

Metastability for the Contact Process on \mathbb{Z} , Part 1.

Following [Sch85]

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Outline

1. Introduction
 - 1.1 Metastability
 - 1.2 Contact Process
2. Crash Course in Contact Process
 - 2.1 Toolbox
 - 2.2 Review
3. Theorem 1
 - 3.1 Overview
 - 3.2 An Excerpt from Schonman Op. 1
 - 3.3 Conclusion

Introduction

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- ▶ As N goes to ∞ , looks like figure 1.

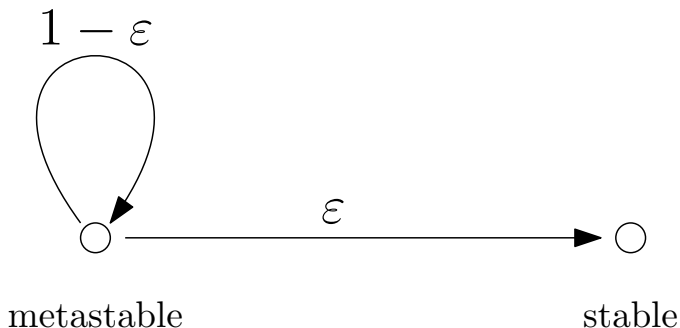


Figure: Coarse Graining as $N \rightarrow \infty$

Formal Metastability

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- ▶ There is a “trap state”, with hitting time T_N
- ▶ Asymptotically as $N \rightarrow \infty$, T_N has an exponential distribution.

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 - ▶ There is a “trap state”, with hitting time T_N
 - ▶ Asymptotically as $N \rightarrow \infty$, T_N has an exponential distribution.
2. Quasi-stationary distribution before hitting time
 - ▶ There is a “approximate invariant distribution” μ
 - ▶ Up until T_N , temporal means of $X_N(t)$ approximate μ

Contact Process, Generator Definition

- ▶ $\xi(t)$ is a Markov process taking values in $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$
- ▶ Characterized by

$$Lf(\eta) = \sum_x c(x, \eta)(f(\eta^x) - f(\eta)) \quad (1)$$

- ▶ With rates

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \\ \lambda(\eta(x-1) + \eta(x+1)) & \text{otherwise} \end{cases} \quad (2)$$

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- ▶ $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \rightarrow (x, t)$ for $y \in A$.

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- ▶ Percolation structure is very useful for proofs

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 - ▶ $\xi_N(t) = \xi_{[-N, N]}(t)$
 - ▶ $\xi_{[-N, \infty)}(t)$
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- ▶ We are interested in metastability for $\xi_N(t)$, with trap state \emptyset .

Crash Course in Contact Process

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- ▶ For $\lambda < \lambda_c$, goes extinct in finite time a.s. (unique ergodic measure δ_\emptyset)
- ▶ For $\lambda > \lambda_c$, two extremal invariant measures: δ_\emptyset and μ
- ▶ This also holds for $\xi_{[-N,\infty)}(t)$ and $\xi_{(-\infty,N]}(t)$

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- ▶ Converges to $\mu_A(\{\eta \mid \eta \cap B \neq \emptyset\})$, where μ_B is the invariant measure of ξ started in state B .

Self-duality

As long as A or B is finite, then for all t .

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Specifically, for A finite

$$\mathbb{P}(\xi^A(t) \neq \emptyset) = \mathbb{P}(\xi(t) \cap A \neq \emptyset) \quad (4)$$

Consequence of Monotone Convergence + Self-duality

$$\begin{aligned}\mathbb{P}(\xi^A(t) \neq \emptyset, \forall t > 0) &= \lim_{s \rightarrow \infty} \mathbb{P}(\xi^A(t) \neq \emptyset, \forall s \geq t > 0) \\ &= \lim_{s \rightarrow \infty} \mathbb{P}(\xi^A(s) \neq \emptyset) \\ &= \lim_{s \rightarrow \infty} \mathbb{P}(\xi(s) \cap A \neq \emptyset) \\ &= \mu(\{\eta \mid \eta \cap A \neq \emptyset\})\end{aligned}$$

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- ▶ Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.
- ▶ It turns out that you can generalize this: if you start out *with at least* $n(\varepsilon)$ elements, you are almost sure to survive.

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- ▶ How to use the percolation structure to construct many related contact processes
- ▶ Some miscellaneous useful facts about the contact process
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 - ▶ How an infection can persist forever with high probability: it has to start out with enough sites infected

Theorem 1

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Goals

Theorem

Let

$$T_N = \inf\{t \mid \xi_N(t) = \emptyset\}$$

Then for $\lambda > \lambda_c$

$$\frac{T_N}{\mathbb{E} T_N} \xrightarrow{w} \text{Exp}(1)$$

That is, it converges to $\text{Exp}(1)$ in distribution.

