Pirogov-Sinai Theory

by Owen Lynch

1 Idea

We want to understand the behavior of lattice systems at very low temperatures. The basic idea is a typical state consists of

- regions of periodic stability

- separated by boundaries

In the case of the Ising model, a region of periodic stability is a region of all up spins/all down spins. More specifically, the stable regions consists of all lattice elements who agree with all of their neighbors. Then boundaries between these regions consist of the lattice elements who disagree with at least one neighbor.

At low temperatures, the regions of periodic stability are large compared to the boundaries, so we can perform a cluster expansion with polymers the boundary elements to understand the system.

2 Definitions

For the remainder of this presentation, we assume that we work with a system on a lattice $\mathbb{Z}^d$, with finite single-spin space Ω_0, and $\Omega_\Lambda := \Omega_0^\Lambda$, $\Omega := \Omega_0^{\mathbb{Z}^d}$. The Hamiltonian

$$\mathcal{H}_\Lambda(\omega) = \sum_{B \cap \Lambda \neq \emptyset} \Phi_B(\omega)$$

on this system is given by a set of potentials $\{\Phi_B\}_{B \subset \mathbb{Z}^d}$, which are of **finite range** ($|B|$ is bounded) and are **invariant under translation** ($\Phi_B \circ \theta_j = \Phi_B$, for θ_j a translation). Thus, for finite Λ, the sum for $\mathcal{H}_\Lambda(\omega)$ is always finite.

We want to characterize minima of the Hamiltonian on the infinite lattice. However, the Hamiltonian is not defined on the infinite lattice. To solve this problem, we introduce the **relative Hamiltonian**. Say that a pair of states $\tilde{\omega}, \omega \in \Omega$ are "equal at infinity", written $\tilde{\omega} \stackrel{\infty}{=\!=} \omega$, if $\{i \in \mathbb{Z}^d | \tilde{\omega}_i \neq \omega_i\}$ is finite. Then for such a pair, define

$$\mathcal{H}_\Phi(\tilde{\omega}|\omega) := \sum_{B \in \mathbb{Z}} \{\Phi_B(\tilde{\omega}) - \Phi_B(\omega)\}$$

Note that this sum is finite. A **ground state** η is then a local minimum in the sense that $\mathcal{H}_\Phi(\omega|\eta) \geq 0$ for all $w \stackrel{\infty}{=\!=} \eta$.

We are mostly interested in *periodic* ground states, which are ground states that are tilings of a finite n-rectangle.

Example 1. The Ising model with zero magnetization has ground states η^+ and η^-, which are constantly $+1$ and -1, respectively.

Example 2. The **Blume-Capel** model has $\Omega_0 = \{-1, 0, 1\}$ and formal Hamiltonian

$$\sum_{\{i,j\} \in \mathcal{E}_{\mathbb{Z}^d}} (\omega_i - \omega_j)^2 - h \sum_{i \in \mathbb{Z}^d} \omega_i - \lambda \sum_{i \in \mathbb{Z}^d} \omega_i^2$$

This has three possible ground states, depending on the values for h and λ.