Sampling basics

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History

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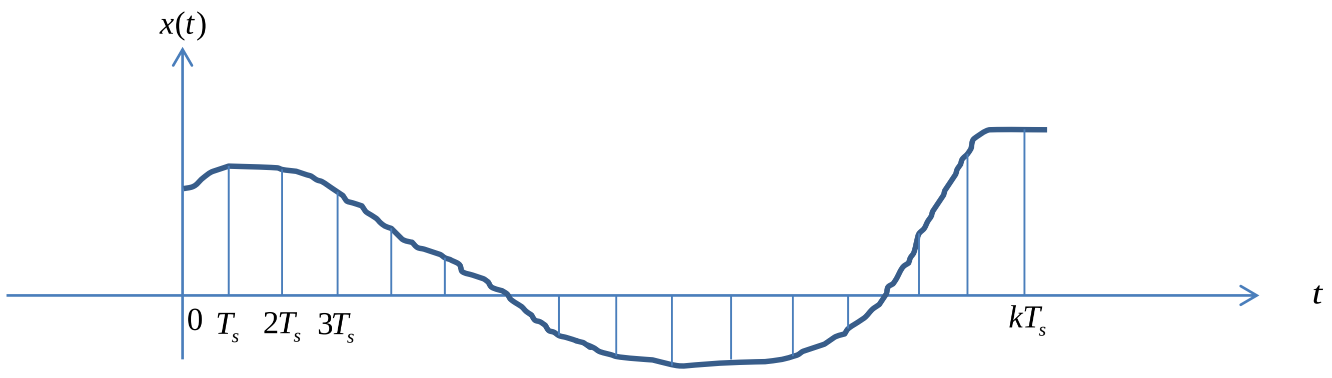
Sampling theory

- Digital sampling of the analog signal x(t):

$$
\begin{aligned}
x_s(t) &= x(t)\,\text{Ш}_{T_s}(t) \\
&= x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s) \\
&= \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s)
\end{aligned}
$$

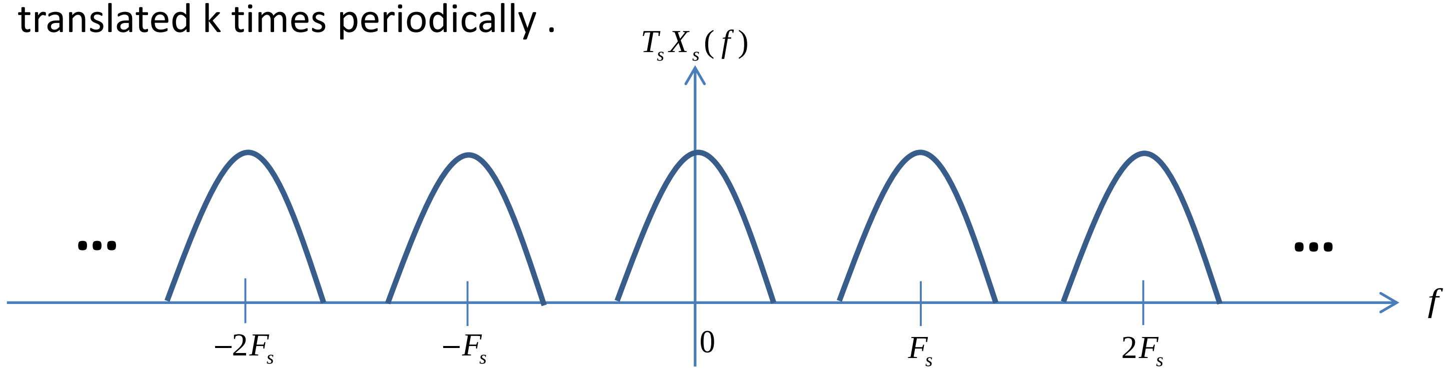


Sampling theory

- Spectral equation of sampled signal:

$$X_s(f) = X(f) * \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta(f - \frac{k}{T_s})$$
$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(f) * \delta(f - \frac{k}{T_s})$$
$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T_s}\right)$$
$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(f - kF_s)$$

- It means that in frequency the **sampled signal spectrum is periodized:** it's a very important information because it shows that the digital spectrum is exactly the analog spectrum translated k times periodically .



