1 Syntax

$$
\begin{array}{lll}
\mathit{program} & ::= & \overline{\mathit{cls}}\ \overline{s} \\
\mathit{cls} & ::= & \texttt{class}\ C\ \{\overline{\mathit{field}}\ \overline{\mathit{method}}\} \\
\mathit{field} & ::= & T\ f; \\
\mathit{method} & ::= & T\ m(T\ x)\ \mathit{contract}\ \{\overline{s}\} \\
\mathit{contract} & ::= & \texttt{requires}\ \phi;\ \texttt{ensures}\ \phi; \\
T & ::= & \texttt{int} \mid C \\
s & ::= & x.f := y; \mid x := e; \mid x := \texttt{new}\ C; \mid x := y.m(z); \\
 & & \mid \texttt{return}\ x; \mid \texttt{assert}\ \phi; \mid \texttt{release}\ \phi; \mid T\ x; \\
\phi & ::= & \texttt{true} \mid e = e \mid e \neq e \mid \texttt{acc}(e.f) \mid x : T \mid \phi * \phi \\
e & ::= & v \mid x \mid e.f \\
x & ::= & \texttt{this} \mid \texttt{result} \mid \langle \mathit{other} \rangle \\
v & ::= & o \mid n \mid \texttt{null} \\
n & \in & \mathbb{Z} \\
 & & \\
H & \in & (o \rightharpoonup (C, \overline{(f \rightharpoonup v)})) \\
\rho & \in & (x \rightharpoonup v) \\
A_s & ::= & \overline{(e, f)} \\
A_d & ::= & \overline{(o, f)} \\
S & ::= & (\rho, A_d, \overline{s}) \cdot S \mid \mathit{nil}
\end{array}
$$

2 Assumptions

All the rules in the following sections are implicitly parameterized over a *programp* that is well-formed.

2.0.1 Well-formed program (*program* OK)

$$\frac{\overline{\mathit{cls}_i\ \texttt{OK}}}{(\overline{\mathit{cls}_i}\ \overline{s})\ \texttt{OK}}\ \textsc{OKProgram}$$

2.0.2 Well-formed class (*cls* OK)

$$\frac{\text{unique } \mathit{field}\text{-names} \qquad \text{unique } \mathit{method}\text{-names} \qquad \overline{\mathit{method}_i\ \texttt{OK in}\ C}}{(\texttt{class}\ C\ \{\overline{\mathit{field}_i}\ \overline{\mathit{method}_i}\})\ \texttt{OK}}\ \textsc{OKClass}$$

2.0.3 Well-formed method (*method* OK in C)

$$\frac{\begin{array}{c} FV(\phi_1) \subseteq \{x, \texttt{this}\} \qquad FV(\phi_2) \subseteq \{x, \texttt{this}, \texttt{result}\} \\ \vdash \{x : T_x * \texttt{this} : C * \phi_1\}\overline{s}\{x : T_x * \texttt{this} : C * \texttt{result} : T_m * \phi_2\} \\ \emptyset \vdash_{\texttt{sfrm}} \phi_1 \qquad \emptyset \vdash_{\texttt{sfrm}} \phi_2 \qquad \overline{\neg \mathit{writesTo}(s_i, x)} \end{array}}{(T_m\ m(T_x\ x)\ \texttt{requires}\ \phi_1;\ \texttt{ensures}\ \phi_2;\ \{\overline{s}\})\ \texttt{OK in}\ C}\ \textsc{OKMethod}$$

3 Static semantics

3.1 Expressions ($A_s \vdash_{\texttt{sfrm}} e$)

$$\frac{}{A \vdash_{\texttt{sfrm}} x} \textsc{WFVar}$$

$$\frac{}{A \vdash_{\texttt{sfrm}} v} \textsc{WFValue}$$

$$\frac{(e, f) \in A \qquad A \vdash_{\texttt{sfrm}} e}{A \vdash_{\texttt{sfrm}} e.f} \textsc{WFField}$$

