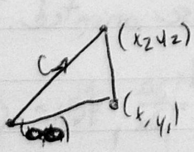
$y(t) = b\sin t$

$A = \int_0^{2\pi} a\cos t \cdot y'(t)\,dt = \int_0^{2\pi} a\cos t\, b\cos t\,dt = \int_0^{2\pi} ab\cos^2 t\,dt$

$= \int_0^{2\pi} ab(1+\cos 2t)\,dt = \int_0^{2\pi} \frac{1}{2}ab\,dt + \cancel{\int_0^{2\pi} \frac{1}{2}ab\cos 2t\,dt} = \frac{1}{2}ab\,2\pi = \pi ab$

$A = \pi ab$

•



$A = \oint_C x\,dy = -\int_C y\,dx$

$2A = \oint_C x\,dy - \int_C y\,dx$

$= \oint_C x\,dy - y\,dx$

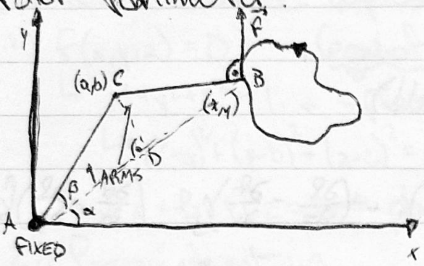
$A = \frac{1}{2}\oint_C x\,dy - y\,dx$

$A = \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \int_{C_3} x\,dy - y\,dx\right]$

$A = \frac{1}{2}\int_{C_2} x\,dy - y\,dx = \frac{1}{2}(x_1 y_2 - y_1 x_2)$

$A = \frac{1}{2}[(a_2-a_1)(b_3-b_1) - (a_3-a_1)(b_2-b_1)]$

- If $\vec{F}$ has the property $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_D dA =$ Area enclosed by C.
- Polar Planimeter.



Distance travelled by wheel

$\hookrightarrow \int_C \vec{F}\cdot d\vec{r}$

$\vec{CB} = \langle x-a,\ y-b\rangle$ Note: (a,b) depends on (x,y)

$\vec{F} = \langle -(y-b),\ x-a\rangle$

($\vec{CB}\cdot\vec{F} = 0$, so $\vec{CB} \perp \vec{F}$)

We need to express a,b in terms of x,y

$\tan\alpha = \frac{y}{x}$ $\quad \alpha = \arctan\left(\frac{y}{x}\right)$

$\cos\beta = \frac{|AD|}{|AC|} = \frac{\frac{1}{2}|AB|}{1} = \frac{1}{2}\sqrt{x^2+y^2} \Rightarrow \beta = \arccos\left(\frac{1}{2}\sqrt{x^2+y^2}\right)$

$\theta = \alpha + \beta = \arctan\left(\frac{y}{x}\right) + \arccos\left(\frac{1}{2}\sqrt{x^2+y^2}\right)$ $\qquad \begin{cases} a = \cos\theta \\ b = \sin\theta \end{cases}$

$a(x,y) = \cos\left(\arctan\left(\frac{y}{x}\right) + \arccos\left(\frac{1}{2}\sqrt{x^2+y^2}\right)\right)$

$b(x,y) = \sin\left(\arctan\left(\frac{y}{x}\right) + \arccos\left(\frac{1}{2}\sqrt{x^2+y^2}\right)\right)$

$\vec{F} = \langle \underbrace{-y+b}_{P},\ \underbrace{x-a}_{Q}\rangle$ $\quad P(x,y) = -y + b(x,y)$ $\quad Q(x,y) = x - a(x,y)$.

Exercise: Show $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

Example of Application of Green's Theorem:

- Compute line integral
  - $\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}$ where C is smooth enclosing $(0,0)$.
  -

  - $\vec{F} = \langle \underbrace{\frac{-y}{y^2+x^2}}_{P},\ \underbrace{\frac{x}{x^2+y^2}}_{Q}\rangle$
  - $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ if $(x,y) \neq 0$
- Remove disk of radius a
  - Greens Theorem for D: $\int_{\text{Boundary of } D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

$\oint_C \vec{F}\cdot d\vec{r} + \oint_{C_1} \vec{F}\cdot d\vec{r} = 0 \Rightarrow$

$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = -\oint_{C_1} \vec{F}\cdot d\vec{r}$

$x = a\cos t \quad x'(t) = -a\sin t$

$y = a\sin t \quad y'(t) = a\cos t$

$\oint_{C_1} \vec{F}\,dr = \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)\,dt + a\cos t(a\cos t)\,dt}{a^2\cos^2 t + a^2\sin^2 t}$

$= \int_0^{2\pi} dt$

$= 2\pi$

$\oint \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi \leftarrow$ independent of C.

Differential Operators (in $\mathbb{R}^3$)

- Recall $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$
- ∇f = gradient of f (scalar field)
- If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$
- $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$
- Line above can be denoted by curl $\vec{F} = \vec{\nabla} \times \vec{F} =$ (sometimes rot $\vec{F}$)
- $\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ ← Divergence of $\vec{F}$
- Line above can be denoted by div $\vec{F} = \nabla \cdot \vec{F}$
- $\vec{\nabla} \cdot \vec{F}$ = a vector field.
- Ex. $\vec{F} = \langle xz, xyz, -y^2 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\hat{i} - (0 - x)\hat{j} + (yz - 0)\hat{k}$$

$$= -y(2+x)\hat{i} + x\hat{j} + yz\hat{k} \leftarrow \text{vector field.}$$

$$\text{div } \vec{F} = \frac{\partial(xz)}{\partial x} + \frac{\partial(xyz)}{\partial y} + \frac{\partial(-y^2)}{\partial z}$$

$$= z + xz \leftarrow \text{scalar}$$

- Notation:

gradient	∇f	grad $\vec{F}$	← produces scalar
curl	$\nabla \times \vec{F}$	curl $\vec{F}$	← produces vector
divergence	$\nabla \cdot \vec{F}$	div $\vec{F}$	← produces scalar

- Identities:
  - curl $\nabla f = \vec{0}$ ← curl of a gradient scalar $= \vec{0}$.
    - If $\vec{F} = \nabla f$ (conservative field) then curl $\vec{F} = \vec{0}$ (conservative fields are curl-free). Converse is true (if P, Q, R have continuous partial derivatives) if $\vec{\nabla} \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative)
- Remark:
  - $\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$
  - $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$
  - curl $\vec{F} \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$
  - Green's Theorem: $\oint_C \vec{F}\, dr = \iint_D \text{curl } \vec{F}\, \hat{k}\, dA$

Surfaces

- Surfaces can be represented by:

1) $f(x,y,z) = 0$ (eqn. of 3 variables?)

↳ $x^2+y^2+z^2=1$ ← sphere of radius 1 centered at $(0,0,0)$.

↳ $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$ ← sphere of radius R centered at (a,b,c)

↳ $Ax+By+Cz=D$ ← plane

2.) Parametization

$$\left.\begin{aligned} x(u,v) &= f(u,v) \\ y(u,v) &= g(u,v) \\ z(u,v) &= h(u,v) \end{aligned}\right\} \text{ where } (u,v) \text{ belong to some set in } \mathbb{R}^2$$

- u, v parameters
- Parametrization of a surface in 3D

Note: $x(t)=f(t)$; $y(t)=g(t)$; $z(t)=h(t)$; parametization of curve in 3D.

- Examples: (1.) Sphere of radius R centered at $(0,0,0)$.

$$\left.\begin{aligned} x &= R\sin\varphi\cos\theta \\ y &= R\sin\varphi\sin\theta \\ z &= R\cos\varphi \end{aligned}\right\} \begin{aligned} &\varphi \in [0,\pi] \\ &\theta \in [0, 2\pi] \end{aligned}$$

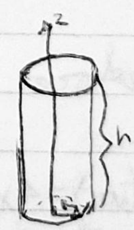
- $x^2+y^2+z^2 = R^2\sin^2\varphi\cos^2\theta + R^2\sin^2\varphi\sin^2\theta + R^2\cos^2\varphi = R^2$
- θ, φ parameters

(2) Cylinder of radius R

$x = R\cos\theta$ R = constant

$y = R\sin\theta$ θ, t ← parameters constant

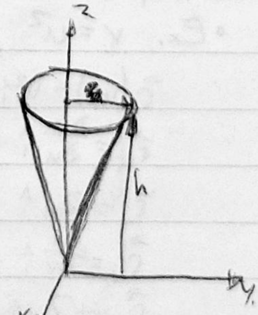
$z = t$ $\theta \in [0, 2\pi]$, $t \in [0, h]$



Note: $x = a\cos\theta$, $y = b\sin\theta$, $z = t$ → elliptical cylinder

(3) Elliptic Cone

$$\left.\begin{aligned} x &= au\cos v \\ y &= bu\sin v \\ z &= u \end{aligned}\right\} \begin{aligned} &a, b \leftarrow \text{constant} \\ &v \in [0, \pi] \\ &u = 1 \end{aligned}$$



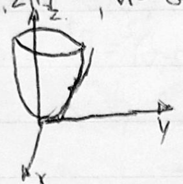
④ ~~Paral~~ Parabola of revolution

- $x = u\cos v$ k- constant

  $y = u\sin v$ u, v - var. pm

  $z = ku^2$ $v \in [0, 2\pi]$, u - as needed.

-

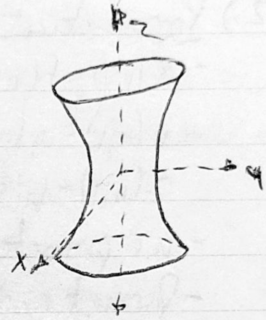


⑤ Hyperboloid of one Sheet

- $x = a\sin u \cosh(v)$ $u \in [0, 2\pi]$

  $y = b\sin u \cosh(v)$ v - as needed

  $z = c\sinh(v)$ a, b, c - constants



• *Memorize the parameterizations*

Goals

- ① How to compute surface area?
- ② How to generalize line integrals into surface integrals?
- ③ Formulate Green's Theorem in 3D.

Surface Area

- $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
- $\vec{r}_u, \vec{r}_v$ define a tangent plane.
- Ex. $x = u^2$, $y = v^2$, $z = u + 2v$. Find Tangent Plane at $(1,1,3)$.

