HW1

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HW1-IE529-F21(1)

IE 529 - Homework 1: Random variables

Due Wednesday, September 22nd

I. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables, with

$$X_n = \begin{cases} n^3 & \text{with prob. } \frac{1}{n^2} \\ 0 & \text{with prob. } 1 - \frac{1}{n^2} \end{cases}$$

Show that $X_n \xrightarrow{w.p.1} 0$, but not *in mean*.

II. Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ where c is a constant (finite).

Show that $X_n Y_n \xrightarrow{d} cX$.

III. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of random variables where $|X_n| \le Y$ for all n and $E(Y) < \infty$.

Show that if $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{m} X$.

NOTE: Problems II and III may be difficult. Please try now; hints will be posted later.

IV. Let $X_1, X_2, \ldots, X_n$ be a set of i.i.d. random variables, with $X_i \in \mathcal{N}(\mu, \sigma^2)$, for all $i = 1, \ldots, n$, μ, σ both finite. Let s^2 denote the sample variance of the $\{X_i\}_{i=1}^n$.

a. Show that the random variable defined by

$$W := \frac{(n-1)s^2}{\sigma^2} \quad \text{is } \chi^2_{(n-1)} - \text{distributed.}$$

b. Show that the random variable defined by

$$U := \frac{\bar{x} - \mu}{s/\sqrt{n}} \quad \text{is } T_{(n-1)} - \text{distributed,}$$

where $\bar{x}$ denotes the sample mean.

V. Let $X_1, X_2, \ldots, X_{n_1}$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu_1, 1)$, and let $Y_1, Y_2, \ldots, Y_{n_2}$ be a set of **i.i.d.** random variables, with $Y_i \in \mathcal{N}(\mu_2, 1)$, and further suppose X_i and Y_j are independent for all i, j. Define a new random variable as

$$W := \sum_{i=1}^{n_1} (X_i - \bar{x})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{y})^2.$$

a. What is the distribution of W?

b. What is $E(W)$? and $\mathrm{Var}(W)$?

VI. Let $\{X_i\}$, $i = 1, 2, \ldots, n, \ldots$ be independent Poisson random variables with respective rates $\{\lambda_i\}$, $i = 1, 2, \ldots, n, \ldots$. Show that if

$$\sum_{i=1}^{\infty} \lambda_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty} X_i \text{ converges a.s.}$$

I. $X_n = \begin{cases} n^3 & \text{with prob. } \frac{1}{n^2} \\ 0 & \text{with prob. } 1 - \frac{1}{n^2} \end{cases}$

$$\lim_{m\to\infty} P(\ |X_n - 0| < \varepsilon\ , \text{for all } n \ge m)\ , \text{for } 1 > \varepsilon > 0$$
$$= \lim_{m\to\infty} P(\ |X_n| < \varepsilon, \text{for all } n \ge m)$$
$$= \lim_{m\to\infty}(\lim_{n\to\infty}(\prod_{k=m}^{n}(1 - \tfrac{1}{n^2})))$$
$$= \lim_{m\to\infty}(\lim_{n\to\infty}(\tfrac{(m-1)(m+1)}{m^2} \cdot \tfrac{m(m+2)}{(m+1)^2} \cdot \ldots \cdot \tfrac{(n-2)n}{(n-1)^2} \cdot \tfrac{(n-1)(n+1)}{n^2}))$$
$$= \lim_{m\to\infty}(\lim_{n\to\infty}(\tfrac{m-1}{m} \cdot \tfrac{n+1}{n}))$$
$$= \lim_{m\to\infty}(\tfrac{m-1}{m})$$
$$= 1$$

thus, it shows that $X_n \xrightarrow{w.p.1} 0$.

$$\lim_{n\to\infty} E(|X_n|^2) = \lim_{n\to\infty}(n^6 \cdot \tfrac{1}{n^2} + 0 \cdot (1 - \tfrac{1}{n^2}))$$
$$= \lim_{n\to\infty}(n^4)$$
$$= \infty$$

Therefore, X_n does not converge in mean.

