Gradual Verification

with Implicit Dynamic Frames

Master Thesis of

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Gradual Verification

(with Implicit Dynamic Frames)

```
int getFour(int i)
    requires ?; // haven't figured that one out, yet
    ensures  result = 4;
{
    i = i + 1;
    return i;
}
```

Motivation

- Program verification (against some specification)
- Two flavors: static & dynamic

```
// spec: callable only if (this.balance >= amount)
void withdrawCoins(int amount)
{
   // business logic
   this.balance -= amount;
}
```

Dynamic Verification

- runtime checks
- testing techniques
- guarantee compliance **at runtime**

```
void withdrawCoins(int amount)
{
   assert this.balance >= amount;
   // business logic
   this.balance -= amount;
}
```

Dynamic Verification – Drawbacks

- runtime checks — runtime overhead
- testing techniques — additional efforts
- guarantee compliance **at runtime** — pot. late detection

```
void withdrawCoins(int amount)
    requires this.balance >= amount;
{
    // business logic
    this.balance -= amount;
}
```

Static Verification

- declarative
- formal logic
- guarantee compliance **in advance**

```
void withdrawCoins(int amount)
    requires this.balance >= amount;
{
    // business logic
    this.balance -= amount;
}
```

Static Verification – Drawbacks

- declarative
- formal logic
- guarantee compliance **in advance**

limited syntax
decidability

annotation pressure

```
void withdrawCoins(int amount)
    requires this.balance >= amount;
    ensures  this.balance = old(this.balance) - amount;
{
    // business logic
    this.balance -= amount;
}
```

Solution? Static + Dynamic

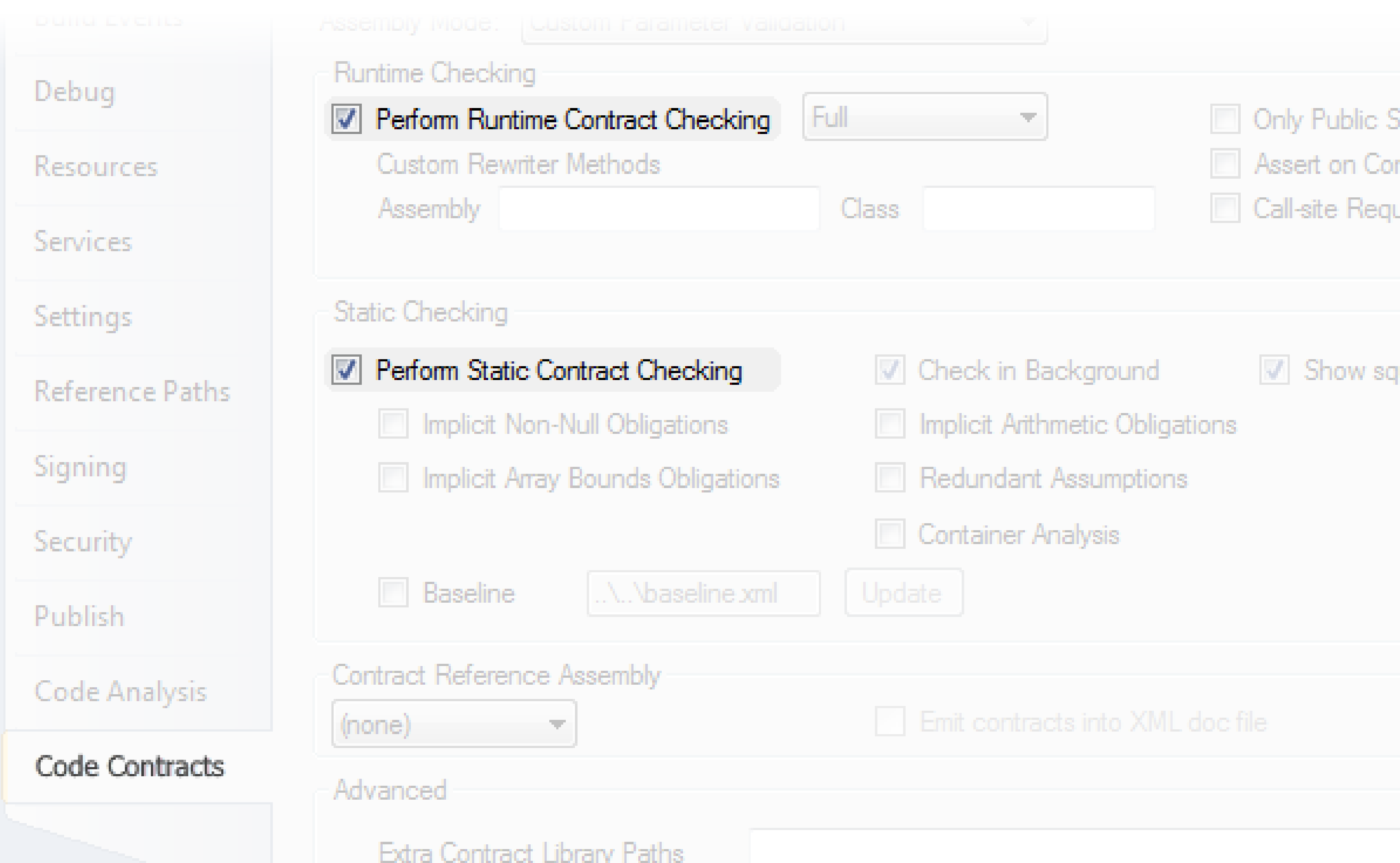
Means:

