SVL with Recursive Predicates

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Contents

1 Grammar

$$
\begin{array}{rcl}
x, y, z & \in & \mathit{VAR} \\
v & \in & \mathit{VAL} \\
e & \in & \mathit{EXPR} \\
s & \in & \mathit{STMT} \\
o & \in & \mathit{LOC} \\
f & \in & \mathit{FIELDNAME} \\
m & \in & \mathit{METHODNAME} \\
C, D & \in & \mathit{CLASSNAME} \\
\alpha & \in & \mathit{PREDNAME} \\
P & ::= & \overline{\mathit{cls}}\ s \\
\mathit{cls} & ::= & \texttt{class}\ C\ \texttt{extends}\ D\ \{\overline{\mathit{field}}\ \overline{\mathit{pred}}\ \overline{\mathit{method}}\} \\
\mathit{field} & ::= & T\ f; \\
\mathit{pred} & ::= & \texttt{predicate}\ \alpha_C(\overline{T\ x}) = \widetilde{\phi} \\
T & ::= & \texttt{int} \mid \texttt{bool} \mid C \mid \top \\
\mathit{method} & ::= & T\ m(\overline{T\ x})\ \texttt{dynamically}\ \mathit{contract}\ \texttt{statically}\ \mathit{contract}\ \{s\} \\
\mathit{contract} & ::= & \texttt{requires}\ \widetilde{\phi}\ \texttt{ensures}\ \widetilde{\phi} \\
\oplus & ::= & + \mid - \mid * \mid \backslash \mid \&\& \mid \ || \\
\odot & ::= & \neq \mid = \mid < \mid > \mid \leq \mid \geq \\
s & ::= & \texttt{skip} \mid s_1\ ;\ s_2 \mid T\ x \mid x := e \mid \texttt{if}\ (e)\ \{s_1\}\ \texttt{else}\ \{s_2\} \\
 & & \mid \texttt{while}\ (e)\ \texttt{invariant}\ \widetilde{\phi}\ \{s\} \mid x.f := y \mid x := \texttt{new}\ C \mid y := z.m(\overline{x}) \\
 & & \mid y := z.m_C(\overline{x}) \mid \texttt{assert}\ \phi \mid \texttt{release}\ \phi \mid \texttt{hold}\ \phi\ \{s\} \mid \texttt{fold}\ A \mid \texttt{unfold}\ A \\
e & ::= & v \mid x \mid e \oplus e \mid e \odot e \mid e.f \\
x & ::= & \texttt{result} \mid \mathit{id} \mid \texttt{old}(\mathit{id}) \mid \texttt{this} \\
v & ::= & n \mid o \mid \texttt{null} \mid \texttt{true} \mid \texttt{false} \\
A & ::= & \alpha(\overline{e}) \mid \alpha_C(\overline{e}) \\
\circledast & ::= & \wedge \mid * \\
\phi & ::= & e \mid A \mid \texttt{acc}(e.f) \mid \phi \circledast \phi \mid (\texttt{if}\ e\ \texttt{then}\ \phi\ \texttt{else}\ \phi) \mid (\texttt{unfolding}\ A\ \texttt{in}\ \phi) \\
\widetilde{\phi} & ::= & \phi \mid ? * \phi
\end{array}
$$

2 Well-formedness

3 Aliasing

3.1 Definitions

An **object variable** is one of the following:

- a class instance variable i.e. a variable v such that $v : C$ for some class C,
- a class instance field reference i.e. a field reference $e.f$ where $e.f : C$ for some class C,
- `null` as a value such that `null` $: C$ for some class C.

Let $\mathcal{O}$ be a set of object variables. An $O \subset \mathcal{O}$ **aliases** if and only if each $o \in O$ refers to the same memory in the heap as each other, written propositionally as

$$\forall o, o' \in O : o = o' \iff \mathsf{aliases}(O)$$

While it is possible to keep track of negated aliasings (of the form $\sim \mathsf{aliases}\{o_\alpha\}$), this will not be needed for either aliasing tree construction or self-framing desicions. So, it will not be tracked i.e. $x \neq y$ does not contribute anything to an aliasing context.

