Metastability for the Contact Process on Z: Part 2

Kacper Urbański

Universiteit van Amsterdam

kacper.urbanski@protonmail.com

December 6, 2020

Overview

1. Forumlating the theorem
   - Natural language definition
   - Towards the rigorous definition
   - Statement of the theorem

2. Sketch of the proof

A bit of terminology

- ξ - 'xi'
- ξ_N - 'xi n'
- $\xi_{[-N,\infty)}$ - 'xi plus inf'
- $\xi_{(-\infty,N]}$ - 'xi minus inf'
- $[-N, N]$ - 'main interval'
- $\xi_N(t) \neq \varnothing$ - 'process is still alive'

Natural language definition of metastability

Recall that a system is metastable if:

1. It stays out of its equilibrium during a memoryless random time
2. During this time in which the system is out of equilibrium it stabilizes

Let's elaborate some more on point 2.

1. Assume that for a given N we have some intermediate timescale such R_N that $R_N << \beta_N$
2. Say we measure a temporal mean of some observable quantity of a system (e.g. particle density) over this timescale
3. We say system has stabilized if this mean is close to the expectation of this observable quantity w.r.t. some fixed probability distribution on $\{0,1\}^{\mathbb{Z}}$

Towards the rigorous definition

To ensure $R_N << \beta_N$, let's require $R_N/\beta_N \to 0$ as $N \to \infty$.

For our purposes, define **observable quantity of a system** as f, such that:

- $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$
- f is local

Define

$$\Lambda(f) := \text{ smallest } B \subset \mathbb{Z} \text{ s.t. } f(A) = f(A \cap B)\ \forall A \subset \mathbb{Z}$$

We can think of $\Lambda(f)$ as the "support" of f.

Towards the rigorous definition

Define **temporal mean** of observable quantity $f(\xi_N(t))$ as:

$$A_R^N(s, f) := R^{-1} \int_s^{s+R} f(\xi_N(t))dt$$

Where:

- s is the time in which we start our measurement
- R is the duration over which we calculate the temporal mean

Towards the rigorous definition

- Take our **fixed probability distribution** to be μ (i.e. the non-zero invariant measure of the contact process in the supercritical regime).
- Then, we can define **expectation of observable quantity** w.r.t to this probability distribtuion as $\mu(f) := \int f d\mu$
- Take convergence in probability as how we understand **closeness**.

Towards the rigorous definition

Let's try to work out a 1st draft of the theorem.

We want to have a sequence R_N, such that:

- $R_N/\beta_N \to 0$ as $N \to \infty$
- for all $\varepsilon > 0$ and all observable quantities f

$$\mathbb{P}\left[|A_{R_N}(s,f) - \mu(f)| \langle \varepsilon\right] \to 0$$

As $N \to \infty$.

Are we done now?

Towards the rigorous definition

Let's try to work out a 1st draft of the theorem.

We want to have a sequence R_N, such that:

- $R_N/\beta_N \to 0$ as $N \to \infty$
- for all $\varepsilon > 0$ and all observable quantities f

$$\mathbb{P}\left[|A_{R_N}(s, f) - \mu(f)| \langle \, \varepsilon\right] \to 0$$

As $N \to \infty$.

Are we done now? **No!** How do we choose s (the starting point of temporal mean measurement)?

Towards the rigorous definition

Define

$$K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$$

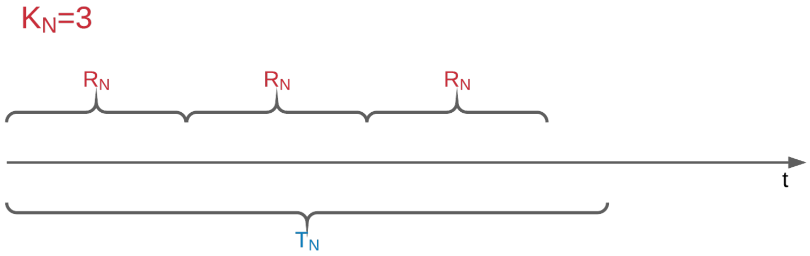

Figure: Example of K_N

Turns out that even if we choose the worst-case interval for averaging, we can still make it highly probable that our temporal mean is as close to the $\mu(f)$ as we want, given that we're free to increase N.

Towards the rigorous definition

In other words, we want to have a sequence R_N, such that:

- $R_N/\beta_N \to 0$ as $N \to \infty$
- for all $\varepsilon > 0$ and all observable quantities f

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| < \varepsilon\right] \to 0$$

As $N \to \infty$.