- “relaxed” static verification (warnings on failure)
- turn contracts into runtime assertions

Notable implementations:

- Java with JML annotations
  - “ESC/Java” for static verification
  - “JML4c” for dynamic verification
- Code Contracts for .NET (by RiSE, MSR)

Solution? Static + Dynamic



Solution! Static ⊕ Dynamic

“Static Typing Where Possible,
Dynamic Typing When Needed” (Erik Meijer)

```
void withdrawCoins(int amount)
   requires this.balance >= amount;
{
   …
}
…
acc.balance = 100;
acc.withdrawCoins(50); // can prove acc.balance >= 50
acc.withdrawCoins(30); // can’t prove acc.balance >= 30
acc.withdrawCoins(30); // can’t prove acc.balance >= 30
```

Approach

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

Gradualization

Syntax

$\widetilde{s} \in \widetilde{\textsc{Stmt}}$

$\widetilde{\phi} \in \widetilde{\textsc{Formula}}$

Program State

$\widetilde{\pi} \in \widetilde{\textsc{ProgramState}}$

Semantics

Static $\widetilde{\vdash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}\}$

Dynamic $\widetilde{\pi} \xrightarrow{\sim} \widetilde{\pi}$

Formula $\widetilde{\pi}\ \widetilde{\vDash}\ \widetilde{\phi}$

Soundness

Gradualization – Starting Point

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

$$e \quad ::= \quad x \mid n \mid e_1 + e_2$$

$$s \quad ::= \quad x \texttt{ := } e \mid \texttt{assert } \phi \mid s_1\texttt{;}\ s_2$$

$$\phi \quad ::= \quad \texttt{true} \mid (e_1 \texttt{ = } e_2) \mid \phi_1 \wedge \phi_2$$

$$= (\textsc{Var} \rightharpoonup \mathbb{N}_0) \times \textsc{Stmt}$$

Gradualization – Starting Point

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

$$\frac{x\ \texttt{:=}\ e}{\vdash \{\phi[e/x]\}\ x\ \texttt{:=}\ e\ \{\phi\}}\ \textsc{HAssign}$$

$$\frac{\phi \Rightarrow \phi_a}{\vdash \{\phi\}\ \texttt{assert}\ \phi_a\ \{\phi\}}\ \textsc{HAssert}$$

$$\frac{\vdash \{\phi_p\}\ s_1\ \{\phi_{q1}\} \qquad \phi_{q1} \Rightarrow \phi_{q2} \qquad \vdash \{\phi_{q2}\}\ s_2\ \{\phi_r\}}{\vdash \{\phi_p\}\ s_1\texttt{;}\ s_2\ \{\phi_r\}}\ \textsc{HSeq}$$

