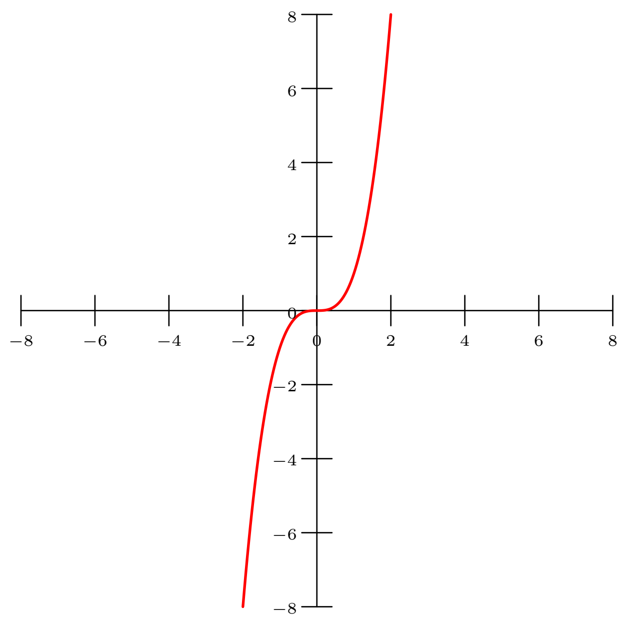
Term 4

Academic Notes

Author:
Dylan Yu

School:
DHS

Date:
May 19, 2020



Math is hard. And all other subjects too.

"Education is not preparation for life; education is life itself." – John Dewey

Contents

I

General

1 Introduction

This is just a collection of notes I made across high school. So far, it only has the notes from online schooling during COVID-19 in 2020 (9th grade year). Hope you enjoy!

§1.1 9th Grade

The following are the classes I took:

1. Precalculus (PreAP)
2. English 1 (PreAP)
3. World Geography (ON Level)
4. Spanish 3 (PreAP)
5. Physics 1 (AP)
6. Computer Science 2 (AP)
7. Biology (PreAP)

It is important to note that there are no notes for **English 1** and **World Geography** because I felt no reason to. The teachers had us do online school on a different website, and I didn't feel like taking notes for them.

It should be noted that I ended up taking a great part of Precalculus notes above all other notes. Sorry about that.

Update: As of 5.19.20, there are **116** examples and **178** problems. To the reader who actually plans to work through some of the problems, do about half of them. No need to fully commit to doing the problems.

II

Precalculus

2 Parametric Equations (4.3.20)

These are my solutions to the problems in the Precalculus Videos. I would recommend watching the actual videos and not just reading my solutions.

§2.1 Eliminating the Parameter

For linear parameters, simply just use elimination or substitution to eliminate the parameter.

Example 2.1.1
$x = 2t - 1$ and $y = 1 - t$

Solution. Multiply the second equation by 2:

$$2y = 2 - 2t,$$

then add this to the first equation:

$$x + 2y = 2 - 2t + 2t - 1 = 1,$$

$$y = -\frac{1}{2}x - \frac{1}{2}.$$

It is easy to graph from here.

Example 2.1.2
$x = \sqrt{t-1}, y = t + 2$

Solution. Square the first equation:

$$x^2 = t - 1,$$

then subtract this from the second equation:

$$y - x^2 = 3,$$

$$y = x^2 + 3.$$

Then plug in values of t and graph.

Example 2.1.3

$x = t^2 - 3, y = 2t$

Solution. Square the second equation, multiply the first by 4, and subtract:

$$y^2 - 4x = 4t^2 - 4(t^2 - 3) = 12,$$

$$y^2 = 4x + 12.$$

We can plug in values of t and solve from here.

With problems with trig functions, use your identities! The most popular one is $\sin^2\theta + \cos^2\theta = 1$.

Example 2.1.4

$x = 3 + 2\cos\theta, y = -1 + 3\sin\theta$

Solution. From moving the variables around, we get

$$\frac{x-3}{2} = \cos\theta,$$

$$\frac{y+1}{3} = \sin\theta.$$

Squaring and adding, we get

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1.$$

Now we can apply our knowledge of conics and graph. In fact, it might be a good idea to plug in values of θ to get the values of x, y.

Example 2.1.5

$x = 2\tan\theta + 3, y = 3\sec\theta - 1$

Solution. Solve for tan and sec:

$$\frac{x-3}{2} = \tan\theta,$$

$$\frac{y+1}{3} = \sec\theta.$$

Square and use $\tan^2\theta + 1 = \sec^2\theta$:

$$\frac{(y+1)^2}{9} - \frac{(x-3)^2}{4} = 1.$$

We can now graph by plugging in θ to get our x, y values.

The following examples are from the "Parametric Equations - The difference between the three graphs" section.

Example 2.1.6

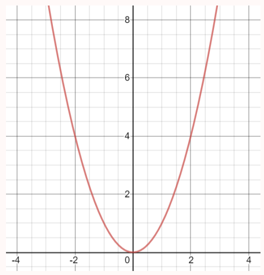
$x = t, y = t^2$

Solution. If we square the first equation, and set them equal, we get

$$x^2 = t^2 = y,$$

$$y = x^2.$$

The graph is the parent parabola, shown below:



Example 2.1.7

$x = e^t, y = e^{2t} - 1$

Solution. If we square the first equation and subtract one, we get

$$x^2 - 1 = e^{2t} - 1 = y,$$

$$y = x^2 - 1.$$

This is the same equation, but **there are restrictions** of the domain. The range of e^t is $(0, \infty)$, so the graph is the **right half of the parabola**.

Example 2.1.8

$x = \sin t, y = \sin^2 t - 1$

Solution. The range of $\sin t$ is $[-1, 1]$, so x is at most 1 and at least -1. Thus, the graph is the **parabola underneath the y-axis**.

§2.2 Graphing Parametric Equations

There are two ways:

1. Convert to rectangular then graph
2. Understand the general shape, then plug in values of t to get the specific shape.

I won't focus too much on graphing, because most graphs aren't new (at least not yet).

§2.3 Homework

Problem 2.3.1 — $x = \sqrt{t+1}, y = t+3$

Solution. If we square the first equation and subtract from the second, we get

$$x^2 = t+1,$$

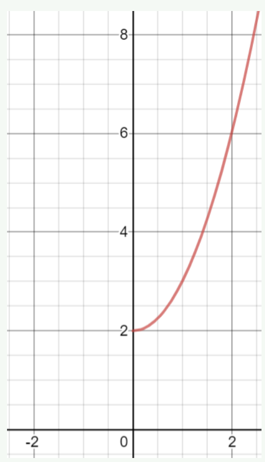
$$y - x^2 = 2,$$

$$y = x^2 + 2.$$

Now let us plug in some nice values for t:

t	x	y
0	1	4
3	2	6
8	3	11

The following is the graph:



Note that the restrictions of $\sqrt{t+1}$ give us only the right side of the parabola.

Problem 2.3.2 — $x = t+1, y = t^2$

Solution. If we solve for t in the first equation and plug it in to the second equation, we get

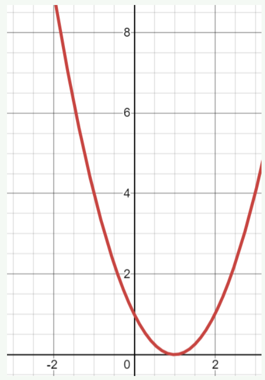
$$x - 1 = t,$$

$$(x-1)^2 = y.$$

Now let us plug in some nice values for t:

t	x	y
0	1	0
1	2	1
2	3	4

The following is the graph:



Problem 2.3.3 — $x = 1 - t^2, y = t + 1$

Solution. If we solve for t in the second equation, and substitute into the first we get

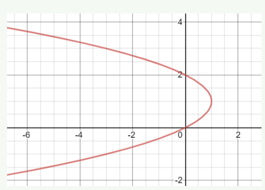
$$y - 1 = t,$$

$$x = 1 - (y - 1)^2.$$

Now let us plug in some nice values for t:

t	x	y
0	1	1
1	0	2
2	-3	3

The following is the graph:



Problem 2.3.4 — $x = t^3, y = \frac{t^2}{2}$

Solution. If we multiply the second equation by 2 and cube it, then square the first one, we get

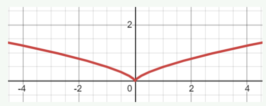
$$x^2 = t^3 = (2y)^3,$$

$$y = \frac{\sqrt[3]{x^2}}{2}.$$

Now let us plug in some nice values for t:

t	x	y
0	0	0
2	8	2
4	64	8

The following is the graph:



Problem 2.3.5 — $x = 2t, y = |t-1|$

Solution. If we divide the first equation by 2 and plug in t to the second, we get

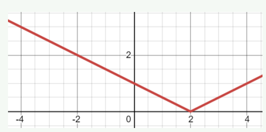
$$\frac{x}{2} = t,$$

$$y = |\frac{x}{2} - 1|.$$

Now let us plug in some nice values for t:

t	x	y
0	0	1
1	2	0
2	4	1

The following is the graph:



Problem 2.3.6 — $x = 3\sin\theta + 1, y = 2\cos\theta - 3$

Solution. Let us solve for sin and cos and use $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{x-1}{3} = \sin\theta,$$

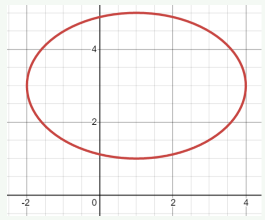
$$\frac{y+3}{2} = \cos\theta,$$

$$\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 1.$$

Now let us plug in some nice values for θ:

θ	x	y
0	1	-1
$\frac{\pi}{2}$	1	-3
π	1	-5

The following is the graph:



Problem 2.3.7 — $x = 2\cos\theta + 4, y = 4\sin\theta - 1$

Solution. Let us solve for sin and cos and use $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{x-4}{2} = \cos\theta,$$

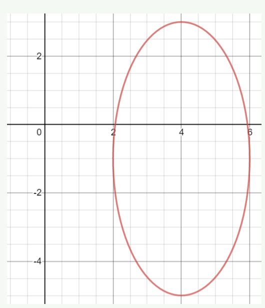
$$\frac{y+1}{4} = \sin\theta,$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1.$$

Now let us plug in some nice values for θ:

θ	x	y
0	6	-1
$\frac{\pi}{2}$	4	3
π	2	-1

The following is the graph:



Problem 2.3.8 — $x = 4\sec\theta - 1, y = 3\tan\theta + 2$

Solution. Let us solve for sec and tan and use $\sec^2\theta = \tan^2\theta + 1$:

$$\frac{x+1}{4} = \sec\theta,$$

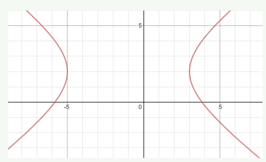
$$\frac{y-2}{3} = \tan\theta.$$

$$\frac{(x+1)^2}{16} - \frac{(y-2)^2}{9} = 1.$$

Now let us plug in some nice values for θ:

θ	$\boldsymbol{x}$	$\boldsymbol{y}$
0	3	2
$\frac{\pi}{4}$	$4\sqrt{2}-$ 1	5
π	-5	2

The following is the graph:



3 Parametric Equations with Restrictions (4.6.20)

§3.1 Examples

Example 3.1.1

$x = 1 + \frac{1}{t}, y = 1 - \frac{1}{t}$

Solution. If we add the equations, we get

$$x + y = 2,$$

$$y = 2 - x.$$

There is a problem though - we have to check for **holes.** We can do this by considering whne $1 + \frac{1}{t}$ and $1 - \frac{1}{t}$ are undefined. This occurs when $t = 0$, meaning there is a hole at $(1, 1)$. Thus, the graph is the line $y = 2 - x$ with a hole at $(1, 1)$.

How I determined that it was $(1, 1)$ that didn't exist is by solving for t using both equations. If we use the first equation, we get

$$t = \frac{1}{x - 1},$$

and the second equation gives us

$$t = \frac{1}{1 - y}.$$

Thus, it becomes apparent $x \neq 1, y \neq 1$, and therefore the hole is at $(1, 1)$.

Example 3.1.2

$x = |t|, y = |t|$

Solution. It is apparent that

$$x = y.$$

However, $|t| \geq 0$, so $x, y \geq 0$. Thus, the graph is just the first quadrant part of $Y = x$.

Note that in both of these examples, we plug in values of t to verify it works. Technically, we could just graph it, but in order to understand how we move with respect to t we must graph the points. In the last example, we moved from the rightmost point to the origin (the leftmost pojnt) and back to infinity (the rightmost point).

Example 3.1.3

$x = \frac{6}{t-2}, y = \frac{-5}{t+3}$

Solution. Let us solve for t in both equations:

$$t = \frac{6}{x} + 2,$$

$$t = -\frac{5}{y} - 3.$$

Both of these show $x, y \neq 0$. Thus, there is a hole at $(0, 0)$. Now, let us set them equal and solve:

$$y = \frac{-5}{\frac{6}{x} + 2 + 3} = \frac{-5x}{6 + 5x}.$$

This is a hyperbola (or more accurately a rotated one such that the asymptotes are vertical and horizontal), so the vertical asymptote is $x \neq -\frac{6}{5}$, and the horizontal asymptote $y \neq -1$. Now we plug in the values and find the orientation of the graph.

Example 3.1.4

Graph the parametric equations below for $-1 \leq t \leq 4$. Make a table, plot the points and identify the type of graph. Check your answer below.

$$x = t + 1,$$

$$y = 3t - 1.$$

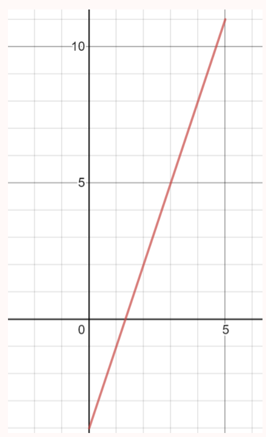
Solution. If we multiply the first equation by 3 and subtract the equations we get

$$3x = 3t + 3,$$

$$3x - y = 4,$$

$$y = 3x - 4.$$

Now for the domain. If we plug in the integral values from -1 to 4 for t, we quickly realize the domain is just $[0, 5]$. Thus, the endpoints are $(0, -4)$ and $(5, 11)$. From here, we can determine the graph as



§3.2 Homework

Problem 3.2.1 — $x = 2 - \frac{1}{t}, y = t + \frac{1}{t}$

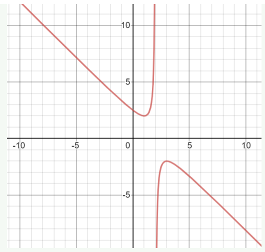
Solution. If we solve for t in the first equation we get

$$t = \frac{1}{2 - x}.$$

If we plug this in the the second equation, we get

$$y = \frac{1}{2 - x} + 2 - x = \frac{5 - 4x + x^2}{2 - x}.$$

Thus, we have a hyperbola with Vertical Asymptote $x = 2$, and slant asymptote $y = -x + 2$. If we graph this, we get



Note that the hole at $t = 0$ actually lies on the asymptote, so we do not need to draw any actual holes.

Problem 3.2.2 — $x = t^2, y = 1 + t$

Solution. If we solve for t in the second equation and plug it into the first we get

$$y - 1 = t,$$

$$x = (y - 1)^2.$$

Thus, the shape is a sideways parabola. This implies the graph is

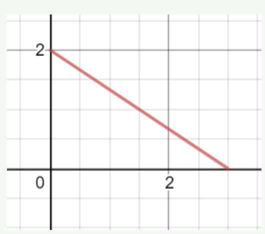
Problem 3.2.3 — $x = 3\cos^2 t, y = 2\sin^2 t$

Solution. If we divide the first equation by 3 and the second by 2 and add them, we get

$$\frac{x}{3} + \frac{y}{2} = 1,$$

$$y = -\frac{2x}{3} + 2.$$

We know that $\sin t, \cos t$ have the range $[-1, 1]$, so the endpoints of the line are $(3, 0)$ and $(0, 2)$. Thus, the graph is

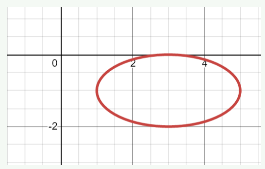


Problem 3.2.4 — $x = 3 + 2\cos t, y = -1 + \sin t$

Solution. If we solve for $\sin t, \cos t$ and square them and add the equations, we get

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{1} = 1,$$

so the graph is an ellipse. However, $\sin t, \cos t$ have the range $[-1, 1]$, which turns out to not affect the outcome. Thus, the graph is



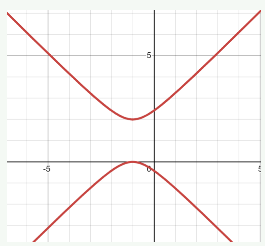
Problem 3.2.5 — $x = -1 + \tan t, y = 1 + \sec t$

Solution. If we solve for $\tan t, \sec t$ and use the $\sec^2 t = \tan^2 t + 1$ formula, we get

$$(y-1)^2 = (x+1)^2 + 1,$$

$$\frac{(y-1)^2}{1} - \frac{(x+1)^2}{1} = 1.$$

Thus, the graph is



Problem 3.2.6 — $x = h + r\cos t, y = k + r\sin t$

Solution. If we solve for $\cos t, \sin t$ we get

$$\frac{x-h}{r} = \cos t,$$

$$\frac{y-k}{r} = \sin t.$$

Using $\cos^2 t + \sin^2 t = 1$, we get

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1,$$

$$(x-h)^2 + (y-k)^2 = 1.$$

Thus, this is simply the general equation of a **circle**.

Problem 3.2.7 — $x = h + a\cos t, y = k + b\sin t$

Solution. If we solve for $\cos t, \sin t$ and apply the formula once again, we get

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

which is the general equation of an **ellipse**. Intuitively, we could've seen this simply because the only thing that changed was the r value became different, so it was simply a streched circle, i.e. an ellipse.

Problem 3.2.8 — $x = h + a\sec t, y = k + b\tan t$

Solution. If we solve for $\sec t, \tan t$ and use $\sec^2 t - \tan^2 t = 1$, we get

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

which is the general equation of a **hyperbola**.

4 Rectangular to Parametric Equations (4.7.20)

§4.1 Examples

Example 4.1.1

$9x^2 + 4y^2 - 54x - 8y + 49 = 0$

Solution. By solving it as we would any other conic, we get

$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{9} = 1.$$

Now, if we let the first part equal $\sin^2 t$ and the second part equal $\cos^2 t$, we get

$$x = 2\sin t + 3, y = 3\cos t + 1.$$

Similarly, we could've done it the other way around:

$$x = 2\cos t + 3, y = 3\sin t + 1.$$

The function remains the same, but how we loop around changes a little (i.e. the direction goes from clockwise to counterclockwise).

Example 4.1.2

The foci are $(3 \pm 2\sqrt{10}, 5)$ and center $(3, 5)$. If $c = 2\sqrt{10}$, determine the equation of the ellipse.

Solution. From othe points we can tell it is a **horizontal ellipse**. Thus, because $c = 2\sqrt{10}$, we can determine b as 3. Using $a^2 = c^2 + b^2$ we get

$$a^2 = (2\sqrt{10})^2 + 3^2 = 49,$$

$$a = 7.$$

Thus, the equation is

$$\frac{(x-5)^2}{49} + \frac{(y-5)^2}{9} = 1.$$

Now, when we solve the equation, we have **8 possibilities**: two ways to order the sin and cos, and each of these has two possibilities for the sign, so $2 \cdot 2 \cdot 2 = 8$ ways. I will not list them all out, but rather just write

$$\frac{x-3}{7} = \pm\sin t, \frac{y-5}{3} = \pm\cos t,$$
$$\frac{x-3}{7} = \pm\cos t, \frac{y-5}{3} = \pm\sin t.$$

Example 4.1.3

A particle moves along a circular path starting at $(2, -9)$ in a counterclockwise direction. The circle has a radius of 5 with a center at $(2, -4)$. Write a parametric equation that describes the motion of the particle.

Solution. We can easily determine the equation of the circle in rectangular coordinates as

$$(x-2)^2 + (y+4)^2 = 5^2,$$

$$\frac{(x-2)^2}{25} + \frac{(y+4)^2}{25} = 1.$$

Now, we plug in the important points ($t = 0, \frac{\pi}{2}$, etc.) and we get

$$x = 5\sin t + 2, -5\cos t - 4.$$

§4.2 Homework

Problem 4.2.1 — $9x^2 + 16y^2 - 18x + 64y - 71 = 0$

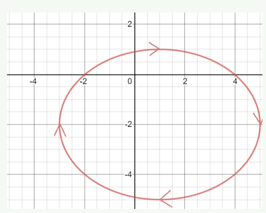
Solution. If we solve this conic, we get

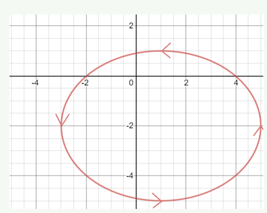
$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1.$$

Thus, the two possibilities are

$$x = 4\sin t + 1, y = 3\cos t + 2,$$

$$x = 4\cos t + 1, y = 3\sin t + 2.$$





Write the equation of the conic in standard form and then convert the equation to two different sets of parametric equations. For each sketch the graph of the path and indicate the orientation.

Problem 4.2.2 — Circle with center $(1, 5)$ and passes through the point $(7, 2)$

Solution. We can easily tell the distance between these two points is $\sqrt{45}$, so $r = \sqrt{45}$. Thus,

$$\frac{(x-1)^2}{45} + \frac{(y-5)^2}{45} = 1.$$

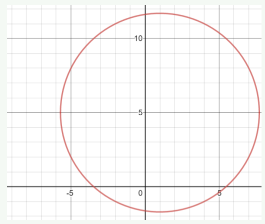
Thus,

$$x = 3\sqrt{5}\sin t + 1, y = 3\sqrt{5}\cos t + 1,$$

or

$$x = 3\sqrt{5}\cos t + 1, y = 3\sqrt{5}\sin t + 1.$$

The former is clockwise and the latter is counterclockwise.



Problem 4.2.3 — Ellipse with foci $(1, 4 \pm 2\sqrt{3})$ and endpoints of the minor axis length of $(3, 4)$ and $(-1, 4)$

Solution. We can easily determine $b = 2, c = 2\sqrt{3}$, so

$$a^2 = b^2 + c^2 = 16,$$

$$a = 4.$$

Thus, the equation is

$$\frac{(x-1)^2}{4} + \frac{(y-4)^2}{16} = 1.$$

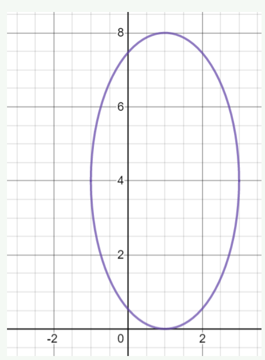
We can now convert:

$$x = 2\sin t + 1, y = 4\cos t + 4,$$

or

$$x = 2\cos t + 1, y = 4\sin t + 4.$$

The former is clockwise and the latter is counterclockwise.

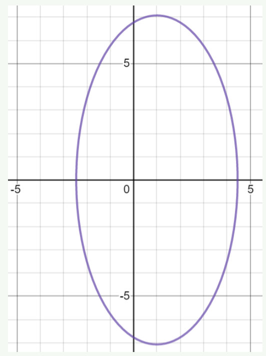


Sketch the graphs of the following parametric equations and identify the orientation. Then convert them to a rectangular equation.

Problem 4.2.4 — $x = -2\sqrt{3}\sin t + 1, y = 5\sqrt{2}\cos t$

Solution. Let us solve for sin and cos and use the formula $\sin^2 t + \cos^2 t = 1$:

$$\frac{(x-1)^2}{12} + \frac{y^2}{50} = 1.$$

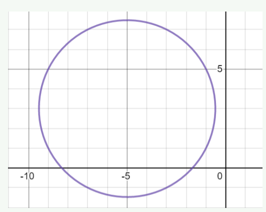


The figure is **counterclockwise**.

Problem 4.2.5 — $x = -2\sqrt{5}\cos t - 5, y = -2\sqrt{5}\sin t + 3$

Solution. Let us solve for sin and cos and use the formula $\sin^2 t + \cos^2 t = 1$:

$$\frac{(x+5)^2}{20} + \frac{(y-3)^2}{20} = 1.$$



The figure is **counterclockwise**.

Write a parametric equation that describes the motion of the particle.

Problem 4.2.6 — A particle moves along a horizontal elliptical path whose major axis is 12 and a minor axis of 8 units long. The circle center of the elliptical path is $(-5, 7)$. The particle starts at the right most point on the path moving counterclockwise.

Solution. We can easily determine

$$a = 6, b = 4.$$

The center is at $(-5, 7)$. Thus, we can solve in terms of t directly:

$$x = -5 + 6\cos t, y = 7 + 4\sin t.$$

Problem 4.2.7 — A circle with a radius of 4 and in the second quadrant is tangent to both x and y axes. A particle moves along the circle clockwise starting on the x-axis.

Solution. The equation of the circle is

$$(x+4)^2 + (y-4)^2 = 16,$$

so we can solve to get

$$x = -4 - 4\sin t, y = 4 - 4\cos t.$$

Problem 4.2.8 — An elliptical path has foci $(0, \pm 3)$ and the eccentricity is $\frac{3}{4}$. A particle starts at the top of the ellipse to travel along the path in a clockwise direction.

Solution. We know that

$$c = 3, e = \frac{3}{4}, a = 4,$$

so solving for b we get

$$b = \sqrt{7}.$$

Thus,

$$\frac{x^2}{7} + \frac{y^2}{16} = 1.$$

From here,

$$x = \sqrt{7}\sin t, y = 4\cos t.$$

5 Parametric Applications - Ferris Wheel (4.8.20)

§5.1 Equations

For a **Ferris wheel** the movement is circular, so we use the parametric form of a circle to determine our position.

Theorem 5.1.1 (Circle Parameterization)
$x = h \pm r\sin\theta, y = h \pm r\cos\theta$ or $x = h \pm r\cos\theta, y = h \pm r\sin\theta$.

