Metastability for the Contact Process on Z: Part 2

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Overview

A bit of terminology

- ξ - 'xi'
- ξ_N - 'xi n'
- $\xi_{[-N,\infty)}$ - 'xi plus inf'
- $\xi_{(-\infty,N]}$ - 'xi minus inf'
- $[-N, N]$ - 'main interval'
- $\xi_N(t) \neq \varnothing$ - 'process is still alive'

Natural language definition of metastability

Recall that a system is metastable if:

1. It stays out of its equilibrium during a memoryless random time
2. During this time in which the system is out of equilibrium it stabilizes

Let's elaborate some more on point 2.

1. Assume that for a given N we have some intermediate timescale such R_N that $R_N << \beta_N$
2. Say we measure a temporal mean of some observable quantity of a system (e.g. particle density) over this timescale
3. We say system has stabilized if this mean is close to the expectation of this observable quantity w.r.t. some fixed probability distribution on $\{0,1\}^{\mathbb{Z}}$

Towards the rigorous definition

To ensure $R_N << \beta_N$, let's require $R_N/\beta_N \to 0$ as $N \to \infty$.

For our purposes, define **observable quantity of a system** as $f(\xi_N(t))$, such that:

- $f : \{0, 1\}^{\mathbb{Z}} \to \mathbb{R}$
- f is local

Towards the rigorous definition

Define **temporal mean** of observable quantity $f(\xi_N(t))$ as:

$$A_R^N(s,f) := R^{-1}\int_s^{s+R} f(\xi_N(t))dt$$

Where:

- s is the time in which we start our measurement
- R is the duration over which we calculate the temporal mean

Towards the rigorous definition

Recall what we said about the 2nd condition for metastability:

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Take our **fixed probability distribution** to be μ (i.e. the non-zero invariant measure of the contact process in the supercritical regime).

Then, we can define **expectation of observable quantity** w.r.t to this probability distribtuion as $\mu(f)$

Take convergence in probability as how we understand **closeness**.

Thus, we want to have a sequence R_N, such that:

- $R_N/\beta_N \to 0$ as $N \to \infty$
- for all $\varepsilon > 0$ and all observable quantities f

$$\mathbb{P}\left[|A_{R_N}(s,f) - \mu(f)|\langle \varepsilon\right] \to 0$$

As $N \to \infty$.

Are we done now?

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As $N \to \infty$.

Are we done now? **No!** How do we choose s (the starting point of temporal mean measurement)?

Obviously, we want to choose s such that the process ξ_N is still alive. Otherwise our temporal mean would indeed be far from the expectation. Thus, we want to start measuring at s such that $R_N + s < T_N$.

Would saying that there exists s such that our convergence in probability holds be sufficient?

Obviously, we want to choose s such that the process ξ_N is still alive. Otherwise our temporal mean would indeed be far from the expectation. Thus, we want to start measuring at s such that $R_N + s < T_N$.

Would saying that there exists s such that our convergence in probability holds be sufficient?

It would be a bit weak. We'd then say that the observer needs to start measuring at a very particular s to observe that the temporal mean is close to the expectation. This is hard to satisfy, so we need something stronger.

Define

$$K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$$

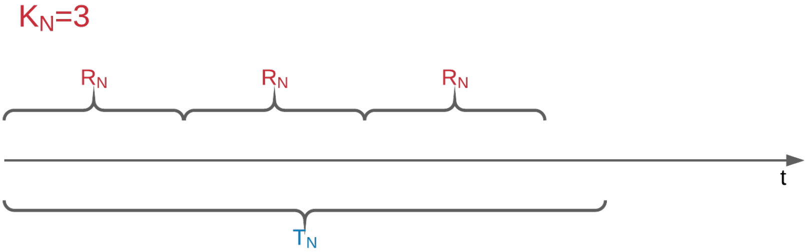

Figure: Example of K_N

Turns out that regardless of which of those intervals we choose for averaging, we can still make it highly probable that our temporal mean is as close to the $\mu(f)$ as we want, given that we're free to increase N.

In other words, we want to have a sequence R_N, such that:

- $R_N/\beta_N \to 0$ as $N \to \infty$
- for all $\varepsilon > 0$ and all observable quantities f

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| < \varepsilon\right] \to 0$$

As $N \to \infty$.

Phew. . . are we done now?

In other words, we want to have a sequence R_N, such that:

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As $N \to \infty$.

Phew... are we done now? Not quite!

Turns out that we need the following technicality: we need to choose $L(\varepsilon, f)$, and we need to have that

- $L < N$
- $\Lambda(f) := supp(f) \subset [-N + L, N - L] \cap \mathbb{Z}$

Notice that L does not depend on N. Thus, having to choose this L doesn't restrict our choice of f - it merely sets the minimum N we can consider and we chose to grow $N \to \infty$.

Theorem 2

We're finally ready to formulate the theorem.

