SPIN-ORBITAL INTERACTIONS AND SUPERPOSITION THEORY

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CERTIFICATION

This to certify that the Project was carried out by SANNI Qudus Lanre with Registration Number PHY/2015/046 as part of the requirements for the award of Bachelor of Science (B.Sc.) Degree in the Department of Physics and Engineering Physics, Faculty of Science, Obafemi Awolowo University, Ile-Ife, Nigeria.

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DEDICATION

This work is dedicated to God Almighty for his mercy over me from my mother's womb till now. Also I'd like to dedicate my project to my loving Mother and Sibling, Young Physicist around the world and also in memory of my Great-Grandma, Maami

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CHAPTER ONE

INTRODUCTION

The discovery of Zeeman effect(1896) and its theoretical interpretation demonstrated that atoms have magnetic dipole moments. However, no constraint was placed on the orientation of the moments by the "classical" explanation of the normal Zeeman effect, in which the spectra lines of some elements in a magnetic field are split into three components. Bohr's theory(1913) of the hydrogen atom assumed circular orbits and required the quantization of angular momentum and by implication, quantization of the associated magnetic moment. Sommerfeld(1916) generalized the Bohr theory to allow elliptical orbits described by quantum numbers: n, l, m.

The number $n = 1, 2, 3..$, called the principal quantum number, corresponded to the quantum number of the Bohr theory. The number, $m = -l, -l+1, ..., +l, m \neq 0$, determined the projection of the vector angular momentum on any prescribed axis, a consequence of the theory that was called space quantization.

Sommerfeld showed that his theory could account for Fine structure of hydrogen atom (now explained in terms of spin-orbit coupling) when relativistic effects on the motion in the elliptical orbits were considered. The theory also provided an alternative explanation of the normal Zeeman effect. In 1922, Stern-Gerlach confirmed the space quantization although the first proposal concerning the spin of electron, made in 1955 by Uhlenbach and Goudsmit, was based on the analysis of atomic spectra.

Later, it was understood that elementary quantum particles can be divided into two; Fermions an Bosons. Mesons and Photons posses integer spin and their spin class can be used as implication to the statistical behavior of its distribution some of this implication also show effects on big thing like the quantum explanation of the existence of a white dwarf and explanations to why neutron stars gets smaller with increase in mass.

1.1 Stern-Gerlach Experiment and Discovery of Spin

The original experimental arrangement took the form of a collimated beam of neutral silver atoms heading in the y-direction, and passing through an inhomogeneous magnetic field directed in the z-direction which was deflected by the force exerted by the field on the magnetic dipole moments of the atoms. If the silver atoms possess a non-zero magnetic moment μ, the magnetic field will have two effects which are:

1. The magnetic field will exert a torque on the magnetic dipole
2. The atoms experience a sideways force given by;

where $U = -\mu \cdot \mathbf{B} = -\mu_z \mathbf{B}$ is the potential energy of the silver atom in the magnetic field. Hence;

$$F_z = \mu_z \frac{\partial U}{\partial z} \tag{1.1}$$

The silver atoms arrived on the screen at two points correspond to magnetic moments of

$$\mu_z = \pm \mu_B;$$

and

$$\mu_B = \frac{e\hbar}{2m_e} \tag{1.2}$$

where μ_B is known as the Bohr magneton.

The magnetic moment due to the intrinsic spin of the electron is given by;

$$\mu_S = -\frac{e}{2m_e} g\mathbf{S} \tag{1.3}$$

where m_e and e are the mass and charge of electron, $\mathbf{S}$ is the spin angular momentum of electron, and g is the gyro-magnetic ratio.

The allowed values for the z component of spin are;

$$S_z = \pm\frac{1}{2}\hbar \tag{1.4}$$

which, with the gyro-magnetic value of two, yields the two values of μ_z given by

$$\mu_z = \pm\mu_B \tag{1.5}$$

Any component of the spin of an electron will have only two values, ie.;

$$S_x = \pm\frac{1}{2}\hbar$$

and

$$S_y = \pm\frac{1}{2}\hbar$$

with $\hat{n}$ as the unit vector specifying some arbitrary direction in space, then;

$$\mathbf{S}\cdot\hat{n} = \pm\frac{1}{2}\hbar \tag{1.6}$$

The positive sign referred to spin up in the $\hat{n}$ direction while the negative sign as spin down in the $\hat{n}$ direction.

