Metastability for the Contact Process on $\mathbb{Z}$, Part 1.

Following [Sch85]

O. Lynch[1]

[1]Department of Mathematics
Universiteit Utrecht

December 8, 2020

Outline

Introduction

Informal Metastability

- Markov process $\xi_N(t) \subseteq \mathbb{Z}$, "infected states at time t"

Informal Metastability

- Markov process $\xi_N(t) \subseteq \mathbb{Z}$, "infected states at time t"
- As N goes to ∞, looks like figure 1.

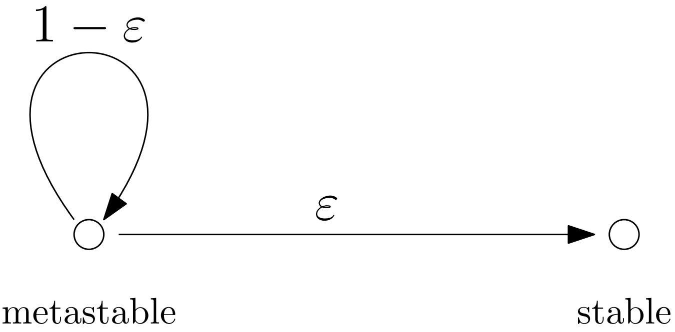


Figure: Coarse Graining as $N \to \infty$

Formal Metastability

1. Exponential hitting time
   - There is a "trap state", with hitting time T_N
   - Asymptotically as $N \to \infty$, T_N has an exponential distribution.

Formal Metastability

1. Exponential hitting time
   - There is a "trap state", with hitting time T_N
   - Asymptotically as $N \to \infty$, T_N has an exponential distribution.
2. Quasi-stationary distribution before hitting time (not discussed in this presentation)
   - There is a "approximate invariant distribution" μ
   - Up until T_N, temporal means of $X_N(t)$ approximate μ

Contact Process, Generator Definition

(Review from before)

- $\xi(t)$ is a Markov process taking values in $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$
- Characterized by

$$Lf(\eta) = \sum_x c(x,\eta)(f(\eta^x) - f(\eta)) \tag{1}$$

- With rates

$$c(x,\eta) = \begin{cases} 1 & \text{if} \quad \eta(x) = 1 \\ \lambda(\eta(x-1) + \eta(x+1)) & \text{otherwise} \end{cases} \tag{2}$$

Contact Process, Percolation Structure Definition

- (show animation 1)

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is...

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is... easier to see intuitively in than write down formally, and already covered by Michael

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is... easier to see intuitively in than write down formally, and already covered by Michael
- Percolation structure is very useful for proofs

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B
- Superscript: "starts in", subscript: "contained in"

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B
- Superscript: "starts in", subscript: "contained in"
- If B is omitted, assumed to be $\mathbb{Z}$. If A is omitted, assumed to be B.

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B
- Superscript: "starts in", subscript: "contained in"
- If B is omitted, assumed to be $\mathbb{Z}$. If A is omitted, assumed to be B.
- Special cases
  - $\xi_N(t) = \xi_{[-N,N]}(t)$
  - $\xi_{[-N,\infty)}(t)$
  - $\xi_{(-\infty,N]}(t)$

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B
- Superscript: "starts in", subscript: "contained in"
- If B is omitted, assumed to be $\mathbb{Z}$. If A is omitted, assumed to be B.
- Special cases
  - $\xi_N(t) = \xi_{[-N,N]}(t)$
  - $\xi_{[-N,\infty)}(t)$
  - $\xi_{(-\infty,N]}(t)$
- We are interested in metastability for $\xi_N(t)$, with trap state $\emptyset$.

Crash Course in Contact Process

Critical λ

- Survival/extinction of $\xi(t)$ depends on infection rate λ

Critical λ

- Survival/extinction of $\xi(t)$ depends on infection rate λ
- For $\lambda < \lambda_c$, goes extinct in finite time a.s. (unique ergodic measure $\delta_\emptyset$)

Critical λ

- Survival/extinction of $\xi(t)$ depends on infection rate λ
- For $\lambda < \lambda_c$, goes extinct in finite time a.s. (unique ergodic measure $\delta_\emptyset$)
- For $\lambda > \lambda_c$, two extremal invariant measures: $\delta_\emptyset$ and μ (upper and lower invariant measures).

Critical λ

- Survival/extinction of $\xi(t)$ depends on infection rate λ
- For $\lambda < \lambda_c$, goes extinct in finite time a.s. (unique ergodic measure $\delta_\emptyset$)
- For $\lambda > \lambda_c$, two extremal invariant measures: $\delta_\emptyset$ and μ (upper and lower invariant measures).
- This also holds for $\xi_{[-N,\infty)}(t)$ and $\xi_{(-\infty,N]}(t)$ with the same λ_c.

Monotone Convergence

- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of t.