Gradualization – Starting Point

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\quad \vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\quad \pi \longrightarrow \pi$

Formula $\quad \pi \vDash \phi$

Soundness

$$\frac{\mathcal{N}_\sigma(e) = n}{\langle \sigma, x\ \texttt{:=}\ e\texttt{;}\ \ s\rangle \longrightarrow \langle \sigma[x \mapsto n], s\rangle}\ \textsc{SsAssign}$$

$$\frac{\langle \sigma, \texttt{assert}\ \ \phi_a\texttt{;}\ \ s\rangle \vDash \phi_a}{\langle \sigma, \texttt{assert}\ \ \phi_a\texttt{;}\ \ s\rangle \longrightarrow \langle \sigma, s\rangle}\ \textsc{SsAssert}$$

Gradualization – Starting Point

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

$$\langle [x \mapsto 3], s\rangle \vDash (\mathtt{x\ =\ 3})$$

$$\langle [x \mapsto 3, y \mapsto 5], s\rangle \vDash (\mathtt{y} \neq \mathtt{x})$$

$$\phi_1 \Rightarrow \phi_2 \overset{\text{def}}{\iff} \forall \pi.\ \pi \vDash \phi_1 \implies \pi \vDash \phi_2$$

$$(\mathtt{a\ =\ b}) \wedge (\mathtt{b\ =\ c}) \Rightarrow (\mathtt{a\ =\ c})$$

Gradualization – Starting Point

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

$$\frac{\vdash \{\phi\}\ s\ \{\phi'\}}{\vDash \{\phi\}\ s\ \{\phi'\}}\ \textsc{Soundness}$$

$$\vDash \{\phi\}\ s\ \{\phi'\}$$

$$\overset{\text{def}}{\iff}$$

$$\forall \pi, \pi'.\ \pi \xrightarrow{s} \pi' \land \pi \vDash \phi \implies \pi' \vDash \phi'$$

Gradualization – Overview

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

Gradualization

Syntax

$\widetilde{s} \in \widetilde{\textsc{S}}\textsc{tmt}$

$\widetilde{\phi} \in \widetilde{\textsc{F}}\textsc{ormula}$

Program State

$\widetilde{\pi} \in \widetilde{\textsc{P}}\textsc{rogramState}$

Semantics

Static $\widetilde{\vdash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}\}$

Dynamic $\widetilde{\pi} \mathrel{\widetilde{\longrightarrow}} \widetilde{\pi}$

Formula $\widetilde{\pi} \mathrel{\widetilde{\vDash}} \widetilde{\phi}$

Soundness

Gradualization – Goal 1/3

Introduction of wildcard formula ?

- placeholder for arbitrary (satisfiable) formula
- enables Hoare deduction despite incomplete information
- enables gradual annotation of programs (? as default)

Formula Precision

$$\widetilde{\phi_1} \sqsubseteq \widetilde{\phi_2} \overset{\text{def}}{\iff} \gamma(\widetilde{\phi_1}) \subseteq \gamma(\widetilde{\phi_2})$$

Gradualization – Goal 2/3

Compatibility with static language

- don't reject source code that was accepted before — static
- observable behavior is not changed — dynamic

$$\textsc{Formula} \subseteq \widetilde{\textsc{Formula}} \qquad \textsc{Stmt} \subseteq \widetilde{\textsc{Stmt}}$$

$$\vdash \{\phi\}\ s\ \{\phi'\} \implies \widetilde{\vdash} \{\phi\}\ s\ \{\phi'\}$$

$$\pi \xrightarrow{s} \pi' \implies \exists \pi''.\ \pi \overset{s}{\widetilde{\longrightarrow}} \pi'' \land (\forall \phi.\ \pi' \vDash \phi \implies \pi'' \vDash \phi)$$