This draft is almost ready - we now only have one more technicality left.

Towards the rigorous definition

We need to choose $L(\varepsilon, f) \in \mathbb{N}$, and we need to have that

- $L < N$
- $\Lambda(f) \subset [-N+L, N-L]$ (call $[-N+L, N-L]$ **inside region**)

Notice that L does not depend on N. Thus, having to choose this L doesn't restrict our choice of f - it merely sets the minimum N we can consider and we chose to grow $N \to \infty$.

Towards the rigorous definition

We need to choose $L(\varepsilon, f) \in \mathbb{N}$, and we need to have that

- $L < N$
- $\Lambda(f) \subset [-N+L, N-L]$ (call $[-N+L, N-L]$ **inside region**)

Notice that L does not depend on N. Thus, having to choose this L doesn't restrict our choice of f - it merely sets the minimum N we can consider and we chose to grow $N \to \infty$.

Additionally, define

- $[-N, -N+L)$ - **left boundary region**
- $(N-L, N]$ - **right boundary region**

We're finally ready to formulate the theorem.

Theorem (Thermalization)

If $\lambda > \lambda^*$ there is a sequence $\{R_N\}_{N\in\mathbb{N}} \subset \mathbb{R}_+$ such that:

- $R_N/\beta_N \to 0$ *as* $N \to \infty$
- *For all* $\varepsilon > 0$ *and observable quantities* f $\exists L(\varepsilon, f) \in \mathbb{N}$ *such that*

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right] \to 0$$

as $N \to \infty$, *where* $K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$ *and* $\Lambda(f) \subset [-N + L, N - L] \cap \mathbb{Z}$

We're finally ready to formulate the theorem.

Theorem (Thermalization)

If $\lambda > \lambda^*$ there is a sequence $\{R_N\}_{N\in\mathbb{N}} \subset \mathbb{R}_+$ such that:

- $R_N/\beta_N \to 0$ *as* $N \to \infty$
- *For all $\varepsilon > 0$ and observable quantities f $\exists L(\varepsilon, f) \in \mathbb{N}$ such that*

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right] \to 0$$

as $N \to \infty$, where $K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$ and $\Lambda(f) \subset [-N + L, N - L] \cap \mathbb{Z}$

The proof of this theorem is quite long. I will be focusing on most interesting details related to proving convergence in probability.

Set

$$B_k^N = \{|A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\}$$

On B_k^N the difference between temporal mean of observable measured over the k-th interval and its expectation is greater than we'd like to.
B_k^N means **failure** (in k-th interval).

We want the probability of having no failures to go to 1

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| \leq \varepsilon\right] = \mathbb{P}\left[\bigcap_{\mathbb{N}_0 \ni k < K_N} \left(B_k^N\right)^C\right] \to 1$$

as $N \to \infty$

It is easy to show that $\mathbb{P}(K_N = 0) \to 0$. Thus, we can safely focus only on the subset of Ω where $K_N \geq 1$.

Combining this with the event from the previous slide and applying some simple algebra, we arrive at the following bound

$$\mathbb{P}\left[K_N \geq 1, \bigcap_{\mathbb{N}_0 \ni k < K_N} \left(B_k^N\right)^C\right] \geq \mathbb{P}[1 \leq K_N \leq m] - m^2 \max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k^N, K_N = j\right]$$

Our objective will be to find a sequence $\{m_N\}_{N \in \mathbb{N}}$ such that:

- $\mathbb{P}[1 \leq K_N \leq m_N] \to 1$
- $m_N^2 \max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k^N, K_N = j\right] \to 0$

As $N \to \infty$

Let's take a closer look at $\mathbb{P}\left[B_k^N, K_N = j\right]$.

We will estimate the difference between the temporal average of $f(\xi_N)$ and it's expectation with a triangle inequality. For $k < j$ we have

$$\mathbb{P}\left[|A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right] \leq$$
$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2 \ \vee\right.$$
$$\left.\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

Let's take the first term in the alternative and try to find a bound for

$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right]$$

It looks like we could almost apply weak convergence here, but we there's something missing...

By weak convergence, as $R \to \infty$, we have

$$R_N^{-1}\int_{kR_N}^{(k+1)R_N} \mathbb{E}\left[f(\xi(t))\right] dt \to \mu(f)$$

Let's take the first term in the alternative and try to find a bound for

$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt-\mu(f)\right|>\varepsilon/2\right]$$

It looks like we could almost apply weak convergence here, but we there's something missing...