II. V1 $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} c$, show that $X_n Y_n \xrightarrow{d} cX$

From $X_n \xrightarrow{d} X$, we know that
$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

From $Y_n \xrightarrow{P} c$, we know that
$\lim_{n\to\infty} P(|Y_n - Y| \ge \varepsilon) = 0$, for all $\varepsilon > 0$

case (i): $c > 0$, Take any $\varepsilon > 0$, we have
$$P(X_n Y_n \le cx) \le P(Y_n \le c(1-\varepsilon)) + P(X_n \le \tfrac{x}{1-\varepsilon})$$

Using these $\varepsilon > 0$ such that $x/(1-\varepsilon)$ is a continuity point of F_X and send n to ∞, we obtain
$$\limsup_{n\to\infty} P(X_n Y_n \le cx) \le F_X(\tfrac{x}{1-\varepsilon}).$$

In the other direction,
$$P(X_n Y_n \le cx) \ge P(X_n \le \tfrac{x}{1+\varepsilon'}) - P(Y_n \ge c(1+\varepsilon') \text{ or } Y_n \le 0)$$

Choose $\varepsilon' > 0$ such that $x/(1+\varepsilon')$ is a continuity point of F_X, we have
$$\liminf_{n\to\infty} P(X_n Y_n \le cx) \ge F_X(\tfrac{x}{1+\varepsilon'})$$

Combine them we conclude that at continuity point x of F_X,
$\lim_{n\to\infty} P(X_n Y_n \le cx) = F_X(x) = F_{cX}(cx)$

case (ii): $c < 0$, follows from taking $-Y_n$.
case (iii): $c = 0$, we need to show that $X_n Y_n \Rightarrow 0$, i.e., $P(|X_n Y_n| \ge \varepsilon) \to 0$.
We have for any $M > 0$,
$$P(|X_n Y_n| \ge \varepsilon) \le P(|Y_n| \ge 1/M) + P(|X_n| \ge M\varepsilon).$$
Choose M such that $\pm M\varepsilon$ are continuity point of F_X, we have
$$\limsup_{n\to\infty} P(|X_n Y_n| \ge \varepsilon) \le F_X(-M\varepsilon) + 1 - F_X(M\varepsilon).$$
Then send M to ∞ we obtain $P(|X_n Y_n| \ge \varepsilon) \to 0$

V2 (Refer to the hint)

Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} c$ where c is a constant.

show $X_n Y_n \xrightarrow{d.} cX$ $\longrightarrow F_{X_nY_n}(x) \to F_{cX}(x)$ as $n \to \infty$

- Cosider $P(X_n Y_n \le z)$ for some z
- Note this can be written as:

$$P(X_n Y_n \le z) = P(X_n(Y_n - c) + X_n c \le z)$$
$$= P(X_n(Y_n - c) + X_n c \le z, |Y_n - c| \le \varepsilon)$$
$$+ P(X_n(Y_n - c) + X_n c \le z, |Y_n - c| > \varepsilon)$$

Now consider assumptions and definitions of covergence in probability and distribution.

$X_n Y_n \xrightarrow{P} cX \Leftarrow$ "$P(|X_n Y_n - cX| \ge \varepsilon) \to 0$"

$\Downarrow$

$X_n Y_n \xrightarrow{D} cX$

$\Uparrow$

$P(|X_n Y_n| \ge cX) \to 0$

$\Uparrow$

$P(|X_n(Y_n - c) + X_n c| \ge cX) \to 0$

$\Uparrow$

$P(|X_n(Y_n - c) + X_n c| \le cX|\,|Y_n - c| \le \varepsilon)$
$+ P(|X_n(Y_n - c) + X_n c| \le cX|\,|Y_n - c| > \varepsilon) \to 0$

$X_n \xrightarrow{d.} X \Rightarrow F_{X_n}(x) \to F_X(x) \Rightarrow E[F_{X_n}(x)] \to E[F_X(x)]$

$X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} c$
$\Downarrow$
$(X_n, Y_n) \xrightarrow{d} (X, c)$

$Y_n \xrightarrow{P} c \Rightarrow P(|Y_n - c| \ge \varepsilon) \to 0$, for all $\varepsilon > 0$