$\vec{r}(u,v) = u^2\hat{i} + v^2\hat{j} + (u+2v)\hat{k}$

$\vec{r}_u = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}$

$\vec{r}_u = 2u\hat{i} + 0\hat{j} + \hat{k} = 2u\hat{i} + \hat{k}$

$\vec{r}_v = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j} + \frac{\partial z}{\partial v}\hat{k}$

$\vec{r}_v = 2v\hat{j} + 2\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\hat{i} - 4u\hat{j} + 4uv\hat{k}$$

at $(1,1,3)$, $u=1$, $v=1$.

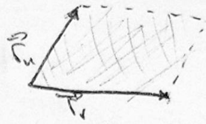
$-2\hat{i} - 4\hat{j} + 4\hat{k} = \vec{n}$

$\vec{n}$ is the normal vector at $(1,1,3)$.

$-2(x-1) - 4(y-1) + 4(z-3) = 0$

$x + 2y - 2z + 3 = 0$

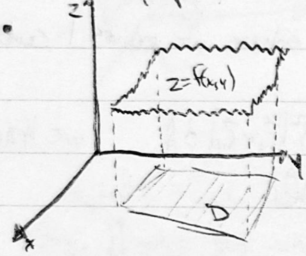
tangent plane.



- Surface Area $= A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

Special Cases

- Surface area of $z = f(x,y)$ lying directly over region D in x-y plane
-



$$\boxed{A = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

Natural Parameterization

$x = u, \ y = v, \ z = f(u,v)$

$\vec{r} = \langle u, v, f(u,v) \rangle$ $\qquad$ $\vec{r}_u = \langle u, v, f_u \rangle$

$\vec{r}_u = \langle 1, 0, \frac{\partial f}{\partial u} \rangle$ $\qquad$ $\vec{r}_v = \langle 0, 1, \frac{\partial f}{\partial v} \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix} = \hat{i}\left(-\frac{\partial f}{\partial u}\right) - \hat{j}\left(\frac{\partial f}{\partial v}\right) + \hat{k} = -\frac{\partial f}{\partial u}\hat{i} - \frac{\partial f}{\partial v}\hat{j} + \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1^2} = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$$\boxed{A = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy}$$

Note: Arc length $\Rightarrow$ $L = \int_a^b \sqrt{1 + f'(x)^2} \, dx$

Example

- Find surface area of a sphere of radius a, $x^2 + y^2 + z^2 = a^2$

Parameterization: $x = a\sin\varphi\cos\theta$ $\qquad$ $D = \{(\varphi, \theta) : 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\}$

$y = a\sin\varphi\sin\theta$

$z = a\cos\varphi$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\varphi\cos\theta\,\hat{i} + a^2\sin^2\varphi\sin\theta\,\hat{j} + a^2\sin\varphi\cos\varphi\,\hat{k}$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\sin^2\varphi\cos^2\varphi} = a^2\sin\varphi$$

since $\sin\varphi$ is positive

$$A = \iint_D a^2\sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi \, d\varphi \, d\theta = 4\pi a^2.$$

Surface Integral of Function $f(x,y,z)$ Over the Surface S.

- Surface Integral - $\iint_S f(x,y,z)\, \underset{\substack{\uparrow \\ \text{Surface} \\ \text{area element}}}{dS} \overset{\text{def}}{=} \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, dA.$

  where $\vec{r}(u,v)$ is parametrization of S, $(u,v) \in D$.

- $\iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, dA = \iint_D f(x(u,v), y(u,v), z(u,v)) |\vec{r}_u \times \vec{r}_v| \, dA.$
- Example: Compute $\iint_S x^2 dS$, where S is a sphere of radius 1 centered at $(0,0,0)$

$f(x,y,z) = x^2$

$x = \sin\varphi\cos\theta$

$y = \sin\varphi\cos\theta$

$z = \cos\varphi$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$\iint_S x^2 dS = \iint_D (\sin\varphi\cos\theta)^2 |\vec{r}_u \times \vec{r}_v| \, dA.$ we know $|\vec{r}_\varphi \times \vec{r}_\theta|$ $\sin\varphi$

$= \iint (\sin\varphi\cos\theta)^2 \sin\varphi \, d\varphi\, d\theta.$

$\circ\circ\circ$

$= \frac{4\pi}{3}.$

Recall Line Integrals

- $\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt.$

Special Case:

- If S is a graph of $z = g(x,y)$ where $(x,y) \in D$ then we use x, y as ~~variables~~ parameters
- $x = x$, $y = y$, $z = g(x,y)$ $\Rightarrow$ $\vec{r} = x\hat{i} + y\hat{j} + g(x,y)\hat{k}$

$\frac{\partial r}{\partial x} = \hat{i} + \frac{\partial g}{\partial x}\hat{k}$

$\frac{\partial r}{\partial y} = \hat{j} + \frac{\partial g}{\partial y}\hat{k}$

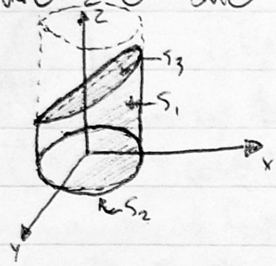
$\vec{r}_x \times \vec{r}_y = -\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k}$

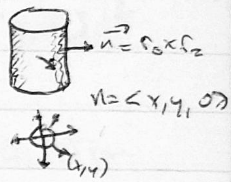
$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + g_x^2 + g_y^2}$

$\iint_S f(x,y,z)\, dS \Rightarrow \iint_D f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1} \, dx\, dy.$

Example:

- Compute $\iint_S z\,dS$, when S consists of surface of cylinder $x^2+y^2=1$ above $z=0$ and below $z=1+x$.





$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

S_1: $x=\cos\theta$, $y=\sin\theta$, $z=z$ $\quad 0\le\theta\le 2\pi$, $\quad 0\le z\le 1+\cos\theta$

lower surface $z=0$, higher surface $z=1+x$

$$r_\theta \times r_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad |r_\theta \times r_z| = 1$$

- $$\iint_{S_1} z\,dS = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} z\cdot 1\cdot dz\,d\theta = \ldots = \frac{3\pi}{2}.$$

- S_2: $\iint_{S_2} z\,ds$ but since $x=..$, $y=..$, $z=0$ $\Rightarrow \iint 0\,ds = 0$.

- S_3: $z=1+x$

$f(x,y,z)=z$

$$\iint_{S_3} z\,dS = \iint_D 1+x\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dA.$$

$$= \iint_D (1+x)\sqrt{1+1+0}\,dA = \iint_D \sqrt{2}(1+x)\,dA.$$

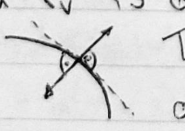
$$= \int_0^{2\pi}\int_0^{1} (1+\cos\theta)\,r\,dr\,d\theta = \ldots = \sqrt{2}\,\pi.$$

D is a projection on (x,y) plane so it is a disk of radius 1 centered at $(0,0)$.

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_3} z\,dS = \frac{3\pi}{2} + \sqrt{2}\,\pi = \left(\frac{3}{2}+\sqrt{2}\right)\pi.$$

Orientation of Surfaces

- $\vec{r}_u \times \vec{r}_v$ is orthogonal to surface



The normal vector could face either way depending on your choosing.

- Choosing direction of normal vector is called orienting the surface

- Example: Surface $z=g(x,y)$

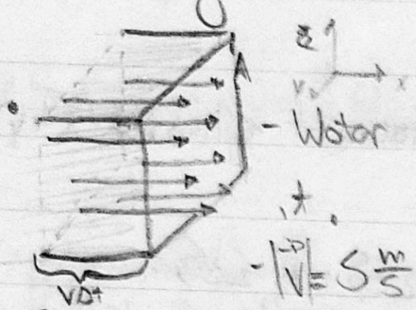
$\vec{r} = \langle x, y, g(x,y)\rangle$

$\vec{r}_x \times \vec{r}_y = -g_x\hat{i} - g_y\hat{j} + \hat{k}$

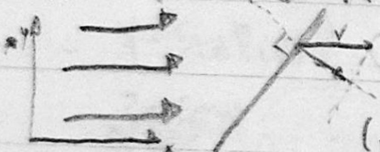
Define unit normal vector to $z=g(x,y)$ oriented up

$$\vec{n} = \frac{-g_x\hat{i} - g_y\hat{j} + \hat{k}}{\sqrt{g_x^2+g_y^2+1}}, \text{ so that } |\vec{n}|=1$$

Surface Integrals of Vector Fields



- Water flowing through some crosssectional area
- $|\vec{v}| = 5\frac{m}{s}$
- Q: How much water (in kg) flows through A per unit time? ($\rho = 1000\frac{kg}{m^3}$)
- Volume passing A in time $\Delta t = A v \Delta t$
- Mass passing A in time $\Delta t = \rho A v \Delta t$
- $\Delta m = \rho A v \Delta t \Rightarrow \frac{\Delta m}{\Delta t} = \rho A v \Rightarrow 1000 \cdot 5 \cdot 2 \left(\frac{kg}{m^3} \cdot \frac{m}{s} \cdot m^2\right) = 10{,}000\frac{kg}{s}$
- But what if A is not perpendicular to $\vec{v}$?