§5.2 Examples

Example 5.2.1
Jane is riding on a Ferris wheel with a radius of 30 feet. The wheel turns counterclockwise at a rate of one revolution every 10 seconds. Assume the lowest point of the Ferris wheel is 10 feet above the ground and that Jane is $\frac{3}{4}$ of the way through the revolution (position 3 o' clock) at time $t = 0$ seconds.

a) Determine the parametric equations that model Jane's path.
b) Determine her position when $t = 22$ seconds.

Solution.
a) She is at $(30, 40)$ when $t = 0$. Thus,

$$x = 0 + 30\cos\theta,$$

$$y = 40 + 30\sin\theta.$$

Note that

x: the horizontal position depends on the changing angle measure related to time.
y: the vertical position depends on the changing angle measure related to time.

Thus, if one revolution takes 10 seconds, then 2π radians takes 10 seconds, so $\Delta\theta = 36°/\text{sec}$. Therefore,

$$x(t) = 0 + 30\cos(\frac{\pi}{5}t),$$

$$y(t) = 40 + 30\sin(\frac{\pi}{5}t).$$

b) We can plug in $t = 22$ to get the answer of $x = 9,271$ ft, $y = 68.532$ ft

Example 5.2.2

A Ferris wheel has a diameter of 70 meters long and its highest point is 74 meters above the ground. The Ferris wheel rotates clockwise at a speed of 13 minutes per revolution. A capybara starts her ride on the Ferris wheel at the lowest point.

a) Find a parametric equation that represents the capybara's position at any time t in minutes.
b) What will be her height above the ground after 35 minutes?
c) And how long does it take her to reach a height of 65 meters?

Solution.

a) We can use the method in the past example to find the parametric equations:

$$\Delta\theta = \frac{2\pi}{13}\text{rad/min},$$

$$x(t) = -35\sin(\frac{2\pi}{13}t),$$

$$y(t) = 39 - 35\cos(\frac{2\pi}{13}t).$$

b) We can determine the height at $t = 35$ minutes by adjusting the domain of t and use `TRACE` $t = 35$. We can also just plug in the value into $y(t)$.

c) To use the `ANALYZE GRAPH` feature to find t when $y = 65m$, rewrite the equations as functions where x represents t (time). We get approximately 4.982 meters.

§5.3 Homework

Problem 5.3.1 — A Ferris wheel with a radius of 16 meters makes one complete revolution in 2 minutes and it is 2 meters off the ground at its lowest point. A rider starts at the bottom of the Ferris Wheel and rotates in the anticlockwise direction.

a) Determine the parametric equations that model the Ferris wheel's path.
b) Find the position of the rider after 10 sec and after 70 sec
c) Find the times, within the first revolution, when the rider is at the height of 10 meters.

Solution. The equation can easily be graphed as

$$\frac{x^2}{256} + \frac{(y-18)^2}{256} = 1.$$

Thus, because the period is $\frac{2\pi}{b}$, we know that $b = \frac{\pi}{60}$ for the sine graph.

a) From our steps above we can easily deduce

$$x = 16\sin(\frac{\pi}{60}t),$$

$$y = -16\cos(\frac{\pi}{60}t) + 18.$$

b) When $t = 10$ sec, we have $x = 8m, y = 4.134m$. When $t = 70$ sec, we have $x = -8m, y = 31.856m$.

c) We must use rectangular equations, so graph the two and find the intersections. We get $t = 20$ sec and $t = 100$ sec.

Problem 5.3.2 — Carter is on a Ferris wheel of radius 35 ft. that turns clockwise at the rate of one revolution every 5 minutes. The lowest point of the Ferris wheel is 15 feet above ground level at the point $(0, 15)$ on a rectangular coordinate system.

a) Find parametric equations for the position of Carter as a function of time t (in minutes) if the Ferris wheel starts with Carter at the point $(35, 50)$.

b) Find the Carter position after 12 minutes.

c) Find the third time, in minute, that Carter's height is at 20 ft.

Solution. We can easily find

$$\frac{x^2}{1225} + \frac{(y-50)^2}{1225} = 1,$$

So if the period is $\frac{2\pi}{b} = 5$, then $b = \frac{2\pi}{5}$.

a) From the equations above we can easily get

$$x = 35\cos(\frac{2\pi}{5}t),$$

$$y = -35\sin(\frac{2\pi}{5}t) + 50.$$

b) When $t = 12$ minutes, $x = -28.316$ ft, $y = 29.428$ ft.

c) The third time (calculator) we get $t = 5.431$ minutes.

6 Projectile Applications of Parametric Equations (4.9.20)

§6.1 Definitions

§6.1.1 Steps to Solving Projectile Motion Questions

1. Create parametric equations using v_0 (the initial velocity), h_0 (the initial height in feet), and θ (the measure of the angle in degrees at which the object is projected)

2. Graph on the calculator

3. Use equations and the graph to answer the questions asked

§6.1.2 Parameterized Parabolas

Theorem 6.1.1 (Parabola in Parametric Form)
Horizontal position is described by the equation $x = (v_0 \cos\theta)t$
Vertical position is described by the equation $y = h_0 + (v_0 \sin\theta)t - 16t^2$
Notice that the equation is written in units of feet and seconds.

§6.2 Examples

Example 6.2.1
Find the maximum height and distance if a rocket is projected from the ground at an angle $\theta = 60°$ and its initial velocity is 88 feet/sec.

Solution. We can solve the equations to get

$$x(t) = 88t \cos 60°,$$

$$y(t) = 0 + 88t \sin 60° - 16t^2.$$

Then we graph (make sure the calculator is set to degrees and that you put multiplication symbols between your variables!) as two separate functions and we get the maximum height as 90.75 ft at approximately 2.382 sec, and the distance travelled horizontally from the initial position is 209.578 ft.

Example 6.2.2

A baseball leaves the bat with an initial velocity of 152 ft/sec at an angle of elevation of 20°. The ball is 3 ft above the ground when it is hit. A 20 ft fence is 400 ft away from home plate.

A) Write the parametric equations to model the motion of the ball.

B) Sketch the graph.

C) Will the ball go over the fence? Explain.

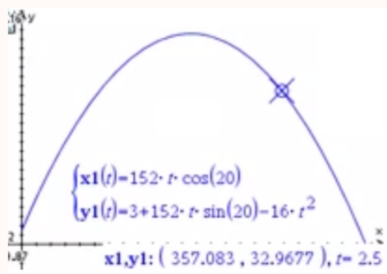
Solution.

A) The equations are pretty simple:

$$x(t) = 152t \cos 20^\circ,$$

$$y(t) = 3 + 152t \sin 20^\circ - 16t^2.$$

B) Using the calculator, we get



C) When the ball is 400 ft away from home plate, we can solve for the time. Then we can plug the time into the $y(t)$ to see if it is above 20 ft. In this case, the answer is yes.

Notice that to find the maximum horizontal distance, we simply set $y(t) = 0$ and solve for the two values of t, then plug them into the $x(t)$ equation and subtract their values. If wanted the maximum height, we would the $-\frac{b}{2a}$ formula to solve for t then plug it in to get the maximum height.

§6.3 Homework

Problem 6.3.1 — A ball hit in golf leaves the club at 110 ft/sec. If the ball leaves the club at an angle of 35° with the horizontal.

a) Set up the parametric equations represent the ball.

b) Find the maximum distance that the ball will travel.

c) When will the ball hit the ground?

d) What is the maximum height that the ball reaches?

e) At what time will the ball each maximum height?

Solution.

a) Using the equations, we get

$$x(t) = 110 \cos 35^\circ t,$$

$$y(t) = 110 \sin 35^\circ t - 16t^2.$$

b) When the ball hits the ground, the height is 0, which occurs at $t = 3.943$ sec, which gives us a distance of 355.318 ft.

c) The ball hits the ground at $t = 3.943$ sec.

d) The maximum height is 62.2 ft.

e) The time to reach the maximum height $t = 1.972$ sec.

Problem 6.3.2 — Catherine hits a baseball at 3 ft above the ground with an initial air speed of 150 ft/sec at an angle of 41°.

a) Find the parametric equations represent the baseball.

b) What is the ball height at 2 sec?

c) There is a 120 foot tall fence that is 400 feet away. Will the ball clear it? Explain your answer.

Solution.

a) The general parametric equations give us

$$x(t) = 150\cos 41^\circ t,$$

$$y(t) = 150\sin 41^\circ t - 16t^2 + 3.$$

b) If we plug in $t = 2$, we get 1335.818 ft.

c) We simply solve for t using

$$t = \frac{400}{150\cos 41^\circ},$$

and we plug this into equation $y(t)$ to get 150.960 ft, meaning the ball clears the fence.

Problem 6.3.3 (My Own Problem) **—** A ball is hit at 50 ft/sec at an angle of 60°. Find the maximum height reached by the ball in feet.

Solution. Using the parametric equations formula, we get

$$x(t) = 50\cos 60^\circ t,$$

$$y(t) = 50\sin 60^\circ t - 16t^2.$$

The maximum height is at $-\frac{b}{2a}$, so

$$-\frac{b}{2a} = -\frac{-16}{2\cdot 50\sin 60^\circ}.$$

Plugging this in, we get

$$y(t) = -\frac{-16}{2\cdot 50\sin 60^\circ}\cdot 50\sin 60^\circ - 16\left(-\frac{-16}{2\cdot 50\sin 60^\circ}\right)^2 = 7.454 \text{ ft}$$

7 Polar Coordinates and Converting (4.14.20)

§7.1 Definitions

The coordinates in the polar plane is written as (r, θ), where r is the distance from the pole (the center), i.e. the radius, and θ is the direction angle with respect to the x-axis.

§7.2 Examples

Example 7.2.1

Determine two additional ways to express this point in polar form.

$$\left(-1, \frac{2\pi}{3}\right)$$

Solution. If we let the radius be positive, we can have

$$\left(1, -\frac{\pi}{3}\right)$$

or

$$\left(1, \frac{5\pi}{3}\right)$$

Theorem 7.2.2 (Polar to Rectangular)

Let (r, θ) be the polar coordinate of a point. Then in rectangular coordinates we have

$$(r\cos\theta, r\sin\theta).$$

Example 7.2.3

Convert the following to rectangular form:

$$\left(-1, \frac{2\pi}{3}\right).$$

Solution. Using the formula, we have

$$(-\cos\frac{2\pi}{3}, -\sin\frac{2\pi}{3}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Theorem 7.2.4 (Rectangular to Polar)
Let (x, y) be the rectangular coordinate of a point. Then in polar coordinates we have

$$\left(\sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x}\right).$$

Example 7.2.5
Convert the following rectangular point below to polar coordinates:

$$(-3, 6).$$

Solution. Plugging the values into the formula, we have

$$(3\sqrt{5}, -1.107 \text{ rad}).$$

Theorem 7.2.6 (Rectangular to Polar Equations)
Simply replace x and y with the polar version $(r \cos\theta, r \sin\theta)$ and solve.

§7.3 Homework

The problem numbers correspond to the ones given in Schoology. Most of these are simply an application of the formula, so not much work is needed.

Problem 7.3.1 (Problem 3) — We can simply either switch the direction in which we count the angle or reverse the direction/sign of the radius:

$$\left(4, -\frac{3\pi}{4}\right), \left(-4, \frac{\pi}{4}\right).$$

Problem 7.3.2 (Problem 8) — This is easily convertible to

$$(2, 210^\circ).$$

Problem 7.3.3 (Problem 10) — Using the formula, we get

$$\left(4, \frac{\pi}{2}\right).$$

Problem 7.3.4 (Problem 14) — Using the formula, we get

$$\left(2, \frac{4\pi}{3}\right).$$

Problem 7.3.5 (Problem 15) — Using the formula, we get

$$\left(4, \frac{2\pi}{3}\right).$$

Problem 7.3.6 (Problem 21) — Using the formula, we get

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Problem 7.3.7 (Problem 22) — Using the formula, we get

$$\left(-\frac{3}{2}, -\frac{3\sqrt{2}}{2}\right).$$

Problem 7.3.8 (Problem 25) — If we plug in $y = r\sin\theta$, we get

$$r\sin\theta = -4,$$

$$r = -4\csc\theta.$$

Problem 7.3.9 (Problem 26) — If we plug in $x = r\cos\theta, y = r\sin\theta$, we get

$$2r\cos\theta - r\sin\theta = 3,$$

$$r = \frac{3}{2\cos\theta - \sin\theta}.$$

Problem 7.3.10 (Problem 30) — If we plug in $x = r\cos\theta, y = r\sin\theta$, we get

$$(r\sin\theta)^2 = 4r\cos\theta,$$

so

$$r = 0$$

or

$$r = -4\cos\theta\csc^2\theta.$$

Problem 7.3.11 (Problem 32) — If we plug in $x = r\cos\theta, y = r\sin\theta$, we get

$$(r\cos\theta)^2 + (r\sin\theta)^2 + 4r\cos\theta = 0,$$

$$r^2 = -4r\cos\theta,$$

so

$$r = 0$$

or

$$r = -4\cos\theta.$$

Problem 7.3.12 (Problem 33) — If we multiply by r, we get

$$r^2 = 2r\cos\theta = 2x,$$

but

$$r^2 = x^2 + y^2,$$

so

$$x^2 + y^2 - 2x = 0$$

is the equation.

Problem 7.3.13 (Problem 36) — If we square it, we get

$$r^2 = 16,$$

but

$$r^2 = x^2 + y^2,$$

so

$$x^2 + y^2 = 16.$$

Problem 7.3.14 (Problem 37) — If we multiply r, we get

$$r^2 = 2r\sin\theta + 2r\cos\theta = 2x + 2y,$$

but

$$r^2 = x^2 + y^2,$$

so

$$x^2 + y^2 - 2x - 2y = 0$$

is the equation.

Problem 7.3.15 (Problem 38) — We know that $r \sin\theta = y$, so

$$y = 6$$

is the equation.

Problem 7.3.16 (Problem 42) — We know that

$$\tan\theta = \frac{y}{x},$$

so the equation must be

$$y = x.$$

8 Converting between Polar and Rectangular Equations (4.15.20)

§8.1 Examples

Example 8.1.1
$y = \frac{1}{6}x^2$
Solution. Let $x = r\cos\theta, y = r\sin\theta$. Then

$$6r\sin\theta = r^2\cos^2\theta,$$

so

$$r - 0$$

or

$$r = 6\tan\theta.$$

Example 8.1.2
$x^2 + y^2 = 4xy$
Solution. Let $x = r\cos\theta, y = r\sin\theta$. Then

$$r^2 = 4(r\cos\theta)(r\sin\theta),$$

so either

$$r = 0$$

or

$$1 = 4\cos\theta\sin\theta,$$

$$1 = 2\sin 2\theta,$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Example 8.1.3

$\sin 2\theta = \cos 2\theta$

Solution. If we bring all variables to one side, we get

$$\tan 2\theta = 1,$$

so

$$\theta = \frac{k\pi}{8},$$

where $k = 1, 5, 9, 13$. Thus, either

$$y = \tan$$

Example 8.1.4

$r = \csc\theta + \sec\theta$

Solution. We can take the common denominator to get

$$r = \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta},$$

$$r(r\sin\theta\cos\theta) = (\cos\theta + \sin\theta)r,$$

$$xy = x + y,$$

$$y = \frac{x}{x-1}.$$

§8.2 Homework

Problem 8.2.1 — $y = -\sqrt{3}x$

Solution. This is equivalent to

$$\frac{y}{x} = -\sqrt{3} = \tan^{-1}\theta,$$

so

$$\theta = \frac{2\pi}{3}, \frac{-\pi}{3}, \frac{5\pi}{3}.$$

Problem 8.2.2 — $x^2 + (y-2)^2 = 4$

Solution. Plugging in $x = r\cos\theta, y = r\sin\theta$, we get

$$r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta = 0,$$

$$r(r - 4\sin\theta = 0,$$

so

$$r = 0$$

or

$$r = 4\sin\theta.$$

Problem 8.2.3 — $y = -\frac{4}{3} + 2$

Solution. Just plug in $x = r\cos\theta, y = r\sin\theta$ to get

$$r = \frac{2}{\sin\theta + \frac{4}{3}\cos\theta}.$$

Problem 8.2.4 — $(x^2 + y^2)^2 = 3x$

Solution. Note that $x^2 + y^2 = r^2$ and $x = r\cos\theta$. Thus,

$$(r^2)^2 - 3r\cos\theta = 0,$$

so

$$r = 0$$

or

$$r = \sqrt[3]{3\cos\theta}.$$

Problem 8.2.5 — $r^2 = \sin 2\theta$

Solution. If we multiply by r^2 and use the formulas above, we get

$$r^4 = 2r\sin\theta \cdot r\cos\theta,$$

$$(x^2 + y^2)^2 = 2xy.$$

Problem 8.2.6 — $r = \csc\theta - \sin\theta$

Solution. If we multiply by $\sin\theta$, we get

$$r\sin\theta = 1 - \sin^2\theta = \cos^2\theta,$$

so multiplying by $r^2 = x^2 + y^2$, we get

$$(x^2 + y^2)y = r^2\cos^2\theta = x^2,$$

so

$$x = \pm\sqrt{\frac{y^3}{1 - y}}.$$

Problem 8.2.7 — $r = -4\csc\theta$

Solution. If we multiply by $\sin\theta$, we get

$$r\sin\theta = -4,$$

so

$$y = -4.$$

Problem 8.2.8 — $r = \frac{1}{1+\cos\theta}$

Solution. If we bring the variables to one side, we get

$$r + r\cos\theta = 1,$$

$$\sqrt{x^2 + y^2} + x = 1,$$

$$y^2 = -2x + 1.$$

9 Graphing Polar: Circles and Limacons

§9.1 Examples

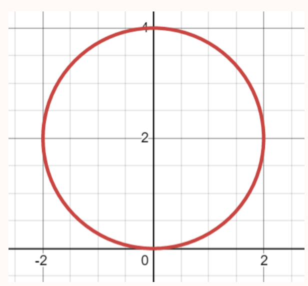
Theorem 9.1.1 (Circle Polar Equation)
A circle has the polar equation
$$r = a\sin\theta$$
or
$$r = a\cos\theta.$$

Example 9.1.2
Graph the polar function $r = 4\sin\theta$. Make a table of values.

Solution. This is the table of values:

θ	$\sin\theta$	r
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{2}$	1	4
$\frac{5\pi}{6}$	$\frac{1}{2}$	2
π	0	0

And here is the graph:



Theorem 9.1.3 (Limacon Polar Equation)
A limacon has polar equation
$$r = a \pm b\sin\theta$$
or
$$r = a \pm b\cos\theta.$$

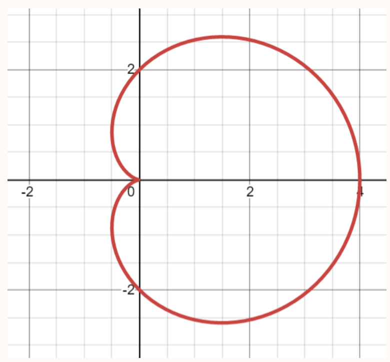
Example 9.1.4

Graph the polar function $r = 2 + 2\cos\theta$. Make a table of values.

Solution. This is the table of values:

θ	$\cos\theta$	r
0	1	4
$\frac{\pi}{3}$	$\frac{1}{2}$	3
$\frac{\pi}{2}$	0	2
$\frac{2\pi}{3}$	$-\frac{1}{2}$	1
π	-1	0
$\frac{4\pi}{3}$	$\frac{1}{2}$	3
2π	1	4

And here is the graph:



Thus, this is a **cardioid** (heart shaped).

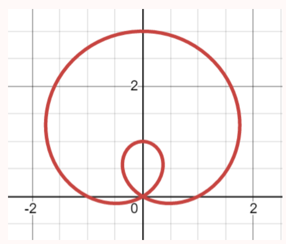
Example 9.1.5

Graph the polar function $r = 1 + 2\sin\theta$. Make a table of values.

Solution. This is the table of values:

θ	$\sin\theta$	r
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{2}$	1	3
$\frac{5\pi}{6}$	$\frac{1}{2}$	2
π	0	1
$\frac{7\pi}{6}$	$-\frac{1}{2}$	0
$\frac{3\pi}{2}$	-1	-1
$\frac{11\pi}{6}$	$-\frac{1}{2}$	0
2π	0	1

And here is the graph:



Thus, this is a **limacon with an inner loop**.

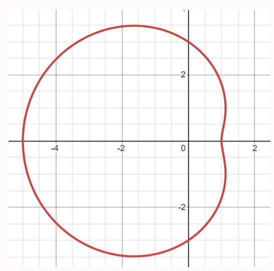
Example 9.1.6

Graph the polar function $r = 3 - 2\cos\theta$. Make a table of values.

Solution. This is the table of values:

θ	$\cos\theta$	r
0	1	1
$\frac{\pi}{3}$	$\frac{1}{2}$	2
$\frac{\pi}{2}$	0	3
$\frac{2\pi}{3}$	$-\frac{1}{2}$	4
π	-1	5
$\frac{4\pi}{3}$	$-\frac{1}{2}$	4
$\frac{3\pi}{2}$	0	3
$\frac{5\pi}{3}$	$\frac{1}{2}$	2
2π	1	1

And here is the graph:



Thus, this is a **convex limacon**. They technically have two types, but for now this is the terminology we will use.

A few notes: look at the values of radians used, and compare it to trigonometric function used. See the similariites?

§9.2 Types of Limacons

A limacon with an inner loop occurs when $\frac{a}{b} < 1$. A cardioid occurs when $\frac{a}{b} = 1$. A convex limacon occurs when $\frac{a}{b} \geq 2$. A limacon that is symmetric with the x-axis (horizontal) occurs when we use cos. A limacon that is symmetric with the y-axis (vertical) occurs when we use sin.

§9.3 Problems

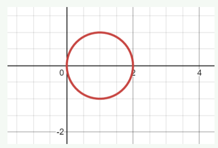
Make a table and graph.

Problem 9.3.1 — $r = 2\cos\theta$

Solution. This is the table of values:

θ	$\cos\theta$	r
0	1	2
$\frac{\pi}{3}$	$\frac{1}{2}$	1
$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$	-1
π	-1	-2

And here is the graph:

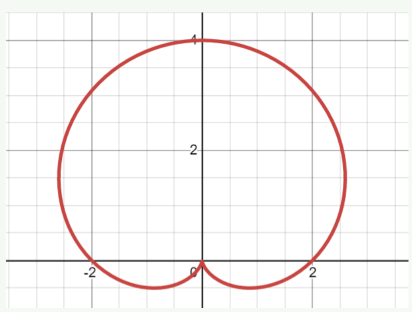


Problem 9.3.2 — $r = 2 + 2\sin\theta$

Solution. This is the table of values:

θ	$\sin\theta$	r
0	0	2
$\frac{\pi}{6}$	$\frac{1}{2}$	3
$\frac{\pi}{2}$	1	4
$\frac{5\pi}{6}$	$\frac{1}{2}$	3
π	0	2
$\frac{7\pi}{6}$	$-\frac{1}{2}$	1
$\frac{3\pi}{2}$	-1	0
$\frac{11\pi}{6}$	$-\frac{1}{2}$	1
2π	0	2

And here is the graph:

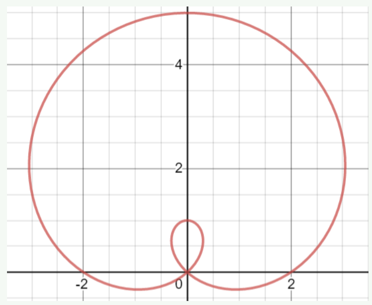


Problem 9.3.3 — $r = 2 + 2\sin\theta$

Solution. This is the table of values:

θ	$\sin\theta$	r
0	0	-2
$\frac{\pi}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\pi}{2}$	1	1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$
π	0	-2
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{7}{2}$
$\frac{3\pi}{2}$	-1	-5
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$-\frac{7}{2}$
2π	0	-2

And here is the graph:

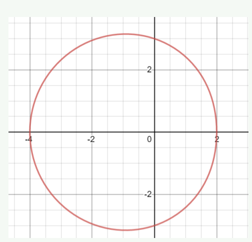


Problem 9.3.4 — $r = 3 - \cos\theta$

Solution. This is the table of values:

θ	$\cos\theta$	r
0	1	2
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{5}{2}$
$\frac{\pi}{2}$	0	3
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{7}{2}$
π	-1	4
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\frac{7}{2}$
$\frac{3\pi}{2}$	0	3
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\frac{5}{2}$
2π	1	2

And here is the graph:



10 Graphing Polar: Roses (4.17.20)

§10.1 Definitions

Definition 10.1.1 (Polar Coordinate of a Rose) — $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$

§10.2 Examples

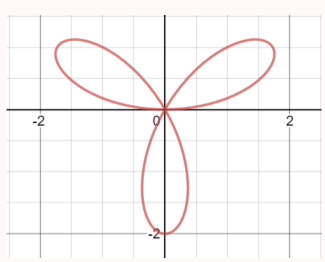
Example 10.2.1

Make a table and graph $r = 3\sin 3\theta$

Solution. This is the table of values:

3θ	$\sin 3\theta$	r
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	1
$\frac{\pi}{2}$	1	2
$\frac{5\pi}{6}$	$\frac{1}{2}$	1
π	0	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$	-1
$\frac{3\pi}{2}$	-1	-2
$\frac{11\pi}{6}$	$-\frac{1}{2}$	-1
2π	0	0

And here is the graph:



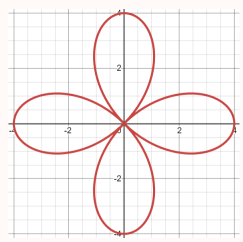
Example 10.2.2

Make a table and graph $r = 4\cos 2\theta$

Solution. This is the table of values:

θ	2θ	$\cos 2\theta$	r
0	0	0	0
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{1}{2}$	2
$\frac{\pi}{4}$	$\frac{\pi}{2}$ 0	0	
$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$-\frac{1}{2}$	-2
$\frac{\pi}{2}$	π	-1	-4
$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$-\frac{1}{2}$	-2
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{6}$	$\frac{5\pi}{3}$	$\frac{1}{2}$	2
π	2π	1	4

And here is the graph:



§10.3 Homework

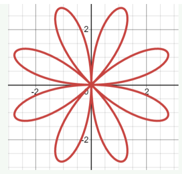
Make a table and graph the following polar functions.

