Stat 361 - Recitation 5

Generating Random Variates

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Question 1:

Generate a random sample of size $n = 10^5$ from Beta(3,9). Try to use only *runif(1)* function to generate random variables.

Density function of Beta(α, β) is

$$f_X(x : \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad x \in (0,1); \alpha, \beta > 0$$

If both αandβ are integers, we can generate random numbers from the beta distribution with parameter α and β by using the following algorithm:

- Generate $\alpha + \beta - 1$ Uniform Random Numbers; $U_1, U_2, ..., U_{\alpha+\beta-1}$.
- Deliver $X = U_{(a)}$, which is the α^{th} order statistic.

Also, show that your sample has the properties of Beta(3,9).

$$E(X) = \frac{\alpha}{\alpha + \beta}, \quad Var(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

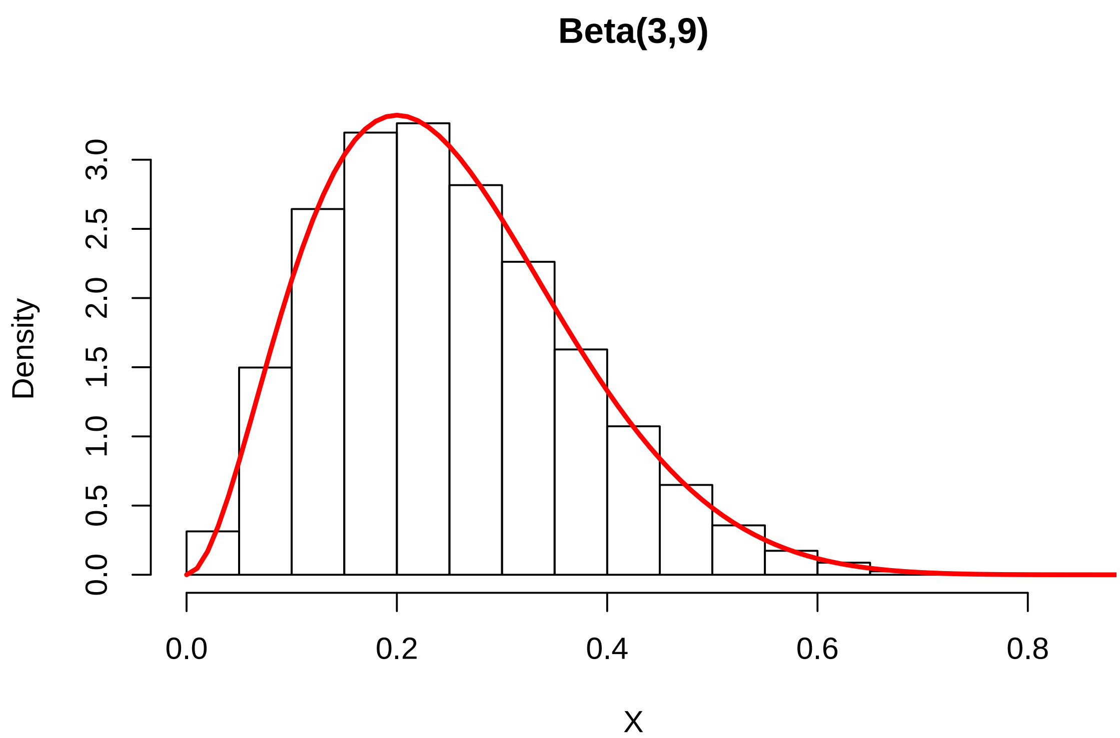
```r
n <- 10^5
alpha <- 3
beta  <- 9
X <- numeric(n)

for(i in 1:n){

  u <- runif(alpha + beta - 1)
  sortu <- sort(u)
  X[i] <- sortu[alpha]

}
```

```r
hist(X, prob = TRUE, main ="Beta(3,9)")
y=seq(0,1,0.01)
lines(y,dbeta(y,alpha,beta),col="red", lwd = 2.5)
```




```r
theoratical_mean <- alpha / (alpha + beta)
theoratical_var  <- (alpha*beta) / ((alpha+beta)^2*(alpha+beta+1))
est_mean <- mean(X)
est_var <- var(X)
```

```r
control <- matrix(round(c(theoratical_mean,theoratical_var,
                  est_mean,est_var),5), nrow = 2, byrow = T)

colnames(control) <- c("Mean","Variance")
rownames(control) <- c("Theoritical", "Estimated")

control
```

```
##                 Mean Variance
## Theoritical 0.25000  0.01442
## Estimated   0.24988  0.01449
```

Question 2:

Generate a random sample of size $n = 10^5$ from Gamma(5,8) by using relation between Exponential and Gamma distributions. Try to use only *runif(1)* function to generate random variables.