Gradualization – Goal 3/3

Gradual guarantee (Siek et al.), adapted

Reducing precision will not

- introduce verification failure static
- change observable behavior dynamic

$$\text{Given} \quad \widetilde{\phi_1} \sqsubseteq \widetilde{\phi_2} \land \widetilde{\phi_1'} \sqsubseteq \widetilde{\phi_2'} \land \widetilde{s_1} \sqsubseteq \widetilde{s_2} \land \widetilde{\pi_1} \sqsubseteq \widetilde{\pi_2}$$

$$\widetilde{\vdash} \{\widetilde{\phi_1}\}\ \widetilde{s_1}\ \{\widetilde{\phi_1'}\} \implies \widetilde{\vdash} \{\widetilde{\phi_2}\}\ \widetilde{s_2}\ \{\widetilde{\phi_2'}\}$$

$$\widetilde{\pi_1} \xrightarrow{\widetilde{s_1}} \widetilde{\pi_1'} \implies \exists \widetilde{\pi_2'}.\ \widetilde{\pi_2} \xrightarrow{\widetilde{s_2}} \widetilde{\pi_2'} \land \widetilde{\pi_1'} \sqsubseteq \widetilde{\pi_2'}$$

Gradual Predicate Lifting

Introduction

$$\forall \phi_1, \phi_2 \in \textsc{Formula}.\ P(\phi_1, \phi_2) \implies \widetilde{P}(\phi_1, \phi_2)$$

Monotonicity

$$\forall \widetilde{\phi_1}, \widetilde{\phi_2}, \widetilde{\phi_1'}, \widetilde{\phi_2'} \in \widetilde{\textsc{Formula}}.\ \widetilde{\phi_1} \sqsubseteq \widetilde{\phi_1'} \wedge \widetilde{\phi_2} \sqsubseteq \widetilde{\phi_2'} \wedge \widetilde{P}(\widetilde{\phi_1}, \widetilde{\phi_2}) \implies \widetilde{P}(\widetilde{\phi_1'}, \widetilde{\phi_2'})$$

$$P(\phi_1, \phi_a, \phi_2) \stackrel{\text{def}}{=} \phi_1 = \phi_2 \wedge \phi_1 \Rightarrow \phi_a$$

$$\widetilde{P}(\widetilde{\phi_1}, \widetilde{\phi_a}, \widetilde{\phi_2}) \stackrel{\text{def}}{=} \widetilde{\phi_1} \approx \widetilde{\phi_2} \wedge \widetilde{\phi_1} \mathrel{\widetilde{\Rightarrow}} \widetilde{\phi_a}$$

$$\widetilde{\phi_1} \approx \widetilde{\phi_2} \stackrel{\text{def}}{=} \widetilde{\phi_1} = \widetilde{\phi_2} \vee \widetilde{\phi_1} = \,? \vee \widetilde{\phi_2} = \,?$$

$$\widetilde{\phi_1} \mathrel{\widetilde{\Rightarrow}} \widetilde{\phi_2} \stackrel{\text{def}}{=} \widetilde{\phi_1} \Rightarrow \widetilde{\phi_2} \vee \widetilde{\phi_1} = \,? \vee \widetilde{\phi_2} = \,?$$

$$\frac{\phi \Rightarrow \phi_a}{\vdash \{\phi\}\ \texttt{assert}\ \ \phi_a\ \{\phi\}}\ \textsc{HAssert}$$

$$\frac{}{\widetilde{\vdash} \{\widetilde{\phi_1}\}\ \texttt{assert}\ \ \widetilde{\phi_a}\ \{\widetilde{\phi_2}\}}\ \widetilde{\text{H}}\textsc{Assert}$$

Gradual Predicate Lifting

Introduction

$$\forall \phi_1, \phi_2 \in \textsc{Formula}.\ P(\phi_1, \phi_2) \implies \widetilde{P}(\phi_1, \phi_2)$$