By weak convergence, as $R\to\infty$, we have

$$R_N^{-1}\int_{kR_N}^{(k+1)R_N}\mathbb{E}\left[f(\xi(t))\right]dt\to\mu(f)$$

If we can now find a way to bind what's below, we're in business!

$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))-\mathbb{E}\left[f(\xi(t))\right]dt\right|>\varepsilon/4\right]$$

Let's take the first term in the alternative and try to find a bound for

$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt-\mu(f)\right|>\varepsilon/2\right]$$

It looks like we could almost apply weak convergence here, but we there's something missing...

By weak convergence, as $R\to\infty$, we have

$$R_N^{-1}\int_{kR_N}^{(k+1)R_N}\mathbb{E}\left[f(\xi(t))\right]dt\to\mu(f)$$

If we can now find a way to bind what's below, we're in business!

$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))-\mathbb{E}\left[f(\xi(t))\right]dt\right|>\varepsilon/4\right]$$

Turns out there is a way to do it, which has to do with the fact that time correlations of $\xi(t)$ decay exponentially fast.

Recall, we've had:

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right] + \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

We managed to bind the 1st term using:

- Weak convergence
- Exponentially decaying correlations

Recall, we've had:

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right] + \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

We managed to bind the 1st term using:

- Weak convergence
- Exponentially decaying correlations

Will this go through for the 2nd term?

Recall, we've had:

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right] +$$
$$\mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

We managed to bind the 1st term using:

- Weak convergence
- Exponentially decaying correlations

Will this go through for the 2nd term?

- Weak convergence is a no-go. $\xi_N(t)$ converges weakly to δ_0, but $\xi(t)$ converges to μ...

Recall, we've had:

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right] + \mathbb{P}\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

We managed to bind the 1st term using:

- Weak convergence
- Exponentially decaying correlations

Will this go through for the 2nd term?

- Weak convergence is a no-go. $\xi_N(t)$ converges weakly to δ_0, but $\xi(t)$ converges to μ...
- However, we know that in our intermediate timescale $\xi_N(t)$ is distributed close to μ, so we should be able, in principle, to find a bound for the 2nd term

Let us first introduce an intermediate result. We will say $\xi_N(t)$ is **wide** at t if it intersects both boundary regions. In other words:

$$\min \xi_N(t) < -N + L \wedge \max \xi_N(t) > N - L$$

We will call the process **narrow** otherwise.

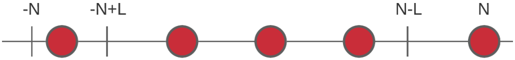

Figure: Snapshot at t of a process wide at t

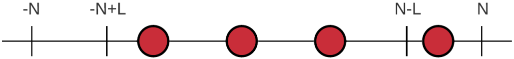

Figure: Snapshot at t of a process narrow at t

Lemma (Shielding by a wide process)

If $\xi_N(t)$ is wide at t, then

$$\xi_N(t) = \xi(t) \text{ on } [-N+L, N-L] \cap \mathbb{Z}$$

In particular we have $f(\xi_N(t)) = f(\xi(t))$

Why would this be true?

Lemma (Shielding by a wide process)

If $\xi_N(t)$ is wide at t, then

$$\xi_N(t) = \xi(t) \text{ on } [-N+L, N-L] \cap \mathbb{Z}$$

In particular we have $f(\xi_N(t)) = f(\xi(t))$

Why would this be true?

Essentially, rightmost and leftmost infected individuals "shield" the entire space between them from outside influence (see gif)

Recall, we were trying to find a bound for

$$P\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

Recall, we were trying to find a bound for

$$P\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

- We know that if $\xi_N(t)$ is wide, $f(\xi_N(t)) = f(\xi(t))$

Recall, we were trying to find a bound for

$$P\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

- We know that if $\xi_N(t)$ is wide, $f(\xi_N(t)) = f(\xi(t))$
- Thus, what we need is to make it highly unlikely for $\xi_N(t)$ to be narrow, i.e. we need to increase L

Recall, we were trying to find a bound for

$$P\left[\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right]$$

- We know that if $\xi_N(t)$ is wide, $f(\xi_N(t)) = f(\xi(t))$
- Thus, what we need is to make it highly unlikely for $\xi_N(t)$ to be narrow, i.e. we need to increase L
- Can we do that such that the value of L is independent of N?

- Notice: in the left boundary region, ξ_N will look similar to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive).

- Notice: in the left boundary region, ξ_N will look similar to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive).
- By the same reasoning, ξ_N will look similar to $\xi_{(-\infty,N]}$ in the right boundary region.