Problem 10.3.1 — $r = 3\sin(4\theta)$

Solution. This is the table of values:

θ	4θ	$\sin 4\theta$	r
0	0	0	0
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{1}{2}$	2
$\frac{\pi}{24}$	$\frac{\pi}{6}$ $\frac{1}{2}$	$\frac{3}{2}$	
$\frac{\pi}{8}$	$\frac{\pi}{2}$	1	3
$\frac{5\pi}{24}$	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{\pi}{4}$	π	0	0
$\frac{7\pi}{24}$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3\pi}{8}$	$\frac{3\pi}{2}$	-1	-3
$\frac{11\pi}{24}$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{\pi}{2}$	2π	0	0

And here is the graph:

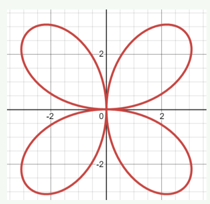


Problem 10.3.2 — $r = -4\sin 2\theta$

Solution. This is the table of values:

θ	2θ	$\sin 2\theta$	r
0	0	0	0
$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{1}{2}$	-2
$\frac{\pi}{4}$	$\frac{\pi}{2}$	1	-4
$\frac{5\pi}{12}$	$\frac{5\pi}{6}$	$\frac{1}{2}$	-2
$\frac{\pi}{2}$	π	0	0
$\frac{7\pi}{12}$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	2
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	-1	4
$\frac{11\pi}{12}$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	2
π	2π	0	0

And here is the graph:

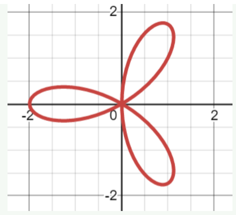


Problem 10.3.3 — $r = -2\cos 3\theta$

Solution. This is the table of values:

θ	3θ	$\cos 3\theta$	r
0	0	1	-2
$\frac{\pi}{9}$	$\frac{\pi}{3}$	$\frac{1}{2}$	-1
$\frac{\pi}{6}$	$\frac{\pi}{2}$	0	0
$\frac{2\pi}{9}$	$\frac{2\pi}{3}$	$-\frac{1}{2}$	1
$\frac{\pi}{3}$	π	-1	2
$\frac{4\pi}{9}$	$\frac{4\pi}{3}$	$-\frac{1}{2}$	1
$\frac{\pi}{2}$	$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{9}$	$\frac{5\pi}{3}$	$\frac{1}{2}$	-1
$\frac{2\pi}{3}$	2π	1	-2

And here is the graph:


2
-2
0
2
-2

11 Graphing Polar: Lemniscate and Spiral of Archimedes (4.20.20)

§11.1 Definition

Definition 11.1.1 (Polar Coordinate of a Lemniscate) — $r^2 = a^2 \sin 2\theta$ or $r^2 = a^2 \cos 2\theta$

Definition 11.1.2 (Polar Coordinate of Spiral of Archimedes) — $r = \theta + a$

§11.2 Examples

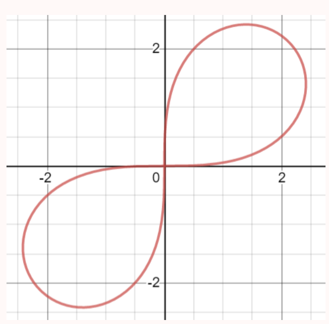
Example 11.2.1

$r^2 = 9 \sin 2\theta$

Solution. This is the table of values:

θ	2θ	r^2	r
0	0	$9(0)$	0
$\frac{\pi}{12}$	$\frac{\pi}{6}$	$9\left(\frac{1}{2}\right)$	$\pm\frac{3}{\sqrt{2}}$
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$9(1)$	± 3
$\frac{5\pi}{12}$	$\frac{5\pi}{6}$	$9\left(\frac{1}{2}\right)$	$\pm\frac{3}{\sqrt{2}}$
$\frac{\pi}{2}$	π	$9(0)$	0
$\frac{7\pi}{12}$	$\frac{7\pi}{6}$	$9\left(-\frac{1}{2}\right)$	DNE
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$9(-1)$	DNE

And here is the graph:



Example 11.2.2

$r^2 = 8\sin^2\theta - 8\cos^2\theta$

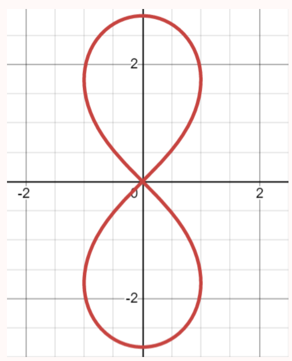
Solution. Note that $\cos^2\theta - \sin^2\theta = \cos 2\theta$. Thus,

$$r^2 = -8\cos 2\theta.$$

This is the table of values:

θ	2θ	r^2	r
0	$\frac{\pi}{2}$	$-8(0)$	0
$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$-8\left(-\frac{1}{2}\right)$	± 2
$\frac{\pi}{2}$	π	$-8(-1)$	$\pm 2\sqrt{2}$
$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$-8\left(-\frac{1}{2}\right)$	± 2
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$-8(0)$	0

And here is the graph:



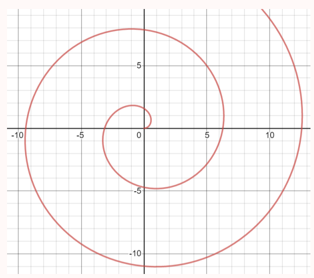
Example 11.2.3

$r = \theta$

Solution. This is the table of values:

θ	r
0	0
$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\frac{\pi}{2}$	$\frac{\pi}{2}$
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$
π	π
$\frac{5\pi}{4}$	$\frac{5\pi}{4}$
$\frac{3\pi}{2}$	$\frac{3\pi}{2}$
$\frac{7\pi}{4}$	$\frac{7\pi}{4}$
2π	2π

And here is the graph:



§11.3 Homework

Problem 11.3.1 — $r^2 = -50 \sin\theta \cos\theta$

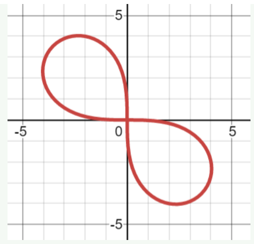
Solution. Note that $2\sin\theta\cos\theta = \sin 2\theta$, so

$$r^2 = -25 \sin 2\theta.$$

This is the table of values:

θ	2θ	r^2	r
$\frac{\pi}{2}$	π	$-25(0)$	0
$\frac{7\pi}{12}$	$\frac{7\pi}{6}$	$-25\left(-\frac{1}{2}\right)$	$\pm\frac{5}{\sqrt{2}}$
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$-25(-1)$	± 5
$\frac{11\pi}{12}$	$\frac{11\pi}{6}$	$-25\left(-\frac{1}{2}\right)$	$\pm\frac{5}{\sqrt{2}}$
π	2π	$-25(0)$	0

And here is the graph:

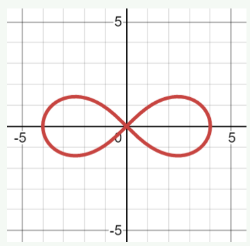


Problem 11.3.2 — $r^2 = 16\cos 2\theta$

Solution. This is the table of values:

θ	2θ	r^2	r
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$16(0)$	0
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$16\left(\frac{1}{2}\right)$	$\pm 2\sqrt{2}$
0	0	$16(1)$	± 4
$\frac{5\pi}{6}$	$\frac{5\pi}{3}$	$16\left(\frac{1}{2}\right)$	$\pm 2\sqrt{2}$
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$-25(0)$	0

And here is the graph:

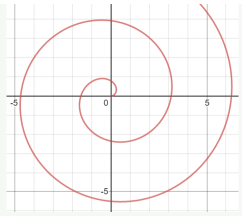


Problem 11.3.3 — $r = \frac{1}{2}\theta$

Solution. This is the table of values:

θ	r
0	0
$\frac{\pi}{4}$	$\frac{\pi}{8}$
$\frac{\pi}{2}$	$\frac{\pi}{4}$
$\frac{3\pi}{4}$	$\frac{3\pi}{8}$
π	$\frac{\pi}{2}$
$\frac{5\pi}{4}$	$\frac{5\pi}{8}$
$\frac{3\pi}{2}$	$\frac{3\pi}{4}$
$\frac{7\pi}{4}$	$\frac{7\pi}{8}$
2π	π

And here is the graph:


-5
0
5
-5

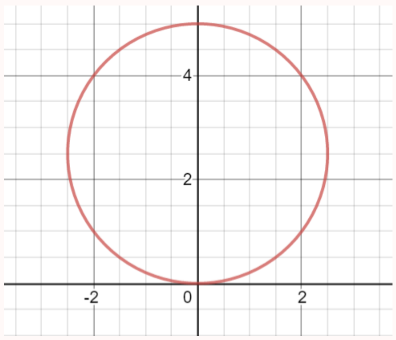
12 Writing Polar Equations from Graphs

§12.1 Examples

§12.1.1 Circle

Example 12.1.1

Find the equation of the following graph:



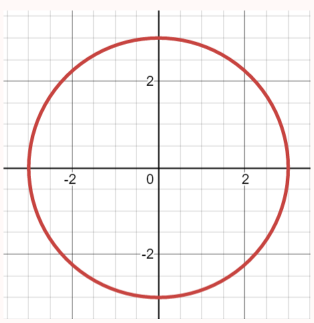
Solution. The center lies on the positive y-axis, and the diameter is 5. Thus,

$$r = 5\sin\theta.$$

Note that it would be -5 on the negative y-axis and cos on the x-axis.

Example 12.1.2

Find the equation of the following graph:



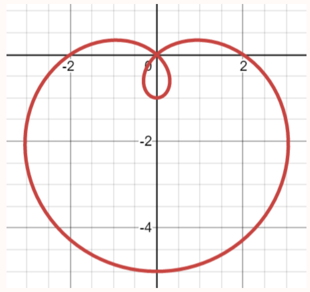
Solution. The radius is at the origin, and extends a distance 3 from the origin. Thus, the answer is

$$r = 3.$$

§12.1.2 Limacon

Example 12.1.3

Find the equation of the following graph:



Solution. Because it lies on the negative y-axis, we must have

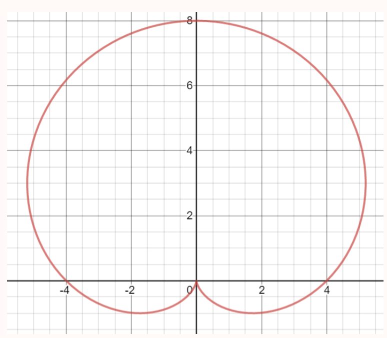
$$r = a - b\sin\theta,$$

for positive a, b. The limacon intersects the x axis at $(2, 0°)$, so $a = 2$. The y-axis is intersected at 1 and 5, so b must be 3. Thus,

$$r = 2 - 3\sin\theta.$$

Example 12.1.4

Find the equation of the following graph:

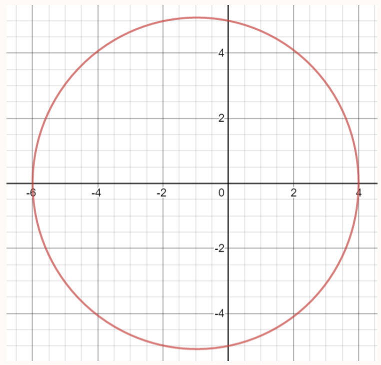


Solution. The limacon is symmetric with the positive y-axis. Furthermore, it intersects it at 0 and 8. Thus, we must have $a = b = 4$. Thus,

$$r = 4 + 4\sin\theta.$$

Example 12.1.5

Find the equation of the following graph:



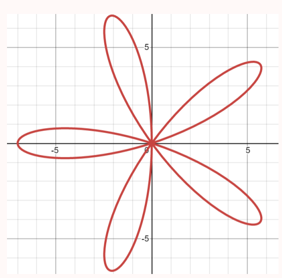
Solution. It is symmetric with the negative x-axis, so we must have $-\cos\theta$. It intersects the y-axis at 5 and -5, so

$$r = 5 - \cos\theta.$$

§12.1.3 Roses

Example 12.1.6

Find the equation of the following graph:

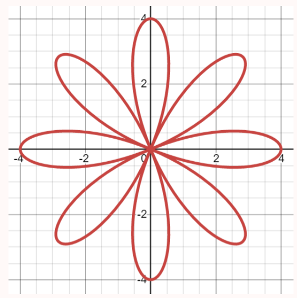


Solution. The rose extends all the way to 7. Furthermore, it is symmetric with the negative x-axis, and has 5 petals. Thus,

$$r = -7\cos(5\theta).$$

Example 12.1.7

Find the equation of the following graph:



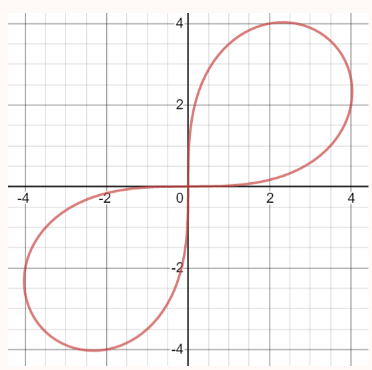
Solution. Regardless of if it is symmetric with the y-axis, if it is symmetric with the x-axis, it is cosine. It has 8 petals, but this translates to a 4 (because even). It extends to 4, so

$$r = 4\cos(4\theta).$$

§12.1.4 Lemniscate

Example 12.1.8

Find the equation of the following graph:



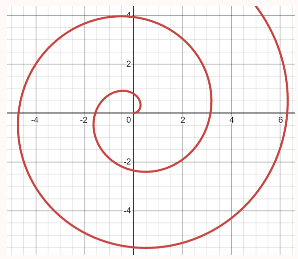
Solution. It is not along the x-axis or y-axis, so it must be sine. Furthermore, it extends a distance of 5, so

$$r^2 = 5^2\sin(2\theta).$$

§12.1.5 Spiral

Example 12.1.9

Find the equation of the following graph:



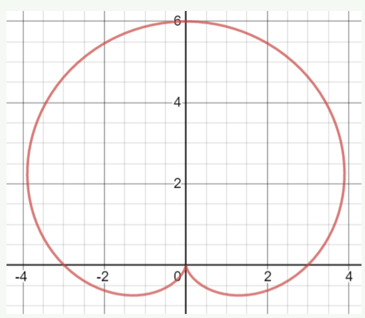
Solution. At 0, which is 2π, it intersects at π. Thus,

$$r = \frac{1}{2}\theta.$$

§12.2 Homework

There was originally 12 problems, but problem 8 didn't exist, so the problem numbering is shifted. (Note: when I say "lies on", I mean is symmetric with.)

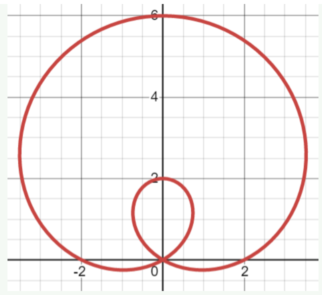
Problem 12.2.1 — Find the equation of the following graph:



Solution. It is a **limacon**, and more specifically, a **cardioid**. It lies on the positive y-axis, hits the x-axis at 3 and -3, so

$$r = 3 + 3\sin\theta.$$

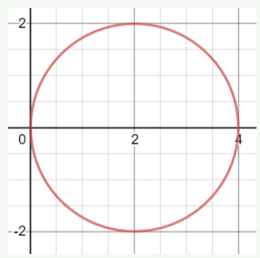
Problem 12.2.2 — Find the equation of the following graph:



Solution. It is a **limacon with an inner loop**. It lies on the positive y-axis, and intersects the y-axis at 2 and 6, so

$$r = 2 + 4\sin\theta.$$

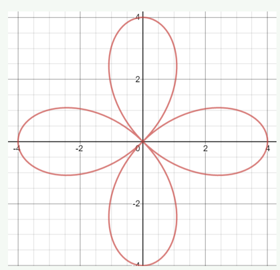
Problem 12.2.3 — Find the equation of the following graph:



Solution. It is a **circle**. It lies on the positive x-axis, and extends to 4, so

$$r = 4\cos\theta.$$

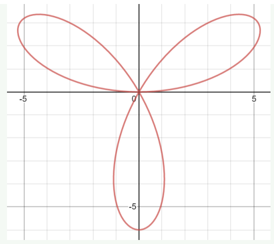
Problem 12.2.4 — Find the equation of the following graph:



Solution. It is a **rose**. It lies on the x-axis, and extends to 4. Thus,

$$r = 4\cos 2\theta.$$

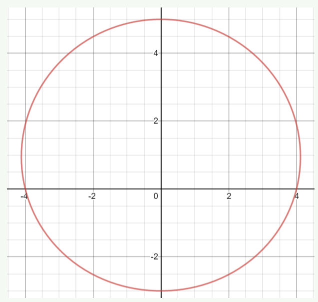
Problem 12.2.5 — Find the equation of the following graph:



Solution. It is a **rose**. It lies on the y-axis, and has 3 petals, so

$$r = 6 \sin 3\theta.$$

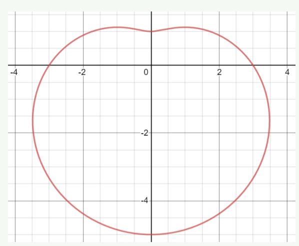
Problem 12.2.6 — Find the equation of the following graph:



Solution. It is a **limacon**, specifically, a **convex limacon**. It lies on the positive y-axis, and intersects the x-axis at 4 and -4, so

$$r = 4 + \sin \theta.$$

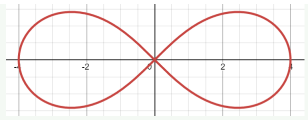
Problem 12.2.7 — Find the equation of the following graph:



Solution. It is a **dimpled limacon**. It lies on the negative y-axis, and intersects the x-axis at 3 and -3, and the y-axis at 1 and -5, so

$$r = 3 - 2 \sin \theta.$$

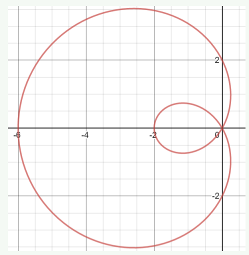
Problem 12.2.8 — Find the equation of the following graph:



Solution. It is a **lemniscate**, and extends to 4. It lies on the x-axis, so

$$r^2 = 16\cos 2\theta.$$

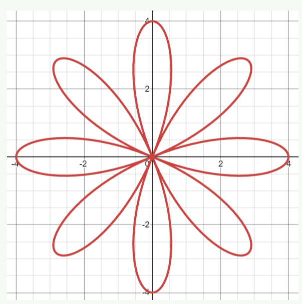
Problem 12.2.9 — Find the equation of the following graph:



Solution. It is a **limacon with an inner loop**. It intersects the x-axis at 2 and -2, lies on the negative x-axis, and intersects it at -2 and -6. Thus,

$$r = 2 - 4\cos\theta.$$

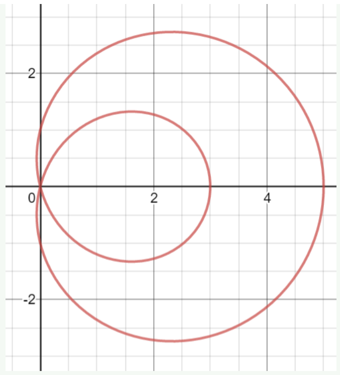
Problem 12.2.10 — Find the equation of the following graph:



Solution. It is a **rose**. It lies on the x-axis, has 8 petals (but we divide by 2), so

$$r = 4\cos 4\theta.$$

Problem 12.2.11 — Find the equation of the following graph:



Solution. It is a **limacon with an inner loop**. It lies on the x-axis, and hits it at 3 and 5, and hits the y-axis at 1 and -1. Thus,

$$r = 1 + 4\cos\theta.$$

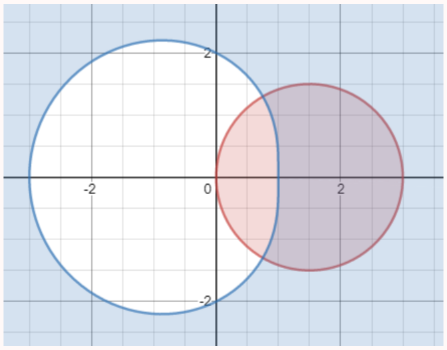
13 Shaded Region of Two Polar Graphs (4.22.20)

§13.1 Examples

Example 13.1.1

Sketch a graph and shade the region. Inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$.

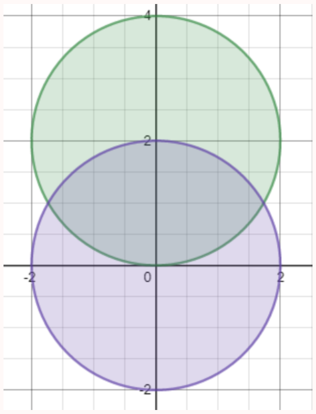
Solution. The first graph is a circle, and the second is a convex limacon. By graphing, we get



Example 13.1.2

Find the common interior of $r = 4\sin\theta$ and $r = 2$.

Solution. These are two circles, so

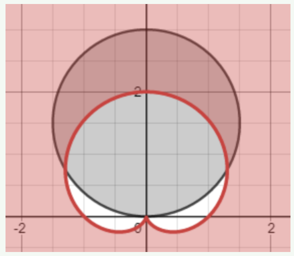


If you would like more accuracy while graphing, make a table of values to more accurately draw your figures.

§13.2 Homework

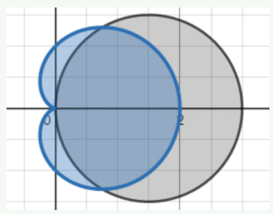
Problem 13.2.1 — Find the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

Solution. We have a circle and cardioid, so



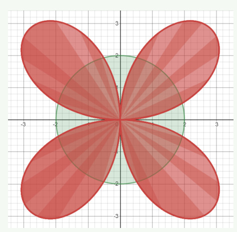
Problem 13.2.2 — Find the common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

Solution. We have a cardioid and a circle, so



Problem 13.2.3 — Find the common interior of $r = 4\sin 2\theta$ and $r = 2$.

Solution. We have a lemniscate and a circle, so



14 Polar Unit Assessment (4.23.20)

§14.1 Assessment

Problem 14.1.1 — Convert the following rectangular equation to a polar equation (solve for r).

$$x^2 + 3y^2 = 2x + y.$$

Solution. Note that $x = r\cos\theta, y = r\sin\theta$. Thus,

$$(r\cos\theta)^2 + 3(r\sin\theta)^2 = 2r\cos\theta + r\sin\theta,$$

$$r^2 + 2r^2\sin^2\theta = 2r\cos\theta + r\sin\theta,$$

$$r(1 + 2\sin^2\theta) = 2\cos\theta + \sin\theta.$$

Thus,

$$\boxed{r = \frac{2\cos\theta + \sin\theta}{1 + 2\sin^2\theta}}.$$

Problem 14.1.2 — Convert the following polar equation to a rectangular equation (solve for y).

$$r = \sec\theta - \frac{2}{\sin\theta}.$$

Note that

$$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r},$$

so

$$r = \frac{r}{x} - \frac{2r}{y},$$

$$1 = \frac{1}{x} - \frac{2}{y},$$

$$\boxed{y = \frac{2x}{1-x}}.$$

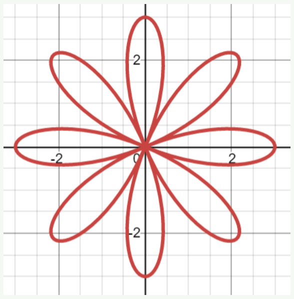
Name the type of polar graph. Include a table of points used to graph and a graph.

Problem 14.1.3 — $r = -3\cos 4\theta$

Solution. This is the table of values:

θ	r
$\frac{\pi}{2}$	-3
$\frac{\pi}{12}$	$-\frac{3}{2}$
$\frac{\pi}{8}$	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	3

And here is the graph:



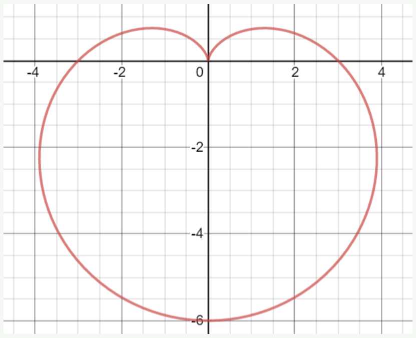
Thus, it is a **rose**.

Problem 14.1.4 — $r = 3 - 3\sin\theta$

Solution. This is the table of values:

θ	r
0	3
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{2}$	0
$\frac{5\pi}{6}$	$\frac{3}{2}$
π	3

And here is the graph:



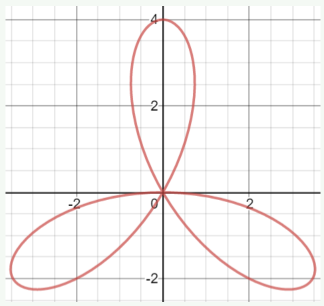
Thus, it is a **cardioid**.

Problem 14.1.5 — $r = 3 - 3\sin\theta$

Solution. This is the table of values:

θ	r
0	0
$\frac{\pi}{6}$	-4
$\frac{\pi}{3}$	0
$\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	-4

And here is the graph:



Thus, it is a **rose**.

15 Sequences and Series (4.27-28.20)

§15.1 Definitions

Definition 15.1.1 (Sequence) — A **sequence** is a function whose domain is the set of positive integers.

