HW2

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I.

a. $f(\mu|x_1,x_2,\dots,x_n) = \dfrac{f(x_1,x_2,\dots,x_n|\mu)\cdot f(\mu)}{f(x_1,x_2,\dots,x_n)} = \dfrac{f(x_1,x_2,\dots,x_n,\mu)}{f(x_1,x_2,\dots,x_n)}$

where $f(x_1,x_2,\dots,x_n) = \int_{-\infty}^{+\infty} f(x_1,x_2,\dots x_n|\mu)f(\mu)\,d\mu$

we can write $f(x_1,x_2,\dots,x_n,\mu) = f(x_1,x_2,\dots,x_n|\mu)\cdot f(\mu)$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}}\left(\frac{1}{\sigma}\right)^n\left(\frac{1}{\rho}\right)^* \exp\left\{-\frac{1}{2}\left(\sum_i \frac{(x_i-\mu)^2}{\sigma^2}+\frac{(\mu-v)^2}{\rho^2}\right)\right\}$$

$$f(x_1,x_2,\dots,x_n,\mu) \propto f(x_1,x_2,\dots,x_n|\mu)\cdot f(\mu) \propto \exp\left\{-\frac{1}{2}\left(\frac{\sum_i(x_i^2-2\mu x_i+\mu^2)}{\sigma^2}+\frac{\mu^2-2\mu v+v^2}{\rho^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{\sum_i x_i^2\rho^2-\sum_i 2\mu x_i\rho^2+n^2\mu^2\rho^2+\mu^2\sigma^2-2\mu v\sigma^2+v^2\sigma^2}{\sigma^2\rho^2}\right)\right\}$$

$$\propto \exp\left\{\frac{-\mu^2(\rho^2+\sigma^2)+2\mu(\sum_i x_i\rho^2+v\sigma^2)-(\sum_i x_i^2\rho^2+v^2\sigma^2)}{2\sigma^2\rho^2}\right\}$$

$$\propto \exp\left\{\frac{-\mu^2+2\mu\cdot\frac{\sum_i x_i\rho^2+v\sigma^2}{\rho^2+\sigma^2}-\left(\frac{\sum_i x_i\rho^2+v^2\sigma^2}{\rho^2+\sigma^2}\right)^2}{2\sigma^2\rho^2/\rho^2+\sigma^2}\right\}\times\exp\left\{-\frac{\sum_i x_i^2\rho^2+v^2\sigma^2}{2\sigma^2\rho^2}\right\}$$

$$\propto \exp\left\{\frac{-\left(\mu-\frac{v^2\sigma^2+\sum_i x_i\rho^2}{\rho^2+\sigma^2}\right)^2}{2\cdot\frac{\sigma^2\rho^2}{\rho^2+\sigma^2}}\right\}$$

Then $\mu^* = \dfrac{v^2\sigma^2+\sum_i x_i\rho^2}{\rho^2+\sigma^2} = \dfrac{v^2\sigma^2+n\bar{x}\rho^2}{\rho^2+\sigma^2} = \left(\dfrac{\rho^2}{\rho^2+\frac{\sigma^2}{n}}\right)\bar{x}+\left(\dfrac{\frac{\sigma^2}{n}}{\rho^2+\frac{\sigma^2}{n}}\right)v$

$\sigma^* = \sqrt{\dfrac{\sigma^2\rho^2}{\rho^2+\sigma^2}} = \sqrt{\dfrac{\rho^2\sigma^2}{n\rho^2+\sigma^2}}$

so that $\mu|x_1,x_2,\dots,x_n \sim N(\mu^*,\sigma^{*2})$

b.

1\. The number of successes has binominal distribution, its PMF is :

p denotes the probability of successes

$$x_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{otherwise} \end{cases}$$

$P\{X_i=1\} = p = 1-P\{X_i=0\} \Rightarrow P\{X_i=x\} = p^x(1-p)^{1-x},\ x=0,1$

2\. $L(p;x_1,x_2,\dots,x_n) := \prod_{i=1}^{n} f(x_i|p) = p^{x_1}(1-p)^{1-x_1}\cdot p^{x_2}(1-p)^{1-x_2}$ $\cdots\cdots p^{x_n}(1-p)^{1-x_n} = p^{x_1+x_2+\cdots+x_n}(1-p)^{n-x_1-x_2\cdots\cdots-x_n}$

$\log L(p;x_1,x_2,\dots,x_n) = (x_1+x_2+\cdots+x_n)\log p+(n-x_1-x_2\cdots\cdots-x_n)\log(1-p)$

3\. $\dfrac{\partial \log L(\cdot;\cdot)}{\partial p} = \dfrac{x_1+x_2+\cdots+x_n}{p}-\dfrac{n-x_1-x_2-\cdots-x_n}{1-p}$

set it to 0 then $\dfrac{\sum_{i=1}^n x_i}{p}-\dfrac{n-\sum_{i=1}^n x_i}{1-p}=0$

$$\sum_{i=1}^n x_i(1-p)-np+\sum_{i=1}^n x_i\cdot p = 0$$

$$\sum_{i=1}^n x_i - np = 0$$

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

if $n=10$, we can directly use $\sum_{i=1}^n x_i$, i.e., the number of successful trial, then the propotion of the observed trials that result in success is $(10\sum_{i=1}^n x_i)\%$.

