Metastability for the Contact Process on $\mathbb{Z}$

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The most basic description of metastability is that it is the presence of "quasi-equlibria" states and rapid transitions between them in a system. Metastability is a common phenomenon in nature, with examples ranging from physics to economics and social sciences. In this paper we will focus on framing the empirical phenomenon of metastability as a rigorous property of an abstract system. We will use a well-known interacting particle system - the Voter Model, for this purpose.

1 Overview - what's metastability?

Take a family of Markov processes $\{\xi_N(t)\}_{N\in\mathbb{N}}$. The essential idea behind metastability is that, if we were to discretize ("coarse-grain") time and increase N, the process under consideration would look more and more like the Markov process in figure **??**.

Figure 1: Metastability in a Nutshell
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The system begins in a so-called "metastable state", and has a very small chance to move to the stable state at any time. The stable state is either absorbing, or close-to absorbing; in the case that we will talk about in this paper, the stable state is absorbing.

In order to make this "coarse-graining" happen, we need two things to be true asymptotically.

1. The hitting time of the stable state is exponentially distributed.
2. Up until the hitting time, temporal means of measurements made to the process converge to a stationary distribution.

Together, these two properties intuitively allow us to approximate the entire process by sampling from the stationary distribution up until the hitting time, and then putting the system in the absorbing state after the hitting time.

The difficulty comes in stating these two properties precisely. To do this, suppose that T_N is the hitting time of the absorbing state, μ is the stationary distribution, and

R_N is a "time scale" parameter that satisfies $R_N/\mathbb{E}\,T_N \to 0$. Then we rewrite the two properties more formally as

1. $T_N/\mathbb{E}\,T_N \to \mathrm{Exp}(1)$ in distribution as $N \to \infty$.

2. For any f,
$$\int_S^{S+R_N} f(\xi_N(t))dt \to \mu(f)$$
as $N \to \infty$, for any $S + R_N < T_N$.

This second statement is still very imprecise, and actually mathematically meaningless as currently posed. Also, it turns out that we want a much stronger statement than that. However, we hope that this first statement should "innoculate" the reader to the precise statement, which is fairly dense on its own.

2 Review of Contact Process

2.1 Basic Definitions

We could succinctly define the Contact Process by using its Markov generator. However, for our purposes we will use an alternative definition that lends itself better to certain constructions that are useful in proofs concerning metastability.

First we construct a "percolation structure", which consists of for each $x \in \mathbb{Z}$

1. A Poisson process P_x with rate 1, which we call the "death" process at x.

2. A Poisson process $P_{x \to x+1}$ with rate λ, which we call the "right infection" process at x.

3. A Poisson process $P_{x \to x-1}$ with rate λ, which we call the "left infection" process at x.

We consider P_x to be a random element of $\mathcal{P}(\mathbb{R})$, i.e. $t \in P_x$ if and only if the Poisson process "ticks" at time t.

Figure 2: Example Percolation Structure

We define a "path" between $(x, s), (y, t) \in \mathbb{Z} \times \mathbb{R}$ with $s \le t$ to be a sequence $(z_0, r_0), \ldots, (z_n, r_n)$ with $r_i \le r_{i+1}$ such that for all $(z_i, r_i), (z_{i+1}, r_{i+1})$, either

1. $r_i = r_{i+1}$, $|z_i - z_{i+1}| = 1$, and $r_i \in P_{z_i \to z_{i+1}}$. In this case, we are jumping laterally by one line at a time of infection.

2. $z_i = z_{i+1}$, and $[r_i, r_{i+1}] \cap P_{z_i} = \emptyset$. In this case, we are moving along a vertical line, uninterrupted by any deaths.

Define $\xi^A(t)$ to be the set of y such that there is a path from $(x,0)$ to (y,t) for some $x \in A$. If the superscript is omitted, then we assume $A = \mathbb{Z}$, i.e. $\xi(t) = \xi^{\mathbb{Z}}(t)$.