§15.2 Examples

Write the first 5 terms of the sequence (start with $n = 1$):

Example 15.2.1

$a_n = 3n + 1$

Solution. We simply plug the numbers in to get

$$a_1 = 4,$$

$$a_2 = 7,$$

$$a_3 = 10,$$

$$a_4 = 13,$$

$$a_5 = 16.$$

Example 15.2.2

$a_n = \left(\frac{1}{2}\right)^n$

Solution. We simply plug in the numbers to get

$$a_1 = \frac{1}{2},$$

$$a_2 = \frac{1}{4},$$

$$a_3 = \frac{1}{8},$$

$$a_4 = \frac{1}{16},$$

$$a_6 = \frac{1}{32}.$$

Example 15.2.3

$a_n = \frac{n}{n+1}$

Solution. We plug in the numbers to get

$$a_1 = \frac{1}{2},$$
$$a_2 = \frac{2}{3},$$
$$a_3 = \frac{3}{4},$$
$$a_4 = \frac{4}{5},$$
$$a_5 = \frac{5}{6}.$$

Example 15.2.4

$a_n = 3$

Solution. The equation is constant, so $a_1 = a_2 = a_3 = a_4 = a_5 = 3$.

Example 15.2.5

$a_n = n^{\frac{1}{3}}$

Solution. We plug in the numbers to get

$$a_1 = 1,$$
$$a_2 = \sqrt[3]{2},$$
$$a_3 = \sqrt[3]{3},$$
$$a_4 = \sqrt[3]{4},$$
$$a_5 = \sqrt[3]{5}.$$

Example 15.2.6

$a_n = (-1)^n \cdot n$

Solution. We plug in the numbers to get

$$a_1 = -1,$$

$$a_2 = 2,$$

$$a_3 = -3,$$

$$a_4 = 4,$$

$$a_5 = -5.$$

Example 15.2.7

Find the 12th term for the sequence given by: $a_n = \frac{4n}{2n^2-3}$.

Solution. We plug in 12 to get

$$a_12 = \frac{48}{285} = \frac{16}{95}.$$

Write an expression for the apparent nth term of the sequence. Let n start with 1.

Example 15.2.8

$3, 7, 11, 15, 19, \dots$

Solution. This looks like an arithmetic sequence with common difference 4, so

$$a_n = 4n - 1.$$

Example 15.2.9

$2, -4, 6, -8, 10, \dots$

Solution. There is an alternating sign, but the first sign is positive, so we have $(-1)^{n+1}$. Furthermore, now we have a sequence of even numbers, so

$$a_n = (-1)^{n+1}(2n).$$

Example 15.2.10

$\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \dots$

Solution. The top is 2^{n-1}, and the bottom is 3^n, so

$$a_n = \frac{2^{n-1}}{3^n}.$$

Example 15.2.11

The bottom is the factorials, and the top is powers of 2. Thus,

$$a_n = \frac{2^{n-1}}{(n-1)!}.$$

§15.3 Homework

Write the first five terms of the sequence.

Problem 15.3.1 — $a_n = 3n - 1$

Solution. Plugging in values, we get

$$2, 5, 8, 11, 14.$$

Problem 15.3.2 — $a_n = 3 + (-1)^n$

Solution. Plugging in values, we get

$$2, 4, 2, 4, 2.$$

Problem 15.3.3 — $a_n = \frac{3^n}{2^n}$

Solution. Plugging in values, we get

$$\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \frac{243}{32}.$$

Problem 15.3.4 — $a_n = 2 - \frac{1}{3^m}$

Solution. Plugging in values, we get

$$\frac{5}{3}, \frac{17}{9}, \frac{53}{27}, \frac{161}{81}, \frac{485}{243}.$$

Problem 15.3.5 — $a_n = \frac{1}{4}n$

Solution. Plugging in values, we get

$$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}.$$

Find the indicated term of the sequence:

Problem 15.3.6 — If $a_n = (-1)^n(5n + 3)$, find a_25.

Solution. Plugging in 25, we get

$$a_25 = -1(128) = -128.$$

Problem 15.3.7 — Find the 12th term of $a_n = \frac{1}{n+1}$.
Solution. Plugging in 12, we get

$$a_{12} = \frac{1}{13}.$$

Problem 15.3.8 — Find the 6th term of $a_n = \frac{n^2}{n^2+2}$.
Solution. Plugging in 6, we get

$$a_6 = \frac{18}{19}.$$

Write an expression for the nth term of the sequence assume that n begins with 1).

Problem 15.3.9 — $1, 4, 7, 10, 13, \ldots$
Solution. This an arithmetic sequence with difference 3, so

$$a_n = 3n - 2.$$

Problem 15.3.10 — $0, 3, 8, 15, 24, \ldots$
Solution. These are all one less than a perfect square, so

$$a_n = n^2 - 1.$$

Problem 15.3.11 — $-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$
Solution. The numerator is always one less than the denominator, and the signs alternate, so,

$$a_n = (-1)^n \frac{n+1}{n+2}.$$

Problem 15.3.12 — $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$
Solution. The numerator is all the positive integers from 2 to infinity, and the denominator is the odd numbers. Thus,

$$a_n = \frac{n+1}{2n-1}.$$

Problem 15.3.13 — $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \ldots$
Solution. The numerator is the odd numbers, and the denominator is the powers of 2. Thus,

$$a_n = \frac{2n-1}{2^n}.$$

Problem 15.3.14 — $1, -1, 1, -1, 1, \ldots$

Solution. The signs are alternating, so

$$a_n = (-1)^{n+1}.$$

Problem 15.3.15 — $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

Solution. The only change is the denominator of the fraction, which is simply all the positive integers. Thus,

$$a_n = 1 + \frac{1}{n} = \frac{n+1}{n}.$$

Problem 15.3.16 — $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

Solution. Note that the denominator is always powers of 2, and the numerator is always one less than the denominator. Thus,

$$a_n = 1 + \frac{2^n - 1}{2^n} = \frac{2^{n+1} - 1}{2^n}.$$

16 Factorials, Recursive Sequences, Series and Summation (4.29-30.20)

§16.1 Factorials

Definition 16.1.1 (Factorial) — The nth **factorial** is defined as

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!.$$

Theorem 16.1.2 (Divisibility of Factorials)
If $m > n$, then $m!$ is divisible by $n!$.

§16.1.1 Examples

Example 16.1.3
Simplify the following.

1. $0!$
2. $\frac{7!}{5!}$
3. $\frac{11!-8!}{10!}$
4. $\frac{n!}{6(n-2)!}$
5. $\frac{(3n+2)!}{(3n)!}$

Solution. The idea is to use the theorem above.

1. $0! = 1$
2. $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 840$
3. $\frac{11 \cdot 10 \cdot 9 \cdot 8! - 8!}{10 \cdot 9 \cdot 8!} = \frac{11 \cdot 10 \cdot 9 - 1}{10 \cdot 9} = \frac{989}{90}$
4. $\frac{n(n-1)(n-2)!}{6(n-2)!} = \frac{n(n-1)}{6}$
5. $\frac{(3n+2)(3n+1)(3n)!}{(3n)!} = (3n+2)(3n+1) = 9n^2 + 9n + 2$

§16.2 Recursive Sequences

Definition 16.2.1 (Recursive Sequences) — A sequence where given one or more of the first few terms, all other terms are defined using the values of the previous terms.

§16.2.1 Examples

Write the first five terms of the sequence defined recursively.

Example 16.2.2

$a_1 = 6, a_{k+1} + a_k + 1$

Solution. By simply plugging the values in, we get

$$a_1 = 6,$$

$$a_2 = 6 + 1 = 7,$$
$$a_3 = 7 + 1 = 8,$$
$$a_4 = 8 + 1 = 9,$$
$$a_5 = 9 + 1 = 10.$$

Example 16.2.3

$a_0 = 1, a_1 = 1, a_k = a_{k-2} + a_{k-1}, k \geq 2$

Solution. By simply plugging the values in, we get

$$a_0 = 1,$$

$$a_1 = 1$$
$$a_2 = a_0 + a_1 = 1 + 1 = 2,$$
$$a_3 = a_1 + a_2 = 1 + 2 = 3,$$
$$a_4 = a_2 + a_3 = 2 + 3 = 5,$$
$$a_5 = a_3 + a_4 = 3 + 5 = 8.$$

Example 16.2.4

Find the first 5 terms and the explicit form: $a_1 = 7, a_{k+1} = 2a_k$

Solution. By simply plugging the values in, we get

$$a_1 = 7,$$

$$a_2 = 2(7) = 14,$$
$$a_3 = 2(14) = 28,$$
$$a_4 = 2(28) = 56,$$
$$a_5 = 2(56) = 112.$$

The explicit form is pretty easy to see - it is some power of 2 dependent on k, and there must be a 7 within the answer. By trial and error, we get

$$a_n = 7 \cdot 2^{n-1}.$$

§16.3 Series and Summation Notation

Definition 16.3.1 (Series) — A **series** is the sum of the set of terms in a finite or infinite sequence.

Definition 16.3.2 (Summation Notation) — **Summation notation or Sigma notation** is a way to express the sum of the terms of a sequence without listing all the terms.

§16.3.1 Examples

List the terms of the series and find the partial sum.

Example 16.3.3

$\sum_{i=0}^{6}(3i-1)$

Solution. Plugging in all values of i, we get

$$-1 + 2 + 5 + 8 + 11 + 14 + 17 = 56.$$

Example 16.3.4

$\sum_{i=2}^{4} \frac{1}{i^3-1}$

Solution. Plugging in all values of i, we get

$$\frac{1}{7} + \frac{1}{26} + \frac{1}{63} = \frac{323}{1638}.$$

Example 16.3.5

$\sum_{i=1}^{8}(-5)^1$

Solution. Plugging in all values of i, we get

$$-5-5-5-5-5-5-5-5=-40.$$

Write using Sigma notation.

Example 16.3.6

$\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+\frac{5}{1+3}+\frac{5}{1+4}+\ldots+\frac{5}{1+15}$

Solution. Each time, a part of the numerator increased by 1, so we can simply write

$$\sum_{i=1}^{15}\frac{5}{1+i}.$$

Example 16.3.7

$36-12+4-\frac{4}{3}+\frac{4}{9}-\ldots$

Solution. Each time, the term is divided by -3, so

$$\sum_{k=0}^{\infty}36\left(-\frac{1}{3}\right)^k.$$

Example 16.3.8

$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots+\frac{1023}{1024}$

Solution. The denominator is a power of 2, and the numerator is one less than a power of 2. Thus,

$$\sum_{n=1}^{10}\frac{2^n-1}{2^n}.$$

§16.4 Homework

Simplify the following.

Problem 16.4.1 — $\frac{99!}{101!}$

Solution. Using the theorem for factorials above, we get

$$\frac{99!}{101\cdot 101\cdot 99!}=\frac{1}{10100}.$$

Problem 16.4.2 — $\frac{8!+6!}{5!}$

Solution. Using the theorem for factorials above, we get

$$\frac{8\cdot 7\cdot 6\cdot 5!+6\cdot 5!}{5!}=8\cdot 7\cdot 6+6=342.$$

Problem 16.4.3 — $\frac{n!(n+1)!}{(n-1)!(n-1)!}$

Solution. Using the theorem for factorials above, we get

$$\frac{n(n-1)!(n+1)n(n-1)!}{(n-1)!(n-1)!}=n(n+1)n=n^3+n^2.$$

Problem 16.4.4 — $\frac{(4n-2)!}{(4n!)}$

Solution. Using the theorem for factorials above, we get

$$\frac{(4n-2)!}{(4n)(4n-1)(4n-2)!}=\frac{1}{4n(4n-1)}=\frac{1}{16n^2-4n}.$$

Problem 16.4.5 — Write the first five terms of the sequence $a_n=n!$.

Solution. If we plug in $n=1,2,3,4,5$, we get

$$1,2,6,24,120.$$

Write the first five terms of the sequence defined recursively and then for 7 and 8 use the pattern to write the nth term as a function of n.

Problem 16.4.6 — $a_1=-6$ and $a_{k+1}=2a_k+3$

Solution. If we plug in the values, we get

$$a_1=-6,$$

$$a_2=2(-6)+3=-9,$$
$$a_3=2(-9)+3=-15,$$
$$a_4=2(-15)+3=-27,$$
$$a_5=2(-27)+3=-51.$$

Problem 16.4.7 — $a_1=7$ and $a_{k+1}=a_k-4$

Solution. If we plug in the values, we get

$$a_1=7,$$

$$a_2=3,$$

$$a_3 = -1,$$

$$a + 4 = -5,$$

$$a_5 = -9.$$

We can tell that this an arithmetic sequence, so

$$a_n = -4n + 11.$$

Problem 16.4.8 — $a_1 = 24$ and $a_{k+1} = \frac{1}{2}a_k$

Solution. If we plug in the values, we get

$$a_1 = 24,$$

$$a_2 = 12,$$

$$a_3 = 6,$$

$$a_4 = 3,$$

$$a_5 = \frac{3}{2}.$$

We can tell that this is a geometric sequence, so

$$a_n = 48\left(\frac{1}{2}\right)^n.$$

List the terms of the series and find the partial sum.

Problem 16.4.9 — $\sum_{k=2}^{6}(2k+1)$

Solution. Plugging in the values, we get

$$5 + 7 + 9 + 11 + 13 = 45.$$

Problem 16.4.10 — $\sum_{k=0}^{4} k^2$

Solution. Plugging in the values, we get

$$0 + 1 + 4 + 9 + 16 = 30.$$

Problem 16.4.11 — $\sum_{k=0}^{3} \frac{1}{k^2+1}$

Solution. Plugging in the values, we get

$$1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{9}{5}.$$

Problem 16.4.12 — $\sum_{k=1}^{4}(-2)^k$
Solution. Plugging in the valeus, we get

$$-2+4-8+16=10.$$

Write using Sigma Notation.

Problem 16.4.13 — $2+7+12+17+\ldots+117$
Solution. This is an arithmetic sequence with common difference 5 and first term 2 and last term 117. Thus,

$$\sum_{k=0}^{23} 5k+2.$$

Problem 16.4.14 — $50+47+44+\ldots+(-7)$
Solution. This is an arithmetic sequence with common difference -3 and first term 50 and last term -7. Thus,

$$\sum_{k=0}^{19} -3k+50.$$

Problem 16.4.15 — $4+2+1+\frac{1}{2}+\ldots+\frac{1}{64}$
Solution. This is a geometric sequence with first term 4, last term $\frac{1}{64}$, and common ratio $\frac{1}{2}$ so

$$\sum_{k=0}^{8} 4\left(\frac{1}{2}\right)^k.$$

Problem 16.4.16 — $7-\frac{9}{4}+\frac{11}{16}-\frac{13}{64}+\frac{15}{256}-\ldots$
Solution. The numerator is an arithmetic sequence, and the denominator is a geometric sequence. Thus,

$$\sum_{k=0}^{\infty} \frac{2k+7}{(-4)^k}.$$

Problem 16.4.17 — $1+4+9+16+\ldots+144$
Solution. These are just the perfect squares summed together. Thus,

$$\sum_{k=1}^{12} k^2.$$

Problem 16.4.18 — $-\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}-\frac{1}{3\cdot 5}+\dots+\frac{1}{10\cdot 12}$

Solution. The bottom is the multiplication of two numbers 2 apart, and the top has alternating signs, so

$$\sum_{n=1}^{10}\frac{(-1)^n}{n(n+2)}.$$

Problem 16.4.19 — $\frac{1}{2}+\frac{2}{[}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\frac{720}{64}$

Solution. The top is the factorials, and the bottom is the powers of 2, so

$$\sum_{n=1}^{6}\frac{n!}{2^n}.$$

17 Arithmetic Sequence and Series (5.1.20)

§17.1 Definitions

Definition 17.1.1 (Arithmetic Sequence) — An **arithmetic sequence** is a sequence whose terms increase/decrease by a common difference (d).

§17.2 Writing Arithmetic Sequences

Theorem 17.2.1 (Arithmetic Sequence Formula)
For all arithmetic sequences,

$$a_n = a_1 + d(n-1),$$

and

$$a_n = a_m + d(n-m),$$

where n, m are the indices, a_i are the terms for positive integer i, and d is the common difference.

§17.3 Examples

Example 17.3.1
$\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots$ What is the common difference?
Solution. By subtracting the first two terms, we get $\frac{1}{2}$, so that is our common difference.

Determine if the following are arithmetic sequences, if yes, find the common difference.

Example 17.3.2
$5, 8, 11, 14, 17, \ldots$
Solution. Each time, the sum increases by 3, so it is an arithmetic sequence, and the common difference is 3.

Example 17.3.3
$2, 4, 8, 16, \ldots$
Solution. The difference is not constant, so no.

Example 17.3.4

$1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \ldots$

Solution. If subtract each term, we get a common difference of $\frac{1}{6}$, so it is an arithmetic sequence

Example 17.3.5

$a_n = 3n - 2$

Solution. We can see that the equation increases linearly, so the common difference is 3, and it is an arithmetic sequence

Example 17.3.6

$a_n = n^3 - 2$

Solution. The equation is cubic, so no,

Write the formula for following arithmetic sequences,

Example 17.3.7

$a_1 = 2, d = \frac{4}{5}$

Solution. Simply plugging into the theorem, we get

$$a_n = 2 + \frac{4}{5}(n-1).$$

Example 17.3.8

$\frac{1}{5}, 1, \frac{9}{5}, \frac{13}{5}, \frac{17}{5}, \ldots$

Solution. The common difference is obviously $\frac{4}{5}$, so

$$a_n = \frac{1}{5} + \frac{4}{5}(n-1).$$

Example 17.3.9

$a_1 = 1, a_5 = 9$

Solution. The common difference is

$$d = \frac{9-1}{5-1} = 2,$$

so

$$a_n = 1 + 2(n-1).$$

Example 17.3.10

$a_4 = 20, a_{13} = 65,$

Solution. The common difference is

$$d = \frac{65 - 20}{13 - 4} = 5,$$

so

$$a_n = 20 + 5(n - 4) = 5n.$$

Example 17.3.11

$a_1 = 8, a_{k+1} = a_k - 3$

Solution. From the implicit form, we know the common difference is -3, so

$$a_n = 8 + (-3)(n - 1).$$

§17.4 Homework

Determine whether the sequence is arithmetic. If so, find the common difference.

Problem 17.4.1 — $10, 8, 6, 4, 2, \ldots$

Solution. Yes, and the common difference is -2.

Problem 17.4.2 — $1, 2, 4, 8, 16, \ldots$

Solution. No, because the common difference changes.

Find a formula for a_n for the arithmetic sequence.

Problem 17.4.3 — $a_1 = 15, d = 4$

Solution. Using the formula, we get

$$a_n = 15 + 4(n - 1).$$

Similarly,

$$a_{k+1} = a_k + 4.$$

Problem 17.4.4 — $a_1 = -y, d = 5y$

Solution. Using the formula, we get

$$a_n = 5yn - 6y.$$

Similarly,

$$a_{k+1} = a_k + 5y.$$

Problem 17.4.5 — $10, 5, 0, -5, -10, \ldots$

Solution. The common difference is -5, so using the formula we get

$$a_n = -5n + 15.$$

Similarly,

$$a_{k+1} = a_k - 5.$$

Problem 17.4.6 — $a_1 = 4, a_4 = 15$

Solution. The common difference is

$$d = \frac{15-4}{4-1} = \frac{11}{3}.$$

Thus,

$$a_n = \frac{11}{3}n + \frac{1}{3}.$$

Similarly,

$$a_{k+1} = a_k + \frac{11}{3}.$$

Problem 17.4.7 — $a_3 = 190, a_{10} = 115$

Solution. The common difference is

$$d = \frac{115-190}{10-5} = -15.$$

Thus,

$$a_n = -15(n-10) + 115.$$

Similarly,

$$a_{k+1} = a_k - 15.$$

18 Arithmetic Sequences and Partial Sums (5.4-5.20)

§18.1 Definitions

Definition 18.1.1 (Infinite Series) — An **infinite series** is a series with infinite terms.

For example, $\sum_{n=1}^{\infty} n$ is an infinite series. Unfortunately, an infinite arithmetic series has no real sum.

Definition 18.1.2 (Partial Sums Series) — A **partial sums series**, or **finite series**, has a finite amount of terms.

For example, $\sum_{n=1}^{10} n$ is a finite series.

Theorem 18.1.3 (Partial Sum Formula)
The sum of a partial sum is

$$S_n = \frac{(a_1 + a_n)n}{2},$$

where S_n is the sum of the first n terms, a_1 is the first term, and a_n is the nth term. Note that this is equivalent to the number of terms multiplied by the average of the terms, which happens to be the median.

§18.2 Examples

Write the first five terms of the sequence.

Example 18.2.1
$a_1 = 4, d = -0.5$

Solution. Simply following the pattern, we get

$$4, 3.5, 3, 2.5, 2.$$

Example 18.2.2

$a_4 = 16, a_10 = 46$

Solution. From these two equations we get that

$$d = \frac{46-16}{10-4} = 5.$$

Thus,

$$a_4 = a_1 + 3d = a_1 + 15 = 16,$$

so $a_1 = 1$. Thus the first 5 terms are

$$1, , 6, 11, 16, 21.$$

Find the indicated term.

Example 18.2.3

$a_1 = 3, a_2 = 12, a_9 = \_\_$

Solution. From this, we know the common difference is $a_2 - a_1 = 12 - 3 = 9$, Thus,

$$a_n = 3 + 9(n-1),$$

so $a_9 = 3 + 9 \cdot 8 = 75$.

Example 18.2.4

$a_1 = 4.2, a_2 = 6.6, a_7 = \_\_$

Solution. From this, we know the common difference is $a_2 - a_1 = 6.6 - 4.2 = 2.4$.Thus,

$$a_n = 4.2 + 2.4(n-1),$$

$$a_7 = 18.6.$$

Find the partial sums of the following.

Example 18.2.5

$\sum_{n=1}^{100} 2n$

Solution. This is equivalent to

$$2 + 4 + 6 + \ldots + 200,$$

and since there are 100 terms and the middle term is $\frac{2+200}{2} = 101$, the answer is 10100.

Example 18.2.6

$\sum_{k=51}^{100} 7k$

Solution. The first term is $7 \cdot 51 = 357$, the 100th term is $7 \cdot 100 = 700$, and the number of terms is 50, so

$$\frac{(357 + 700)50}{2} = 26425.$$

Example 18.2.7

$\sum_{i=25}^{30} i - \sum_{i=1}^{24} i$

Solution. The first series has first term 25, last term 30, and 6 terms in total. Thus,

$$\sum_{i=25}^{30} i = \frac{(25 + 30)6}{2} = 165.$$

Similarly, the second series has first term 1, last term 24, and 24 terms in total. Thus,

$$\sum_{i=1}^{24} i = \frac{(1 + 24)24}{2} = 300.$$

Thus, $165 - 300 = -135$.

Example 18.2.8

An auditorium is shaped like a triangle, with the stage at the wide end. Determine the total number of seats in the auditorium if there are 20 rows and the first row has 50 seats, the second row has 48 seats, and the third row has 46 seats and so on?

Solution. The common difference is -2, and the first term is 50. Thus,

$$a_{20} = 50 - 2(20 - 1) = 12.$$

Thus, the total number of seats is

$$12 + 14 + 16 + \ldots + 50 = \frac{(12 + 50)20}{2} = 620.$$

Example 18.2.9

If Aunt Millie sent you \$100 on your first birthday and then increased each successive birthday present by \$10, how much will you receive on your 21st birthday?

Solution. Using the formula, we get

$$a_{21} = 100 + 10(21 - 1) = 300.$$

§18.3 Homework

Problem 18.3.1 — Find the 18th term of $7, 2, -3, -8, \ldots$

Solution. The common difference is $2 - 7 = -5$, so

$$a_{18} = 7 - 5(18 - 1) = -68.$$

Write the first five terms of the arithmetic sequence. Find the common difference and write the nth term of the sequence as a function of n.

Problem 18.3.2 — $a_1 = 6, a_{k+1} = a_k - 10$

Solution. The common difference is -10, so

$$a_n = 6 - 10(n - 1) = -10n + 16.$$

Problem 18.3.3 — $a_1 = \frac{5}{8}, a_{k+1} = a_k - \frac{1}{8}$

Solution. The common difference is $-\frac{1}{8}$, so

$$a_n = \frac{5}{8} - \frac{1}{8}(n - 1) = -\frac{1}{8}n + \frac{3}{4}.$$

Find the indicated nth partial sum of the arithmetic sequence.

Problem 18.3.4 — $9 + 4 + (-1) + (-6) + \ldots + (-86)$

Solution. The number of terms is

$$\frac{-86 - 9}{4 - 9} + 1 = 20.$$

Thus,

$$\frac{(9 - 86)20}{2} = -770.$$

Problem 18.3.5 — $a_1 = 100, a_{25} = 200, n = 25$

Solution. Using the formula, we get

$$\frac{(100 + 200)25}{2} = 3750.$$

Problem 18.3.6 — $a_1 = 2, d = -2, n = 40$

Solution. The 40th term is

$$a_{40} = 2 - 2(40 - 1) = -76.$$

Thus,

$$\frac{(2 - 76)40}{2} = -1480.$$

Problem 18.3.7 — $a_8 = 26, a_{12} = 42, n = 40$

Solution. The common difference is

$$d = \frac{42 - 26}{12 - 8} = 4.$$

Thus,

$$26 = a_1 + 4(8 - 1),$$

$$a_1 = -2.$$

Similarly, $a_{40} = 154$. Therefore,

$$\frac{(-2 + 154)40}{2} = 3040.$$

Find the partial sums of the following.

Problem 18.3.8 — $\sum_{n=2}^{12} 3n - 1$

Solution. The first term is 5, the last term is 35, so

$$\frac{(5 + 35)11}{2} = 220.$$

Problem 18.3.9 — $\sum_{n=1}^{10} 3n + \sum_{n=20}^{30} n$

Solution. Using the formula, we get

$$\frac{(3 + 30)10}{2} + \frac{(20 + 30)11}{2} = 165 + 275 = 440.$$

Problem 18.3.10 — $5, 9, 13, 17, \ldots \quad n = 20$

Solution. The common difference is 4, so

$$a_n = 5 + 4(n - 1) = 4n + 1,$$

$$a_{20} = 4 \cdot 20 + 1 = 81.$$

Using the formula, we get

$$\frac{(5 + 81)20}{2} = 860.$$

Problem 18.3.11 — A calculus professor fell from a plane while (hoping) to go on vacation. Luckily, he has math to keep him busy until he hits the ground. During the first second of the fall, he fell 4.9 meters, during the second; he fell 14.7 meters, the third 24.5 meters, and fourth 34.3 meters. He realized that this is an arithmetic sequence. If the pattern continues, how far did he fall when he hit the ground at 10 seconds?