3.2 Aliasing Context

Let ϕ be a formula. The **aliasing context** $\mathcal{A}$ of ϕ is a tree of set of aliasing proposition about aliasing of object variables that appear in ϕ. $\mathcal{A}$ needs to be a tree because the conditional sub-formulas that may appear in ϕ allow for branching aliasing contexts not expressible flatly at the top level. Each node in the tree corresponds to a set of aliasing propositions, and each branch refers to a branch of a unique conditional in ϕ. The parts of the tree are labeled in such a way that modularly allows a specified sub-formula of ϕ to be matched to the unique aliasing sub-context that corresponds to it. For example, consider the following formula:

$$\begin{aligned}
\phi := {} & (o_1 = o_2) * \\
& (\texttt{if } (b_1) \\
& \quad \texttt{then } (\\
& \qquad (o_1 \neq o_3) * \\
& \qquad (\texttt{if } (b_2) \\
& \qquad\quad \texttt{then } (o_1 = o_4) \\
& \qquad\quad \texttt{else } (b_3))) \\
& \quad \texttt{else } (o_1 = o_3)) * \\
& (o_1 = o_4)
\end{aligned}$$

where b_1, b_2 are arbitrary boolean expressions that do not assert aliasing propositions. ϕ has a formula-structure represented by the tree in figure 3.2. The formula-structure tree for

Figure 1: Formula structure tree for ϕ.

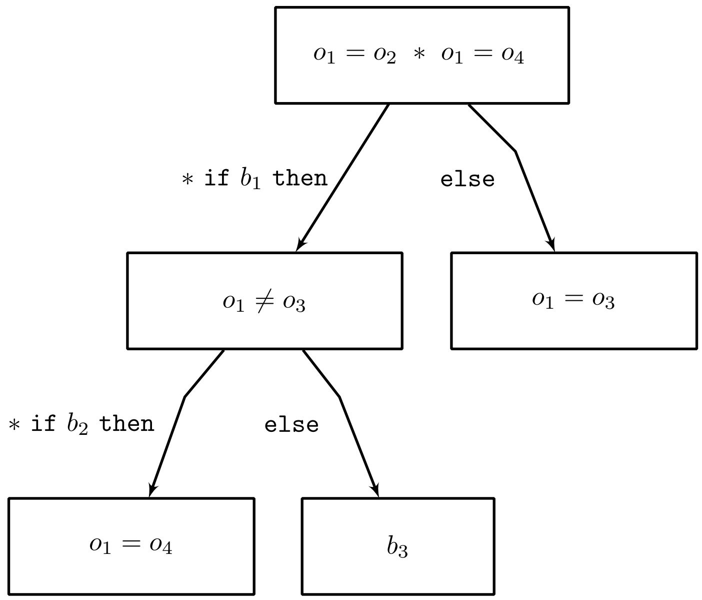
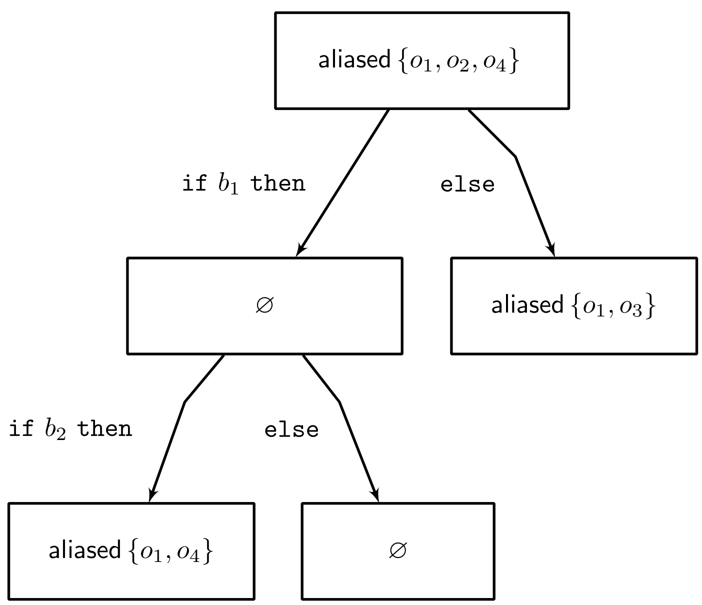


Figure 2: $\mathcal{A}(\phi)$, the aliasing context tree for ϕ.



ϕ corresponds node-for-node and edge-for-edge to the aliasing context tree in figure 3.2.

More generally, for ϕ a formula and ϕ' a sub-formula of ϕ, write $\mathcal{A}_\phi(\phi')$ as the **total aliasing context** of ϕ' which includes aliasing propositions inherited from its ancestors in the aliasing context tree of ϕ. These aliasing contexts are combined via $\sqcup$ which will be defined in the next section. For example, the total aliasing context at the sub-formula $(o_1 = o_4)$ of ϕ is:

$$\mathcal{A}_\phi(o_1 = o_4) := \{\mathsf{aliased}\,\{o_1, o_2, o_4\}\}$$

along with the fact that it has no child branches. Usually $\mathcal{A}_{\phi_{\mathsf{root}}}(\phi')$ is abbreviated to $\mathcal{A}(\phi')$ when the top level formula ϕ is implicit and ϕ' is a sub-formula of ϕ_{root}.

