some title

January 6, 2018

Regression

Correlation

The most crucial part in defining correlation geometrically is definig the dot product as it enables to compute the length of a vecotr:

$$|\vec{a}| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

and the angle between any two vectors:

$$\cos(\vec{a}, \vec{b}) = \frac{\langle \vec{a}, \vec{a} \rangle}{|\vec{a}||\vec{b}|}$$

Now we define scalar product of two random vectors as covariation between them:

$$\langle X, Y \rangle = \mathrm{Cov}(X, Y)$$

The main characteristics of a random vector are its length and direction. So, we introduce the length

$$\sqrt{\mathrm{Cov}\, X, X} = \sqrt{\mathrm{Var}(X)} = \sigma_X$$

and the angle between two random vectors

$$\cos(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \mathrm{Corr}(X, Y)$$

Note that from the definition of the angle it follows that correlation can range from -1 to 1.

The most widespread definition of correlation is

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

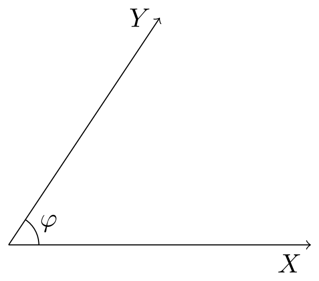


Figure 1: Random vector X of length σ_X and random vector Y of length σ_Y, $\cos\varphi$ is the angle between X and Y.

Another important geometrical tool is projection. Recall that for any two vectors the scalar product $\langle \vec{a}, \vec{b} \rangle$ can be interpreted as the length of projected $\vec{b}$ multuplied by the length of $\vec{a}$. The projection itself is $cos(\vec{a}, \vec{b})\vec{b}$. Same holds for random vectors. The projection of a random vector Y onto $\{cX | c \in \mathbb{R}\}$ is $\hat{Y} = \mathrm{Corr}(X, Y) \cdot Y$.

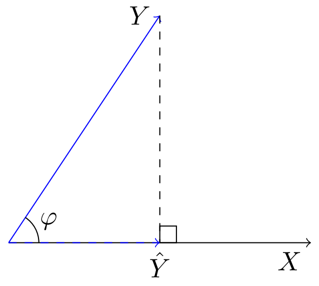


Figure 2: Random vector Y projected onto random vector X.

Looking at Figure , we can interpret the square of correlation coefficient. Using the fact that $cos^2\varphi$ is the squared ratio of the leg adjacent to φ to hypotenuse, we can write

$$\mathrm{Corr}^2(X, Y) = \frac{\mathrm{Var}(\hat{Y})}{\mathrm{Var}(Y)}$$

as the variance of a random vecor is associated with the square of its length. Thus, the correlation coefficient squared shows the fraction of variance in Y which can be explained with the most similar random variable proportional to X.

Sample correlation coefficient in simple linear regression

Theorem 1. *A linear regression model with one explanatory variable and constant term*

$$y = \beta_1 + \beta_2 x + \varepsilon$$

has the property

$$\mathrm{sCorr}(y, \hat{y}) = sign(\hat{\beta}_2)\,\mathrm{sCorr}(y, x)$$

Assuming the underlying relationship between x and y to be

$$y_i = \beta_1 + \beta_2 x_i + \varepsilon_i \quad i = 1, \ldots, n$$

where ε_i is an error term the following holds

$$\begin{aligned}
\mathrm{sCorr}(y, \hat{y}) &= \frac{\mathrm{sCov}(y)\,\mathrm{sCov}(\hat{y})}{\sqrt{\mathrm{sVar}(y)\,\mathrm{sVar}(\hat{y})}} \\
&= \frac{\mathrm{sCov}(y)\,\mathrm{sCov}(\hat{\beta}_1 + \hat{\beta}_2 x)}{\sqrt{\mathrm{sVar}(y)\,\mathrm{sVar}(\hat{\beta}_1 + \hat{\beta}_2 x)}} \\
&= \frac{\mathrm{sCov}(y)\,\mathrm{sCov}(\hat{\beta}_2 x)}{\sqrt{\mathrm{sVar}(y)\,\mathrm{sVar}(\hat{\beta}_2 x)}} \\
&= \frac{\hat{\beta}_2\,\mathrm{sCov}(y)\,\mathrm{sCov}(x)}{|\hat{\beta}_2|\sqrt{\mathrm{sVar}(y)\,\mathrm{sVar}(x)}} \\
&= sign(\hat{\beta}_2)\frac{\mathrm{sCov}(y)\,\mathrm{sCov}(x)}{\sqrt{\mathrm{sVar}(y)\,\mathrm{sVar}(x)}}
\end{aligned}$$

Proof. Firstly, we consider the case when $\hat{\beta}_2 > 0$ so the main picuture is of the form depicted in Figure . It has been shown earlier that the correlation coefficient represents the angle betweem two random vectors. So in order to complete the proof we need to find the appropriate angles and compare them.

