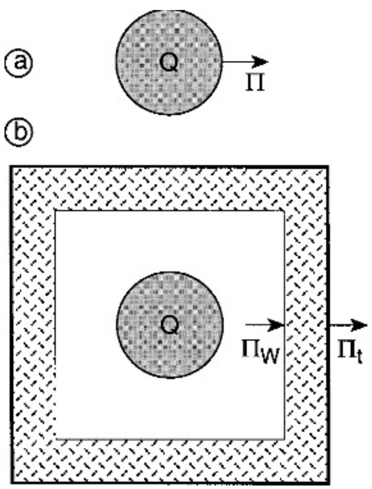
L Capsules and Cabins

Usually capsules and cabins are combined in one chapter, like here, although their tasks and the analytical methods are quite different. The task of a capsule is to reduce the sound pressure level in the environment from an inside noise source; the task of cabins is to produce a quiet space in a noisy environment. Suppose we have a noise source with constant sound power output, whatever the sound field around the source may be, and suppose we have a capsule surrounding the source with some transmission loss of its walls, but with no sound absorption, either inside or in the walls. The sound pressure level inside the capsule will rise until all the sound power produced is radiated by the capsule again. Thus sound absorption in a capsule plays an important role.

L.1 The Energetic Approximation for the Efficiency of Capsules

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

Consider two arrangements of a source Q: (a) The source is placed in free space and radiates the (effective) sound power Π. (b) The source is surrounded by a capsule; the question is, what sound power Π_t will be radiated?



The capsule efficiency is measured by its insertion loss

$$D_e = -10 \cdot \lg(\tau_e) = -10 \cdot \lg(\Pi_t/\Pi)\ [\mathrm{dB}]\,. \tag{1}$$

A widely used proposal by Goesele evaluates

$$D_e = R - 10 \cdot \lg(1/\alpha)\,, \tag{2}$$

where $R = -10 \cdot \lg(\Pi_t/\Pi_w)$ is the sound transmission loss of the capsule wall and α is the sound absorption coefficient of the interior side of the capsule wall (measured with a hard backing of the wall). This proposal produces $D_e \to -\infty$ for $\alpha \to 0$.

By identical transformations:

$$\tau_e = \frac{\Pi_t}{\Pi} = \frac{\Pi_w}{\Pi} \cdot \frac{\Pi_t}{\Pi_w} = c_w \cdot \tau_{VS}\,; \quad c_w = \frac{\Pi_w}{\Pi}\,; \quad \tau_{VS} = \frac{\Pi_t}{\Pi_w}\,, \tag{3}$$

where τ_{VS} is the sound transmission coefficient of the capsule wall (possibly a combination of an interior absorber layer and an outer tight wall). If r_{VS} is the symbol for the interior reflection factor of the capsule wall (including its radiation to the outside), the sound power Π_V loss inside the capsule is $\Pi_V = (1 - |r_{VS}|^2) \cdot \Pi_W$. The "energetic approximation", which was proposed by Hennig, assumes that at the equilibrium $\Pi = \Pi_V$; thus:

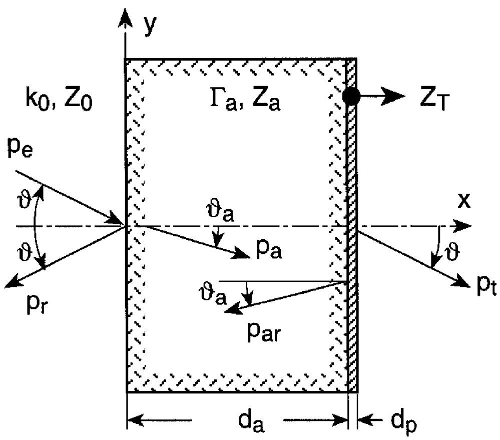
$$\tau_e = \frac{\tau_{VS}}{1 - |r_{VS}|^2} = \frac{\tau_{VS}}{\alpha_{VS}}\,; \quad \alpha_{VS} = 1 - |r_{VS}|^2\,. \tag{4}$$

The difference from Goesele's proposal is the fact that the absorption coefficient α_{VS} now also contains the radiated power. If the sound transmission factor t_{VS} through the wall of the capsule is used, with $\tau_{VS} = |t_{VS}|^2$, the insertion power ratio becomes:

$$\tau_e = \frac{|t_{VS}|^2}{1 - |r_{VS}|^2}\,. \tag{5}$$

Simple example: *porous interior layer and outer metal sheet*

It is assumed that the interior sound field can be described by plane waves p_e incident on the capsule wall at a polar angle ϑ. The capsule wall consists of an interior porous layer of thickness d_a and with characteristic values Γ_a, Z_a of its material plus an outer metal sheet with partition impedance Z_T.



In total, it is assumed that the capsule is large and has plane walls. Field formulations:

$$\begin{aligned}
p_e(x,y) &= P_e \cdot e^{jk_0 y \sin\vartheta} \cdot e^{-jk_0 x \cos\vartheta}, \\
p_r(x,y) &= P_r \cdot e^{jk_0 y \sin\vartheta} \cdot e^{+jk_0 x \cos\vartheta}, \\
p_t(x,y) &= P_t \cdot e^{jk_0 d_a \cos\vartheta} \cdot e^{jk_0 y \sin\vartheta} \cdot e^{-jk_0 x \cos\vartheta},
\end{aligned} \tag{6}$$

$$\begin{aligned}
p_a(x,y) &= P_a \cdot e^{\Gamma_a\, y \sin\vartheta_a} \cdot e^{-\Gamma_a\, x\cos\vartheta_a}, \\
p_{ar}(x,y) &= P_{ar} \cdot e^{-\Gamma_a\, d_a\, \cos\vartheta_a} \cdot e^{\Gamma_a\, y\sin\vartheta_a} \cdot e^{+\Gamma_a\, x\cos\vartheta_a}
\end{aligned} \tag{7}$$

with interior angle in the porous layer: $\sin\vartheta_a = \dfrac{jk_0}{\Gamma_a} \cdot \sin\vartheta$ (8)

and partition impedance Z_T:

$$\frac{Z_T}{Z_0} = 2\pi Z_m F[\eta F^2 \cdot \sin^4\vartheta_a + j(1 - F^2 \cdot \sin^4\vartheta_a)]\,; \quad Z_m = \frac{f_{cr} d_p}{Z_0}\rho_p; \quad F = \frac{f}{f_{cr}}, \tag{9}$$

where η is the bending loss factor of the sheet, ρ_p the density of its material and f_{cr} the critical frequency.

The boundary conditions give the following system of equations:

$$\begin{pmatrix} 1 & -1 & -e^{-a} & 0 \\ 1 & b & -be^{-a} & 0 \\ 0 & e^{-a} & 1 & -(1 + Z_{Tn}\cos\vartheta) \\ 0 & be^{-a} & -b & -1 \end{pmatrix} \cdot \begin{pmatrix} P_r \\ P_a \\ P_{ar} \\ P_t \end{pmatrix} = \begin{pmatrix} -P_e \\ P_e \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

with the abbreviations

$$a = \Gamma_a\, d_a \cdot \cos\vartheta_a = k_0 d_a \sqrt{\Gamma_{an}^2 + \sin^2\vartheta}\,; \quad b = \frac{Z_0 \cos\vartheta_a}{Z_a\, \cos\vartheta} = \frac{1}{\Gamma_{an} Z_{an}} \frac{\sqrt{\Gamma_{an}^2 + \sin^2\vartheta}}{\cos\vartheta} \tag{11}$$

($\Gamma_{an} = \Gamma_a/k_0$; $Z_{an} = Z_a/Z_0$). It can be solved with Kramer's rule:

$$\begin{aligned}
\det &= -(1+b)^2 + (1-b)^2 \cdot e^{-2a} - b\left(1 + b + (1-b)e^{-2a}\right) Z_T/Z_0\,, \\
\det 1 &= -(1-b^2)(1 - e^{-2a}) + b\left(1 - b + (1+b)e^{-2a}\right) Z_T/Z_0\,, \\
\det 2 &= -2\,(1 + b(1 + Z_T/Z_0))\,, \\
\det 3 &= 2\,(1 - b(1 + Z_T/Z_0))\, e^{-a}\,, \\
\det 4 &= -4b \cdot e^{-a}\,,
\end{aligned} \tag{12}$$

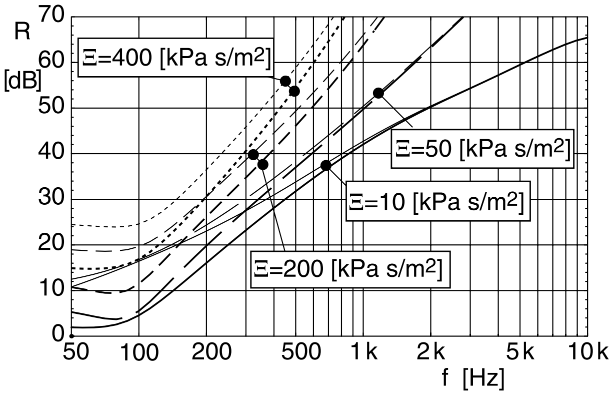
and the desired amplitude ratios are:

$$\frac{P_r}{P_e} = r_{VS} = \frac{\det 1}{\det}; \quad \frac{P_t}{P_e} = t_{VS} = \frac{\det 4}{\det}; \quad \frac{P_a}{P_e} = \frac{\det 2}{\det}; \quad \frac{P_{ar}}{P_e} = \frac{\det 3}{\det}. \tag{13}$$

The insertion coefficient for a single incident plane wave (with angle ϑ) becomes:

$$\tau_e(\vartheta) = \frac{\tau_{VS}}{1-|r_{VS}|^2} = \frac{|t_{VS}|^2}{1-|r_{VS}|^2} = \frac{|\det 4|^2}{|\det|^2 - |\det 1|^2}$$
$$= |4b\cdot e^{-a}|^2 \cdot \Big[\, \big|(1+b)^2-(1-b)^2\cdot e^{-2a} + b\cdot\big(1+b+(1-b)\cdot e^{-2a}\big)\cdot Z_T/Z_0\big|^2 \tag{14}$$
$$-\; \big|(1-b^2)(1-e^{-2a}) + b\cdot\big(1-b+(1+b)\cdot e^{-2a}\big)\cdot Z_T/Z\big|^2\Big]^{-1}.$$

The example shows the sound transmission loss $R = -10\cdot \lg(\tau_{VS})$ of a capsule wall (thin curves) and the insertion loss of a capsule (thick curves) with these walls, for three flow resistivity values Ξ of the absorber layer (sound incidence under $\vartheta = 45°$).