- In general, $\frac{\Delta m}{\Delta t} = \rho \vec{v} \cdot \vec{n} A$ where $\vec{n}$ is the normal unit vector to A

(if $\vec{v} \parallel n$, $\vec{v} \cdot \vec{n} = |\vec{v}| = v$)

- If instead of flat A we have surface S

$$\frac{\Delta m}{\Delta t} = \iint_S \underbrace{\rho \vec{v} \cdot \vec{n}}_{\text{scalar}} \, dS$$

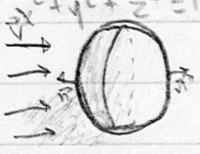
- In general, for vector fields $\vec{F}(x,y,z)$ the flux of $\vec{F}$ across surface S is defined as $\iint_S \vec{F} \cdot \vec{n} \, dS$ (normal vector to S)
- Remark: If $\vec{F} = \rho \vec{v}$, then we have flux of the fluid
  - $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$
- Notation: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, d\vec{S}$ ← Surface Integral of Vector Field (Type II)
- Before: $\iint_S f(x,y,z) \, dS$ ← Surface Integral of Scalar Field (Type I)
- Note Anallogy: $\int_C \vec{f}(x,y,z) \, ds$ ← Line Integral of Scalar field (Type I (L))

  $\int_C \vec{F} \cdot d\vec{r}$ ← Line Integral of Vector field (Type II (L))

Example:

- Find the flux of $\vec{F}(x,y,z) = z\hat{i} + y\hat{j} + x\hat{k}$ across the unit sphere

$x^2+y^2+z^2=1$



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,dS = \iint_D \vec{F}\cdot\underbrace{\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}}_{\text{normal}}\cdot\underbrace{|\vec{r}_u\times\vec{r}_v|\,dA}_{dS}$$

$$= \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA \quad \text{where } dA = du\,dv.$$

- $\vec{r}_u\times\vec{r}_v = \sin^2\varphi\cos\theta\,\hat{i} + \sin^2\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\varphi\,\hat{k}$

* Parameterization $r(\varphi,\theta) = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\cos\theta\,\hat{j} + \cos\varphi\,\hat{k}$

- $\vec{F}$ in parameterized form
- $\vec{F}(\vec{r}(\varphi,\theta)) = \cos\varphi\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\theta\,\hat{k}$
- $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\varphi,\theta))\cdot\vec{r}_\varphi\times\vec{r}_\theta\,d\varphi\,d\theta$

$$= \iint_D (\cos\varphi\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\theta\,\hat{k})\cdot(\sin^2\varphi\cos\theta\,\hat{i} + \sin^2\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\varphi\,\hat{k})\,d\varphi\,d\theta$$

$$= \iint_D \cos\varphi\sin^2\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j}\,\sin^2\varphi\sin\theta + \sin\varphi\cos\theta\,\sin\varphi\cos\varphi\,\hat{k}\,d\varphi\,d\theta$$

$$= \iint_D 2\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta\,d\varphi\,d\theta$$

$$= 2\int_0^{\pi}\sin^2\varphi\cos\varphi\,d\varphi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\varphi\,d\varphi\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= \frac{4\pi}{3}$$

Applications

- 1) Electric Field $\vec{E}(x,y,z)$
  - Gauss Law — Closed Surface, Q - Electric Charge in S.

$$Q = \varepsilon_0 \underbrace{\iint_S \vec{E}\,d\vec{S}}_{\text{FLUX OF ELECTRIC FIELD}}, \quad \varepsilon_0 = 8.85\,\frac{C^2}{Nm^2}$$

- 2.) Heat Flow
  - $u(x,y,z)$ - temperature at (x,y,z)
  - If u is constant heat will not flow from high to low
  - Heat Flow - amount of heat (energy) flowing across surface S per unit time
  - Field: $\vec{F} = -k\nabla u$ — thermal conductivity constant
  - Heat Flow $= \iint_S \vec{F}\cdot d\vec{S} = -k\iint_S \nabla u\cdot d\vec{S}$

Example

- $u(x,y,z) = \overset{\text{constant}}{C}(x^2+y^2+z^2)$
- $\vec{F} = -K\nabla u = -KC(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$
- Let S - sphere of radius a, $x^2+y^2+z^2=a^2$
- $\iint_S \vec{F}\cdot d\vec{S} = ?$
- Note: Recall that for sphere $\vec{n} = \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$

  $|\vec{n}| = 1$
- $\vec{F}\cdot\vec{n} = -\frac{2KC}{a}(x^2+y^2+z^2) = -\frac{2KCa^2}{a} = -2KCa$
- $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = -2KCa\iint_S dS = -2KCa[4\pi a^2]$

  $= -8KCa^3\pi$

Stokes Theorem

- We define $\iint_S \vec{F}\, d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$ normal unit vector on S
- Stokes Theorem: Let S be oriented and piecewise-smooth, bounded by a closed piecewise-smooth curve, C with positive orientation. Let $\vec{F}$ be a vector field with continuous partial derivatives on some region $\mathbb{R}^3$ containing S. Then,

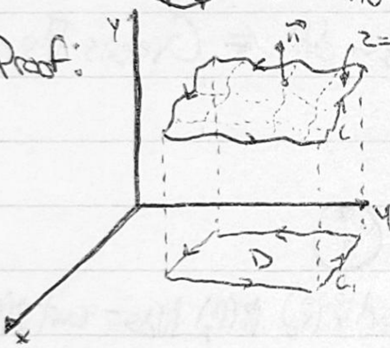
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

piece piece

- Can use righthand rule to figure out orientation
- Remark: C is a boundary of S $\Rightarrow \iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$

∂S is a symbol for boundary of S.

- Proof:



D is a projection of S on x-y plane

$$\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_S \operatorname{curl}\vec{F} \cdot \vec{n}\, dS$$

$$\iint_D \Big[\underbrace{-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)}_{\text{x component of curl}}\frac{\partial z}{\partial x} - \underbrace{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)}_{\text{y component of curl}}\frac{\partial z}{\partial y} + \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{z component of curl}}\Big] dA \Leftrightarrow$$

Remark: $\iint_S \vec{F}\, d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

- Suppose C_1 is parameterized by $x = x(t)$, $y = y(t)$. Then parameterization of C is $x = x(t)$, $y = y(t)$, & $z = g(x(t), y(t))$, for $t \in [a,b]$

- $\int_C \vec{F}\, d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt$
- Then $\int_a^b \Big[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\underbrace{\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)}_{\text{By chain Rule}}\Big] dt$.

$$\int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \Rightarrow \int_{C_1} \left(P + R\frac{\partial z}{\partial y}\right)dx + \left(Q + R\frac{\partial z}{\partial y}\right)dy$$

By Greens Theorem $\rightarrow$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P - R\frac{\partial z}{\partial x}\right)\right] dA$$

$$= \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y} + \cancel{\frac{\partial R}{\partial z}\frac{\partial x}{\partial x}\frac{\partial z}{\partial y}}\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\right] dA$$

- What is left is identical to
- Thus $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$

Interpretation and Remarks

- Greens Theorem is a special case of Stokes
  - Take $\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$, where $P = P(x,y)$ and $Q = Q(x,y)$
  - $\operatorname{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial Q}{\partial z}\right) - \hat{j}\left(\frac{\partial 0}{\partial z} - \frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

    $= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$
  - Let S be in (x,y) plane, $\vec{n} = \vec{k}$

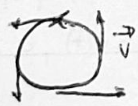
  - $\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$

    becomes

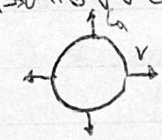
    $\int_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \cdot \hat{k}\, dS$

    $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ $(dS = dA)$. ← Green's Theorem
- Interpretation - What is $\operatorname{curl}\vec{F}$?
  - Let $\vec{F} = \vec{v}$ (velocity of fluid)
  - $\int_C \vec{v}\, d\vec{r} = \iint_S \operatorname{curl}\vec{v} \cdot \vec{n}\, dS$

    $\approx \iint_S \operatorname{curl}\vec{v}(P_0) \cdot \vec{n}(P_0)\, dS = \operatorname{curl}\vec{v}(P_0)\, \vec{n}(P_0) \iint dS = \operatorname{curl}\vec{v}(P_0)\hat{n}(P_0)\pi a^2$
  - Therefore $\operatorname{curl}\vec{v}(P_0) \cdot \hat{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int \vec{v}\, d\vec{r}$
  -



$\int \vec{v} \cdot d\vec{r} \neq 0$



$\int \vec{v}\, d\vec{r} = 0$

  - Curl - measure of rotation per unit area.

START —

Stoke's Theorem

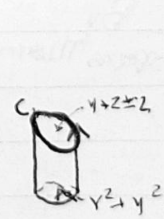
If S is given by $z = g(x,y)$:

$\iint \vec{F} \cdot dS$

$= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

for $\vec{F} = \langle P, Q, R \rangle$

- ① $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$ ②
- Applications: Either ① or ② is easier to complete.
- Example: Compute $\int_C \vec{F} \cdot d\vec{r}$, $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C is the intersection $x^2 + y^2 = 1$ and $y + z = 2$.



$\iint_C \vec{F} \cdot d\vec{r} = \iint \operatorname{curl}\vec{F} \cdot d\vec{S}$

$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y^2)}{\partial y}\right)\hat{k} = (1+2y)\hat{k}$

$g(x,y) = z = 2 - y$, $\vec{G} = \operatorname{curl}\vec{F}$

$\iint_S \vec{G} \cdot d\vec{S} = \iint_D \Big(\underbrace{-0 \cdot 0}_{P\frac{\partial g}{\partial x}} - \underbrace{0 \cdot (-1)}_{Q\frac{\partial g}{\partial y}} + \underbrace{(1+2y)}_{R}\Big) dA$

$= \iint_D (1 + 2y)\, dA$

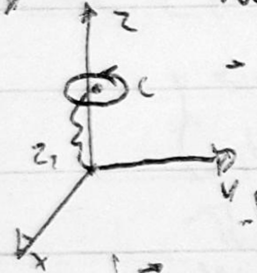
$= \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$

...

$= \pi$

- Example. Compute $\iint_S \operatorname{curl} \vec{F}\, d\vec{S}$ for $\vec{F} = \langle xz, zy, xy \rangle$, where S is part of $x^2+y^2+z^2=4$ inside cylinder $1 = x^2+y^2$.

- $\iint_S \operatorname{curl} \vec{F}\, d\vec{S} = \oint_C \vec{F}\, d\vec{r}$
- Need parameterization of C (circle)



- C satisfies $\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow z^2=3,\ z=\pm\sqrt{3}.$
- Parameterization of C: $x(t)=\cos t$, $y(t)=\sin t$, $z(t)=\sqrt{3}$; $x'(t)=-\sin t$, $y'(t)=\cos t$, $z'(t)=0$.

$$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} P\,dx + Q\,dy + R\,dz = \int_0^{2\pi} xz\,dx + yz\,dy + xy\,dz =$$

$$= \int_0^{2\pi} x(t)z(t)x'(t)\,dt + y(t)z(t)y'(t)\,dt + x(t)y(t)\,dt$$

$$= 0.$$

The Divergence Theorem

- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, $\vec{F} = \langle Q, R, \ldots$
- $\mathrm{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
- Theorem: Let E be a solid which is x-simple, y simple, z simple; Let S be the surface of E, with positive (outward) orientation. Let $\vec{F}$ be a smooth vector field.