Monotone Convergence

- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of t.
- Converges to $\mu_A(\{\eta \mid \eta \cap B \neq \emptyset\})$, where μ_A is the invariant measure of ξ started in state B.

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

"The probability of reaching somewhere in B starting from A is the same as the probability of reaching somewhere in A starting from B"

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

"The probability of reaching somewhere in B starting from A is the same as the probability of reaching somewhere in A starting from B"
Specifically, for A finite

$$\mathbb{P}(\xi^A(t) \neq \emptyset) = \mathbb{P}(\xi(t) \cap A \neq \emptyset) \tag{4}$$

Consequence of Monotone Convergence + Self-duality

$$\begin{aligned}
\mathbb{P}(\xi^A(t) \neq \emptyset, \forall t > 0) &= \lim_{s \to \infty} \mathbb{P}(\xi^A(t) \neq \emptyset, \forall s \geq t > 0) \\
&= \lim_{s \to \infty} \mathbb{P}(\xi^A(s) \neq \emptyset) \\
&= \lim_{s \to \infty} \mathbb{P}(\xi(s) \cap A \neq \emptyset) \\
&= \mu(\{\eta \mid \eta \cap A \neq \emptyset\})
\end{aligned}$$

Characterizing μ

- μ is translation invariant

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta|\eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta|\eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$
- Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta|\eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$
- Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.
- It turns out that you can generalize this: if you start out *with at least* $n(\varepsilon)$ elements, you are almost sure to survive.

What have we learned so far?

- An intuition for metastability — two key properties

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
    - Critical λ
    - Monotone convergence

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - Monotone convergence
  - Self-duality

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - Monotone convergence
  - Self-duality
  - How an infection can persist forever with high probability: it has to start out with enough sites infected

Theorem 1

Goals

Theorem

Let

$$T_N = \inf\{t \mid \xi_N(t) = \emptyset\}$$

Then for $\lambda > \lambda_c$

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to Exp(1) *in distribution.*

Goals

Theorem

Let

$$T_N = \inf\{t \mid \xi_N(t) = \emptyset\}$$

Then for $\lambda > \lambda_c$

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to $\mathrm{Exp}(1)$ *in distribution.*

- Not feasible to give whole proof

Goals

Theorem

Let

$$T_N = \inf\{t \mid \xi_N(t) = \emptyset\}$$

Then for $\lambda > \lambda_c$

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to Exp(1) *in distribution.*

- Not feasible to give whole proof
- Instead, will give strategy and then give detailed proof of just one part

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be the tail distribution for T_N/β_N

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be the tail distribution for T_N/β_N
- (note that $\{\frac{T_N}{\beta_N} > t\} = \{\xi_N(\beta_N t) \neq \emptyset\}$)

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be the tail distribution for T_N/β_N
- (note that $\{\frac{T_N}{\beta_N} > t\} = \{\xi_N(\beta_N t) \neq \emptyset\}$)
- Prove that

$$\lim_{N\to\infty} |G_N(t)G_N(s) - G_N(t+s)| = 0$$

- The only function with $f(t+s) = f(t)f(s)$ and $\int_0^\infty f(t)\,\mathrm{d}t = 1$ is $f(t) = e^{-t}$.

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions
- If we are not in F_b, then we are most likely already extinct. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of “sufficiently dense” initial conditions
- If we are not in F_b, then we are most likely already extinct. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$
- Starting from $A \in F_b$ is “just as good” as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions
- If we are not in F_b, then we are most likely already extinct. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$
- Starting from $A \in F_b$ is "just as good" as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.
- Put together, these two imply that $G_N(t)G_N(s) \sim G_N(t + s)$

Into the Unknown

- Show that $\mathbb{P}[T_N = T_N^A] > 1 - \varepsilon$ when $A \in F_b$, for sufficiently large N and b. (i.e., $A \in F_b$ is "just as good" as $[-N, N]$)
- Let
$$F_b = \{A \in \mathbb{Z} \mid \frac{|A \cap [-b, -1]|}{b} \geq \frac{\rho}{2}, \frac{|A \cap [1, b]|}{b} \geq \frac{\rho}{2}\}$$
- "sufficiently dense on both sides"
- $\rho = \mathbb{P}[\xi^{\{0\}}(t) \neq \emptyset, \forall t] = \mu(\eta \mid \eta(0) = 1)$.

