SVL with Recursive Predicates

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1 Grammar

$$
\begin{array}{rcl}
x, y, z & \in & \mathit{VAR} \\
v & \in & \mathit{VAL} \\
e & \in & \mathit{EXPR} \\
s & \in & \mathit{STMT} \\
o & \in & \mathit{LOC} \\
f & \in & \mathit{FIELDNAME} \\
m & \in & \mathit{METHODNAME} \\
C, D & \in & \mathit{CLASSNAME} \\
\alpha & \in & \mathit{PREDNAME} \\
P & ::= & \overline{\mathit{cls}}\ s \\
\mathit{cls} & ::= & \texttt{class}\ C\ \texttt{extends}\ D\ \{\overline{\mathit{field}}\ \overline{\mathit{pred}}\ \overline{\mathit{method}}\} \\
\mathit{field} & ::= & T\ f; \\
\mathit{pred} & ::= & \texttt{predicate}\ \alpha_C(\overline{T\ x}) = \widetilde{\phi} \\
T & ::= & \texttt{int} \mid \texttt{bool} \mid C \mid \top \\
\mathit{method} & ::= & T\ m(\overline{T\ x})\ \texttt{dynamically}\ \mathit{contract}\ \texttt{statically}\ \mathit{contract}\ \{s\} \\
\mathit{contract} & ::= & \texttt{requires}\ \widetilde{\phi}\ \texttt{ensures}\ \widetilde{\phi} \\
\oplus & ::= & + \mid - \mid * \mid \backslash \mid \&\& \mid \ || \\
\odot & ::= & \neq \mid = \mid < \mid > \mid \leq \mid \geq \\
s & ::= & \texttt{skip} \mid s_1\ ;\ s_2 \mid T\ x \mid x := e \mid \texttt{if}\ (e)\ \{s_1\}\ \texttt{else}\ \{s_2\} \\
 & & \mid \texttt{while}\ (e)\ \texttt{invariant}\ \widetilde{\phi}\ \{s\} \mid x.f := y \mid x := \texttt{new}\ C \mid y := z.m(\overline{x}) \\
 & & \mid y := z.m_C(\overline{x}) \mid \texttt{assert}\ \phi \mid \texttt{release}\ \phi \mid \texttt{hold}\ \phi\ \{s\} \mid \texttt{fold}\ A \mid \texttt{unfold}\ A \\
e & ::= & v \mid x \mid e \oplus e \mid e \odot e \mid e.f \\
x & ::= & \texttt{result} \mid \mathit{id} \mid \texttt{old}(\mathit{id}) \mid \texttt{this} \\
v & ::= & n \mid o \mid \texttt{null} \mid \texttt{true} \mid \texttt{false} \\
A & ::= & \alpha(\overline{e}) \mid \alpha_C(\overline{e}) \\
\circledast & ::= & \wedge \mid * \\
\phi & ::= & e \mid A \mid \texttt{acc}(e.f) \mid \phi \circledast \phi \mid (\texttt{if}\ e\ \texttt{then}\ \phi\ \texttt{else}\ \phi) \mid (\texttt{unfolding}\ A\ \texttt{in}\ \phi) \\
\widetilde{\phi} & ::= & \phi \mid ? * \phi
\end{array}
$$

2 Well-formedness

3 Aliasing

3.1 Definitions

An **object variable** is one of the following:

- a class instance variable i.e. a variable v such that $v : C$ for some class C,
- a class instance field reference i.e. a field reference $e.f$ where $e.f : C$ for some class C,
- `null` as a value such that `null` $: C$ for some class C.

Let $\mathcal{O}$ be a set of object variables. An $O \subset \mathcal{O}$ **aliases** if and only if each $o \in O$ refers to the same memory in the heap as each other, written propositionally as

$$\forall o, o' \in O : o = o' \iff \mathsf{aliases}(O)$$

While it is possible to keep track of negated aliasings (of the form $\sim \mathsf{aliases}\,\{o_\alpha\}$), this will not be needed for either aliasing tree construction or self-framing desicions. So, it will not be tracked i.e. $x \neq y$ does not contribute anything to an aliasing context.