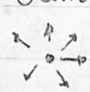
  Then $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \mathrm{div}\,\vec{F}\, dV$
- Remark: ① $\iiint_E \mathrm{div}\,\vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$ (∂E is boundary of E)

  ② $\iint_S \mathrm{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F}\, d\vec{r}$

  ③ $\int_a^b \frac{\partial f}{\partial x}\, dx = f(b) - f(a)$ (FTC)

  $\int_{[a,b]} \frac{\partial f}{\partial x}\, dx = \int_{\partial [a,b]} f$
- Interpretation: If $\vec{F}$ such that $\mathrm{div}\,\vec{F} = 0$, then $\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{flux over } S} = 0$
  - Think of electric field E where $\iint_S \vec{E} \cdot d\vec{S} = 0$ for closed surface (no charge in S) $\Rightarrow$ no source of field inside.
  - $\mathrm{div}\,\vec{F} = 0$ - source free vector field.
  - If at (x_0, y_0, z_0), we have $\mathrm{div}\,\vec{F}(x,y,z) \neq 0$ then $\begin{cases} \mathrm{div}\,F > 0 & \text{source} \\ \mathrm{div}\,F < 0 & \text{sink} \end{cases}$
  - $\vec{F} = \langle P, Q \rangle$

    Source

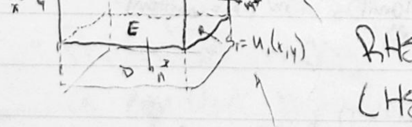
    Sink

    Source free

- Proof: Simplified Region E



$\vec{F} = \langle P, Q, R \rangle$

RHS: $= \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \underbrace{\iiint_E \frac{\partial P}{\partial x}\, dV}_{①} + \underbrace{\iiint_E \frac{\partial Q}{\partial y}\, dV}_{②} + \underbrace{\iiint_E \frac{\partial R}{\partial z}\, dV}_{③}$

LHS: $= \iint_S \vec{F} \cdot d\vec{S} = \iint (P\hat{i} + Q\hat{j} + R\hat{k}) \cdot \vec{n}\, dS = \underbrace{\iint_S P\hat{i} \cdot \vec{n}\, dS}_{ⓐ} + \underbrace{\iint_S Q\hat{j} \cdot \vec{n}\, dS}_{ⓑ} + \underbrace{\iint_S R\hat{k} \cdot \vec{n}\, dS}_{ⓒ}$

① = ⓐ ② = ⓑ ③ = ⓒ

③ $= \iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1}^{u_2} \frac{\partial R}{\partial z}\, dz\right] dA = \overbrace{\iint R(x,y,u_2) - R(x,y,u_1)\, dA}^{(*)}$

ⓒ $= \iint_S R\hat{k} \cdot \vec{n}\, dS = \iint_{S_1} R\hat{k} \cdot \vec{n}\, dS + \iint_{S_2} R\hat{k} \cdot \vec{n}\, dS + \iint_{S_3} R\underbrace{\hat{k} \cdot \vec{n}}_{0}\, dS$ since $\hat{k} \perp \vec{n}$

use surface integral formula over $z = g(x,y)$

$\vec{G} = \langle P_G, Q_G, R_G \rangle$

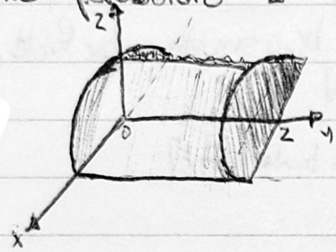
$\vec{G} = R\hat{k} = \langle 0, 0, R \rangle$

$\iint_S \vec{G} \cdot d\vec{S} = \iint_D \left(-P_G \frac{\partial g}{\partial x} - Q_G \frac{\partial g}{\partial y} + R\right) dA \Rightarrow$ ⓒ $= -\iint_{S_1} R(x,y,u_1(x,y))\, dA + \iint R(x,y,u_2(x,y))\, dA$

③ = ⓒ

Example of Application

- $\vec{F} = \langle x+\cos y,\ y+\sin z,\ z+e^x \rangle$
- Compute flux over surface, S, bounded by $z=0$, $y=0$, $y=2$, planes and paraboloid $z = 1-x^2$



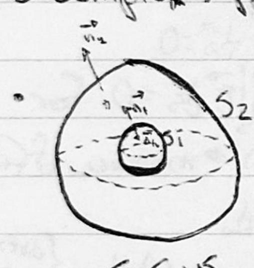
$\iint_S \vec{F}\cdot d\vec{S} =$

$\operatorname{div}\vec{F} = 1+1+1 = 3$

$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 3\,dV = \int_{-1}^{1}\int_0^2\int_0^{1-x^2} 3\,dz\,dy\,dx = 8$

If $\operatorname{div}\vec{F}$ is "simpler" than $\vec{F}$, then $\iint_S \to \iiint_E$

More Complex Applications



- S_1 is inside of S_2

S_1 is empty inside.

$S = S_1 \cup S_2$

$\iiint_E \operatorname{div}\vec{F} = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot(-\vec{n}_1)\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS$

$= -\iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS$

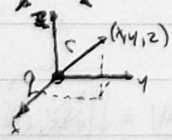
If $\operatorname{div} F = 0$, then $\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS$

We can replace complicated S_1 by simple S_2 (sphere).

$\iint E\cdot d\vec{S}$ (any surface enclosing (0,0,0)) $= \iint \vec{E}\,d\vec{S}$ (sphere, S at rad. 1) $= \iint \vec{E}\cdot\vec{n}\,dS$

- Electric Field

We know $\vec{E} = \frac{Kq}{|\vec{r}|^3}\cdot\vec{r}$, where $\vec{r} = \langle x, y, z\rangle$



$\vec{E} = \frac{Kq}{r^3}\cdot\vec{r}$ ← Coulomb's law (K = constant.)

$E = \frac{Kq}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot(x\hat{i}+y\hat{j}+z\hat{k})$

New identity for scalar field F, and vector field G

$\nabla\cdot(FG) = F(\nabla\cdot\vec{G}) + \nabla F\cdot\vec{G}$ or $F\operatorname{div}\vec{G} + \operatorname{grad}F\cdot\vec{G}$

$\nabla\vec{r} = \nabla\langle x,y,z\rangle = 3$

$\nabla F = \nabla(x^2+y^2+z^2)^{-\frac{3}{2}}$

$= -\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\langle 2x, 2y, 2z\rangle$

- Compute flux of $\vec{E}$ over S enclosing q

$\operatorname{div}\vec{E} = Kq\left[\frac{1}{r^3}(\nabla\cdot\vec{r}) + \left(\nabla\frac{1}{r^3}\right)\cdot\vec{r}\right] = Kq\left[\frac{3}{r^3} - \frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\langle 2x,2y,2z\rangle\cdot\langle x,y,z\rangle\right]$

$= Kq\left[\frac{3}{r^3} - 3r^{-5}(x^2+y^2+z^2)\right] = Kq\left[\frac{3}{r^3} - 3r^{-5}r^2\right] = Kq\left[\frac{3}{r^3} - 3r^{-3}\right] = 0$

if $r \neq 0$, $\operatorname{div}\vec{E} = 0$

$\vec{n} = \frac{\vec{r}}{|\vec{r}|} \Rightarrow \vec{E}\cdot\vec{n} = \frac{Kq}{r^3}\vec{r}\cdot\frac{\vec{r}}{r} = \frac{Kq}{r^4}\vec{r}\cdot\vec{r} = \frac{Kq}{r^4}r^2 = \frac{Kq}{r^2}$

$\iint_S \vec{E}\,\vec{n}\,dS = \iint_S Kq\,dS = Kq\iint_S dS = Kq\,4\pi = 4\pi Kq$

Flux of $\vec{E}$ produced by single charge q, is $\iint_S \vec{E}\cdot d\vec{S} = 4\pi Kq$ for any S surrounding the charge

Gauss's Law

Remarks About Divergence

- 
  - $S_r \to$ Sphere of radius, r
  - $B_r \to$ Solid ball bounded by S_r
  - What is the meaning of $\operatorname{div}\vec{F}$ at P?
- $\iint_{S_r} \vec{F} \cdot dS \overset{\text{div}}{\underset{\text{thm}}{=}} \iiint_{B_r} \operatorname{div}\vec{F}\, dV \approx \iiint_{B_r} \underbrace{\operatorname{div}\vec{F}(p)}_{\text{divergence of } \vec{F} \text{ at } p} dV = \operatorname{div}\vec{F}(p) \iiint_{B_r} dV = \operatorname{div}\vec{F}(p)\, V_r$

  $\underbrace{\quad}_{\text{flux of } \vec{F}}$

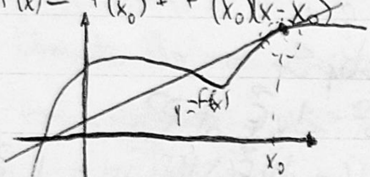
  $\Rightarrow \operatorname{div}\vec{F}(p) = \frac{1}{V}\iint_S \vec{F} \cdot ds \Rightarrow \operatorname{div}\vec{F}(p) = \lim_{r \to 0} \iint_{S_r} \vec{F} \cdot ds$
-  $\operatorname{div}\vec{F}(p) > 0$, P is a source of $\vec{F}$
-  $\operatorname{div}\vec{F}(p) < 0$, P is a sink of $\vec{F}$
- Physical Applications:
  - $\vec{E}$ = electric field ①
  - $\operatorname{div}\vec{E} = 0$ at any point where there is no charge difference
  - $\operatorname{div}\vec{E} = \frac{e}{\varepsilon_0}$ otherwise (e = charge, ε_0 = constant)
  - $\vec{B}$ = Magnetic field ②
  - $\operatorname{div}\vec{B} = 0$ everywhere (no single magnetic charge or mono pole.
  - By divergence theorem, ①, ② can be written in integral form
    - $\iint_S \vec{E}\, ds = \frac{1}{\varepsilon_0} \iiint e\, dV$ (Gauss' Law)
    - $\iint_S \vec{B}\, ds = 0$ (Gauss' Law for magnetism)
- Some identities of vector field theory Laplacian
  - $\nabla \cdot (\underbrace{\nabla f}_{\text{field}})^{\text{Laplacian}} = \nabla \cdot \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$
- Problem of finding f such that $\nabla^2 f = 0$ ← Laplace Equation is equivalent to finding potential of vector field with zero divergence
- Greens Identities $\to \frac{\partial f}{\partial n} \overset{\text{def}}{=} \nabla f \cdot \vec{n}$ $\quad \vec{n}$ = directional derivative.