- For more simplicity, the amplitude axis will be represented just with $X_s(f)$ for digital spectrum and with $X(f)$ for analog spectrum

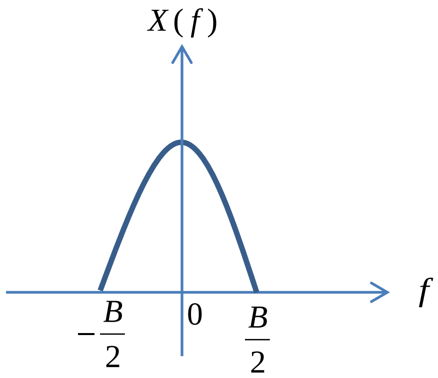
Shannon theorem

- Now, how we can sample correctly without information loss because this periodization will introduce **aliasing** in our bandwidth of interest if it is not chosen correctly?

- The response is given by the Shannon theorem where the minimum sampling frequency must be:

$$F_s > 2F_{\max}$$

- Consider a baseband signal with a bandwidth B centered around 0, so we have:

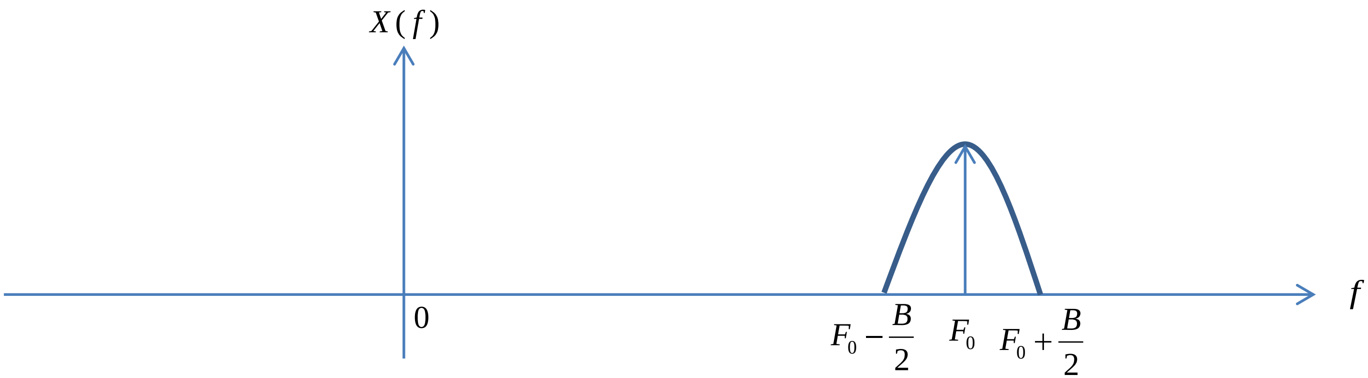


$$\begin{aligned} F_s &\geq 2F_{\max} \\ &\geq 2\frac{B}{2} \\ &\geq B \end{aligned}$$

- It means that to sample correctly this signal in order to avoid aliasing, we must choose a sampling frequency at least equal B (in practice we go from 2xB to 10xB).

Shannon theorem

- Now, how can we sample correctly a wideband signal around a carrier F_0 (it can be an intermediate frequency for ex) ?