An aliasing context $\mathcal{A}$ may entail $\mathsf{aliased}(O)$ for some $O \subset \mathcal{O}$. Since $\mathcal{A}$ is efficiently represented as a set of propositions about sets, it may be the case that $\mathsf{aliased}(O) \notin \mathcal{A}$ yet still the previous judgement holds. For example, this is true when $\exists O' \subset \mathcal{O}$ such that $O \subset O'$ and $\mathsf{aliased}(O') \in \mathcal{A}$. So, the explicit definition for making this judgement is as follows:

$$\mathcal{A} \vdash \mathsf{aliased}(O) \iff \exists O' \subset \mathcal{O} : (O \subset O') \;\wedge\; (\mathsf{aliased}(O') \in \mathcal{A})$$

The notations $\mathsf{aliased}(O) \in \mathcal{A}$ is a little misleading because $\mathcal{A}$ is in fact a tree and not just a set. To be explicit, $\mathsf{aliased}(O) \in \mathcal{A}$ is defined to be set membership of the set of aliasing propositions in the total aliasing context at $\mathcal{A}$.

3.3 Constructing an Aliasing Context

An aliasing context of a formula ϕ is a tree, where nodes represent local aliasing contexts and branches represent the branches of conditional sub-formulas nested in ϕ. So, an aliasing context is defined structurally as

$$\mathcal{A} \;::= \langle A, \{e_\alpha : \mathcal{A}_\alpha\}\rangle$$

where A is a set of propositions about aliasing and the $e_\alpha : \mathcal{A}_\alpha$ are the nesting aliasing contexts that correspond to the then and else branches of conditionals directly nested in ϕ, the e_α indicating the condition for branching to $\mathcal{A}_\alpha$. For the purposes of look-up, each $\mathcal{A}$ is labeled by the sub-formula it corresponds to.

Given a root formula ϕ_{root}, the aliasing context of ϕ_{root} is written $\mathcal{A}(\phi_{\mathsf{root}})$. With the root invariant, the following recursive algorithm constructs $\mathcal{A}(\phi)$ for any sub-formula of ϕ_{root}

(including $\mathcal{A}(\phi_{\textsf{root}})$).

$$
\begin{array}{rcll}
\mathcal{A}(\phi) & := & \textsf{match } \phi \textsf{ with} & \\
 & & v & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & x & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & e_1 \ \&\& \ e_2 & \mapsto \mathcal{A}(e_1) \sqcup \mathcal{A}(e_2) \\
 & & e_1 \ || \ e_2 & \mapsto \mathcal{A}(e_1) \sqcap \mathcal{A}(e_2) \\
 & & e_1 \oplus e_2 & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & o_1 = o_2 & \mapsto \langle \{\textsf{aliases}\, \{o_1, o_2\}\}, \varnothing \rangle \\
 & & e_1 \odot e_2 & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & e.f & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & \texttt{acc}(e.f) & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & \phi_1 * \phi_2 & \mapsto \mathcal{A}(\phi_1) \sqcup \mathcal{A}(\phi_2) \\
 & & \phi_1 \wedge \phi_2 & \mapsto \mathcal{A}(\phi_1) \sqcup \mathcal{A}(\phi_2) \\
 & & \alpha_C(e_1, \ldots, e_k) & \mapsto \langle \varnothing, \varnothing \rangle \\
 & & \texttt{if } e \texttt{ then } \phi_1 \texttt{ else } \phi_2 & \mapsto \langle \varnothing, \{\mathcal{A}(e) \sqcup \mathcal{A}(\phi_1), (\mathcal{A}(\sim e)) \sqcup \mathcal{A}(\phi_2)\} \rangle \\
 & & \texttt{unfolding } \alpha_C(e_1, \ldots, e_k) \texttt{ in } \phi' & \mapsto \mathcal{A}(\phi')
\end{array}
$$

Note that the $\mathcal{A}(\sim e)$ result of the rule for $\mathcal{A}(\texttt{if } e \texttt{ then } \phi_1 \texttt{ else } \phi_2)$ means to negate the boolean expression of e and then take the aliasing context of that. As examples,

$$
\begin{aligned}
\mathcal{A}(\sim (x = y)) &= \mathcal{A}(x \neq y) = \langle \varnothing, \varnothing \rangle \\
\mathcal{A}(\sim (x \neq y)) &= \mathcal{A}(x = y) = \langle \{\textsf{aliased}\, \{x, y\}\}, \varnothing \rangle
\end{aligned}
$$

