Stat 361 - Recitation 6

Monte Carlo Integration

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12-13 / 03 / 2020

Monte Carlo Integration:

Consider the problem of estimating $\theta = \int_0^1 g(x)dx$. If $X_1, ..., X_m$ is a random Uniform(0,1) sample then

$$\hat{\theta} = \overline{g_m(X)} = \frac{1}{m}\sum_{i=1}^{m} g(X_i)$$

converges to E[g(X)] = θ with probability 1, by Strong Law of Large Numbers (SLLN). The simple Monte Carlo estimator of $\int_0^1 g(x)dx$ is $\overline{g_m(X)}$.

To summarize, the simple Monte Carlo estimator of the integral $\theta = \int_a^b g(x)dx$ is computed as follows;

- Generate $X_1, ..., X_m$ i.i.d. from Uniform(a,b).
- Compute $\overline{g(X)}) = \frac{1}{m}g(X_1)$.
- $\hat{\theta} = (b-a)\overline{g_(X)}$.

The generic problem: Evaluate

$$\int_X g(x)f(x)dx = E_f[g(x)]$$

The Convergence;

$$\overline{g_m(X)} = \frac{1}{m}\sum_{i=1}^{m} g(X_i) \to \int_X g(x)f(x)dx = E_f[g(x)]$$

is valid by **Strong Law of Large Numbers.**

Question 1:

Based on the function given above

$$f(x) = x^3 + 4x \quad 0 < x < 1$$

Find an approximation to the integral

$$I = \int_0^1 x^3 + 4xdx$$

```
f1 <- function(x) x^3 + 4*x

a1 <- 0; b1 <- 1
n <- 10^5

x1 <- runif(n,a1,b1)

estimate1 <- mean(sapply(x1,f1)) * (b1-a1)
estimate1
```