1.2 Spinors

Space of angular momentum states for spin $s = \frac{1}{2}$ is 2 dimensional

$$\left| s = \frac{1}{2}, m_s = \frac{1}{2} \right\rangle = |\uparrow\rangle \tag{1.7}$$

and

$$\left| s = -\frac{1}{2}, m_s = \frac{1}{2} \right\rangle = |\downarrow\rangle \tag{1.8}$$

General spinor state of spin can be written as linear combination,

$$\alpha |\uparrow\rangle + \beta |\downarrow\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Operators acting on spinors are 2×2 matrices. from definition of spinor, z-component spin represented as, $s_z = \frac{1}{2}\hbar\sigma_z$, where σ_z is;

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. S_z has eigenvalues $\pm\frac{\hbar}{2}$ corresponding to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1.2.1 Spin Operators and Pauli Matrices

From raising/lowering operators;

$$\hat{J}_+ |j, m\rangle = \left(\sqrt{j(j+1) - m(m+1)}\right) \hbar |j, m+1\rangle \tag{1.9}$$

and

$$\hat{J}_{-}\left|j,m\right\rangle = \left(\sqrt{j\left(j+1\right)-m\left(m-1\right)}\right)\hbar\left|j,m-1\right\rangle \tag{1.10}$$

with $S_{\pm} = S_x \pm iS_y$ and $s = \frac{1}{2}$, we have

$$S_{+}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{1.11}$$

and

$$S_{-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle \tag{1.12}$$

which in matrix form gives

$$S_x + iS_y = S_{+} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.13}$$

and

$$S_x - iS_y = S_{-} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.14}$$

This leads to Pauli matrix representation for spin $\frac{1}{2}$, $S = \frac{1}{2}\hbar\sigma$ as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.15}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that

$$[\sigma_i, \sigma_j] = 2i \in_{ijk} \sigma_k \tag{1.16}$$

and the total spin;

$$\begin{aligned} S^2 &= \frac{1}{4}\hbar^2\sigma^2 \\ &= \frac{1}{4}\hbar^2 \sum_i \sigma_i^2 \\ &= \frac{3}{4}\hbar^2 \mathrm{I} \\ &= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \mathrm{I} \end{aligned} \tag{1.17}$$

Total state is constructed from direct product as

$$|\psi\rangle = \int d^3x \left(\psi_+(x)\,|x\rangle \otimes |\uparrow\rangle + \psi_-(x)\,|x\rangle \otimes |\downarrow\rangle\right) \equiv \begin{pmatrix} |\psi_+\rangle \\ |\psi_-\rangle \end{pmatrix} \tag{1.18}$$

In a weak magnetic field, the electron Hamiltonian can then be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(r) + \mu_B \left(\frac{\hat{L}}{\hbar} + \sigma\right) \cdot \mathbf{B} \tag{1.19}$$

CHAPTER TWO

CLASSICAL SPIN PRECESSION IN A MAGNETIC FIELD

For a magnetized object spinning about center of mass with an angular momentum $\mathbf{L}$ and magnetic moment $\mu = \gamma \mathbf{L}$ where γ gyro-metric ratio,a magnetic field $\mathbf{B}$ will impose a torque

$$\mathbf{T} = \mu \times \mathbf{B} = \gamma \mathbf{L} \times \mathbf{B} = \partial_t \mathbf{L} \tag{2.1}$$

with $\mathbf{B} = B\hat{e}_z$, and $L_+ = L_x + iL_y$, $\partial_t L_z = -\gamma B L_+$

Angular momentum vector $\mathbf{L}$ precesses about magnetic field direction with angular velocity $\omega_0 = -\gamma \mathbf{B}$

