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Course: Asymptote — Date: December 7, 2013

1. 1 Given the system of ODES

$$x' = -x + y^2 \tag{1}$$

$$y' = -x^2 - y^3 \tag{2}$$

since the system is not in the desired state, then we change to have

$$y' = -x^2 - y^3 \tag{3}$$

$$x' = -x + y^2 \tag{4}$$

the jacobian matrix is

$$J(y,x) = \begin{pmatrix} -3y^2 & 2x \\ 2y & -1 \end{pmatrix}$$

since the rest point is at origin then

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

we calculate for the eigenvalue of the jacobian matrix at the rest point

$$\begin{vmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

so, we have

$$(\lambda + \lambda^2) = 0$$

therefore $\lambda = 0$ and -1.

Since atleast one eigenvalue is zero (0) then the rest point at origin is non- hyperbolic.

We calculate eigenvector corresponding to $\lambda = 0$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the eigenvector corresponding to $\lambda = 0$ is

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E^c = Sp\{c\begin{pmatrix}1\\0\end{pmatrix}|\lambda = 0\}$$

similarly, we calculate eigenvector corresponding to $\lambda = -1$

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}y\\x\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

the eigenvector corresponding to $\lambda = 0$ is

$$\begin{pmatrix}y\\x\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$E^s = Sp\{c\begin{pmatrix}0\\1\end{pmatrix}|\lambda = -1\}$$

we know that

$$y' = Ay + g_1(y, x) \tag{5}$$

$$x' = Bx + g_2(y, x). \tag{6}$$

but equation (3) and (4) are now in the desired form, $A = 0$, $B = -1$, $g_1(y, x) = x^2 - y^3$, $g_2(y, x) = y^2$. Let

$$x = h(y)$$

$$x' = Dh(y)y'$$

substitute for x' and x in equation (6) we get

$$Dh(y)y' = Bh(y) + g_2(y, h(y)) \tag{7}$$

substitute equation (5) into (7), we get

$$Dh(y)[Ay + g_1(y, h(y)] = Bh(y) + g_2(y, h(y))$$

for any function u

$$(Mu)(y) := Du(y)[Ay + g_1(y, h(y))] - Bu(y) - g_2(y, h(y)) \tag{8}$$

then

$$h'(y)[(h(y))^2 - y^3] + h(y) - y^2 = 0. \tag{9}$$

Now we shall see how the approximation algorithm works. Assume that our approximation to the centre manifold is

$$u(y) = ay^2 + by^3 + cy^4$$

then

$$u'(y) = 2ay + 3by^2 + 4cy^3$$

put this answer into (9) we have

$$(2ay + 3by^2 + 4cy^3)[(ay^2 + by^3 + cy^4)^2 - y^3] + ay^2 + by^3 + cy^4 - y^2 = 0$$

imply that

$$y^2(a-1) + by^3 + y^4(c-2a) + y^5(d-3b+2a^3) = 0.$$

Then we have from (8)

$$(Mu)(y) = ay^2 - y^2 + O(y^3).$$

Therefore, appealing to the approximation theorem, to approximate the centre manifold to $O(y^3)$ we take $a = 1$. That is all there is to it. Though this is sufficient for stability, let go further on. If $a = 1$, we have from (8)

$$(Mu)(y) = by^3 + O(y^4).$$

Thus to approximate the centre manifold to $O(y^4)$ we take $b = 0$. Going on , we have

$$(Mu)(y) = cy^4 - 2ay^4 + O(y^5).$$

Thus we choose c=2a and the approximation to the centre manifold obtained is

$$u(y) = y^2 + 2y^4 - 2y^5 + O(y^6)$$

The stability of (0,0) of the systems is the same as the stability of

$$y' = (u(y))^2 - y^3 = (y^2)^2 - y^3 + O(y^5)$$

$$y' = y^4 - y^3 + O(y^5)$$

Clearly the system is stable.