However, it seems to be difficult to compare the angles in the three dimensional space. That is why we start with projecting both x and y onto the space perpendicular to the vector of all ones $\mathbf{1}$ as shown in Figure 4(a). We denote this space as $Lin^{\perp}(\mathbf{1})$. The resulting vectors are $x - \bar{x} \cdot \mathbf{1}$ and $y - \bar{y} \cdot \mathbf{1}$ respectively since projection of any vector $\vec{a}$ on the line given by a vector of all ones yields the vector of averages $\vec{\bar{a}}$.

In order to get the angle between y and $\hat{y}$ we should start with regressing y on $Lin(x, \mathbf{1})$. Then the only thing thing left is to project $\hat{y}$ onto $Lin^{\perp}\mathbf{1}$ since the y vector has already been projected. Note that the projected $\hat{y}$ falls

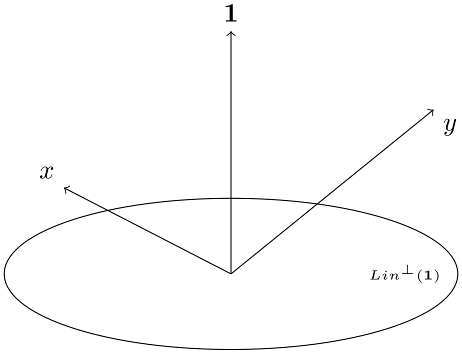
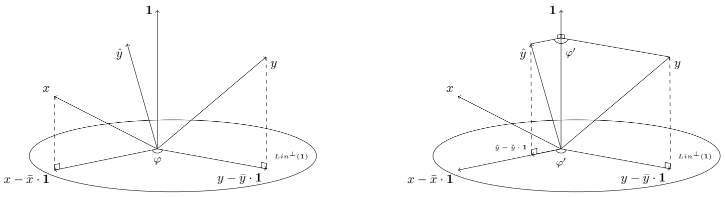


Figure 3: Vectors x, y and $\mathbf{1}$.

onto tha span of vector $x - \bar{x} \cdot \mathbf{1}$ as it can be decomposed into a sum $ax + b\mathbf{1}$ where $a, b \in \mathbb{R}$. ax is projected in the same way as x and $b\mathbf{1}$ yields zero when projected onto the orthogonal space. The result of this step is shown in Figure 4(b).


$\mathbf{1}$
$\hat{y}$
y
x
φ
$Lin^{\perp}(\mathbf{1})$
$x - \bar{x} \cdot \mathbf{1}$
$y - \bar{y} \cdot \mathbf{1}$
$\mathbf{1}$
$\hat{y}$
φ'
y
x
$\hat{y} - \bar{\hat{y}} \cdot \mathbf{1}$
φ'
$Lin^{\perp}(\mathbf{1})$
$x - \bar{x} \cdot \mathbf{1}$
$y - \bar{y} \cdot \mathbf{1}$

(a)

(b)

Figure 4: (a): 'Centred' x and y, i.e., projected onto $Lin^{\perp}(\mathbf{1})$; (b): 'Centred' $\hat{y}$, i.e., projected onto $Lin^{\perp}(\mathbf{1})$.

Since the projection of $\hat{y}$ lies exactly on the span of vector $x - \bar{x} \cdot \mathbf{1}$, we can conclude that $cos\varphi = \cos\varphi'$ and to put it another way sCorr$(x, y) =$ sCorr$(y, \hat{y})$.

Now consider the case when $\hat{\beta}_2 < 0$. Note that the sign of β_1 does not influence the correlation coefficient sign. The only difference is that now $\hat{y}$ is projected onto the span of $x - \bar{x} \cdot \mathbf{1}$ not on this vector itself while the projections of x and y remain the same. Looking at Figure we deduce that the angle betwween y and $\hat{y}$ is compelement to the angle between x and y. Using trigonometric properties, we simplify $\cos(180° - \varphi) = -\cos\varphi$ which in turn implies sCorr$(x, y) = -$ sCorr$(y, \hat{y})$.