Sound transmission loss R of capsule walls and insertion loss D_e of a capsule for three flow resistivity values Ξ of the porous layer. Parameters: $\vartheta = 45°$; $d_a = 0.05$ [m]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$

Special cases:

$$\tau_e(\vartheta) \xrightarrow[\text{no abs.}]{} 4\Big/\big[|2+Z_T/Z_0|^2 - |Z_T/Z_0|^2\big] \xrightarrow[\eta\to 0]{} 1\,,$$
$$\tau_e(\vartheta) \xrightarrow[|Z_T|\to\infty]{} \frac{|t_{vs}|^2}{1-|r_A|^2} = \frac{|\det 4|^2}{|\det|^2\big[1-|r_A|^2\big]}\,,$$
$$r_{VS} \xrightarrow[|Z_T|\to\infty]{} r_A = \frac{(1+b)e^{-2a}+1-b}{(1-b)e^{-2a}+1+b}\,, \tag{15}$$
$$\tau_e(\vartheta) \xrightarrow[|a|\gg 1]{} \frac{|\det 4|^2}{|\det|^2_{|a|\gg 1} - |\det 1|^2_{|a|\gg 1}} = \big[|4b\cdot e^{-a}|^2\big]$$
$$\Big/\big[4\mathrm{Re}\{b\}\cdot|1+b\cdot(1+Z_T/Z_0)|^2\big]\,.$$

For three-dimensional diffuse sound incidence:

$$\tau_{3-dif} = \frac{2}{\sin^2\vartheta_{hi}}\int_0^{\vartheta_{hi}} \tau_e(\vartheta)\cos\vartheta\sin\vartheta\, d\vartheta\,. \tag{16}$$

For two-dimensional diffuse sound incidence:

$$\tau_{2-\mathrm{dif}} = \frac{1}{\sin\vartheta_{hi}} \int_0^{\vartheta_{hi}} \tau_e(\vartheta)\cos\vartheta\, d\vartheta\,. \tag{17}$$

L.2 Absorbent Sound Source in a Capsule

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

Sound absorption inside a capsule may be produced not only by an absorber layer on the capsule walls, but also by the source itself. This effect will be illustrated with a model in which the capsule and the source are two-dimensional; the source offers at its surface an impedance Z_i to incident waves. Let Z_F be the field impedance at the source surface; then the sound pressure and particle velocity at its surface are given by:

$$p(x_{Qu}) = \frac{1}{1+Z_i/Z_F}\cdot p_{Qu}\,;\quad v(x_{Qu}) = \frac{1}{1+Z_F/Z_i}\cdot v_{Qu}\,, \tag{1}$$

where p_{Qu}, v_{Qu} are generally used to characterise a source and belong to the special cases:

$$p(x_{Qu}) \xrightarrow[Z_F\to\infty]{} p_{Qu}\,;\quad v(x_{Qu}) \xrightarrow[Z_F\to 0]{} v_{Qu}\,. \tag{2}$$

The relation $p_{Qu} = Z_i \cdot v_{Qu}$ (Helmholtz's source theorem) exists and

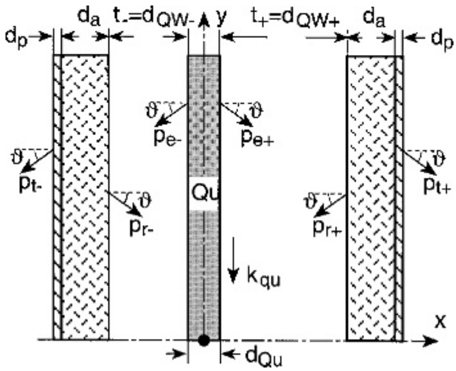
$$\frac{p(x_{Qu})}{v(x_{Qu})} = \frac{1+Z_F/Z_i}{1+Z_i/Z_F}\cdot\frac{p_{Qu}}{v_{Qu}} = \frac{Z_F}{Z_i}\cdot\frac{p_{Qu}}{v_{Qu}}\,. \tag{3}$$

Because of the finite interior impedance Z_i of the source, the condition of the energetic approximation of the previous *Sect. L.1*, that the source power Π is constant for whatever exterior sound field, no longer holds.

The model consists of a plane sound source Qu which radiates a plane wave towards both sides at an angle ϑ, which is given by the wave number k_{qu} along the source surface with $\sin\vartheta = k_{qu}/k_0$. The walls of the capsule are equal on both sides (for simplicity) but possibly have different distances $t_\pm$ to the source. They consist of a porous layer and a metal sheet or plate. The source will not be transmissive for incident sound (such as big machines). Thus both sides of the source are independent of each other; sound fields will be written only for one side; the fields on the other side follow by simple substitutions. Because of this independence, the source thickness can be taken to be $d_{Qu} = 0$ (or the co-ordinate x is shifted correspondingly).

Field formulations:

$$\begin{aligned}
p_{e+}(x,y) &= P_{e+}\cdot e^{jk_0 y\sin\vartheta}\cdot e^{-jk_0(x-t_+)\cos\vartheta}\,,\\
p_{r+}(x,y) &= P_{r+}\cdot e^{jk_0 y\sin\vartheta}\cdot e^{+jk_0(x-t_+)\cos\vartheta}\,,\\
p_{t+}(x,y) &= P_{t+}\cdot e^{jk_0 d_a\cos\vartheta}\cdot e^{jk_0 y\sin\vartheta}\cdot e^{-jk_0(x-t_+)\cos\vartheta}\,,\\
p_{a+}(x,y) &= P_{a+}\cdot e^{\Gamma_a y\sin\vartheta_a}\cdot e^{-\Gamma_a (x-t_+)\cos\vartheta_a}\,,\\
p_{ar+}(x,y) &= P_{ar+}\cdot e^{-\Gamma_a d_a\cos\vartheta_a}\cdot e^{\Gamma_a y\sin\vartheta_a}\cdot e^{+\Gamma_a (x-t_+)\cos\vartheta_a}
\end{aligned} \tag{4}$$



with interior angle in the porous layer $\sin\vartheta_a = \frac{jk_0}{\Gamma_a}\cdot\sin\vartheta$ (5)

and partition impedance Z_T of the metal sheet:

$$\frac{Z_T}{Z_0} = 2\pi Z_m F[\eta F^2\cdot\sin^4\vartheta_a + j(1-F^2\cdot\sin^4\vartheta_a)]\,; \quad Z_m = \frac{f_{cr}d_p}{Z_0}\rho_p; \quad F = \frac{f}{f_{cr}}\,, \tag{6}$$

where η is the bending loss factor of the sheet, ρ_p the density of its material and f_{cr} the critical frequency. The amplitudes P_{e+}, P_{r+}, P_{a+} are "defined" at $x = t_+$, the amplitudes P_{ar+}, P_{t+} at $x = t_+ + d_a$. The source strength is described by its surface velocity profile:

$$v_{Qu}(y) = V_{Qu}\cdot e^{jk_q y\sin\vartheta}\,. \tag{7}$$

The condition at $x = x_{Qu} = 0$: $p(0,y) + Z_i\cdot v(0,y) = Z_i\cdot v_{Qu}$, together with the boundary conditions, leads to the following system of equations:

$$\begin{pmatrix} 1 & -1 & -e^{-a} & 0 & 1 \\ 1 & b & -be^{-a} & 0 & -1 \\ 0 & e^{-a} & 1 & -(1+Z_T/Z_0) & 0 \\ 0 & be^{-a} & -b & -1 & 0 \\ d & 0 & 0 & 0 & c \end{pmatrix}\cdot\begin{pmatrix} P_{r+} \\ P_{a+} \\ P_{ar+} \\ P_{t+} \\ P_{e+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Z_i\cdot V_{Qu} \end{pmatrix} \tag{8}$$

with the abbreviations ($\Gamma_{an} = \Gamma_a/k_0$; $Z_{an} = Z_a/Z_0$):

$$\begin{aligned} &a := \Gamma_a\, d_a\ \cdot\cos\vartheta_a = k_0 d_a\sqrt{\Gamma_{an}^2+\sin^2\vartheta}\,; \quad b := \frac{Z_0\cos\vartheta_a}{Z_a\ \cos\vartheta} = \frac{1}{\Gamma_{an}Z_{an}}\frac{\sqrt{\Gamma_{an}^2+\sin^2\vartheta}}{\cos\vartheta}\,, \\ &c := (1+\cos\vartheta\cdot Z_i/Z_0)\cdot e^{+jk_0 t_+\cos\vartheta}\,; \quad d := (1-\cos\vartheta\cdot Z_i/Z_0)\cdot e^{-jk_0 t_+\cos\vartheta}\,. \end{aligned} \tag{9}$$

The transmission and reflection factors of the capsule walls are:

$$t_{VS} = \frac{P_{t\pm}}{P_{e\pm}}\,; \quad r_{VS} = \frac{P_{r\pm}}{P_{e\pm}}\,. \tag{10}$$

The sound intensities I_+ radiated by the capsule (on one side) and I_0 by the source, if it is in the free space ($Z_F(0) = Z_0/\cos\vartheta$), are:

$$\begin{aligned} I_+ &= \frac{\cos\vartheta}{2Z_0}|P_{t+}|^2 , \\ I_0 &= \frac{\cos\vartheta}{2Z_0}|p_+(0)|^2 = \frac{\cos\vartheta}{2Z_0} \cdot \left|\frac{Z_i \cdot Z_F(0)}{Z_i + Z_F(0)}\right|^2 \cdot |V_{Qu}|^2 = \frac{\cos\vartheta}{2Z_0} \frac{|Z_i V_{Qu}|^2}{|Z_i/Z_0 \cdot \cos\vartheta + 1|^2} . \end{aligned} \tag{11}$$

Thus the insertion power coefficient for one side becomes:

$$\tau_{e+} = \frac{I_+}{I_0} = |1 + Z_i/Z_0 \cdot \cos\vartheta|^2 \cdot \left|\frac{P_{t+}}{Z_i V_{Qu}}\right|^2 , \tag{12}$$

and for both sides together:

$$\tau_e = \frac{I_+ + I_-}{2I_0} = \frac{|1 + Z_i/Z_0 \cdot \cos\vartheta|^2}{2} \cdot \left[\left|\frac{P_{t+}}{Z_i V_{Qu}}\right|^2 + \left|\frac{P_{t-}}{Z_i V_{Qu}}\right|^2\right] . \tag{13}$$

