Metastability for the Contact Process on $\mathbb{Z}$

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Metastability is the coexistence of an equilibrium and a "quasi-equilibrium" (metastable) state in a system. If we look at a metastable system in time domain, it initially appears to have stabilized in the "quasi-equilibrium" state, until it suddenly relaxes to the true equilibrium. Metastability is a common phenomenon in nature, with examples ranging from physics to economics and social sciences. This work is a summary of a paper by R. Schonmann [Sch85], which frames this phenomenon as a rigorous property of an abstract interacting particle system - the Contact Process.

1 Overview - what's metastability?

Take a family of Markov processes $\{\xi_N(t)\}_{N\in\mathbb{N}}$. The essential idea behind metastability is that, if we were to discretize ("coarse-grain") time, increasing N would make the process under consideration look more and more like the Markov process in figure 1.
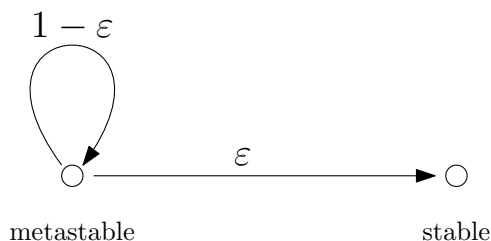

Figure 1: Metastability in a Nutshell

In other words, the system begins in a so-called "metastable state", and has a very small chance to move to the stable state at any time. The stable state is either absorbing, or close-to absorbing; in the case that we will talk about in this paper, the stable state is absorbing.

In order to make this "coarse-graining" happen, we need two things to be true asymptotically.

1. The hitting time of the stable state is exponentially distributed.

2. Up until the hitting time, temporal means of measurements made to the process converge to an expectation of those measurements with respect to some stationary distribution.

Together, these two properties intuitively allow us to approximate the entire process by sampling from the stationary distribution up until the hitting time, and then putting the system in the absorbing state.

The difficulty comes in stating these two properties precisely. To do this, suppose that T_N is the hitting time of the absorbing state, μ is the stationary distribution, and R_N is a "time scale" parameter that satisfies $R_N / \mathbb{E}\, T_N \to 0$. Then we rewrite the two properties more formally as

1. $T_N / \mathbb{E}\, T_N \to \mathrm{Exp}(1)$ in distribution as $N \to \infty$.

2. For any f cylindrical,
$$\int_S^{S+R_N} f(\xi_N(t))dt \to \mu(f)$$
as $N \to \infty$, for any $S + R_N < T_N$.

This second statement is still very imprecise, and actually mathematically meaningless as currently posed. Also, it turns out that we want a much stronger statement than that. However, we hope that this first statement should "innoculate" the reader to the precise statement, which is fairly dense on its own.

2 Review of Contact Process

2.1 Basic Definitions

We could succinctly define the Contact Process by using its Markov generator. However, for our purposes we will use an alternative definition that lends itself better to certain constructions that are useful in proofs concerning metastability.

First we construct a "percolation structure", which consists of for each $x \in \mathbb{Z}$

1. A Poisson process P_x with rate 1, which we call the "death" process at x.

2. A Poisson process $P_{x \to x+1}$ with rate λ, which we call the "right infection" process at x.

3. A Poisson process $P_{x \to x-1}$ with rate λ, which we call the "left infection" process at x.

We consider P_x to be a random element of $\mathcal{P}(\mathbb{R})$, i.e. $t \in P_x$ if and only if the Poisson process "ticks" at time t.

We define a "path" between $(x, s), (y, t) \in \mathbb{Z} \times \mathbb{R}$ with $s \leq t$ to be a sequence $(z_0, r_0), \ldots, (z_n, r_n)$ with $r_i \leq r_{i+1}$ such that for all $(z_i, r_i), (z_{i+1}, r_{i+1})$, either

1. $r_i = r_{i+1}$, $|z_i - z_{i+1}| = 1$, and $r_i \in P_{z_i \to z_{i+1}}$. In this case, we are jumping laterally by one line at a time of infection.