Context union, $\sqcup$, and context intersection, $\sqcap$, are operations that combine aliasing contexts and are defined below.

$$
\begin{aligned}
&\langle A_1, \{e_\alpha : \mathcal{A}_\alpha\}\rangle \sqcup \langle A_2, \{e_\beta : \mathcal{A}_\beta\}\rangle := \\
&\qquad \langle \{\textsf{aliased}\, \{o' \mid \forall o' : (A_1 \vdash \textsf{aliased}\, \{o, o'\}) \ \vee \ (A_2 \vdash \textsf{aliased}\, \{o, o'\})\} \mid \forall o\}, \\
&\qquad \{e_\alpha : \mathcal{A}_\alpha\} \cup \{e_\beta : \mathcal{A}_\beta\}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle A_1, \{e_\alpha : \mathcal{A}_\alpha\}\rangle \sqcap \langle A_2, \{e_\beta : \mathcal{A}_\beta\}\rangle := \\
&\qquad \langle \{\textsf{aliased}\, \{o' \mid \forall o' : (A_1 \vdash \textsf{aliased}\, \{o, o'\}) \ \wedge \ (A_2 \vdash \textsf{aliased}\, \{o, o'\})\} \mid \forall o\}, \\
&\qquad \{e_\alpha : \mathcal{A}_\alpha\} \cap \{e_\beta : \mathcal{A}_\beta\}\rangle
\end{aligned}
$$

4 Framing

4.1 Definitions

For framing, a formula is considered inside a **permission context**, a set of permissions, where a **permission** π is to do one of the following:

- to reference $e.f$, written $\mathsf{accessed}(e.f)$.
- to assume $\alpha_C(\overline{e})$, written $\mathsf{assumed}(\alpha_C(\overline{e}))$. This allows the a single unrolling of $\alpha_C(\overline{e})$. Explicitly, an instance of $\mathsf{assumed}(\alpha_C(\overline{e}))$ in a set of permissions Π may be expanded into $\Pi \cup \mathsf{granted}(\dots)$ where $\dots$ is replaced with a single unrolling of the body of $\alpha_C(\overline{e})$ with the arguments substituted appropriately[1].

Let ϕ be a formula. ϕ may **require** a permission π. For example, the formula $e.f = 1$ requires $\mathsf{accessed}(e.f)$, because it references $e.f$. The set of all permissions that ϕ requires is called the **requirements** of ϕ. ϕ may also **grant** a permission π. For example, the formula $\mathtt{acc}(e.f)$ grants the permission $\mathsf{accessed}(e.f)$.

Altogether, ϕ is **framed** by a set of permissions Π if all permissions required by ϕ are either in Π or granted by ϕ. The proposition that Π frames ϕ is written

$$\Pi \vDash_I \phi$$

Of course, ϕ may grant some of the permissions it requires but not all. The set of permissions that ϕ requires but does not grant is called the **footprint** of ϕ. The footprint of ϕ is written

$$\lfloor \phi \rfloor$$

Finally, a ϕ is called **self-framing** if and only if for any set of permissions Π, $\Pi \vDash_I \phi$. The proposition that ϕ is self-framing is written

$$\vdash_{\mathsf{frm}I} \phi$$

Note that $\vdash_{\mathsf{frm}I} \phi \iff \varnothing \vDash_I \phi$, in other words ϕ is self-framing if and only if it grants all of the permissions it requires. Or in other words still, $\lfloor \phi \rfloor = \varnothing$.

[1] As demonstrated by this description, $\mathsf{assumed}$ predicates are really just a useful shorthand and not a fundamentally new type of permission. The only kind fundamental kind of permission is $\mathsf{accessed}$.