3.2 Aliasing Context

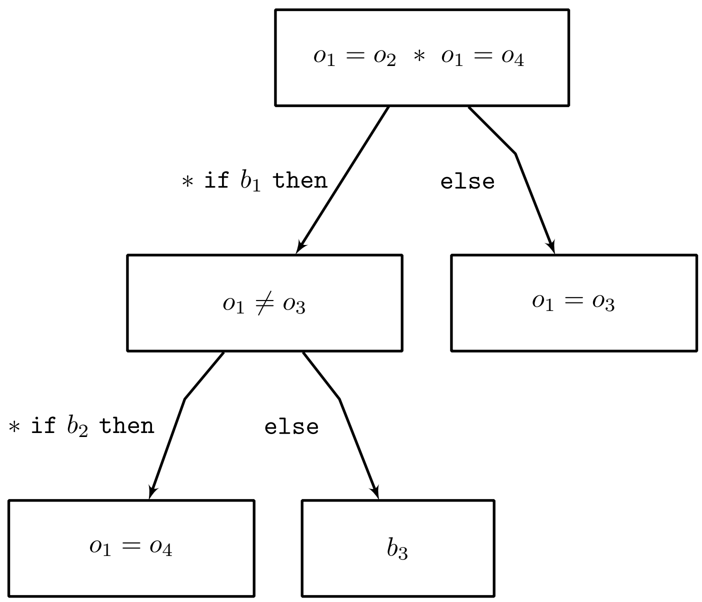
Let ϕ be a formula. The **aliasing context** $\mathcal{A}$ of ϕ is a tree of set of aliasing proposition about aliasing of object variables that appear in ϕ. $\mathcal{A}$ needs to be a tree because the conditional and unfolding sub-formulas that may appear in ϕ allow for branching aliasing contexts not expressible flatly at the top level. In the case of conditionals i.e. sub-formulas of the form `if` e `then` ϕ_1 `else` ϕ_2, two branches sprout from the original context. In the case of unfoldings i.e. sub-formulas of the form `unfolding` $\alpha_C(\overline{e})$ `in` ϕ, one branch sprouts from the original context. Each node in the tree corresponds to a set of aliasing propositions, and each branch refers to a branch of a unique conditional in ϕ. The parts of the tree are labeled in such a way that modularly allows a specified sub-formula of ϕ to be matched to the unique aliasing sub-context that corresponds to it. For example, consider the following formula:

$$\begin{aligned}
\phi := {} & (o_1 = o_2) * \\
& (\texttt{if } (b_1) \\
& \quad \texttt{then } (\\
& \qquad (o_1 \neq o_3) * \\
& \qquad (\texttt{if } (b_2) \\
& \qquad\quad \texttt{then } (o_1 = o_4) \\
& \qquad\quad \texttt{else } (b_3))) \\
& \quad \texttt{else } (o_1 = o_3)) * \\
& (o_1 = o_4)
\end{aligned}$$

where b_1, b_2 are arbitrary boolean expressions that do not assert aliasing propositions. ϕ has a formula-structure represented by the tree in figure 3.2. The formula-structure tree for ϕ corresponds node-for-node and edge-for-edge to the aliasing context tree in figure 3.2.

More generally, for ϕ a formula and ϕ' a sub-formula of ϕ, write $\mathcal{A}_\phi(\phi')$ as the **total**

Figure 1: Formula structure tree for ϕ.



aliasing context of ϕ' which includes aliasing propositions inherited from its ancestors in the aliasing context tree of ϕ. These aliasing contexts are combined via $\sqcup$ which will be defined in the next section. For example, the total aliasing context at the sub-formula $(o_1 = o_4)$ of ϕ is:

$$\mathcal{A}_\phi(o_1 = o_4) := \{\mathsf{aliased}\,\{o_1, o_2, o_4\}\}$$

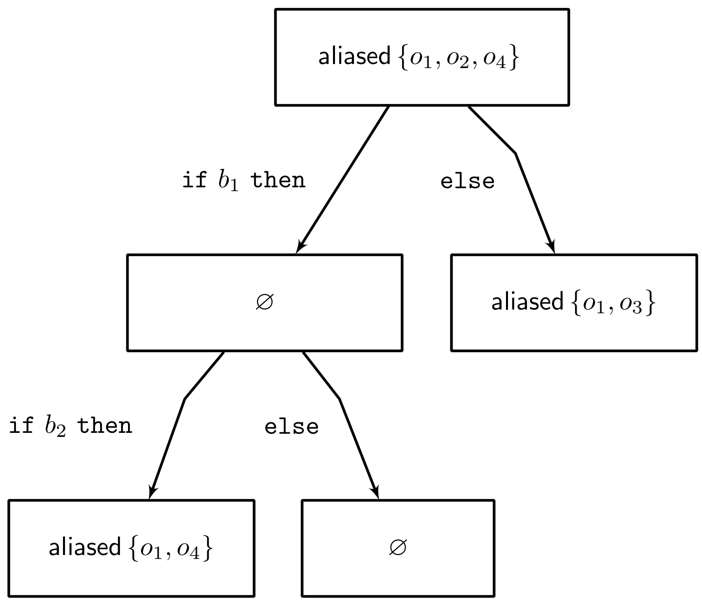
along with the fact that it has no child branches. Usually $\mathcal{A}_{\phi_{\mathsf{root}}}(\phi')$ is abbreviated to $\mathcal{A}(\phi')$ when the top level formula ϕ is implicit and ϕ' is a sub-formula of ϕ_{root}.