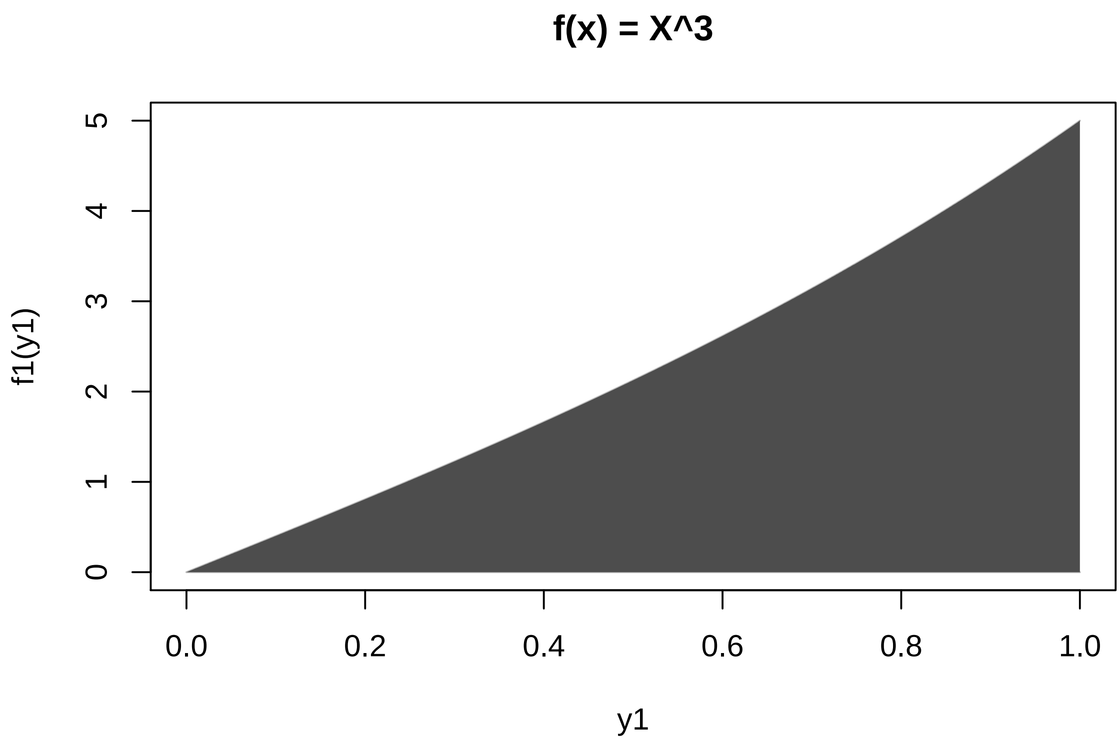
```
## [1] 2.254319
```

Is it so ?

```
integrate(f1,a1,b1)
```

```
## 2.25 with absolute error < 2.5e-14
```

```
y1 <- seq(a1,b1,0.001)
y.low1 <- rep(0,times = length(y1))

plot(y1, f1(y1), type = "n",
     main = "f(x) = X^3",
     lwd = 2.5)

lines(y1,f1(y.low1), col = 'grey')
lines(y1,f1(y1), col = 'grey')
polygon(c(y1, rev(y1)), c(f1(y1), rev(f1(y.low1))),
     col = "grey30", border = NA)
```



Question 2:

Based on the function given above

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} \quad 0 < x < 1$$

Find an approximation to the integral

$$I = \int_0^1 \sqrt{x + \sqrt{x + \sqrt{x}}} dx$$

```
f2 <- function(x) sqrt(x + sqrt(x + sqrt(x)))

a2 <- 0; b2 <- 1
n <- 10^5

x2 <- runif(n,a2,b2)

estimate2 <- mean(f2(x2)) * (b2-a2)
estimate2
```

