Metastability for the Contact Process on $\mathbb{Z}$, Part 1.

O. Lynch[1]

[1]Department of Mathematics
Universiteit Utrecht

December 6, 2020

Outline

Informal Metastability

- Markov process $X_N(t)$

Informal Metastability

- Markov process $X_N(t)$
- As N goes to ∞, looks like figure 1.

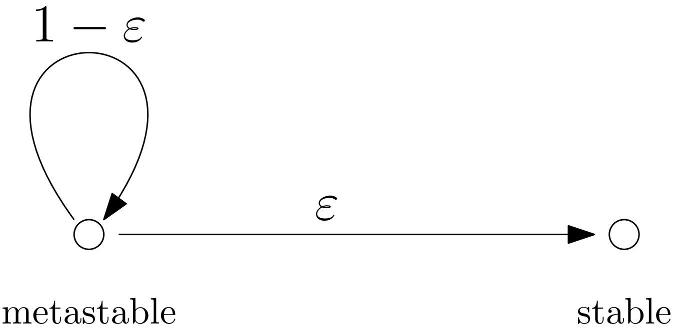

Figure: Coarse Graining as $N \to \infty$

Formal Metastability

1. Exponential hitting time
   - There is a “trap state”, with hitting time T_N
   - Asymptotically as $N \to \infty$, T_N has an exponential distribution.

Formal Metastability

1. Exponential hitting time
   - There is a "trap state", with hitting time T_N
   - Asymptotically as $N \to \infty$, T_N has an exponential distribution.
2. Quasi-stationary distribution before hitting time
   - There is a "approximate invariant distribution" μ
   - Up until T_N, temporal means of $X_N(t)$ approximate μ

Contact Process, Generator Definition

- $\xi(t)$ is a Markov process taking values in $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$
- Characterized by

$$Lf(\eta) = \sum_x c(x,\eta)(f(\eta^x) - f(\eta)) \tag{1}$$

- With rates

$$c(x,\eta) = \begin{cases} 1 & \text{if} \quad \eta(x) = 1 \\ \lambda(\eta(x-1) + \eta(x+1)) & \text{otherwise} \end{cases} \tag{2}$$

Contact Process, Percolation Structure Definition

- (show animation 1)

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is...

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is... easier to see intuitively in than write down formally

Contact Process, Percolation Structure Definition

- (show animation 1)
- 3 processes for each site $x \in \mathbb{Z}$: P_x (rate 1), $P_{x \to x+1}$ and $P_{x \to x-1}$ (both rate λ).
- $\xi^A(t)$ is the set of $x \in \mathbb{Z}$ such that there is a path $(y, 0) \to (x, t)$ for $y \in A$.
- A path is... easier to see intuitively in than write down formally
- Percolation structure is very useful for proofs

Smaller Processes

- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y, 0) \to (x, t)$ with $y \in A$ and the path not leaving B
- Special cases
  - $\xi_N(t) = \xi_{[-N,N]}(t)$
  - $\xi_{[-N,\infty)}(t)$
  - $\xi_{(-\infty,N]}(t)$
- Last two also have invariant measure μ, for $\lambda > \lambda_c$.

Critical λ

Proposition

There exists λ_c such that for $\lambda < \lambda_c$, $\xi(t)$ has only one invariant measure, which is concentrated at $\emptyset$. For $\lambda > \lambda_c$, there is also another extremal invariant measure, which we call μ, which is obtained by time-averaging $\xi(t)$.

Fundamental Lemma of the Percolation Structure

Lemma

- *Suppose there is a path from* $(y_1, 0)$ *to* (x_1, t) *and a path from* $(y_2, 0)$ *to* (x_2, t)*, and* $y_1 < y_2$*,* $x_1 > x_2$

Fundamental Lemma of the Percolation Structure

Lemma

- *Suppose there is a path from* $(y_1, 0)$ *to* (x_1, t) *and a path from* $(y_2, 0)$ *to* (x_2, t)*, and* $y_1 < y_2$*,* $x_1 > x_2$
- *Then* $\exists$ *a path from* $(y_1, 0)$ *to* (x_2, t) *and a path from* $(y_2, 0)$ *to* (x_1, t)*.*

Fundamental Lemma of the Percolation Structure

Lemma

- *Suppose there is a path from* $(y_1, 0)$ *to* (x_1, t) *and a path from* $(y_2, 0)$ *to* (x_2, t)*, and* $y_1 < y_2$*,* $x_1 > x_2$
- *Then* $\exists$ *a path from* $(y_1, 0)$ *to* (x_2, t) *and a path from* $(y_2, 0)$ *to* (x_1, t)*.*

Proof.