□

RSS + ESS = TSS

Theorem 2. *A linear regression model with n observations and k explanatory variables including a constant unit vector*

$$y = X\beta + \varepsilon$$

Consider a regresion model with n observations and k explanatory variables including a constant unit vector

$$y = X\beta + \varepsilon$$

The OLS estimator for the vector of coefficients β is

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

and the residual vector is

$$\begin{aligned} \hat{e} &= y - \hat{y} \\ &= y - X\hat{\beta} \\ &= y - X(X^T X)^{-1} X^T y \end{aligned}$$

Then we define residual sum of squares (RSS), explained sum of squares (ESS) and total sum of squares (TSS) as follows:

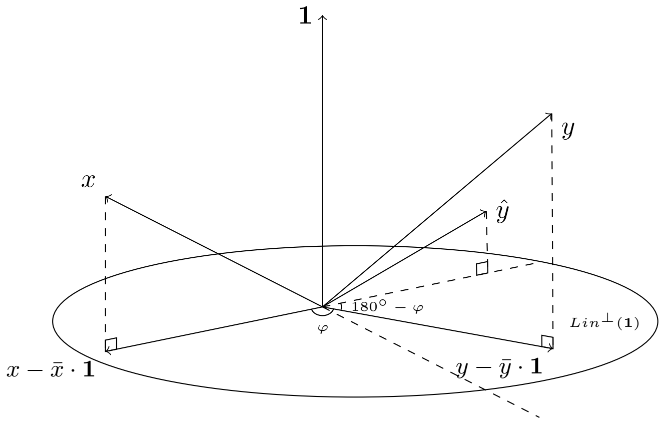

$\mathbf{1}$
y
x
$\hat{y}$
$180^\circ - \varphi$
φ
$Lin^{\perp}(\mathbf{1})$
$x - \bar{x} \cdot \mathbf{1}$
$y - \bar{y} \cdot \mathbf{1}$

Figure 5: Case of $\beta_2 < 0$.

has the following property

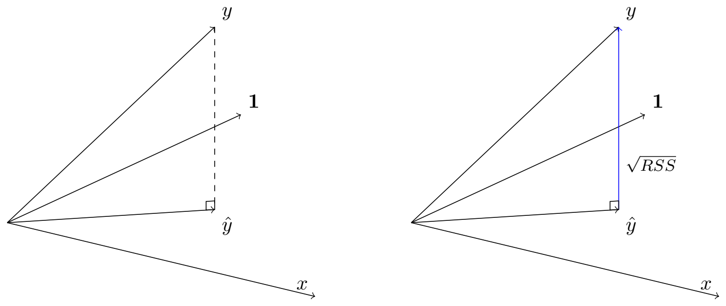
$$RSS + ESS = TSS$$

where $RSS = \|y - \hat{y}\|_2^2, ESS = \|\hat{y} - \bar{y}\|_2^2, TSS = \|y - \bar{y}\|$.

Proof. The proof will be presented for the case of two regressor x and $\mathbf{1}$ in order for the picture to be clear. However, the same logic applies for the case of k regressors.

We start with depicting the vectors $y \in \mathbb{R}^{n-2}$ and $x, \mathbf{1} \in \mathbb{R}^2$. Then we project y onto $Lin(x, \mathbf{1})$ and obtain $\hat{y}$ which is shown in Figure 6(a).

From this picture we can immediately derive $\sqrt{RSS}$ as by definition this is the squared difference between y and $\hat{y}$.



(a)

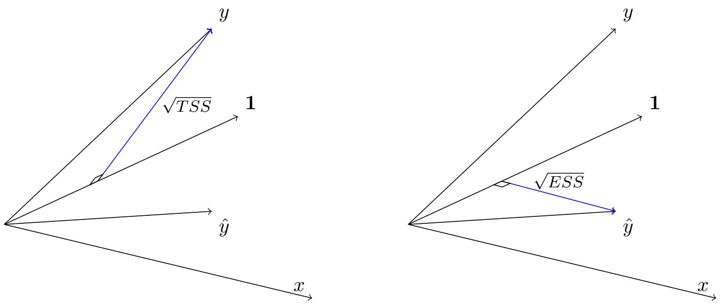
(b)

Figure 6: (a): Vectors $y \in \mathbb{R}^{n-2}$ and $\hat{y} \in Lin(x, \mathbf{1})$; (b): Residual sum of squares.