2.0.2 Quantum Spin Precession in a Magnetic Field

From the magnetic moment of an electron

$$\mu_{orbital} = -\frac{e}{2m_e}\mathbf{L} \tag{2.2}$$

The intrinsic electron spin impart an additional contribution

$$\mu_{spin} = \gamma \hat{S} \tag{2.3}$$

where the gyro-magnetic ratio,γ is

$$\gamma = -g(\frac{e}{2m_e}) \tag{2.4}$$

and g (known as the Landé g-factor) where $g \approx 2$

These components combine to give the total magnetic moment

$$\mu = -\frac{e}{2m_e}(\hat{L} + g\hat{S})$$

In a magnetic field, the interaction of the dipole moment is given by

$$\hat{H}_{int} = -\mu \cdot B \tag{2.5}$$

Focusing on the spin contribution alone we have

$$\begin{aligned} \hat{H}_{int} &= -\gamma \hat{S} \cdot B \\ &= -\frac{\gamma}{2}\sigma \cdot B \end{aligned} \tag{2.6}$$

from the time-evolution operator,we inferred the spin dynamics, where

$$\begin{aligned} \hat{U} &= e^{-i\frac{\hat{H}_{int}t}{\hbar}} \\ &= e^{\frac{i}{2}\gamma\sigma \cdot Bt} \end{aligned} \tag{2.7}$$

Therefore, for initial spin configuration,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\varphi}{2}} \cos\left(\frac{\theta}{2}\right) \\ e^{\frac{i\varphi}{2}} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{2.8}$$

With $\mathbf{B} = B\hat{e}_z$,

$$\hat{U}_{(t)} = e^{\frac{i}{2}\gamma B t \sigma_z}$$

$$\left|\psi_{(t)}\right\rangle = \hat{U}_{(t)} \left|\psi_{(0)}\right\rangle$$

and

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \begin{pmatrix} e^{-\frac{i}{2}\omega_0 t} & 0 \\ 0 & e^{\frac{i}{2}\omega_0 t} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{i}{2}(\varphi+\omega_0 t)} \cos\left(\frac{\theta}{2}\right) \\ e^{\frac{i}{2}(\varphi+\omega_0 t)} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \end{aligned} \tag{2.9}$$

i.e. spin precessed with **angular frequency**;

$$\omega_0 = -\gamma \mathbf{B} = -g\omega_c \hat{e}_z$$

where $\omega_c = \frac{eB}{2m_e}$ is cyclotron frequency, ($\frac{\omega_c}{B} \approx 10^{11} rads^{-1}T^{-1}$)

CHAPTER THREE

BOSONS AND FERMIONS

Particles may be classified according to there interactions and also they can be group in terms of their spins into either Bosons (integer or zero spins) or Fermions (half-integer spins).

3.1 Bosons

Bosons were named by Paul Dirac to commemorate the contribution of the Indian physicist, Satyendra Nath Bose and Albert Einstein in developing and theorizing the statistics characteristics of elementary particles (ie Bose-Einstein statistics). There are various examples of bosons, such as;

1. The Elementary Bosons; which usually serves as messenger particles for interaction fields. According to the standard model the elementary boson are sub-divided, as a function of their spin into;

    a) Scalar bosons i.e for spin $= 0$ e.g H^0.

    b) Vector bosons i.e for spin $= 1$ e.g γ (photon), g (gluon, 8 types with their color charges), Z^0, $W^\pm$.

    c) Tensor bosons i.e for spin $= 2$ e.g G (graviton)

2. The Composite Bosons; they are boson that are made of composite particles which may or may not be boson themselves eg in case of mesons which are made of two quarks (each of spin-$\frac{1}{2}$) adding up to make integer spin or in case of $He-4$, $Pb-208$ and deuterium and some quasi-particle (eg. Cu-pairs, Plasmons and Phonon).

3.2 Fermions

Named by Paul Dirac after Enrico Fermi, Fermions are particles that exhibits half-integer spin and obey Pauli exclusion principle. In addition to spin quantum number, Fermions also possess other quantum numbers such as Baryon number and Lepton number. Like Bosons, Fermions can also be either elementary or composite;

a) Elementary Fermion: quarks and leptons are the only elementary Fermions, making up 24 of these (ie 6 quarks, 6 leptons and each of there antiparticles), but mathematically we can have; -Weyl fermions (massless) -Dirac fermions (massive) -Majorana fermions (fermions that are their own antiparticle)

b) Composite Fermions: they are fermions made of composites (or sub-particle) which are either (odd) bosons or fermions depending on their constituents, spin and statistical relation examples of these are Baryon (e.g Proton and neutrons containing 3-quarks).