$\Downarrow$

$P(cX|Y_n - c| \ge \varepsilon) \to 0$

$P(|X_n(Y_n - c) + X_n c| \le 0) \to 0$

$\Uparrow$

$X_n(Y_n - c) \nrightarrow -X_n c$, $n \to \infty$
$Y_n - c \nrightarrow -c$
$Y_n \nrightarrow 0$, $n \to \infty$

III. $X_n \xrightarrow{m} X$ is equivalently written $E(|X_n - X|) \to 0$ as $n \to \infty$

We know that

$$(X_n - X)_+ = \max\{0, (X_n - X)\}$$

and $(X_n - X)_- = \max\{0, (X - X_n)\}$

and that $E(|X_n - X|) \le E(X_n - X)_+ + E(X_n - X)_-$

From $X_n \xrightarrow{P} X$, we know that $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$, for all $\varepsilon > 0$

$$E(X_n - X)_+ = P_{X_n - X > 0} \cdot (X_n - X) = P_{X_n > X}(X_n - X)$$

$$E(X_n - X)_- = P_{X > X_n}(X - X_n) = -P_{X > X_n}(X_n - X)$$

then $E(X_n - X)_+ + E(X_n - X)_-$

$$= (P_{X_n > X} - P_{X > X_n})(X_n - X)$$

With $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$, for all $\varepsilon > 0$.

we can obtain that $P_{X_n > X} \to 0$, $P_{X > X_n} \to 0$

Therefore $P_{X_n - X} - P_{X > X_n} \to 0$, then $E(X_n - X)_+ + E(X_n - X)_- \to 0$

And $0 \le E(|X_n - X|) \le E(X_n - X)_+ + E(X_n - X)_- \to 0$

Thus we can conclude that $E(|X_n - X|) \to 0$ as $n \to \infty$,

which proves that $X_n \xrightarrow{m} X$

IV.

a. $W := \frac{(n-1)s^2}{\sigma^2}$, $s^2 = \frac{\sum(X - \bar{X})^2}{n-1}$

$$W := \frac{(n-1)\sum(X_i - \bar{X})^2}{(n-1)\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$

$$\sum(X_i - \mu)^2 = \sum(\{X_i - \bar{x}\} + \{\bar{x} - \mu\})^2$$

$$= \sum(\{X_i - \bar{x}\}^2 + 2\{\bar{x} - \mu\}\{X_i - \bar{x}\} + \{\bar{x} - \mu\}^2)$$

$$= \sum\{X_i - \bar{x}\}^2 + n\{\bar{x} - \mu\}^2$$

which follows from the fact that $\{\bar{x} - \mu\}\sum\{X_i - \bar{x}\} = 0$

then we can obtain that $\sum \frac{(X_i - \mu)^2}{\sigma^2} = \sum \frac{(X_i - \bar{x})^2}{\sigma^2} + \frac{n(\bar{x} - \mu)^2}{\sigma^2}$

with $\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$, which MGF is $\frac{1}{(1-2t)^{n/2}}$

$\frac{n(\bar{x} - \mu)^2}{\sigma^2} \sim \chi^2(1)$, which MGF is $\frac{1}{(1-2t)^{\frac{1}{2}}}$

$M_X(t) \overset{def}{=}$ the MGF of the random virable x.

then $M_W(t) = \frac{1}{(1-2t)^{\frac{1}{2}}} \Big/ \frac{1}{(1-2t)^{n/2}} = \frac{1}{(1-2t)^{(n-1)/2}}$

so we can conclude that W is $\chi^2(n-1)$ distributed.

b. $\overset{T(n-1)}{U} := \frac{\bar{x} - \mu}{s/\sqrt{n}}$ $\qquad T = \frac{Z}{\sqrt{Y/n}} = \frac{Z}{\sqrt{Y}/\sqrt{n}}$, $Z \sim N(0,1)$, $Y \sim \chi^2(n)$

define $Z := \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$, then $Z \sim N(0,1)$.