The determinants needed in Kramer's rule for the solution of the system of equations are:

$$\begin{aligned} \det &= -(1+b)^2c - (1-b^2)d + \left((1-b)^2c - (1-b^2)d\right) \cdot e^{-2a} \\ &\quad - b \cdot Z_T/Z_0 \cdot \left[((1-b)c + (1+b)d) \cdot e^{-2a} + (1+b)c + (1-b)d\right] , \end{aligned} \tag{14}$$

$$\begin{aligned} \det 1 &= Z_i V_{Qu} \left[-(1-b^2) - b(1-b)Z_T/Z_0 + (1+b)\left(1 - b(1+Z_T/Z_0) \cdot e^{-2a}\right)\right] , \\ \det 2 &= Z_i V_{Qu} \left[-2 \cdot (1 + b(1 + Z_T/Z_0))\right] , \\ \det 3 &= Z_i V_{Qu} \left[2 \cdot (1 - b(1 + Z_T/Z_0)) \cdot e^{-a}\right] , \\ \det 4 &= Z_i V_{Qu} \left[-4 \cdot b \cdot e^{-a}\right] , \\ \det 5 &= Z_i V_{Qu} \left[-(1+b)^2 - b(1+b)Z_T/Z_0 + (1-b)\left(1 - b(1+Z_T/Z_0)\right) \cdot e^{-2a}\right] , \end{aligned} \tag{15}$$

and the required ratio:

$$\begin{aligned} \frac{P_{t\pm}}{Z_i V_{Qu}} &= \frac{\det 4}{\det} \\ &= [4 \cdot b \cdot e^{-a}] \cdot \Big\{(1+b)^2 c_\pm + (1-b^2)d_\pm - \left((1-b)^2 c_\pm + (1-b^2)d_\pm\right) \cdot e^{-2a} \\ &\quad + b \cdot Z_T/Z_0 \cdot \left[(1+b)c_\pm + (1-b)d_\pm + ((1-b)c_\pm + (1+b)d_\pm) \cdot e^{-2a}\right]\Big\}^{-1} . \end{aligned} \tag{16}$$

Special case: *the source is a pressure source*, i. e. $Z_i \to 0$:

$$\begin{aligned} \tau_{e+} \to \tau_{ep+} = |4b \cdot e^{-a}|^2 \cdot \Big| &\Big[(1+b)^2 - (1-b)^2 \cdot e^{-2a} \\ &+ b \cdot Z_T/Z_0 \left(1 + b + (1-b) \cdot e^{-2a}\right)\Big] \cdot e^{+jk_0 t_+ \cos\vartheta} \\ &+ \left[(1-b^2) \cdot (1 - e^{-2a}) + b \cdot Z_T/Z_0 \left(1 - b + (1+b) \cdot e^{-2a}\right)\right] \\ &\cdot e^{-jk_0 t_+ \cos\vartheta}\Big|^{-2} . \end{aligned} \tag{17}$$

Special case: *the source is a velocity source*, i. e. $Z_i \to \infty$:

$$\begin{aligned}\tau_{e+} \to \tau_{ev+} = |4b \cdot e^{-a}|^2 \cdot \Big| & \big[(1+b)^2 - (1-b)^2 \cdot e^{-2a} \\ & + b \cdot Z_T/Z_0 \left(1 + b + (1-b) \cdot e^{-2a}\right)\big] \cdot e^{+jk_0 t_+ \cos\vartheta} \\ & - \big[(1-b^2)(1-e^{-2a}) + b \cdot Z_T/Z_0 \left(1 - b + (1+b) \cdot e^{-2a}\right)\big] \\ & \cdot e^{-jk_0 t_+ \cos\vartheta}\Big|^{-2} .\end{aligned} \tag{18}$$

Special case: *no absorber layer*, i. e. $d_i \to 0$; $a \to 0$; $b \to 1$:

$$\begin{aligned}\tau_{e+} \to \tau_{e0+} &= 4\frac{|1 + Z_i/Z_0 \cdot \cos\vartheta|^2}{|2c + (c+d) \cdot Z_{Tnx}|^2} \\ &= 4\frac{|1 + Z_i/Z_0 \cdot \cos\vartheta|^2}{\left|2 + \left(1 + \dfrac{(1 - Z_i/Z_0)}{(1 + Z_i/Z_0)} \cdot e^{-2jk_0 t_+ \cos\vartheta}\right) \cdot Z_T/Z_0\right|^2} .\end{aligned} \tag{19}$$

If Z_i of the source is large:

$$\tau_{e0+} \to 4\frac{|1 + Z_i/Z_0 \cdot \cos\vartheta|^2}{\left|2 + (1 + e^{-2jk_0 t_+ \cos\vartheta}) \cdot Z_T/Z_0\right|^2} . \tag{20}$$

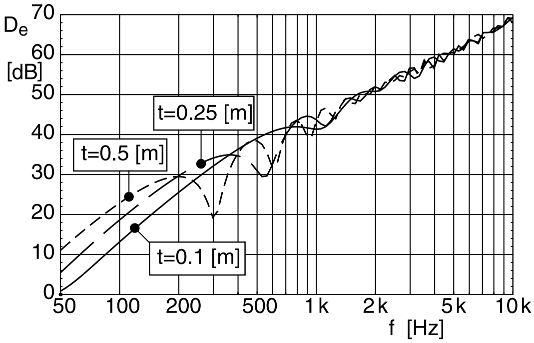
This quantity oscillates strongly with frequency and/or distance t_+:

$$\tau_{e0+,\max} \to |1 + Z_i/Z_0 \cdot \cos\vartheta|^2 ; \quad \tau_{e0+,\min} \to \frac{|1 + Z_i/Z_0 \cdot \cos\vartheta|^2}{|1 + Z_T/Z_0|^2} . \tag{21}$$

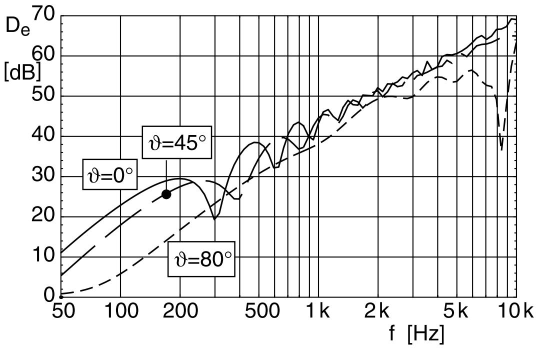
Special case of *a narrow capsule*, i. e. $t_\pm \ll \lambda_0$:

$$\begin{aligned} c &\to (1 + \cos\vartheta \cdot Z_i/Z_0) \cdot (1 + jk_0 d_+ \cos\vartheta) , \\ d &\to (1 - \cos\vartheta \cdot Z_i/Z_0) \cdot (1 - jk_0 d_+ \cos\vartheta) ,\end{aligned} \tag{22}$$

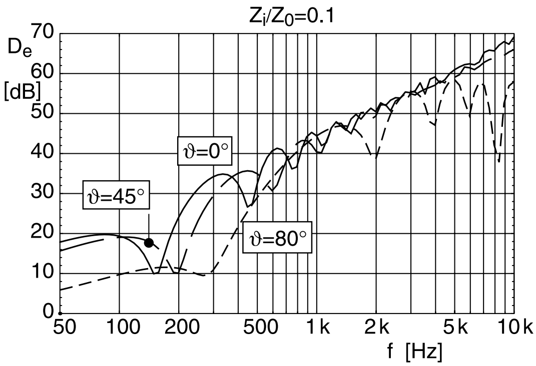
$$\begin{aligned}\tau_{en+} = {} & |4b(1 + Z_i/Z_0 \cdot \cos\vartheta) \cdot e^{-a}|^2 \\ & \cdot \Big|(1 - Z_i/Z_0 \cdot \cos\vartheta) \cdot (1 - jk_0 t_+ \cos\vartheta) \big[(1-b^2)(1-e^{-2a}) \\ & + b \cdot Z_T/Z_0 \left(1 - b + (1+b) \cdot e^{-2a}\right)\big] \\ & + (1 + Z_i/Z_0 \cdot \cos\vartheta) \cdot (1 + jk_0 t_+ \cos\vartheta) \big[(1+b)^2 - (1-b)^2 e^{-2a} \\ & + b \cdot Z_T/Z_0 \left(1 + b + (1-b) \cdot e^{-2a}\right)\big]\Big|^{-2} .\end{aligned} \tag{23}$$



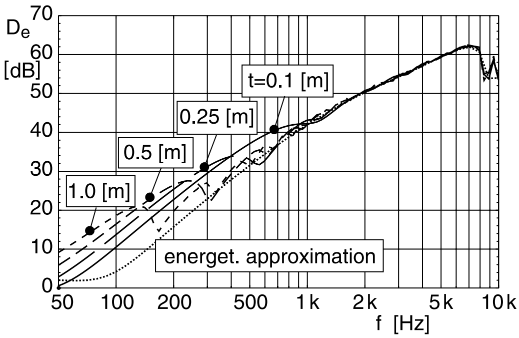
Insertion loss D_e of a capsule, for only normal incidence $\vartheta = 0$, with a rather "high-ohmic" source $Z_i/Z_0 = 10$, for three distances t between source and wall. Parameters: $\vartheta = 0$; $Z_i/Z_0 = 10$; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 1.5$ [mm]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



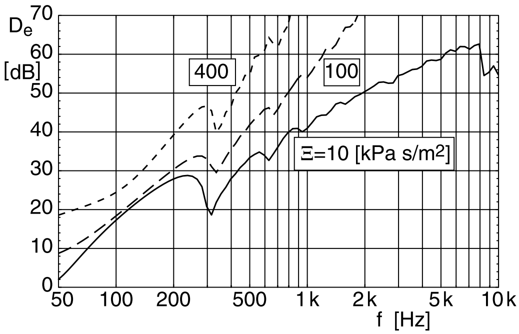
As before, but with one distance t = 0.5 [m] and three angles of incidence ϑ



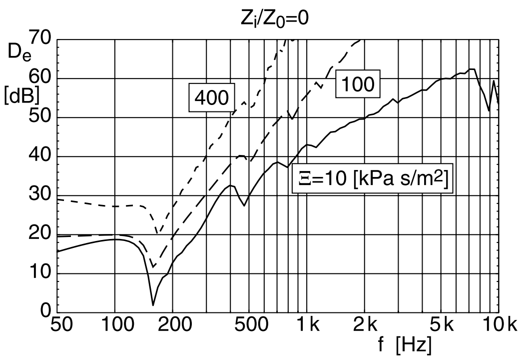
As before, but with a "low-ohmic" source Z_i/Z_0



Insertion loss D_e of a capsule for diffuse sound incidence, with three distances t of source and wall. The energetic approximation is shown for comparison. Parameters: ϑ = diff; $Z_i/Z_0 = 10$; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 1.5$ [mm]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



Insertion loss D_e of a capsule for diffuse sound incidence, with three flow resistivity values Ξ of the porous layer material, for a velocity source. Parameters: ϑ = diff; $Z_i/Z_0 = \infty$; $t = 0.5$ [m]; $d_a = 0.05$ [m]; Ξ = var.; $d_p = 1.5$ [mm]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



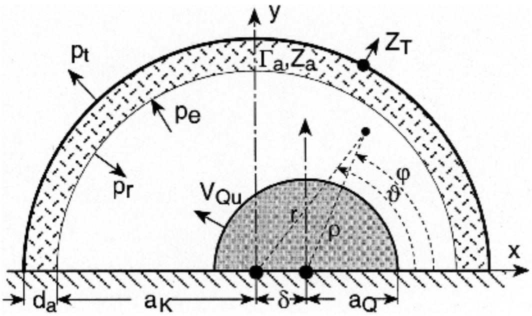
As before, but for a pressure source

L.3 Semicylindrical Source and Capsule

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

A better approach to a capsule than the flat model of the previous section may be a semicylindrical model.