3.2 Formulas ($A_s \vdash_{\texttt{sfrm}} \phi$)

$$\frac{}{A \vdash_{\texttt{sfrm}} \texttt{true}} \textsc{WFTrue}$$

$$\frac{A \vdash_{\texttt{sfrm}} e_1 \qquad A \vdash_{\texttt{sfrm}} e_2}{A \vdash_{\texttt{sfrm}} (e_1 = e_2)} \textsc{WFEqual}$$

$$\frac{A \vdash_{\texttt{sfrm}} e_1 \qquad A \vdash_{\texttt{sfrm}} e_2}{A \vdash_{\texttt{sfrm}} (e_1 \neq e_2)} \textsc{WFNEqual}$$

$$\frac{A \vdash_{\texttt{sfrm}} e}{A \vdash_{\texttt{sfrm}} \texttt{acc}(e.f)} \textsc{WFAcc}$$

$$\frac{}{A \vdash_{\texttt{sfrm}} x : T} \textsc{WFType}$$

$$\frac{A_s \vdash_{\texttt{sfrm}} \phi_1 \qquad A_s \cup \lfloor \phi_1 \rfloor \vdash_{\texttt{sfrm}} \phi_2}{A_s \vdash_{\texttt{sfrm}} \phi_1 * \phi_2} \textsc{WFSepOp}$$

3.2.1 Implication ($\phi_1 \stackrel{.}{\Longrightarrow} \phi_2$)

Conservative approx. of $\phi_1 \implies \phi_2$.

3.3 Footprint ($\lfloor \phi \rfloor = A_s$)

$$
\begin{aligned}
&\lfloor \texttt{true} \rfloor && = \emptyset \\
&\lfloor e_1 = e_2 \rfloor && = \emptyset \\
&\lfloor e_1 \neq e_2 \rfloor && = \emptyset \\
&\lfloor \texttt{acc}(e.f) \rfloor && = \{(e, f)\} \\
&\lfloor \phi_1 * \phi_2 \rfloor && = \lfloor \phi_1 \rfloor \cup \lfloor \phi_2 \rfloor
\end{aligned}
$$

3.4 Type ($\phi \vdash e : T$)

$$\frac{}{\phi \vdash n : \texttt{int}}\ \textsc{STValNum}$$

$$\frac{}{\phi \vdash \texttt{null} : T}\ \textsc{STValNull}$$

$$\frac{\phi \implies (x : T)}{\phi \vdash x : T}\ \textsc{STVar}$$

$$\frac{\phi \vdash e : C \qquad \vdash C.f : T}{\phi \vdash e.f : T}\ \textsc{STField}$$

3.5 Hoare ($\vdash \{\phi\}\overline{s}\{\phi\}$)

$$\frac{\vdash \{\phi_p\}s_1\{\phi_{q1}\} \qquad \phi_{q1} \implies \phi_{q2} \qquad \emptyset \vdash_{\texttt{sfrm}} \phi_{q2} \qquad \vdash \{\phi_{q2}\}s_2\{\phi_r\}}{\vdash \{\phi_p\}s_1; s_2\{\phi_r\}}\ \textsc{HSec}$$

$$\frac{\phi \implies \phi' \qquad \emptyset \vdash_{\texttt{sfrm}} \phi' \qquad x \notin FV(\phi') \qquad \phi \vdash x : C \qquad \texttt{fields}(C) = \overline{f}}{\vdash \{\phi\}x := \texttt{new}\ C\{\overline{\texttt{acc}(x, f_i)*}x : C * (x \neq \texttt{null}) * \phi'\}}\ \textsc{HNewObj}$$

$$\frac{\begin{array}{c}\phi \implies \texttt{acc}(x.f) * (x \neq \texttt{null}) * \phi' \\ \emptyset \vdash_{\texttt{sfrm}} \phi' \qquad \phi \vdash x : C \qquad \phi \vdash y : T \qquad \vdash C.f : T\end{array}}{\vdash \{\phi\}x.f := y\{x : C * \texttt{acc}(x.f) * (x \neq \texttt{null}) * (x.f = y) * \phi'\}}\ \textsc{HFieldAssign}$$

$$\frac{\begin{array}{c}\phi \implies \phi' \\ \emptyset \vdash_{\texttt{sfrm}} \phi' \qquad x \notin FV(\phi') \qquad x \notin FVe(e) \qquad \phi \vdash x : T \qquad \phi \vdash e : T \qquad \lfloor \phi' \rfloor \vdash_{\texttt{sfrm}} e\end{array}}{\vdash \{\phi\}x := e\{\phi' * (x = e)\}}\ \textsc{HVarAssign}$$