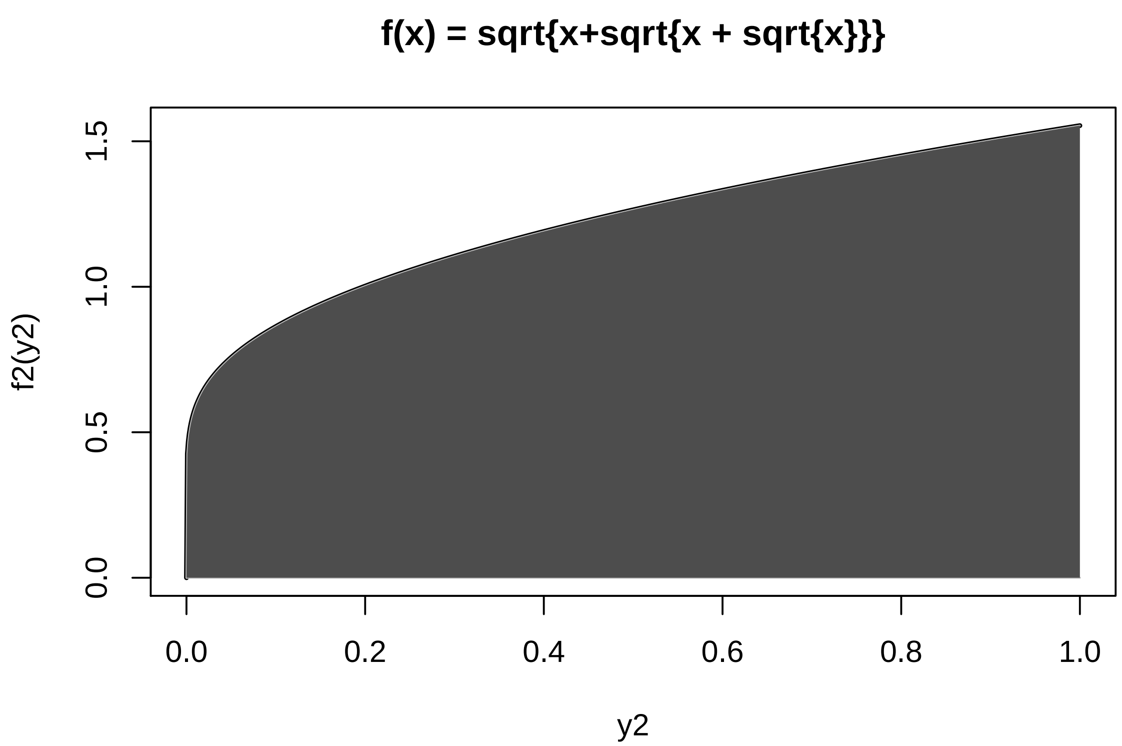
```
## [1] 1.219212
```

Is it so ?

```r
integrate(f2,a2,b2)$value
```

```
## [1] 1.219005
```

```r
y2 <- seq(a2,b2,0.001)
y.low2 <- rep(0,times = length(y2))

plot(y2, f2(y2), type = "l",
     main = "f(x) = sqrt{x+sqrt{x + sqrt{x}}}",
     lwd = 2.5)

lines(y2,f2(y.low2), col = 'grey')
lines(y2,f2(y2), col = 'grey')
polygon(c(y2, rev(y2)), c(f2(y2), rev(f2(y.low2))),
     col = "grey30", border = NA)
```



Question 3:

Based on the function given above

$$f(x) = x^2 + 2x + 5, \quad -1 < x < 3$$

Compute the integral of

$$I = \int_{-1}^{3} x^2 + 2x + 5dx$$

```
f3 <- function(x) x^2 + 2*x + 5

n <- 10^5
a3 <- -1; b3 <- 3

x3 <- runif(n,a3,b3)

estimate3 <- mean(sapply(x3,f3)) * (b3-a3)
estimate3
```

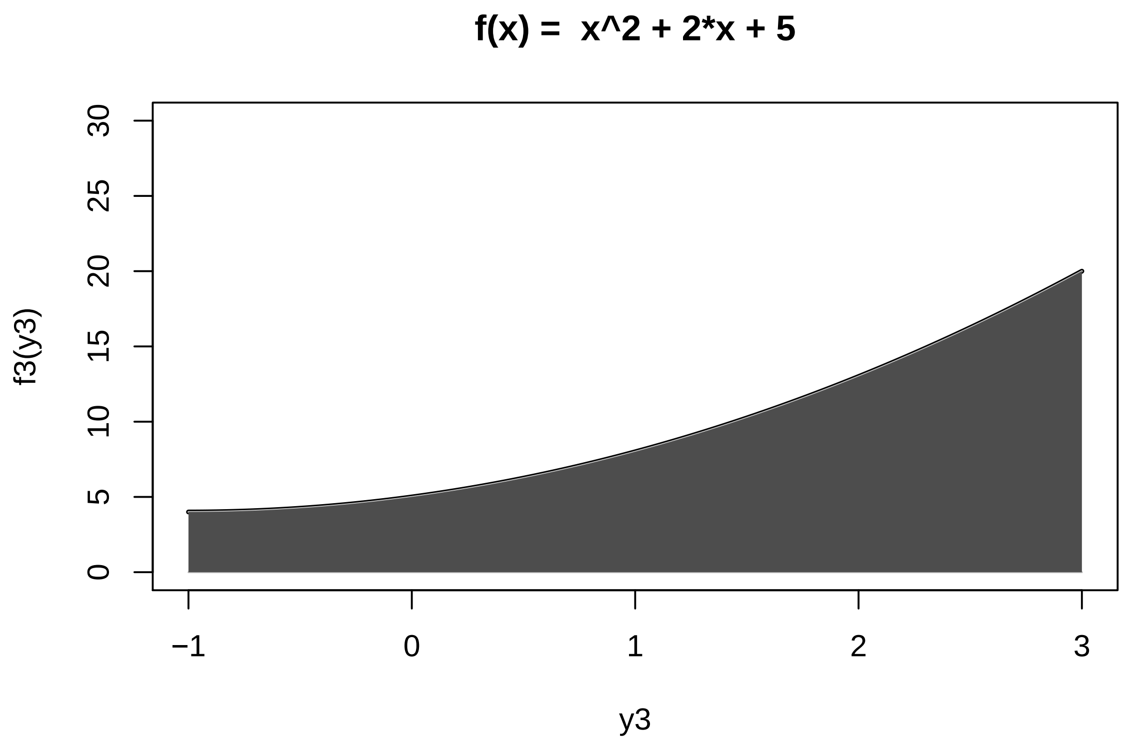
```
## [1] 37.32193
```

Is it so ?

```
integrate(f3,a3,b3)$value
```

```
## [1] 37.33333
```

```
y3 <- seq(a3,b3,0.001)
y.low3 <- rep(0,times = length(y3))

plot(y3, f3(y3), type = "l",
     main = "f(x) =  x^2 + 2*x + 5",
     lwd = 2.5, ylim = c(0,30))

lines(y3,y.low3, col = 'grey')
lines(y3,f3(y3), col = 'grey')
polygon(c(y3, rev(y3)), c(f3(y3), rev(y.low3)),
     col = "grey30", border = NA)
```