Solution. The common difference is 9.8, so

$$a_10 = 4.9 + 9.8(10 - 1) = 93.1.$$

Thus,

$$\frac{(4.9 + 93.1)10}{2} = 490.$$

Problem 18.3.12 — Suppose the starting salary on a job is \$30000 and you get an annual raise of \$1200. Determine the salary during the 8th year of employment. Determine the total compensation from the company through 8 full years of employment.

Solution. The common difference is 1200, so

$$a_8 = 30000 + 1200(8 - 1) = 38400.$$

Thus,

$$\frac{(30000 + 38400)8}{2} = \$273600.$$

19 Geometric Sequences and Partial and Infinite Sums (5.6-8.20, 5.11-15.20)

§19.1 Introduction and Definitions

Definition 19.1.1 (Geometric Sequence) — A **geometric sequence** is a sequence whose terms increase/decrease by a *common ratio* r.

Theorem 19.1.2 (Common Ratio)
For all terms a_k where k and $k+1$ are defined over the range,

$$r = \frac{a_{k+1}}{a_k}.$$

Theorem 19.1.3 (Explicit Formula of Geometric Sequences)
If the sequence is geometric, you can express the sequence explicitly by using the formula:

- $a_n = a_1 \cdot r^{n-1}$, or
- $a_n = a_m \cdot r^{n-m}$,

where a_1 is the first term, a_n and a_m are random terms in the sequence such that $n > m$, and r is the common ratio.

Theorem 19.1.4 (Sum of Finite Geometric Series)
If the geometric series is finite (terminates), then

$$\sum_{i=1}^{n} a_1 \cdot r^{i-1} = S_n = a_1 \left(\frac{1-r^n}{1-r} \right).$$

Theorem 19.1.5 (Sum of Infinite Geometric Series)
If the geometric series is infinite, then if $|r| < 1$, the series **converges** to

$$S_\infty = \frac{a_1}{1-r}.$$

If $|r| \geq 1$, the series **diverges** and the sum is undefined.

§19.2 Examples

Determine whether the following sequences are geometric. If so, find their common ratio.

Example 19.2.1

$\frac{2}{3}, -\frac{4}{9}, \frac{8}{27}, -\frac{16}{81}, \ldots$

Solution. Each time we multiply by $-\frac{2}{3}$, so that is our common ratio.

Example 19.2.2

$\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \ldots$

Solution. The first two terms give us a ratio of $\frac{9}{5}$, and the next two give us a ratio of $\frac{25}{21}$. Thus, this is not a geometric sequence.

Example 19.2.3

$64, -16, 4, -1, \ldots$

Solution. Each time we divide by -4, so our common ratio is $-\frac{1}{4}$.

Example 19.2.4

$2, 0.2, 0.02, 0.002, \ldots$

Solution. Each time we divide by 10, so our common ratio is $\frac{1}{10}$.

Example 19.2.5

$a_n = 2n^{n+1}$

Solution. We can easily find the first few terms:

$$a_1 = 2(1)^2 = 2,$$

$$a_2 = 2(2)^3 = 16,$$

$$a_3 = 2(3)^4 = 162.$$

The first two terms have a ratio of 8, and the next two have a ratio of $\frac{81}{8}$, so this is not a geometric sequence.

Write the formula for the following geometric sequences using both the explicit and implicit recursive form.

Example 19.2.6

$a_1 = -\frac{1}{6}, r = \frac{3}{2}$

Solution. Using the formula, we get

$$a_n = \left(-\frac{1}{6}\right)\left(\frac{3}{2}\right)^{n-1}.$$

The implicit form is simply

$$a_{k+1} = \frac{3}{2}a_k,$$

where $a_1 = -\frac{1}{6}$.

Example 19.2.7

$12, 3, 0.75, 0.1875, \ldots$

Solution. Each time we divide by 4, so

$$a_n = 12\left(\frac{1}{4}\right)^n.$$

The implicit form is simply

$$a_{k+1} = \frac{1}{4}a_k,$$

where $a_1 = 12$.

Example 19.2.8

$a_1 = 48, a_{k+1} = -\frac{1}{2}a_k$

Solution, Note that this equation is already recursive. Thus,

$$a_n = 48\left(-\frac{1}{2}\right)^{n-1}.$$

Example 19.2.9

$a_1 = 3, a_5 = 75$

Solution. The common ratio is

$$r = \sqrt[4]{\frac{75}{3}} = \pm\sqrt{5}.$$

Thus,

$$a_n = 3(\pm\sqrt{5})^{n-1}.$$

The implicit form is then

$$a_{k+1} = \pm\sqrt{5}a_k,$$

where $a_1 = 3$.

Example 19.2.10

$a_4 = \frac{1}{30}, a_7 = -\frac{36}{5}$

Solution. The common ratio is

$$r = \sqrt[3]{\frac{-\frac{36}{5}}{\frac{1}{30}}} = -6,$$

so the first term is

$$a_1 = \frac{\frac{1}{30}}{(-6)^3} = -\frac{1}{-6480}.$$

Thus,

$$a_n = -\frac{1}{6480}(-6)^{n-1}.$$

The implicit form is then

$$a_{k+1} = -\frac{36}{5}a_k,$$

where $a_1 = -\frac{1}{6480}$.

Write the first five terms of the geometric sequence.

Example 19.2.11

$a_1 = -2, r = \frac{3}{4}$

Solution. Simply using the formula, we get

$$-2, -\frac{3}{2}, -\frac{9}{8}, -\frac{27}{32}, -\frac{81}{128}.$$

Example 19.2.12

$a_4 = -160, a_8 = -40960$

Solution. Using the formula, we have

$$a_8 = a_4 r^{8-4},$$

$$r = \pm 4.$$

Thus,

$$a_4 = a_1 r^{4-1},$$

$$a_1 = \pm\frac{5}{2}.$$

Thus, we either have

$$-\frac{5}{2}, -10, -40, -160, -640$$

or

$$\frac{5}{2}, -10, 40, -160, 640.$$

Find the indicated term of the geometric sequence.

Example 19.2.13

$3, 36, 432, \ldots \quad a_8 = \_\_$

Solution. The common ratio is $\frac{36}{3} = 12$, so

$$a_8 = a_1 \cdot r^{11} = 3 \cdot 12^{11} = 107495424.$$

Example 19.2.14

$a_2 = -\frac{3}{2}, a_5 = \frac{3}{250}, a_7 = \_\_$

Solution. Using the formula, we get

$$a_5 = a_2 r^{5-2},$$

$$r = \frac{1}{5}.$$

Thus,

$$a_7 = -\frac{3}{250}\left(\frac{1}{5}\right)^{7-5} = -\frac{3}{6250}.$$

Example 19.2.15

Find the sum of the following geometric series:

$$\sum_{i=1}^{4} 2^i.$$

Solution. If we plug in the values i can take on, we get

$$2 + 4 + 8 + 16 = 30.$$

Find the partial sum of the following:

Example 19.2.16

$\sum_{i=1}^{12} 4(-0.25)^{i-1}$

Solution. Using the formula, we get

$$S_{12} = 4 \cdot \frac{1 - (-0.25)^{12}}{1 - (-0.25)} = \frac{3355443}{1048576}.$$

Example 19.2.17

$\sum_{n=0}^{10} -2\left(\frac{2}{3}\right)^n$

Solution. The first term is $a_1 = -2$, and the common ratio is $r = \frac{2}{3}$. Thus,

$$S_{11} = -2 \cdot \frac{1 - \left(\frac{2}{3}\right)^{11}}{1 - \frac{2}{3}} = -\frac{350198}{59049}.$$

Example 19.2.18

$5 + 15 + 45 + \ldots + 3645$

Solution. The first term is $a_1 = 5$, and the common ratio is $r = 3$. Thus,

$$3645 = 5(3)^{n-1},$$

$$n = 7.$$

Thus,

$$S_7 = 5 \cdot \frac{1 - 3^7}{1 - 3} = 5465.$$

Find the sum of each of the following infinite series, if possible. Indicate whether the series converges or diverges.

Example 19.2.19

$\sum_{i=0}^{\infty} 5\left(-\frac{1}{2}\right)^i$

Solution. Because $\left|-\frac{1}{2}\right| < 1$, we have

$$S_\infty = \frac{5}{1-\left(-\frac{1}{2}\right)} = \frac{10}{3}.$$

Example 19.2.20

$\sum_{n=1}^{\infty} -4(0.06)^n$

Solution. Because $|0.06| < 1$, the series converges. Thus,

$$S_\infty = \frac{-0.24}{1-0.06} = -0.255319.$$

Example 19.2.21

$\sum_{n=0}^{\infty} 2\left(\frac{3}{2}\right)^n$

Solution. Because $\left|\frac{3}{2}\right| \geq 1$, the series diverges, and the sum is undefined.

Example 19.2.22

$-9+3-1+\frac{1}{3}-\frac{1}{9}+\dots$

Solution. Because $\left|-\frac{1}{3}\right| < 1$, the series converges, so

$$S_\infty = \frac{-9}{1-\left(-\frac{1}{3}\right)} = -\frac{27}{4}.$$

Example 19.2.23

If Uncle Jeff sent you \$1 on your first birthday, \$ on your second birthday, \$4 on your third birthday, and kept doubling the amount each day,

1. How much will you receive on your 21st birthday?
2. How much money have you received for your birthday from Uncle Jeff over the 21 years? Write the series that models this situation in sigma notation then determine the partial sum.

Solution.

1. The 21st term is $1 \cdot 2^{20} = \$1048576$.
2. In sigma notation, we have
$$\sum_{n=1}^{21} 2^{n-1}.$$
Using our formula, we get
$$S_{21} = \frac{1 - 2^{21}}{1 - 2} = \$2097150.$$

Example 19.2.24

A ball is dropped from a height of 200 ft. Every time it hits the ground, it bounces back to $\frac{3}{4}$ of its previous height. Find the total vertical distance the ball travels. Write the series that models this situation in sigma notation then determine the infinite sum.

Solution. Let the infinite sum when the ball just touches the ground from the first time. This means the total distance is 200 ft plus this infinite sum. Furthermore, note that for every bounce up, there is a bounce down, implying we can find half of the sum and simply multiply by two afterwards. Thus, in sigma notation, we have
$$S_\infty = \sum_{n=1}^{\infty} 150 \left(\frac{3}{4}\right)^{n-1},$$
which equals
$$\frac{150}{1 - \frac{3}{4}} = 600.$$
Thus, the total distance is $200 + 2 \cdot S_\infty = 1400$ ft.

§19.3 Homework

Determine whether the following sequences are geometric. If so, find their common ratio.

Problem 19.3.1 — $14, \frac{7y}{2}, \frac{7y^2}{8}, \frac{7y^3}{32}, \frac{7y^4}{128}$

Solution. We multiply by $\frac{y}{4}$ each time, so that is our common ratio.

Problem 19.3.2 — $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$
Solution. The first two terms have a ratio of $\frac{6}{5}$, and the second two have a ratio of $\frac{15}{14}$, so this is not a geometric sequence.

Problem 19.3.3 — $-2, 2e, -2e^2, 2e^3, -2e^4$
Solution. We multiply by $-e$ each time, so that is our common ratio.

Write the formula for the following geometric sequences using both the explicitly and implicit (recursive) form.

Problem 19.3.4 — $a_1 = -27, r = \frac{5}{3}$
Solution, Using the formula, we get

$$a_n = -27\left(\frac{5}{3}\right)^{n-1}.$$

Thus, the implicit form is

$$a_{k+1}\frac{5}{3}a_k,$$

where $a_1 = -27$.

Problem 19.3.5 — $5xy, -\frac{y}{2}, \frac{y}{20x}, -\frac{y}{200x^2}, \frac{y}{2000x^3}, \ldots$
Solution. The common ratio is $-\frac{1}{10x}$, so

$$a_n = 5xy\left(-\frac{1}{10x}\right)^{n-1}.$$

Thus, the implicit form is

$$a_{k+1} = \left(-\frac{1}{10x}\right)a_k,$$

where $a_1 = 5xy$.

Problem 19.3.6 — $a_1 = -\frac{1}{3}, a_7 = -576$
Solution. We know that

$$a_7 = a_1 r^{7-1},$$

so

$$r = \pm 2\sqrt{3}.$$

Thus,

$$a_n = -\frac{1}{3}(2\sqrt{3})^{n-1}.$$

Thus, the implicit form is

$$a_{k+1} = \pm 2\sqrt{3}a_k,$$

where $a_1 = -\frac{1}{3}$.

Problem 19.3.7 — $a_4 = 18, a_6 = 2$

Solution. We know that

$$a_6 = a_4 r^{6-4},$$

so

$$r = \pm\frac{1}{3}.$$

Thus,

$$a_1 = \pm 486.$$

Therefore, we have

$$a_n = 486\left(\frac{1}{3}\right)^{n-1},$$

or

$$a_n = -486\left(-\frac{1}{3}\right)^{n-1}.$$

Thus, the implicit form is

$$a_{k+1} = \frac{1}{3}a_k,$$

where $a_1 = 486$, or

$$a_{k+1} = -\frac{1}{3}a_k,$$

where $a_1 = -486$.

Problem 19.3.8 — $a_3 = 800, a_6 = -1562.5$

Solution. Using the formula, we get

$$a_6 = a_3 r^{6-3},$$

$$r == \frac{5}{4}.$$

Thus,

$$a_1 = 512.$$

Therefore,

$$a_n = 512\left(-\frac{5}{4}\right)^{n-1}.$$

Thus, the implicit form is

$$a_{k+1} = -\frac{5}{4}a_k,$$

where $a_1 = 512$.

Write the first terms of the geometric sequence.

Problem 19.3.9 — $a_1 = -9, r = \frac{2}{3}$

Solution. Using the formula, we get

$$-9, -6, 4, \frac{8}{3}, \frac{16}{9}.$$

]

Problem 19.3.10 — $a_3 = 45, a_6 = 1215$

Solution. Using the formula, we get

$$a_6 = a_3 r^{6-3},$$

$$r = 3.$$

Thus, the sequence is

$$5, 15, 45, 135, 405.$$

Find the indicated term of the geometric sequence.

Problem 19.3.11 — $0.1 + 0.4 + 1.6 + \dots \quad a_7 = \_\_$

Solution. The common ratio is $\frac{0.4}{0.1} = 4$, so

$$a_7 = 0.1 \cdot 0.4^6 = 4.096.$$

Problem 19.3.12 — $a_1 = 6, a_3 = \frac{27}{2}, a_8 = \_\_$

Solution. The common ratio is

$$a_3 = a_1 r^{3-1},$$

$$r = \pm\frac{3}{2}.$$

Thus,

$$a_8 = 6\left(\pm\frac{3}{2}\right)^7 = \pm\frac{6561}{64}.$$

Problem 19.3.13 — $a_3 = 10, a_7 = 40, a_{12} = \_\_$

Solution. Using the formula, we get

$$a_7 = a_3 r^{7-3},$$

$$r = \pm\sqrt{2}.$$

Thus,

$$a_{12} = 10 \cdot (\pm\sqrt{2})^{12-3} = \pm 160\sqrt{2}.$$

Problem 19.3.14 — $a_2 = -3, a_5 = -\frac{3}{64}, a_{10} = \_\_$

Solution. Using the formula, we get

$$a_5 = a_2 r^{5-2},$$

$$r = \frac{1}{4}.$$

Thus,

$$a_{10} = -3 \cdot \left(\frac{1}{4}\right)^{10-2} = -\frac{3}{65536}.$$

Write the series in sigma notation and find the partial sum.

Problem 19.3.15 — $-3 + 15 - 75 + 375 - \ldots \quad n = 8$

Solution. The first term is $a_1 = -3$, and the common ratio is $r = -5$, so the sum in sigma notation is

$$\sum_{n=1}^{8} -3(-5)^{n-1},$$

and

$$S_8 = -3 \cdot \frac{1-(-5)^8}{1-(-5)} = 195312.$$

Problem 19.3.16 — $2 + \frac{1}{2} + \frac{1}{8} + \ldots + \frac{1}{2048}$

Solution. The first term is $a_1 = 2$, and the common ratio is $r = \frac{1}{4}$, and there are 7 terms, so the sum in sigma notation is

$$\sum_{n=1}^{7} 2(\frac{1}{4})^{n-1},$$

and

$$S_7 = 2 \cdot \frac{1-\left(\frac{1}{4}\right)}{1-\frac{1}{4}} = \frac{5461}{2048}.$$

Find the partial sum of the following.

Problem 19.3.17 — $\sum_{n=0}^{9} 2\left(\frac{5}{2}\right)^{n-1}$

Solution. The first term is $a_1 = \frac{4}{5}$, the common ratio is $r = \frac{5}{2}$, so

$$S_{10} = \frac{3254867}{640}.$$

Problem 19.3.18 — $\sum_{k=1}^{8} -6\left(-\frac{3}{4}\right)^k$

Solution. The first term is $a_1 = -6$ and the common ratio is $r = -\frac{3}{4}$, so

$$S_8 = \frac{75825}{32768}.$$

Find the sum of each of the following infinite series, if possible. Indicate whether the series converges or diverges.

Problem 19.3.19 — $\sum_{k=0}^{\infty} -7\left(\frac{9}{10}\right)^k$

Solution. Since $\left|\frac{9}{10}\right| < 1$, the series converges. Thus,

$$S_\infty = \frac{-7}{1-\frac{9}{10}} = -70.$$

Problem 19.3.20 — $\sum_{k=1}^{\infty} 2\left(-\frac{2}{3}\right)^k$

Solution. Since $\left|-\frac{2}{3}\right| < 1$, the series converges. Thus,

$$S_\infty = \frac{-\frac{4}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{4}{5}.$$

Problem 19.3.21 — $\sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{7}{2}\right)^{n-1}$

Solution. Since $\left|\frac{7}{2}\right| \geq 1$, the series diverges, and the sum is undefined.

Problem 19.3.22 — Sophie wants to throw a party at her house. She invites 3 friends and tells them to invite 3 friends. The three friends also each invite 3 friends. This invitation process goes on for 5 "stages" of invitations. Including herself, how many people can Amanda expect at her party? Write the series that models this situation in sigma notation then determine the partial sum.

Solution. In sigma notation, we have

$$\sum_{n=1}^{6} 3^{n-1},$$

which gives us

$$S_6 = \frac{1-3^6}{1-3} = 364.$$

Problem 19.3.23 — Because of air resistance, the length of each swing of a certain pendulum is 95% of the length of the previous swing. If the length traveled of the first swing of the pendulum is 40 cm, find the total length the pendulum will swing before coming to rest. Write the series that models this situation in sigma notation then determine the infinite sum.

Solution. Since $|0.95| < 1$, the sum converges. This means that in sigma notation, we have

$$\sum_{n=1}^{\infty} 40(0.95)^{n-1},$$

so the sum is

$$S_\infty = \frac{40}{1-0.95} = 800.$$

Problem 19.3.24 — A ball is dropped from a height of 36 meters. Each time it strikes the ground, it rebounds to a height seven-tenths of its previous height.

1. How high does the ball rise on the 6th bounce?
2. If the ball continues to bounce "forever", what will be the total distance that the ball traveled?

Solution.

1. Let a_i be the height on the ith bounce. Then

$$a_7 = a_1 r^{7-1} = 36\left(\frac{7}{10}\right)^6 = 4.235.$$

2. The total distance is modeled by

$$36 + 2\sum_{n=2}^{\infty} 36\left(\frac{7}{10}\right)^{n-1} = 36 + 2 \cdot \frac{\frac{126}{5}}{1-\frac{7}{10}} = 36 + 2 \cdot 84 = 204.$$

20 Binomial Expansion (5.18.20)

§20.1 Combinations

§20.1.1 Definition

Definition 20.1.1 (Combination) — A **combination** is the number of ways to choose a certain subset of elements from a set of elements where order does not matter.

Theorem 20.1.2 (Combination)
For all integer n, r, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

§20.1.2 Combinations on the Calculator

On a `TI-nSpire CX`, to compute $\binom{n}{r}$ type

`nCr(n,r)`.

For example, `nCr(20,4)` returns `4845`.

§20.2 Pascal's Triangle

Pascal's triangle is as follows:

$$\begin{array}{l ccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}$$

While here the 0th row to the 6th row is displayed, in reality it extends onto infinity. Here is Pascal's triangle in terms of combinations (i.e. binomial coefficients).

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

§20.3 Binomial Theorem

Theorem 20.3.1

For real or complex a, b, and non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

§20.4 Examples

Example 20.4.1

Expand, using Pascal's triangle, $(3x^2 - 2y)^4$.

Solution. The 4th row has $1, 4, 6, 4, 1$, implying the answer is

$$81x^8 - 216x^6y + 216x^4y^2 - 96x^2y^3 + 16y^4.$$

Example 20.4.2

Expand, using Binomial Theorem, $(2x - y)^5$

Solution. We know that the terms follow the general structure

$$\binom{n}{r} a^{n-r} b^r,$$

so the answer is

$$32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5.$$

Example 20.4.3

Find the third term of $(2x^2 + 3y)^6$.

Solution. The 3rd term is of the form

$$\binom{6}{2}(2x^2)^{6-2}(3y)^2,$$

so we get $2160x^8y^2$.

Example 20.4.4

Find the b^3 term of $(4a - b)^5$.

Solution. The term with b^3 will be

$$\binom{5}{3}(4a)^{5-3}(-b)^3,$$

so we get $-160a^2b^3$.

Example 20.4.5

Find the first three terms of $(a^2 - b^3)^7$.

Solution. The 7th row of Pascal's triangle is $1, 7, 21, 35, 35, 21, 7, 1$. Thus, we get

$$a^{14} - 7a^{12}b^3 + 21a^{10}b^6.$$

§20.5 Homework

Find the value of the following.

Problem 20.5.1 — $\binom{7}{3}$

Solution. Using the formula, we get

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35.$$

Problem 20.5.2 — $\binom{50}{48}$

Solution. This is equivalent to

$$\binom{50}{48} = \frac{50!}{48!2!} = 1225.$$

Expand and simplify.

Problem 20.5.3 — $(3x^2+2y)^5$

Solution. We know the coefficients are $1, 5, 10, 10, 5, 1$, so we get

$$243x^{10} + 810x^8y + 1080x^6y^2 + 720x^4y^3 + 240x^2y^4 + 32y^5.$$

Problem 20.5.4 — $(x^3-3y^2)^4$

Solution. Using the Binomial Theorem, we get

$$x^{12} - 12x^4y^2 + 54x^6y^4 - 108x^3y^6 + 81y^8.$$

Problem 20.5.5 — $(2x^4-5y^2)^3$

Solution. Using the Binomial Theorem, we get

$$8x^{12} - 60x^8y^2 + 150x^4y^4 - 125y^6.$$

Problem 20.5.6 — Find the third term of $(x^2-2y)^5$.

*Solution.*The third term occurs when $n=5$ and $r=2$. Thus,

$$\binom{5}{2}(x^2)^{5-2}(-2y)^2 = 40x^6y^2.$$

Problem 20.5.7 — Find the fourth term of $(2m^2-k^3)^9$.

Solution. Using the Binomial Theorem, we get

$$\binom{9}{3}(2m^2)^6(-k^3)^3 = -5376m^{12}k^9.$$

Problem 20.5.8 — Find the x^6 term of $(5x-3y)^8$.

Solution. Since $n=8$ and $r=2$, we get

$$\binom{8}{2}(5x)^{8-2}(-3y)^2 = 3937500x^6y^2.$$

Problem 20.5.9 — Find the first three terms of $(x^2-y^5)^7$.

Solution. Using the Binomial Theorem, we get

$$x^{14} - 7x^{12}y^5 + 21x^{10}y^{10}.$$

III

Spanish 3

21 Problemas Ambientales (4.7.20)

§21.1 "Problemas Ambientales" Article

§21.1.1 Instrucciones

Read the article "20 ejemplos de problemas ambientales" and then:

- On a sheet of paper write 10 words in Spanish and English that you think are key to understanding the article and are also relevant to the topic of the environment. Keep this sheet handy so you can continue to add words to it.
- Choose the three problems that in your opinion pose the gravest danger to the environment/people and write a 1-2 sentence summary of them in English. You can't just translate; you need to use your own words. The turn them in here.

Link: https://www.ejemplos.co/20-ejemplos-de-problemas-ambientales/#ixzz58JMPQTwW

§21.1.2 Vocabulario

1. **ambientales**: environmental
2. **la capa de ozono**: ozone layer
3. **deforestación**: deforrestation
4. **cambio climático**: climate change
5. **contaminación del aire/agua**: air/water contamination
6. **sustancias químicas**: chemical substances
7. **agotamiento de los suelos:** soil depletion
8. **desecho radiactivo:** radioactive waste
9. **biodegradable**: biodegradable
10. **sobrepoblación**: overpopulation

§21.2 Tres Problemas Más Grandes

Yo creo que los problemas más peligrosos son sobrepoblación, contaminación del aire/agua, y contaminación fotoquímica. Debido a la avaricia humana, la única forma en que los humanos ayudarán el medio ambiente es si los humanos se ayudan a sí mismos. Por lo tanto, nosotros necesitamos resolver la sobrepoblación primero, luego pasar a otros problemas. A continuación, nosotros debemos resolver el problema con nuestros recursos naturales, como agua y aire. Sin esos dos, nadie estaría vivo ahora. Además, esto significa que debemos solucionar el problema con contaminación fotoquímica, porque es lo que causa la contaminación del aire. Por estas razones, estos problemas representan el mayor peligro para la sociedad.