$$\frac{\phi \implies \phi' \qquad \emptyset \vdash_{\texttt{sfrm}} \phi' \qquad \texttt{result} \notin FV(\phi') \qquad \phi \vdash x : T \qquad \phi \vdash \texttt{result} : T}{\vdash \{\phi\}\texttt{return}\ x\{\texttt{result} : T * (\texttt{result} = x) * \phi'\}}\ \textsc{HReturn}$$

$$\frac{\begin{array}{c}\phi \vdash y : C \qquad \texttt{mmethod}(C, m) = T_r\ m(T_p\ z)\ \texttt{requires}\ \phi_{pre};\ \texttt{ensures}\ \phi_{post};\ \{\_\} \\ \phi \vdash x : T_r \qquad \phi \vdash z' : T_p \qquad \phi \implies (y \neq \texttt{null}) * \phi_p * \phi_r \\ \emptyset \vdash_{\texttt{sfrm}} \phi_r \qquad x \notin FV(\phi_r) \qquad @listDistinct(x, x \cdot y \cdot z' \cdot \emptyset) \\ \phi_p = \phi_{pre}[y, z'/\texttt{this}, z] \qquad \phi_q = \phi_{post}[y, z', x/\texttt{this}, z, \texttt{result}]\end{array}}{\vdash \{\phi\} x := y.m(z') \{\phi_q * \phi_r\}} \text{ HApp}$$

$$\frac{\phi_1 \implies \phi_2}{\vdash \{\phi_1\} \texttt{assert}\ \phi_2 \{\phi_1\}} \text{ HAssert}$$

$$\frac{\phi_1 \implies \phi_2 * \phi_r \qquad \emptyset \vdash_{\texttt{sfrm}} \phi_r}{\vdash \{\phi_1\} \texttt{release}\ \phi_2 \{\phi_r\}} \text{ HRelease}$$

$$\frac{\phi \implies \phi' \qquad \emptyset \vdash_{\texttt{sfrm}} \phi' \qquad x \notin FV(\phi')}{\vdash \{\phi\} T\ x \{x : T * (x = \texttt{defaultValue}(T)) * \phi'\}} \text{ HDeclare}$$

Note: issue with HApp and z' in the post-condition: the substitution reflects **any changes** made to z onto z' which is wrong in general (except we make z' a by-ref parameter in the small-step semantics)

4 Dynamic semantics

4.1 Expressions ($H, \rho \vdash e \Downarrow v$)

$$\frac{}{H, \rho \vdash x \Downarrow \rho(x)} \text{ EEVar}$$

$$\frac{}{H, \rho \vdash v \Downarrow v} \text{ EEValue}$$

$$\frac{H, \rho \vdash e \Downarrow o}{H, \rho \vdash e.f \Downarrow H(o)(f)} \text{ EEAcc}$$

4.2 Formulas ($H, \rho, A \vDash \phi$)

$$\frac{}{H, \rho, A \vDash \texttt{true}} \text{ EATrue}$$

$$\frac{H, \rho \vdash e_1 \Downarrow v_1 \qquad H, \rho \vdash e_2 \Downarrow v_2 \qquad v_1 = v_2}{H, \rho, A \vDash (e_1 = e_2)} \text{ EAEqual}$$

$$\frac{H, \rho \vdash e_1 \Downarrow v_1 \qquad H, \rho \vdash e_2 \Downarrow v_2 \qquad v_1 \neq v_2}{H, \rho, A \vDash (e_1 \neq e_2)} \text{ EANEqual}$$

$$\frac{H, \rho \vdash e \Downarrow o \qquad (o, f) \in A}{H, \rho, A \vDash \texttt{acc}(e.f)} \text{ EAAcc}$$

$$\frac{\rho(x) = v \qquad H \vdash v : T}{H, \rho, A \vDash x : T} \text{ EAType}$$

$$\frac{A_1 = A \backslash A_2 \qquad H, \rho, A_1 \vDash \phi_1 \qquad H, \rho, A_2 \vDash \phi_2}{H, \rho, A \vDash \phi_1 * \phi_2} \text{ EASepOp}$$

We give a denotational semantics of formulas as $[\![\phi]\!] = \{\ (H, \rho, A) \mid H, \rho, A \vDash \phi\ \}$
Note: ϕ satisfiable $\iff [\![\phi]\!] \neq \emptyset$