3.3 Bose-Einstein Statistics

In Bose-Einstein statistics there is no restriction to the number of particles associated with a particular wave-function. The number of ways of distributing n_i particles in the ith level which is g_i-fold degenerate is equivalent to the distribution of n_i indistinguishable particles in a box divided into g_i components, without any restriction in the number in each component. The g_i compartment of the box can be obtained by $(g_i - 1)$ partitions. The total number of permutations of the n_i particles and the $(g_i - 1)$ partitions is

$$\alpha = (n_i + g_i - 1)! \tag{3.1}$$

since the n_i particles are indistinguishable, total permutation among themselves which is $n!$ in number, do not produce a new arrangement. Hence the total number we evaluated has to be divided by $n_i!$ in the same way the permutation of $(g_i - 1)$ partitions do not alter the fact that there still are g_i section. The divisions of the total number by

$$A = (g_i - 1)! \tag{3.2}$$

is also needed, then the number of distinct ways of distributing n_i distinguishable particles among the g_i wave-function is;

$$\frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \tag{3.3}$$

Similar expressions hold good for other levels as well the total number G of ways (eigenstates) for the whole system with the specified distribution of the the n element is;

$$G = \frac{(n_1 + g_1 - 1)!}{n_1!\,(g_1 - 1)!}\frac{(n_2 + g_2 - 1)!}{n_2!\,(g_2 - 1)!} \cdots \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} = \prod \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \tag{3.4}$$

Where $\prod$ indicates the product of a series of similar terms.

The probability W of the system having the particular distribution is proportional to the total number of eigenstates, Hence;

$$W = \prod_i \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \tag{3.5}$$

taking logarithm of both sides;

$$\ln W = \sum_i \left[\ln (n_i + g_i - 1)! - \ln (g_i - 1)!\right] + constant \tag{3.6}$$

using Stirling approximation;

$$\ln (g_i - 1)! = (g_i - 1) \ln (g_i - 1) - (g_i - 1) \approx g_i \ln g_i - g_i \tag{3.7}$$

also

$$\ln (n_i + g_i - 1)! \approx (n_i + g_i) \ln (n_i + g_i) - \ln (n_i + g_i) \tag{3.8}$$

substitute these expressions into equation 3.6, we get;

$$\ln W = \sum_i \left[(n_i + g_i) \ln (n_i + g_i) - (n_i + g_i) + n_i \ln n_i - n_i - g_i \ln g_i + g_i\right] \tag{3.9}$$

collecting like and similar term, we get;

$$\ln W = \sum_i \left[(n_i + g_i) \ln (n_i + g_i) - (n_i + g_i) + (n_i + g_i) - n_i \ln n_i - g_i \ln g_i\right] + constant \tag{3.10}$$

i.e

$$\ln W = \sum_i \left[(n_i + g_i) \ln (n_i + g_i) - n_i \ln n_i - g_i \ln g_i\right] + constant \tag{3.11}$$

the condition for the probability to be maximum is $d\,(\ln W) = 0$

Using this condition and the fact that n_i is a continuous variety then equation (11) reduces to;

$$d\left(\ln W\right)=\sum_{i}\left[\ln\left(n_{i}+g_{i}\right)+\left(n_{i}+g_{i}\right)\frac{1}{\left(n_{i}+g_{i}\right)}-\ln n_{i}-n_{i}\frac{1}{n_{i}}\right]dn_{i}=0 \tag{3.12}$$

then;

$$\sum_{i}\left[\ln\left(n_{i}+g_{i}\right)-\ln n_{i}-n_{i}\right]dn_{i}=0 \tag{3.13}$$

then;

$$\sum_{i}\left[\ln\left(n_{i}+g_{i}\right)-\ln n_{i}-\alpha-\beta E_{i}\right]dn_{i}=0 \tag{3.14}$$

$$\sum_{i}\left[\ln\left(\frac{n_{i}+g_{i}}{n_{i}}\right)-\alpha-\beta E_{i}\right]dn_{i}=0 \tag{3.15}$$