2. $z_i = z_{i+1}$, and $[r_i, r_{i+1}] \cap P_{z_i} = \emptyset$. In this case, we are moving along a vertical line, uninterrupted by any deaths.

Define $\xi^A(t)$ to be the set of y such that there is a path from $(x, 0)$ to (y, t) for some $x \in A$. If the superscript is omitted, then we assume $A = \mathbb{Z}$, i.e. $\xi(t) = \xi^{\mathbb{Z}}(t)$.

For any $A \subseteq B$, we define $\xi_B^A(t)$ to be the set of y such that there is a path from $(x, 0)$ to (y, t) for some $x \in A$ that stays entirely within B. As a special case, we let $\xi_N^A(t) = \xi_{[-N,N]}^A$, for $N \in \mathbb{N}$. Note that ξ_B takes values exclusively in $\mathcal{P}(B)$.

2.2 Useful Properties

One of the most important facts about the contact process is that there is a critical value of λ, λ_c. For $\lambda < \lambda_c$, $\xi(t)$ has a unique extremal invariant measure $\delta_\emptyset$. At $\lambda = \lambda_c$ the system undergoes a phase transition - for $\lambda > \lambda_c$, another extremal invariant measure μ appears, with the property that

$$\mu(f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} f(\xi(t)) \, dt$$

This measure is not concentrated at $\emptyset$.

For $\xi_N(t)$, the only invariant measure is $\delta_\emptyset$, because $\emptyset$ is a trap and ξ_N takes values on a finite state space. However, as mentioned before, for $\lambda > \lambda_c$ an analogue of a phase transition takes place. The system starts being metastable, and stays distributed *approximately* as μ, before a sudden fluctuation takes it to $\emptyset$.

3 Hitting Time of Stable State

Theorem 1 (from [Sch85]). *If $T_N = \inf\{t > 0 \mid \xi_N(t) \neq \emptyset\}$, then*

$$\frac{T_N}{\mathbb{E} \, T_N} \to \mathrm{Exp}(1)$$

in distribution.

Without further ado, we get down to the proof. First of all, we replace $\mathbb{E} \, T_N$ by the unique (by monotonicity) β_N such that $\mathbb{P}(T_N > \beta_N) = \mathrm{e}^{-1}$. At the end, we will show that $\frac{\mathbb{E} \, T_N}{\beta_N} \to 1$.

Now let $G_N(t) = \mathbb{P}[\frac{T_N}{\beta_N} > t]$, the CDF of T_N/β_N. Similarly, let $G_N^A(t) = \mathbb{P}[\frac{T_N^A}{\beta_N} > t]$. Note that $G_N^A(t) \leq G_N(t)$ To show that T_N/β_N converges in distribution to $\mathrm{Exp}(1)$, we must show that $G_N(t) \to \mathrm{e}^{-t}$. This can be accomplished by showing that

$$\lim_{N \to \infty} |G_N(t+s) - G_N(t)G_N(s)| = 0$$

for all $t, s > 0$.

It is at this point that we introduce a curious little construction, which seems to not make much sense at first but turns out to be the key to the entire proof. Define F_b for $b > 0$ by

$$F_b = \left\{ A \subseteq \mathbb{Z} \mid \frac{|A \cap [-b, -1]|}{b} > \frac{\rho}{2}, \frac{|A \cap [1, b]|}{b} > \frac{\rho}{2} \right\}$$

For $A \in F_b$ we have

$$G_N(t + s) \geq \min_{A \in F_b} G_N^A(t) \, \mathbb{P}[\xi_N(\beta_N s) \in F_b]$$

(We assume that $A \subseteq [-N, N]$ when such an assumption is necessary).