4.2.1 Implication ($\phi_1 \implies \phi_2$)

$$\phi_1 \implies \phi_2 \qquad \iff \qquad [\![\phi_1]\!] \subseteq [\![\phi_2]\!]$$

4.3 Footprint ($\lfloor \phi \rfloor_{H,\rho} = A_d$)

$$\begin{aligned}
\lfloor \texttt{true} \rfloor_{H,\rho} &= \emptyset \\
\lfloor e_1 = e_2 \rfloor_{H,\rho} &= \emptyset \\
\lfloor e_1 \neq e_2 \rfloor_{H,\rho} &= \emptyset \\
\lfloor \texttt{acc}(x.f) \rfloor_{H,\rho} &= \{(o, f)\} \text{ where } H, \rho \vdash x \Downarrow o \\
\lfloor \phi_1 * \phi_2 \rfloor_{H,\rho} &= \lfloor \phi_1 \rfloor_{H,\rho} \cup \lfloor \phi_2 \rfloor_{H,\rho}
\end{aligned}$$

4.4 Small-step ($(H, S) \rightarrow (H, S)$)

$$\frac{H, \rho \vdash x \Downarrow o \qquad H, \rho \vdash y \Downarrow v_y \qquad (o, f) \in A \qquad H' = H[o \mapsto [f \mapsto v_y]]}{(H, (\rho, A, x.f := y; \overline{s}) \cdot S) \rightarrow (H', (\rho, A, \overline{s}) \cdot S)} \text{ ESFieldAssign}$$

$$\frac{H, \rho \vdash e \Downarrow v \qquad \rho' = \rho[x \mapsto v]}{(H, (\rho, A, x := e; \overline{s}) \cdot S) \rightarrow (H, (\rho', A, \overline{s}) \cdot S)} \text{ ESVarAssign}$$

$$\frac{\begin{array}{c} o \notin \texttt{dom}(H) \qquad \texttt{fields}(C) = \overline{T}\ \overline{f} \\ \rho' = \rho[x \mapsto o] \qquad A' = A * \overline{(o, f_i)} \qquad H' = H[o \mapsto [\overline{f \mapsto \texttt{defaultValue}(T)}]] \end{array}}{(H, (\rho, A, x := \texttt{new}\ C; \overline{s}) \cdot S) \rightarrow (H', (\rho', A', \overline{s}) \cdot S)} \text{ ESNewObj}$$

$$\frac{H, \rho \vdash x \Downarrow v_x \qquad \rho' = \rho[\texttt{result} \mapsto v_x]}{(H, (\rho, A, \texttt{return } x; \overline{s}) \cdot S) \rightarrow (H, (\rho', A, \overline{s}) \cdot S)} \text{ ESReturn}$$

$$\frac{\begin{array}{c} H, \rho \vdash y \Downarrow o \qquad H, \rho \vdash z \Downarrow v \\ H(o) = (C, \_) \qquad \texttt{mmethod}(C, m) = T_r \ m(T \ w) \ \texttt{requires } \phi; \ \texttt{ensures } \_; \ \{\overline{r}\} \\ \rho' = [\texttt{result} \mapsto \texttt{defaultValue}(T_r), \texttt{this} \mapsto o, w \mapsto v] \qquad H, \rho', A \vDash \phi \qquad A' = \lfloor \phi \rfloor_{H,\rho'} \end{array}}{(H, (\rho, A, x := y.m(z); \overline{s}) \cdot S) \rightarrow (H, (\rho', A', \overline{r}) \cdot (\rho, A \setminus A', x := y.m(z); \overline{s}) \cdot S)} \text{ ESApp}$$

$$\frac{\begin{array}{c} H, \rho \vdash y \Downarrow o \qquad H(o) = (C, \_) \\ \texttt{mpost}(C, m) = \phi \qquad H, \rho', A' \vDash \phi \qquad A'' = \lfloor \phi \rfloor_{H,\rho'} \qquad H, \rho' \vdash \texttt{result} \Downarrow v_r \end{array}}{(H, (\rho', A', \emptyset) \cdot (\rho, A, x := y.m(z); \overline{s}) \cdot S) \rightarrow (H, (\rho[x \mapsto v_r], A * A'', \overline{s}) \cdot S)} \text{ ESAppFinish}$$

$$\frac{H, \rho, A \vDash \phi}{(H, (\rho, A, \texttt{assert } \phi; \overline{s}) \cdot S) \rightarrow (H, (\rho, A, \overline{s}) \cdot S)} \text{ ESAssert}$$