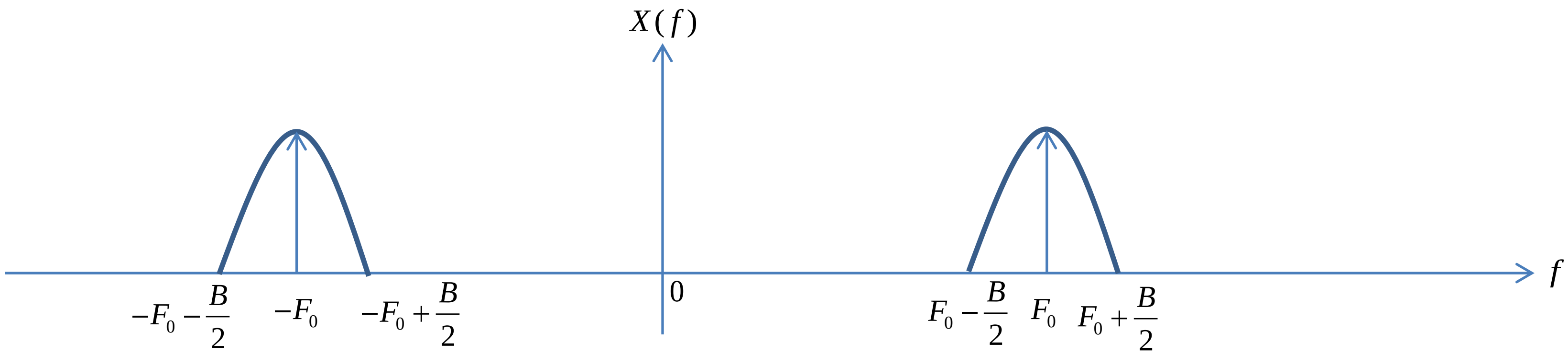
- According to Shannon, we have:

$$
\begin{aligned}
F_s &\geq 2F_{\max} \\
&\geq 2\left(F_0 + \frac{B}{2}\right) \\
&\geq 2F_0 + B
\end{aligned}
$$

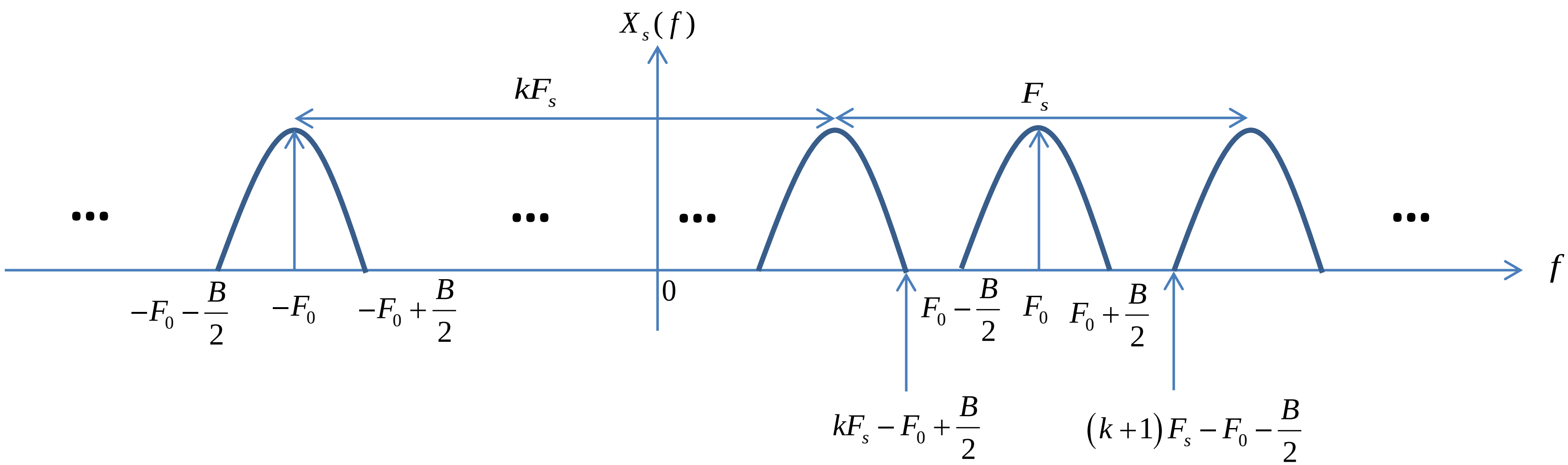
- If the IF F_0 is too high we will need an expensive ADC (or it might be impossible if it is a higher RF frequency): in this case we will see the under sampling principle.