After some simple algebra we get the following bound:

$$G_N(t) G_N(s) - G_N(t + s) \leq G_N(t) - \min_{A \in F_b} \mathbb{P}[\xi_N^A(\beta_N t) \neq \emptyset] + \mathbb{P}[\xi_N(\beta_N s) \neq \emptyset, \xi_N(\beta_N s) \notin F_b]$$

We will have finished if for any $\varepsilon > 0$, we can find $b(\varepsilon)$ and $N(\varepsilon) > b(\varepsilon)$ such that for $N \geq N(\varepsilon)$ and $A \in F_b$, we have both

$$G_N(t) - G_N^A(t) < \varepsilon \tag{1}$$

$$\mathbb{P}[\xi_N(\beta_N s) \neq \emptyset, \xi_N(\beta_N s) \notin F_b] < \varepsilon \tag{2}$$

We tackle (1) first. Remember that $\xi_N(t)$ and ξ_N^A are defined on the same percolation structure. Therefore, $\xi_N(t) \supset \xi_N^A(t)$, so we have

$$\mathbb{P}[\xi_N(\beta_N t) \neq \emptyset] - \mathbb{P}[\xi_N(\beta_N t) \neq \emptyset] = \mathbb{P}[\xi_N(\beta_N t) \neq \emptyset, \xi_N^A(\beta_N t) = \emptyset] \leq \mathbb{P}[T_N \neq T_N^A]$$

Set $E = \{T_N^A = T_N\}$. As $\lambda > \lambda_c$, we can find $n(\varepsilon)$ such that

$$\mu(B \mid B \cap [1, n(\varepsilon)] = \emptyset) \leq \frac{\varepsilon}{2}$$

Then take $b = b(\varepsilon)$ such that $n(\varepsilon) \leq b \cdot \rho/2$. For $A \in F_b$, $|A \cap [-b, -1]| \geq b \cdot \rho/2 \geq n(\varepsilon)$, so

$$\mathbb{P}[T_{[-N,\infty)}^{A \cap [-b,-1]} = \infty] \geq \mathbb{P}[T_{[-N,\infty]}^{[-N,\ldots,-N+n(\varepsilon)]} = \infty] \geq 1 - \frac{\varepsilon}{2}$$

First inequality follows from the fact that $A \cap [-b, -1]$ is at least as spread out as $[-N, \ldots, -N + n(\varepsilon)]$. A process which is more spread out has more chances of infecting healthy individuals, and so survives longer. The second inequality follows from duality and monotone convergence.

Similarly, $\mathbb{P}[T_{(-\infty,N]}^{A \cap [1,b]} = \infty] \geq 1 - \frac{\varepsilon}{2}$. Let E' be the event that both of these stopping times are infinite; clearly $P(E') \geq 1 - \varepsilon$. It remains to show that $E \supset E'$.

Define stopping times

$$U = \inf\{t > 0 \mid N \in \xi_{[-N,\infty)}^{A \cap [-b,-1]}(t)\}$$

$$V = \inf\{t > 0 \mid -N \in \xi_{(-\infty,N]}^{A \cap [1,b]}(t)\}$$

Until U, $\xi_N^{A\cap[-b,-1]}(t) = \xi_{[-N,\infty)}^{A\cap[-b,-1]}(t)$. Therefore, on E', $\xi_N^{A\cap[-b,-1]}(t)$ is alive up until U, whence ξ_N^A is alive up until U. Similarly, ξ_N^A is alive up until V. Therefore, on E', $T_N \geq T_N^A > \max(U, V)$.

However, for $t > \max(U, V)$, I claim that $\xi_N(t) = \xi_N^A(t)$. To see this, any path to $x \in \xi_N(t)$ must either cross the path that goes from $A \cap [-b, -1]$ at time 0 to N at time U or the path that goes from $A \cap [1, b]$ at time 0 to $-N$ at time V. Therefore, there is a path from A at time 0 to x at time t, whence $x \in \xi_N^A(t)$. Therefore, on E', $T_N = T_N^A$, and we have shown (1).

Equation 2 also relies crucially on the percolation structure. Let $D_b = F_b^C \setminus \{\emptyset\}$. Then we must show that $\mathbb{P}(\xi_N(\beta_N s) \in D_b) < \varepsilon$. To do this, note that as long as $\xi_N(t)$ is alive, we have
$$\xi_N(t) = \xi(t) \cap [\min \xi_N(t), \max \xi_N(t)]$$
This is because any path to $x \in \xi(t) \cap [\min \xi_N(t), \max_N(t)]$ must either stay inside $[\min \xi_N(t), \max_N(t)]$, in which case $x \in \xi_N(t)$, or pass through the path to $\min \xi_N(t)$ or $\max \xi_N(t)$, in which case we also have $x \in \xi_N(t)$.