$$\frac{H, \rho, A \vDash \phi \qquad A' = A \setminus \lfloor \phi \rfloor_{H,\rho}}{(H, (\rho, A, \texttt{release } \phi; \overline{s}) \cdot S) \rightarrow (H, (\rho, A', \overline{s}) \cdot S)} \text{ ESRelease}$$

$$\frac{\rho' = \rho[x \mapsto \texttt{defaultValue}(T)]}{(H, (\rho, A, T \ x; \overline{s}) \cdot S) \rightarrow (H, (\rho', A, \overline{s}) \cdot S)} \text{ ESDeclare}$$

5 Gradualization

5.1 Syntax

5.1.1 Gradual formula

$$\widetilde{\phi} \quad ::= \quad \phi \mid ? * \phi$$

Note: consider ? in other positions as “self-framing delimiter”, but with semantically identical meaning.
As long as ? is only legal in the front though: $\phi_1 * \widetilde{\phi_2}$ propagates the ? to the very left in case $\widetilde{\phi_2}$ contains one.

5.2 Concretization

Syntax $\hat{\phi}$:= self-framed and satisfiable ϕ

$$\begin{array}{ll} \gamma(\hat{\phi}) & = \{ \ \hat{\phi} \ \} \\ \gamma(? * \phi') & = \{ \ \hat{\phi} \mid \hat{\phi} \implies \phi' \ \} \ \text{ if } \phi' \text{ satisfiable} \\ \gamma(\phi) \text{ undefined otherwise} & \end{array}$$

5.3 Abstraction

$$\begin{array}{ll}
\alpha(\emptyset) \text{ undefined} & \\
\alpha(\{\ \phi\ \}) & = \phi \\
\alpha(\overline{\phi} \text{ with maximum element } \phi) & = \ ? * \phi \\
\alpha(\overline{\phi}) & = \ ? \quad \text{otherwise}
\end{array}$$

5.4 Gradual Lifting

5.4.1 Self framing

$$\frac{A \vdash_{\texttt{sfrm}} \phi}{A \mathrel{\widetilde{\vdash_{\texttt{sfrm}}}} \phi}\ \textsc{GSfrmNonGrad}$$

$$\frac{}{A \mathrel{\widetilde{\vdash_{\texttt{sfrm}}}} ?\ *\ \phi}\ \textsc{GSfrmGrad}$$

5.4.2 Implication

$$\frac{\phi_1 \implies \phi_2}{\phi_1 \mathrel{\widetilde{\implies}} \widetilde{\phi_2}}\ \textsc{GImplNonGrad}$$

$$\frac{\hat{\phi_m} \implies \phi_2 \qquad \hat{\phi_m} \implies \phi_1}{?\ *\ \phi_1 \mathrel{\widetilde{\implies}} \widetilde{\phi_2}}\ \textsc{GImplGrad}$$

$\hat{\phi_m}$ is evidence!

Consistent transitivity
While $\implies$ is transitive, $\widetilde{\implies}$ is generally not.
But maybe not even necessary with smarter hoare rules?

5.4.3 Hoare and evidence

Discussion/Considerations:

- The post-condition-ϕ seems to inherit its gradual-ness from implication, which itself does not care about whether its second argument is gradual or not...
- Gradual

Example:

$$\frac{\begin{array}{c} \epsilon \vdash \widetilde{\phi} \mathrel{\widetilde{\implies}} \widetilde{\phi}' \qquad \emptyset \vdash_{\texttt{sfrm}} \widetilde{\phi}' \\ x \notin FV(\widetilde{\phi}') \qquad x \notin FV(e) \qquad \epsilon \vdash \widetilde{\phi} \vdash x : T \qquad \epsilon \vdash \widetilde{\phi} \vdash e : T \qquad \epsilon \vdash \lfloor \widetilde{\phi}' \rfloor \vdash_{\texttt{sfrm}} e \end{array}}{\vdash \{\widetilde{\phi}\} x := e \{\widetilde{\phi}' * (x = e)\}}\ \textsc{GHVarAssign}$$