Monotonicity

$$\forall \widetilde{\phi_1}, \widetilde{\phi_2}, \widetilde{\phi'_1}, \widetilde{\phi'_2} \in \widetilde{\textsc{Formula}}.\ \widetilde{\phi_1} \sqsubseteq \widetilde{\phi'_1} \land \widetilde{\phi_2} \sqsubseteq \widetilde{\phi'_2} \land \widetilde{P}(\widetilde{\phi_1}, \widetilde{\phi_2}) \implies \widetilde{P}(\widetilde{\phi'_1}, \widetilde{\phi'_2})$$

(Optimality)

$\widetilde{P}$ is smallest predicate closed under above rules

$\Leftrightarrow$

$$\widetilde{P}(\widetilde{\phi_1}, \widetilde{\phi_2}) \iff \exists \phi_1 \in \gamma(\widetilde{\phi_1}), \phi_2 \in \gamma(\widetilde{\phi_2}).\ P(\phi_1, \phi_2)$$

Gradual Predicate Lifting

$$
\begin{aligned}
&\widetilde{\pi} \mathrel{\widetilde{\vDash}} \widetilde{\phi} \\
\iff\ & \exists \pi \in \gamma(\widetilde{\pi}), \phi \in \gamma(\widetilde{\phi}).\ \pi \vDash \phi \\
\iff\ & \exists \phi \in \gamma(\widetilde{\phi}).\ \widetilde{\pi} \vDash \phi \\
\iff\ & \widetilde{\pi} \vDash \widetilde{\phi} \lor \widetilde{\phi} = \mathbf{?}
\end{aligned}
$$

$$
\frac{\widetilde{\pi} \vDash \phi}{\widetilde{\pi} \mathrel{\widetilde{\vDash}} \phi}\ \textsc{EvalPhiStatic}
\qquad\qquad
\frac{}{\widetilde{\pi} \mathrel{\widetilde{\vDash}} \mathbf{?}}\ \textsc{EvalPhiStatic}
$$

$$
\widetilde{P}(\widetilde{\phi_1}, \widetilde{\phi_2}) \iff \exists \phi_1 \in \gamma(\widetilde{\phi_1}), \phi_2 \in \gamma(\widetilde{\phi_2}).\ P(\phi_1, \phi_2)
$$

Gradual Function Lifting

Introduction

$$\forall \phi \in \textsc{Formula}.\ f(\phi) \sqsubseteq \widetilde{f}(\phi)$$

Monotonicity

$$\forall \widetilde{\phi_1}, \widetilde{\phi_2} \in \widetilde{\textsc{Formula}}.\ \widetilde{\phi_1} \sqsubseteq \widetilde{\phi_2} \implies \widetilde{f}(\widetilde{\phi_1}) \sqsubseteq \widetilde{f}(\widetilde{\phi_2})$$

(Optimality)

$\widetilde{f}$ has most precise return values among all liftings

⇔ $\widetilde{f}(\widetilde{\phi}) = \alpha(\overline{f}(\gamma(\widetilde{\phi})))$ where $\langle \alpha, \gamma \rangle$ is $\{\ \overline{f}\ \}$-partial Galois connection

Gradual Partial Function Lifting

Introduction

$$\forall \phi \in \textsc{Formula} \cap \mathsf{dom}(f).\ f(\phi) \sqsubseteq \widetilde{f}(\phi)$$

Monotonicity

$$\forall \widetilde{\phi_1}, \widetilde{\phi_2} \in \widetilde{\textsc{F}}\textsc{ormula}.\ \widetilde{\phi_1} \sqsubseteq \widetilde{\phi_2} \wedge \widetilde{\phi_1} \in \mathsf{dom}(\widetilde{f}) \implies \widetilde{f}(\widetilde{\phi_1}) \sqsubseteq \widetilde{f}(\widetilde{\phi_2})$$

(Optimality)

$\widetilde{f}$ has smallest domain and most precise return values among all liftings