An aliasing context $\mathcal{A}$ may entail $\mathsf{aliased}(O)$ for some $O \subset \mathcal{O}$. Since $\mathcal{A}$ is efficiently represented as a set of propositions about sets, it may be the case that $\mathsf{aliased}(O) \notin \mathcal{A}$ yet still the previous judgement holds. For example, this is true when $\exists O' \subset \mathcal{O}$ such that $O \subset O'$ and $\mathsf{aliased}(O') \in \mathcal{A}$. So, the explicit definition for making this judgement is as follows:

$$\mathcal{A} \vdash \mathsf{aliased}(O) \iff \exists O' \subset \mathcal{O} : (O \subset O') \ \wedge \ (\mathsf{aliased}(O') \in \mathcal{A})$$

The notations $\mathsf{aliased}(O) \in \mathcal{A}$ is a little misleading because $\mathcal{A}$ is in fact a tree and not just a set. To be explicit, $\mathsf{aliased}(O) \in \mathcal{A}$ is defined to be set membership of the set of aliasing propositions in the total aliasing context at $\mathcal{A}$.

Figure 2: $\mathcal{A}(\phi)$, the aliasing context tree for ϕ.



3.3 Constructing an Aliasing Context

An aliasing context of a formula ϕ is a tree, where nodes represent local aliasing contexts and branches represent the branches of conditional sub-formulas nested in ϕ. So, an aliasing context is defined structurally as

$$\mathcal{A} ::= \langle A,\ \{l_\alpha : \mathcal{A}_\alpha\}\rangle$$

where A is a set of propositions about aliasing and the $l_\alpha : \mathcal{A}_\alpha$ are the nesting aliasing contexts that correspond to the branches of conditionals and unfoldings directly nested in ϕ, the l_α being labels for each child context.

Given a root formula ϕ_{root}, the aliasing context of ϕ_{root} is written $\mathcal{A}(\phi_{\mathsf{root}})$. With the root invariant, the following recursive algorithm constructs $\mathcal{A}(\phi)$ for any sub-formula of ϕ_{root}

(including $\mathcal{A}(\phi_{\mathsf{root}})$).

$$
\begin{array}{lll}
\mathcal{A}(\phi) & := & \mathsf{match}\ \phi\ \mathsf{with} \\
 & & v \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & x \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & e_1\ \&\&\ e_2 \quad\mapsto\quad \mathcal{A}(e_1) \sqcup \mathcal{A}(e_2) \\
 & & e_1 \mid\mid e_2 \quad\mapsto\quad \mathcal{A}(\texttt{if}\ e_1\ \texttt{then}\ \texttt{true}\ \texttt{else}\ e_2) \\
 & & e_1 \oplus e_2 \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & o_1 = o_2 \quad\mapsto\quad \langle\{\mathsf{aliases}\,\{o_1, o_2\}\}\,,\ \varnothing\rangle \\
 & & e_1 \odot e_2 \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & e.f \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & \texttt{acc}(e.f) \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & \phi_1 * \phi_2 \quad\mapsto\quad \mathcal{A}(\phi_1) \sqcup \mathcal{A}(\phi_2) \\
 & & \phi_1 \wedge \phi_2 \quad\mapsto\quad \mathcal{A}(\phi_1) \sqcup \mathcal{A}(\phi_2) \\
 & & \alpha_C(\overline{e}) \quad\mapsto\quad \langle\varnothing,\ \varnothing\rangle \\
 & & \texttt{if}\ e\ \texttt{then}\ \phi_1\ \texttt{else}\ \phi_2 \quad\mapsto\quad \langle\varnothing,\ \{e : \mathcal{A}(e) \sqcup \mathcal{A}(\phi_1),\ \sim e : (\mathcal{A}(\sim e)) \sqcup \mathcal{A}(\phi_2)\}\rangle \\
 & & \texttt{unfolding}\ \alpha_C(\overline{e})\ \texttt{in}\ \phi' \quad\mapsto\quad \langle\varnothing,\ \{\texttt{unfolding}(\alpha_C(\overline{e})) : \mathcal{A}(\mathsf{unfold}\ \alpha_C(\overline{e})) \sqcup \mathcal{A}(\phi')\}\rangle
\end{array}
$$

Note the following:

- $\mathcal{A}(\phi_{\mathsf{root}})$ is implicitly unioned with the discrete aliasing context $\{\{o\} : o \in \mathcal{O}\}$. This convention yields that each $o \in \mathcal{O}$ is always considered an alias of itself.
- The $\sim e$ expression in the result of the rule for $\mathcal{A}(\texttt{if}\ e\ \texttt{then}\ \phi_1\ \texttt{else}\ \phi_2)$ means to negate the boolean expression of e
- The $e_1 \mid\mid e_2$ expression is translated into $\texttt{if}\ e_1\ \texttt{then}\ \texttt{true}\ \texttt{else}\ e_2$ for the purpose of aliasing. So, boolean or operations in forumals yield branching just like conditional expressions.
- The $\mathsf{unfold}\ \alpha_C(\overline{e})$ expression in the result of the rulle for $\mathcal{A}(\texttt{unfolding}\ \alpha_C(\overline{e})\ \texttt{in}\ \phi')$ is translated to a single unfolding of the body of $\alpha_C(\overline{e})$ with the arguments substituted appropriately.