  $\to \iint_S \frac{\partial f}{\partial n} dS = \underbrace{\iint_S \nabla f \cdot \vec{n}\, dS}_{\text{flux of } \nabla f} = \iint_S \nabla f\, d\vec{S} = \iiint_T \nabla \cdot \nabla f\, dV = \iiint_T \nabla^2 f\, dV$
- First Greens Identity $\to \iint_S f \frac{\partial f}{\partial n} dS = \iiint_T |\nabla f|^2 dV$

  $\to$ Proof: $\iint_S f \nabla f \cdot \vec{n}\, dS = \iint_S f \cdot \nabla f \cdot d\vec{S} = \iiint_T \nabla \cdot (f \nabla f) dV = \iiint_T (\nabla f \cdot \nabla f + f \nabla \cdot \nabla f) dV$

  $= \iiint_T (|\nabla f|^2 + f \nabla^2 f) dV = \iiint_T |\nabla f|^2 dV$
- Second Greens Identity $\to \iint_S f \frac{\partial g}{\partial n} ds = \iiint_T (f \nabla^2 g - g \nabla^2 f)\, dV$

Interpretation of Vector Field Operators (div, curl) in the Light of Linear Algebra

- We know that (for smooth fxns) $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots$
- $f(x) \approx f(x_0) + f'(x_0)(x-x_0)$ for $x \in (x_0-\epsilon, x_0+\epsilon)$ for small ϵ
- 

  $y = f(x_0) + f'(x_0)(x-x_0)$ ← linear function

- Above is local linear approximation
- Vector field is a function $\mathbb{R}^2 \to \mathbb{R}^2$ (in 2D) or $\mathbb{R}^3 \to \mathbb{R}^3$ (in 3D)
- In 2D, $F(x,y) = P\hat{i} + Q\hat{j} = \langle P, Q \rangle$
- We can write F as "column vector function" $F(x,y) = \begin{bmatrix} P \\ Q \end{bmatrix}$
  - $F(x,y) = F(x_0,y_0) + DF(x_0,y_0)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) + \dots$

    where $DF(x,y) = \begin{vmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{vmatrix}$ ← jacobian matrix
- If $\begin{bmatrix} x \\ y \end{bmatrix}$ is close to $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then
  - $F(x,y) \approx F(x_0,y_0) + \begin{bmatrix} P_x & P_y \\ Q_x & Q_y \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right)$ ← linear approximation of $\vec{F}$

Theorem

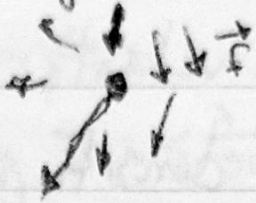
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A = \underbrace{A_{SKEW}}_{\text{skew-symmetric Matrix}} + \overbrace{A_{DIAG}}^{\text{Diagonal}} + \underbrace{A_{SYMM,0}}_{\text{symmetric with trace 0.}}$

  where $A_{SKEW} = \frac{1}{2}(A - A^T) = \begin{pmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{pmatrix}$, $A_{DIAG} = \text{trace}(A) I_{2\times2} = \begin{pmatrix} \frac{1}{2}(a+d) & 0 \\ 0 & \frac{1}{2}(a+d) \end{pmatrix}$

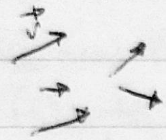
  and $A_{sym,0} = \frac{1}{2}(A + A^T) = \begin{pmatrix} \frac{a-d}{2} & \frac{b+c}{2} \\ \frac{b+c}{2} & \frac{d-a}{2} \end{pmatrix}$
- $DF(x,y) = \frac{1}{2}\begin{bmatrix} 0 & P_y - Q_x \\ Q_x - P_y & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} P_x + Q_y & 0 \\ 0 & P_x \end{bmatrix} + A_{sym}$
- Note: In 3D, we could say
  - $\vec{F} = \langle P, Q, 0 \rangle$
    - $\text{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial 0}{\partial z} = P_x + Q_y$
    - $\text{curl} F = 0\hat{i} + 0\hat{j} + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = (Q_x - P_y)\hat{k}$
- $DF(x,y) = \underbrace{\frac{1}{2}\begin{bmatrix} 0 & -\text{curl} F \\ \text{curl} F & 0 \end{bmatrix}}_{\text{Part of linearization responsible for local rotation}} + \underbrace{\frac{1}{2}\begin{bmatrix} \text{div} F & 0 \\ 0 & \text{div} F \end{bmatrix}}_{\text{Part responsible for contraction/expansion}} + \underbrace{A_{sym,0}}_{\text{Remaining Part.}}$

• Visualization - let P, Q be components of fluid velocity at (x,y)

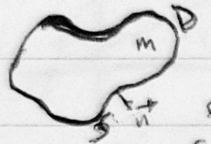
$$
\begin{cases}
\frac{\partial x}{dt} = P(x,y) \\
\frac{\partial y}{dt} = Q(x,y)
\end{cases}
$$



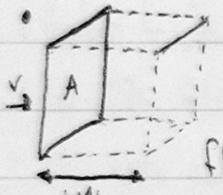
Fluid Dynamics (Application of Vector Calculus)



- $\vec{v}(x,y,z)$ - velocity at (x,y,z)
  $\delta(x,y,z)$ - density of mass at (x,y,z) ← Scalar function
  - $\delta(x,y,z) = \lim_{\Delta V \to} \frac{\Delta m}{\Delta V}$



- $m = \iiint \delta(x,y,z)\,dV = \iiint \delta\, dV$
  rate of change of $m = \frac{\partial}{\partial t}\left(\iiint \delta\, dV\right) = \iiint \frac{\partial \delta}{\partial t}\, dV$ ← This is due to only flow in/out of D
- flow across D = flux of $\delta \vec{v} = \iint_S \delta \vec{v} \cdot d\vec{S}$



- $V = A v \Delta t$
  $m = \delta A v \Delta t$

flow per area $= \delta v \Delta t$

flow rate per area $= \delta v$

note sign convention

By conservation of mass ⇒ $\iiint_D \frac{\partial \delta}{\partial t}\, dV = -\iint_S \delta \vec{v} \cdot d\vec{S}$

surface S

- $\iint_S \delta \cdot d\vec{S} = \iiint_D \operatorname{div}(\delta \vec{v})\, dV$ (Divergence theorem)
- $\iiint_D \frac{\partial \delta}{\partial t}\, dV = -\iiint_D \operatorname{div}(\delta \vec{v})\, dV$
- $\iiint_D \frac{\partial \delta}{\partial t} + \operatorname{div}(\delta \vec{v})\, dV = 0$
- True for any region, D. This can happen:
  → - $\boxed{\frac{\partial \delta}{\partial t} + \operatorname{div}(\delta \vec{v}) = 0}$ ← continuity equation
- Other eqn is Navier-Stokes eqn, which is derived starting from conservation of momentum
  - $\delta \frac{\partial \vec{v}}{\partial t} + \delta(\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \delta \vec{F}$

    pressure   force
- If δ is constant in time and space, $\frac{\partial \delta}{\partial t} = 0$, $\frac{\partial \delta}{\partial x} = 0 = \frac{\partial \delta}{\partial y} = \frac{\partial \delta}{\partial z}$
  (incompressible fluid)
  - $0 + \delta \operatorname{div} \vec{v} = 0$ $\delta > 0$

    $\boxed{\operatorname{div} \vec{v} = 0}$

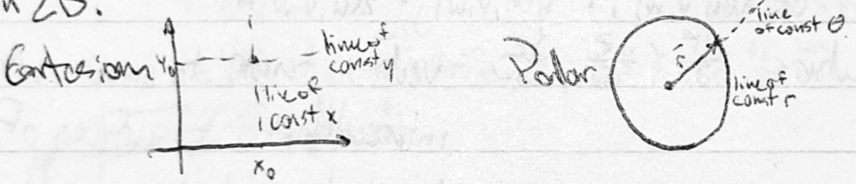
Differential Operators and Vector Calculus in Nonlinear Coordinates.

- $\vec{F}(x,y,z)$ in cartesian coordinates
- $\vec{\nabla}\times F$, $\vec{\nabla}\cdot\vec{F}$ are computed using $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$
- If $\vec{F} = \vec{F}(\rho, \theta, \varphi)$, is given in spherical coordinates, what is ∇?
- General Theory: Suppose we have (u, v, w) coordinate system

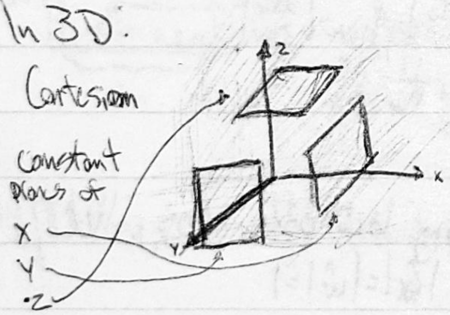
Cylindrical - r, θ, z

Spherical - ρ, θ, φ

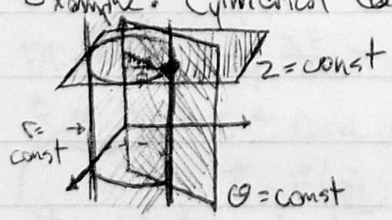
- In 2D:



- In 3D:



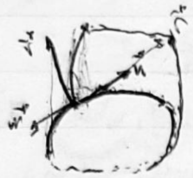
- In general, $u = u_0$, $w = w_0$, $v = v_0$ represent surfaces (in 3D) of constant coordinates, passing through (u_0, w_0, v_0). If planes tangent to them are perpendicular then we call the system (u,v,w) orthogonal
- Example: Cylindrical Coordinates



- For spherical coordinates, the same is true:

$$\left.\begin{array}{ll} \rho = \rho_0 & \text{sphere} \\ \varphi = \varphi_0 & \text{cone} \\ \theta = \theta_0 & \text{semi-plane} \end{array}\right\} \text{are orthogonal}$$

Curvilinear Coordinates

- 

  $x = x(u, v, w)$ $\quad$ x, y, z - cartesian

  $y = y(u, v, w)$ $\quad$ u, v, w - curvilinear

  $z = z(u, v, w)$

  $\vec{r} = x(u,v,w)\,\hat{i} + y(u,v,w)\,\hat{j} + z(u,v,w)\,\hat{k}$

- Consider vectors: $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}$ - vectors tangent to curves which are intersections of surfaces of constant coordinates.