⇔ $\widetilde{f}(\widetilde{\phi}) = \alpha(\overline{f}(\gamma(\widetilde{\phi})))$ where $\langle \alpha, \gamma \rangle$ is $\{\ \overline{f}\ \}$-partial Galois connection

Gradual Verification - Approach

Syntax

$s \in \textsc{Stmt}$

$\phi \in \textsc{Formula}$

Program State

$\pi \in \textsc{ProgramState}$

Semantics

Static $\vdash \{\phi\}\ s\ \{\phi\}$

Dynamic $\pi \longrightarrow \pi$

Formula $\pi \vDash \phi$

Soundness

Gradualization

Syntax

$\widetilde{s} \in \widetilde{\textsc{Stmt}}$

$\widetilde{\phi} \in \widetilde{\textsc{Formula}}$

Program State

$\widetilde{\pi} \in \widetilde{\textsc{ProgramState}}$

Semantics

Static $\widetilde{\vdash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}\}$

Dynamic $\widetilde{\pi} \xrightarrow{\sim} \widetilde{\pi}$

Formula $\widetilde{\pi}\ \widetilde{\vDash}\ \widetilde{\phi}$

Soundness

Gradual Verification - Approach

Syntax		**Syntax**
$s \in \textsc{Stmt}$		$\widetilde{s} \in \widetilde{\textsc{Stmt}}$
$\phi \in \textsc{Formula}$	syntax extension abstract interpretation	$\widetilde{\phi} \in \widetilde{\textsc{Formula}}$
Program State		**Program State**
$\pi \in \textsc{ProgramState}$		$\widetilde{\pi} \in \widetilde{\textsc{ProgramState}}$
Semantics		**Semantics**
Static $\vdash \{\phi\}\ s\ \{\phi\}$	predicate lifting	Static $\widetilde{\vdash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}\}$
Dynamic $\pi \longrightarrow \pi$	partial function lifting	Dynamic $\widetilde{\pi} \ \widetilde{\longrightarrow}\ \widetilde{\pi}$
Formula $\pi \vDash \phi$	predicate lifting	Formula $\widetilde{\pi}\ \widetilde{\vDash}\ \widetilde{\phi}$
Soundness		**Soundness**

Gradual Soundness

$$\frac{\vdash \{\phi\}\ s\ \{\phi'\}}{\vDash \{\phi\}\ s\ \{\phi'\}}\ \textsc{Soundness}$$

$$\vDash \{\phi\}\ s\ \{\phi'\}$$

$$\overset{\text{def}}{\iff}$$

$$\forall \pi, \pi'.\ \pi \xrightarrow{s} \pi' \wedge \pi \vDash \phi \implies \pi' \vDash \phi'$$

$$\frac{\widetilde{\vdash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}'\}}{\widetilde{\vDash} \{\widetilde{\phi}\}\ \widetilde{s};\ \texttt{assert}\ \widetilde{\phi}'\ \{\widetilde{\phi}'\}}\ \widetilde{\textsc{S}}\textsc{oundness}$$

$$\widetilde{\vDash} \{\widetilde{\phi}\}\ \widetilde{s}\ \{\widetilde{\phi}'\}$$

$$\overset{\text{def}}{\iff}$$

$$\forall \widetilde{\pi}, \widetilde{\pi}'.\ \widetilde{\pi} \xrightarrow{\widetilde{s}} \widetilde{\pi}' \wedge \widetilde{\pi}\ \widetilde{\vDash}\ \widetilde{\phi} \implies \widetilde{\pi}'\ \widetilde{\vDash}\ \widetilde{\phi}'$$

$$\widetilde{\vdash} \{?\}\ \texttt{y := 4}\ \{(\texttt{x = 2}) \wedge (\texttt{y = 4})\}$$

$$\widetilde{\vDash} \{?\}\ \texttt{y := 4; assert (x = 2)}\ \{(\texttt{x = 2}) \wedge (\texttt{y = 4})\}$$