(*Hint:* Summation of t random variables from Exponential(λ) is distributed as Gamma(t,λ))

Also, show that your sample has the properties of Gamma(5,8).

$$E(X) = \frac{t}{\lambda}, \quad Var(X) = \frac{t}{\lambda^2}$$

Generating Random Variables from Gamma Distribution.

Let X is a random variable from Gamma Distribution(t,λ) and its PDF is

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(t)\lambda^t} x^{t-1} e^{\frac{-x}{\lambda}} \quad x > 0; \ \ t, \lambda > 0$$

Additionally, CDF of X is given as

$$F_x(x) = \frac{1}{\Gamma(t)\lambda^t} \int_0^x x^{t-1} e^{\frac{-x}{\lambda}} \, dx \quad x > 0$$

The inverse transform method cannot be applied in this case since there is not a closed form of solution for its inverse.

We know that the sum of t independent Exponential(λ) is distributed as Gamma(t,λ). This leads the following transformation based on t uniform random numbers.

Remember from Recitation 3 we had shown that the inverse transform of Exponential distribution is

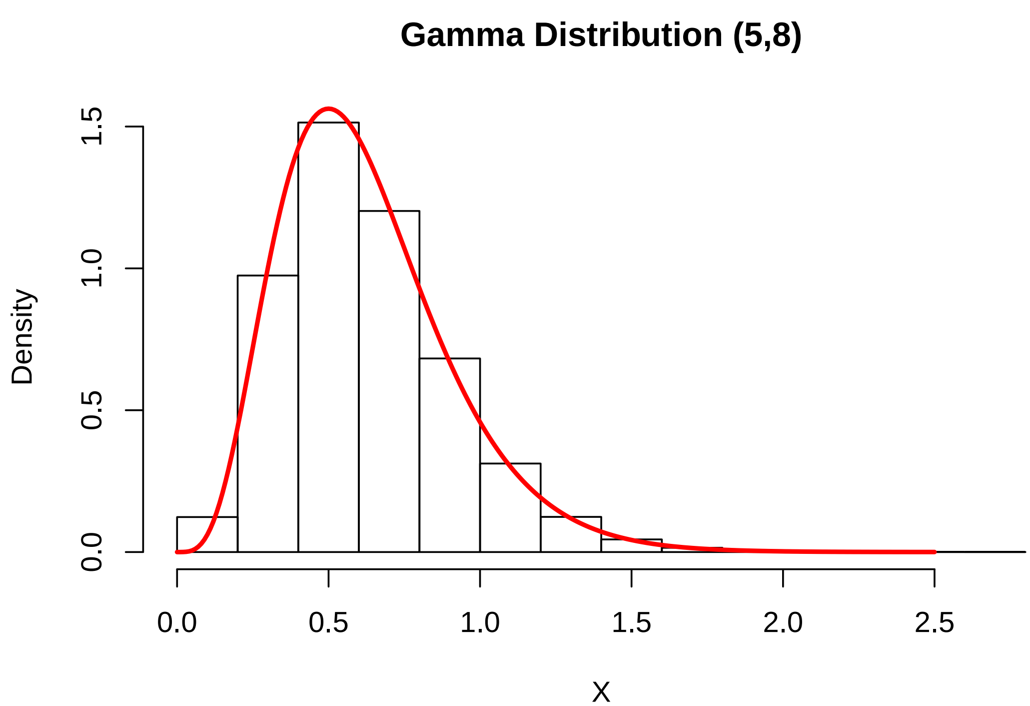
$$x = -\frac{1}{\lambda} log(U)$$

Then, to generate a random sample from Gamma Distribution we can use inverse transform method for exponential distribution;

$$X = -\frac{1}{\lambda}\log(U_1) - ... - \frac{1}{\lambda}\log(U_t)$$

```
set.seed(361)
n <- 10^5
t <- 5
lambda <- 8
U <- matrix(runif(n*t),nrow = t, ncol = n)
logU <- -log(U) / lambda # inverse transform method
X <- apply(logU,2,sum) # col sums of matrix logU
```

```
hist(X, prob=TRUE, main ="Gamma Distribution (5,8)")
y = seq(0,2.5,0.01)
lines(y,dgamma(y,t,lambda),col="red",lwd = 2.5)
```



```r
theoratical_mean <- t/lambda
theoratical_var  <- t/(lambda^2)
est_mean <- mean(X)
est_var <- var(X)

control <- matrix(round(c(theoratical_mean,theoratical_var,
                  est_mean,est_var),5), nrow = 2, byrow = T)

colnames(control) <- c("Mean","Variance")
rownames(control) <- c("Theoritical", "Estimated")

control
```

```
##                Mean Variance
## Theoritical 0.62500  0.07812
## Estimated   0.62576  0.07857
```

Question 3:

Generate random sample of size 10^5 from χ^2 distribution with 7 degrees of freedom by using the relation between Chi-Square and standard normal distribution. Check the histogram and compare estimated mean and variance with the theoritical expectation and variance of $\chi^2(v)$ distribution (v=7).