Since the variation dn_i are independent, equation (3.15) will be satisfied only if the coefficient of each term is summation is zero that is;

$$\ln\left(\frac{n_{i}+g_{i}}{n_{i}}\right)-\alpha-\beta E_{i}=0 \tag{3.16}$$

therefore ;

$$\ln\left(\frac{n_{i}+g_{i}}{n_{i}}\right)=\alpha+\beta E_{i} \tag{3.17}$$

taking exponential on both sides;

$$\frac{n_{i}+g_{i}}{n_{i}}=e^{\alpha+\beta E_{i}} \tag{3.18}$$

so that

$$\frac{g_i}{n_i} = e^{\alpha+\beta E_i} - 1 \tag{3.19}$$

therefore;

$$n_i = \frac{g_i}{e^{\alpha+\beta E_i} - 1} \tag{3.20}$$

equation 3.20 gives the probability distribution of the particle of a system obeying Bose-Einstein statistics among the various energy levels.

3.4 Fermi-Dirac Statistics

Similar to Bose-Einstein statistics, in the Fermi-Dirac statistics n indistinguishable particles constitute the system. Let the total number of particle n and total energy E be constants and the n particles are to be divided so that n_1 particles are in level 1, n_2 in level 2, n_3 in level 3, and so on, the degeneracy of the ith level be g_i since the particle are fermions, Pauli's principle allows only one particle in each of the g_i wave-functions, therefore g_i must be greater than or equal to n_i because there must be at least one wave function for every particle in the group.

The required number of eigenstates in the ith level is the same as the number of combinations of g_i articles taken n_i at a time, this is given by;

$${}^{g_i}C_{n_i} = \frac{g_i}{n_i!\,(g_i - n_i)!} \tag{3.21}$$

the total number of eigenstates G for the whole system is;

$$G = \frac{g_1!}{n_1!\,(g_1 - n_1)!}.\frac{g_2!}{n_2!\,(g_2 - n_2)!}...\frac{g_i!}{n_i!\,(g_i - n_i)!} = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!} \tag{3.22}$$

The probability W of the system with the specified distribution is proportional to the total number of eigenstates;

$$W = \prod_i \frac{g_i!}{n_i!\,(g_i - n_i)!} \times constant \tag{3.23}$$

taking logarithm of both sides

$$\ln W = \sum_i \left[\ln g_i! - \ln n_i! - \ln (g_i - n_i)!\right] + constant \tag{3.24}$$

Assuming the n_i,g_i and $(g_i - n_i)$ are large and using the Stirling approximation;

$$\ln W = \sum_i \left[g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln (g_i - n_i)\right] + constant \tag{3.25}$$

For the probability to be maximum $d\,(\ln W) = 0$, such that equation 3.25 becomes

$$\sum_i \left[\ln \left(\frac{g_i - n_i}{n_i}\right) dn_i\right] = 0 \tag{3.26}$$

This condition is subject to the conditions in equations (3.1) and (3.2) to solve by the method of Lagrange multipliers, so again we multiply equations (3.1) and (3.2) by constants $-\alpha$ and $-\beta$ respectively and added to equation (3.26) we have;

$$\sum_i \left[\ln \left(\frac{g_i - n_i}{n_i}\right) - \alpha - \beta E_i\right] dn_i = 0 \tag{3.27}$$

as the dn_i independent, we get;

$$\sum_i \ln \left(\frac{g_i - n_i}{n_i}\right) - \alpha - \beta E_i = 0 \tag{3.28}$$

or

$$\ln \left(\frac{g_i - n_i}{n_i}\right) = \alpha + \beta E_i \tag{3.29}$$

taking exponentials on both sides;

$$\frac{g_i - n_i}{n_i} = e^{\alpha + \beta E_i} \tag{3.30}$$

or

$$\frac{g_i}{n_i} = e^{\alpha + \beta E_i} + 1 \tag{3.31}$$

therefore;

$$n_i = \frac{g_i}{e^{\alpha + \beta E_i} + 1} \tag{3.32}$$

Which gives the most probable distribution of n-distinguishable particles among the various energy levels of a system obeying the Fermi-Dirac statistics