Therefore, as long as $\min \xi_N(\beta_N s) < -N + L$, $\max \xi_N(\beta_N s) > N - L$ and $\xi_N(\beta_N s) \neq \emptyset$, we have $\xi(\beta_N s) \in D_b$ iff $\xi_N(\beta_N s) \in D_b$.

We can minimize $\xi(\beta_N s) \in D_b$ by picking $b > b''(\varepsilon)$ sufficiently large. In order to use this, we use the following decomposition

$$\begin{aligned}
\mathbb{P}[\xi_N(\beta_N s) \in D_b] \leq{} & \mathbb{P}[\xi_N(\beta_N s) \in D_b, \min \xi_N(\beta_N s) < -N + L, \max \xi_N(\beta_N s) > N - L] \\
& + \mathbb{P}[\min \xi_N(\beta_N s) \geq -N + L, \xi_N(\beta_N s) \neq \emptyset] \\
& + \mathbb{P}[\max \xi_N(\beta_N s) \leq N - L, \xi_N(\beta_N s) \neq \emptyset]
\end{aligned}$$

By what we noted earlier, the first term is less than $\mathbb{P}(\xi(\beta_N s) \in D_b)$; pick b such that this is less than $\frac{\varepsilon}{3}$.

It remains to minimize the last two terms; by symmetry we only show how to minimize the first. Using a percolation structure argument, it is easy to show that as long as $\xi_N(\beta_N s) \neq \emptyset$, $\min \xi_N(t) = \min \xi_{[-N,\infty)}$. Therefore,

$$\begin{aligned}
\mathbb{P}[\min \xi_N(\beta_N s) \geq -N + L, \xi_N(\beta_N) \neq \emptyset] \geq{} & \mathbb{P}[\min \xi_{[-N,\infty)} \geq -N + L] \\
& \geq \mu_{[-N,\infty)}\{A \in [-N, \infty) \mid A \cap [-N, -N + L - 1] = \emptyset\}
\end{aligned}$$

Finally can pick L large enough to make that last term less than $\frac{\varepsilon}{3}$, and we have shown that everything can be made as small as we like, so we are done.

4 Quasi-stationarity up to the Hitting Time

Recall our initial description of Quasi-stationarity in Section 1. To make it more precise, we will define

- **intermediate timescale** (for a given N) as $R_N \in \mathbb{R}_+$ with $R_N / \mathbb{E}[T_N] \to 0$

- **observable quantity** as a cylindrical (local) $f : \{0, 1\}^{\mathbb{Z}} \to \mathbb{R}$

- **temporal mean** of observable quantity f as

$$A_{R_N}^N(s, f) := R_N^{-1} \int_s^{s+R_N} f(\xi_N(t))dt$$

- **fixed probability distribution** as μ (upper invariant measure of the Contact Process on $\mathbb{Z}$)

Note that $A_{R_N}^N$ is a random variable. We will say that this quantity **close** to $\mu(f)$ if the two converge in probabilty when $N \to \infty$.

It seems like the definitions given above would be sufficient to make our natural language definition of metastability formal. However, it turns out that an additional technical condition is needed. Define

$$\Lambda(f) := \text{ smallest } B \subset \mathbb{Z} \text{ s.t. } f(A) = f(A \cap B) \ \forall A \subset \mathbb{Z}$$

although this is not strictly true, we can think of $\Lambda(f)$ as the "support" of f.

For our theorem to hold, we need to choose $L(\varepsilon, f) \in \mathbb{N}$, and we need to have that

$$L < N \text{ and } \Lambda(f) \subset [-N + L, N - L]$$

Notice that L does not depend on N. Thus, since we chose to grow $N \to \infty$, having to choose this L doesn't restrict our choice of f - it merely sets the minimum N we can consider.

