Let $S = \{n \in \mathbf{N} \mid P(n) \text{ is true}\}$.
If (i) $1 \in S$ and
if (ii) for all $n \in \mathbf{N}$, $n \in S \Rightarrow n+1 \in S$,
then $S = \mathbf{N}$. (That is, for all $n \in \mathbf{N}$ the statement $P(n)$ is true.)

Example 6.1.1 illustrates how to use the Fundamental Theorem of Mathematical Induction to prove for all $n \in \mathbf{N}$ that the statement $P(n)$ is true. Notice in this example, we first prove that (i) $1 \in S$ and then we prove that (ii) for any $n \in \mathbf{N}$, if $n \in S$, then $n+1 \in S$.

Example 6.1.1 Prove for all $n \in \mathbf{N}$ that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: Let $P(n)$ be the statement $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ and let

$$S = \{n \in \mathbf{N} \mid P(n) \text{ is true}\}.$$

(i) *Remark:* First we prove that $1 \in S$. For $n = 1$, the left-hand side of the statement $P(1)$ is 1 and the right-hand side is $\dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$. Consequently, $P(1)$ is true and $1 \in S$.

(ii) *Remark:* Next, we prove that for $n \in \mathbf{N}$, $n \in S \Rightarrow n+1 \in S$. Let $n \in \mathbf{N}$ and assume that $n \in S$. Hence, $P(n)$, which is

$$(1) \qquad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

is assumed to be true. The statement $P(n+1)$, which we must prove to be true using (1), is

$$(2) \qquad 1 + 2 + \cdots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Grouping the summands on the left-hand side of (2) as shown below and then substituting from (1), we find

$$\begin{aligned}(1 + 2 + \cdots + n) + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}.\end{aligned}$$

That is, if (1) is true, it follows that (2) is true and therefore $n+1 \in S$.

Since the hypotheses (i) and (ii) of the Fundamental Theorem of Mathematical Induction are true, the conclusion $S = \mathbf{N}$ is true. Consequently, the statement $P(n)$ is true for all $n \in \mathbf{N}$—that is, $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbf{N}$. ■

Instead of defining S to be the set of all natural numbers n such that the statement $P(n)$ is true, proofs by mathematical induction are usually presented in the following logically equivalent, but less formal, form.