§21.3 Energía Nuclear

> **Problem 21.3.1 —** In Spanish, a 3-4 sentence paragraph with your personal opinion about nuclear energy.

Yo creo que la energía nuclear no es buena para el medio ambiente. El plutonio que se genera de la energía nuclear causa daños mucho más peligrosos que otras formas de energía. Las plantas de energía nuclear generan mucho poder, pero si hay un accidente o error humano, el daño será gigantesco como el caso de Chernobyl y Fukushima . Por lo tanto, yo pienso que es mejor que nosotros usemos una forma de energía más limpia y segura.

22 Calentamiento Global (4.14.20)

Si tu piensas que los problemas con el calentamiento global son de ciencia ficción, estas equivocado porque es un problema real. La tierra está protegida por una capa llamada “atmósfera”, la cual atrapa algunos de los rayos de sol para mantener una temperatura adecuada para la vida planetaria. Sin embargo, los humanos producimos gases en nuestra vida diaria que han causado el efecto invernadero, si los niveles de los gases tóxicos se incrementan los rayos del sol no pueden escapar y la temperatura aumenta. Existen varios factores que incrementan el nivel de gases en la atmósfera. Los arboles absorben CO2. Sin embargo, los estamos destruyendo con la deforestación. También, la sobrepoblación hace que se produzcan más alimentos se necesita más agua y más energía, esto genera un incremento en la temperatura. Aunado con la tecnología obsoleta para la producción de energía en base al carbón y la cantidad de carros y transportes que generan más de dos mil millones de toneladas de CO2 que se van directo a nuestra atmósfera. La consecuencia de esto es lo que genera el calentamiento global. También, hay otros problemas en el medio ambiente. Por ejemplo, la destrucción de la capa de ozono que causa daños al medio ambiente y a la vida planetaria. Además, la deforestación es un problema grave porque los árboles renuevan al oxígeno. Para concluir, es nuestra trabajo a ayudar y cuidar el medio ambiente y nuestro hogar.

23 Un Problema Ambiental Grande (4.21.20)

Problem 23.0.1 — You are going to create a video where you talk about the biggest environmental concern in your neighborhood/city/region/state, you can use your own knowledge or do a little research on the matter. Include as much detail as you can as well as your opinion about it. Your video should be about 1:30-2:00 minutes long.

La contaminación del aire es un gran problema en Houston. Houston es la segunda ciudad más contaminada en Tejas. La razón mayor de contaminación del aire es el metano que se quema en las refinerías y los pozos petroleros. Más de nueve millones de libras de CO2 se lanzan al aire en Houston cada año también dióxido de azufre, que produce lluvia ácida. La mayor contaminación del aire proveniente de la industria química especialmente la petrolera, más la gran cantidad de vehículos que cada año aumenta, causa problemas de salud a largo término en la población. Un ejemplo de estos problemas de salud son cáncer, enfermedades cardíacas y pulmonares. Las compañías en Houston y sus alrededores están obligadas a reportar todas sus emisiones al gobierno del estado para controlar la calidad del medio ambiente. Sin embargo, no todas lo hacen. Esto sin considerar los accidentes como las fugas y las explosiones que tuvimos el año pasado en donde grandes cantidades de benceno y otros contaminantes fueron lanzadas al aire sin control, causando una mala calidad del aire por muchos meses. Yo creo que la contaminación del aire debe detenerse, porque si continúa, nosotros no podremos salir de nuestras casas. Si no lo detenemos, estaremos viviendo en una cuarentena permanente.

24 Proteger El Medio Ambiente (4.28.20)

Problem 24.0.1 — Think of a way that you could protect the environment by changing a daily habit. Create a persuasive video encouraging others to follow your lead.

Una de cada tres personas deja la llave abierta cuando se baña, o lava los trastes. Esto podría desperdiciar hasta veinticuatro (24) litros de agua por día por persona. Al reducir la cantidad de agua desperdiciada, estarías ayudando a los pobres como a ti mismo. Las personas en las comunidades más pobres podrían tener agua limpia diario. También reutilizar el agua de la ducha y de la lavadora para los baños, o para el riego. Con estos simples hábitos diarios, podemos ahorrar agua y cuidar el planeta. Les voy a dar unos consejos para cuidar el agua en casa. Uno - cierra la llave mientras te lavas los dientes. Dos - dúchate con agua fría y hazlo por menos tiempo. Tres - cierra la llave mientras te enjabonas. Cuatro - utiliza un balde o un cubo para regar las plantas y lavar tu carro en lugar de usar la manguera. Cinco - cuando laves la ropa, utiliza las cargas completas. No laves pocas prendas. Seis - verifica que no tengas fugas de agua, si es así llama al plomero para repararlas de inmediato. Recuerda que el agua es un recurso no renovable y es vital para la subsistencia de humanos y animales. ¡Cierra la llave y salva vidas!

25 Tarea de Español

A few of the smaller homeworks for Spanish 3.

§25.1 5.5.20

1. Antes uno tenía que llevar si máquina de escribir Olivetti.
2. Antes tenías que ir a la biblioteca a hacer investigaciones.
3. Antes hacías los trabajos con recortes de periódicos y revistas.
4. Antes uno tenía que memorizar las direcciones o usar mapas impresos.
5. Antes tenían que escribir mucho porque no tenían computadores.

As you can tell, there isn't much explanation for these. Sorry.

1. Ahora, tan solo podemos investigar en Google.
2. Ahora, nosotros pagamos mucho dinero por un almuerzo o lo llevamos de nuestras casas.
3. Ahora, todos los estudiantes tenemos una laptop.
4. Ahora, podemos usar e-mail para comunicarnos, y podemos usar Google Docs para trabajar juntos con otros estudiantes.
5. Ahora, podemos tener clases en línea, por ejemplo, en esta pandemia que no podemos ir a la escuela, pero no perdemos el curso.

§25.2 5.12.20

De acuerdo con la prueba de tecnología, yo soy un tecnomaniático. Yo estoy de acuerdo con la prueba, porque yo uso mis electronicos mucho y yo entiendo mucho sobre tecnología. Aunque yo no compro mis propios aparatos electrónicos, estoy bien informado sobre los últimos avances en tecnología. Por otro lado, mi madre es una ingeniera en computación, por lo que también está bien informada y capacitada en tecnología. Mi casa está llena con aparatos electrónicos, así que yo soy un tecnomaniático.

§25.2.1 Lunes

Hoy salí a caminar y vi una caja negra con personas adentro, pero había otras personas mirándolos. Los de adentro en la caja parecían pequeños humanos corriendo detrás de un objeto redondo que rodaba a gran velocidad y todos los humanitos querían golpearlo con sus pies. Las personas que estaban afuera de la caja gritaron algo parecido a "gol", y todo festejaron. ¡Que interesante!

§25.2.2 Martes

¡Hoy fue un día de muchas sorpresas, pero la mayor fue ver un objeto grande con personas adentro! Iba a gran velocidad y tenía cuatro cosas que giraban y lo movían, parecía el barco de faraón flotando sobre agua, pero no había agua. También, había un hombre que movía algo redondo con dos manos para girar esta extraña nave. !Lo más raro fue, que tenía puertas!

§25.2.3 Miercoles

Hoy ha sido el día más emocionante. He visto a un pájaro gigantesco. Yo pensaba que era una nave extraterrestre y que nos desintegraria, lo he seguido pero vuela muy rápido. Está vez ha superado todas las locuras que he visto en este tiempo. Es como la nave del faraón pero no hay agua ni tierra. ¡He tenido muchas sorpresas! ¡Me regreso a mi pirámide! ¡Ya no quiero ser eterno!

IV

Physics 1

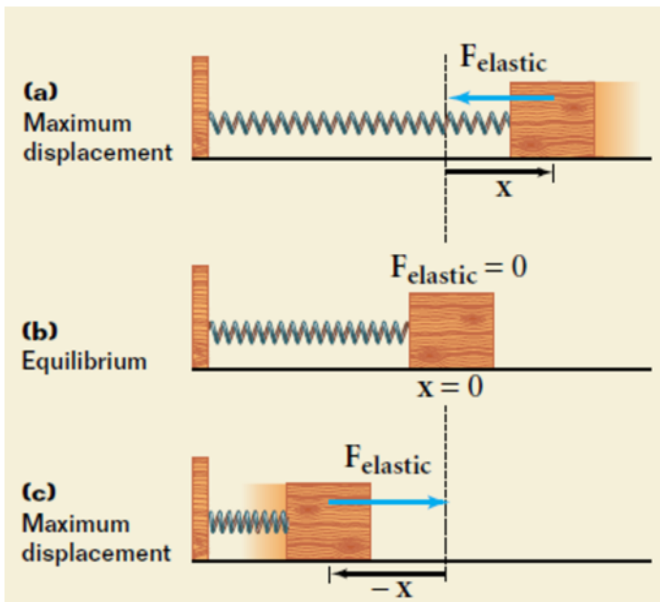
26 Simple Harmonic Motion (4.3.20)

§26.1 Introduction and Definitions

A **repeated motion,** such as that of an acrobat swinging on a trapeze, is called a periodic motion.

Other periodic motions include those made by a child on a playground swing, a wrecking ball swaying to and fro, and the pendulum of a grandfather clock or a metronome. In each of these cases, **the periodic motion is back and forth over the same path**.

One of the simplest types of back-and-forth periodic motion is a mass attached to a spring. Let us assume that the mass moves on a frictionless horizontal surface.



When the spring is stretched or compressed and then released, it vibrates back and forth around its unstretched position.

We will begin by considering this example, and then we will apply our conclusions to the swinging motion of a trapeze acrobat.

In the figure below, the spring is stretched away from its unstretched, or equilibrium, position ($x = 0$).

When released, the spring exerts a force on the mass toward the equilibrium position. This spring force decreases as the spring moves towards the equilibrium position, and it reaches 0 at equilibrium - the mass's acceleration also becomes 0 at equilibrium.

Though the spring force and the acceleration decrease as the mass moves toward the equilibrium position, the velocity of the mass increases. At the equilibrium position, when the acceleration reaches 0, velocity reaches a maximum.

At that point, although no net force is acting on the mass, the mass's momentum causes it to overshoot the equilibrium position and compress the spring.

As the mass moves beyond equilibrium, the spring force and the acceleration increase. But the direction of the spring force and the acceleration (**toward equilibrium**) is opposite the mass's direction of motion (**away from equilibrium**), and the mass begins to slow down.

When the spring's compression is equal to the distance the spring was originally stretched away from the equilibrium position, the mass is at maximum displacement, and the spring force and acceleration of the mass reach a maximum.

At point (c), the velocity of the mass becomes 0.

The spring force acting to the right causes the mass to change its direction, and the mass begins moving back toward the equilibrium position. Then the entire process begins again, and the mass continues to oscillate back and forth over the same path.

In an ideal system, the mass-spring system would oscillate indefinitely. But in the physical world, friction slows the motion of the vibrating mass, and the mass-spring system eventually comes to rest. This effect is called **damping**.

In most cases, the effect of damping is minimal over a short period of time, so the ideal mass-spring system provides an approximation for the motion of a physical mass-spring system.

As you have seen, the spring force always pushes or pulls the mass back toward its original equilibrium position. For this reason, it is sometimes called a **restoring force**. Measurements show that **the restoring force is directly proportional to the displacement of the mass**.

Any periodic motion that is the result of a restoring force that is proportional to displacement is described by the term **simple harmonic motion**. Because simple harmonic motion involves a restoring force, every simple harmonic motion is a back-and-forth motion over the same path.

In 1678, Robert Hooke found that most mass-spring systems obey a simple relationship between force and displacement. For small displacements from equilibrium, the following equation describes the relationship:

Theorem 26.1.1 (Hooke's Law)

$F_{\text{elastic}} = -kx$, where F_{elastic} is the spring force, k is the spring constant, and x is the displacement from the equilibrium position. The SI units of k is $\frac{N}{m}$. The negative sign in the equation signifies that the direction of the spring force is always opposite the direction of the mass's displacement from equilibrium. In other words, the negative sign shows that the spring force will tend to move the object back to its equilibrium position.

The term k is a positive constant called the spring constant. The value of the spring constant is a measure of the **stiffness** of the spring. A greater value of k means a stiffer spring because a greater force is needed to stretch or compress that spring.

A stretched or compressed spring stores elastic potential energy. To see how mechanical energy is conserved in an ideal mass-spring system, consider an archer shooting an arrow from a bow. Bending the bow by pulling back the bowstring is analogous to stretching a spring. To simplify this situation, we will disregard friction and internal energy. Once the bowstring has been pulled back, the bow stores elastic potential energy. Because the bow, arrow, and bowstring (the system) are now at rest, the kinetic energy of the system is zero, and the mechanical energy of the system is solely elastic potential energy. When the bowstring is released, the bow's elastic potential energy is converted to the kinetic energy of the arrow. At the moment the arrow leaves the bowstring, it gains most of the elastic potential energy originally stored in the bow. Thus, once the arrow has been released, the mechanical energy of the bow-and-arrow system is solely kinetic. Because mechanical energy must be conserved, the kinetic energy of the bow, arrow, and bowstring is equal to the elastic potential energy originally stored in the bow.

§26.2 Pre-Investigation for SHM

Pre-Investigation:

1. For the following questions, use the figure above.
   a. What happens to the mass on the spring when it reaches its *equilibrium position*?
      i. No force is acting on the mass.
      ii. The mass's momentum causes it to overshoot & compress the spring.
      iii. Therefore, velocity & momentum are maximum.
      iv. Acceleration is zero.

   b. What happens to the mass on the spring when it reaches its maximum displacement?
      i. The acceleration reaches a maximum.
      ii. Therefore, velocity is zero.
      iii. The spring force acting to the right causes the mass to change its direction.
      iv. This happens because the spring force is a restoring force.

2. Using your electronic device, research the term **oscillate.** the repeated back and forth movement of something between two positions or states.

3. What would cause a system—like a mass on a bobbing spring—to stop oscillating? An external force acting on the spring

4. Using your electronic device, research the term **damping** (as used in simple harmonic motion).
   a. restraining of vibratory motion
   b. Provide 2 examples of damping.
      i. shock absorbers
      ii. putting your hands on a cello to stop it from vibrating

5. Why is simple harmonic motion *periodic*? because the restoring forcec allows it to return back to its original position.

6. Draw a *sine wave* below & label its *equilibrium position*.

§26.3 Standing Waves and Wave Action

Link: https://shaverphysics.weebly.com/uploads/2/2/9/4/22945204/standing_waves_and_wave_action_key.pdf

27 Vibrational Motion (4.7.20)

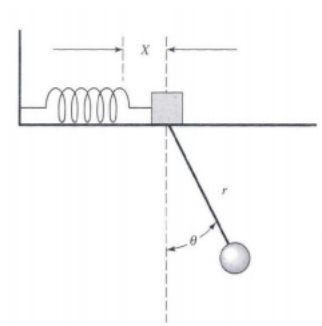
§27.1 Vibrational Motion Activity

Link: https://www.physicsclassroom.com/class/waves/Lesson-0/Vibrational-Motion

1. What is a simple harmonic oscillator? A simple harmonic oscillator is a system that, when displaced from equilibrium, experiences a restoring force that can bring it back to equilibrium.

2. How does vibrational motion contrast with translational motion? Translational motion often results in the object being permanently displaced, but vibrational motion can be restored to its original position.

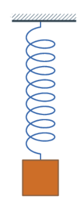
3. How does the back-and-forth motion of a box on a spring mirror the motion of a pendulum? It is similar in that they both have an equilibrium position, and each has a restoring force (in the figure, one is caused by the elastic force and the other by gravity) that allows it to return to equilibrium.



28 Periodic Motion (4.9.20)

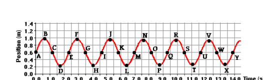
§28.1 Periodic Motion Activity

A vibrating object is wiggling about a fixed position. Like the mass on a spring in the animation at the right, a vibrating object is moving over the same path over the course of time. Its motion repeats itself over and over again. If it were not for damping, the vibrations would endure forever (or at least until someone catches the mass and brings it to rest). The mass on the spring not only repeats the same motion, it does so in a regular fashion. The time it takes to complete one back and forth cycle is always the same amount of time. If it takes the mass 3.2 seconds for the mass to complete the first back and forth cycle, then it will take 3.2 seconds to complete the seventh back and forth cycle. It's like clockwork. It's so predictable that you could set your watch by it. In Physics, a motion that is regular and repeating is referred to as a **periodic motion**. Most objects that vibrate do so in a regular and repeated fashion; their vibrations are periodic.



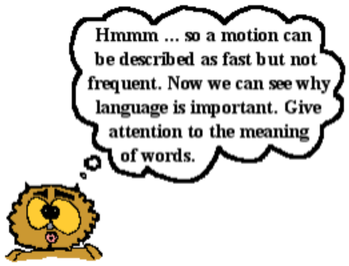
§28.1.1 Sinusoidal Nature of a Vibration

Suppose that a motion detector was placed below a vibrating mass on a spring in order to detect the changes in the mass's position over the course of time. And suppose that the data from the motion detector could represent the motion of the mass by a position vs. time plot. The graphic below depicts such a graph. For discussion sake, several points have been labeled on the graph to assist in the follow-up discussion.



§28.1.2 Work from Activity

1. Justify the owl's statement in the illustration below. The owl's statement can be justified with an example: how many times you go on a walk does not equal how fast you complete the walk. **Frequency** represents the number of times, and **fast** represents the speed during those times. Thus, language is important to being clear about your statements.



2. At the bottom of the article, there is a section called We Would Like to Suggest... Please visit the Mass on a Spring Interactive. Answer the question below.

   Experiment with the masses and various amounts of stiffness for one spring. Does the stiffness (remember, this is the k factor in Hooke's Law) affect the way the spring vibrates? Yes if so, how? The displacement is increased because x = -F/k, and because the force (gravity) remains the same, the displacement is inversely proportional to the stiffness.

3. What factors affect the period of oscillation for a simple pendulum & a mass oscillating on a spring? The period of oscillation for a simple pendulum is it's length and gravity, whereas in a mass oscillating on a spring, mass and the stiffness affect the period.

29 Newton's Universal Law of Gravitation (4.14.20)

§29.1 Apple and the Moon

Both the moon and the apple are attracted by the Earth - but only the apple ever actually reaches the Earth.

Theorem 29.1.1 (Newton's Law of Gravitation)
For two objects m_1, m_2 a distance r apart, their gravitational attraction to each other is

$$F = -G\frac{m_1 m_2}{r^2}\hat{r}.$$

Note that G is the Universal Gravitational Constant of value $6.67 \cdot 10^{-11} m^3 kg^2 s^{-2}$, and $\hat{r}$ represents the direction. For example, if we go from one object to another, the force of gravity on the first object is opposite in direction, and pulls it to the second object. This is why the minus sign and $\hat{r}$ are usually excluded.

Theorem 29.1.2 (Acceleration from Law of Gravitation)
The acceleration a of an object due to another object of mass M and a distance R from the object is

$$a = -G\frac{M}{R^2}\hat{r}.$$

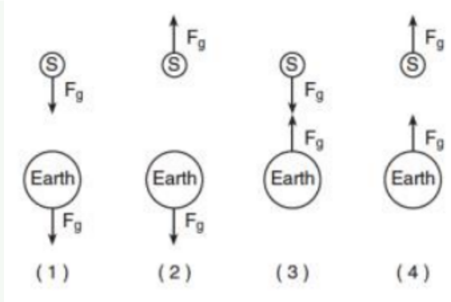
Notice that R represents the distance of the object's center to the center of the other object, because **gravity acts on the center of mass of the object**. Also note that $a = g$ on Earth, so

$$g = G\frac{M_E}{R_E^2},$$

where M_E is the mass of the earth, and R_E is the distance from the Earth to the object. The object of the other mass has no affect on the gravitational acceleration.

§29.2 Universal Gravitation Exercise

Problem 29.2.1 — Which diagram best represents the gravitational forces, F_g, between a satellite, S, and Earth?



Solution. The answer is (3), because gravity makes objects attract each other.

Problem 29.2.2 — Suppose that two objects attract each other with a gravitational force of 16 units. If the distance between the two objects is reduced in half, then what is the new force of attraction between the two objects?

Solution. Notice that

$$F \propto \frac{1}{r^2}$$

when the masses are constant. Thus, halving the distance increases the force by 4 times, so the answer is 64 units.

Problem 29.2.3 — Suppose that two objects attract each other with a gravitational force of 16 units. If the mass of both objects was tripled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?

Solution. Tripling both the masses will result in the force being 9 times stronger, and doubling the distance will weaken the force 4-fold. Thus, $\frac{9}{4} = 2.25$ times stronger.

Problem 29.2.4 — Having recently completed her first Physics course, Dawn Well has devised a new business plan based on her teacher's Physics for Better Living theme. Dawn learned that objects weigh different amounts at different distances from Earth's center. Her plan involves buying gold by the weight at one altitude and then selling it at another altitude at the same price per weight. Should Dawn buy at a high altitude and sell at a low altitude or vice versa?

Solution. She should buy at the high altitude, because the distance is increased and therefore the force of gravity is less, and sell at the low altitude, because the distance is decreased and therefore the force of gravity is more.

Problem 29.2.5 — When comparing mass and size data for the planets Earth and Jupiter, it is observed that Jupiter is about 300 times more massive than Earth. One might quickly conclude that an object on the surface of Jupiter would weigh 300 times more than on the surface of the Earth. For instance, one might expect a person who weighs 500 N on Earth would weigh 150000 N on the surface of Jupiter. Yet this is not the case. In fact, a 500-N person on Earth weighs about 1500 N on the surface of Jupiter. Explain how this can be.

Solution. This is due to the size of Jupiter - because Jupiter is bigger, the distance from the center is increased. There is also a difference in density, but that is a result of the larger volume. From here, we can actually calculate the ratio of the radius of Jupiter to the radius of the Earth.

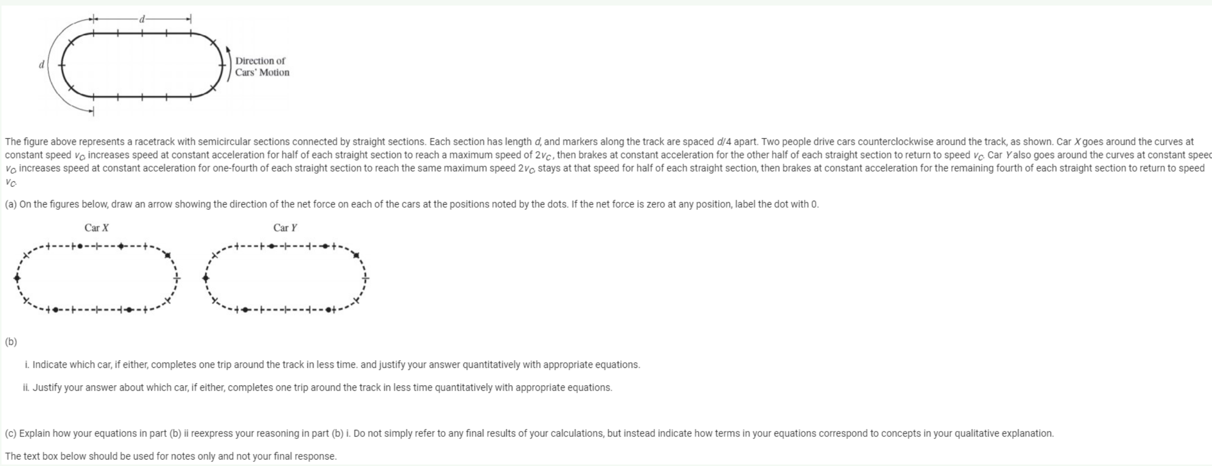
30 AP Physics FRQ (4.16-19.20)

The following are my solutions to the Unit FRQs on College Board for AP Students.

§30.1 Unit 1 FRQ

§30.1.1 FRQ 1

Problem 30.1.1 — A screenshot of the problem (apologies for the resolution):


d
d
Direction of Cars' Motion

The figure above represents a racetrack with semicircular sections connected by straight sections. Each section has length d, and markers along the track are spaced $d/4$ apart. Two people drive cars counterclockwise around the track, as shown. Car X goes around the curves at constant speed v_C, increases speed at constant acceleration for half of each straight section to reach a maximum speed of $2v_C$, then brakes at constant acceleration for the other half of each straight section to return to speed v_C. Car Y also goes around the curves at constant speed v_C, increases speed at constant acceleration for one-fourth of each straight section to reach the same maximum speed $2v_C$, stays at that speed for half of each straight section, then brakes at constant acceleration for the remaining fourth of each straight section to return to speed v_C.

(a) On the figures below, draw an arrow showing the direction of the net force on each of the cars at the positions noted by the dots. If the net force is zero at any position, label the dot with 0.

Car X Car Y

(b)

i. Indicate which car, if either, completes one trip around the track in less time. and justify your answer quantitatively with appropriate equations.

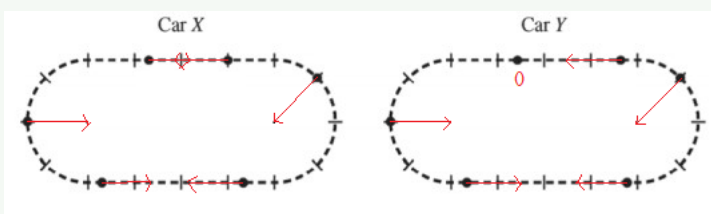
ii. Justify your answer about which car, if either, completes one trip around the track in less time quantitatively with appropriate equations.

(c) Explain how your equations in part (b) ii reexpress your reasoning in part (b) i. Do not simply refer to any final results of your calculations, but instead indicate how terms in your equations correspond to concepts in your qualitative explanation.

The text box below should be used for notes only and not your final response.

Solution.

(a) The following portrays how outside the circular regions, no force is applied, whereas inside the circular regions the net force is centripetal force, which points towards the center.



(b)

i. Because both cars go around the curved region at the same speed, and each region is the same length, we can ignore the times of these regions. Thus, we only need to take into consideration the straight regions.

