Quadratic Interpolation

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History

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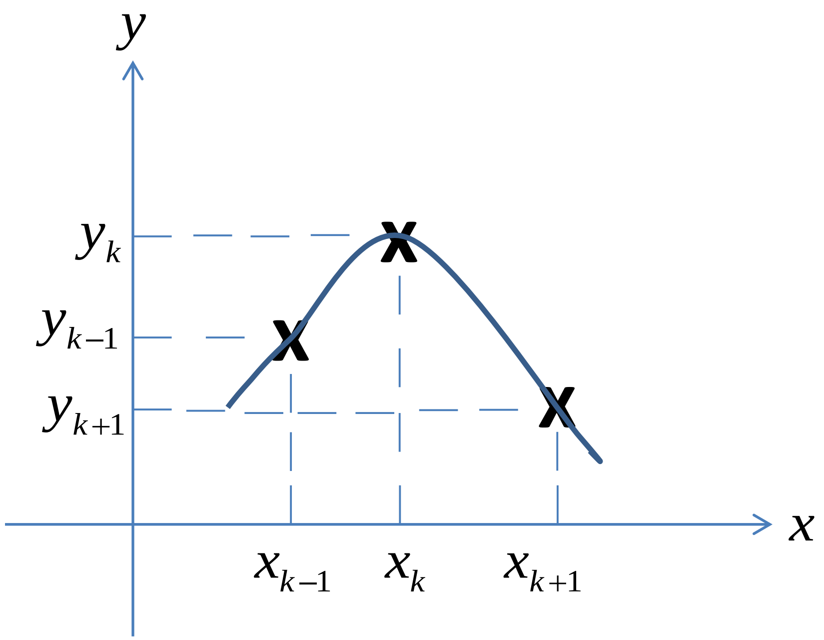
Quadratic interpolation

- The idea is to find interpolated point based on a parabolic function defined by the polynomial equation:

$$y = ax^2 + bx + c$$

- In this case we need 3 points of the parabola:



- The next point, interpolated, is given directly by the polynomial equation after finding the unknown variables: a, b and c

Quadratic interpolation

- The system of linear equations would be:

$$\begin{cases} y_{k-1} = ax_{k-1}^2 + bx_{k-1} + c \\ y_k = ax_k^2 + bx_k + c \\ y_{k+1} = ax_{k+1}^2 + bx_{k+1} + c \end{cases}$$

- There are many solutions types for this system (Cramer, Gauss, …).

- Here we will show the straight solution procedure based on eliminations/substitutions.

- At the end we will detail a practical example based on a FFT peak

Quadratic interpolation

- By eliminating c (retrieve the 1st line), one gets:

$$\begin{cases} y_{k-1} = ax_{k-1}^2 + bx_{k-1} + c \\ y_k - y_{k-1} = a(x_k^2 - x_{k-1}^2) + b(x_k - x_{k-1}) \\ y_{k+1} - y_k = a(x_{k+1}^2 - x_k^2) + b(x_{k+1} - x_k) \end{cases}$$

- Isolate b (by dividing lines 2 and 3):

$$\begin{cases} y_{k-1} = ax_{k-1}^2 + bx_{k-1} + c \\ \dfrac{y_k - y_{k-1}}{x_k - x_{k-1}} = a\,\dfrac{x_k^2 - x_{k-1}^2}{x_k - x_{k-1}} + b \\ \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k} = a\,\dfrac{x_{k+1}^2 - x_k^2}{x_{k+1} - x_k} + b \end{cases}$$

- As x points are distinct, here we don’t have 0 division

Quadratic interpolation

- By eliminating b (retrieve the 2nd line), one gets:

$$\begin{cases} y_{k-1} = a x_{k-1}^2 + b x_{k-1} + c \\ \dfrac{y_k - y_{k-1}}{x_k - x_{k-1}} = a\,\dfrac{x_k^2 - x_{k-1}^2}{x_k - x_{k-1}} + b \\ \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k} - \dfrac{y_k - y_{k-1}}{x_k - x_{k-1}} = a\left(\dfrac{x_{k+1}^2 - x_k^2}{x_{k+1} - x_k} - \dfrac{x_k^2 - x_{k-1}^2}{x_k - x_{k-1}}\right) \end{cases}$$

- Here a is given by the 3rd line and then we can find easily b using 2nd line so finally c is deduced in the 1st line.

Quadratic interpolation FFT example

- On this example we suppose that we have a FFT result in the frequency domain where we are in presence of a peak (sinusoidal wave in time of a Doppler signal for ex)

- The goal is to find the maximum of the peak. Instead of taking the maximum between 3 points of the peak, we use quadratic interpolation for a better accuracy of this maximum.

- As x step here is 1 because x are digital indexes (at sampling frequency), we can write the previous linear system:

$$\begin{cases} y_{k-1} = ax_{k-1}^2 + bx_{k-1} + c \\ y_k - y_{k-1} = a(x_k^2 - x_{k-1}^2) + b \\ y_{k+1} + y_{k-1} = a(x_{k+1}^2 + x_{k-1}^2) \end{cases}$$

Quadratic interpolation FFT example

- Or, because of 1 step on x points, one gets:

$$
\begin{aligned}
x_k^2 - x_{k-1}^2 &= (x_k + x_{k-1}) \bullet (x_k - x_{k-1}) \\
&= x_k + x_{k-1}
\end{aligned}
$$

- So the 2nd line in the system is simplified by:

$$
\begin{cases}
y_{k-1} = a x_{k-1}^2 + b x_{k-1} + c \\
y_k - y_{k-1} = a(x_k + x_{k-1}) + b \\
y_{k+1} + y_{k-1} = a(x_{k+1}^2 + x_{k-1}^2)
\end{cases}
$$

Quadratic interpolation FFT example

- So the solution is:

$$\begin{cases} a = \dfrac{y_{k+1} + y_{k-1}}{x_{k+1}^2 + x_{k-1}^2} \\ b = y_k - y_{k-1} - a(x_k + x_{k-1}) \\ c = y_{k-1} - a x_{k-1}^2 - b x_{k-1} \\ \end{cases}$$

- Or the derivative function on the maximum is 0, so one gets:

$$\left.\frac{dy}{dx}\right|_{x=x_{\max}} = 2a x_{\max} + b = 0$$

$$\Rightarrow \begin{cases} x_{\max} = -\dfrac{b}{2a},\ for\ a \neq 0 \\ y_{\max} = a x_{\max}^2 + b x_{\max} + c \end{cases}$$