Also, define $Y := \frac{(n-1)s^2}{\sigma^2}$, then $Y \sim \chi^2(n-1)$

$$\frac{Z}{\sqrt{Y/(n-1)}} = \frac{\frac{\bar{x} - \mu}{\cancel{\sigma}/\sqrt{n}}}{\frac{\sqrt{n-1}\cdot s}{\cancel{\sigma}} \cdot \sqrt{n-1}} = \frac{\bar{x} - \mu}{\sqrt{n} \cdot s}$$

then we can conclude that U is $T(n-1)$ distributed.

V. $X_i \in N(\mu_1, 1)$, $Y_i \in N(\mu_2, 1)$

then $\sum_{i=1}^{n_1}(X_i - \bar{X})^2 \sim \chi^2(n_1 - 1)$, $\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2 \sim \chi^2(n_2 - 1)$

the MGF of $\chi^2(n)$: $M_X(t) = (1-2t)^{-n/2}$

$$M_W(t) = E(e^{Wt}) = E(e^{(X+Y)t}) = E(e^{Xt} \cdot e^{Yt})$$

Then $M_W(t) = M_X(t) \cdot M_Y(t)$

$$= (1-2t)^{-(n_1-1)/2} \cdot (1-2t)^{-(n_2-1)/2}$$

$$= (1-2t)^{-(n_1+n_2-2)/2}$$

Therefore, we can conclude that $W \sim \chi^2(n_1 + n_2 - 2)$.

VI. V1

Kolmogorov's one series Theorem

Let a sequence $\{x_i\}$ of independent random variables, each of which has finite mean and variance, satisfy $E(X_i)=0$ and $\sum_{i=1}^{\infty} Var(X_i) < \infty$, then $S(w) = \sum_{i=1}^{\infty} x_i(w)$ converges with probability 1.

Kolmogorv's inequality

Let $x_1, \dots, x_n : \Omega \to R$ be independent random variables defined on a common probability space (Ω, F, Pr), with expected value $E[X_k]=0$ and variance $Var[X_k] < +\infty$ for $k=1, \dots, n$.
Then for each $\lambda > 0$.

$$Pr(\max_{1 \le k \le n} |S_k| \ge \lambda) \le \frac{1}{\lambda^2} Var[S_n] = \frac{1}{\lambda^2} \sum_{k=1}^{n} Var[X_k] = \frac{1}{\lambda^2} \sum_{k=1}^{n} E[X_k^2],$$

where $S_k = x_1 + \cdots + x_k$

$$\lim_{\substack{n\to\infty \\ m\to\infty}} P\{\sup_{m \le k \le n} |S_k - S_m| \ge \delta\} \le \lim_{\substack{n\to\infty \\ m\to\infty}} \frac{1}{\delta^2} \sum_{v=m+1}^{n} E(X_i^2) = \lim_{\substack{n\to\infty \\ m\to\infty}} \frac{1}{\delta^2} \sum_{v=m+1}^{n} Var(X_i) = 0$$

Therefore $\lim_{\substack{n\to\infty \\ m\to\infty}} P\{\sup_{m \le k \le n} |S_k - S_m| \ge \delta\} \le 0$

According to Chebyshev's inequality, $\lim_{\substack{n\to\infty \\ m\to\infty}} P\{|S_n - S_m| \ge \delta\} = 0$, and then apply Levy's Theorem to get almost sure convergence.

We can prove Kolomogorov's first series theorem by above steps, then set $Y_i = X_i - \lambda_i$, we can obtain $\sum_{i=1}^{\infty} X_i$ converges a.s..

V2 (some deriation process refering to the hint)

PMF of poisson : $\frac{\lambda^k e^{-\lambda}}{k!}$

Consider $S_n = \sum_{i=1}^{n} x_i$

$$S_n = \sum_{i=1}^{n} Pois(\lambda_n) \sim Pois(\sum_{i=1}^{n} \lambda_i)$$

$$E(S_n) = \sum_{i=1}^{n} \lambda_i$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} Pois(\sum_{i=1}^{n} \lambda_i) = Pois(\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i)$$

$$\lim_{n\to\infty} E(S_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i$$

$$\|$$

$$E(\lim_{n\to\infty} S_n) = E(Pois(\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i)) = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i$$