For any $A \subseteq B$, we define $\xi_B^A(t)$ to be the set of y such that there is a path from $(x,0)$ to (y,t) for some $x \in A$ that stays entirely within B. As a special case, we let $\xi_N^A(t) = \xi_{[-N,N]}^A$, for $N \in \mathbb{N}$. Note that ξ_B takes values exclusively in $\mathcal{P}(B)$.

2.2 Useful Properties

One of the most important facts about the contact process is that there is a critical value of λ, λ_c. For $\lambda < \lambda_c$, $\xi(t)$ has a unique extremal invariant measure $\delta_\emptyset$. At $\lambda = \lambda_c$ the system undergoes a phase transition - for $\lambda > \lambda_c$, another extremal invariant measure μ appears, with the property that

$$\mu(f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\, f(\xi(t))\, \mathrm{d}t$$

This measure is not concentrated at $\emptyset$.

For $\xi_N(t)$, the only invariant measure is $\delta_\emptyset$, because $\emptyset$ is a trap and ξ_N takes values on a finite state space. However, as mentioned before, for $\lambda > \lambda_c$ an analogue of a phase transition takes place. The system starts being metastable, and stays distributed *approximately* as μ, before a sudden fluctuation takes it to $\emptyset$.

3 Hitting Time of Stable State

Theorem 1. *If $T_N = \inf\{t > 0 \mid \xi_N(t) \neq \emptyset\}$, then*

$$\frac{T_N}{\mathbb{E}\, T_N} \to \mathrm{Exp}(1)$$

in distribution.

Without further ado, we get down to the proof. First of all, we replace $\mathbb{E}\, T_N$ by the unique (by monotonicity) β_N such that $\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$. At the end, we will show that $\frac{\mathbb{E}\, T_N}{\beta_N} \to 1$.

Now let $G_N(t) = \mathbb{P}[\frac{T_N}{\beta_N} > t]$, the CDF of T_N/β_N. Similarly, let $G_N^A(t) = \mathbb{P}[\frac{T_N^A}{\beta_N} > t]$. Note that $G_N^A(t) \leq G_N(t)$ To show that T_N/β_N converges in distribution to Exp(1), we must show that $G_N(t) \to \mathrm{e}^{-t}$. This can be accomplished by showing that

$$\lim_{N \to \infty} |G_N(t+s) - G_N(t)G_N(s)| = 0$$

for all $t, s > 0$.

Note that $G_N(t) = \mathbb{P}[\xi_N(t) \neq \emptyset]$, as ξ_N is "alive" at time t if and only if $T_N > t$. Thus,

$$\begin{aligned}
G_N(t+s) &= \mathbb{P}[\xi_N(t+s) \neq \emptyset] \\
&= \sum_{A \neq \emptyset} \mathbb{P}[\xi_N(t+s) \neq \emptyset | \xi_N(t) = A]\, \mathbb{P}[\xi_N(t) = A] \\
&= \sum_{A \neq \emptyset} G_N^A(s)\, \mathbb{P}[\xi_N(t) = A] \\
&\leq G_N(s) \sum_{A \neq \emptyset} \mathbb{P}[\xi_N(t) = A] \\
&= G_N(s) G_N(t)
\end{aligned}$$

Therefore, we are looking to show that $G_N(s)G_N(t) - G_N(t+s) \to 0$ (we can forget the absolute value signs).

It is at this point that we introduce a curious little construction, which seems to not make much sense at first but turns out to be the key to the entire proof. Define F_b for $b > 0$ by

$$F_b = \{A \subseteq \mathbb{Z} \mid \frac{|A \cap [-b,-1]|}{b} > frac\rho 2, \frac{|A \cap [1,b]|}{b} > frac\rho 2\}$$

Recall that ρ was defined in **??** to be the probability that $\xi^{\{0\}}$ never dies. The intuition that the reader should hold for F_b is that it is the set of A that are "sufficiently dense" on $[-b,-1]$ and $[1,b]$ so that ξ_N^A behaves just like ξ_N. The reason we have this "two-sided" condition is that we will use the processes $\xi_{(-\infty,N)}$ and $\xi_{(-N,\infty)}$, which have invariant measure μ as well, to talk about ξ_N.