Gradual Verification – Put to the Test

$$\frac{\widetilde{\vdash} \{?\}\ \texttt{y := 2}\ \{\widetilde{\phi_1}\} \qquad \widetilde{\phi_1} \mathrel{\widetilde{\Rightarrow}} \widetilde{\phi_2} \qquad \widetilde{\vdash} \{\widetilde{\phi_2}\}\ \texttt{x := 3}\ \{(\texttt{x = 3}) \land (\texttt{y = 2})\}}{\widetilde{\vdash} \{?\}\ \texttt{y := 2; x := 3}\ \{(\texttt{x = 3}) \land (\texttt{y = 2})\}}\ \widetilde{\text{HSEQ}}$$

a) $\widetilde{\phi_1} = (\texttt{y = 2})$
$\widetilde{\phi_2} = (\texttt{y = 2})$

"good" (Hoare triples even valid)
….but only one option

b) $\widetilde{\phi_1} = ?$
$\widetilde{\phi_2} = ?$

"too weak" (no information forwarded)
idea: try to be more precise

c) $\widetilde{\phi_1} = (\texttt{y = 2}) \land (\texttt{x = 4})$
$\widetilde{\phi_2} = (\texttt{y = 2})$

"too strict" (unnecessary assumption)
idea: try to produce valid Hoare triple

Deterministic Lifting

- Idea: treat static Hoare logic as (multivalued) function

$$\vdash \{\cdot\} \cdot \{\cdot\} \subseteq \textsc{Formula} \times \textsc{Stmt} \times \textsc{Formula}$$

$$\vdash \{\cdot\} \cdot \{\cdot\} \; : \; \textsc{Formula} \times \textsc{Stmt} \to \mathcal{P}^{\textsc{Formula}}$$

- lift that function (rules similar to partial function)

$$\vec{\vdash} \{\cdot\} \cdot \{\cdot\} \; : \; \widetilde{\textsc{Formula}} \times \widetilde{\textsc{Stmt}} \to \widetilde{\textsc{Formula}}$$

- Properties
  - can derive gradual lifting
  - stronger, assertion-free notion of soundness
  - deterministic verifier
  - free transitivity (no assertions to justify premises of $\widetilde{\textsc{HSeq}}$)

Deterministic Lifting

Introduction $\forall \phi_1, \phi_2.\ P(\phi_1, \phi_2) \implies \phi_1 \in \mathsf{dom}(\vec{P})$

Strength $\forall \widetilde{\phi_1}, \widetilde{\phi_2}.\ \vec{P}(\widetilde{\phi_1}) = \widetilde{\phi_2} \implies \forall \phi_1 \in \gamma(\widetilde{\phi_1}),\ \phi.\ P(\phi_1, \phi)$

$$\implies \exists \phi_2 \in \gamma(\widetilde{\phi_2}).\ P(\phi_1, \phi_2)\ \wedge\ (\phi_2 \Rightarrow \phi)$$

Monotonicity

$$\forall \widetilde{\phi_1}, \widetilde{\phi_2} \in \widetilde{\text{F}}\textsc{ormula}.\ \widetilde{\phi_1} \sqsubseteq \widetilde{\phi_2} \wedge \widetilde{\phi_1} \in \mathsf{dom}(\vec{P}) \implies \vec{P}(\widetilde{\phi_1}) \sqsubseteq \vec{P}(\widetilde{\phi_2})$$

(Optimality)

$\vec{P}$ has smallest domain and most precise return values among all liftings

Deterministic Lifting

Obtaining a Gradual Lifting

Let

$$\vec{\vdash} \{\cdot\} \cdot \{\cdot\} \;:\; \widetilde{\text{FORMULA}} \times \widetilde{\text{STMT}} \to \widetilde{\text{FORMULA}}$$

be a deterministic lifting of

$$\vdash \{\cdot\} \cdot \{\cdot\} \subseteq \text{FORMULA} \times \text{STMT} \times \text{FORMULA}$$