As examples,

$$
\begin{aligned}
\mathcal{A}(\sim (x = y)) &= \mathcal{A}(x \neq y) = \langle\varnothing,\ \varnothing\rangle \\
\mathcal{A}(\sim (x \neq y)) &= \mathcal{A}(x = y) = \langle\{\mathsf{aliased}\,\{x, y\}\}\,,\ \varnothing\rangle
\end{aligned}
$$

Context union, $\sqcup$, and context intersection, $\sqcap$, are operations that combine aliasing contexts and are defined below.

$$
\begin{aligned}
&\langle A_1,\ \{l_\alpha : \mathcal{A}_\alpha\}\rangle \sqcup \langle A_2,\ \{l_\beta : \mathcal{A}_\beta\}\rangle := \\
&\qquad \langle\{\mathsf{aliased}\,\{o' \mid \forall o' : (A_1 \vdash \mathsf{aliased}\,\{o, o'\})\ \vee\ (A_2 \vdash \mathsf{aliased}\,\{o, o'\})\} \mid \forall o\}\,, \\
&\qquad \{l_\alpha : \mathcal{A}_\alpha\} \cup \{l_\beta : \mathcal{A}_\beta\}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle A_1,\ \{l_\alpha : \mathcal{A}_\alpha\}\rangle \sqcap \langle A_2,\ \{l_\beta : \mathcal{A}_\beta\}\rangle := \\
&\qquad \langle\{\mathsf{aliased}\,\{o' \mid \forall o' : (A_1 \vdash \mathsf{aliased}\,\{o, o'\})\ \wedge\ (A_2 \vdash \mathsf{aliased}\,\{o, o'\})\} \mid \forall o\}\,, \\
&\qquad \{l_\alpha : \mathcal{A}_\alpha\} \cap \{l_\beta : \mathcal{A}_\beta\}\rangle
\end{aligned}
$$

4 Framing

4.1 Definitions

For framing, a formula is considered inside a **permission context**, a set of permissions, where a **permission** π is to do one of the following:

- to reference $e.f$, written $\mathsf{accessed}(e.f)$.
- to assume $\alpha_C(\overline{e})$, written $\mathsf{assumed}(\alpha_C(\overline{e}))$. This allows the a single unrolling of $\alpha_C(\overline{e})$. Explicitly, an instance of $\mathsf{assumed}(\alpha_C(\overline{e}))$ in a set of permissions Π may be expanded into $\Pi \cup \mathsf{granted}(\dots)$ where $\dots$ is replaced with a single unrolling of the body of $\alpha_C(\overline{e})$ with the arguments substituted appropriately[1].

Let ϕ be a formula. ϕ may **require** a permission π. For example, the formula $e.f = 1$ requires $\mathsf{accessed}(e.f)$, because it references $e.f$. The set of all permissions that ϕ requires is called the **requirements** of ϕ. ϕ may also **grant** a permission π. For example, the formula $\mathtt{acc}(e.f)$ grants the permission $\mathsf{accessed}(e.f)$.

Altogether, ϕ is **framed** by a set of permissions Π if all permissions required by ϕ are either in Π or granted by ϕ. The proposition that Π frames ϕ is written

$$\Pi \vDash_I \phi$$

Of course, ϕ may grant some of the permissions it requires but not all. The set of permissions that ϕ requires but does not grant is called the **footprint** of ϕ. The footprint of ϕ is written

$$\lfloor \phi \rfloor$$

Finally, a ϕ is called **self-framing** if and only if for any set of permissions Π, $\Pi \vDash_I \phi$. The proposition that ϕ is self-framing is written

$$\vdash_{\mathsf{frm}I} \phi$$

Note that $\vdash_{\mathsf{frm}I} \phi \iff \varnothing \vDash_I \phi$, in other words ϕ is self-framing if and only if it grants all of the permissions it requires. Or in other words still, $\lfloor \phi \rfloor = \varnothing$.

[1] As demonstrated by this description, $\mathsf{assumed}$ predicates are really just a useful shorthand and not a fundamentally new type of permission. The only kind fundamental kind of permission is $\mathsf{accessed}$.

4.2 Deciding Framing

Deciding $\Pi \vDash_I \phi$ must take into account the requirements, granteds, and aliases contained in Π and the sub-formulas of ϕ. The following recursive algorithm decides $\Pi \vDash_I \phi_{root}$, where $\mathcal{A}$ is implicitly assumed to be the top-level aliasing context (where the top-level in this context is the level that ϕ_{root} exists at in the program).