1b. Given the system of ODES

$$x' = -x^2 \tag{10}$$

$$y' = -y + x^2. \tag{11}$$

We know that

$$x' = Ax + g_1(x, y) \tag{12}$$

$$y' = By + g_2(x, y) \tag{13}$$

The system in (10) and (11) is already in the desired form with $A = 0$,
$B = -1$, $g_1(x, y) = -x^2$ and $g_2(x, y) = x^2$. Let

$$y = h(x) \tag{14}$$

then

$$y' = Dh(x)x' \tag{15}$$

Substitute (14) and (15) into (13) we have

$$Dh(x)x' = Bh(x) + g_2(x, h(x)) \tag{16}$$

by substituting (14) into (12) we also have

$$x' = Ax + g_1(x, h(x)). \tag{17}$$

Thus, substitute (17) into (16)

$$Dh(x)(Ax + g_1(x, h(x))) = Bh(x) + g_2(x, h(x)) \tag{18}$$

for any function u

$$(Mu)(x) := Du(x)(Ax + g_1(x, u(x))) - Bu(x) - g(x, u(x)) \tag{19}$$

note that $(M(h))(x) = 0$ After substituting, equation (18) becomes

$$h'(x)[-x^2] + h(x) - x^2 = 0 \tag{20}$$

if

$$u(x) = ax^2 + bx^3 + cx^4 + dx^5 + ex^6$$

then

$$u'(x) = 2ax + 3bx^2 + 4cx^3 + 5dx^4 + 6ex^5$$

put both in equation (20) then we have

$$(2ax + 3bx^2 + 4cx^3 + 5dx^4 + 6ex^5)(-x^2) + ax^2 + bx^3 + cx^4 + dx^5 + ex^6 - x^2 = 0$$

$$-2ax^3 - 3bx^4 - 4cx^5 - 5dx^6 - 6ex^7 + ax^2 + bx^3 + cx^4 + dx^5 + ex^6 - x^2 = 0.$$

Then we have from (19)

$$(Mu)(x) = ax' - x^2 + O(x^3)$$

if $a = 1$

$$(Mu)(x) = x^2 - x^2 + O(x^3)$$
$$(Mu)(x) = O(x^3).$$

Therefore, appealing to the approximation theorem, to approximate the centre manifold $O(x^3)$ we take $a = 1$. That is all there is to it. Let go further, if $a = 1$, we have from(19)

$$(Mu)(x) = -2x^3 + bx^3 + O(x^4).$$

Thus to approximate the centre manifold to $O(x^4)$ we take $b = 2$.

$$(Mu)(x) = O(x^4).$$

for $O(x^5)$, if $a = 1$ and $b = 2$, we have from (19)

$$(Mu)(x) = -3bx^4 + cx^4 + O(x^5)$$

to approximate the centre manifold to $O(x^5)$ we take $c = 3b = 6$ since $b = 2$.

$$(Mu)(x) = O(x^5).$$

Similarly for $O(x^6)$ we have from (19)

$$(Mu)(x) = -4cx^5 + dx^5 + O(x^6)$$

to approximate the centre manifold to $O(x^6)$ we take $d = 4c = 24$ since $c = 6$.

$$(Mu)(x) = O(x^6).$$

$$(Mu)(x) = -5dx^6 + ex^6 + O(x^7)$$

thus we choose $e = 5d = 120$ since $d = 24$ the above becomes

$$(Mu)(x) = O(x^7).$$

So,

$$h(x) = ax^2 + bx^3 + cx^4 + dx^5 + ex^6 + O(x^7)$$

but $a = 1$, $b = 2$, $c = 6$, $d = 24$, $e = 120$ then

$$h(x) = x^2 + 2x^3 + 6x^4 + 24x^5 + 120x^6 + O(x^7)$$

we realise that it is a pattern of the form 1!, 2!, 3!, 4!, 5!, and so on.
Therefore

$$h(x) = \sum_{n=1}^{\infty}(n)!x^{n+1}.$$

Radius of convergence

$$\sum_{n=1}^{\infty}(n)!x^{n+1}$$

$$\begin{aligned}
&\frac{(n+1)!|x^{n+2}|}{(n)!|x^{n+1}|} \\
&= \frac{(n+1)!|x^{n+2}|}{(n)!|x^{n+1}|} \\
&= \frac{(n+1)|x^n| \times |x^2|}{|x^n||x|} \\
&= (n+1)|x|
\end{aligned}$$

Radius of convergence is

$$\frac{1}{R} = \lim_{n\to\infty}(n+1)$$

$$R = 0$$

$$\lim_{n\to\infty}(n+1)|x| = 0.|x| = 0$$

Therefore the asymptotic expansion of the centre manifold have zero radius of convergence.