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- ▶ Not feasible to give whole proof
- ▶ Instead, will give strategy and then give detailed proof of just one part

Strategy Part 1

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- ▶ (note that $\{\frac{T_N}{\beta_N} > t\} = \{\xi_N(\beta_N t) \neq \emptyset\}$)
- ▶ Prove that

$$\lim_{N \rightarrow \infty} |G_N(t)G_N(s) - G_N(t+s)| = 0$$

- ▶ The only function with $f(t+s) = f(t)f(s)$ and $\int_0^\infty f(t) dt = 1$ is $f(t) = e^{-t}$.

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- ▶ Starting from $A \in F_b$ is “just as good” as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.

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- ▶ Put together, these two imply that $G_N(t)G_N(s) \sim G_N(t + s)$

Into the Unknown

- ▶ Show that $\mathbb{P}[T_N = T_N^A] > 1 - \varepsilon$ when $A \in F_b$, for sufficiently large N and b . (i.e., $A \in F_b$ is “just as good” as $[-N, N]$)

- ▶ Let

$$F_b = \{A \in \mathbb{Z} \mid \frac{|A \cap [-b, -1]|}{b} \geq \frac{\rho}{2}, \frac{|A \cap [1, b]|}{b} \geq \frac{\rho}{2}\}$$

- ▶ “sufficiently dense on both sides”
- ▶ $\rho = \mathbb{P}[\xi^{\{0\}}(t) \neq \emptyset, \forall t] = \mu(\eta \mid \eta(0) = 1).$

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- ▶ Let E be the event that $\xi_{[-N, \infty)}^{A \cap [-b, -1]}(t)$ and $\xi_{(-\infty, N]}^{A \cap [1, b]}(t)$ persist forever (the invariant distribution for $[-N, \infty)$ is the same as that for \mathbb{Z})

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- ▶ $\mathbb{P}(E) > 1 - \varepsilon$
- ▶ Therefore, we are done if in E , $T_N = T_N^A$.

- Define two stopping times

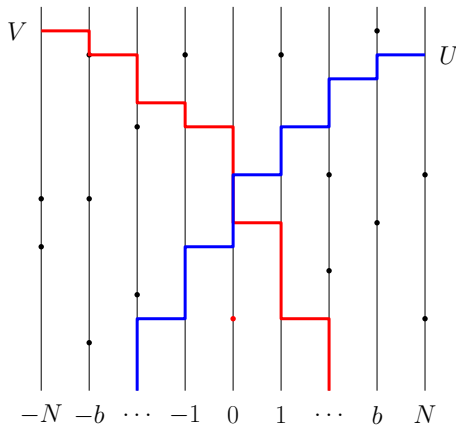
$$U = \inf\{t \mid N \in \xi_{[-N, \infty)}^{A \cap [-b, -1]}(t)\}$$

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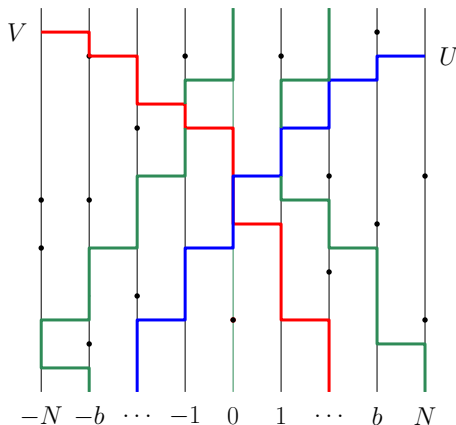
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- ▶ U and V are almost surely finite on E .
- ▶ At time U , ξ_N^A is alive, because $\xi_N^A(U) \supset \xi_{[-N, \infty)}^{A \cap [-b, -1]}(t)$, and similarly for V , so $T_N \geq T_N^A > \max(U, V)$

- After U and V , there is a path from $[-b, -1] \cap A$ to N , and a path from $[1, b] \cap A$ to $-N$. Intuitively, one of those paths intersects any path from $x \in [-N, N]$ to $y \in \xi_N(t)$.



- ▶ For $t > \max(U, V)$, $\xi_N(t) = \xi_N^A(t)$
- ▶ Therefore, $T_N = T_N^A$ on E , and we have shown that $\mathbb{P}(T_N = T_N^A) > 1 - \varepsilon$.

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- ▶ Basic properties of the contact process
- ▶ A rough sketch of how the contact process is metastable (part 1)
- ▶ What working with the contact process in practice looks like

Citations

- [Sch85] Roberto H. Schonmann. “Metastability for the contact process.” In: *Journal of Statistical Physics* 41.3 (Nov. 1, 1985), pp. 445–464. ISSN: 1572-9613. DOI: 10.1007/BF01009017. URL: <https://doi.org/10.1007/BF01009017> (visited on 11/12/2020).