CHAPTER FOUR

SPIN-ORBITAL INTERACTION

4.1 Spin Commutativity

In quantum mechanics, we use an important operator called commutator, written and defined as;

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{4.1}$$

such that if $\left[\hat{A}, \hat{B}\right] = 0$, then we say the operators $\hat{A}$ and $\hat{B}$ commutes such that $\hat{A}\hat{B} = \hat{B}\hat{A}$ and $\hat{A}$ and $\hat{B}$ are compatible. One of the properties of commutator is that any operator $\hat{A}$ always commute with its own square ;

$$\left[\hat{A}, \hat{A}^2\right] = 0 \tag{4.2}$$

similarly, spin $\hat{S}^2$ always commute with spin along a specific direction;

$$\left[S_z, S^2\right] = \left[S^2, S_z\right] = 0 \tag{4.3}$$

also applies to x and y such that we got;

$$\left[S_x, S^2\right] = \left[S^2, S_x\right] = 0 \tag{4.4}$$

and

$$\left[S_y, S^2\right] = \left[S^2, S_y\right] = 0 \tag{4.5}$$

where S_x, S_y and S_z do not commute mutually with one another and we have;

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \tag{4.6}$$

where S_i and S_j are spin along direction i and j respectively and ε_{ijk} is the Levi-Civita pseudo-tensor.

Any operator $\hat{A}^2$ commutes with any other operator that is a function of $\hat{A}^2$ as;

$$\left[\hat{A}, f(\hat{A})\right] = 0 \tag{4.7}$$

by this property, we say spin S and full angular momentum J commutes since $J = L+S$ i.e J is a function of spin and orbital angular momentum $J(S, L)$ and since S and L commutes then S commutes with J as;

$$\left[\hat{S}, \hat{J}\right] = 0 \tag{4.8}$$

and spin S also commutes with Hamiltonian $\hat{H}$ as;

$$\left[\hat{S}, \hat{H}\right] = 0 \tag{4.9}$$

4.2 Pauli's Exclusion Principle and Heisenberg Uncertainty Principle

When two operators commute we know that they are compatible and we can measure both observable with perfect precision simultaneously. But when they do not commute then we can not measure them both with precision simultaneously. if two operators $\hat{A}$ and $\hat{B}$ do not commute such that;

$$\left[\hat{A}, \hat{B}\right] = iC$$

then the uncertainty in measurement of A and B is given by;

$$\Delta A \Delta B = \frac{1}{2} \left| \langle C \rangle \right| \tag{4.10}$$

the uncertainty is given in standard deviation such as in case of $\hat{x}$ and $\hat{P}_x$ as ;

$$\left[\hat{x}, \hat{P}_x\right] = i\hbar \tag{4.11}$$

where $C = \hbar$. therefore we get;

$$\sigma_x \sigma_{P_x} = \frac{\hbar}{2} \tag{4.12}$$

applying this to the measurement of spins along different directions simultaneously, then we have;

$$\sigma_{S_i} \sigma_{S_j} = \epsilon_{ijk} \frac{\hbar S_k}{2} \tag{4.13}$$

4.3 Dirac Matrices

When describing relativistic quantum mechanics, Paul Dirac constructed his own set of matrices from the Pauli matrices. The Dirac matrices γ^μ satisfy the following anti-commutative relations;

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{4.14}$$

for $\mu = \nu$

where the 4×4 representation of these is;

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.15}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{4.16}$$

Where each of the matrix element is a 2×2 matrix, and σ_i is are Pauli matrices.