Under sampling principle

- Consider the following wideband signal (eg: BPSK modulation with an IF of F_0):



- Let's represent the digital spectrum after sampling at F_s (periodization):



- We can easily see that we can under sample in order to take the signal close to DC, by this way we reduce the processor burden. Now let's find the right conditions to do this without aliasing.

Under sampling principle

- In order to avoid spectrum aliasing, we deduce the following system equations:

$$\begin{cases} kF_s - F_0 + \dfrac{B}{2} < F_0 - \dfrac{B}{2} \\ (k+1)F_s - F_0 - \dfrac{B}{2} > F_0 + \dfrac{B}{2} \end{cases}$$

- So it gives the **sampling frequency expression**:

$$\frac{2\left(F_0 + \dfrac{B}{2}\right)}{k+1} < F_s < \frac{2\left(F_0 - \dfrac{B}{2}\right)}{k}$$

- Knowing that:

$$\frac{2\left(F_0 + \dfrac{B}{2}\right)}{k+1} < \frac{2\left(F_0 - \dfrac{B}{2}\right)}{k}$$

- We get the following condition:

$$k < \frac{F_0 - \dfrac{B}{2}}{B}$$

Under sampling example: signal with IF

- Eg: BPSK modulation with the following parameters:

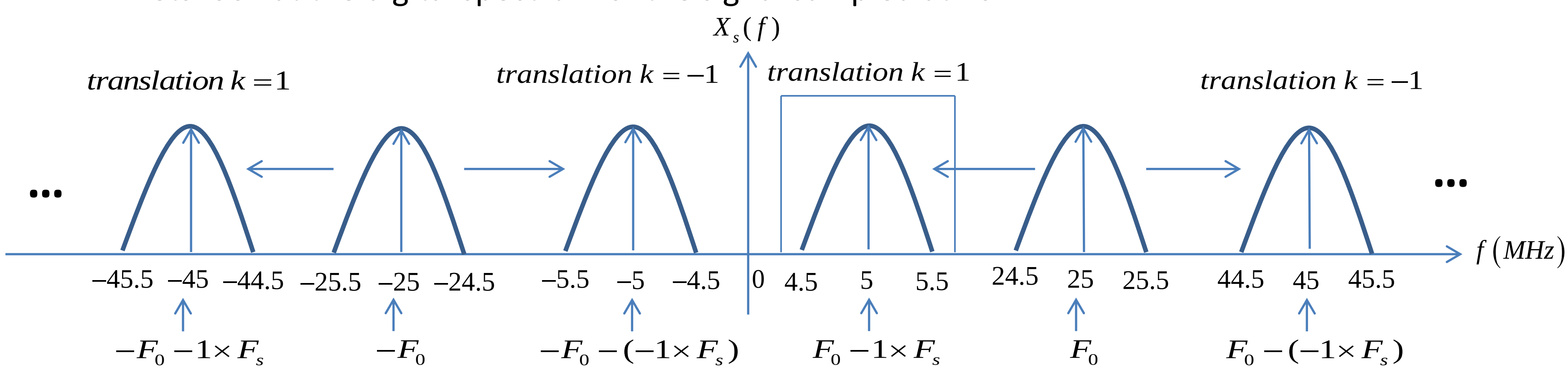
$$F_0 = 25\,MHz$$
$$B = 1\,MHz$$

- We can deduce the sampling frequency by the under sampling formula:

$$\frac{2(25+0.5)}{k+1} < F_s < \frac{2(25-0.5)}{k}$$

- $For\ k = 2 \Rightarrow 17 < F_s < 24.5$, so we can choose for ex a **sampling frequency equal to 20 MHz** (which is lower than Shannon theoretical frequency that must be greater than 2x25.5 = 55.5 MHz).