4.2 Deciding Framing

Deciding $\Pi \vDash_I \phi$ must take into account the requirements, granteds, and aliases contained in Π and the sub-formulas of ϕ. The following recursive algorithm decides $\Pi \vDash_I \phi_{root}$, where $\mathcal{A}$ is implicitly assumed to be the top-level aliasing context (where the top-level in this context is the level that ϕ_{root} exists at in the program).

$$
\begin{array}{rcll}
\Pi \vDash_I \phi & \iff & \textsf{match } \phi \textsf{ with} & \\
& & v & \mapsto \top \\
& & x & \mapsto \top \\
& & e_1 \oplus e_2 & \mapsto \Pi \vDash_I e_1, e_2 \\
& & e_1 \odot e_2 & \mapsto \Pi \vDash_I e_1, e_2 \\
& & e.f & \mapsto (\Pi \vDash_I e) \ \wedge \ (\Pi \vdash \textsf{accessed}_\phi(e.f)) \\
& & \texttt{acc}(e.f) & \mapsto (\Pi \vDash_I e) \\
& & \phi_1 \circledast \phi_2 & \mapsto (\Pi \cup \textsf{granted}(\phi_2) \vDash_I \phi_1) \ \wedge \\
& & & \quad (\Pi \cup \textsf{granted}(\phi_1) \vDash_I \phi_2) \\
& & \alpha_C(e_1, \dots, e_k) & \mapsto \Pi \vDash_I e_1, \dots, e_2 \\
& & \texttt{if } e \texttt{ then } \phi_1 \texttt{ else } \phi_2 & \mapsto \Pi \vDash_I e, \phi_1, \phi_2 \\
& & \texttt{unfolding } \alpha_C(\overline{e}) \texttt{ in } \phi' & \mapsto (\Pi \vdash \textsf{assumed}_\phi(\alpha_C(\overline{e}))) \ \wedge \ (\Pi \vDash_I \phi') \\
\textsf{granted}(\phi) & := & \textsf{match } \phi \textsf{ with} & \\
& & e & \mapsto \varnothing \\
& & \texttt{acc}(e.f) & \mapsto \{\textsf{accessed}(e.f)\} \\
& & \phi_1 \circledast \phi_2 & \mapsto \textsf{granted}(\phi_1) \cup \textsf{granted}(\phi_2) \\
& & \alpha_C(\overline{e}) & \mapsto \{\textsf{assumed}(\alpha_C(\overline{e}))\} \\
& & \texttt{if } e \texttt{ then } \phi_1 \texttt{ else } \phi_2 & \mapsto \textsf{granted}(\phi_1) \cap \textsf{granted}(\phi_2) \\
& & \texttt{unfolding } \alpha_C(\overline{e}) \texttt{ in } \phi' & \mapsto \textsf{granted}(\phi')
\end{array}
$$

Where $\textsf{accessed}_\phi$ and $\textsf{assumed}_\phi$ indicate the respective propositions considered within the total alias context (including inherited aliasing contexts). More explicitly,

$$
\begin{array}{rl}
\Pi \vdash \textsf{accessed}_\phi(o.f) & \iff \exists o' \in O : (\mathcal{A}(\phi) \vdash \textsf{aliased}\,\{o, o'\}) \ \wedge \ (\textsf{accessed}(o'.f) \in \Pi) \\
\Pi \vdash \textsf{assumed}_\phi(\alpha_C(e_1, \dots, e_k)) & \iff (\forall i : e_i = e'_i \ \vee \ \exists (o, o') = (e_i, e'_i) : \mathcal{A}(\phi) \vdash \textsf{aliased}\,\{o, o'\}) \\
& \qquad \wedge \ (\textsf{assumed}(\alpha_C(e'_1, \dots, e'_k)) \in \Pi)
\end{array}
$$

4.2.1 Notes

- TODO: explain how non-object-variable expressions cannot alias to anything (thus the e.f case in granted and required)

4.3 Examples

In the following examples, assume that the considered formulas are well-formed.

Example 1

Define

$$\phi_{\mathsf{root}} := x = y * \mathtt{acc}(x.f) * \mathtt{acc}(y.f).$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \{\mathsf{aliased}\,\{x, y\}\}\,, \varnothing \rangle.$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} \iff\ & \varnothing \vDash_I \phi_{\mathsf{root}} \\
\iff\ & \varnothing \vDash_I x = y * \mathtt{acc}(x.f) * \mathtt{acc}(y.f) \\
\iff\ & \varnothing \vDash_I (x = y) * (\mathtt{acc}(x.f) * \mathtt{acc}(y.f)) \\
\iff\ & (\mathsf{granted}((\mathtt{acc}(x.f) * \mathtt{acc}(y.f))) \vDash_I x = y) \land \\
& (\mathsf{granted}(x = y) \vDash_I \mathtt{acc}(x.f) * \mathtt{acc}(y.f)) \\
\iff\ & \top \ \land (\varnothing \vDash_I \mathtt{acc}(x.f) * \mathtt{acc}(y.f)) \\
\iff\ & (\mathsf{granted}(y.f) \vDash_I \mathtt{acc}(x.f)) \ \land \ (\mathsf{granted}(x.f) \vDash_I \mathtt{acc}(y.f)) \\
\iff\ & ((\{\mathsf{accessed}(y.f)\} \vDash_I x) \land \\
& \qquad \sim ((\{\mathsf{accessed}(y.f)\} \vDash_I x) \vdash \mathsf{accessed}_{(\mathtt{acc} x.f)}(x.f))) \land \qquad (\star) \\
& (\mathsf{granted}(\mathtt{acc}(x.f)) \vDash_I \mathtt{acc}(y.f)) \\
\iff\ & \bot.
\end{aligned}$$