In the first half of one of the straight lines for Car X, we have the equation

$$\Delta x = v_i t + \frac{1}{2}at^2,$$

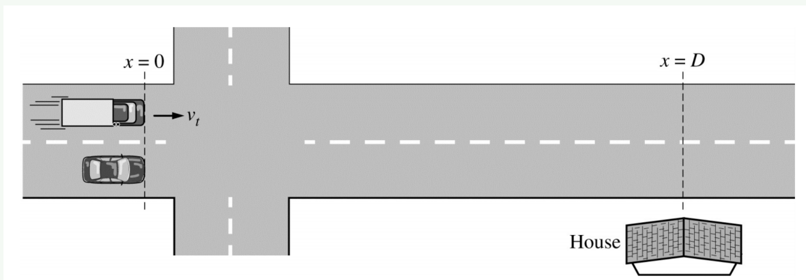
which implies

$$\Delta x = v_C t \dots$$

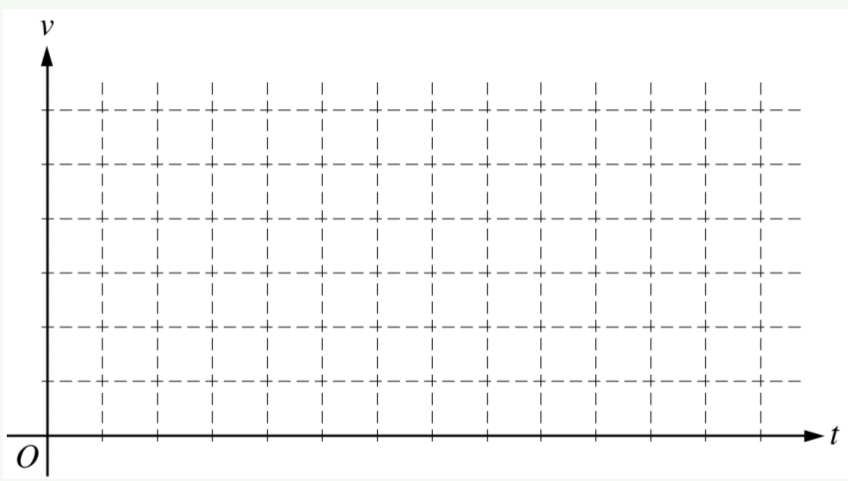
Note: This solution was *never completed.* The rest of the solution is left as an exercise to the reader.

§30.1.2 FRQ 2

Problem 30.1.2 — A car is stopped at a traffic light. The light turns green, and at time $t = 0$ the car starts moving and travels with a constant acceleration. At that instant a truck traveling at constant speed v_t is alongside the car, with the front of each vehicle at position $x = 0$, as shown above. The truck passes the car, but the car later catches up to the truck in front of a house, such that at time t_D the front of each vehicle is at position $x = D$.



a. On the axes below, sketch and label graphs of the velocity of the car and the velocity of the truck as a function of time. Indicate any important velocities or times.



b. Two students are discussing how the speed of the car compares to the speed of the truck when both vehicles are in front of the house.

Student 1 says, "The distance traveled by the car and the truck is the same, and the time is the same, so they must have the same speed."

Student 2 says, "I don't see how that can be. The car catches up to the truck, so the car has to be going faster."

i. Which aspects of Student 1's reasoning, if any, are correct? Support your answer in terms of relevant features of your graphs in part (a).

ii. Which aspects of Student 2's reasoning, if any, are correct? Support your answer in terms of relevant features of your graphs in part (a).

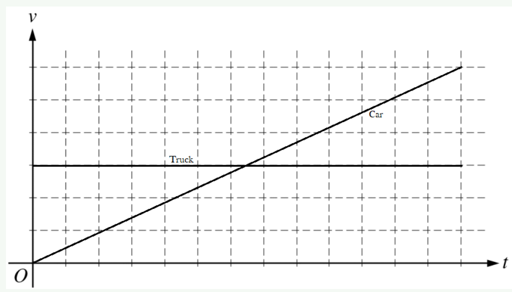
c. Derive an expression for the acceleration of the car. Express your answer in terms of D and v_t.

d. Determine the time at which the speed of the car is equal to the speed v_t of the truck. Express your answer in terms of t_D. Justify your answer.

The text box below should be used for notes only and not your final response.

Solution.

a. This is the graph for (a):



b.

i. The distance and time travelled is the same, since the area is identical at $t = t_D$, but this doesn't correlate to the same speed, since the speed of the car changes.

ii. The idea that the car catches up is correct, because the area under the lines are the same.

c. The truck covers a distance D with a constant velocity of v_t. Thus, it took a time of

$$t_D = \frac{D}{v_t}.$$

The car also covered a distance D, and the initial velocity is 0. Let's say the final velocity is v_f. Then

$$\frac{1}{2}(0 + v_f)t_D = D.$$

Since

$$\frac{D}{t_D} = v_t,$$

we have

$$\frac{1}{2}v_f = v_t,$$

$$v_f = 2v_t.$$

Thus, the acceleration is

$$\frac{v_f}{t_D} = \frac{2v_t}{t},$$

and $t_D = \frac{D}{v_t}$, so the acceleration is $\boxed{\frac{2v_t^2}{D}}$.

d. They are equal when they intersect (because then the y-value is identical), and the equation of the car is

$$v = \frac{2v_t^2}{D}t,$$

and the equation of the truck is

$$v = v_t,$$

so

$$v_t = \frac{2v_t^2}{D}t,$$

$$t = \boxed{D}2v_t.$$

V

Computer Science 2

31 ArrayList (4.3.20)

§31.1 College Board AP Computer Science ArrayList Notes

§31.1.1 AnimalTester

```java
// Anatomy of a Class and Anatomy of an AL<> Sample Code
// Jill Westerlund | 3-29-2020

import java.util.ArrayList;

public class AnimalTester
{
   public static void main(String[] args)
   {

   ArrayList<Animal> MyList = new ArrayList<Animal>();
   MyList.add(new Animal("Felis Domesticus", "Cat"));
   MyList.add(new Animal("Camelus dromedarius", "Camel"));
   MyList.add(new Animal("Chamaeleonidae", "Chameleon"));

   System.out.println(MyList);
   System.out.println("Size: " + MyList.size());
   System.out.println();

   // traversal using for loop
   System.out.println("Traversal 1: Traditional for loop");
   for (int k = 0; k < MyList.size(); k++)
     System.out.println(MyList.get(k) + " ");
   System.out.println();
   System.out.println("--------------");

   System.out.println("Traversal 2: Enhanced for Loop");
   // traversal using enhanced for loop
   for (Animal item : MyList)
     System.out.println(item + " ");
   System.out.println("--------------");

   System.out.println("Get Random Animal");
   // Extract random element from MyList
   int rand = (int)(Math.random() * MyList.size());
   System.out.println(MyList.get(rand));
   System.out.println();
   System.out.println();
   System.out.println("the end--------------");
   }
```

```
}
```

Listing 31.1: AnimalTester Code

§31.1.2 Animal

```
//-----------Animal Class starts here-----------
public class Animal
{
   private String sciName;
   private String comName;

  public Animal(String sN, String cN)  //constructor for Animal Class
  {
    sciName = sN;
      comName = cN;
  }

  public String getSciName() { return sciName; }

   public String getComName() { return comName; }

   public String toString() {
     return "" + sciName + " " + comName; }
}
```

Listing 31.2: Animal Code

§31.1.3 ConstructorAnimalSample

```
// ConstructorsAnimalSample.java
// Jill Westerlund March 26 2020
// This program demonstrates inheritance at three levels.
// Sources: https://en.wikipedia.org/wiki/Hippopotamus,
   https://www.livescience.com/27339-hippos.html

public class ConstructorAnimalSample
{
  public static void main(String[] args)
  {
    Hippo henrietta = new Hippo("Hippopotamus
       amphibius","Hippopotamus","River,Swamp,Lake",125000,false);
    System.out.println();
    System.out.println("Animal scientific name:   " + henrietta.getSciName());
    System.out.println("Animal habitat: " + henrietta.getHabitat());
    System.out.println("Animal tagged:     " + henrietta.getTaggedStatus());
  }
}

class Animal
{
  private String sciName;
   private String comName;
```

```

  public Animal(String sN, String cN)  //constructor for Animal Class
  {
    sciName = sN;
      comName = cN;
      System.out.println("Animal Constructor Called");
  }

  public String getSciName() { return sciName; }

   public String getComName() { return comName; }
}


class Wild extends Animal                //2nd level Class in hierarchy
{
  private String habitat;
   private int estPop;

  public Wild(String sN, String cN, String hab, int estP)   //constructor for Wild Class
  {
    super(sN,cN);                        //passing PIVs from superclass; must be first line
    habitat = hab;
      estPop = estP;
    System.out.println("Wild Constructor Called");
  }

    public String getHabitat() { return habitat; }

      public int getEstimatedPop() { return estPop; }
}

class Hippo extends Wild               //3rd level Class in hierarchy
{
  private boolean isTagged = false;     //optional: set a default value for isTagged

  public Hippo(String sN, String cN, String hab, int estP, boolean tag)
  {
    super(sN,cN,hab,estP);
      isTagged = tag;
    System.out.println("Hippo Constructor Called");
  }

  public boolean getTaggedStatus() { return isTagged; }

}
```

Listing 31.3: ConstructorAnimalSample Code

§31.1.4 DogTester

```java
/*
 * Date: 3/22/2020
 * Programmer: Rob Schultz
 * Purpose:  Demonstrate the use of super and sub-classes
 */

public class DogTester{

   public static void main(String[] args){

      BassetHound JW_Dog = new BassetHound();

      JW_Dog.sleep();

      MixedBreed RS_Dog = new MixedBreed();

      RS_Dog.sleep();


      /*
      Dog[] ourDogs = new Dog[3];

      //IMPORTANT: Remember that, at this point, we have an Array of
      //reference variables that can point to Dog objects...but each
      //variable at this point is NULL.  You must assigne each specific
      //element in the Array to a Dog object. The next three lines will
      //resolve this issue!

      ourDogs[0] = new BassetHound();
      ourDogs[1] = new MixedBreed();
      ourDogs[2] = new BassetHound();

      for (int index = 0; index < ourDogs.length; index++){
         System.out.println("");
         System.out.println(ourDogs[index].toString());
      }
      */

   }
}
```

Listing 31.4: DogTester Code

§31.1.5 Dog

```java
/*
 * Date: 3/22/2020
 * Programmer: Rob Schultz
 * Purpose: Demonstrate a Parent (Super) Class)
 */

```

```java
 public class Dog{

   private String name;
   private int age;
   private double weight;

   public Dog(){
      name = "Bingo";
      age = 3;
      weight = 20.5;
   }

   public String getName(){
      return name;
   }

   public void breath(){
      System.out.println("Inhale...Exhale...Inhale...Exhale...Inhale...");
   }

   public void sleep(){
      System.out.println("ZZZZZZzzzzzzz.....");
   }

   public String toString(){
      String str = "Name: " + name
                 + "\nAge: " + age
                 + "\nWeight: " + weight;
      return str;
   }
}
```

Listing 31.5: Dog Code

§31.2 ArrayList Methods

ArrayList Class	
`int size()`	Returns the number of elements in the list
`boolean add(E obj)`	Appends `obj` to end of list; returns `true`
`void add(int index, E obj)`	Inserts `obj` at position `index` `(0 <= index <= size)`, moving elements at position `index` and higher to the right (adds 1 to their indices) and adds 1 to size
`E get(int index)`	Returns the element at position `index` in the list
`E set(int index, E obj)`	Replaces the element at position `index` with `obj`; returns the element formerly at position `index`
`E remove(int index)`	Removes element from position `index`, moving elements at position `index + 1` and higher to the left (subtracts 1 from their indices) and subtracts 1 from size; returns the element formerly at position `index`

32 Unit 6 Progress Check: FRQ (4.6.20)

§32.1 Question 1

Problem 32.1.1 — Assume that the classes listed in the Java Quick Reference have been imported where appropriate. Unless otherwise noted in the question, assume that parameters in method calls are not and that methods are called only when their preconditions are satisfied. In writing solutions for each question, you may use any of the accessible methods that are listed in classes defined in that question. Writing significant amounts of code that can be replaced by a call to one of these methods will not receive full credit.

An array of String objects, words, has been properly declared and initialized. Each element of words contains a String consisting of lowercase letters (a–z).

Write a code segment that uses an enhanced for loop to print all elements of words that end with "ing". As an example, if words contains "ten", "fading", "post", "card", "thunder", "hinge", "trailing", "batting", then the following output should be produced by the code segment.

fading
trailing
batting

Write the code segment as described above. The code segment must use an enhanced for loop to earn full credit.

Solution.

```
    for(String s : words){
            int length = s.length();
            if(s.substring(length-3,length).equals("ing")){
                System.out.println(s);
            }
     }
```

Listing 32.1: Unit 6 FRQ Code 1

§32.2 Question 2

Problem 32.2.1 — (Problem not included due to length.)

Solution.

```
    double sum = 0;
      for(double i : itemsSold){
            sum+=i;
      }
      double length = (double) itemsSold.length;
      System.out.println(sum/length);
```

Listing 32.2: Unit 6 FRQ Code 2a

```java
    public void computeWages(double fixedWage, double perItemWage){
      double bonus = computeBonusThreshold();
      for(int i = 0; i < itemsSold.length; i++){
         if(bonus > itemsSold[i]){
            wages[i] = fixedWage + perItemWage * itemsSold[i];
         }
         else{
            wages[i] = (fixedWage + perItemWage * itemsSold[i])*1.1;
         }
      }
    }
```

Listing 32.3: Unit 6 FRQ Code 2b

VI

Biology

33 Pathogens and Immunity (4.3.20)

§33.1 Introduction and Definition

With coronavirus, the topic on everyone's mind is pathogens (or more specifically, viruses).

Definition 33.1.1 (Pathogens) — any agent that cause disease. The etymology of the word **pathogen** stems from "path", meaning bad, and "gen", meaning origin or creation.
Pathogens can include bacteria, viruses, parasitic worms, fungi, protists and prions.

We will be focusing on bacteria and viruses, but let us briefly describe some of the rest.
Some fungal disease include:

- Yeast infection
- Ring worm
- Other skin infections

Protists cause dissease including

- Malaria
- Dysentery
- Giardia

Parasitic worms include things like

- Heartworms in valves
- Hookworms
- Tapeworms

Prions can cause *Mad Cow Disease*.

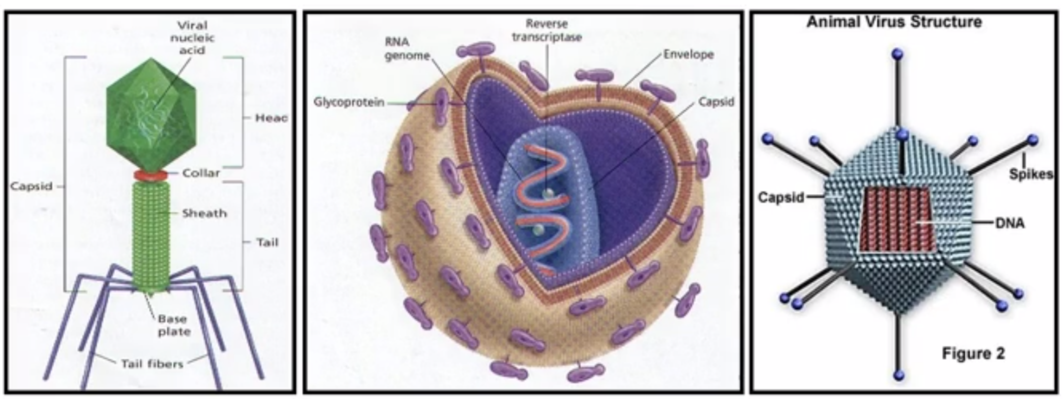
§33.2 Kingdom of Viruses

Example 33.2.1
What kingdom do **viruses** belong to?
Solution. Trick question - they don't belong to any of the kingdoms, because they are **not living organisms**!

Definition 33.2.2 (8 Characteristics of Living Things) — The 8 characterisitics include

1. Made of cells
2. Can reproduce
3. Contain a universal genetic code
4. Grow and develop
5. Obtain and use materials and energy
6. Respond to the enviironment
7. Maintain homeostasis
8. Change over time

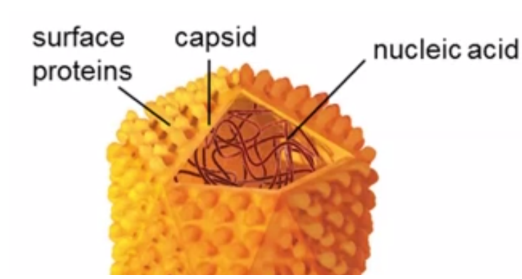
Viruses only satisfy **3, 6, and 8**. Viruses are **not** made of cells. They are hundreds of times smaller than cells. All viruses have the **same major parts**, but each virus infects a **specific** type of host.



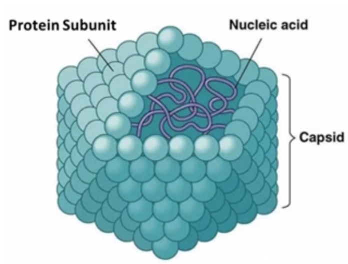
§33.3 Structure of a Virus

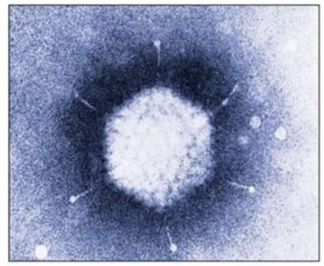
All viruses have a very simple structure.

- **Genetic material:** DNA *or* RNA (not both!)
- **Capsid:** *protein* coat that determines the shape and infection process and houses nucleic acids.

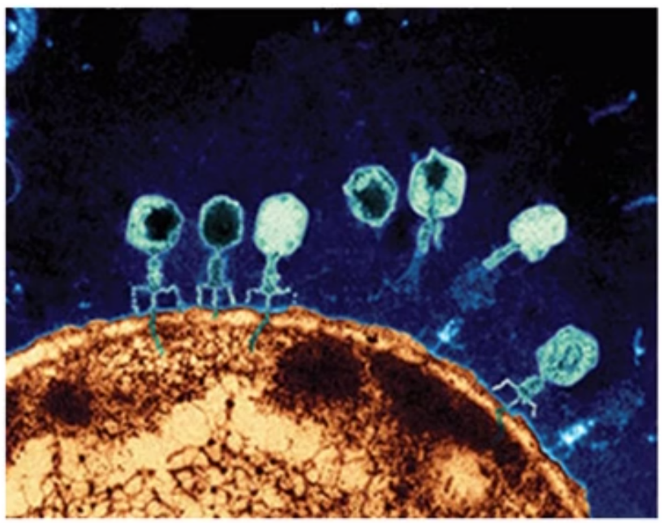


The **capsid** provides *protection* for the virus' genes and provides a way for the virus to enter a host cell.





Viruses are not cells, so they don't have the machinery to reproduce on their own. They need a **host cell** to copy their viral genes for them.



§33.3.1 RNA Viruses

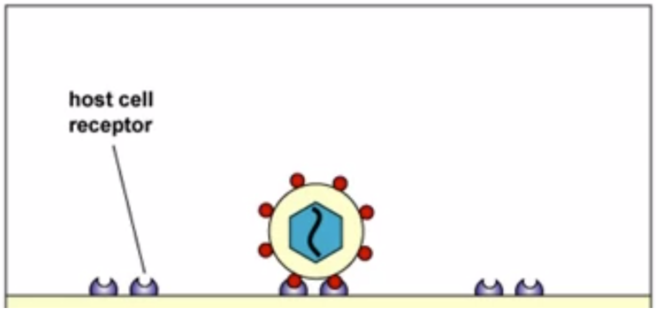
RNA viruses do not have built in proofreading, so they mutate and evolve **quickly**! A few examples:

- HIV
- Chicken Pox

A virus' capsid plays an important role in **how it infects a host cell**. Viruses "trick" the host cell into letting the virus in by **mimicking molecules** the host cell needs.

§33.3.2 Viral Attachment

Viruses have special proteins called **spike** that stick out of their capsid and **match** with **receptor proteins** on a host cell membrane.



Viruses are species-specific. Viruses attach to cells by recognizing certain proteins on cell membranes, kind of like a **lock and key**. In fact, they are specific to a tissue pipe in that species. For example, viruses that trick our nerves do not trick our lungs.

Viruses that can jump from one species to another are called **zoonotic diseases**. Viruses can cross-species if they mutate. For example, SARS and bird flu. More recently, COVID-19 jumped from bats (not in soup!) to an intermediate larger species, then to us. These mutations are **random**, not with an end goal in mind.

Example 33.3.1
Can we kill viruses?

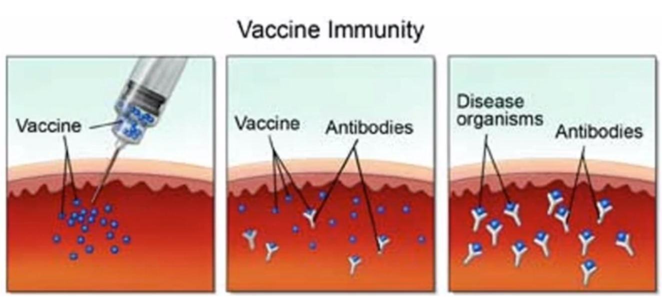
Solution. No, because they are **not living**. However, we can destroy their **structure**. By using soap or alcohol, we can break their structure and alleviate any symptoms they could cause.

§33.4 Prevention

There are two ways to fight off viruses:

1. Antibody Immune Response, something your body naturally does
2. Vaccinations, which help speed up the immune response

However, it takes 12-18 months to develop a vaccine. Vaccines are **preventative**. They do not treat people who are already sick, but rather give our immune systems a chance to experience it beforehand.

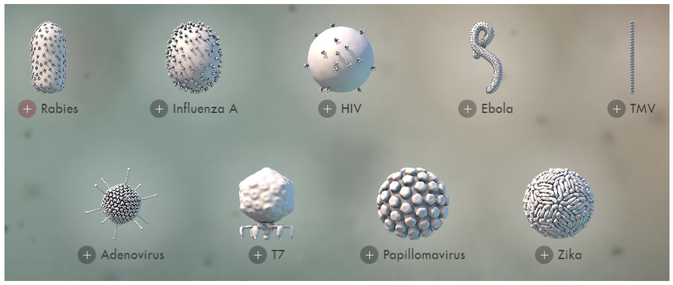


§33.5 Cell Size and Scale

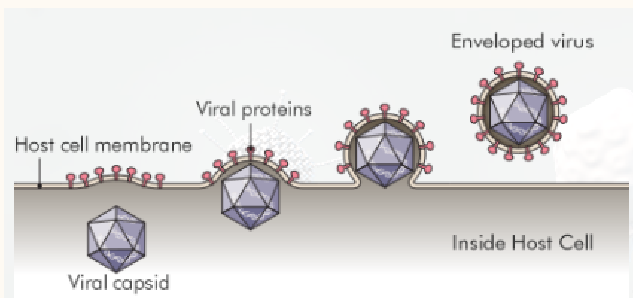
Link: https://learn.genetics.utah.edu/content/cells/scale/

§33.6 Virus Explorer

Link: https://media.hhmi.org/biointeractive/click/virus-explorer/index.html

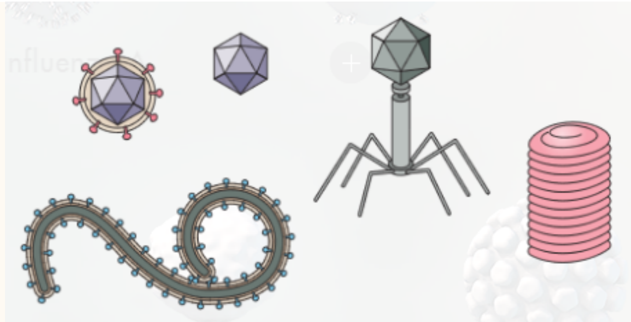


Definition 33.6.1 (Envelope) — Viruses consist of genetic material (DNA or RNA) enclosed in a layer of viral proteins. Some viruses exit their host cell by budding from its surface. In the process, **part of the host cell membrane envelops the virus particle forming an outer layer, called the envelope.** The envelope contains host and viral proteins embedded within it. Some of these proteins (colored red in the illustrations) serve to bind to the host cells. Viruses whose replication does not involve budding from the host cell surface do not have an envelope and are referred to as "naked" or "non-enveloped."



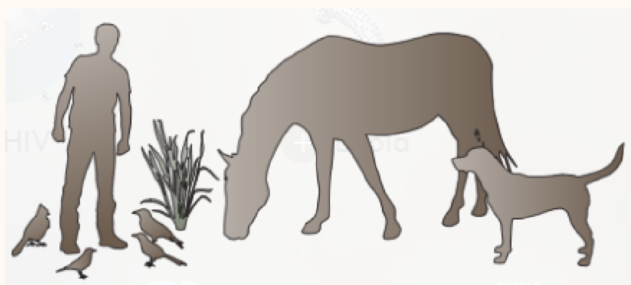
The sequence of steps leading to the formation of the viral envelope as a newly assembled virus buds from a host cell.

Definition 33.6.2 (Structure) — **The structure of a virus is typically described based on the overall shape of the protein layer that surrounds the virus genetic material.** This layer is called the **capsid**, or the core in enveloped viruses. **The shape of the capsid or core is determined by the arrangement of many individual proteins and is typically symmetrical.** The core structure is not visible in the 3D models for enveloped viruses.



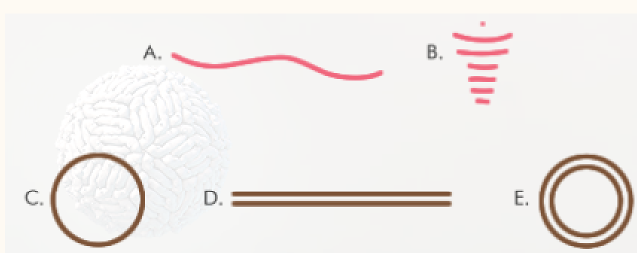
Some of the many shapes of viruses.