- Lets look at the digital spectrum of the signal sampled at 20 MHz:



- For ex, we can take the signal of interest which is located around 5MHz and digitally we can shift it into baseband with a NCO based on SIN/COS LUT.

Under sampling example: signal with IF

- For more readability we have not represented the translations of:

  - k = 2 for F_0 => spectrum around 25-2x20 = -15 MHz
  - k = -2 for $-F_0$ => spectrum around -25-(-2x20) = 15 MHz
  …

- It's important to understand here that we are not under Shannon's conditions concerning our original spectrum around F_0 but of course we always follow Shannon here because we consider a translated period of our original spectrum which is located around 5 MHz (< 20 MHz).

- After baseband shift with NCO, the spectrum of interest will be from -0.5 to 0.5 MHz:

$$y_{NCO}\left(nT_s\right) = e^{-j2\pi f_0 nT_s}$$

$$where\ f_0 = 5\ MHz\ and\ T_s = \frac{1}{F_s}$$

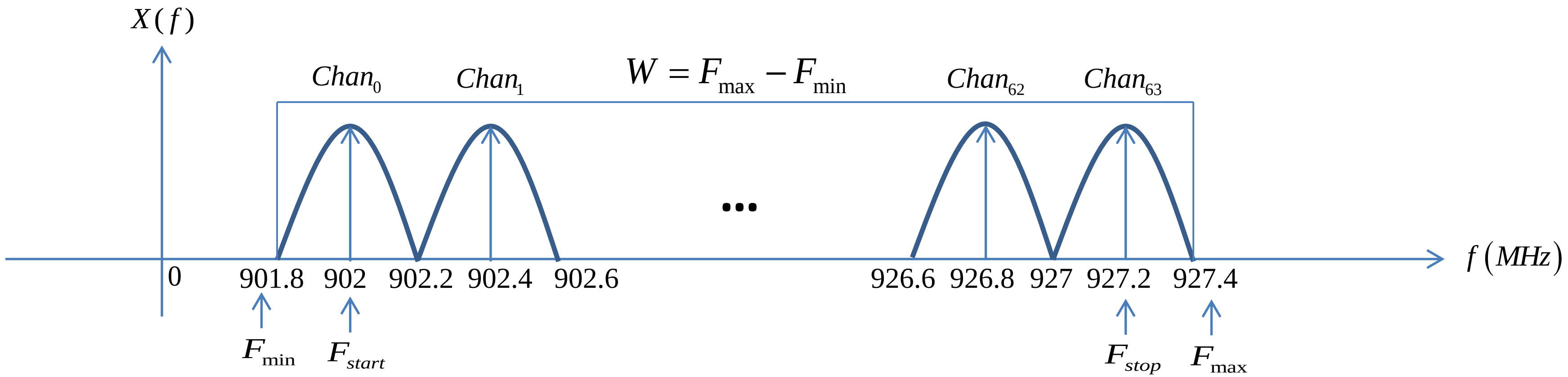
- We must be carefully if we want to decimate because of the aliasing (new spectrum periodization due to decimation): for ex if we decimate by 5 after baseband shift, it means that we are working now with a new sampling frequency:

$$F_{dec} = \frac{F_s}{5} = 4\ MHz$$

- That's why we must filter (low pass filtering for ex with a cutoff at $\frac{F_{dec}}{2}$) before decimation.