The model is now a semicircular capsule and a similar source. In principle the source can have an eccentric position. The radiated wave can be expanded in cylindrical harmonics in the coordinate system (ρ, φ) of the source. These harmonics, in turn, can be expanded, by the addition theorem for Bessel functions, in cylindrical harmonics with the co-ordinates (r, ϑ). For simplicity this double expansion is avoided here; the source is concentric with the capsule.



The strength of the source is described by its radial particle velocity profile v_{Qu}, which, if necessary, is expanded as:

$$v_{Qu}(\varphi, z) = \sum_n V_n \cdot \cos(n\varphi) \cdot \cos(k_{zn} z) . \tag{1}$$

The interior source impedance Z_i is assumed to be constant in z, φ. The following relation holds:

$$p(\rho = a_Q, \varphi, z) + Z_i \cdot v_r(\rho = a_Q, \varphi, z) = Z_i \cdot v_{Qu}(\varphi, z) . \tag{2}$$

Let the field in the interspace between source and capsule wall be $p_i = p_e + p_r$, in the absorber layer p_a, and in the outer space p_t. Let the floor of the capsule be hard.

Field formulations:

$$\begin{aligned}
p_i(r, \vartheta, z) &= \sum_{n\geq 0} \left[P_{e,n} \cdot \frac{H_n^{(2)}(k_r r)}{H_n^{(2)}(k_r a_Q)} + P_{r,n} \cdot \frac{H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r a_K)} \right] \\
&\quad \cdot \cos(n\vartheta) \cdot \cos(k_{zn} z) , \\
p_a(r, \vartheta, z) &= \sum_{n\geq 0} \left[P_{a,n} \cdot \frac{H_n^{(2)}(k_{a.r} r)}{H_n^{(2)}(k_{a.r} a_K)} + Q_{a,n} \cdot \frac{H_n^{(1)}(k_{a.r} r)}{H_n^{(1)}(k_{a.r}(a_K + d_a))} \right] \\
&\quad \cdot \cos(n\vartheta) \cdot \cos(k_{zn} z) , \\
p_t(r, \vartheta, z) &= \sum_{n\geq 0} P_{t,n} \cdot \frac{H_n^{(2)}(k_r r)}{H_n^{(2)}(k_r(a_K + d_a))} \cdot \cos(n\vartheta) \cdot \cos(k_{zn} z)
\end{aligned} \tag{3}$$

$$\text{with} \quad k_r^2 + k_{zn}^2 = k_0^2; \quad k_{a,r}^2 + k_{zn}^2 = k_a^2 = -\Gamma_a^2. \tag{4}$$

This relation defines a modal angle of incidence Θ_n:

$$\left(\frac{k_r}{k_0}\right)^2 + \left(\frac{k_{zn}}{k_0}\right)^2 = 1 = \cos^2\Theta_n + \sin^2\Theta_n\,, \tag{5}$$

which is zero for conphase excitation along the z axis. The relevant angle χ_n for the evaluation of the partition impedance Z_T of the outer shell of the capsule is given by:

$$\sin\chi_n = \frac{1}{k_0}\sqrt{k_{zn}^2 + (n/a)^2} = \sqrt{\sin^2\Theta_n + (n/k_0(a_K + d_a))^2}\,. \tag{6}$$

The boundary conditions with a concentric source

$$\begin{aligned}
&p_i(a_K) \stackrel{!}{=} p_a\,(a_K); \quad v_{ir}(a_K) \stackrel{!}{=} v_{ar}(a_K)\,,\\
&p_a\,(a_K + d_a) - p_t(a_K + d_a) \stackrel{!}{=} Z_T \cdot v_{tr}(a_K + d_a)\,,\\
&v_{ar}(a_K + d_a) \stackrel{!}{=} v_{tr}(a_K + d_a)\,,\\
&p_i(a_Q) + Z_i \cdot v_{i\rho}(a_Q) \stackrel{!}{=} Z_i \cdot v_{Qu}
\end{aligned} \tag{7}$$

hold term-wise and produce the following system of equations:

$$\begin{pmatrix} \vdots\cdots & \cdots & \cdots\vdots \\ \vdots & \ddots \quad \iddots & \vdots \\ \vdots & a_{i,k} & \vdots \\ \vdots & \iddots \quad \ddots & \vdots \\ \vdots\cdots & \cdots & \cdots\vdots \end{pmatrix} \cdot \begin{pmatrix} P_{e,n} \\ P_{r,n} \\ P_{a,n} \\ Q_{a,n} \\ P_{t,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Z_i V_n \end{pmatrix} \tag{8}$$

with coefficients (a prime indicates the derivative):

$$\begin{aligned}
&a_{11} = \frac{H_n^{(2)}(k_r a_K)}{H_n^{(2)}(k_r a_Q)}; \quad a_{12} = 1; \quad a_{13} = -1; \quad a_{14} = -\frac{H_n^{(1)}(k_{a,r} a_K)}{H_n^{(1)}(k_{a,r}(a_K + d_a))};\\
&a_{15} = 0\,,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&a_{21} = jk_r a_K \cdot \frac{H_n'^{(2)}(k_r a_K)}{H_n^{(2)}(k_r a_Q)}; \quad a_{22} = jk_r a_K \cdot \frac{H_n'^{(1)}(k_r a_K)}{H_n^{(1)}(k_r a_K)}\,;\\
&a_{23} = \frac{k_{a,r} a_K}{\Gamma_{an} Z_{an}} \frac{H_n'^{(2)}(k_{a,r} a_K)}{H_n^{(2)}(k_{a,r} a_K)}\,; \quad a_{24} = \frac{k_{a,r} a_K}{\Gamma_{an} Z_{an}} \frac{H_n'^{(1)}(k_{a,r} a_K)}{H_n^{(1)}(k_{a,r}(a_K + d_a))}; \quad a_{25} = 0\,,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&a_{31} = a_{32} = 0; \quad a_{33} = -\frac{H_n^{(2)}(k_{a,r}(a_K + d_a))}{H_n^{(2)}(k_{a,r} a_K)}; \quad a_{34} = -1\,;\\
&a_{35} = 1 + j\frac{k_r Z_T}{k_0 Z_0} \frac{H_n'^{(2)}(k_r(a_K + d_a))}{H_n^{(2)}(k_r(a_K + d_a))}\,,
\end{aligned} \tag{11}$$

$$a_{41} = a_{42} = 0\,; \quad a_{43} = \frac{k_{a,r}a_K}{\Gamma_{an}Z_{an}} \frac{H_n'^{(2)}(k_{a,r}(a_K+d_a))}{H_n^{(2)}(k_{a,r}a_K)}\,;$$
$$a_{44} = \frac{k_{a,r}a_K}{\Gamma_{an}Z_{an}} \frac{H_n'^{(1)}(k_{a,r}(a_K+d_a))}{H_n^{(1)}(k_{a,r}(a_K+d_a))}\,; \quad a_{45} = jk_r a_K \cdot \frac{H_n'^{(2)}(k_r(a_K+d_a))}{H_n^{(2)}(k_r(a_K+d_a))}\,, \tag{12}$$

$$a_{51} = 1 + j\frac{k_r Z_i}{k_0 Z_0} \frac{H_n'^{(2)}(k_r a_Q)}{H_n^{(2)}(k_r a_Q)}\,; \quad a_{52} = \frac{H_n^{(1)}(k_r a_Q)}{H_n^{(1)}(k_r a_K)} + j\frac{k_r Z_i}{k_0 Z_0}\frac{H_n'^{(1)}(k_r a_Q)}{H_n^{(1)}(k_r a_K)}\,;$$
$$a_{53} = a_{54} = a_{55} = 0\,. \tag{13}$$

The radiated (effective) power is the sum of the modal powers. The radiated modal power of the free source (radius $a = a_Q$) without z factors (they cancel in τ) is:

$$\Pi_n^{(0)} = \frac{\pi a_Q}{2\delta_n Z_0} \cdot \frac{\operatorname{Re}\left\{Z_n(a_Q)/Z_0\right\}}{|Z_i/Z_0 + Z_n(a_Q)/Z_0|^2} \cdot |Z_i V_n|^2\,. \tag{14}$$

The radiated (effective) modal power of the capsule (radius $a = a_K + d_a$) is:

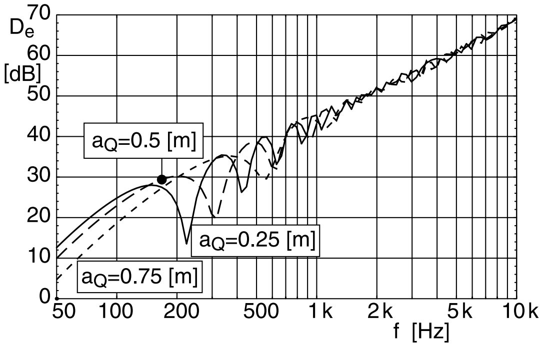
$$\Pi_n = \frac{a_K + d_a}{2} Z_0 \cdot \operatorname{Re}\{Z_n(a_K+d_a)/Z_0\} \cdot \int_0^{\pi} |v_{tr,n}(a_K+d_a,\vartheta)|^2\, d\vartheta$$
$$= \frac{\pi(a_K+d_a)}{2\delta_n} Z_0 \cdot \operatorname{Re}\{Z_n(a_K+d_a)/Z_0\} \cdot \frac{|k_r P_{t,n}|^2}{(k_0 Z_0)^2} \left| \frac{H_n'^{(2)}(k_r(a_K+d_a))}{H_n^{(2)}(k_r(a_K+d_a))} \right|^2\,. \tag{15}$$

Thus the insertion power coefficient of the capsule becomes:

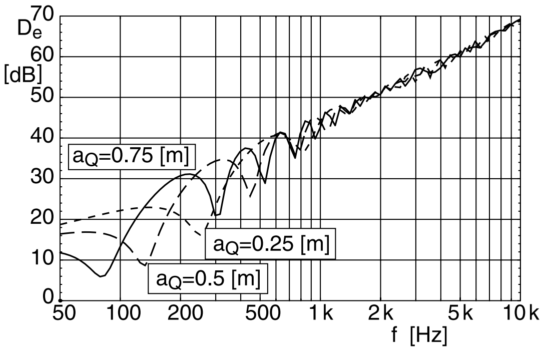
$$\tau_K = \frac{\sum\limits_{n\geq 0} \Pi_n}{\sum\limits_{n\geq 0} \Pi_n^{(0)}}$$
$$= \frac{a_K + d_a}{a_Q} \cdot \frac{\sum\limits_{n\geq 0} \frac{1}{\delta_n} \operatorname{Re}\{Z_n(a_K+d_a)/Z_0\} \cdot \left| \frac{k_r}{k_0} \cdot \frac{H_n'^{(2)}(k_r(a_K+d_a))}{H_n^{(2)}(k_r(a_K+d_a))} \right|^2 \cdot \left|P_{t,n}\right|^2}{\sum\limits_{n} \frac{1}{\delta_n} \frac{\operatorname{Re}\{Z_n(a_Q)/Z_0\}}{\left|(Z_i + Z_n(a_Q))/Z_0\right|^2} \cdot |Z_i V_n|^2} \tag{16}$$

with $\delta_0 = 1\,;\ \delta_{n>0} = 2$ and the modal radiation impedances

$$\frac{Z_n(a)}{Z_0} = -j\frac{k_0}{k_r} \frac{H_n^{(2)}(k_r a)}{H_n'^{(2)}(k_r a)}\,. \tag{17}$$



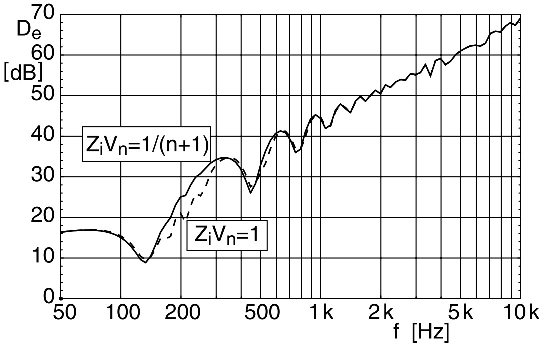
Insertion loss D_e for monomodal excitation, $n = 0$, by a high-ohmic source, $Z_i/Z_0 = 10$, with three source radii a_Q. Parameters: $\Theta = 0°$; $n = 0$; $Z_i/Z_0 = 10$; $a_K = 1$ [m]; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



As before, but for a low-ohmic source $Z_i/Z_0 = 0.1$

One needs information about the Z_iV_n for further valuation if the source pattern is not mono-modal.

Heuristic assumptions about the source mode amplitudes could be Z_iV_n = const or $Z_iV_n \sim 1/(n + 1)$. The following diagram shows the influence of such assumptions on D_e for a multimodal excitation with the mode orders $n = 0, \ldots, 4$ (with a low-ohmic source $Z_i/Z_0 = 0.1$).

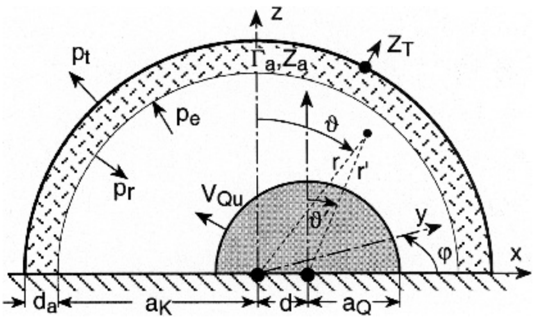


Insertion loss D_e for a multimodal excitation, $n = 0, \ldots, 4$, with two assumptions about the modal strength. Parameters: $\Theta = 0°$; $n = 0 - 4$; $Z_i/Z_0 = 0.1$; $a_K = 1$ [m]; $a_Q = 0.5$ [m]; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa $\cdot$ s/m^2]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz $\cdot$ m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$

L.4 Hemispherical Source and Capsule

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

The object is similar to the object of the previous *Sect. L.3*, but now the source and the capsule are hemispherical. An eccentric source could again be treated with the addition theorem for Bessel functions, but, for simplicity, a concentric source will mainly be considered below. The source strength is described by a surface radial velocity profile v_{Qu}.



The sound field inside the capsule is $p_i = p_e + p_r$, the sound field in the interior absorber layer is p_a, and the radiated sound outside the capsule is represented by p_t.

Field formulation:

$$v_{Qu}(\vartheta, \varphi) = \sum_{n \geq 0} V_n \cdot P_n^m(\cos\vartheta) \cdot \cos(m\varphi)\,, \tag{1}$$

(for ease of writing, only one azimuthal mode $m \geq 0$ is assumed to exist; in the final equations below a sum of azimuthal modes will be considered)

$$p_i(r,\vartheta,\varphi) = \sum_{n\geq 0}\left[P_{e,n}\cdot\frac{h_n^{(2)}(k_0 r)}{h_n^{(2)}(k_0 a_Q)} + P_{r,n}\cdot\frac{h_n^{(1)}(k_0 r)}{h_n^{(1)}(k_0 a_K)}\right]\cdot P_n^m(\cos\vartheta)\cdot\cos(m\varphi)\,,$$

$$p_a(r,\vartheta,\varphi) = \sum_{n\geq 0}\left[P_{a,n}\cdot\frac{h_n^{(2)}(k_a r)}{h_n^{(2)}(k_a a_K)} + Q_{a,n}\cdot\frac{h_n^{(1)}(k_a r)}{h_n^{(1)}(k_a(a_K+d_a))}\right] \cdot P_n^m(\cos\vartheta)\cdot\cos(m\varphi)\,, \tag{2}$$

$$p_t(r,\vartheta,\varphi) = \sum_{n\geq 0} P_{t,n}\cdot\frac{h_n^{(2)}(k_0 r)}{h_n^{(2)}(k_0(a_K+d_a))}\cdot P_n^m(\cos\vartheta)\cdot\cos(m\varphi)\,.$$

$h_n^{(\alpha)}(z);\ \alpha = 1, 2;$ are spherical Hankel functions $\quad h_n^{(\alpha)}(z) = \sqrt{\dfrac{\pi}{2z}}H_n^{(\alpha)}(z).$ (3)

$P_n^m(z)$ are associate Legendre functions with special values:

$$P_n^m(z) \equiv 0;\quad m > n\,,$$
$$P_n^m(0) \xrightarrow[n+m=odd]{} 0\,;\qquad \frac{dP_n^m(z)}{dz} \xrightarrow[n+m=even]{} 0\,. \tag{4}$$

For $m = 0$ they go over to the Legendre polynomes $P_n(z)$:

$$P_n^m(z) \xrightarrow[m=0]{} P_n(z)\,,$$
$$P_n^m(z) = (-1)^m\cdot(1-z^2)^{m/2}\frac{d^m P_n(z)}{dz^m}\,;\quad z = \text{real},\quad |z|\leq 1\,. \tag{5}$$

The partition impedance Z_T of the outer, elastic shell follows from the bending wave equation as:

$$Z_T := \frac{\delta p}{v_\perp} = \frac{B}{j\omega}\frac{\left[\Delta_{\vartheta,\varphi}\Delta_{\vartheta,\varphi} - k_B^4\right]v_\perp(\vartheta,\varphi)}{v_\perp(\vartheta,\varphi)} = \frac{B}{j\omega}\frac{\left[\Delta_{\vartheta,\varphi}\Delta_{\vartheta,\varphi} - k_B^4\right]T(\vartheta)\cdot P(\varphi)}{T(\vartheta)\cdot P(\varphi)}$$
$$= \frac{B}{j\omega}\left(k_{trace}^4 - k_B^4\right) = j\omega m\left[1-(k_{trace}/k_B)^4\right] = j\omega m\left[1-(f/f_{cr})^2\sin^4\Theta\right]\,, \tag{6}$$

where B is the bending modulus, k_B the free bending-wave number, f_{cr} the critical frequency of the shell (if it were a plane plate), k_{trace} the wave number of the trace of the exciting wave along the shell, and Θ the polar angle of incidence on the shell with $k_{trace} = k_0\cdot\sin\Theta$. It is ($a$ = shell radius):

$$k_{trace}^4 = \frac{n(n^2-1)(2+n)}{a^4};\quad \left(\frac{k_{trace}}{k_B}\right)^4 = \frac{n(n^2-1)(2+n)}{(k_0 a)^4}\left(\frac{f}{f_{cr}}\right)^2\,, \tag{7}$$

and therefore $\quad \sin^4\Theta = \dfrac{n(n^2-1)(2+n)}{(k_0 a)^4}.$ (8)

The radiated effective power of the source into free space is the sum of modal powers:

$$\Pi_{m,n}^{(0)} = \frac{a_Q^2 N_{m,n}}{4Z_0}\cdot\frac{\mathrm{Re}\left\{Z_n(a_Q)/Z_0\right\}}{|Z_i/Z_0 + Z_n(a_Q)/Z_0|^2}\cdot|Z_i V_n|^2 \tag{9}$$

with the mode norms $N_{m,n}$:

$$\frac{a^2}{2} N_{m,n} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} a^2 \sin\vartheta \cdot (P_n^m(\cos\vartheta))^2 \cdot \cos^2(m\varphi)\, d\vartheta = \frac{2\pi a^2}{\delta_m} \frac{1}{2n+1} \frac{(n+m)!}{(n-m)!} \tag{10}$$

($\delta_0 = 1$; $\delta_{n>0} = 2$), and with the modal radiation impedance (a prime denotes the derivative):

$$\frac{Z_n(a)}{Z_0} = -j \frac{h_n^{(2)}(k_0 a)}{h_n'^{(2)}(k_0 a)} . \tag{11}$$

The modal effective power radiated by the capsule is correspondingly:

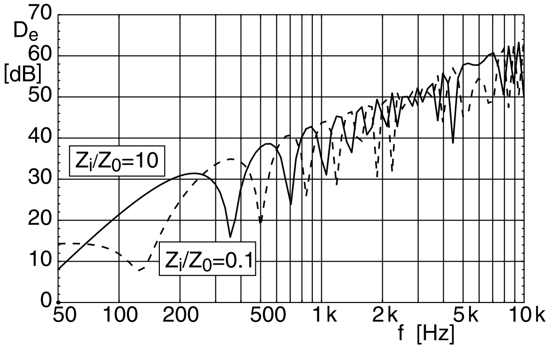
$$\Pi_{m,n} = \frac{(a_K + d_a)^2 N_{m,n}}{4 Z_0} \cdot \mathrm{Re}\,\{Z_n(a_K + d_a)/Z_0\} \cdot \left| \frac{h_n'^{(2)}(k_0(a_K + d_a))}{h_n^{(2)}(k_0(a_K + d_a))} \right|^2 |P_{t,n}|^2 . \tag{12}$$

The insertion power coefficient for multimodal excitation finally becomes:

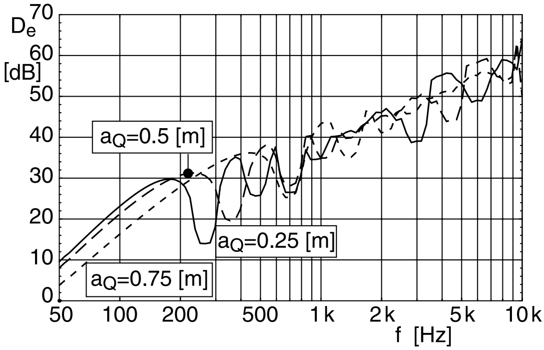
$$\tau_e = \frac{\sum\limits_{m,n\geq 0} \Pi_{m,n}}{\sum\limits_{m,n\geq 0} \Pi_{m,n}^{(0)}} = \frac{(a_K + d_a)^2}{a_Q^2} \cdot \frac{\sum\limits_{m,n\geq 0} N_{m,n} \mathrm{Re}\{Z_n(a_K + d_a)/Z_0\} \cdot \left| \frac{h_n'^{(2)}(k_0(a_K + d_a))}{h_n^{(2)}(k_0(a_K + d_a))} \right|^2 \cdot \left|P_{t,m,n}\right|^2}{\sum\limits_{m,n\geq 0} N_{m,n} \frac{\mathrm{Re}\{Z_n(a_Q)/Z_0\}}{\left|(Z_i + Z_n(a_Q))/Z_0\right|^2} \cdot \left|Z_i V_{m,n}\right|^2} . \tag{13}$$

The amplitudes $P_{t,m,n}$ therein follow from the system of equations of the previous Sect. L.3 after substitution $H_n^{(\alpha)}(k_r r) \to h_n^{(\alpha)}(k_0 r)$ in the coefficients a_{ik} of that system, and $Z_i V_n \to Z_i V_{m,n}$ on the right-hand side.

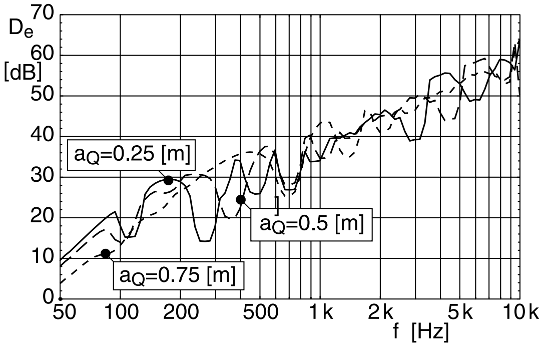
The examples shown below illustrate the influence of the source interior impedance Z_i and of the source radius a_Q (with a fixed capsule radius $a_K = 1$ [m]) for monomodal and multimodal excitations.



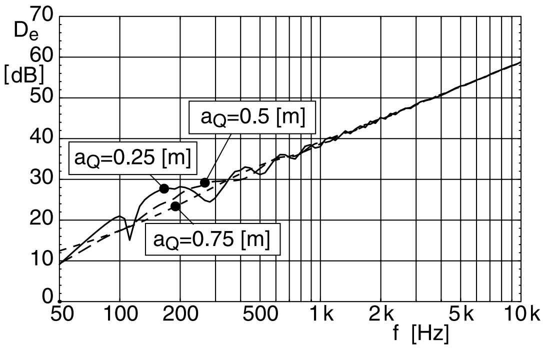
Insertion loss D_e for monomodal excitation with two source impedances Z_i. Parameters: $m = 0$; $n = 0$; $Z_i/Z_0 =$ var; $a_K = 1$ [m]; $a_Q = 0.5$ [m]; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa$\cdot$s/m^2]; $d_p = 0.0015$ [m]; $f_{cr} d_p = 12.3$ [Hz $\cdot$ m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



Insertion loss D_e for monomodal excitation with three source radii a_Q. Parameters: $m = 0$; $n = 0$; $Z_i/Z_0 = 10$; $a_K = 1$ [m]; a_Q = var; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



Insertion loss D_e for multimodal excitation and three source radii a_Q. The excitation mode amplitudes decay as $Z_iV_n = 1/(n+1)$. Parameters: $m = 0$; $n = 0 - 5$; $Z_i/Z_0 = 10$; $a_K = 1$ [m]; a_Q = var; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



As before, but for a source which is "matched" to the exterior field, $Z_I/Z_0 = 1$

L.5 Cabins, Semicylindrical Model

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

Cabins are exposed to an exterior sound field p_e. A suitable quantity for the qualification of the efficiency of the cabin is the sound pressure level difference (*sound protection measure*):

$$\Delta L = -10 \cdot \lg \frac{\langle |p_i|^2 \rangle_{V_K}}{\langle |p_e|^2 \rangle_{V_K}} \text{ [dB]}, \tag{1}$$

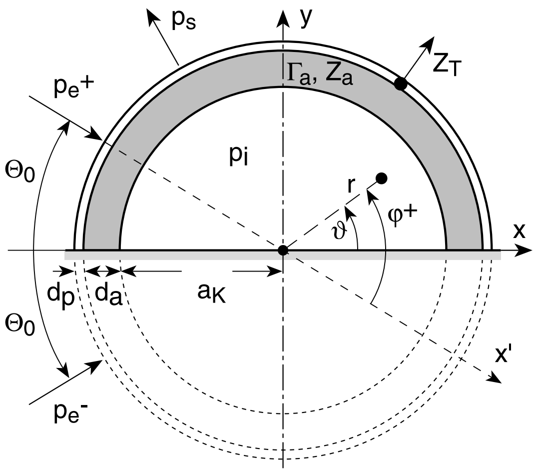
where p_i is the interior sound field, and $\langle \ldots \rangle_{V_K}$ indicates the spatial average over the volume V_K of the cabin. However, because cabins are often relatively small with a low mode density in them, the measurement of the average may be difficult. Alternatively a sound pressure level in some defined point r_0 in the cabin could be used:

$$\Delta L_0 = -10 \cdot \lg \frac{|p_i(r_0)|^2}{|p_e(r_0)|^2} \text{ [dB]}. \tag{2}$$

This definition reduces the amount of numerical computation considerably.

The model to be treated below is a semicylindrical cabin on a hard floor. It consists of an outer elastic shell with partition impedance Z_T and an interior layer of porous material with characteristic constants Γ_a, Z_a (or, in normalised form, $\Gamma_{an} = \Gamma_a/k_0$, $Z_{an} = Z_a/Z_0$).

A plane wave p_e^+ is assumed as incident wave (see below for diffuse sound incidence); a mirror-reflected wave p_e^- simulates the hard floor.



A variation of the incident wave in the z direction could be $\cos(k_z z)$, $\sin(k_z z)$, $e^{\pm jk_z z}$ or a linear combination thereof. It would influence the radial wave number k_r by $k_z^2 + k_r^2 = k_0^2$. The field factor in the z direction can be dropped because it is the same in all field parts.

The exciting field $p_e = p_e^+ + p_e^-$ is: (3)

$$
\begin{aligned}
p_e(r,\vartheta) &= P_e \sum_{n\geq 0} \delta_n(-j)^n \cdot J_n(k_r r) \cdot [\cos(n(\vartheta+\Theta_0)) + \cos(n(\vartheta-\Theta_0))] \\
&= 2P_e \sum_{n\geq 0} \delta_n(-j)^n \cdot \cos(n\Theta_0) \cdot J_n(k_r r) \cdot \cos(n\vartheta) \\
&:= \sum_{n\geq 0} P_{en} \cdot J_n(k_r r) / J_n(k_r(a_K + d_a)) \cdot \cos(n\vartheta)\,.
\end{aligned}
\tag{4}
$$

The last form uses the abbreviation

$$P_{en} = 2P_e \cdot \delta_n(-j)^n \cdot \cos(n\Theta_0) \cdot J_n(k_r(a_K + d_a)) \tag{5}$$

with $\delta_0 = 1$; $\delta_{n>0} = 2$.

The interior field p_i, in the absorber layer p_a, and the exterior scattered field p_s are formulated as:

$$
\begin{aligned}
p_i(r,\vartheta) &= \sum_{n\geq 0} P_{in} \cdot J_n(k_r r) / J_n(k_r a_K) \cdot \cos(n\vartheta)\,, \\
p_a(r,\vartheta) &= \sum_{n\geq 0} \left[P_{an} \cdot \frac{H_n^{(2)}(k_{ar} r)}{H_n^{(2)}(k_{ar} a_K)} + Q_{an} \cdot \frac{H_n^{(1)}(k_{ar} r)}{H_n^{(1)}(k_{ar}(a_K + d_a))} \right] \cdot \cos(n\vartheta)\,, \\
p_s(r,\vartheta) &= \sum_{n\geq 0} P_{sn} \cdot H_n^{(2)}(k_r r) / H_n^{(2)}(k_r(a_K + d_a)) \cdot \cos(n\vartheta)\,.
\end{aligned}
\tag{6}
$$

The radial wave number k_{ar} in the absorber layer is given by $k_{ar}^2 + k_z^2 = k_a^2 = -\Gamma_a^2$. The unknown amplitudes $P_{in}, P_{sn}, P_{an}, Q_{an}$ follow from the boundary conditions which hold term-wise:

$$
\begin{aligned}
&p_{in}(a_K) \stackrel{!}{=} p_{an}(a_K); \quad v_{irn}(a_K) \stackrel{!}{=} v_{arn}(a_K)\,, \\
&p_{an}(a_K + d_a) - p_{en}(a_K + d_a) - p_{sn}(a_K + d_a) \stackrel{!}{=} Z_{Tn} \cdot v_{arn}(a_K + d_a)\,, \\
&v_{arn}(a_K + d_a) \stackrel{!}{=} v_{ern}(a_K + d_a) + v_{srn}(a_K + d_a)\,.
\end{aligned}
\tag{7}
$$

The modal partition impedance Z_{Tn} of the shell is evaluated from:

$$
\begin{aligned}
&\frac{Z_{Tn}}{Z_0} = 2\pi Z_m F[\eta F^2 \cdot \sin^4\chi_n + j(1 - F^2 \cdot \sin^4\chi_n)]; \quad Z_m = \frac{f_{cr} d_p}{Z_0}\rho_p; \quad F = \frac{f}{f_{cr}}\,, \\
&\sin\chi_n = \frac{1}{k_0}\sqrt{k_z^2 + (n/(a_K + d_a))^2}
\end{aligned}
\tag{8}
$$

(d_p = shell thickness, ρ_p = shell material density, f_{cr} = critical frequency of the shell as a plane plate, η = bending loss factor of the shell).

