# Metastability for the Contact Process on $\mathbb{Z}$, Part 1.

## Following [Sch85]

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# Outline

# Introduction

# Informal Metastability

- Markov process $X_N(t)$

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- As $N$ goes to $\infty$, looks like figure 1.

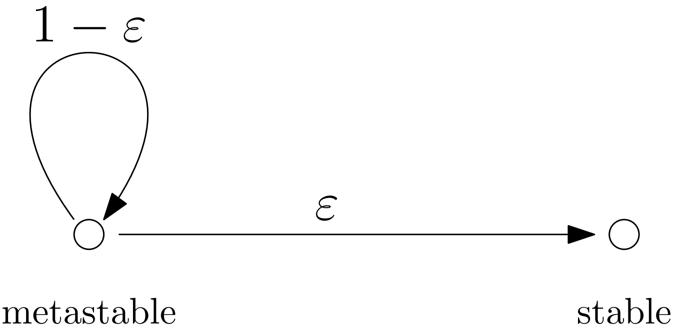


Figure: Coarse Graining as $N \to \infty$

# Formal Metastability

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   - There is a “trap state”, with hitting time $T_N$
   - Asymptotically as $N \to \infty$, $T_N$ has an exponential distribution.
2. Quasi-stationary distribution before hitting time
   - There is a “approximate invariant distribution” $\mu$
   - Up until $T_N$, temporal means of $X_N(t)$ approximate $\mu$

# Contact Process, Generator Definition

- $\xi(t)$ is a Markov process taking values in $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$
- Characterized by

$$Lf(\eta) = \sum_x c(x, \eta)(f(\eta^x) - f(\eta)) \tag{1}$$

- With rates

$$c(x, \eta) = \begin{cases} 1 & \text{if} \quad \eta(x) = 1 \\ \lambda(\eta(x-1) + \eta(x+1)) & \text{otherwise} \end{cases} \tag{2}$$

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- Percolation structure is very useful for proofs

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- $\xi_B^A(t)$ is the set of $x \in B$ such that there is a path $(y,0) \to (x,t)$ with $y \in A$ and the path not leaving $B$
- Special cases
  - $\xi_N(t) = \xi_{[-N,N]}(t)$
  - $\xi_{[-N,\infty)}(t)$
  - $\xi_{(-\infty,N]}(t)$

# Crash Course in Contact Process

# Critical $\lambda$

### Proposition

*There exists $\lambda_c$ such that for $\lambda < \lambda_c$, $\xi(t)$ has only one invariant measure, which is concentrated at $\emptyset$. For $\lambda > \lambda_c$, there is also another extremal invariant measure, which we call $\mu$, which is obtained by time-averaging $\xi(t)$. This also holds for $\xi_{[-N,\infty)}$ and $\xi_{(-\infty,N]}$*

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- $\mathbb{P}(\xi^A(t) \cap B \neq \emptyset)$ is a nonincreasing function of $t$.
- Converges to $\mu_B(\eta \mid \eta \cap A \neq \emptyset)$, where $\mu_B$ is the invariant measure of $\xi$ started in state $B$.

# Self-duality

As long as $A$ or $B$ is finite, then for all $t$.

$$\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\xi^B(t) \cap A \neq \emptyset) \tag{3}$$

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"The probability of reaching somewhere in $B$ starting from $A$ is the same as the probability of reaching somewhere in $A$ starting from $B$"
Specifically, for $A$ finite

$$\mathbb{P}(\xi^A(t) \neq \emptyset) = \mathbb{P}(\xi(t) \cap A \neq \emptyset) \tag{4}$$

# Consequence of Monotone Convergence + Self-duality

$$
\begin{aligned}
\mathbb{P}(\xi^A(t) \neq \emptyset, \forall t > 0) &= \lim_{t \to \infty} \mathbb{P}(\xi^A(t) \neq \emptyset) \\
&= \lim_{t \to \infty} \mathbb{P}(\xi(t) \cap A \neq \emptyset) \\
&= \mu(\eta \mid \eta \cap A \neq \emptyset)
\end{aligned}
$$

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- Therefore, we can find $n(\varepsilon)$ so that if you start out in $[1, n(\varepsilon)]$, you are almost sure to survive.
- It turns out that you can generalize this: if you start out *with at least* $n(\varepsilon)$ elements, you are almost sure to survive.

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  - Self-duality

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- How to use the percolation structure to construct many related contact processes
- Some miscellaneous useful facts about the contact process
  - Critical $\lambda$
  - Monotone convergence
  - Self-duality
  - How to stay alive in the cruel, hard world of the contact process: be big!