The two paths must intersect at some point (z, s). So then there are paths $(y_1, 0) \to (z, s)$, $(y_2, 0) \to (z, s)$, $(z, s) \to (x_1, t)$, and $(z, s) \to (x_2, t)$. Compose these paths to get our answer. □

Monotone Convergence

- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of t.

Monotone Convergence

- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of t.
- Converges to $\mu_B(\eta \mid \eta \cap A \neq \emptyset)$, where μ_B is the invariant measure of ξ started in state B.

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

"The probability of reaching somewhere in B starting from A is the same as the probability of reaching somewhere in A starting from B"

Self-duality

As long as A or B is finite, then for all t.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

"The probability of reaching somewhere in B starting from A is the same as the probability of reaching somewhere in A starting from B"
Specifically, for A finite

$$\mathbb{P}(\xi^A(t) \neq \emptyset) = \mathbb{P}(\xi(t) \cap A \neq \emptyset) \tag{4}$$

Consequence of Monotone Convergence + Self-duality

$$\begin{aligned}
\mathbb{P}(\xi^A(t) \neq \emptyset, \forall t > 0) &= \lim_{t \to \infty} \mathbb{P}(\xi^A(t) \neq \emptyset) \\
&= \lim_{t \to \infty} \mathbb{P}(\xi(t) \cap A \neq \emptyset) \\
&= \mu(\eta \mid \eta \cap A \neq \emptyset)
\end{aligned}$$

Characterizing μ

- μ is translation invariant

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta|\eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta | \eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$
- Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.

Characterizing μ

- μ is translation invariant
- For $\lambda > \lambda_c$, given ε, we can find $n(\varepsilon)$ such that $\mu(\eta|\eta \cap [1, n(\varepsilon)] \neq \emptyset) > 1 - \varepsilon$
- Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.
- It turns out that you can generalize this: if you start out *with at least* $n(\varepsilon)$ elements, you are almost sure to survive.

What have we learned so far?

- An intuition for metastability — two key properties

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - How percolation structure relates different contact processes

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - How percolation structure relates different contact processes
  - Monotone convergence

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - How percolation structure relates different contact processes
  - Monotone convergence
  - Self-duality

What have we learned so far?

- An intuition for metastability — two key properties
- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical λ
  - How percolation structure relates different contact processes
  - Monotone convergence
  - Self-duality
  - How to stay alive in the cruel, hard world of the contact process: be big!

Goals

Theorem

If T_N is hitting time of $\emptyset$, then

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to Exp(1) *in distribution.*

Goals

Theorem

If T_N is hitting time of $\emptyset$, then

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to $\mathrm{Exp}(1)$ *in distribution.*

- Not feasible to give whole proof

Goals

Theorem

If T_N is hitting time of $\emptyset$, then

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

That is, it converges to Exp(1) *in distribution.*

- Not feasible to give whole proof
- Instead, will give strategy and then give detailed proof of just one part

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that
$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$
- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be CDF for T_N/β_N

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be CDF for T_N/β_N
- (note that $\{\frac{T_N}{\beta_N} > t\} = \{\xi_N(\beta_N t) \neq \emptyset\}$)

Strategy Part 1

- Use β_N instead of $\mathbb{E}\, T_N$, where β_N is unique number such that

$$\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$$

- Let $G_N(t) = \mathbb{P}\left[\frac{T_N}{\beta_N} > t\right]$ be CDF for T_N/β_N
- (note that $\{\frac{T_N}{\beta_N} > t\} = \{\xi_N(\beta_N t) \neq \emptyset\}$)
- Prove that

$$\lim_{N\to\infty} |G_N(t)G_N(s) - G_N(t+s)| = 0$$

- The only function with $f(t+s) = f(t)f(s)$ and $\int_0^\infty f(t)\,\mathrm{d}t = 1$ is $f(t) = e^{-t}$.