- Notice: in the left boundary region, ξ_N will look similar to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive).
- By the same reasoning, ξ_N will look similar to $\xi_{(-\infty,N]}$ in the right boundary region.

Set L such that

$$\mu_{[-N,\infty)}(\{A : A \cap [-N, -N + L] = \varnothing\}) \leq \varepsilon/(16\|f\|)$$

This will make all configurations not intersecting left (and by symmetry - right) boundary region very unlikely.

- Notice: in the left boundary region, ξ_N will look similar to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive).
- By the same reasoning, ξ_N will look similar to $\xi_{(-\infty,N]}$ in the right boundary region.

Set L such that

$$\mu_{[-N,\infty)}(\{A : A \cap [-N, -N + L] = \varnothing\}) \leq \varepsilon/(16\|f\|)$$

This will make all configurations not intersecting left (and by symmetry - right) boundary region very unlikely.

But that is equivalent, by translation invariance, to setting L such that

$$\mu_{[0,\infty)}(\{A : A \cap [0, L] = \varnothing\}) \leq \varepsilon/(16\|f\|)$$

- Notice: in the left boundary region, ξ_N will look similar to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive).
- By the same reasoning, ξ_N will look similar to $\xi_{(-\infty,N]}$ in the right boundary region.

Set L such that

$$\mu_{[-N,\infty)}\left(\{A : A \cap [-N, -N + L] = \varnothing\}\right) \leq \varepsilon/(16\|f\|)$$

This will make all configurations not intersecting left (and by symmetry - right) boundary region very unlikely.

But that is equivalent, by translation invariance, to setting L such that

$$\mu_{[0,\infty)}\left(\{A : A \cap [0, L] = \varnothing\}\right) \leq \varepsilon/(16\|f\|)$$

Defining L in this way makes it independent of N, as we wanted. Surprising, eh?

We've arrived at

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\Gamma_k^{R_N}\right] + \mathbb{P}\left[\Psi_k^{R_N,L}\right]$$

Where

$$\Gamma_k^{R_N} := \left\{\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right\}$$

$$\Psi_k^{R_N,L} := \left\{\left|R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right\}$$

Formally, our previous considerations give us (for appropriate R_N, L)

$$\mathbb{P}\left[B_k^N, K_N = j\right] \leq \mathbb{P}\left[\Gamma_k^{R_N}\right] + \mathbb{P}\left[\Psi_k^{R_N,L}\right] \leq C/R_N$$

Recall we needed to find m_N, R_N such that

- $\mathbb{P}[1 \leq K_N \leq m_N] \to 1$
- $m_N^2 \max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k^N, K_N = j\right] \to 0$

We can bind

- $\max_{1 \leq j} \max_{0 \leq k < j} \mathbb{P}\left[B_k^N, K_N = j\right] \leq C/R_N$

Another lemma shows that $N/\beta_N \to 0$. We can then set

$$m_N = \beta_N^{1/5}/N^{1/5}, \quad R_N = \beta_N^{9/10} N^{1/10}$$

As our solution.

Originally, the paper also considers shifted versions of f, but I thought this would only introduce more confusing indices. If $\tau_i,\ i \in \mathbb{Z}$ is the shift operator we can also prove (using exactly the same techniques) that

$$\mathbb{P}\left[\max_i \max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, \tau_i f) - \mu(f)| > \varepsilon\right] \to 0$$

We must restrict i to be such that $\Lambda(\tau_i f) \subset [-N+L, N-L]$, though.

Originally, the paper also considers shifted versions of f, but I thought this would only introduce more confusing indices. If $\tau_i,\ i \in \mathbb{Z}$ is the shift operator we can also prove (using exactly the same techniques) that

$$\mathbb{P}\left[\max_i \max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, \tau_i f) - \mu(f)| > \varepsilon\right] \to 0$$

We must restrict i to be such that $\Lambda(\tau_i f) \subset [-N+L, N-L]$, though.

As a corollary, same statements also holds for spatial means of observable quantities

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, \bar{f}) - \mu(f)| > \varepsilon\right] \to 0$$

Where $\bar{f} = \frac{1}{\#i} \sum_i \tau_i f$ and $\#i$ is the total number of all is allowed.

Spare slides

Define

$$h_L(\eta) = I_{\{\xi:\xi\cup[-N,-N+L]=\varnothing\}}$$

Notice that h_L can tell us whether a process is wide or not. If we take S to be a horizontal flip operator, then $h_L(\xi_N(t))h_L(S\xi_N(t))$ is the desired indicator.