# Theorem 1

# Goals

**Theorem**

*If $T_N$ is hitting time of $\emptyset$, then*

$$\frac{T_N}{\mathbb{E}\, T_N} \xrightarrow{w} \mathrm{Exp}(1)$$

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- Not feasible to give whole proof
- Instead, will give strategy and then give detailed proof of just one part

# Strategy Part 1

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- Prove that
$$\lim_{N\to\infty} |G_N(t)G_N(s) - G_N(t+s)| = 0$$
- The only function with $f(t+s) = f(t)f(s)$ and $\int_0^\infty f(t)\,\mathrm{d}t = 1$ is $f(t) = e^{-t}$.

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- Starting from $A \in F_b$ is "just as good" as starting from $[-N, N]$, so if we end up in $A \in F_b$ at time $\beta_N t$, then we have approximately $G_N(s)$ chance of still being alive at time $\beta_N(s + t)$.

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- Put together, these two imply that $G_N(t)G_N(s) \sim G_N(t + s)$

# Into the Unknown

- Show that $\mathbb{P}[T_N = T_N^A] > 1 - \varepsilon$ when $A \in F_b$, for sufficiently large $N$ and $b$. (i.e., $A \in F_b$ is "just as good" as $[-N, N]$)
- Let
$$F_b = \{A \in \mathbb{Z} \mid \frac{|A \cap [-b, -1]|}{b} \geq \frac{\rho}{2}, \frac{|A \cap [1, b]|}{b} \geq \frac{\rho}{2}\}$$
- "sufficiently dense on both sides"
- $\rho = \mathbb{P}[\xi^{\{0\}}(t) \neq \emptyset, \forall t] = \mu(\eta \mid \eta(0) = 1)$.

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- Let $E$ be the event that $\xi^{A\cap[-b,-1]}_{[-N,\infty)}(t)$ and $\xi^{A\cap[1,b]}_{(-\infty,N]}(t)$ survive forever (the invariant distribution for $[-N, \infty)$ is the same as that for $\mathbb{Z}$)

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- $\mathbb{P}(E) > 1 - \varepsilon$
- Therefore, we are done if in $E$, $T_N = T^A_N$.

- Define two stopping times

$$U = \inf\{t \mid N \in \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)\}$$
$$V = \inf\{t \mid -N \in \xi_{(-\infty,N]}^{A\cap[1,b]}(t)\}$$

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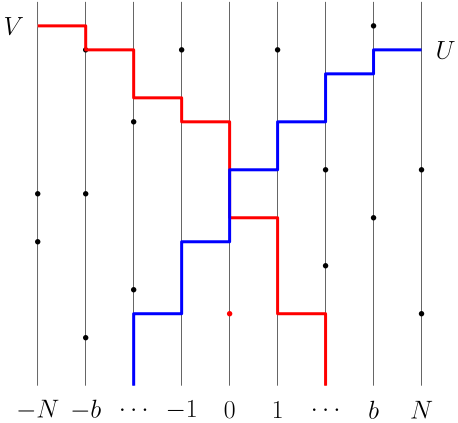
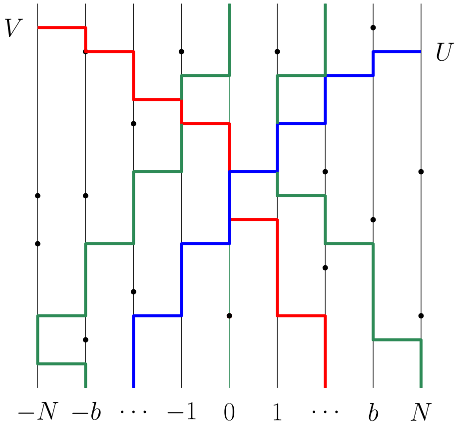


- $U$ and $V$ are almost surely finite on $E$.
- At time $U$, $\xi_N^A$ is alive, because $\xi_N^A(U) \supset \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$, and similarly for $V$, so $T_N \geq T_N^A > \max(U, V)$

- After $U$ and $V$, there is a path from $[-b,-1]\cap A$ to $N$, and a path from $[1,b]\cap A$ to $-N$. Intuitively, one of those paths intersects any path from $x \in [-N,N]$ to $y \in \xi_N(t)$.



- For $t > \max(U, V)$, $\xi_N(t) = \xi_N^A(t)$
- Therefore, $T_N = T_N^A$ on $E$, and we have shown that $\mathbb{P}(T_N = T_N^A) > 1 - \varepsilon$.

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- Basic properties of the contact process
- A rough sketch of how the contact process is metastable (part 1)
- What working with the contact process in practice looks like

# Citations

[Sch85] Roberto H. Schonmann. "Metastability for the contact process." In: *Journal of Statistical Physics* 41.3 (Nov. 1, 1985), pp. 445–464. ISSN: 1572-9613. DOI: 10.1007/BF01009017. URL: https://doi.org/10.1007/BF01009017 (visited on 11/12/2020).