$$
\begin{array}{lcll}
\Pi \vDash_I \phi & \iff & \mathsf{match}\ \phi\ \mathsf{with} & \\
 & & v & \mapsto\ \top \\
 & & x & \mapsto\ \top \\
 & & e_1 \oplus e_2 & \mapsto\ \Pi \vDash_I e_1, e_2 \\
 & & e_1 \odot e_2 & \mapsto\ \Pi \vDash_I e_1, e_2 \\
 & & e.f & \mapsto\ (\Pi \vDash_I e)\ \wedge\ (\Pi \vdash \mathsf{accessed}_\phi(e.f)) \\
 & & \mathtt{acc}(e.f) & \mapsto\ (\Pi \vDash_I e) \\
 & & \phi_1 \circledast \phi_2 & \mapsto\ (\Pi \cup \mathsf{granted}(\phi_2) \vDash_I \phi_1)\ \wedge \\
 & & & \ \ (\Pi \cup \mathsf{granted}(\phi_1) \vDash_I \phi_2) \\
 & & \alpha_C(e_1, \ldots, e_k) & \mapsto\ \Pi \vDash_I e_1, \ldots, e_2 \\
 & & \mathtt{if}\ e\ \mathtt{then}\ \phi_1\ \mathtt{else}\ \phi_2 & \mapsto\ \Pi \vDash_I e, \phi_1, \phi_2 \\
 & & \mathtt{unfolding}\ \alpha_C(\overline{e})\ \mathtt{in}\ \phi' & \mapsto\ (\Pi \vdash \mathsf{assumed}_\phi(\alpha_C(\overline{e})))\ \wedge\ (\Pi \vDash_I \phi') \\
\mathsf{granted}(\phi) & := & \mathsf{match}\ \phi\ \mathsf{with} & \\
 & & e & \mapsto\ \varnothing \\
 & & \mathtt{acc}(e.f) & \mapsto\ \{\mathsf{accessed}(e.f)\} \\
 & & \phi_1 \circledast \phi_2 & \mapsto\ \mathsf{granted}(\phi_1) \cup \mathsf{granted}(\phi_2) \\
 & & \alpha_C(\overline{e}) & \mapsto\ \{\mathsf{assumed}(\alpha_C(\overline{e}))\} \\
 & & \mathtt{if}\ e\ \mathtt{then}\ \phi_1\ \mathtt{else}\ \phi_2 & \mapsto\ \mathsf{granted}(\phi_1) \cap \mathsf{granted}(\phi_2) \\
 & & \mathtt{unfolding}\ \alpha_C(\overline{e})\ \mathtt{in}\ \phi' & \mapsto\ \mathsf{granted}(\phi')
\end{array}
$$

Where $\mathsf{accessed}_\phi$ and $\mathsf{assumed}_\phi$ indicate the respective propositions considered within the total alias context (including inherited aliasing contexts). More explicitly,

$$
\begin{array}{rl}
\Pi \vdash \mathsf{accessed}_\phi(o.f) & \iff \exists \mathsf{accessed}(o'.f) \in \Pi : \mathcal{A}(\phi) \vdash \mathsf{aliased}\ \{o, o'\} \\
\Pi \vdash \mathsf{assumed}_\phi(\alpha_C(e_1, \ldots, e_k)) & \iff \exists \mathsf{assumed}(\alpha_C(e'_1, \ldots, e'_k)) \in \Pi : \forall i : \mathcal{A}(\phi) \vdash \mathsf{aliased}\ \{e_i, e'_i\}
\end{array}
$$

4.3 Examples

In the following examples, assume that the considered formulas are well-formed.

Example 1

Define

$$\phi_{\mathsf{root}} := x = y * \mathsf{acc}(x.f) * \mathsf{acc}(y.f).$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \{\mathsf{aliased}\,\{x, y\}\}\,,\ \varnothing \rangle.$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} \iff & \varnothing \vDash_I \phi_{\mathsf{root}} \\
\iff & \varnothing \vDash_I x = y * \mathsf{acc}(x.f) * \mathsf{acc}(y.f) \\
\iff & (\mathsf{granted}(\mathsf{acc}(x.f) * \mathsf{acc}(y.f)) \vDash_I x = y) \wedge \\
& (\mathsf{granted}(x = y * \mathsf{acc}(y.f)) \vDash_I \mathsf{acc}(x.f)) \wedge \\
& (\mathsf{granted}(x = y * \mathsf{acc}(x.f)) \vDash_I \mathsf{acc}(y.f)) \\
\iff & \top \wedge \\
& (\mathsf{granted}(x = y * \mathsf{acc}(y.f)) \vDash_I x) \wedge \\
& (\mathsf{granted}(x = y * \mathsf{acc}(x.f)) \vDash_I y) \\
\iff & \top \ \wedge \ \top \ \wedge \ \top \\
\iff & \top
\end{aligned}$$

Example 2

Define

$$\phi_{\mathsf{root}} := \mathtt{acc}(x.f) \ * \ (\mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathit{true}\ \mathtt{else}\ \mathtt{acc}(x.f))$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{x.f = 1 : \langle \varnothing,\ \varnothing \rangle,\ x.f \neq 1 : \langle \varnothing,\ \varnothing \rangle\} \rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \mathtt{acc}(x.f) \ * \ (\mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathit{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \\
&\iff (\mathsf{granted}(\mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \vDash_I \mathtt{acc}(x.f)) \wedge \\
&\qquad (\mathsf{granted}(\mathtt{acc}(x.f)) \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \\
&\iff (\mathsf{granted}(\mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \vDash_I x) \wedge \\
&\qquad (\mathsf{granted}(\mathtt{acc}(x.f)) \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \\
&\iff \top \ \wedge \ (\mathsf{granted}(\mathtt{acc}(x.f)) \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \\
&\iff \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{acc}(x.f)) \\
&\iff \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I x.f = 1) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I \mathtt{true}) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vDash_I \mathtt{acc}(x.f) \\
&\iff \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi_{\mathsf{root}}}(x.f)) \wedge \\
&\qquad \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I x) \\
&\iff \top \ \wedge \ \top \ \wedge \ \top \ \wedge \ \top \\
&\iff \top
\end{aligned}$$