So as to visualize ESS and TSS we first need to visualize vector of averages $\bar{y}$. Geometrically this means projecting a vector onto a line spanned by vector $\mathbf{1}$. It can be shown that the matrix corresponding to projecting onto the line spanned by a vector of all ones has the following form

$$\frac{\mathbf{1}^T\mathbf{1}}{\mathbf{1}\mathbf{1}^T} = \frac{\begin{pmatrix}1 & \dots & 1\end{pmatrix}\begin{pmatrix}1\\ \dots \\ 1\end{pmatrix}}{\sum_{i=1}^n 1} = \begin{pmatrix}\frac{1}{n} & \dots & \frac{1}{n}\\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \dots & \frac{1}{n}\end{pmatrix}$$

Now we both project y and $\hat{y}$ onto $\mathbf{1}$ and following the definition obtain $\sqrt{TSS}$ as the difference vector $y - \bar{y}$ and $\sqrt{ESS}$ as the vector $\hat{y} - \bar{y}$.



(a)

(b)

Figure 7: (a): Total sum of squares; (b): Explained sum of squares.

The final step is to put everything together. Note that since $y - \hat{y}$ is perpendicular to $lin(x, \mathbf{1})$ it is also perpendicular to $\hat{y} - \bar{y}$ and $\mathbf{1}$ as these vectoros are in $lin(x, \mathbf{1})$. Then, applying the theorem of three perpendiculars we conclude that vecotr foot of vector $y - \bar{y}$ is the same point as the foot of the vector $\hat{y} - \bar{y}$. Thus, we obtain a right angle triangle and can apply the Pythagorean theorem for the catheti $\sqrt{RSS}$ and $\sqrt{ESS}$ and the hypothenuse $\sqrt{TSS}$:

$$\sqrt{RSS} + \sqrt{ESS} = \sqrt{TSS}$$

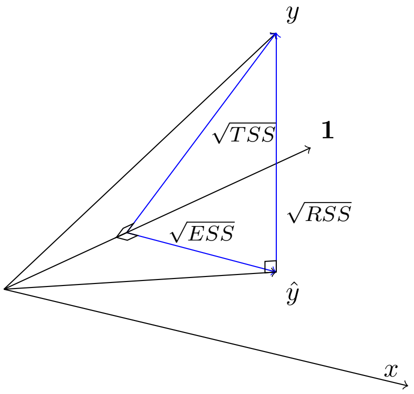


Figure 8: $\sqrt{RSS} + \sqrt{ESS} = \sqrt{TSS}$

□

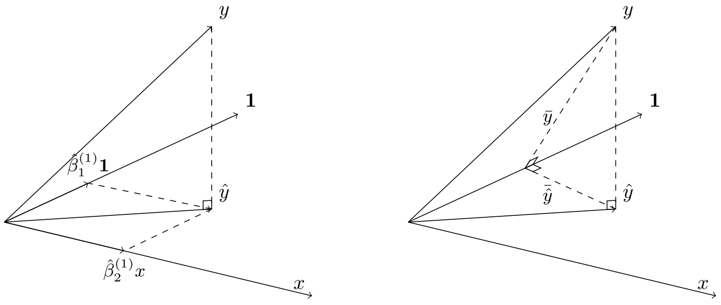
Determination coefficient

Regression line and point of averages

Theorem 3. *The point of averages lies on the estimated regression line.*

Proof. For the geometrical proof it suffices to show that $\hat{y}$ is a linear combination of the regressors, which is true by construction, and that $\frac{1}{n}\sum_{i=1}^{n}\hat{y}_i = \frac{1}{n}\sum_{i=1}^{n} y$. In order for the pictures to be more clear the proof will be presented for the case of two regressors.

The first step is regressing y on $Lin(\mathbf{1}, x)$. As shown in Figure 9(a), we obtain $\hat{y}$ as a linear combination of $\mathbf{1}$ and x. The next step is to regress both y and $\hat{y}$ on $\mathbf{1}$ which results in $\bar{y}$ and $\bar{\hat{y}}$ correspondingly. By the theorem of three perpendiculars, $\bar{y} = \bar{\hat{y}}$ which is shown in Figure 9(b).