- Scale factors are: $h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|$, $h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|$, $h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$
  - $\hat{u} = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u}$, $\hat{v} = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}$, $\hat{w} = \frac{1}{h_w}\frac{\partial \vec{r}}{\partial w}$
  - Triad of unit vector forming local base, also called local base vectors $|\hat{u}| = |\hat{v}| = |\hat{w}| = 1$
  - Any point in $\mathbb{R}^3$ can be expressed as linear combination of $\hat{u}, \hat{v}, \hat{w}$.

Important: $\hat{u}, \hat{v}, \hat{w}$ are dependent on position (Except cartesian's constant vectors)

- If $\hat{u} \perp \hat{v}$, $\hat{v} \perp \hat{w}$, & $\hat{u} \perp \hat{w}$ then $(\hat{u}, \hat{v}, \hat{w})$ is orthogonal
  - $\hat{u}\cdot\hat{v} = 0$, $\hat{v}\cdot\hat{w} = 0$, $\hat{u}\cdot\hat{w} = 0$

- Ex - ~~Col~~ unit

$$\vec{r} = \rho\sin\varphi\cos\theta\,\hat{i} + \rho\sin\varphi\sin\theta\,\hat{j} + \rho\cos\varphi\,\hat{k}$$

$$\frac{\partial r}{\partial \rho} = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$$

$$\frac{\partial r}{\partial \varphi} = \rho\cos\varphi\cos\theta\,\hat{i} + \rho\cos\varphi\sin\theta\,\hat{j} - \rho\sin\varphi\,\hat{k}$$

$$\frac{\partial r}{\partial \theta} = -\rho\sin\varphi\sin\theta\,\hat{i} + \rho\sin\varphi\cos\theta\,\hat{j}$$

Scale factor $\quad h_\rho = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{(\sin\varphi\cos\theta)^2 + (\sin\varphi\sin\theta)^2 + \cos^2\varphi} = 1$

$$h_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \sqrt{\rho^2(\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi)} = \rho$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \rho\sin\varphi$$

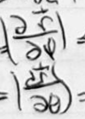
Local unit vector

$$\hat{\rho} = \frac{1}{h_\rho}\frac{\partial \vec{r}}{\partial \rho} = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$$

$$\hat{\varphi} = \frac{1}{h_\varphi}\frac{\partial r}{\partial \varphi} = \cos\varphi\cos\theta\,\hat{i} + \cos\varphi\sin\theta\,\hat{j} - \sin\varphi\,\hat{k}$$

$$\hat{\theta} = \frac{1}{h_\theta}\frac{\partial r}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

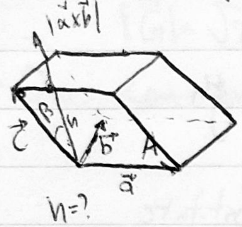
$$\hat{\rho} \times \hat{\varphi} = \hat{\theta}$$



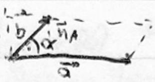
Alternative Notation:

spherical	cartesian
$\hat{e}_\theta = \hat{\theta}$	$\hat{e}_x = \hat{i}$
$\hat{e}_\varphi = \hat{\varphi}$	$\hat{e}_y = \hat{j}$
$\hat{e}_\rho = \hat{\rho}$	$\hat{e}_z = \hat{k}$

Applications of Triple Integrals



- Volume $= V = Ah$



$A = |\vec{a}| \cdot h_A$

$h_A = |\vec{b}| \sin\alpha$

$h = |\vec{c}| \cos\beta$

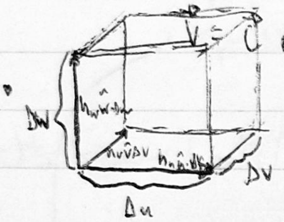
$A = |\vec{a}||\vec{b}| \sin\alpha$

$A = |\vec{a} \times \vec{b}|$

$V = hA$

$= |\vec{c}| \cdot |\vec{a} \times \vec{b}| \cdot \cos\beta \qquad \beta = \measuredangle(\vec{c}, \vec{a}\times\vec{b})$

$V = \vec{c} \cdot (\vec{a} \times \vec{b})$



- $\Delta V = h_w \hat{w} \Delta w \cdot (h_u \hat{u} \Delta u \times h_v \hat{v} \Delta v)$

$\Delta V = \underbrace{h_u h_v h_w}_{\text{Jacobian}} \Delta u \Delta v \Delta w$

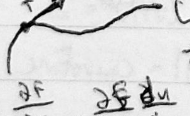
- $\iiint_T f(x,y,z)\,dV = \iiint_T f(x(uvw), y(uvw), z(uvw))\, h_u h_v h_w \, du\,dv\,dw$

- Change of ~~variables~~ coordinates in triple integrals

Transformation of Differential Operator

- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$



$C \leftarrow$ Curve in 3D.

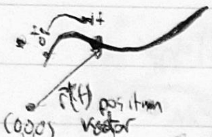
$\vec{r} = \vec{r}(s)$ arc length-parameterization

- $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial u}\frac{du}{ds} + \frac{\partial f}{\partial v}\frac{dv}{ds} + \frac{\partial f}{\partial w}\frac{dw}{ds}$
- $\frac{\partial f}{\partial s}$ = rate of change in direction of tangent unit vector $\hat{T} = \nabla f \cdot \hat{T}$
- $\hat{T} = \frac{\partial \vec{r}}{\partial s} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial \vec{r}}{\partial w}\frac{\partial w}{\partial s}$

$= h_u \hat{u} \frac{\partial u}{\partial s} + h_v \hat{v} \frac{\partial v}{\partial s} + h_w \hat{w} \frac{\partial w}{\partial s}$

- $\frac{\partial f}{\partial s} = \nabla f \cdot \hat{T} = f_u h_u \frac{\partial u}{\partial s} + f_v h_v \frac{\partial v}{\partial s} + f_w h_w \frac{\partial w}{\partial s}$
- $\nabla f = f_u \hat{u} + f_v \hat{v} + f_w \hat{w}$
- $\frac{\partial f}{\partial u} = f_u h_u, \; \frac{\partial f}{\partial v} = f_v h_v, \; \frac{\partial f}{\partial w} = f_w h_w$
- $f_u = \frac{1}{h_u}\frac{\partial f}{\partial u}, \; f_v = \frac{1}{h_v}\frac{\partial f}{\partial v}, \; f_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$
- $\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$ ← Gradient in Curvilinear Coordinates.

Frenet Frame



$\vec{r}(t) \to$ reparametrize to arclength

$\to S(t) \to \int_{t_0}^{t} |\vec{r}\,'(u)|\,du$

Solve for t in terms of ~~S~~ and s substitute, obtain $\vec{r}(s)$

$\vec{T}(s) = \vec{r}\,'(s) = \frac{d}{ds}(\vec{r}(s))$ tangent vector, $|\vec{T}(s)| = 1$

$\vec{N}(s) = \frac{\vec{T}'(s)}{|\vec{T}'(s)|}$ normal vector, $\vec{T} \perp \vec{N}$

- Binomial vector $= \vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$
- We have triad $\vec{T}, \vec{N}, \vec{B}$, $|\vec{T}| = |\vec{N}| = |\vec{B}| = 1$
  - $\vec{T} \cdot \vec{N} = \vec{N} \cdot \vec{B} = \vec{T} \cdot \vec{B} = 0$
  - They form base in $\mathbb{R}^3$
- Frenet frame - Orthogonal coordinate system with unit vectors $\vec{T}, \vec{N}, \vec{B}$
- Frenet changes with S (and also t), moving frame
- How $\vec{T}, \vec{N}, \vec{B}$ change with s? Compute Derivatives
  - Any vector $\vec{A} = a\vec{T} + b\vec{B} + c\vec{N}$

    $\vec{A} \cdot \vec{T} = \underbrace{a\vec{T} \cdot \vec{T}}_{1} + \underbrace{b\vec{B} \cdot \vec{T}}_{0} + \underbrace{c\vec{N} \cdot \vec{T}}_{0}$

  - $a = \vec{A} \cdot \vec{T}$, $b = \vec{A} \cdot \vec{B}$, $c = \vec{N} \cdot \vec{A}$
  - Now: $\vec{T}' = a\vec{T} + b\vec{B} + c\vec{N}$ $\leftarrow \vec{T}' = |\vec{T}'(s)|\vec{N}$

    $\vec{T}'(s) = K(s)\vec{N}(s)$ $\quad K = |\vec{T}'(s)| =$ curvature
  - ~~[illegible]~~ $\vec{N}' \cdot \vec{T} = 0 \Rightarrow \vec{N}' \cdot \vec{T} + \vec{N} \cdot \vec{T}' = 0$

    $\vec{N}' \cdot \vec{T} = -\vec{N} \cdot \vec{T}' = -\vec{N} \cdot K\vec{N} = -K \Rightarrow \vec{N}' \cdot \vec{T} = -K$
  - $\vec{N}'(s) \cdot \vec{B}(s) = ?$ Define $\tau(s) = \vec{N}'(s) \cdot \vec{B}(s) =$ torsion the deviation of the curve being planar
  - $\vec{N}' \cdot \vec{B} = \tau \Rightarrow \vec{N}' = 0 \cdot \vec{N} - K\vec{T} + \tau\vec{B}$
- Frenets First Formula - $\vec{T}'(s) = K(s)\vec{N}(s)$
- Frenets Second Formula - $\vec{N}'(s) = -K\vec{T}(s) + \tau\vec{B}(s)$
- Frenets Third formula - $\vec{B}'(s) = -\tau\vec{N}$
- $\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$ Matrix form is skew symmetric

• Def'n: $\vec{G} = \tau\vec{T} + K\vec{B}$ ← Darboux Vector

$|\vec{G}| = \sqrt{\tau^2 + K^2}$

• One can show
$$\left.\begin{aligned}\vec{T}' &= \vec{G} \times \vec{T}\\ \vec{N}' &= \vec{G} \times \vec{N}\\ \vec{B}' &= \vec{G} \times \vec{B}\end{aligned}\right\}$$
Darboux form of Frenet's equations.