Example 3

Define

$$\phi_{\mathsf{root}} := \mathsf{acc}(x.f) \;*\; x = y \;*\; y.f = 1$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \{\mathsf{aliased}\,\{x, y\}\}\,,\ \varnothing \rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} \iff{} & \varnothing \vDash_I \phi_{\mathsf{root}} \\
\iff{} & \varnothing \vDash_I \mathsf{acc}(x.f) \;*\; x = y \;*\; y.f = 1 \\
\iff{} & (\mathsf{granted}(x = y * y.f = 1) \vDash_I \mathsf{acc}(x.f)) \wedge \\
& (\mathsf{granted}(\mathsf{acc}(x.f) * y.f = 1) \vDash_I x = y) \wedge \\
& (\mathsf{granted}(\mathsf{acc}(x.f) * x = y) \vDash_I y.f = 1) \\
\iff{} & (\mathsf{granted}(x = y * y.f = 1) \vDash_I x) \wedge \\
& (\mathsf{granted}(\mathsf{acc}(x.f) * y.f = 1) \vDash_I x, y) \wedge \\
& (\mathsf{granted}(\mathsf{acc}(x.f) * x = y) \vDash_I y.f) \\
\iff{} & \top \;\wedge\; \top \wedge\; (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi_{\mathsf{root}}}(y.f)) \\
\iff{} & \top \;\wedge\; \top \wedge\; \top \\
\iff{} & \top
\end{aligned}$$

Example 4

Define

$$
\begin{array}{l}
\texttt{class}\ \mathsf{List}\ \{ \\
\quad \texttt{int}\ \mathit{head}; \\
\quad \mathsf{List}\ \mathit{tail}; \\
\quad \texttt{predicate}\ \mathsf{List}(l) = \\
\quad\quad l \neq \texttt{null}\ *\ \texttt{acc}(l.\mathit{head})\ *\ \texttt{acc}(l.\mathit{tail})\ * \\
\quad\quad \texttt{if}\ l.\mathit{tail} = \texttt{null}\ \texttt{then}\ \texttt{true}\ \texttt{else}\ \mathsf{List}(l.\mathit{tail}); \\
\}
\end{array}
$$

$$\phi_{\mathsf{root}} := \mathsf{List}(l)\ *\ \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{head} = 1$$

Then

$$
\begin{aligned}
\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{&\texttt{unfolding}(\mathsf{List}(l)) : \\
&\{\langle \varnothing,\ \{t.\mathit{tail} = \texttt{null} : \langle \{\mathsf{aliased}\,\{t.\mathit{tail}, \texttt{null}\}\}\,,\ \varnothing\rangle,\ t.\mathit{tail} \neq \texttt{null} : \langle \varnothing,\ \varnothing\rangle\}\rangle\}\}\rangle
\end{aligned}
$$

And so,

$$
\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \mathsf{List}(l)\ *\ \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{head} = 1 \\
&\iff (\mathsf{granted}(\texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{head} = 1) \vDash_I \mathsf{List}(l)) \wedge \\
&\qquad (\mathsf{granted}(\mathsf{List}(l)) \vDash_I \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{head} = 1) \\
&\iff \top\ \wedge\ (\{\mathsf{assumed}(\mathsf{List}(l))\} \vDash_I \texttt{unfolding}\ \mathsf{List}(l)\ \texttt{in}\ l.\mathit{head} = 1) \\
&\iff \top\ \wedge\ (\mathsf{granted}(l \neq \texttt{null}\ *\ \texttt{acc}(l.\mathit{head})\ *\ \texttt{acc}(l.\mathit{tail})\ * \\
&\qquad\quad \texttt{if}\ l.\mathit{tail} = \texttt{null}\ \texttt{then}\ \texttt{true}\ \texttt{else}\ \mathsf{List}(l.\mathit{tail})) \vDash_I \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(expansion of } \mathsf{assumed}(\mathsf{List}(l))) \\
&\qquad l.\mathit{head} = 1) \\
&\iff \top\ \wedge\ (\{\mathsf{accessed}(l.\mathit{head}), \mathsf{accessed}(l, \mathit{tail})\}) \vDash_I l.\mathit{head} = 1) \\
&\iff \top\ \wedge\ (\{\mathsf{accessed}(l.\mathit{head}), \mathsf{accessed}(l, \mathit{tail})\}) \vdash \mathsf{accessed}_{\phi_{\mathsf{root}}}(l.\mathit{head})) \\
&\iff \top\ \wedge\ \top \\
&\iff \top
\end{aligned}
$$