Question 4:

Based on the function given above

$$f(x) = (cos(50x) + sin(20x))^2, \quad 0 < x < 1$$

Compute the integral of

$$I = \int_0^1 (cos(50x) + sin(20x))^2 dx$$

```
f4 <- function(x) (cos(50*x) + sin(20*x))^2

n <- 10^5
a4 <- 0; b4 <- 1

x4 <- runif(n,a4,b4)

estimate4 <- mean(sapply(x4,f4)) * (b4-a4)
estimate4
```

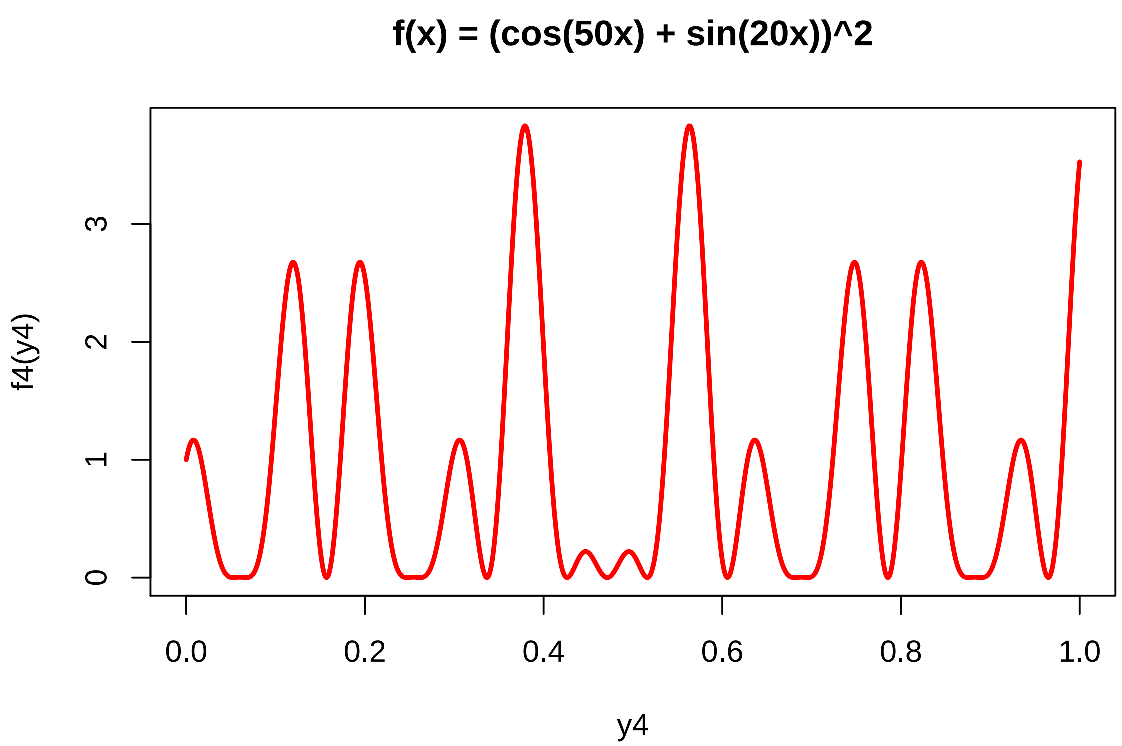
```
## [1] 0.9651935
```

Is it so ?

```r
integrate(f4,a4,b4)$value
```

```
## [1] 0.9652009
```

```r
y4 <- seq(a4,b4,0.001)
plot(y4, f4(y4), type = "l",
     main = "f(x) = (cos(50x) + sin(20x))^2",
     lwd = 2.5, col = "Red")
```



Question 5:

Find the area of the unit circle (with radius 1) by using Monte Carlo integration.