Theorem (Thermalization)

If $\lambda > \lambda^*$ there is a sequence $\{R_N\}_{N\in\mathbb{N}} \subset \mathbb{R}_+$ such that:

- $R_N/\beta_N \to 0$ *as* $N \to \infty$
- *For all* $\varepsilon > 0$ *and observable quantities* f $\exists L(\varepsilon, f) \in \mathbb{N}$ *such that*

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right] \to 0$$

  as $N \to \infty$, *where* $K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$ *and* $\Lambda(f) \subset [-N+L, N-L] \cap \mathbb{Z}$

Proof

$$K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$$

As a consequence of Theorem 1, and the requirement $R_N/\beta_N \to 0$ we have that $\mathbb{P}(K_N = 0) \to 0$.

Thus, we can safely focus only on the subset of Ω where $K_N \geq 1$.

In other words we're almost guaranteed to be able to fit at least one measurement between time 0 and when the process dies.

Proof

Set

$$B_k^N = \{|A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\}$$

On B_k^N the difference between temporal mean of observalbe measured over the k-th interval and its expectation is greater than we'd like to.
B_k^N means **failure** (in k-th interval).

We want the probability of having failure in at least 1 interval to go to 0

$$\left\{\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right\} = \bigcup_{\mathbb{N}_0 \ni k < K_N} B_k$$

In other words, we want the probability of having no failures to go to 1

$$\left\{\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| \leq \varepsilon\right\} = \bigcap_{\mathbb{N}_0 \ni k < K_N} B_k^C$$

Proof

Recall that we can safely focus on $K_N \geq 1$. After some simple algebra, we we get the following bound

$$\mathbb{P}\left[K_N \geq 1, \bigcap_{\mathbb{N}_0 \ni k < K_N} B_k^C\right] \geq \mathbb{P}[1 \leq K_N \leq m] - m^2 \max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k, K_N = j\right]$$

Our objective will be to find a sequence $\{m_N\}_{N \in \mathbb{N}}$ such that:

- $\mathbb{P}[1 \leq K_N \leq m_N] \to 1$
- $m_N^2 \max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k, K_N = j\right] \to 0$

As $N \to \infty$

Proof

We will start with the second term. However, first we need some intermediate results. We will say $\xi_N(t)$ is **wide** at t if

$$\min \xi_N(t) < -N + L \wedge \max \xi_N(t) > N - L$$

Call

- $[-N, -N + L]$ to be **left boundary region**
- $[N - L, N]$ **right boundary region**
- $[-N + L, N - L]$ **inside region**

A process is wide at t if it intersects both boundary regions.

Lemma (Shielding by a wide process)

If $\xi_N(t)$ is wide at t, $\xi_N(t) = \xi(t)$ on $[-N + L, N - L] \cap \mathbb{Z}$.
In particular we have $f(\xi_N(t)) = f(\xi(t))$

Why?

Essentially, rightmost and leftmost infected individuals "shield" the entire space between them from outside influence. GIF SHOWING WHAT I MEAN

Proof

Moreover, define $h_L(\eta) = I_{\{\xi:\xi\cup[-N,-N+L]=\varnothing\}}$. $h_L(\eta)$ tells us whether a process intersects the left boundary region. h_L can also tell us if a process intersects the right boundary region: we simply need to flip the process before feeding it to h_L.

Lemma (Shielding of the left boundary region)

$$\{T_N > t\} \subset \{h_L(\xi_N(t)) = h_L(\xi_{[-N,\infty)}(t))\}$$

Why would this be true?

- If $\xi_N(t)$ intersects left boundary region, $\xi_{[-N,\infty)}$ intersects it too
- If $\xi_N(t)$ does not intersect the left boundary region, but has at least one node still alive, this node shields the left boundary region from outside influence. Moreover, no influence can propagate from $-N$. Hence, they need to agree on the left boundary region.

GIF SHOWING WHAT I MEAN

Proof

Now we're ready to proceed. Recall, we want

- $m_N^2 \max_{1\leq j} \max_{0\leq k<j} \mathbb{P}\left[B_k, K_N = j\right] \to 0$

Let's take a closer look at $\mathbb{P}\left[B_k, K_N = j\right]$. We will estimate the difference between the temporal average of $f(\xi_N)$ and it's expectation with a triangle inequality. For $k < j$ we have

$$|A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon \to$$

$$\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| + \left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon$$

$$\to \left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2 \;\; \vee$$

$$\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2$$

Proof

Let's look a the first term in the alternative.

$$\left| R_N^{-1} \int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt \right| > \varepsilon/2$$

Recall that if ξ_N is wide at t, we have $f(\xi_N(t)) = f(\xi(t))$. Thus only t for which $\xi_N(t)$ is not wide will contribute to the intergral. Hence the former implies

$$2\|f\|R_N^{-1} \int_{kR_N}^{(k+1)R_N} \varphi(f(\xi_N(t)))dt > \varepsilon/2$$

Where φ is the indicator taking value 1 when $\xi_N(t)$ is not wide. A process which is not wide does not intersect neither left nor right boundary region. Thus we have

$$2\|f\|R_N^{-1} \int_{kR_N}^{(k+1)R_N} h_L(\xi_N(t)) + h_L(S\xi_N(t))dt > \varepsilon/$$

Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Blocks of Highlighted Text

Block 1

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Block 2

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Block 3

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Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

Table

Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
\frametitle{Theorem}
\begin{theorem}[Mass--energy equivalence]
$E = mc^2$
\end{theorem}
\end{frame}
```

Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

The End