II.

a. ① a_i must be positive, $|x_i|\geq 0 \Rightarrow \max_i\{a_i|x_i|\}\geq 0$

② $\because |x_i|\geq 0,\ a_i>0 \Rightarrow \max_i\{a_i|x_i|\}=0$ if and only if $|x_i|=0$

$|x_i|=0 \Rightarrow |x_1|=0,\ |x_2|=0,\ \dots\dots,\ |x_n|=0$

③ $\|cx\| = \max_i\{a_i|cx_i|\} = |c|\max_i\{a_i|x_i|\}$

④ $\|x+y\| = \max_i\{a_i|x_i+y_i|\} \leq \max_i\{a_i|x_i|\}+\max_i\{a_i|y_i|\} = \|x\|+\|y$

b. (i) $\|y\|_1 = \max_{\|u\|_\infty=1}(|y^*u|)$

$\max_{\|u\|_\infty=1}(|y^*u|) = \max_i(\sum_j|y_{ij}|)$

Suppose $y=[y_1,y_2,\dots,y_n]$, $y_i\in C^m$, $i=1,2,\dots,n$

To any vector $u=[u_1,u_2,\dots,u_n]^T\in C^n$, Suppose $y = [y_1, y_2, \dots, y_n]$, $y_i \in C^m$, $i = 1, 2, \dots, n$

To any vector $u = [u_1, u_2, \dots, u_n]^T \in C^n$,

$$\|u\|_1 = \sum_{i=1}^{n} |u_i| = 1$$

we have

$$\begin{aligned} |y^* u| &= |y_1 u_1 + y_2 u_2 + \cdots + y_n u_n| \\ &\le |y_1 u_1| + |y_2 u_2| + \cdots + |y_n u_n| \\ &\le \sum_i |y_i||u_i| \\ &= \sum_i |y_i| = \|y\|_1 \end{aligned}$$

Thus $\max_{\|u\|_\infty = 1} (|y^* u|) = \|y\|_1$

(ii) $\|y\|_\infty = \max_{\|u\|_1 = 1} (|y^* u|)$

$\|y\|_\infty = \max_i \{|y_i|\}$

$\|u\|_1 = \sum_i |u_i| = 1 \Rightarrow |y^* u| = \sum_i |y_i u_i| \le \sum_i |y_i||u_i| = \sum_i |y_i|$

thus $\|y\|_\infty = \max_{\|u\| = 1} (|y^* u|)$

c. $\|x\|_F^2 = [Tr(x^H x)]^{1/2}$

$\|Ux\|_F^2 = [Tr((Ux)^* (Ux))]^{1/2} = [Tr(x^H U^* U x)]^{1/2} = [Tr(x^H x)]^{1/2}$

thus $\|x\|_2 = \|Ux\|_2$

d.

$$\sup_{\|x\|_2 = 1} \|Ax\|_2 = \sup_{\|x\|_2 = 1} \|U \Sigma V^* x\|_2 = \sup_{\|x\|_2 = 1} \|\Sigma V^* x\|_2$$

U is unitary, we can obtain that,

$$\|U x_0\|_2^2 = x_0^T U^T U x_0 = x_0^T x_0 = \|x_0\|_2^2$$

Then let $y = V^* x$. Similarity, we have

$$\|y\|_2 = \|V^* x\|_2 = \|x\|_2 = 1$$

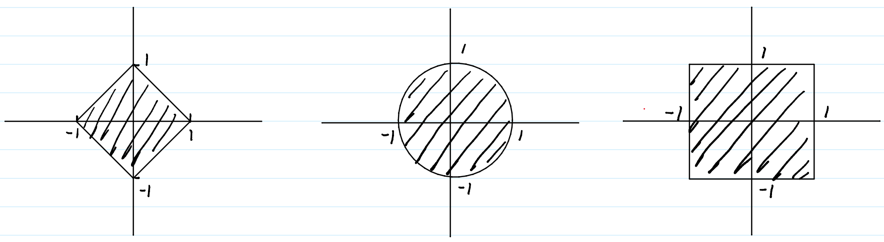
since V is unitary.

$$\sup_{\|x\|_2 = 1} \|\Sigma V^* x\|_2 = \sup_{\|x\|_2 = 1} \|\Sigma y\|_2$$

$\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$, where σ_1 is the largest singular value. The max for the above, i.e., $\max \|Ax\|_2^2$, is attained when $y = (1, \dots, 0)^T$.

Then we can conclude that $\max \|Ax\|_2^2$ is given by the maximum singular value, $\sigma(A)$.

e.



$B_1 = \{x : \|x\|_1 \le 1\}$ $B_2 = \{x : \|x\|_2 \le 1\}$ $B_3 = \{x : \|x\|_\infty \le 1\}$