Shannon example: sniffer receiver

- Suppose that we need to do a sniffer on 64 channels with a channel bandwidth of B = 400 kHz.
- Start frequency is set to 902 MHz and stop frequency is set to 927.2 MHz.
- Let's represent the spectrum on TX side :



- So we have a wideband of:

$$
\begin{aligned}
W &= F_{\max} - F_{\min} \\
&= \left(F_{stop} + \frac{B}{2}\right) - \left(F_{start} - \frac{B}{2}\right) \\
&= 927.4 - 901.8 \\
&= 25.6\,MHz
\end{aligned}
$$

Shannon example: sniffer receiver

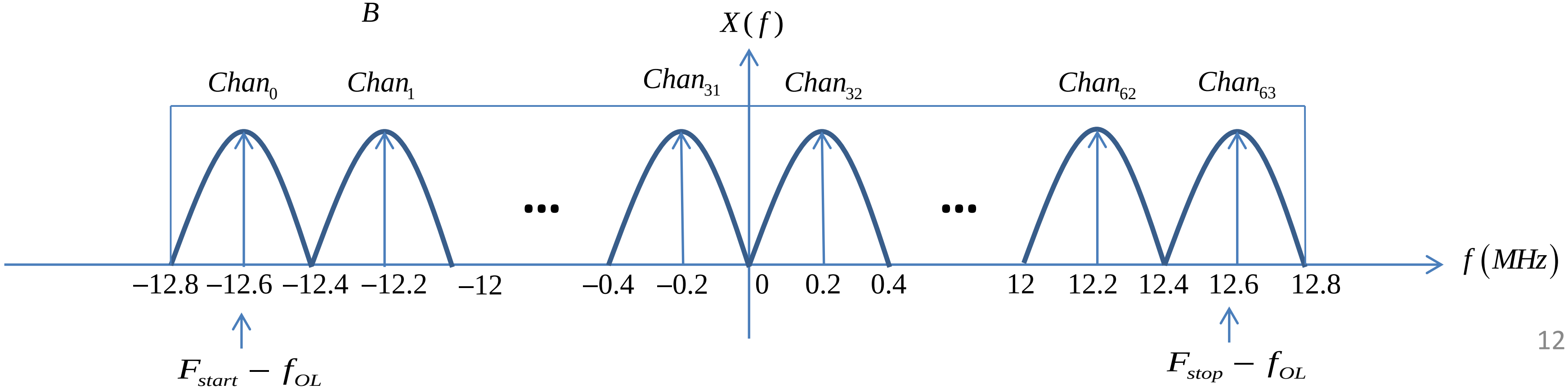
- In order to have the lowest sampling frequency , we are going to be at Shannon limit frequency i.e. at 2x maximum frequency.

- To perform this let's choose on the RX side the OL RF frequency at:

$$f_{OL} = F_{start} + \frac{F_{stop} - F_{start}}{2}$$
$$= 902 + \frac{927.2 - 902}{2}$$
$$= 914.6\,MHz$$

- Note that with this RX OL frequency we avoid a baseband spectrum around DC so by this way the DC is not affected (which can deteriorate the baseband signal). Let's look at the RX spectrum:

$$W = 25.6\,MHz$$
$$\Rightarrow N = \frac{W}{B} = 64\,channels$$



Shannon example: sniffer receiver

- According to Shannon, we have:

$$F_s \geq 2F_{\max}$$
$$\geq 2 \times 12.8$$
$$\geq 25.6\,MHz$$

- So here we are going to sample **at 25.6 MHz** which is exactly our wideband.

- Before ADC sampling, wee need a band-pass analog anti-aliasing filter from -12.8 to 12.8 MHz (=> symmetrical low-pass)

- Then, ADC samples are sent into a FPGA for ex and the channels are deduced by the following flow:

  - decimation by 64
  - polyphase FIR filtering
  - complex FFT of 64 points

- Finally, for each baseband channel of B = 400 kHz (+/-200 kHz) we are with a sampling frequency at:

$$F_s^{'} = \frac{F_s}{N}$$
$$= \frac{25.6}{64}$$
$$= 400\,kHz$$

- Of course, if higher sampling frequencies are needed for baseband channels, we can interpolate $F_s^{'}$ to higher rates (cubic interpolator for ex with a rate of 1.66675 in order to go from 400 kHz to 666.7 kHz).