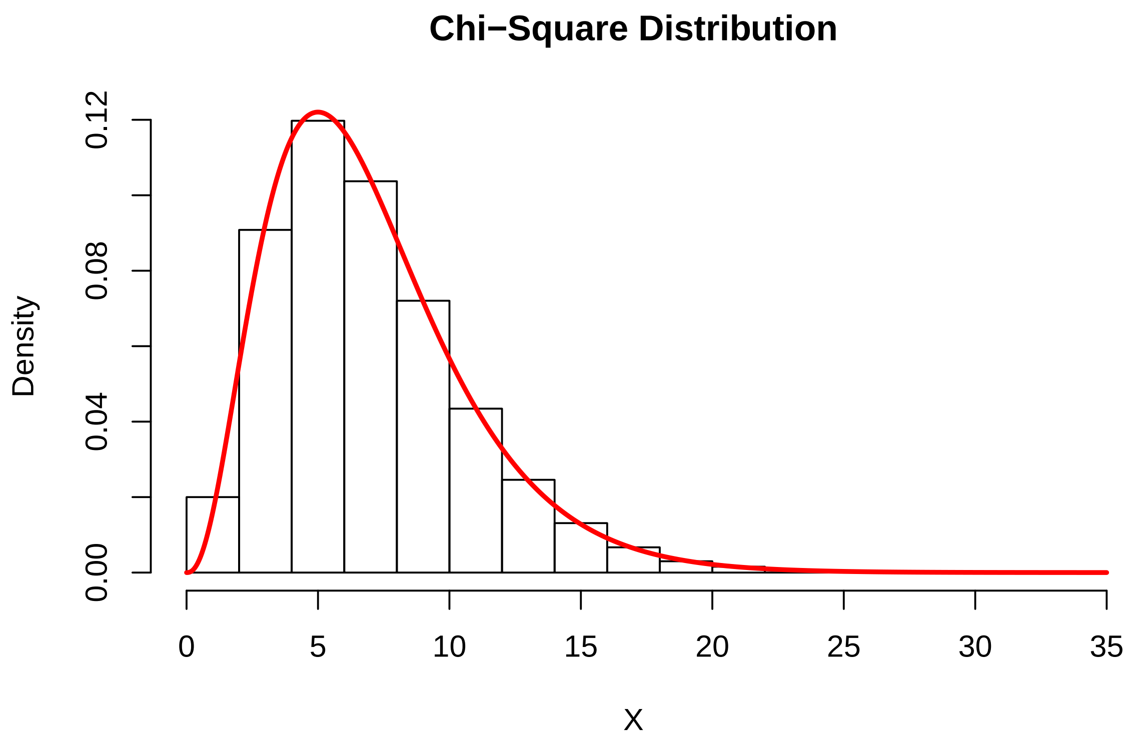
$$E(X) = v \quad Var(X) = 2v$$

We can use the fact that the chi-square distribution with V degrees of freedom is the sum of V squared independent standard normal;

$$X = \sum_{i=1}^{v} Z_i^2$$

```r
set.seed(361)
n <- 10^5
v <- 7
Z <- matrix(rnorm(v*n,0,1),nrow = v, ncol = n)
SquaredZ <- Z^2
X <- colSums(SquaredZ)
```

```r
hist(X, prob=TRUE, main ="Chi-Square Distribution")
y = seq(0,35,0.01)
lines(y,dchisq(y,v),col="red",lwd = 2.5)
```




```
theoratical_mean <- v
theoratical_var  <- 2*v
est_mean <- mean(X)
est_var <- var(X)

control <- matrix(round(c(theoratical_mean,theoratical_var,
                  est_mean,est_var),5), nrow = 2, byrow = T)

colnames(control) <- c("Mean","Variance")
rownames(control) <- c("Theoritical", "Estimated")

control
```

```
##                Mean Variance
## Theoritical 7.00000  14.0000
## Estimated   6.98805  14.0141
```

Question 4:

Generate random sample of size 10^5 from Student-T distribution with 40 degrees of freedom by using the following transformation method. Compare the histogram with the Student-T density curve.

Hint: If $Z \sim N(0,1)$ and $V \sim \chi^2(n)$ are independent, then $\frac{Z}{\sqrt{(V/n)}}$ has the Student-T distribution with n degrees of freedom. Generate χ^2 random variables from standard normal distribution.

```r
set.seed(361)
n <- 10^5
v <- 40

Z1 <- rnorm(n)
Z2 <- matrix(rnorm(n*v),n,v)
Z2_sqr <- Z2^2
V <- rowSums(Z2_sqr)

t <- Z1 / (sqrt(V / v))
length(t)
```

```
## [1] 100000
```

```r
hist(t, prob = TRUE, main = "Student-T Distribution")
y <- seq(-4,4,0.01)
lines(y, dt(y,v),col = "Red", lwd = 2.5)
```