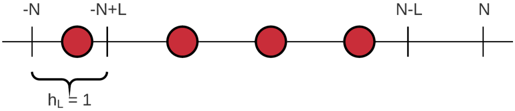

Figure: $h_L(\xi_N(t)) = 1$

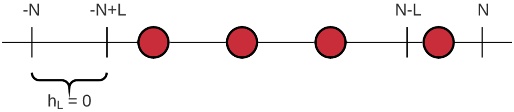

Figure: $h_L(S\xi_N(t)) = 0$

Lemma (Shielding of the left boundary region)

$$\{T_N > t\} \subset \{h_L(\xi_N(t)) = h_L(\xi_{[-N,\infty)}(t))\}$$

In other words, if $\xi_N(t)$ is still alive, it fully determines whether $\xi_{[-N,\infty)}(t)$ intersects the left boundary region.

Why would this be true?

- If $\xi_N(t)$ intersects left boundary region, $\xi_{[-N,\infty)}$ intersects it too
- If $\xi_N(t)$ does not intersect the left boundary region, but has at least one node still alive, this node shields the left boundary region from outside influence. Moreover, no influence can propagate from $-N$. Hence, they need to agree on the left boundary region.

Wait, didn't we talk about a situation where $f(\xi_N(t))$ is actually equal to $f(\xi(t))$ before?

$$\left| R_N^{-1} \int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt \right| > \varepsilon/2$$

Recall that

- If ξ_N is wide at t, we have $f(\xi_N(t)) = f(\xi(t))$ (shielding by a wide process).
- We can use h_L to tell whether a process is wide or not (flipping trick).
- ξ_N is alive for all t we're considering. Hence $h_L(\xi_N(t)) = h_L(\xi_{[-N,\infty)}(t))$ (shielding of the left boundary region).

We get that the condition above implies

$$2\|f\| R_N^{-1} \int_{kR_N}^{(k+1)R_N} h_L(\xi_{[-N,\infty)}(t)) + h_L(S\xi_{(-\infty,N]}(t))dt > \varepsilon/2$$

Indeed, we can find a bound on variance, and thus, by Chebyshev, on exceedance probability. This has to do with the fact that time correlations of $\xi(t)$ decay exponentially fast:

Theorem (Exponentially decaying correlations)

For any $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ local, there are constants C, γ such that

$$|\mathbb{C}ov\,[f(\xi(t)), f(\xi(s))]| \leq Ce^{-\gamma|s-t|}$$

Indeed, we can find a bound on variance, and thus, by Chebyshev, on exceedance probability. This has to do with the fact that time correlations of $\xi(t)$ decay exponentially fast:

Theorem (Exponentially decaying correlations)

For any $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ local, there are constants C, γ such that

$$|\mathbb{C}ov\left[f(\xi(t)), f(\xi(s))\right]| \leq Ce^{-\gamma|s-t|}$$

Now, consider the variance of the expression from the previous slide:

$$\mathbb{E}\left[\left(R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t)) - \mathbb{E}\left[f(\xi(t))\right] dt\right)^2\right]$$

Indeed, we can find a bound on variance, and thus, by Chebyshev, on exceedance probability. This has to do with the fact that time correlations of $\xi(t)$ decay exponentially fast:

Theorem (Exponentially decaying correlations)

For any $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ local, there are constants C, γ such that

$$|\mathbb{C}ov\left[f(\xi(t)), f(\xi(s))\right]| \leq Ce^{-\gamma|s-t|}$$

Now, consider the variance of the expression from the previous slide:

$$\mathbb{E}\left[\left(R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t)) - \mathbb{E}\left[f(\xi(t))\right] dt\right)^2\right]$$

- Looks very much like we could get an expression for a covariance from this!

Indeed, we can find a bound on variance, and thus, by Chebyshev, on exceedance probability. This has to do with the fact that time correlations of $\xi(t)$ decay exponentially fast:

Theorem (Exponentially decaying correlations)

For any $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ local, there are constants C, γ such that

$$|\mathbb{C}ov\left[f(\xi(t)), f(\xi(s))\right]| \leq Ce^{-\gamma|s-t|}$$

Now, consider the variance of the expression from the previous slide:

$$\mathbb{E}\left[\left(R_N^{-1}\int_{kR_N}^{(k+1)R_N} f(\xi(t)) - \mathbb{E}\left[f(\xi(t))\right] dt\right)^2\right]$$

- Looks very much like we could get an expression for a covariance from this!
- Then, Chebyshev will finish out our job for us.