We know that we could find the area of unit circle by using following integral:

$$\text{Area} = 4\int_0^1 \sqrt{1-x^2}dx$$

```r
f5 <- function(x) sqrt(1-x^2)

n <- 10^5
a5 <- 0; b5 <- 1

x5 <- runif(n,a5,b5)

estimate5 <- 4 * mean(sapply(x5,f5)) * (b5-a5)
estimate5
```

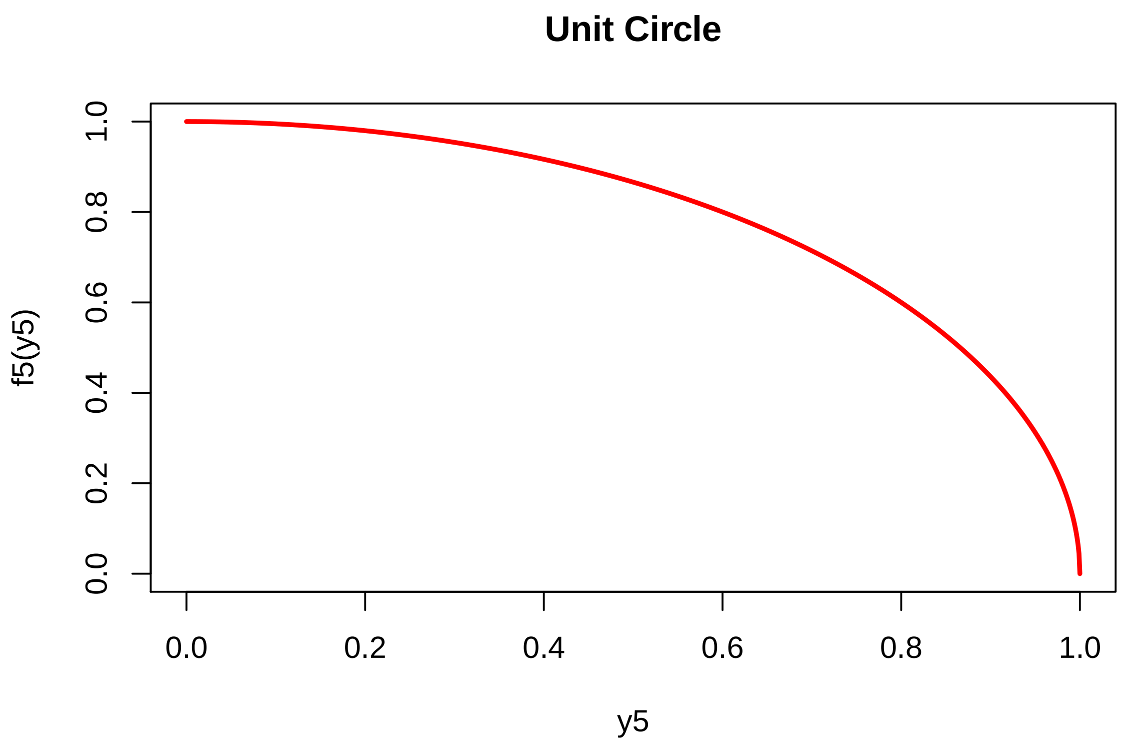
```
## [1] 3.145139
```

Is it so ?

```r
integrate(f5,a5,b5)$value * 4
```

```
## [1] 3.141593
```

```r
y5 <- seq(a5,b5,0.001)
plot(y5, f5(y5), type = "l",
     main = "Unit Circle",
     lwd = 2.5, col = "Red")
```



Additional Example:

Based on the function given above

$$f(x) = e^{sin(x)} + x^5 + ln(x^2) \quad 0 < x < \frac{\pi}{2}$$

Compute the integral of

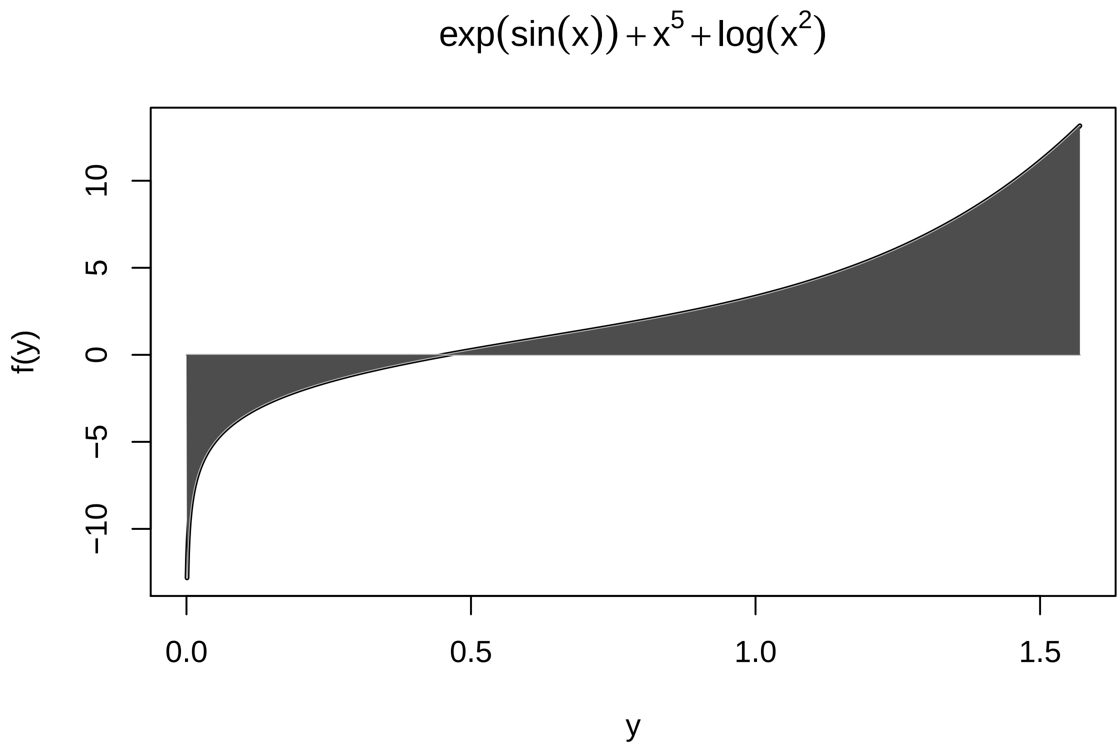
$$I = \int_0^{\frac{\pi}{2}} e^{sin(x)} + x^5 + ln(x^2)dx$$

Now, write an R function which calculates the Monte Carlo Integration for a given funtion and given upper and lower bounds of x.

```
Monte_Carlo_Int <- function(f, lower, upper, n = 10^5, graph = F){

  x <- runif(n, lower, upper)

  estimate <- mean(sapply(x,f)) * (upper-lower)
  exact <- integrate(f, lower, upper)$value

  out <- list(Function = body(f),
              Range = round(c(lower,upper),5),
              Monte_Carlo_Integral = round(estimate,5),
              Exact_Value = exact,
              Error = abs(estimate - exact))

  if(graph){

    y <- seq(lower,upper,0.001)
    y.low <- rep(0,times = length(y))

    plot(y, f(y), type = "l", lwd = 2.5,
         main = body(f))

    lines(y,y.low, col = 'grey')
    lines(y,f(y), col = 'grey')
    polygon(c(y, rev(y)), c(f(y), rev(y.low)),
        col = "grey30", border = NA)
  }

  return(out)
```

```
}
```

```r
Monte_Carlo_Int(f = function(x) exp(sin(x)) + x^5 + log(x^2),
                lower = 0, upper = pi/2, graph = T)
```




```
## $Function
## exp(sin(x)) + x^5 + log(x^2)
##
## $Range
## [1] 0.0000 1.5708
##
## $Monte_Carlo_Integral
## [1] 3.8658
##
## $Exact_Value
## [1] 3.885093
##
## $Error
## [1] 0.01929065
```