$(\star)$ is decided to be $\bot$, thus yielding the entire conjunct to be decided $\bot$, because in the sub-formula $\phi := \mathtt{acc}(x.f)$,

$$(\mathcal{A}(\phi) \vdash \mathsf{aliased}\,\{x, y\}) \ \vdash \ (\{\mathsf{accessed}(y.f)\} \vdash \mathsf{accessed}_{\phi}(x.f))$$

contradicts the requirement of ϕ that

$$\sim (\{\mathsf{accessed}(y.f)\} \vDash_I x) \vdash \mathsf{accessed}_{\phi}(x.f))$$

Example 2

Define

$$\phi_{\mathsf{root}} := \texttt{acc}(x.f) \ * \ (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))$$

Then

$$\begin{aligned}
\mathcal{A}(\phi_{\mathsf{root}}) &= \langle \varnothing, \{b : \mathcal{A}(x.f = 1), \sim b : \mathcal{A}(\texttt{acc}(x.f))\}\rangle \\
\mathcal{A}(x.f = 1) &= \langle \varnothing, \varnothing \rangle \\
\mathcal{A}(\texttt{acc}(x.f)) &= \langle \varnothing, \varnothing \rangle
\end{aligned}$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} \iff & \varnothing \vDash_I \phi_{\mathsf{root}} \\
\iff & \varnothing \vDash_I (\texttt{acc}(x.f)) \ * \ (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f)) \\
\iff & (\mathsf{granted}(\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f)) \vDash_I (\texttt{acc}(x.f))) \wedge \\
& (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))) \\
\iff & ((\mathsf{granted}(x.f = 1) \cap \mathsf{granted}(\texttt{acc}(x.f)) \vDash_I (\texttt{acc}(x.f)) \wedge \\
& (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))) \\
\iff & (\varnothing \vDash_I \texttt{acc}(x.f)) \wedge \\
& (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))) \\
\iff & \top \ \wedge \ (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))) \\
\iff & \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I (\texttt{if } b \texttt{ then } x.f = 1 \texttt{ else } \texttt{acc}(x.f))) \\
\iff & \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I (b), (x.f = 1), (\texttt{acc}(x.f))) \\
\iff & \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I b) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I x.f = 1) \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I x.f = 1) \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ ((\{\mathsf{accessed}(x.f)\} \vDash_I x) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I x.f)) \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ (\top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}(x.f))) \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ (\top \ \wedge \ \top) \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ \top \wedge \\
& (\{\mathsf{accessed}(x.f)\} \vDash_I \texttt{acc}(x.f)) \\
\iff & \top \ \wedge \ \top \ \wedge \ \top \wedge \\
& ((\{\mathsf{accessed}(x.f)\} \vDash_I x) \wedge \ \sim (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{(\texttt{acc}(x.f))}(x.f))) \\
\iff & \top \ \wedge \ \top \ \wedge \ \top \wedge \\
& (\top \wedge \ \sim (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{(\texttt{acc}(x.f))}(x.f))) \qquad (\star) \\
\iff & \top \ \wedge \ \top \ \wedge \ \top \ \wedge \ (\top \ \wedge \ \bot) \\
\iff & \bot
\end{aligned}$$

($\star$) is decided to be $\perp$ because in the sub-formula $\phi := \texttt{acc}(x.f)$,