Example 5

Define

$$\phi_{\mathsf{root}} := \texttt{if } x = \texttt{null} \texttt{ then } \texttt{true} \texttt{ else } (\texttt{acc}(x.f) * x.f = 1)$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{x = \texttt{null} : \langle \{\mathsf{aliased}\,\{x, \texttt{null}\}\}\,,\ \varnothing\rangle, x \neq \texttt{null} : \langle \varnothing,\ \varnothing\rangle\}\rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}\,I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \texttt{if } x = \texttt{null} \texttt{ then } \texttt{true} \texttt{ else } (\texttt{acc}(x.f) * x.f = 1) \\
&\iff (\varnothing \vDash_I x = \texttt{null}) \ \wedge\ (\varnothing \vDash_I \texttt{true}) \ \wedge\ (\varnothing \vDash_I \texttt{acc}(x.f) * x.f = 1) \\
&\iff \top \ \wedge\ \top \ \wedge (\mathsf{granted}(x.f = 1) \vDash_I \texttt{acc}(x.f)) \ \wedge (\mathsf{granted}(\texttt{acc}(x.f)) \vDash_I x.f = 1) \\
&\iff \top \ \wedge\ \top \ \wedge (\varnothing \vDash_I \texttt{acc}(x.f)) \ \wedge (\{\mathsf{accessed}(x.f)\} \vDash_I x.f = 1) \\
&\iff \top \ \wedge\ \top \ \wedge (\varnothing \vDash_I x) \ \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{x.f}(x.f)) \ \wedge (\{\mathsf{accessed}(x.f)\} \vDash_I 1) \\
&\iff \top \ \wedge\ \top \ \wedge\ \top \ \wedge\ \top \ \wedge\ \top \\
&\iff \top
\end{aligned}$$

Example 6

Use the definition of List from example 4. Define

$$\begin{aligned}
\phi_{\mathsf{root}} &:= \mathtt{acc}(x.f) \;*\; \phi_1 \;*\phi_2 \\
\phi_1 &:= \mathtt{if}\ x.f = 1\ \mathtt{then}\ x = y\ \mathtt{else}\ \mathtt{true} \\
\phi_2 &:= \mathtt{if}\ x.f = 1\ \mathtt{then}\ y.f = 1\ \mathtt{else}\ \mathtt{true}
\end{aligned}$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{x.f = 1 : \langle \{\mathsf{aliased}\,\{x, y\}\}\,,\ \varnothing\rangle,\ x.f \neq 1 : \langle \varnothing,\ \varnothing\rangle\}\rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \mathtt{acc}(x.f) \;*\; \phi_1 \;*\phi_2 \\
&\iff (\mathsf{granted}(\phi_1 * \phi_2) \vDash_I \mathtt{acc}(x.f)) \;\wedge\; (\mathsf{granted}(\mathtt{acc}(x.f) * \phi_2) \vDash_I \phi_1) \wedge \\
&\qquad (\mathsf{granted}(\mathtt{acc}(x.f) * \phi_1) \vDash_I \phi_2) \\
&\iff (\mathsf{granted}(\phi_1 * \phi_2) \vDash_I x) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ x = y\ \mathtt{else}\ \mathtt{true}) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vDash_I \mathtt{if}\ x.f = 1\ \mathtt{then}\ y.f = 1\ \mathtt{else}\ \mathtt{true}) \\
&\iff \top \;\wedge\; (\{\mathsf{accessed}(x.f)\} \vDash_I (x.f = 1), (x = y), (\mathtt{true})) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vDash_I (x.f = 1), (y.f = 1), (\mathtt{true})) \\
&\iff \top \;\wedge\; (\{\mathsf{accessed}(x.f)\} \vDash_I x.f) \;\wedge\; (\{\mathsf{accessed}(x.f)\} \vDash_I x.f) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vDash_I y.f) \\
&\iff \top \;\wedge\; (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{\phi_{\mathsf{root}}}(x.f)) \wedge \\
&\qquad (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{y.f=1}(y.f)) \\
&\iff \top \;\wedge\; \top \;\wedge\; \top \qquad (\star) \\
&\iff \top
\end{aligned}$$

$(\star)$: $\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{y.f=1}(y.f) \iff \top$ since $\mathcal{A}(y.f = 1) \vdash \mathsf{aliased}\,\{x, y\}$ because $\mathcal{A}(y.f = 1)$ and $\mathcal{A}(x = y)$ are combined into a single branch of $\mathcal{A}(\phi_{\mathsf{root}})$, as they have the same conditions.