We're now ready to give a simplified formulation of what we mean by "quasi-stationarity"

Theorem 2 (from [Sch85]). *If $\lambda > \lambda^*$ there is a sequence $\{R_N\}_{N \in \mathbb{N}} \subset \mathbb{R}_+$ such that:*

- *$R_N/\beta_N \to 0$ as $N \to \infty$*

- *For all $\varepsilon > 0$ and $f : \{0, 1\}^{\mathbb{Z}} \to \mathbb{R}$ cylindrical, $\exists L(\varepsilon, f) \in \mathbb{N}$ such that*

$$\mathbb{P}\left[\max_{\mathbb{N}_0 \ni k < K_N} |A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right] \to 0$$

as $N \to \infty$, where $K_N = \max\{k \in \mathbb{N}_0 : kR_N < T_N\}$ and $\Lambda(f) \subset [-N+L, N-L] \cap \mathbb{Z}$

Proof of this statement is beyond the scope of this article - we will merely give the reader some insight into its key points.

Central to all this business seems to be an issue of being able to make the exceedance probability (given below) arbitrarily small.

$$\mathbb{P}\left[|A_{R_N}(kR_N, f) - \mu(f)| > \varepsilon\right]$$

We ought to do so in a way which is uniform across all values of K_N and $\mathbb{N}_0 \ni k < K_N$. We estimate this probabilty with a triangle inequality.

$$\mathbb{P}\left[\left|R_N^{-1} \int_{kR_N}^{(k+1)R_N} f(\xi(t))dt - \mu(f)\right| > \varepsilon/2\right] + \mathbb{P}\left[\left|R_N^{-1} \int_{kR_N}^{(k+1)R_N} f(\xi_N(t)) - f(\xi(t))dt\right| > \varepsilon/2\right] \quad (3)$$

The first term can be made arbitrarily small, as a consequence of $\xi(t) \rightharpoonup \mu(f)$ and exponentially decaying temporal correlations of $\xi(t)$.

To shrink the second term in (3) we make extensive use of the graphical model construction of the Contact Process. We will say $\xi_N(t)$ is **wide** at t if $\min \xi_N(t) < -N + L \wedge \max \xi_N(t) > N - L$. By the fact that we construct all processes on the same percolation structure, and that interactions are between nearest neighbours, we have

Lemma 3 (Shielding by a wide process). *If $\xi_N(t)$ is wide at t, then*

$$\xi_N(t) = \xi(t) \ on \ [-N + L, N - L] \cap \mathbb{Z}$$

In particular, since $\Lambda(f) \subset [-N + L, N - L]$, we have $f(\xi_N(t)) = f(\xi(t))$

Thus, the second term of (3) will go to 0 if we're able to make it arbitrarily likely for ξ_N to be wide. We can do so by setting L, but is it possible to do so such that its value is independent of N?

Here we again make use of the percolation structure. Note that near $-N$, ξ_N will "look similar" to $\xi_{[-N,+\infty)}$ (this is true as long as ξ_N is alive). By the same reasoning, ξ_N will "look similar" to $\xi_{(-\infty,N]}$ near N. Hence we set L such that

$$\mu_{[-N,\infty)} \left(\{A : A \cap [-N, -N + L] = \varnothing\} \right) \leq \varepsilon/(16\|f\|)$$

This makes all possible states of ξ_N that do not intersect $[-N, -N+L]$ very unlikely (by "looking similar" to $\xi_{[-N,\infty)}$). By symmetry, this also makes states that do not intersect $[N - L, N]$ very unlikely. As a final stroke, we use translation invariance to notice that we will get exactly the same effect if we set L

$$\mu_{[0,\infty)} \left(\{A : A \cap [0, L] = \varnothing\} \right) \leq \varepsilon/(16\|f\|)$$

This concludes our overview of the proof of Theorem 2.

References

[Sch85]   Roberto H. Schonmann. "Metastability for the contact process." In: *Journal of Statistical Physics* Vol. 41, Issue 3 (1985).