4.4 Dirac Spinor

The 4-component Dirac particle u and antiparticle v spinors are defined by the relation;

$$\begin{aligned} u(p_\mu, s) &= \sqrt{E_r + m} \begin{pmatrix} 1 \\ \frac{\sigma \cdot p_\mu}{E_r + m} \end{pmatrix} \chi^{(s)} \\ v(p_\mu, s) &= \sqrt{E_r + m} \begin{pmatrix} \frac{\sigma \cdot p_\mu}{E_r + m} \\ 1 \end{pmatrix} \left[C\chi^{(s)}\right] \end{aligned} \tag{4.17}$$

where $C = -i\sigma_2$, E_r is relativistic energy,$E_r = \sqrt{m^2 + p^2}$, $\mu =$ 1,2,3. The 2-component spinors χ are given by;

$$\begin{aligned} \chi^{\left(\frac{1}{2}\right)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \chi^{\left(-\frac{1}{2}\right)} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{4.18}$$

the antiparticle 2-component spinor is sometimes denoted by η such that

$$\eta^{(s)} = -i\sigma_2\chi^{(s)} \tag{4.19}$$

Hence;

$$
\begin{aligned}
\eta^{\left(-\frac{1}{2}\right)} &= -i\sigma_2\chi^{\left(\frac{1}{2}\right)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\eta^{\left(\frac{1}{2}\right)} &= -i\sigma_2\chi^{\left(-\frac{1}{2}\right)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\end{aligned}
\tag{4.20}
$$

the sign(phase) convention of η is introduced so that the spinors are charge conjugates of one another ie.;

$$
\begin{aligned}
\bar{u}(p_\mu, s) &= u^+(p_\mu, s)\gamma^0 \\
\bar{v}(p_\mu, s) &= v^+(p_\mu, s)\gamma^0
\end{aligned}
\tag{4.21}
$$

from this definition, the spinor satisfy the following normalization and orthogonality condition;

$$
\begin{aligned}
\bar{u}(p_\mu, s)u(p_\mu, s') &= 2m\delta_{ss'} \\
\bar{v}(p_\mu, s)v(p_\mu, s') &= 2m\delta_{ss'} \\
\bar{u}(p_\mu, s)v(p_\mu, s') &= 0 \\
\bar{v}(p_\mu, s)u(p_\mu, s') &= 0
\end{aligned}
\tag{4.22}
$$

the completeness relations are expressed in terms of the positive energy and negative energy projection operators;

$$
\begin{aligned}
\sum_{s=1,2} u(p_\mu, s)\bar{u}(p_\mu, s) &= \gamma^\mu p_\mu + m = 2m\Lambda_+(p) \\
\sum_{s=1,2} v(p_\mu, s)\bar{v}(p_\mu, s) &= \gamma^\mu p_\mu - m = -2m\Lambda_-(p)
\end{aligned}
\tag{4.23}
$$

u and v spinor are related by charge conjugation;

$$
\begin{aligned}
C\bar{u}^+(p_\mu, s) &= v(p_\mu, s) \\
C\bar{v}^+(p_\mu, s) &= u(p_\mu, s)
\end{aligned}
\tag{4.24}
$$

CHAPTER FIVE

SUPERPOSITION THEORY

The principle of quantum superposition states that if a quantum system can be in one of many quantum states $|n\rangle$ then the most general state $|\psi\rangle$ is a linear combination of the states.

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle + ... + c_n |n\rangle \tag{5.1}$$

where c is the combination coefficient (complex number) this implies that any linear combination of states is another valid state and any state can be expressed as linear combination of other possible states.

So for electrons with two possible spin(up or down) the general state $|\psi\rangle$ is given by;

$$|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle \tag{5.2}$$

and the coefficients c_1 and c_2 gives the probability of having each state, the probability p is given by square of the absolute value of the coefficients and addition of the probability of all possible state (according to statistic) equal to 1 ie; $p_\uparrow = |c_1|^2$ and $p_\downarrow = |c_2|^2$ such that;

$$p_{\uparrow or \downarrow} = p_\uparrow + p_\downarrow = |c_1|^2 + |c_2|^2 = 1 \tag{5.3}$$

so that before the spin of an electron is measured, the electron spin is in superposition of

$|\uparrow\rangle$ and $|\downarrow\rangle$, and once it is measured, it gets a specific spin either $|\uparrow\rangle$ or $|\downarrow\rangle$ but can still be described by equation 5.2 except that this time one of the coefficients goes to zero.

REFERENCES