Definition 33.6.3 (Host) — **A host is the organism that a virus infects and replicates in.** Viruses can only replicate inside a host cell. Viral hosts include animals, plants, bacteria, fungi, and archaea. Many viruses have evolved to infect multiple kinds of hosts, while others have a more limited host range.



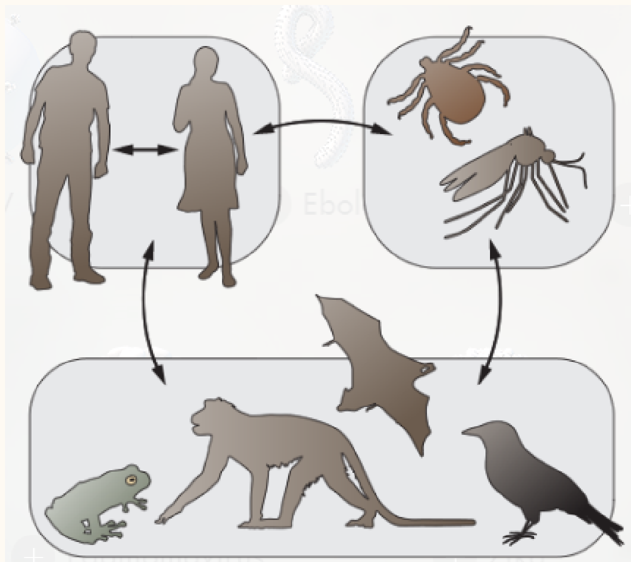
Some of the different types of host organisms viruses can infect.

Definition 33.6.4 (Genome Type) — All viruses contain genetic material (the viral genome) that encodes one or more proteins. **Viral genomes vary by the type of nucleic acid (RNA or DNA), number of strands of nucleic acid (ss, single-stranded, or ds, double-stranded), the sense, or polarity, of the strands (+, positive, or -, negative), and the structure (circular or linear).** Most viral genomes consist of a single continuous sequence either with the two ends joined together (circular genome) or with the two ends not joined (linear). However, some viruses have segmented genomes, made up of multiple independent nucleic acid segments.



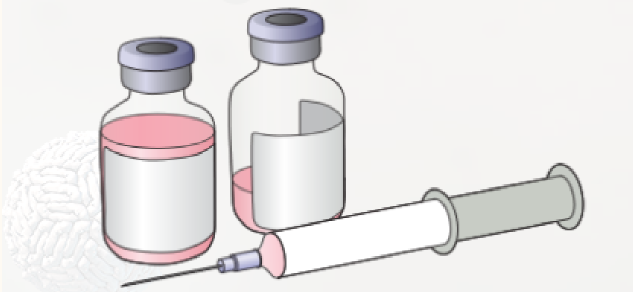
A selection of some types of viral genomes. A: linear ssRNA; B: segmented ssRNA; C: circular ssDNA; D: linear dsDNA; E: circular dsDNA.

Definition 33.6.5 (Transmission) — The mechanism by which a virus passes from one host to another depends on several factors, including **which organisms the virus is able to infect, which types of cells the virus infects, and how the virus is released from an organism (for example, through sneezed droplets or bodily fluids).** Some viruses can easily be passed from person to person, whereas others depend on an intermediate organism, like an arthropod (a mosquito or tick), to transmit it. An organism that serves to transmit a virus from one host to another is called a vector. A virus that is transmitted from a vertebrate animal (for example, a rodent or bat) to humans is considered a **zoonotic** virus.



Some possible routes of transmission of viruses, including between people, people and animals, and arthropod vectors.

Definition 33.6.6 (Vaccine Availability) — **A <u>vaccine</u> is a substance that, when taken into the body, should induce a protective immune response to a virus.** When an individual who has been vaccinated against a virus comes into contact with that virus, the body should already be prepared to fight the infection. Scientists have developed vaccines that protect humans and some animals from diseases caused by several virus infections. Human infectious diseases for which we have effective vaccines include smallpox, polio, and seasonal influenza.



Most vaccines are given by injection, and are available for many human viruses.

§33.7 Virus Explorer Assignment (4.3.20)

1. Locate the "i" next to each viral characteristic tab across the top. Click on these icons and answer the questions below associated with each viral characteristic.

a. Envelope: Not all viruses have an envelope. If a virus has this outer layer, explain how it forms.

The viral capsid moves out of the host cell, and proteins from the host cell bind to the capsid. They use **budding** to exit the cell's surface.

b. Structure: What determines the shape of the capsid, or core?.

The arrangment of proteins determines the shape.

c. Host(s): From the virus' perspective, why is the host important?

Viruses can't reproduce on their own, and therefore need a host to reproduce.

d. Transmission: Define the terms "vector" and "zoonotic."

A **vector** transmits viruses from one host to another. A **zoonotic** virus is a virus transmitted from a vertebrate animal to a human.

e. Vaccine: What is one advantage of being vaccinated against a particular virus?

When vaccinated, the body is more prepared to fight the virus.

2. Virus Scavenger Hunt: Use the home page of the Virus Explorer and the various viral characteristic tabs across the top to answer the questions below.

a. What is one difference between the rabies virus and the influenza virus?

A vaccine is available for both, but the rabies virus doesn't really change, and so only one is needed, whereas the influenza virus is seasonal and therefore needs a new one every year.

b. Of the nine viruses shown, which is the only one that infects plants?

TMV (Tobacco Mosaic Virus) infects plants.

c. Which two viruses infect all the vertebrates included in the interactive?

Adenovirus and Papillomavirus

d. Of the nine viruses shown, which is the only one that infects bacteria?

T7 Virus

e. Read over HIV and Ebola. How are they similar? How are they different?

No vaccine is available (Ebola is being tested, but not available yet). They both use RNA instead of DNA. However, Ebola has been controlled so that it is no longer extremely popular, but HIV still infects millions every year.

3. Locate the + next to each virus name. Click on these icons and answer the questions below associated with selected viruses.

a. Rabies virus: People often associate rabies virus with dogs. Why is this incomplete?

Rabies also infects humans, rodents, and other mammals. If symtoms occur inside a human, infection is fatal.

b. HIV: HIV infects immune cells. Why is this a disadvantage to the infected person?

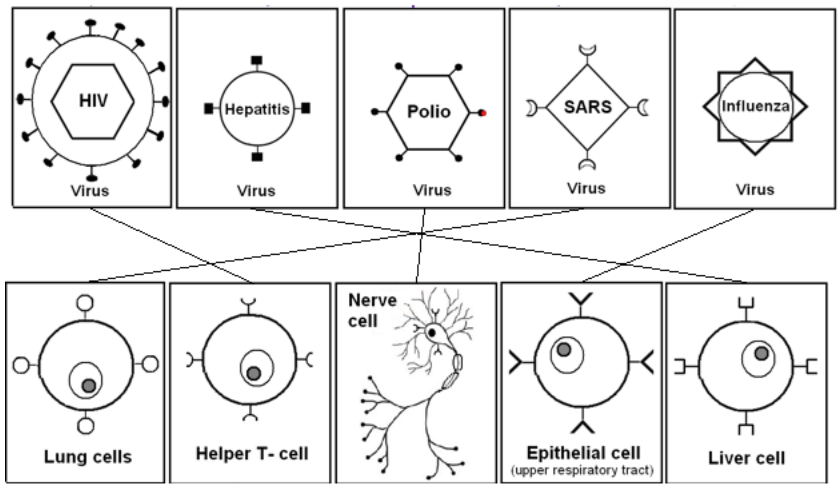
It makes it harder to fight off the disease, because not only are you weakened, the only thing that can make you stronger is also weakened.

c. HIV: Where in the world is HIV most prevalent?

It is most prevalent in Africa.

d. Ebola virus: What animal is associated with Ebola virus outbreaks?

Bats are associated with Ebola.



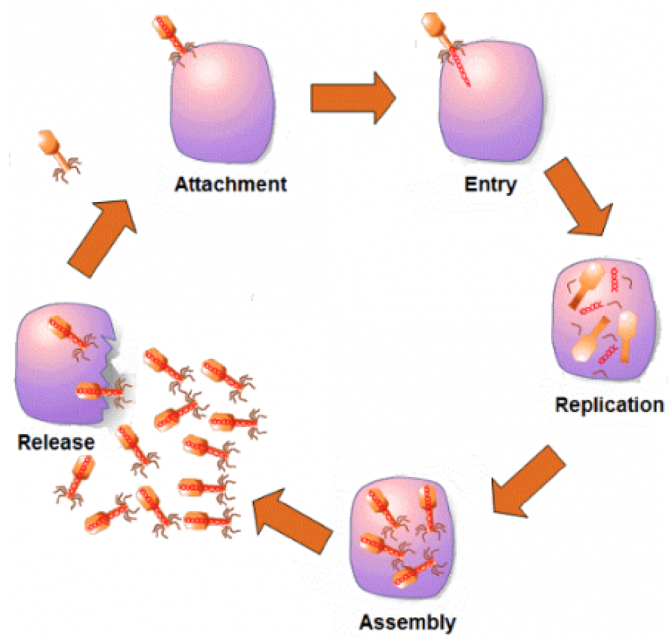
Next lesson: how pathogens and disease spread.

34 Virus Cycles (4.7.20)

§34.1 Lytic Cycle

Most viruses reproduce through a process called lytic infection. During lytic infection, a virus enters the host cell, makes a copy of itself, and causes the cell to burst, or lyse. A bacteriophage, which is a virus that infects and replicates within a bacterium, attaches itself and infects the host cell (in the video I watched).

1. **Attachment**: Virus attaches to the host cell.
2. **Entry**: Genetic material is injected into the host cell.
3. **Replication**: The virus takes over the cell's metabolism, causing the creation of new proteins and nucleic acids by the host cell's organelles.
4. **Assembly**: Proteins and nucleic acids are assembled into new viruses.
5. **Release**: Virus enzymes cause the cell to burst and viruses are released from the host cell. These new viruses can infect other cells.



Source: Adapted from Lytic Cycle, Discovery Health

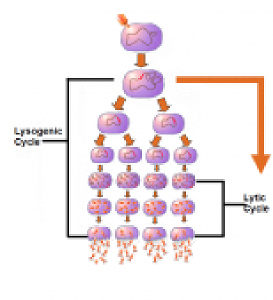
§34.2 Lysogenic Cycle

Unlike a lytic virus, a lysogenic virus does not cause the host cell to lyse away. A lysogenic virus can remain inactive for a period of time. In lysogenic infection, viral DNA gets integrated with the host cell's

DNA, where it is copied along with the host cell's DNA when the host cell replicates. Viral DNA multiplies as the host cell multiplies. Each new daughter cell created is infected with the virus' DNA.

Viral DNA that becomes embedded in a bacterial host cell's DNA is called a prophage. Viral DNA that becomes embedded in an eukaryotic cell's DNA is called a provirus. The prophage/provirus may remain part of the DNA of the host cell for many generations. Influences from the environment, such as radiation, heat, and certain chemicals, trigger the prophage/provirus to become active. It then removes itself from the host cell's DNA and enters the lytic cycle.

1. **Attachment**: Virus attaches to the host cell.
2. **Entry**: Genetic material is injected into the host cell.
3. **Integration**: Viral DNA integrates into the host cell's genome.
4. **Replication** (lysogenic cycle): When the host cell replicates, viral DNA is copied along with host cell DNA. Each new daughter cell is infected with the virus.
5. **Induction**: When the infected cells are exposed to certain environmental conditions, viral DNA is activated and enters the lytic cycle.
6. **Replication** (lytic cycle): The virus takes over the cell's metabolism, causing the creation of new proteins and nucleic acids by the host cell's organelles.
7. **Assembly**: Proteins and nucleic acids are assembled into new viruses.
8. **Release**: Virus enzymes cause the cell to burst and viruses are released from the host cell. These new viruses can infect other cells.



35 Bacteria (4.14.20)

Bacteria can be good and bad. We mostly think of bad bacteria, but some actually help you. An example of a bad bacteria is Streptococcus, which causes strep throat. There are a few shapes for bacteria, including:

- Coccus (cocci) - sphere shaped
- Spirillum (spirilla) - spiral shaped
- Bacillus (bacilli) - rod shaped

The immune system can fight off many bad bacteria, but sometimes it needs **antibiotics** to help defeat the bacteria.

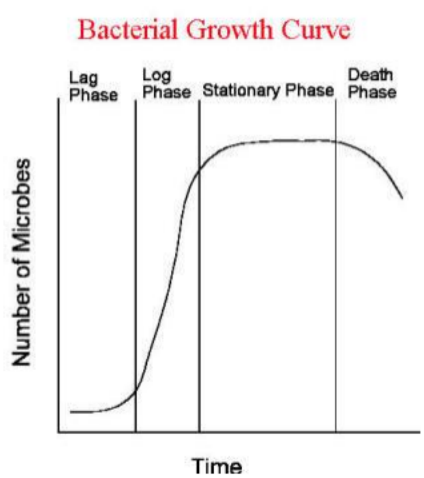
Prokaryotes have no nucleus, and no membrane-bound organelles. It is still living, because it still has DNA just floating inside its body.

Most antibiotics destroy both good and bad bacteria. There is more bacteria cells in your body than human cells! Note that if you destroy the bacteria, the yeast do no need to compete anymore, and therefore you can get a yeast infection from taking too much antibiotics.

§35.1 Bacterial Growth Curves

Bacteria (microbes) grow and divide at a really fast rate compared to larger organisms. Some can completely divide in 20 minutes! We can see the effect of nutrient (food) availability by counting the number of bacteria present. When bacteria have lots of food, they can divide and divide and divide. It seems like slow growth at first but then the population size shoots up!

At some point, the nutrients become a limiting factor- there is a limited supply. Here the death rate and growth rate are even and the population size does not change. The nutrients (food) eventually run out and we see a decrease in bacterial numbers.



Graph

- Relationship:
  - Increasing
  - Constant
  - Decreasing
- Equation:
  - Exponential relationship ($y = a^x$)
  - Direct relationship ($y = mx + b$)
  - Constant relationship (y = constant number)

36 Growing Bacteria (4.16.20)

§36.1 Antibiotics, Antivirals, and Vaccines

Link to Amoeba Sisters video: https://www.youtube.com/watch?v=uVUf_pt7Sh0.

Bacteria, viruses, protists, parasitic worms, and fungi can all be dangerous to your body. Not all are, but it still a a good idea to be cautious of what you touch / ingest.

The skin cells are the first layer of defense against pathogens, not just as a barrier, but there are also helpful microbes that colonize on your skin and help keep intruders out. Mucus membranes are also part of your first line of defense.

The second line of defense include macrophages (phagocytic white blood cells) by engulfing the intruder and warning other cells.

The third line of defense includes B and T cells (lymphocytes - specific type of white blood cells). Unlike the first two, it is more of a specific defense in that it targets specific pathogens. Some examples include white blood cells known as B-Cells and T-Cells.

Antigens can be found on pathogens, but antigens are technically anything foreign in your body, including pollen - someone with an allergy to pollen reacts to pollen antigens, causing harm to the body for no reason. An antigen can activate a response from a B- or T-Cell.

There also exist **memory** B- and T-Cell. If a certain pathogen that has already been "seen" by a B- or T-Cell returns, these memory B-/T-Cells can multiply to kill off the pathogen. We will come back to this when discussing vaccines.

Antibiotics target one type of pathogen - bacteria. Not all bacteria are bad, but antibiotics target them anyway, which is why you shouldn't take to much of them (refer to the notes from last week - you could get a yeast infection!). While your body is naturally defending against the invasion, antibiotics can help destroy bacteria by: a) smashing against the cell wall or b) block the proteins the pathogen needs. Antibiotics can be prescribed as a pill, or injected, or even in an IV. Notice that *anti* means against, *bio* means life, and this is important because two other words, antigen (described before as the stuff on a pathogen), and antibody (proteins made by the immune cells to help fight pathogens). Some antibodies bind to the pathogen and make the pathogen stop working. They can also mark pathogens so they can get eaten by a macrophage.

The memory cells can produce antibodies against pathogens your body has already encountered, and this is where vaccines are used. Vaccines expose your body to an inactive or weakened form of a pathogen, which prevents you from actually getting the disease, but allows your body to simulate an attack so that next time the memory cells already have a plan. This is known as *immunity* against the pathogen. Note that some types of vaccines are contraindicated in vulnerable populations, like someone sick, or an infant, or during pregnancy. Those people rely on *herd immunity*, which means if others around them are vaccinated, they have a degree of protection by the others.

Just as antibiotics attack bacteria, **antivirals** attack viruses. Many of them have to be given in a certain time frame after contracting the virus in order to be effective, because many work by affecting virus replication, which is extremely rapid and uses your own cells to replicate. Thus, in order to minimize harm to your own body, antivirals try to cease production of the protein the viruses lock on to, but in certain cases, this is bad, since the protein could be critical to your survival.

Pathogens can mutate and evolve. For example, influenza changes frequently, and scientists must work hard to figure out which virus will be most prevalent next year.

§36.2 Bacteria and Antibiotics Lab

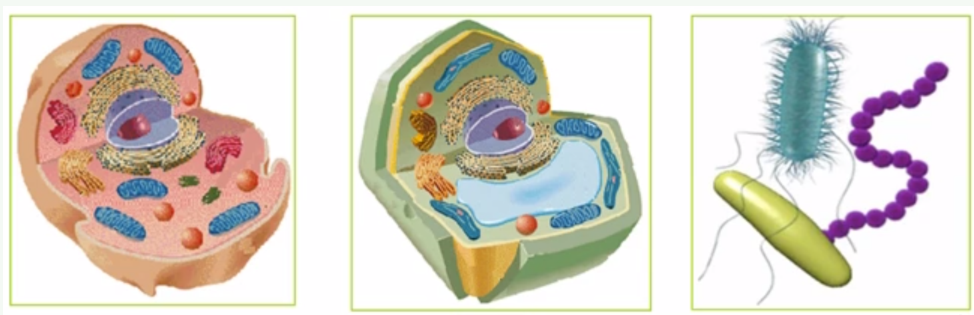
Link: `http://www.glencoe.com/sites/common_assets/science/virtual_labs/LS08/LS08.html`

37 Kingdom Plantae (4.17.20)

§37.1 Review

Plants are extremely important, because they are the base of the food chain and produce the oxygen we need. Plants are **eukaryotic**.

Problem 37.1.1 — Which of the following is a plant cell, and why?



Solution. The one in the middle is a plant, because:

1. There is a cell wall (green layer)
2. There is a large central vacuole
3. There are chloroplasts

Problem 37.1.2 — What are plant cell walls made of?
Solution. The answer is not peptidoglycan or chitin. It is **cellulose**.

§37.2 Characteristics of Plants

- **Eukaryotic** and multicuellular
- Cell walls made of **cellulose**
- Contain **chloroplasts** with chlorophyll (absorb light to do photosynthesis)

§37.3 Photosynthesis

Theorem 37.3.1 (Photosynthesis Equation)
Carbon dioxide and water go through a chemical process with energy supplied by water to create glucose (its food, sugar) and oxygen, which humans breathe. In other words,

$$6CO_2 + 6H_2O \overset{\text{sunlight}}{\longrightarrow} C_6H_12O_6 + 6O_2.$$

§37.4 Cellular Respiration

Plants can make their own sugar, but that sugar still needs to be made into ATP (adenosine triphosphate).

Theorem 37.4.1 (Cellular Respiration Equation)
Sugar and oxygen are converted into carbon dioxide (which plants use), water, and energy. The energy is what allows to function! In other words,

$$C_6H_12O_6 \longrightarrow 6CO_2 + 6H_2O + \text{ Energy (ATP)}.$$

Notice how the equation is the reverse of photosynthesis.

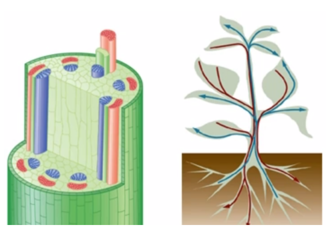
To make and consume energy, plant cells need *both* chloroplasts and mitochondria!

§37.5 Plant Evolution

Plants evolved from a **green algae** ancestor (**protist**) that lived in water. Early plants evolved traits that allowed them to live on land, where there was more sunlight, abundant CO_2 and few herbivores (i.e. less competition).

§37.5.1 Vascular Tissue

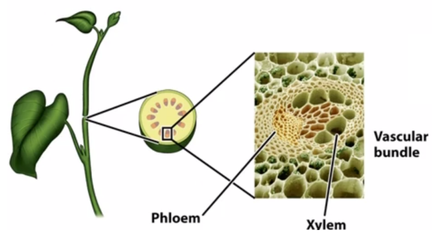
The **vascular tissue** is a tissue that moves water and nutrients through the plant against the force of gravity. This allows plants to be **taller** than just a few centimeters.



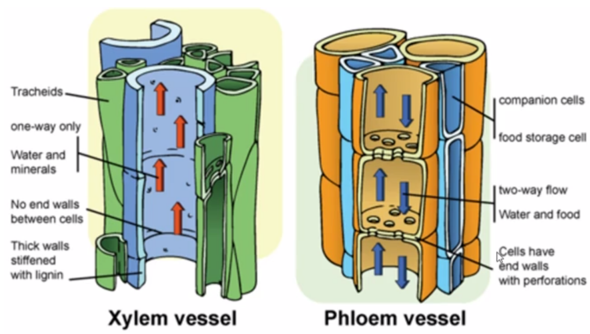
Vascular tissues are somewhat similar to our arteries and veins, but they don't circulate blood. Instead, they have two parts:

- **Xylem:** transports water upwards from roots using capillary action and transpiration (evaporation from leaves)

- **Phloem:** moves sugar from the leaves through the plant



A trick to remember them: W X Y Z (**W**ater is carried by the **XY**lem, and it sounds like a **Z**). **Ph**loem carries **ph**ood (food).

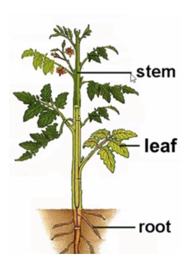


In the **xylem**, water flows upward form the roots to the leaves. In the **phloem**, sugars move up/down the plant from the leaves to other plant cells. The strings in the cross section of a celery stalk are actually vascular tissues! Also, if you put celery into colored water, it will become that color.

§37.6 Organs in Plants

Just like us, plants have organs that are made of tissues, and tissues that are made of cells. Plants can be divided into 3 main parts:

- Stems
- Leaves
- Roots
- Some have flowers as well



§37.6.1 Stems

- Support the plant
- Transport of water and nutrients through xylem and phloem

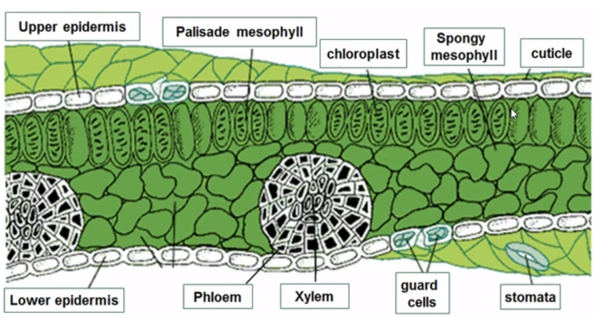
§37.6.2 Roots

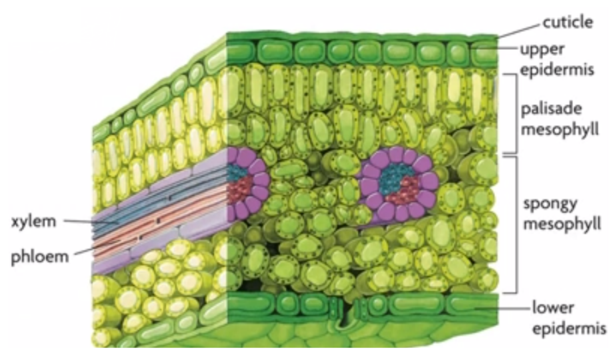
- Anchor the plant into the ground
- Take in water and nutrients from the soil

The xylem drives the water using gravity, surface tension, and osmosis.

§37.6.3 Leaves

Leaves are the site of photosynthesis. Here are some cross sections of the leaf:





The parts of the leaf include:

- **Cuticle:** waxy coating that helps prevent water loss
- **Epidermis:** skin of the leaf
- **Mesophyll Layers:** where photosynthesis takes place
- **Stomata:** openings that allow H_2O and O_2 out, and CO_2 in. They are controlled by guard cells, and they look like mouths. Plants regulate the opening and closing of their stomata to balance water loss with rates of photosynthesis. A stoma opens or closes in response to the changes in pressure within the guard cells that surround the opening. When the guard cells are swollen with water, the stoma is open. When the guard cell lose water, the opening closes, limiting further water loss.

§37.7 Transpiration

Transpiration begins in the leaves. The upper epidermis provides a protective covering. Below that layer is the palisade mesophyll layer. It is where food and water gets made. The spongy mesophyll is a loose layer where gaps aid in gas exchange and passage of water vapor from the leaves. The lower epidermal tissue contains stomata, which are small openings flanked by guard cells in which gases can pass into and out of the leaf through these openings. When water vapor comes out of these openings, it is a processs known and **transpiration**. The spongy mesophyll layer contains arrangements of vascular tissue (xylem and phloem). The water potential determines the direction in which water moves throughout the plant. Water follows into the cell by osmosis. The aquaporin channels in the membrane enhances osmosis allowing bulk flow of water from the soil to the roots. Inside the phloem, there is sieve-tube members and sieve cells. Most angiosperms have sieve-tube members. Both types of cells have pores known as sieve areas. These structures aid in the movement of carbohydrates, known as translocation. Turgor pressure increases in the sieve tube as sucros from surrounding cells is brought into the phloem by active transport. Thus, the processs works like heart using transpiration, water potential, and translocation to move water, nutrients, and mineral to all cells of the plant.

VII

Appendix

A Appendix A: List of Theorems and Definitions

List of Theorems

List of Definitions