2. Given

$$\sin \pi x = \epsilon x^3. \tag{21}$$

If $x = 0$, $\sin \pi x = 0$ and the smallest value of x for this to hold is $x = 1$. Let ϵy be a perturbation of $x = 1$, then

$$x = 1 + \epsilon y \tag{22}$$

Substitute (22) into the LHS of (21) we have that

$$\sin \pi(1 + \epsilon y) = \sin(\pi + \epsilon y\pi)$$

Using Taylor series expansion we have that

$$\sin(\pi + \epsilon y\pi) = \epsilon\pi y\cos\pi - \frac{(\epsilon y\pi)^3\cos\pi}{3!} + O(\epsilon^4) \tag{23}$$

and similarly substitute (22) into RHS of (21) we have that

$$\begin{aligned}\epsilon x^3 &= \epsilon(1+\epsilon y)^2 \\ &= \epsilon(1 + 3\epsilon y + 3\epsilon^2 y^2 + \epsilon^3 y^3) \\ &= (\epsilon + 3\epsilon^2 y + 3\epsilon^3 y^2 + \epsilon^4 y^3) \\ \epsilon(1+\epsilon y)^2 &= (\epsilon + 3\epsilon^2 y + 3\epsilon^3 y^2 + \epsilon^4 y^3)\end{aligned} \tag{24}$$

equating RHS of (23) and RHS of (24) we have

$$-\epsilon\pi y + \frac{(\epsilon y\pi)^3}{3!} + O(\epsilon^4) = (\epsilon + 3\epsilon^2 y + 3\epsilon^3 y^2 + \epsilon^4 y^3). \tag{25}$$

We consider asymptotic expansion of y we have

$$y = y_o + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3)$$

substitute this immediate above equation into (25) we have

$$-\epsilon\pi(y_o+y_1\epsilon+y_2\epsilon^2) + \frac{(\epsilon\pi)^3}{6}(y_o+y_1\epsilon+y_2\epsilon^2) = \epsilon + 3\epsilon^3(y_o+y_1\epsilon+y_2\epsilon^2) + 3\epsilon^2(y_o+y_1\epsilon+y_2\epsilon^2)^2.$$

Equating terms of the same order we have

$$O(\epsilon) : -\pi y_o = 1 \quad \Rightarrow \quad y_o = -\frac{1}{\pi}$$

$$O(\epsilon^2) : -\pi y_1 = 3y_o \quad \Rightarrow \quad y_o = \frac{3}{\pi^2}$$

$$O(\epsilon^3) : -\pi y_2 + \frac{(\pi^3 y_o^3)}{6} = \quad \Rightarrow \quad y_o = -\frac{9}{\pi^2} - \frac{3}{\pi^2} - \frac{1}{6} \quad \Rightarrow \quad y_2 = -\frac{(72+\pi^2)}{6\pi}$$

Substitute the values of y_o, y_1, y_2 into the asymptotic expansion we have

$$y = -\frac{1}{\pi} + \frac{3\epsilon}{\pi^2} - \frac{(72+\pi^2)\epsilon^3}{6\pi^2} + O(\epsilon^4)$$

substitute this immediate value of y into (22) we have that

$$x = 1 - \frac{\epsilon}{\pi} + \frac{3\epsilon^2}{\pi^2} - \frac{(72+\pi^2)}{6\pi^2} + O(\epsilon^4)$$