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions
- If we are not in F_b, then we are most likely already dead. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of “sufficiently dense” initial conditions
- If we are not in F_b, then we are most likely already dead. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$
- Starting from $A \in F_b$ is “just as good” as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.

Strategy Part 2

- Use a set $F_b \subseteq \mathcal{P}([-N, N])$ of "sufficiently dense" initial conditions
- If we are not in F_b, then we are most likely already dead. Therefore, the probability that we end up in F_b at time $\beta_N t$ is approximately $G_N(t)$
- Starting from $A \in F_b$ is "just as good" as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.
- Put together, these two imply that $G_N(t)G_N(s) \sim G_N(t + s)$

Into the Unknown

- Show that $\mathbb{P}[T_N = T_N^A] > 1 - \varepsilon$ when $A \in F_b$, for sufficiently large N and b.
- Let
$$F_b = \{A \in \mathbb{Z} \mid \frac{|A \cap [-b,-1]|}{b} \geq \frac{\rho}{2}, \frac{|A \cap [1,b]|}{b} \geq \frac{\rho}{2}\}$$
- "sufficiently dense on both sides"
- $\rho = \mathbb{P}[\xi^{\{0\}}(t) \neq \emptyset, \forall t] = \mu(\eta \mid \eta(0) = 1)$.

- Let $n = n(\varepsilon)$ large enough so that starting out with n elements will let you survive forever with probability $1 - \varepsilon/2$

- Let $n = n(\varepsilon)$ large enough so that starting out with n elements will let you survive forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, starting in $[1, b] \cap A$ or $[-b, -1]$ will let you survive forever with probability $1 - \varepsilon/2$

- Let $n = n(\varepsilon)$ large enough so that starting out with n elements will let you survive forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, starting in $[1, b] \cap A$ or $[-b, -1]$ will let you survive forever with probability $1 - \varepsilon/2$
- Let E be the event that $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$ and $\xi_{(-\infty,N]}^{A\cap[1,b]}(t)$ survive forever

- Let $n = n(\varepsilon)$ large enough so that starting out with n elements will let you survive forever with probability $1 - \varepsilon/2$
- Then if $b\rho/2 > n(\varepsilon)$, starting in $[1, b] \cap A$ or $[-b, -1]$ will let you survive forever with probability $1 - \varepsilon/2$
- Let E be the event that $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$ and $\xi_{(-\infty,N]}^{A\cap[1,b]}(t)$ survive forever
- $\mathbb{P}(E) > 1 - \varepsilon$

- By symmetry,

$$\mathbb{P}[\xi_{(-\infty,N]}^{A\cap[1,b]}(t) \neq \emptyset, \forall t] > 1 - \frac{\varepsilon}{2}$$

- Let E be the event that $\xi_{(-\infty,N]}^{A\cap[1,b]}(t) \neq \emptyset, \forall t$ and $\xi_{[-N,\infty)}^{A\cap[-b,-1]}(t) \neq \emptyset, \forall t$. We have shown $\mathbb{P}(E) > 1 - \varepsilon$.
- It remains to show that $T_N^A = T_N$ on E.

- Define two stopping times

$$U = \inf t \mid N \in \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$$
$$V = \inf t \mid -N \in \xi_{(-\infty,N]}^{A\cap[1,b]}(t)$$

- These are almost surely finite on E.
- At time U, ξ_N^A is alive, because $\xi_N^A(U) \supset \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$, and similarly for V, so $T_N \geq T_N^A > \max(U, V)$

- After U and V, there is a path from $[-b, -1] \cap A$ to N, and a path from $[1, b] \cap A$ to $-N$. Intuitively, one of those paths intersects any path from $x \in [-N, N]$ to $y \in \xi_N(t)$. (draw picture)
- Therefore, for $t > \max(U, V)$, $\xi_N(t) = \xi_N^A(t)$
- Therefore, $T_N = T_N^A$ on E, and we have shown that $\mathbb{P}(T_N = T_N^A) > 1 - \varepsilon$.