Now, by similar reasoning as before,

$$\begin{aligned}
G_N(t+s) &= \sum_{A \neq \emptyset} G_N^A(t)\, \mathbb{P}[\xi_N(\beta_N s) = A] \\
&\geq \sum_{A \in F_b} G_N^A(t)\, \mathbb{P}[\xi_N(\beta_N s) = A] \\
&\geq \min_{A \in F_b} G_N^A(t)\, \mathbb{P}[\xi_N(\beta_N s) \in F_b]
\end{aligned}$$

(We assume that $A \subseteq [-N,N]$ when such an assumption is necessary).

Therefore,

$$\begin{aligned}
G_N(t)G_N(s) - G_N(t+s) &\leq G_N(t)G_N(s) - \min_{A \in F_b} G_N^A(t)\, \mathbb{P}[\xi_N(\beta_N s) \in F_b] \\
&= G_N(s)(G_N(t) - \min_{A \in F_b} G_N^A(t)) \\
&\quad + \min_{A \in F_b} G_N^A(t)(G_N(s) - \mathbb{P}[\xi(\beta_N s) \in F_b]) \\
&\leq (G_N(t) - \min_{A \in F_b} \mathbb{P}[\xi_N^A(\beta_N t) \neq \emptyset]) \\
&\quad + \mathbb{P}[\xi_N(\beta_N s) \neq \emptyset, \xi_N(\beta_N s) \notin F_b]
\end{aligned}$$

We will have finished if for any $\varepsilon > 0$, we can find $b(\varepsilon)$ and $N(\varepsilon) > b(\varepsilon)$ such that for $N \geq N(\varepsilon)$ and $A \in F_b$, we have both

$$G_N(t) - G_N^A(t) < \varepsilon \tag{1}$$

$$\mathbb{P}[\xi_N(\beta_N s) \neq \emptyset, \xi_N(\beta_N s) \notin F_b] < \varepsilon \tag{2}$$

We tackle (1) first. Remember that $\xi_N(t)$ and ξ_N^A are defined on the same percolation structure. Therefore, $\xi_N(t) \supset \xi_N^A(t)$, so we have

$$\mathbb{P}[\xi_N(\beta_N t) \neq \emptyset] - \mathbb{P}[\xi_N(\beta_N t) \neq \emptyset] = \mathbb{P}[\xi_N(\beta_N t) \neq \emptyset, \xi_N^A(\beta_N t) = \emptyset] \leq \mathbb{P}[T_N \neq T_N^A]$$

The intuition for why the right hand side of this is small is that once $\xi_N(t_0) = \xi_N^A(t_0)$, then for all $t > t_0$, $\xi_N(t) = \xi_N^A(t)$. Therefore, as long as the event $\xi_N(t_0) = \xi_N^A(t_0)$ happens before T_N^A, we will have $T_N^A = T_N$. Let E be this event; we must show that $\mathbb{P}(E) > 1 - \varepsilon$.

As $\lambda > \lambda_c$, we can find $n(\varepsilon)$ such that

$$\mu(B \mid B \cap [1, n(\varepsilon)] = \emptyset) \leq \frac{\varepsilon}{2}$$

Then take $b = b(\varepsilon)$ such that $n(\varepsilon) \leq b \cdot \rho/2$. For $A \in F_b$, $|A \cap [-b, -1]| \geq b \cdot \rho/2 \geq n(\varepsilon)$, so

$$\mathbb{P}[T_{[-N,\infty)}^{A \cap [-b,-1]} = \infty] \geq \mathbb{P}[T_{[-N,\infty]}^{[-N,\ldots,-N+n(\varepsilon)]} = \infty] \geq 1 - \frac{\varepsilon}{2}$$

TODO give argument for the first inequality. The second inequality follows from duality and monotone convergence.

Similarly, $\mathbb{P}[T_{(-\infty,N]}^{A \cap [1,b]} = \infty] \geq 1 - \frac{\varepsilon}{2}$. Let E' be the event that both of these stopping times are infinite; clearly $P(E') \geq 1 - \varepsilon$. It remains to show that $E \supset E'$.