$$\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi}(x.f)$$

contradicts the requirement of ϕ that

$$\sim (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi}(x.f))$$

Example 3

Define

$$\phi_{\mathsf{root}} := \mathtt{acc}(x.f) \ * \ x = y \ * \ y.f = 1$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \{\mathsf{aliased}\,\{x, y\}\}\,, \varnothing \rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \mathtt{acc}(x.f) \ * \ x = y \ * \ y.f = 1 \\
&\iff \varnothing \vDash_I x = y \ * \ \mathtt{acc}(x.f) \ * \ y.f = 1 && (* \text{ is commutative}) \\
&\iff (\mathsf{granted}(\mathtt{acc}(x.f) \ * \ y.f = 1) \vDash_I x = y) \wedge \\
&\qquad (\mathsf{granted}(x = y) \vDash_I \mathtt{acc}(x.f) \ * \ y.f = 1) \\
&\iff \top \ \wedge \ (\mathsf{granted}(x = y) \vDash_I \mathtt{acc}(x.f) \ * \ y.f = 1) \\
&\iff \top \ \wedge \ (\varnothing \vDash_I \mathtt{acc}(x.f) \ * \ y.f = 1) \\
&\iff \top \ \wedge \ (\mathsf{granted}(y.f = 1) \vDash_I \mathtt{acc}(x.f)) \ \wedge \ (\mathsf{granted}(\mathtt{acc}(x.f)) \vDash_I y.f = 1) \\
&\iff \top \ \wedge \ (\varnothing \vDash_I \mathtt{acc}(x.f)) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I y.f = 1) \\
&\iff \top \ \wedge \ ((\varnothing \vDash_I e) \wedge \ \sim (\varnothing \vdash \mathsf{accessed}(x.f))) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I y.f = 1) \\
&\iff \top \ \wedge \ (\top \ \wedge \ \top) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I y.f = 1) \\
&\iff \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{x.f}(x.f)) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I 1) \\
&\iff \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{x.f}(x.f)) \ \wedge \ \top && (\star) \\
&\iff \top \ \wedge \ \top \ \wedge \ (\top) \ \wedge \ \top \\
&\iff \top
\end{aligned}$$

$\star$ is decided to be $\top$ because in the sub-formula $\phi := x.f$,

$$\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi}(x.f)$$

is true since

$$(\mathcal{A}(\phi) \vdash \mathsf{aliased}\,\{x, y\}) \ \vdash \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi}(x.f))$$

l != null List(l) unfolding List(l) in l.tail == null

Example 4

Define

$$
\begin{array}{l}
\texttt{class}\ \mathsf{List}\ \{ \\
\quad \texttt{int}\ \mathit{head}; \\
\quad \mathsf{List}\ \mathit{tail}; \\
\quad \texttt{predicate}\ \mathsf{List}(l) = \\
\qquad \texttt{acc}(l.\mathit{tail}) * \\
\qquad \texttt{if}\ l.\mathit{tail} = \texttt{null} \\
\qquad\quad \texttt{then}\ \texttt{true} \\
\qquad\quad \texttt{else}\ \mathsf{List}(l.\mathit{tail}); \\
\quad \vdots \\
\}
\end{array}
$$

$$\phi_{\mathsf{root}} := l \neq \texttt{null}\ *\ \mathsf{List}(l)\ *\ \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null}$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \{\sim \mathsf{aliased}\,\{l, \texttt{null}\}\,, \mathsf{aliased}\,\{l.\mathit{tail}, \texttt{null}\}\}\,, \varnothing \rangle$$

And so,

$$
\begin{array}{rl}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} & \iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
& \iff \varnothing\ \vDash_I\ (l \neq \texttt{null})\ *\ \mathsf{List}(l)\ *\ (\texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null}) \\
& \iff (\mathsf{granted}(\mathsf{List}(l)\ *\ (\texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null}))\ \vDash_I\ l \neq \texttt{null}) \wedge \\
& \quad\ \ (\mathsf{granted}((\texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null})\ *\ (l \neq \texttt{null}))\ \vDash_I\ \mathsf{List}(l)) \wedge \\
& \quad\ \ (\mathsf{granted}((l \neq \texttt{null})\ *\ \mathsf{List}(l))\ \vDash_I\ \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null}) \\
& \iff (\{\mathsf{assumed}(\mathsf{List}(l))\}\ \vDash_I\ l \neq \texttt{null}) \wedge \\
& \quad\ \ (\varnothing\ \vDash_I\ \mathsf{List}(l)) \wedge \\
& \quad\ \ (\{\mathsf{assumed}(\mathsf{List}(l))\}\ \vDash_I\ \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{tail} = \texttt{null}) \\
& \iff (\mathsf{granted}(\texttt{acc}(l.\mathit{tail}) * \texttt{if}\ l.\mathit{tail} = \texttt{null}\ \texttt{then}\ \texttt{true}\ \texttt{else}\ \mathsf{List}(l.\mathit{tail}))\ \vDash_I\ l \neq \texttt{null}) \wedge \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(expansion of } \mathsf{assumed} \text{ permission)} \\
& \quad\ \ \top \wedge \\
& \quad\ \ ((\{\mathsf{assumed}(\mathsf{List}(l))\} \vdash \mathsf{assumed}_\phi(\mathsf{List}(l)))\ \wedge\ (\{\mathsf{assumed}(\mathsf{List}(l))\} \vdash l.\mathit{tail} = \texttt{null}) \\
& \iff (\{\texttt{acc}(l.\mathit{tail})\}\ \vDash_I\ l \neq \texttt{null}) \wedge \\
& \quad\ \ \top \wedge \\
& \quad\ \ (\top\ \wedge\ ((\mathsf{granted}(\texttt{acc}(l.\mathit{tail}) * \texttt{if}\ l.\mathit{tail} = \texttt{null}\ \texttt{then}\ \texttt{true}\ \texttt{else}\ \mathsf{List}(l.\mathit{tail})) \vdash l.\mathit{tail} = \texttt{null}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(expansion of } \mathsf{assumed} \text{ permission)} \\
& \iff \top\ \wedge\ \top\ \wedge\ (\top\ \wedge\ (\{\texttt{acc}(l.\mathit{tail})\} \vdash l.\mathit{tail} = \texttt{null}) \\
& \iff \top
\end{array}
$$