• Def'n: span[T, N] - oscillating plane
span[T, B] - rectifying plane
span[N, B] = normal plane.
} Frenet Trihedron

Frenet's Frame Cont.

• In other parameterizations (non-arc length)

• $\vec{\alpha}(t)$ parametrization

• After re-parameterization to s, $\vec{\beta}(s)$

• $\vec{\beta}(s) = \vec{\beta}(s(t)) = \vec{\alpha}(t)$

• $\vec{\alpha}'(t) = \frac{\partial}{\partial t}(\vec{\beta}(s(t))) = \beta'(s(t))\, s'(t) = \vec{T}(s(t))v(t)$ $\quad s'(t) = v(t)$

• $\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = K\vec{N}v = vK\vec{N}$

• $\vec{\alpha}''(t) = \frac{\partial}{\partial t}\vec{\alpha}'(t) = \frac{\partial}{\partial t}(T(s(t))v(t)) = T(s(t))\,v'(t) + \frac{d[T(s(t))]}{dt}v(t)$

$= v'(t)T(s(t)) + v(t)T'(s(t))s'(t) = v'(t)T(s(t)) + v(t)^2 K(s(t))\vec{N}(s(t))$

$\vec{\alpha}'' = v'T + Kv^2\vec{N}$ ← acceleration vector

$\vec{\alpha}' = \vec{T}v$ ← ~~speed~~ velocity vector

$\vec{\alpha}$ ← position vector.

• K - curvature $\frac{1}{K}$ largest circle on K

One can show that

• for any parameterization $K = \frac{|\vec{\alpha}' \times \vec{\alpha}''|}{|\vec{\alpha}'|^3}$

• Proof: $\vec{\alpha}' \times \vec{\alpha}'' = (v\vec{T}) \times (v'\vec{T} + Kv^2\vec{N}) = \cancel{v\vec{T} \times v'\vec{T}}^{\,0} + v\vec{T} \times v^2 K\vec{N}$

$= v^3 K \vec{N} \times \vec{T} = K v^3 \vec{B}$

$|\vec{\alpha}' \times \vec{\alpha}''| = K v^3 |\vec{B}| = K v^3$

$K = \frac{|\vec{\alpha}' \times \vec{\alpha}''|}{|\vec{\alpha}'|^3} = \frac{|\alpha' \times \alpha''|}{v^3}$, but $v = |\alpha'|$.

Note: $\vec{\alpha}(t) = \vec{\sigma}(t)$

• We can perform similar calculations for $\tau, \vec{T}, \vec{N}, \vec{B}$

• Results: $\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$ $\qquad K = \frac{|\alpha' \times \alpha''|}{|\alpha'|^2}$

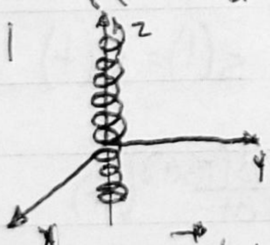
$\vec{T} = \frac{\alpha'}{|\alpha'|}$ $\qquad \vec{B} = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$

$\vec{N} = \vec{B} \times \vec{T} = \frac{(\alpha' \times \alpha'')}{|\alpha' \times \alpha''|} \times \frac{\alpha'}{|\alpha'|}$

• Ex $\vec{r}(t) = \langle a\cos t, a\sin t, 0 \rangle$

$K = \frac{1}{a}, \tau = 0$

• Ex $\vec{r}(s) = \langle a\cos(\frac{s}{c}), a\sin(\frac{s}{c}), b(\frac{s}{c}) \rangle$ where $a > 0$, $c = \sqrt{a^2 + b^2}$



$\vec{T}(s) = r'(s) = \frac{1}{c}\langle -a\sin(\frac{s}{c}), a\cos(\frac{s}{c}), b \rangle$

$\vec{T}'(s) = \underbrace{\frac{a}{c^2}}_{K} \underbrace{\langle -\cos(\frac{s}{c}), -\sin(\frac{s}{c}), 0 \rangle}_{\text{unit vector must be } \vec{N} \text{ by 1st frenet}}$

$\vec{N}(s) = \langle -\cos(\frac{s}{c}), -\sin(\frac{s}{c}) \rangle$; $|\vec{N}| = |\vec{T}| = 1$

$B(s) = \frac{1}{c}\langle b\sin(\frac{s}{c}), -b\cos(\frac{s}{c}), a \rangle$ $\quad \cancel{\vec{B}(s) = \vec{T} \times \vec{N} = \frac{1}{c}\langle b\cos(\frac{s}{c}), b\sin(\frac{s}{c}), a \rangle}$

$B'(s) = \frac{1}{c^2}\langle b\cos(\frac{s}{c}), b\sin(\frac{s}{c}), 0 \rangle$ $\quad \cancel{= \frac{b}{c}\langle -\cos(\frac{s}{c}), -\sin(\frac{s}{c}), a \rangle} = \frac{b}{c}$

$\cancel{\vec{B}' = -\frac{b}{c}\langle \cos(\frac{s}{c}), \sin(\frac{s}{c}), 0 \rangle}$

$\tau = \frac{b}{c^2} = \frac{b}{a^2 + b^2}$ $\qquad K = \frac{a}{c^2} = \frac{a}{a^2 + b^2}$

If $b = 0$, $\vec{r}(s)$ = circle and $\tau = 0$, and $K = \frac{1}{a}$.

~~[illegible]~~ Proposition

- A space curve is a straight line if and only if $\kappa = 0$ everywhere (curvature)

($\Rightarrow$) $\vec{r}(s) = s\vec{v} + \vec{c}$ where $\vec{v}$ & $\vec{c}$ are constant vectors

$\vec{T}(s) = \vec{r}\,'(s) = v$ constant $\Rightarrow \kappa = 0$

- ($\Leftarrow$) Assume $\kappa = 0$, $\vec{T}(s) = \vec{T}_0$ $\therefore \vec{r}\,'(s) = \vec{T}_0$ constant.

$\vec{r}\,'(s) = \vec{T}(s)$ $\therefore \vec{r}(s) = \underbrace{\vec{T}_0 s + \vec{C}}_{\text{Straight line}}$

- Remark: A space curve is planar if and only if $\tau = 0$ everywhere (torsion)

The only planar curve with $\kappa =$ constant $\neq 0$ is a circle

- The Fundamental Theorem of Space Curves
  - Let C_1 & C_2 be two curves, both which have the same nonvanishing curvature function $\kappa(s)$ and the same torsion function $\tau(s)$. Then the curves are congruent. That is, one can be moved rigidly (translated & rotated) so as to coincide exactly with the other.

- Proof
  - We require $\kappa \neq 0$ because $\hat{N}$ & $\hat{B}$ are not defined where $\kappa = 0$. Move C_2 rigidly so their Frenet frames are the same. Let $\hat{T}_1, \hat{T}_2, \hat{N}_1, \hat{N}_2, \hat{B}_1$ & $\hat{B}_2$ be unit tangents, normals, and binormals for the curves.
  - $f(s) = \hat{T}_1(s) \cdot \hat{T}_2(s) + \hat{N}_1(s) \cdot \hat{N}_2(s) + \hat{B}_1(s) \cdot \hat{B}_2(s)$
  - We take the derivative of this

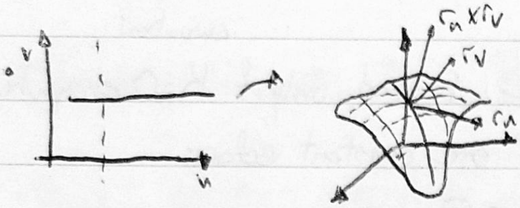
$$f'(s) = \hat{T}_1'(s) \cdot \hat{T}_2(s) + \hat{T}_1(s) \cdot \hat{T}_2'(s) + \hat{N}_1'(s) \cdot \hat{N}_2(s) + \hat{N}_1(s) \cdot \hat{N}_2'(s) + \hat{B}_1'(s) \cdot \hat{B}_2(s) + \hat{B}_1(s) \cdot \hat{B}_2'(s)$$

$$= \kappa \hat{N}(s) \cdot \hat{T}_2(s) + \kappa \hat{T}_1 \cdot \hat{N}_2 - \kappa \hat{T}_1 \cdot \hat{N}_2 + \tau \hat{B}_1 \cdot \hat{N}_2 - \kappa \hat{N}_1 \cdot \hat{T}_2 + \tau \hat{N}_1 \cdot \hat{B}_2 - \tau \hat{N}_1 \cdot \hat{B}_2 - \tau \hat{B}_1 \cdot$$

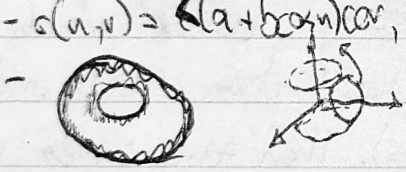
$$= 0$$

  - $\therefore f(s)$ is constant since frames coincident at $s = 0$. The constant fact must be:
  - $\hat{T}_1(s) \cdot \hat{T}_2(s) + \hat{N}_1(s) \cdot \hat{N}_2(s) + \hat{B}_1(s) \cdot \hat{B}_2(s) = 3$
  - $\dfrac{dr_1}{ds} = \hat{T}_1(s) = \hat{T}_2(s) = \dfrac{dr_2}{ds}$,

Parameterized Surfaces



- Definition: A regular parameterization of a subset $M \subset \mathbb{R}^3$ is a (C^3) one to one function
  $r: U \to M \subset \mathbb{R}^3$ so that $r_u \times r_v \neq 0$
  for some open set $U \subset \mathbb{R}^2$. A connected subset $M \subset \mathbb{R}^3$ is called a surface if each point has a neighbourhood that is regularly parameterized.
- Torus – $r(u,v) = \langle (a+b\cos u)\cos v, (a+b\sin u)\sin v, b\sin u \rangle$ $0 \leq u, v \leq 2\pi$
  –



- Unit Sphere – $r(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$ $0 < u < \pi$, $0 \leq v < 2\pi$
- Surface of Revolution – $r(u,v) = \langle f(u)\cos v, f(u)\sin v, g(u) \rangle$, $u \in I$, $0 \leq v \leq 2\pi$
- For a surface $\vec{r}(u,v)$ with a regular parameterization, tangent plane at $P = \vec{r}(u_0, v_0)$, is the plane $T_p = \text{span}\{\vec{r}_u, \vec{r}_v\}$ evaluated at u_0, v_0

  $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ - normal unit vector

Define $E = \vec{r}_u \cdot \vec{r}_u$; $F = \vec{r}_u \cdot \vec{r}_v$; $G = \vec{r}_v \cdot \vec{r}_v$.