Define stopping times

$$U = \inf\{t > 0 \mid N \in \xi_{[-N,\infty)}^{A \cap [-b,-1]}(t)\}$$

$$V = \inf\{t > 0 \mid -N \in \xi_{(-\infty,N]}^{A \cap [1,b]}(t)\}$$

Until U, $\xi_N^{A \cap [-b,-1]}(t) = \xi_{[-N,\infty)}^{A \cap [-b,-1]}(t)$. Therefore, on E', $\xi_N^{A \cap [-b,-1]}(t)$ is alive up until U, whence ξ_N^A is alive up until U. Similarly, ξ_N^A is alive up until V. Therefore, on E', $T_N \geq T_N^A > \max(U, V)$.

However, for $t > \max(U, V)$, I claim that $\xi_N(t) = \xi_N^A(t)$. To see this, any path to $x \in \xi_N(t)$ must either cross the path that goes from $A \cap [-b, -1]$ at time 0 to N at time U or the path that goes from $A \cap [1, b]$ at time 0 to $-N$ at time V. Therefore, there is a path from A at time 0 to x at time t, whence $x \in \xi_N^A(t)$. Therefore, on E', $T_N = T_N^A$, and we have shown (1).

Equation 2 also relies crucially on the percolation structure. Let $D_b = F_b^C \setminus \{\emptyset\}$. Then we must show that $\mathbb{P}(\xi_N(\beta_N s) \in D_b) < \varepsilon$. To do this, note that as long as $\xi_N(t)$ is alive, we have

$$\xi_N(t) = \xi(t) \cap [\min \xi_N(t), \max \xi_N(t)]$$

This is because any path to $x \in \xi(t) \cap [\min \xi_N(t), \max_N(t)]$ must either stay inside $[\min \xi_N(t), \max_N(t)]$, in which case $x \in \xi_N(t)$, or pass through the path to $\min \xi_N(t)$ or $\max \xi_N(t)$, in which case we also have $x \in \xi_N(t)$.

Therefore, as long as $\min \xi_N(\beta_N s) < -N+L$, $\max \xi_N(\beta_N s) > N-L$ and $\xi_N(\beta_N s) \neq \emptyset$, we have $\xi(\beta_N s) \in D_b$ iff $\xi_N(\beta_N s) \in D_b$.

We can minimize $\xi(\beta_N s) \in D_b$ by picking $b > b''(\varepsilon)$ sufficiently large. In order to use this, we use the following decomposition

$$\begin{aligned}\mathbb{P}[\xi_N(\beta_N s) \in D_b] \leq\ & \mathbb{P}[\xi_N(\beta_N s) \in D_b, \min \xi_N(\beta_N s) < -N+L, \max \xi_N(\beta_N s) > N-L] \\ &+ \mathbb{P}[\min \xi_N(\beta_N s) \geq -N+L, \xi_N(\beta_N s) \neq \emptyset] \\ &+ \mathbb{P}[\max \xi_N(\beta_N s) \leq N-L, \xi_N(\beta_N s) \neq \emptyset]\end{aligned}$$

By what we noted earlier, the first term is less than $\mathbb{P}(\xi(\beta_N s) \in D_b)$; pick b such that this is less than $\frac{\varepsilon}{3}$.

It remains to minimize the last two terms; by symmetry we only show how to minimize the first. Using a percolation structure argument, it is easy to show that as long as $\xi_N(\beta_N s) \neq \emptyset$, $\min \xi_N(t) = \min \xi_{[-N,\infty)}$. Therefore,

$$\begin{aligned}\mathbb{P}[\min \xi_N(\beta_N s) \geq -N+L, \xi_N(\beta_N) \neq \emptyset] &\geq \mathbb{P}[\min \xi_{[-N,\infty)} \geq -N+L] \\ &\geq \mu_{[-N,\infty)}\{A \in [-N,\infty) \mid A \cap [-N, -N+L-1] = \emptyset\}\end{aligned}$$

Finally can pick L large enough to make that last term less than $\frac{\varepsilon}{3}$, and we have shown that everything can be made as small as we like, so we are done.

4 Almost-Ergodicity up to Hitting Time