Example 7

Define

$$\phi_{\mathsf{root}} := \texttt{if}\ x = y\ \texttt{then}\ \texttt{acc}(x.f)\ \texttt{else}\ x.f = 2$$

Then

$$\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{x = y : \langle \{\mathsf{aliased}\,\{x, y\}\}\,,\ \varnothing\rangle,\ x \neq y : \langle \varnothing,\ \varnothing\rangle\}\rangle$$

And so,

$$\begin{aligned}
\vdash_{\mathsf{frm}I}\ \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\
&\iff \varnothing \vDash_I \texttt{if}\ x = y\ \texttt{then}\ \texttt{acc}(x.f)\ \texttt{else}\ x.f = 2 \\
&\iff \varnothing \vDash_I (x = y), (\texttt{acc}(x.f)), (x.f = 2) \\
&\iff \top\ \wedge\ (\varnothing \vDash_I x)\ \wedge\ (\varnothing \vDash_I x.f) \\
&\iff \top\ \wedge\ \top\ \wedge\ (\varnothing \vdash \mathsf{accessed}_{x.f=2}(x.f)) \\
&\iff \top\ \wedge\ \top\ \wedge\ \bot \\
&\iff \bot
\end{aligned}$$

Example 8

Define

$$\texttt{predicate}\ \mathsf{aliasChoice}(x,y,z) := \ x = y \mid\mid x = z$$

$$\phi_{\mathsf{root}} := \texttt{acc}(x.f) \ * \ \mathsf{aliasChoice}(x,y,z) \ * \ \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1$$
$$\phi_1 := y.f = 1 \mid\mid z.f = 1$$

Then

$$\begin{aligned}\mathcal{A}(\phi_{\mathsf{root}}) = \langle \varnothing,\ \{&\texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) : \\ &\{x = y : \langle \{\mathsf{aliased}\,\{x,y\}\}\,,\ \varnothing\rangle, \\ &\ x \neq y : \langle \{\mathsf{aliased}\,\{x,z\}\}\,,\ \varnothing\rangle, \\ &\ y.f = 1 : \langle \varnothing,\ \varnothing\rangle, \\ &\ y.f \neq 1 : \langle \varnothing,\ \varnothing\rangle\}\}\rangle\}\end{aligned}$$

Note that the $x = y \mid\mid x = z$ in the body of aliasChoice is translated to $\texttt{if}\ x = y\ \texttt{then}\ \texttt{true}\ \texttt{else}\ x = z$ when construction $\mathcal{A}(\phi_1)$. And so,

$$\begin{aligned}\vdash_{\mathsf{frm}\,I} \phi_{\mathsf{root}} &\iff \varnothing \vDash_I \phi_{\mathsf{root}} \\ &\iff \varnothing \vDash_I \texttt{acc}(x.f) \ * \ \mathsf{aliasChoice}(x,y,z) \ * \ \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1 \\ &\iff (\mathsf{granted}(\mathsf{aliasChoice}(x,y,z) \ * \ \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1) \vDash_I \texttt{acc}(x.f)) \ \wedge \\ &\qquad (\mathsf{granted}(\texttt{acc}(x.f) \ * \ \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1) \vDash_I \mathsf{aliasChoice}(x,y,z)) \ \wedge \\ &\qquad (\mathsf{granted}(\texttt{acc}(x.f) \ * \ \mathsf{aliasChoice}(x,y,z)) \vDash_I \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1) \\ &\iff \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I \mathsf{aliasChoice}(x,y,z)) \ \wedge \\ &\qquad (\{\mathsf{accessed}(x.f), \mathsf{assumed}(\mathsf{aliasChoice}(x,y,z))\} \vDash_I \texttt{unfolding}(\mathsf{aliasChoice}(x,y,z)) \ \texttt{in}\ \phi_1) \\ &\iff \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I y.f = 1 \mid\mid z.f = 1) \\ &\iff \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I y.f) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vDash_I z.f) \\ &\iff \top \ \wedge \ \top \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{y.f=1}(y.f)) \ \wedge \ (\{\mathsf{accessed}(x.f)\} \vdash \mathsf{accessed}_{z.f=1}(z.f)) \\ &\qquad\qquad (\star) \\ &\iff \top \wedge \top \wedge \bot \wedge \bot \\ &\iff \bot\end{aligned}$$

($\star$): The acccessed to $y.f, z.f$ are not framed because it is statically undetermined which branch of $x = y \mid\mid x = z$ will be taken. The case could arise that $x = z$ and then when checking the condition $y.f = 1$ there is not access to $y.f$. The idea of the original formula can be correctly captured in one of the following revisions:

$$\phi'_{\mathsf{root}} := \texttt{acc}(x.f) \ * \ (x = y \mid\mid x = z) \ * \ \texttt{if}\ x = y\ \texttt{then}\ y.f = 1\ \texttt{else}\ z.f = 1$$
$$\phi'_{\mathsf{root}} := \texttt{acc}(x.f) \ * \ \texttt{if}\ x = y\ \texttt{then}\ y.f = 1\ \texttt{else}\ (\texttt{if}\ x = z\ \texttt{then}\ z.f = 1\ \texttt{else}\ \texttt{false})$$

For example. the $z.f = 1$ will be framed because the aliasing context of the $x \neq y$ branch of $(x = y \mid\mid x = z)$ will be combined with the aliasing context of the $x \neq y$ branch of $(\texttt{if}\ x = y\ \texttt{then}\ y.f = 1\ \texttt{else}\ z.f = 1)$, yielding $\mathsf{aliased}\,\{x,z\}$ in $z.f = 1$. The similar case holds for the $x = y$ branches combining to allow the aliasing to frame $y.f = 1$.

5 Satisfiability

6 Implication

7 Weakest Predonditions