Collapsing (hidden) gradual implications into a single one:

$$\frac{\epsilon \vdash \widetilde{\phi} \widetilde{\Longrightarrow} (x : T) * [\![e]\!]_{acc} * [e : T] * \widetilde{\phi}' \quad \emptyset \vdash_{\texttt{sfrm}} [\![e]\!]_{acc} * \widetilde{\phi}' \qquad x \notin FV(\widetilde{\phi}') \qquad x \notin FV(e) \qquad [e : T]}{\vdash \{\widetilde{\phi}\} x := e \{[\![e]\!]_{acc} * \widetilde{\phi}' * (x = e)\}} \text{GHVarAssign}$$

When shifting implication responsibility to GHSec:

$$\frac{x \notin FV(\widetilde{\phi}') \qquad x \notin FV(e) \qquad [e : T]}{\vdash \{(x : T) * [\![e]\!]_{acc} * [e : T] * \widetilde{\phi}'\} x := e \{[\![e]\!]_{acc} * \widetilde{\phi}' * (x = e)\}} \text{GHVarAssign}$$

Example derivation:

$\{(x : T) * \texttt{acc}(y.a) * \texttt{acc}(y.a.b) * \texttt{acc}(y.a.b.c) * (y : C_y) * (y.a.b \neq \texttt{null}) * \widetilde{\phi}'\}$

$\{(x : T) * [\![y.a.b.c]\!]_{acc} * [y.a.b.c : T] * \widetilde{\phi}'\}$

$x := y.a.b.c;$

$x \notin FV(\widetilde{\phi}')$
$x \notin FV(y.a.b.c)$
$\vdash C_y.a : C_a$
$\vdash C_a.b : C_b$
$\vdash C_b.c : T$

$\{[\![y.a.b.c]\!]_{acc} * \widetilde{\phi}' * (x = y.a.b.c)\}$

$\{\texttt{acc}(y.a) * \texttt{acc}(y.a.b) * \texttt{acc}(y.a.b.c) * \widetilde{\phi}' * (x = y.a.b.c)\}$

5.5 Theorems

5.5.1 Soundness of α

$$\forall \overline{\phi} : \overline{\phi} \subseteq \gamma(\alpha(\overline{\phi}))$$

5.5.2 Optimality of α

$$\forall \overline{\phi}, \widetilde{\phi} : \overline{\phi} \subseteq \gamma(\widetilde{\phi}) \implies \gamma(\alpha(\overline{\phi})) \subseteq \gamma(\widetilde{\phi})$$

6 Theorems

6.1 Invariant $invariant(H, \rho, A_d, \phi)$

6.1.1 Phi valid

$$\vdash_{\texttt{sfrm}} \phi$$

6.1.2 Phi holds

$$H, \rho, A_d \models \phi$$

6.1.3 Types preserved

$$
\begin{aligned}
&\forall e, T : \phi \vdash e : T \\
&\implies H, \rho \vdash e : T
\end{aligned}
$$

6.1.4 Heap consistent

$$
\begin{aligned}
\forall o, C, \mu, f, T :\ & H(o) = (C, \mu) \\
&\implies \texttt{fieldType}(C, f) = T \\
&\implies H, \rho \vdash \mu(f) : T
\end{aligned}
$$

6.1.5 Heap not total

$$
\begin{aligned}
&\exists o_{min} : \\
&\forall o \geq o_{min} : o \notin \texttt{dom}(H) \\
&\qquad\quad \wedge \forall f, (o, f) \notin A
\end{aligned}
$$

6.2 Soundness

6.2.1 Progress

$$
\begin{aligned}
\forall \ldots :\quad & \vdash \{\phi_1\} s' \{\phi_2\} \\
&\implies \mathit{invariant}(H_1, \rho_1, A_1, \phi_1) \\
&\implies \exists H_2, \rho_2, A_2 : (H_1, (\rho_1, A_1, s'; \overline{s}) \cdot S) \to^* (H_2, (\rho_2, A_2, \overline{s}) \cdot S)
\end{aligned}
$$

6.2.2 Preservation

$$
\begin{aligned}
\forall \ldots :\quad & \vdash \{\phi_1\} s' \{\phi_2\} \\
&\implies \mathit{invariant}(H_1, \rho_1, A_1, \phi_1) \\
&\implies (H_1, (\rho_1, A_1, s'; \overline{s}) \cdot S) \to^* (H_2, (\rho_2, A_2, \overline{s}) \cdot S) \\
&\implies \mathit{invariant}(H_2, \rho_2, A_2, \phi_2)
\end{aligned}
$$