3. Considering the polynomial equations

$$x^4 - x\epsilon + x^2\epsilon - \epsilon^3 = 0 \tag{26}$$

and from the equation (26) we realise we have coordinate $(4,0)$, $(1,1)$, $(2,1)$ and $(0,3)$. If we plot the graph of these coordinate we will have two scaling: One from coordinate $(0,3)$ and $(1,1)$ which gives $n = 1$ and $m = -2$ and the other from coordinate $(1,1)$ and $(4,0)$ which gives $n = 3$ and $m = -\frac{1}{3}$. Thus, using

$$x(\epsilon) = \epsilon^{-m} y(\epsilon)$$

Case 1 when $m = -2$ we have

$$x = \epsilon^2 y \tag{27}$$

substitute (27) into (26) we have

$$(\epsilon^2 y)^4 - (\epsilon^2 y)\epsilon + (\epsilon^2 y)^2 \epsilon - \epsilon^3 = 0$$

$$\epsilon^8 y^4 - \epsilon^3 y + \epsilon^5 y^2 - \epsilon^3 = 0$$

dividing through by ϵ^3

$$\epsilon^5 y^4 - y + \epsilon^2 y^2 - 1 = 0 \tag{28}$$

if $\epsilon \to 0$ then

$$-y - 1 = 0$$

$$y = -1$$

so,

$$y = -1 + \epsilon y_1 + O(\epsilon^2) \tag{29}$$

substitute (29) into (28) we have

$$1 - \epsilon y_1 + \epsilon^2(-1 + \epsilon y_1)^2 - 1 = 0$$

$$1 - \epsilon y_1 + \epsilon^2(1 + 2\epsilon y_1 + \epsilon^2 y_1^2) - 1 = 0$$

$$1 - \epsilon y_1 + \epsilon^2 + 2\epsilon^3 y_1 + \epsilon^4 y_1^2 - 1 = 0 \tag{30}$$

since we know $x = \epsilon^2 y$ then equation (30) becomes

$$1 - \epsilon y_1 - 1 = 0 \quad \Rightarrow y_1 = 0.$$

So,

$$y = -1 + O(\epsilon^2)$$

substitute this into equation (27) we have that

$$x = \epsilon^2(-1 + O(\epsilon^2))$$

$$x = -\epsilon^2 + O(\epsilon^4).$$

Case 2 when $m = -\frac{1}{3}$ we have

$$x = \epsilon^{\frac{1}{3}} y \tag{31}$$

substitute (28) into (25) we have

$$(\epsilon^{\frac{1}{3}}y)^4 - (\epsilon^{\frac{1}{3}})\epsilon + (\epsilon^{\frac{1}{3}}y)\epsilon - \epsilon^3 = 0$$

$$y^4\epsilon^{\frac{4}{3}} - y\epsilon^{\frac{4}{3}} + y^2\epsilon^{\frac{5}{3}} - \epsilon^3 = 0$$

dividing through by $\epsilon^{\frac{4}{3}}$ then we have

$$y^4 - y + \epsilon^{\frac{1}{3}}y - \epsilon^{\frac{5}{3}} = 0 \tag{32}$$

when $\epsilon \to 0$

$$y^4 - y = 0$$

$$y(y^3 - 1) = 0$$

We choose only the real root of the equation then we have $y = 1$. So, the asymptotic expansion is

$$y = 1 + \epsilon^{\frac{1}{3}}y_1 + \epsilon^{\frac{3}{2}}y_2 + O(\epsilon)$$

substitute immediate above equation into (32) we have

$$(1+\epsilon^{\frac{1}{3}}y_1+\epsilon^{\frac{3}{2}}y_2+O(\epsilon))^4+\epsilon^{\frac{1}{3}}(1+\epsilon^{\frac{1}{3}}y_1+\epsilon^{\frac{3}{2}}y_2+O(\epsilon))^2-(1+\epsilon^{\frac{1}{3}}y_1+\epsilon^{\frac{3}{2}}y_2+O(\epsilon))+\epsilon^{\frac{5}{2}} = 0 \tag{33}$$

equating term of the same order

$$O(\epsilon^{\frac{1}{3}}): \quad y_1 - 1 = 0 \quad \Rightarrow \quad y_1 = 1$$

$$O(\epsilon^{\frac{2}{3}}): \quad 4y_1^2 - y_2 = 0 \quad \Rightarrow \quad y_1 = 2$$

therefore

$$y = 1 + \epsilon^{\frac{1}{3}} + 2\epsilon^{\frac{2}{3}} + O(\epsilon)$$

substitute this into equation (27) we have

$$x = \epsilon^{\frac{1}{3}} + \epsilon^{\frac{2}{3}} + 2\epsilon + O(\epsilon^{\frac{4}{3}})$$