(a)

(b)

□

If the regression contains the intercept, the following equation holds:

$$\hat{y} = X\hat{\beta} = X(X^TX)^{-1}X^Ty$$
$$= X(X^TX)^{-1}X^TX\beta + X(X^TX)^{-1}X^T\varepsilon$$

Premultiplying both sides by X^T, we obtain:

$$X^T\hat{y} = X^TX(X^TX)^{-1}X^TX\beta$$
$$+ X^TX(X^TX)^{-1}X^T\varepsilon$$
$$= X^TX\beta + X^T\varepsilon$$

This is a system of equations. The first row of X^T is $\mathbf{1}$ vector, so we can write out the first equation:

$$\sum_{i=1}^{n}\hat{y}_i = \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}\beta_j$$

From the first equation in the system

$$X^T\hat{y} = X^Ty$$

we obtain

$$\sum_{i=1}^{n}\hat{y}_i = \sum_{i=1}^{n} y$$

And this finishes the proof:

$$\frac{1}{n}\sum_{i=1}^{n} y = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}\beta_j$$

Figure 9: (a): Regression of y on $Lin(\nVdash, x)$; (b): Regression of y and $\hat{y}$ on $\nVdash$.

Frisch–Waugh–Lovell theorem

Theorem 4. *Consider regression*

$$y = X_1\beta_1 + X_2\beta_2 + u \qquad (1)$$

where $X_{n\times k} = [X_1 X_2]$, i.e. X_1 consists of first k_1 columns of X and X_2 consists of remaining k_2 columns of X, β_1 and β_2 are comfortable, i.e. $k_1 \times 1$ and $k_2 \times 1$ vectors. Consider another regresison

$$M_1y = M_1X_2\beta_2 + M_1u \qquad (2)$$

where $M_1 = I - P_1$ projects onto the orthogonal complement of the column space of X_1 and $P_1 = X_1(X_1^TX_1)^{-1}X_1^T$ is the projection onto the column space of X_1. Then the estimate of β_2 from regression 1 will be the same as the estimate from regression 2.

Proof. Geometrical proof will be presented for the following model:

$$y_i = \beta_1x_i + \beta_2z_i + u_i \qquad (3)$$

From regresison 2 we get the following estimator:

$$\hat{\beta}_2 = ((M_1X_2)^TM_1X_2)^{-1}(M_1X_2)^TM_1y$$
$$= (X_2^TM_1^TM_1X_2)^{-1}X_2^TM_1^TM_1y$$
$$= (X_2^TM_1X_2)^{-1}X_2^TM_1y$$

As for regresison 1, let us note that due to $y = \hat{y} + \hat{u}$ y can be decomposed as follows:

$$y = Py + My = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + My$$

Premultiplying both sides by $X_2^TM_1$, we obtain:

$$X_2^TM_1y = X_2^TM_1X_1\hat{\beta}_1 + X_2^TM_1X_2\hat{\beta}_2 + X_2^TM_1My$$
$$= X_2^TM_1X_2\hat{\beta}_2 + X_2^TM_1My$$
$$= X_2^TM_1X_2\hat{\beta}_2$$

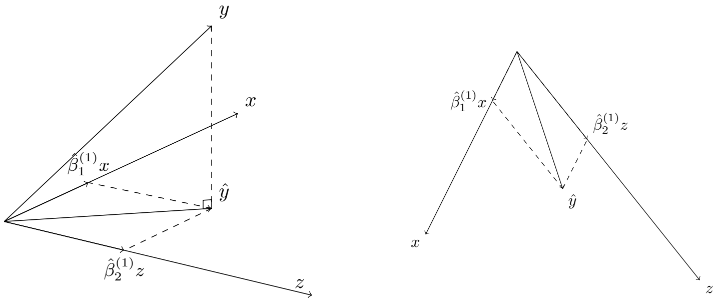
On the last step we used the fact that

$$(X_2^TM_1My)^T = y^TM^TM_1^TX_2$$
$$= y^TMM_1X_2 = y^TMX_2 = 0^T$$

Assuming $X_2^TM_1X_2$ is invertible, we get the same estimator

$$\hat{\beta}_2 = (X_2^TM_1X_2)^{-1}X_2^TM_1y$$

We start with regression 'all-at-once' and will distinct its coefficients with index (1). The only step in obtaining $\beta_1^{(1)}$ is regressing y on $Lin(x, z)$ and then expanding $\hat{y}$ as a linear combination of basis vectors x and z, which is shown in Figure 10(a). Figure 10(b) depicts $Lin(x, z)$.