The system of equations of the boundary conditions has the following form, with the abbreviation

$$C_n = k_r(a_K + d_a)\frac{J_n'(k_r(a_K + d_a))}{J_n(k_r(a_K + d_a))} \tag{9}$$

(a prime indicates the derivative):

$$\begin{pmatrix} 1 & 0 & -1 & a_{1,4} \\ a_{2,1} & 0 & a_{2,3} & a_{2,4} \\ 0 & -1 & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \cdot \begin{pmatrix} P_{in} \\ P_{sn} \\ P_{an} \\ Q_{an} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ P_{en} \\ C_n \cdot P_{en} \end{pmatrix} . \tag{10}$$

The matrix coefficients are:

$$\begin{aligned}
a_{1,4} &= -\frac{H_n'^{(1)}(k_r a_K)}{H_n^{(1)}(k_r(a_K+d_a))}, \\
a_{2,1} &= k_r a_K \frac{J_n'(k_r a_K)}{J_n(k_r a_K)}; \quad a_{2,3} = \frac{-jk_{ar} a_K}{\Gamma_{an} Z_{an}} \frac{H_n'^{(2)}(k_{ar} a_K)}{H_n^{(2)}(k_{ar} a_K)}; \\
a_{2,4} &= \frac{-jk_{ar} a_K}{\Gamma_{an} Z_{an}} \frac{H_n'^{(1)}(k_{ar} a_K)}{H_n^{(1)}(k_{ar}(a_K+d_a))}, \\
a_{3,3} &= \frac{H_n^{(2)}(k_{ar}(a_K+d_a))}{H_n^{(2)}(k_{ar} a_K)} + \frac{k_{ar}/k_0 \cdot Z_{Tn}/Z_0}{\Gamma_{an} Z_{an}} \frac{H_n'^{(2)}(k_{ar}(a_K+d_a))}{H_n^{(2)}(k_{ar} a_K)}; \\
a_{3,4} &= 1 + \frac{k_{ar}/k_0 \cdot Z_{Tn}/Z_0}{\Gamma_{an} Z_{an}} \frac{H_n'^{(1)}(k_{ar}(a_K+d_a))}{H_n^{(1)}(k_{ar}(a_K+d_a))}, \\
a_{4,2} &= -k_r(a_K+d_a) \frac{H_n'^{(2)}(k_r(a_K+d_a))}{H_n^{(2)}(k_r(a_K+d_a))}; \quad a_{4,3} = \frac{jk_{ar}(a_K+d_a)}{\Gamma_{an} Z_{an}} \frac{H_n'^{(2)}(k_{ar}(a_K+d_a))}{H_n^{(2)}(k_{ar} a_K)}; \\
a_{4,4} &= \frac{jk_{ar}(a_K+d_a)}{\Gamma_{an} Z_{an}} \frac{H_n'^{(1)}(k_{ar}(a_K+d_a))}{H_n^{(1)}(k_{ar}(a_K+d_a))} .
\end{aligned} \tag{11}$$

C_n and some matrix coefficients have the form: $$z\frac{Z_n'(z)}{Z_n(z)} = -z\frac{Z_{n+1}(z)}{Z_n(z)} + n \tag{12}$$

with $Z_n(z)$ some cylinder function.

The desired quantity P_{in} is:

$$P_{in} = -\frac{(a_{1,4} \cdot a_{2,3} + a_{2,4}) \cdot (a_{4,2} + C_n)}{(a_{1,4} \cdot a_{2,1} - a_{2,4}) \cdot (a_{3,3} \cdot a_{4,2} + a_{4,3}) + (a_{2,1} + a_{2,3}) \cdot (a_{3,4} \cdot a_{4,2} + a_{4,4})} P_{en} . \tag{13}$$

With this, the field inside the cabin is known. Factors with a variation in the z direction will cancel in the ratio for ΔL after averaging; therefore the average over the area $A_K = \pi\, a_K^2$ is sufficient. The average of the exterior field is (with $k_y = k_r \cdot \sin\Theta_0$):

$$\langle |p_e|^2 \rangle_{A_K} = \frac{2}{A_K} \int_{-a_K}^{+a_K} dx \int_0^{y(x)} |p_e|^2\, dy = 2|P_e|^2 \left(1 + \frac{J_1(2k_y a_K)}{2k_y a_K}\right); \quad y(x) = \sqrt{a_K^2 - x^2} . \tag{14}$$

The average inside the cabin is:

$$\langle |p_i|^2 \rangle_{A_K} = \sum_{n \ge 0} \frac{1}{\delta_n} |P_{in}|^2 \left(1 - \frac{J_{n-1}(k_r a_K) \cdot J_{n+1}(k_r a_K)}{J_n^2(k_r a_K)}\right) . \tag{15}$$

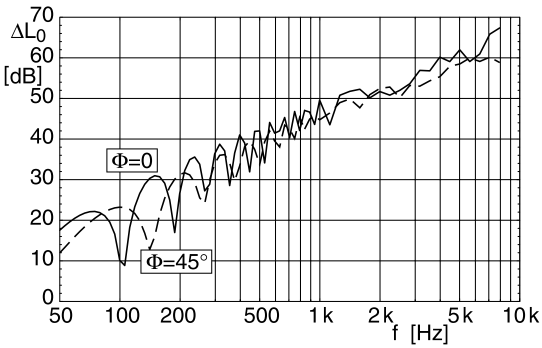
The sound protection measure, based on the average squared pressure magnitudes, for a single incident plane wave p_e^+ is, finally:

$$\Delta L = -10 \cdot \lg \frac{\sum\limits_{n\geq 0} \frac{1}{\delta_n} \left| \frac{P_{in}}{P_e} \right|^2 \left(1 - \frac{J_{n-1}(k_r a_K) \cdot J_{n+1}(k_r a_K)}{J_n^2(k_r a_K)} \right)}{2 \left(1 + \frac{J_1(2k_y a_K)}{2k_y a_K} \right)} . \tag{16}$$

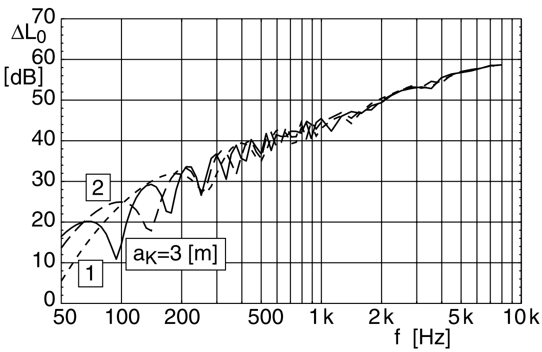
The sound protection measure, based on the level difference in the cabin centre, is:

$$\Delta L_0 = -10 \cdot \lg \left[\frac{1}{4} \left| \frac{P_{i0}}{P_e} \right|^2 \frac{1}{J_0^2(k_r a_K)} \right] . \tag{17}$$

Because the angle of incidence Θ_0 is not contained, it also holds for two-dimensional diffuse incidence.



Sound protection measure ΔL_0 for sound incidence normal and oblique to the cabin axis. Parameters: Φ = var.; $a_K = 2$ [m]; $d_a = 0.05$ [m]; $\Xi = 10$ [kPa · s/m^2]; d_p = 0.0015 [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$

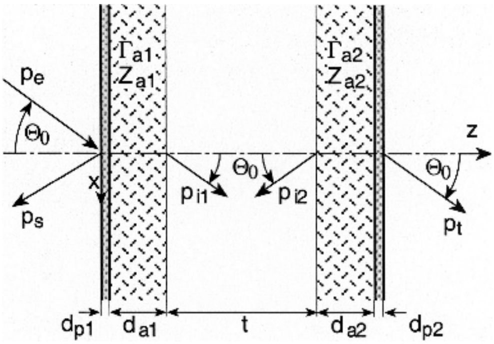


Sound protection measure ΔL_0 for oblique sound incidence and three cabin radii a_K. Parameters: $\Phi = 45°$; a_K = var.; $d_a = 0.1$ [m]; $\Xi = 10$ [kPa · s/m^2]; $d_p = 0.0015$ [m]; $f_{cr}d_p = 12.3$ [Hz · m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$

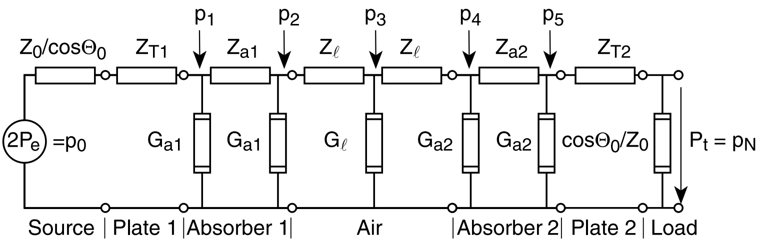
L.6 Cabin with Plane Walls

► *See also:* Mechel, Vol. III, Ch. 20 (1998)

The model is two-dimensional: two walls consisting of an exterior plate and an interior porous absorber layer. This model is an approximation to reality if the lateral dimension of the cabin in the x direction is large (at least compared to t) and if sound incidence comes mainly from the half space in front of one wall.



Under these conditions the cabin is just a multilayer absorber; the desired interior sound field is the field in one layer. This view of the task permits the application of the equivalent network method (see Ch. C).



The source pressure is $p_0 = 2P_e$ with the amplitude P_e of the incident plane wave. The interior impedance of the source and the load impedance are $Z_0/\cos\Theta_0$.

The sound pressure inside the cabin, i. e. in the layer named "Air" above, is:

$$p_i(x,y) = \left[P_{i1} \cdot e^{-jk_x(x-d_{a1})} + P_{i2} \cdot e^{+jk_x(x-d_{a1}-t)}\right] \cdot e^{-jk_y y}\,, \qquad k_x = k_0 \cos\Theta_0; \quad k_y = k_0 \sin\Theta_0 \tag{1}$$

with the relation between amplitudes and circuit node pressures:

$$\begin{aligned} p_i(x=d_{a1}) &= \left[P_{i1} + P_{i2} \cdot e^{-jk_x t}\right] \stackrel{!}{=} 2P_e \cdot p_2\,, \\ p_i(x=d_{a1}+t) &= \left[P_{i1} \cdot e^{-jk_x t} + P_{i2}\right] \stackrel{!}{=} 2P_e \cdot p_4 \end{aligned} \tag{2}$$

or

$$P_{i1} = \frac{p_2 - p_4 \cdot e^{-jk_x t}}{1 - e^{-2jk_x t}} \cdot 2P_e\,; \qquad P_{i2} = \frac{p_4 - p_2 \cdot e^{-jk_x t}}{1 - e^{-2jk_x t}} \cdot 2P_e\,. \tag{3}$$

If one includes in the exciting wave the ground-reflected wave, the factor $e^{-jk_y y}$ simply changes to $\cos(k_y y)$; it cancels anyway in the averages for ΔL and ΔL_0.