$I_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ depends on u, v

Define $\vec{U} = a\vec{r}_u + b\vec{r}_v$; $\vec{V} = c\vec{r}_u + d\vec{r}_v$ $\in T_p$

$\vec{U} \cdot \vec{V} = E(ac) + F(ad+bc) + G(bd)$ ← First normal form

$\vec{U} \cdot \vec{V} \overset{\text{def}}{=} I_p(\vec{U}, \vec{V}) = [c\ d] \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

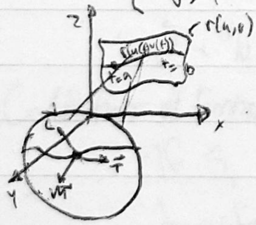
$|\vec{U}|^2 = |\vec{U}||\vec{U}| = U \cdot U = I_p(U,U) = Ea^2 + 2Fab + Gb^2$

- Result: Remember that surface area given by $\int_S |r_u \times r_v|^2 \, du\,dv$
  One can show that $|r_u \times r_v| = \sqrt{EG - F^2}$

Characterization of Surfaces - Local Theory.

- $\vec r(u,v)$ - parametrization on a surface
- $S = \iint_D |\vec r_u \times \vec r_v|\,du\,dv$ $r_u = \frac{\partial \vec r}{\partial u}$, $r_v = \frac{\partial \vec r}{\partial v}$
- Laplace $\to$ $(\vec a\times\vec b)\cdot(\vec c\times\vec d) = (\vec a\cdot\vec c)(\vec b\cdot\vec d) - (\vec a\cdot\vec d)(\vec b\cdot\vec c)$

  $(\vec a\times\vec b)\cdot(\vec a\times\vec b) = |a|^2|b|^2 - (\vec a\cdot\vec b)^2$ ← (special case)
- $|\vec r_u\times\vec r_v|^2 = (\vec r_u\times\vec r_v)\cdot(\vec r_u\times\vec r_v) = |\vec r_u|^2|\vec r_v|^2 - (r_u\cdot r_v)^2$
- Recall first fundamental (quadratic):
  - $I_p = \begin{vmatrix} E & F\\ F & G\end{vmatrix}$, where $E = r_u\cdot r_u$, $F = \vec r_u\cdot\vec r_v$, & $G = \vec r_v\cdot\vec r_v$
  - $|\vec r_u\times\vec r_v|^2 = EG - F^2$, $|\vec r_u\times\vec r_v| = \sqrt{EG-F^2}$
- Surface Area: $S = \iint_D \sqrt{EG-F^2}\,du\,dv = \iint_D\sqrt{\det I_p}\,du\,dv$.
- Arc Length of a Curve Lying on Surface $c(u,v)$, Let the curve be given by $u=u(t)$, $v=v(t)$. The curve has parameterization $\vec r(u(t),v(t))$
- Example $\vec r(u,v) = \langle u^2, u+v, v^3\rangle$ - surface S

  $\begin{cases} u = t+1 \\ v = t^2 \end{cases}$ $\vec r(t) = \langle (t+1)^2, t+1+t^2, t^6\rangle$ - curve on surface S.
- 

  Length of C for $t\in[a,b]$

  $L_{a-b} = \int_a^b \left|\frac{d}{dt}(r(u(t),v(t)))\right|dt = \dots$

  $= \int_a^b |\vec r_u u'(t) + \vec r_v v'(t)|\,dt$

  $= \int_a^b \sqrt{(\vec r_u u' + \vec r_v v')\cdot(\vec r_u u' + \vec r_v v')}\,dt$

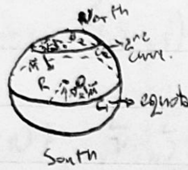
  $= \int_a^b\sqrt{r_u^2u'^2 + 2r_ur_vu'v' + r_v^2v'^2}\,dt$

  $L_{a-b} = \int_a^b\sqrt{Eu'^2+2Fu'v'+Gv'^2}\,dt = \int_a^b\sqrt{[u'\ v']\begin{bmatrix}E&F\\F&G\end{bmatrix}\begin{bmatrix}u'\\v'\end{bmatrix}}$
- Recall for curves in 3D, we have Frenet frame $\vec T,\vec N,\vec B$
- For surface, use normal vector $\to$ $\vec M = \frac{\vec r_u\times\vec r_v}{|\vec r_u\times\vec r_v|}$. Let $\vec N$ to be a

  normal vector to a curve C lying on a surface S, and $\vec T$ the tangent vector of the curve
- Define: $\vec L = \vec M\times\vec T$; $\vec T,\vec M,\vec L$ are unit orthogonal vectors at point P of S&C.
- Let $\vec r(s)$ - arclength parameter of C.
- $\vec T = \frac{d\vec r}{ds}$, Recall $\frac{d^2\vec r}{ds^2}\perp\vec T$
- Therefore $\frac{d^2r}{ds^2}$ is a linear combination of $\vec M$ & $\vec L$
  - $\vec r''(s) = \dots\vec M + \dots\vec L \Rightarrow K_n\vec M + K_g\vec L$
    normal curvature   geodesic curvature
  - $\vec r''(s) = \vec T' = K\vec N$
  - $K\vec N = K_n\vec M + K_g\vec L$
  - Therefore, $K = \sqrt{K_n^2+K_g^2}$

- K_n - curvature of "normal section", that is, curve obtained by intersecting S along the plane of $\vec{M}$ and $\vec{T}$

- K_g - measure of deviation of C from geodesic line.



For C_1: $K = \frac{1}{R}$ $\to$ one can show $K_n = K = \frac{1}{R}$, $K_g = 0$

For C_2: $K = \frac{1}{r}$ $\to$ but $K_g > 0$, $K_n \neq K$

Second Fundamental Form

- Recall: $K^2 = K_n^2 + K_g^2$ ← geodesic curvature; K_n: normal curvature. $\vec{r} = \vec{r}(u,v)$ surface.

  - $\vec{r} = \vec{r}(u(s), v(s))$ ← Curve C

  $\vec{r}\,' = \vec{r}_u u' + \vec{r}_v v'$

  $\vec{r}\,'' = \vec{r}_{uu} u'^2 + 2\vec{r}_{uv} + \vec{r}_{vv} v'^2 + \vec{r}_u u'' + \vec{r}_v v''$

- We want component of $\vec{r}\,''(s)$ along $\vec{M}$ (normal to surface)

  $\vec{r}\,'' \cdot \vec{M} = ?$

  $\vec{r}_u \cdot M = 0$, $\vec{r}_v \cdot M = 0$

  $K_n = \vec{r}\,'' \cdot \vec{M} = \vec{r}_{uu} \cdot \vec{M} u'^2 + 2 r_{uv} \cdot \vec{M} u' v' + \vec{r}_{vv} \vec{M} v'^2$

  Define: $L = \vec{r}_{uu} \cdot \vec{M}$, $m = \vec{r}_{uv} \cdot \vec{M}$, $n = \vec{r}_{vv} \cdot \vec{M}$

  $K_n = L u'^2 + 2m u'v' + n v'^2$

  $$K_n = [u' \; v'] \begin{bmatrix} L & m \\ m & n \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

  $\mathrm{II}_p = \begin{bmatrix} L & m \\ m & n \end{bmatrix} \Rightarrow$ Second fundamental form.

  $$K_n = \frac{L\,du^2 + 2m\,du\,dv + n\,dv^2}{ds^2} = L\left(\frac{du}{ds}\right)^2 + 2m\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + n\left(\frac{dv}{ds}\right)^2$$

- Recall from 1st form: $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

  $$K_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

  $$K_n = \frac{L + 2M\frac{dv}{du} + N\left(\frac{dv}{du}\right)^2}{E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2}$$

  $$K_n = \frac{L + 2M\mu + N\mu^2}{E + 2F\mu + G\mu^2}$$

  E, F, G, L, M, N depend on surface along. μ depends on curve C_μ.

• We want to find max & min of k_n as a function of μ.

$$\frac{dk_n}{d\mu}=0=\left(\frac{L+2M\mu+N\mu^2}{E+2F\mu+G\mu^2}\right)'=\frac{(2M+2N\mu)(E+2F\mu+G\mu^2)-(L+2M\mu+N\mu^2)(2F+2G\mu)}{(E+2F\mu+G\mu^2)^2}$$

numerator $=0 \Rightarrow \dfrac{L+2M\mu+N\mu^2}{E+2F\mu+G\mu^2}=\dfrac{M+N\mu}{F+G\mu}$

at the critical point $k_n=\dfrac{M+N\mu}{F+G\mu}$ ← solve for μ & substitute into equation; numerator $=0$.

This yields: $(EG-F^2)k_n^2-(EN-2FM+GL)k_n+LN-M^2=0$.

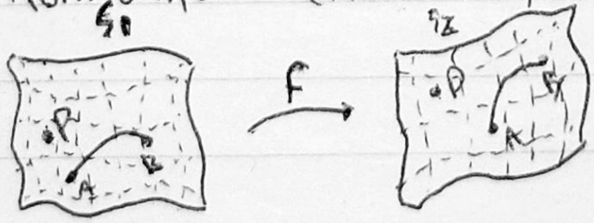
LN has solution k_1^{min} & k_2^{max}

$$K=k_1k_2=\frac{LN-M^2}{EG-F^2}$$ ← Gauss Curvature

$$H=\frac{k_1+k_2}{2}=\frac{EN-2FM+GL}{2(EG-F^2)}$$ ← Average Curvature.

• Theorema Egregium: Curvature $K=k_1k_2$ depends only on EFG and their partial derivatives of first & second order.

• Corollary: - E, F, G appear in distance formula
- If S_1, S_2 are related by distance-preserving transformation (isometric), they have the same I_p (same EFG).

• 

then P and its image have the same Gaussian Curvature.

- Therefore, plane & sphere cannot be related by isometric transformation.