(a) (b)

Figure 10: (a): Regression of y on $Lin(x, z)$; (b): $Lin(x, z)$.

As for the model 2, where several regressions are performed consecutively, we start with regressing y on z, resulting in $\tilde{y}$, which we will refer to as "cleansed" y.

$$\begin{aligned} y &= \alpha z + \varepsilon \\ \hat{\alpha} &= \frac{y^T z}{z^T z} \\ \tilde{y} &= \hat{\varepsilon} = y - \frac{y^T z}{z^T z} z \end{aligned} \tag{4}$$

Following that, x is regressed on z, resulting in $\tilde{x}$ – "cleansed" x.

$$\begin{aligned} x &= \gamma z + \nu \\ \hat{\gamma} &= \frac{x^T z}{z^T z} \\ \tilde{x} &= \hat{\nu} = x - \frac{x^T z}{z^T z} z \end{aligned} \tag{5}$$

Geometric results of these two steps are presented in 11(a).

Finally, 'cleansed' y must be regressed on 'cleansed' x. However, it cannot be performed immediately as $\tilde{y}$ and $\tilde{x}$ are skew lines. So at first, we fix this problem by translation and after taht obtain $\hat{\beta}_1^{(2)}\tilde{x}$ (see Figure 11(b)).

Now, let us picture all the results in one figure and mark some main points.

In Figure 12(b) segments AF and BH = DG stand for $\hat{\beta}_1^{(1)}x$ and $\hat{\beta}_1^{(2)}\tilde{x}$ respectively, while segments AC and BC represent x and $\tilde{x}$. Having two congruent angles, triangles ABC and FHC are simillar. Then, it follows:

$$\frac{AF}{AC} = \frac{BH}{BC} \Leftrightarrow \frac{\hat{\beta}_1^{(1)}x}{x} = \frac{\hat{\beta}_1^{(2)}\tilde{x}}{\tilde{x}} \Leftrightarrow \hat{\beta}_1^{(1)} = \hat{\beta}_1^{(2)}$$

□

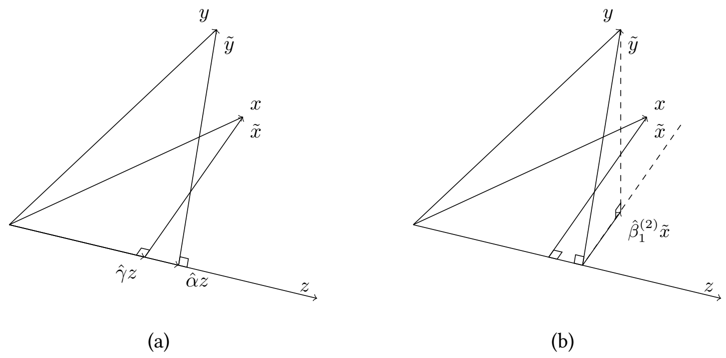


Figure 11: (a): Regression of y on z and of x on z; (b): Translation of $\tilde{x}$.

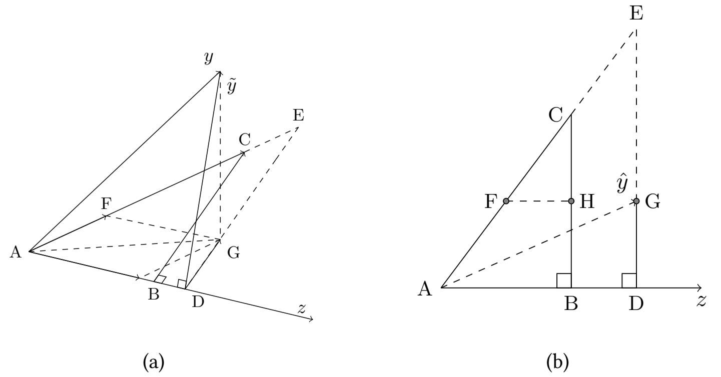


Figure 12: (a): Point A stands for the origin, B – $\hat{\gamma}z$, C – x, D – $\hat{\alpha}z$, E – intersection of vector x and line parallel to $\tilde{x}$, F – $\hat{\beta}_1^{(1)}x$, G – $\hat{\beta}_1^{(2)}\tilde{x}$; (b): $Lin(x, z)$.