Metastability for the Contact Process on $\mathbb{Z}$, Part 1.

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Outline

Informal Metastability

- Markov process $X_N(t)$

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- As N goes to ∞, looks like figure 1.

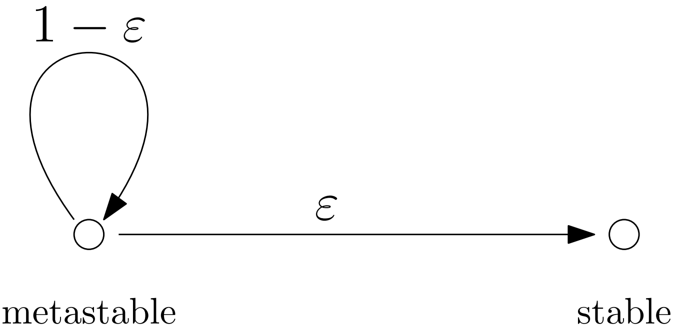

Figure: Coarse Graining as $N \to \infty$

Formal Metastability

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   - Asymptotically as $N \to \infty$, T_N has an exponential distribution.
2. Quasi-stationary distribution before hitting time
   - There is a "approximate invariant distribution" μ
   - Up until T_N, temporal means of $X_N(t)$ approximate μ

Contact Process, Generator Definition

- $\xi(t)$ is a Markov process taking values in $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$
- Characterized by

$$Lf(\eta) = \sum_{x} c(x,\eta)(f(\eta^x) - f(\eta)) \tag{1}$$

- With rates

$$c(x,\eta) = \begin{cases} 1 & \text{if} \quad \eta(x) = 1 \\ \lambda(\eta(x-1) + \eta(x+1)) & \text{otherwise} \end{cases} \tag{2}$$

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- Percolation structure is very useful for proofs

Critical λ

Proposition

There exists λ_c such that for $\lambda < \lambda_c$, $\xi(t)$ has only one invariant measure, which is concentrated at $\emptyset$. For $\lambda > \lambda_c$, there is also another extremal invariant measure, which is obtained by time-averaging $\xi(t)$.

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path from $y \in A$ at time 0 to x at time t that does not go out of B
- Special cases
  - $\xi_N(t) = \xi_{[-N,N]}(t)$
  - $\xi_{[-N,\infty)}$
  - $\xi_{(-\infty,N]}$
- Last two also have invariant measure μ, for $\lambda > \lambda_c$.

Fundamental Lemma of the Percolation Structure

Lemma

- *Suppose there is a path from* $(y_1, 0)$ *to* (x_1, t) *and a path from* $(y_2, 0)$ *to* (x_2, t)*, and* $y_1 < y_2$, $x_1 > x_2$

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- *Then* $\exists$ *a path from* $(y_1, 0)$ *to* (x_2, t) *and a path from* $(y_2, 0)$ *to* (x_1, t)*.*

Proof.

The two paths must intersect at some point (z, s). So then there are paths $(y_1, 0) \to (z, s)$, $(y_2, 0) \to (z, s)$, $(z, s) \to (x_1, t)$, and $(z, s) \to (x_2, t)$. Compose these paths to get our answer. $\square$

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- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of t.
- Converges to $\mu_B(\eta \mid \eta \cap A \neq \emptyset)$, where μ_B is the invariant measure of ξ started in state B.

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

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Specifically, for A finite

$$\mathbb{P}(\xi^A(t) \neq \emptyset) = \mathbb{P}(\xi(t) \cap A \neq \emptyset) \tag{4}$$

Goals

- Not feasible to give whole proof
- Instead, will give strategy and then give detailed proof of just one part

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

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- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be CDF for T_N/β_N
- Prove that

$$\lim_{N\to\infty} |G_N(t)G_N(s) - G_N(t+s)| = 0$$

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- If we are not in F_b, then we are most likely already dead. Therefore, the probability that we end up in F_b at time t is approximately $G_N(t)$
- Starting from $A \in F_b$ is "just as good" as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time t, then we have approximately $G_N(s)$ chance of still being alive at time $s + t$.

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- Put together, these two imply that $G_N(t)G_N(s) \sim G_N(t + s)$

- Show that $\mathbb{P}[T_N = T_N^A] > 1 - \varepsilon$ when $A \in F_b$, for sufficiently large N and b.
- Let
$$F_b = \{A \in \mathbb{Z} \mid \frac{|A \cap [-b,-1]|}{b} \geq \frac{\rho}{2}, \frac{|A \cap [1,b]|}{b} \geq \frac{\rho}{2}\}$$
- $\rho = \mathbb{P}[\xi^{\{0\}}(t) \neq \emptyset, \forall t] = \mu(\eta \mid \eta(0) = 1)$.

- Let $n = n(\varepsilon)$ large enough so that

$$\mu(\eta \mid \eta \cap [1, n]) > 1 - \frac{\varepsilon}{2}$$

- Then choose b such that $b\rho/2 > n$, so that for $A \in F_b$,

$$|A \cap [-b, -1]| \geq b\rho/2 \geq n$$

- Then,

$$\begin{aligned}
\mathbb{P}[\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t) \neq \emptyset, \forall t] &\geq \mathbb{P}[\xi_{[-N,\infty)}^{[-N,-N+n]}(t) \neq \emptyset, \forall t] \\
&= \lim_{t\to\infty} \mathbb{P}[\xi_{[-N,\infty)}^{[-N,-N+n]} \neq \emptyset] \\
&= \lim_{t\to\infty} \mathbb{P}[\xi_{[-N,\infty)} \cap [-N, -N+n] \neq \emptyset] \\
&= \mu(\eta \mid \eta \cap [-N, -N+n] \neq \emptyset) \\
&> 1 - \frac{\varepsilon}{2}
\end{aligned}$$

- By symmetry,

$$\mathbb{P}[\xi_{(-\infty,N]}^{A\cap[1,b]}(t) \neq \emptyset, \forall t] > 1 - \frac{\varepsilon}{2}$$

- Let E be the event that $\xi_{(-\infty,N]}^{A\cap[1,b]}(t) \neq \emptyset, \forall t$ and $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t) \neq \emptyset, \forall t$. We have shown $\mathbb{P}(E) > 1 - \varepsilon$.
- It remains to show that $T_N^A = T_N$ on E.

- Define two stopping times

$$U = \inf t \mid N \in \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$$
$$V = \inf t \mid -N \in \xi_{(-\infty,N]}^{A\cap[1,b]}(t)$$

- These are almost surely finite on E.
- At time U, ξ_N^A is alive, because $\xi_N^A(U) \supset \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$, and similarly for V, so $T_N \geq T_N^A > \max(U, V)$

- After U and V, there is a path from $[-b,-1]\cap A$ to N, and a path from $[1,b]\cap A$ to $-N$. Intuitively, one of those paths intersects any path from $x\in[-N,N]$ to $y\in\xi_N(t)$. (draw picture)
- Therefore, for $t>\max(U,V)$, $\xi_N(t)=\xi_N^A(t)$
- Therefore, $T_N=T_N^A$ on E, and we have shown that $\mathbb{P}(T_N=T_N^A)>1-\varepsilon$.