- Let $n = n(\varepsilon)$ large enough so that starting out with n infected sites will let the epidemic persist forever with probability $1 - \varepsilon/2$

- Let $n = n(\varepsilon)$ large enough so that starting out with n infected sites will let the epidemic persist forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, for A in F_b starting in $[1, b] \cap A$ or $[-b, -1] \cap A$ will persist forever with probability $1 - \varepsilon/2$

- Let $n = n(\varepsilon)$ large enough so that starting out with n infected sites will let the epidemic persist forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, for A in F_b starting in $[1, b] \cap A$ or $[-b, -1] \cap A$ will persist forever with probability $1 - \varepsilon/2$
- Let E be the event that $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$ and $\xi_{(-\infty,N]}^{A\cap[1,b]}(t)$ persist forever (the invariant distribution for $[-N, \infty)$ is the same as that for $\mathbb{Z}$)

- Let $n = n(\varepsilon)$ large enough so that starting out with n infected sites will let the epidemic persist forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, for A in F_b starting in $[1, b] \cap A$ or $[-b, -1] \cap A$ will persist forever with probability $1 - \varepsilon/2$
- Let E be the event that $\xi^{A\cap[-b,-1]}_{[-N,\infty)}(t)$ and $\xi^{A\cap[1,b]}_{(-\infty,N]}(t)$ persist forever (the invariant distribution for $[-N, \infty)$ is the same as that for $\mathbb{Z}$)
- $\mathbb{P}(E) > 1 - \varepsilon$

- Let $n = n(\varepsilon)$ large enough so that starting out with n infected sites will let the epidemic persist forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, for A in F_b starting in $[1, b] \cap A$ or $[-b, -1] \cap A$ will persist forever with probability $1 - \varepsilon/2$
- Let E be the event that $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$ and $\xi_{(-\infty,N]}^{A\cap[1,b]}(t)$ persist forever (the invariant distribution for $[-N, \infty)$ is the same as that for $\mathbb{Z}$)
- $\mathbb{P}(E) > 1 - \varepsilon$
- Therefore, we are done if in E, $T_N = T_N^A$.

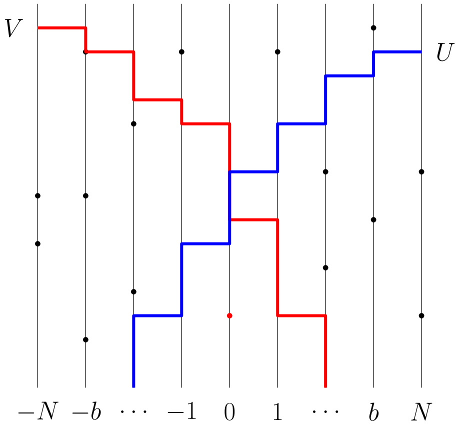
- Define two stopping times

$$U = \inf\{t \mid N \in \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)\}$$
$$V = \inf\{t \mid -N \in \xi_{(-\infty,N]}^{A\cap[1,b]}(t)\}$$

- Define two stopping times

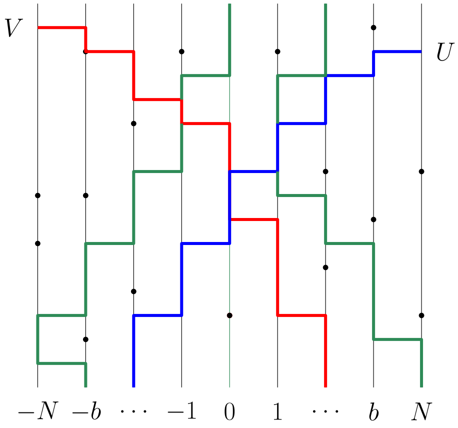
$$U = \inf\{t \mid N \in \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)\}$$

$$V = \inf\{t \mid -N \in \xi_{(-\infty,N]}^{A\cap[1,b]}(t)\}$$



- U and V are almost surely finite on E.
- At time U, ξ_N^A is alive, because $\xi_N^A(U) \supset \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$, and similarly for V, so $T_N \geq T_N^A > \max(U, V)$

- After U and V, there is a path from $[-b,-1]\cap A$ to N, and a path from $[1,b]\cap A$ to $-N$. Intuitively, one of those paths intersects any path from $x \in [-N,N]$ to $y \in \xi_N(t)$.



- For $t > \max(U, V)$, $\xi_N(t) = \xi_N^A(t)$
- Therefore, $T_N = T_N^A$ on E, and we have shown that $\mathbb{P}(T_N = T_N^A) > 1 - \varepsilon$.

Summary

We have learned

Summary

We have learned

- What metastability is

Summary

We have learned

- What metastability is
- Basic properties of the contact process

Summary

We have learned

- What metastability is
- Basic properties of the contact process
- A rough sketch of how the contact process is metastable (part 1)

Summary

We have learned

- What metastability is
- Basic properties of the contact process
- A rough sketch of how the contact process is metastable (part 1)
- What working with the contact process in practice looks like

Citations

[Sch85] Roberto H. Schonmann. "Metastability for the contact process." In: *Journal of Statistical Physics* 41.3 (Nov. 1, 1985), pp. 445–464. ISSN: 1572-9613. DOI: 10.1007/BF01009017. URL: https://doi.org/10.1007/BF01009017 (visited on 11/12/2020).