Example 5

Define

$$\phi_{\mathsf{root}} := \texttt{if } x = \texttt{null then true else } (\texttt{acc}(x.f) * x.f = 1)$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing, \{x = \texttt{null} : \langle \{\mathsf{aliased}\,\{x, \texttt{null}\}\}, \varnothing\rangle, x \neq \texttt{null} : \langle \{\sim \mathsf{aliased}\,\{x, \texttt{null}\}\}, \varnothing\rangle\}\rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \texttt{if } x = \texttt{null then true else } (\texttt{acc}(x.f) * x.f = 1) \\
&\iff (\varnothing \vDash_I x = \texttt{null}) \ \wedge \ (\varnothing \vDash_I \texttt{true}) \ \wedge \ (\varnothing \vDash_I \texttt{acc}(x.f) * x.f = 1) \\
&\iff \top \ \wedge \ \top \ \wedge (\mathsf{granted}(x.f = 1) \vDash_I \texttt{acc}(x.f)) \ \wedge (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I x.f = 1) \\
&\iff \top \ \wedge \ \top \ \wedge (\varnothing \vDash_I \texttt{acc}(x.f)) \ \wedge (\{\mathsf{accessed}(x.f)\} \vDash_I x.f = 1) \\
&\iff \top \ \wedge \ \top \ \wedge (\varnothing \vDash_I x) \wedge \ \sim (\varnothing \vdash \mathsf{accessed}_{\texttt{acc}(x.f)}(x.f)) \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{x.f}(x.f)) \ \wedge (\{\mathsf{accessed}(x.f)\} \vDash_I 1) \\
&\iff \top \ \wedge \ \top \ \wedge \top \ \wedge \ \top \ \top \ \wedge \top \\
&\iff \top
\end{aligned}$$

Example 6

Use the definition of List from example 4. Define

$$
\begin{aligned}
\phi_{\mathsf{root}} &:= (\texttt{if } l = \texttt{null then true else } \phi_1) * \\
&\quad\ (\texttt{if } l = \texttt{null then true else } \phi_2) \\
\phi_1 &:= \texttt{acc}(l.head) * \texttt{acc}(l.tail) * \mathsf{List}(l) \\
\phi_2 &:= l.head = 5
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{A}(\phi_{\mathsf{root}}) =& \langle \varnothing, \{l = \texttt{null} : \mathcal{A}(true) \sqcup \mathcal{A}(true),\ l \neq \texttt{null} : \mathcal{A}(\phi_1) \sqcup \mathcal{A}(\phi_2)\}\rangle \\
\mathcal{A}(\phi_1) &= \varnothing \\
\mathcal{A}(\phi_2) &= \varnothing
\end{aligned}
$$

And so,

$$
\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} \iff& \varnothing \vDash_I \phi_{\mathsf{root}} \\
\iff& \varnothing \vDash_I (\texttt{if } l = \texttt{null then true else } \phi_1) * \\
& (\texttt{if } l = \texttt{null then true else } \phi_2) \\
\iff& (\varnothing \vDash_I l = \texttt{null}) \ \wedge\ (\varnothing \vDash_I \texttt{true}) \ \wedge\ (\varnothing \vDash_I \texttt{acc}(l.head) * \texttt{acc}(l.tail) * \mathsf{List}(l)) \ \wedge\ (\varnothing \vDash_I)
\end{aligned}
$$

5 Satisfiability

6 Implication

7 Weakest Predonditions