Let

$$\widetilde{\vdash} \{\cdot\} \cdot \{\cdot\} \;:\; \widetilde{\text{FORMULA}} \times \widetilde{\text{STMT}} \times \widetilde{\text{FORMULA}}$$

be defined as

$$\widetilde{\vdash} \{\widetilde{\phi_1}\} \, \widetilde{s} \, \{\widetilde{\phi_2}\} \quad \stackrel{\text{def}}{\iff} \quad \exists \widetilde{\phi_2'}.\; \vec{\vdash} \{\widetilde{\phi_1}\} \, \widetilde{s} \, \{\widetilde{\phi_2'}\} \land \widetilde{\phi_2'} \mathrel{\widetilde{\Rightarrow}} \widetilde{\phi_2}$$

Then $\widetilde{\vdash} \{\cdot\} \cdot \{\cdot\}$ is a gradual lifting of $\vdash \{\cdot\} \cdot \{\cdot\}$

Deterministic Lifting

Soundness

$$\frac{\vec{\vdash} \; \{\widetilde{\phi}\} \; \widetilde{s} \; \{\widetilde{\phi'}\}}{\widetilde{\vDash} \; \{\widetilde{\phi}\} \; \widetilde{s} \; \{\widetilde{\phi'}\}} \; \vec{\mathrm{S}}\textsc{oundness}$$

- satisfiable (but not a tautology)
- very helpful for optimizations

Deterministic Lifting – $\textsc{HAssign}$

$$\frac{x \; := \; e}{\vdash \{\phi[e/x]\}\; x \; := \; e \; \{\phi\}}\;\textsc{HAssign}$$

$$\frac{x \notin \mathsf{FV}(\phi) \qquad x \notin \mathsf{FV}(e)}{\vec{\vdash} \{\phi\}\; x \; := \; e \; \{\phi \wedge (x = e)\}}\;\vec{\mathrm{H}}\textsc{Assign1}$$

$$\frac{\vec{\mathrm{H}}\textsc{Assign1}\ \textit{does not apply}}{\vec{\vdash} \{\widetilde{\phi}\}\; x \; := \; e \; \{?\}}\;\vec{\mathrm{H}}\textsc{Assign2}$$

Deterministic Lifting – $\textsc{HAssert}$

$$\frac{\phi \Rightarrow \phi_a}{\vdash \{\phi\}\ \texttt{assert}\ \ \phi_a\ \{\phi\}}\ \textsc{HAssert}$$

$$\frac{\phi \Rightarrow \phi_a}{\vec{\vdash} \{\phi\}\ \texttt{assert}\ \ \phi_a\ \{\phi\}}\ \vec{\text{H}}\textsc{Assert1}$$

$$\frac{\phi_a \in \textsc{SatFormula}}{\vec{\vdash} \{?\}\ \texttt{assert}\ \ \phi_a\ \{?\}}\ \vec{\text{H}}\textsc{Assert2}$$

Deterministic Lifting – $\textsc{HSeq}$

$$\frac{\vdash \{\phi_p\}\ s_1\ \{\phi_{q1}\} \qquad \phi_{q1} \Rightarrow \phi_{q2} \qquad \vdash \{\phi_{q2}\}\ s_2\ \{\phi_r\}}{\vdash \{\phi_p\}\ s_1;\ s_2\ \{\phi_r\}}\ \textsc{HSeq}$$

$$\frac{\vec{\vdash} \{\widetilde{\phi_p}\}\ \widetilde{s_1}\ \{\widetilde{\phi_{q1}}\} \qquad \widetilde{\phi_{q1}} \mathrel{\vec{\Rightarrow}} \widetilde{\phi_{q2}} \qquad \vec{\vdash} \{\widetilde{\phi_{q2}}\}\ \widetilde{s_2}\ \{\widetilde{\phi_r}\}}{\vec{\vdash} \{\widetilde{\phi_p}\}\ \widetilde{s_1};\ \widetilde{s_2}\ \{\widetilde{\phi_r}\}}\ \vec{\mathrm{H}}\textsc{Seq}$$

Demo

http://olydis.github.io/GradVer/impl/HTML5/