The equivalent circuit elements are (i = 1, 2):

$$\frac{Z_{\alpha,i}}{Z_0} = Z_{an,i}\frac{\sinh(\Gamma_{a,i}d_{ai}\cdot\cos\Theta_{\alpha,i})}{\cos\Theta_{\alpha,i}};\quad Z_0G_{\alpha,i} = \frac{\cos\Theta_{\alpha,i}}{Z_{an,i}}\frac{\cosh(\Gamma_{a,i}d_{ai}\cdot\cos\Theta_{\alpha,i})-1}{\sinh(\Gamma_{a,i}d_{ai}\cdot\cos\Theta_{\alpha,i})},$$
$$\cos\Theta_{\alpha,i} = \frac{\sqrt{\Gamma_{an,i}^2+\sin^2\Theta_0}}{\Gamma_{an,i}};\quad \Gamma_{a,i}d_{ai}\cdot\cos\Theta_{\alpha,i} = k_0d_{ai}\sqrt{\Gamma_{an,i}^2+\sin^2\Theta_0}, \tag{4}$$

$$\frac{Z_\ell}{Z_0} = j\frac{1-\cos(k_0t\cdot\cos\Theta_0)}{\cos\Theta_0\cdot\sin(k_0t\cdot\cos\Theta_0)};\quad Z_0G_\ell = j\cos\Theta_0\cdot\sin(k_0t\cdot\cos\Theta_0), \tag{5}$$

$$\frac{Z_{T,i}}{Z_0} = 2\pi\cdot Z_{m,i}\cdot F_i\left[\eta_iF_i^2\cdot\sin^4\Theta_0 + j\left(1-F_i^2\cdot\sin^4\Theta_0\right)\right] \tag{6}$$

$$\text{with}\quad Z_{m,i} = \frac{f_{cr,i}d_i}{Z_0}\rho_i;\quad F_i = \frac{f}{f_{cr,i}} = \frac{f\cdot d_i}{f_{cr,i}d_i} \tag{7}$$

(d_i = plate thickness, ρ_i = plate material density, $f_{cr,i}$ = critical frequency, η_i = bending loss factor)

With the abbreviations

$$z1 = 1+\frac{Z_{T,2}}{Z_0}\cos\Theta_0;\quad \alpha = \cos\Theta_0 + Z_0G_{\alpha,2}\cdot z1,$$
$$z2 = z1+\frac{Z_{a,2}}{Z_0}\cdot\alpha;\quad \beta = \alpha + Z_0G_{\alpha,2}\cdot z2, \tag{8a}$$

$$z3 = \frac{Z_\ell}{Z_0}\cdot\beta;\quad z4 = z2+z3;\quad \gamma = \beta + Z_0G_\ell\cdot z4,$$
$$z5 = z4+\frac{Z_\ell}{Z_0}\cdot\gamma;\quad \delta = \gamma + Z_0G_{\alpha,1}\cdot z5;\quad z6 = z5+\frac{Z_{a,1}}{Z_0}\cdot\delta, \tag{8b}$$

one gets

$$p2 = \frac{z5}{\left(z6 + (\delta + Z_0G_{\alpha,1}\cdot z6)\left(\frac{Z_{T,1}}{Z_0}+\frac{1}{\cos\Theta_0}\right)\right)}, \tag{9}$$

$$\begin{aligned} p_4 = z2\cdot\Bigg\{ & z5 + \frac{Z_{a,1}}{Z_0}\cdot\delta + \left[\frac{Z_{T,1}}{Z_0}+\frac{1}{\cos\Theta_0}\right] \\ & \cdot\Bigg[\delta + Z_0G_{\alpha,1}\cdot\Bigg(z5+\frac{Z_{a,1}}{Z_0}\cdot\Bigg(\gamma + Z_0G_{\alpha,1}\cdot\Bigg(z2+\frac{Z_\ell}{Z_0}\cdot\Bigg(\alpha + Z_0G_{\alpha,2}\cdot\Bigg(z1 \\ & + \frac{Z_{a,2}}{Z_0}\cdot\alpha\Bigg)\Bigg)\Bigg) + \frac{Z_\ell}{Z_0}\cdot\Bigg(\beta + Z_0G_\ell\cdot\Bigg(z2+\frac{Z_\ell}{Z_0}\cdot\Bigg(\alpha+Z_0G_{\alpha,2}\cdot\Bigg(z1 \\ & + \frac{Z_{a,1}}{Z_0}\cdot(\cos\Theta_0 + Z_0G_{\alpha,2}\cdot z_1^2)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg]\Bigg\}^{-1}. \end{aligned}$$

Thus the square of the sound pressure magnitude in the cabin will be (k_x, k_y real):

$$|p_i(x,y)|^2 = \frac{4|P_e|^2}{\sin^2(k_x t)} \left[|p_2|^2 \cdot \sin^2(k_x(x-d_{a1}-t)) + |p_4|^2 \cdot \sin^2(k_x(x-d_{a1})) \right. \\ \left. - 2\mathrm{Re}\{p_2 p_4^*\} \cdot \sin(k_x(x-d_{a1}-t)) \cdot \sin(k_x(x-d_{a1})) \right] \tag{10}$$

with an average

$$\langle |p_i(x,y)|^2 \rangle = \frac{|P_e|^2}{\sin^2(k_x t)} \left[2\left(|p_2|^2 + |p_4|^2\right)\left(1 - \frac{\sin(2k_x t)}{2k_x t}\right) \right. \\ \left. - 4\mathrm{Re}\{p_2 p_4^*\}\left(\cos(k_x t) - \frac{\sin(k_x t)}{k_x t}\right) \right] . \tag{11}$$

The sound protection measure ΔL is, finally:

$$\Delta L = -10 \cdot \lg \left\{ \frac{1}{\sin^2(k_x t)} \left[2\left(|p_2|^2 + |p_4|^2\right)\left(1 - \frac{\sin(2k_x t)}{2k_x t}\right) \right. \right. \\ \left. \left. - 4\mathrm{Re}\{p_2 p_4^*\}\left(\cos(k_x t) - \frac{\sin(k_x t)}{k_x t}\right) \right] \right\} . \tag{12}$$

With the sound pressure in the cabin centre:

$$|p_i(x = d_{a1} + t/2, y)|^2 = \frac{4|P_e|^2 \sin^2(k_x t/2)}{\sin^2(k_x t)} \left[|p_2|^2 + |p_4|^2 + 2\mathrm{Re}\{p_2 p_4^*\} \right] , \tag{13}$$

the sound protection measure ΔL_0 becomes:

$$\Delta L_0 = -10 \cdot \lg \frac{2\,(1-\cos(k_x t))}{\sin^2(k_x t)} \left[|p_2|^2 + |p_4|^2 + 2\mathrm{Re}\{p_2 p_4^*\} \right] . \tag{14}$$

Until now it was tacitly assumed that the shadow field of the cabin, i. e. the field behind the cabin which is generated there by scattering of p_e, was much lower than the sound pressure of the incident wave at the front side of the cabin. A different extreme situation would be a scattered sound field behind the cabin, which would be strong enough to inhibit the radiation of the sound which has traversed the cabin at the back side of the cabin. Then the equivalent network ends in the node with p_5. One gets in this case with the abbreviations

$$\alpha = 1 + G_{\alpha,2} \cdot Z_{\alpha,2} ; \quad g1 = Z_0 G_{\alpha,2} \cdot (1+\alpha) , \\ \beta = \alpha + \frac{Z_\ell}{Z_0} \cdot g1 ; \quad g2 = g1 + Z_0 G_\ell \cdot \beta , \\ \gamma = \beta + \frac{Z_\ell}{Z_0} \cdot g2 ; \quad g3 = g2 + Z_0 G_{\alpha,1} \cdot \gamma \quad ; \quad \delta = \gamma + \frac{Z_{\alpha,1}}{Z_0} \cdot g3 \tag{15}$$

the node pressures:

$$p_2 = \gamma \cdot \left[\delta + \left(g3 + Z_0 G_{\alpha,1}\left(\gamma + Z_0 G_{\alpha,1} \cdot g3\right)\right)\left(\frac{Z_{T1}}{Z_0} + \frac{1}{\cos\Theta_0}\right) \right]^{-1} , \\ p_4 = \alpha \cdot \left[\delta + \left(g3 + Z_0 G_{\alpha,1} \cdot \delta\right)\left(\frac{Z_{T1}}{Z_0} + \frac{1}{\cos\Theta_0}\right) \right]^{-1} . \tag{16}$$

The sound protection measures then follow as above. Numerical checks in a number of examples have shown that the results agree with the former results within the precision of graphical representation.

A further special case can be easily treated: a *coherent* sound incidence with equal strength takes place on both sides of the cabin. Then the equivalent circuit ends in the node with p_3 after substitution $t \to t/2$. The required node pressures are:

$$
\begin{aligned}
p_2 &= \alpha \cdot \left[\beta + \left(g1 + Z_0 G_{\alpha,1} \cdot \beta \right) \left(\frac{Z_{T1}}{Z_0} + \frac{1}{\cos\Theta_0} \right) \right]^{-1} , \\
p_4 &= \left[\beta + \left(g1 + Z_0 G_{\alpha,1} \cdot \beta \right) \left(\frac{Z_{T1}}{Z_0} + \frac{1}{\cos\Theta_0} \right) \right]^{-1}
\end{aligned}
\tag{17}
$$

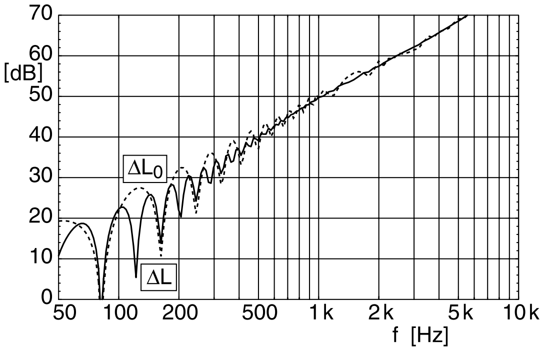
with the abbreviations

$$
\alpha = 1 + G_\ell \cdot Z_\ell ; \quad g1 = Z_0 G_\ell + Z_0 G_{\alpha,1} \cdot \alpha ; \quad \beta = \alpha + \frac{Z_{\alpha,1}}{Z_0} \cdot g1 . \tag{18}
$$

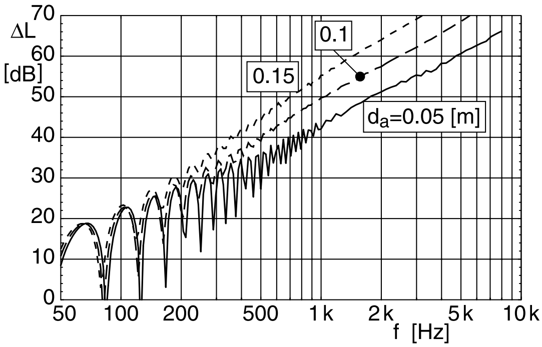
ΔL follows as before, except for an additional factor $1/2$ in the argument of the logarithm.

With *incoherent* sound incidence from both sides the contributions of each side to the interior sound pressure magnitude are evaluated separately and added.

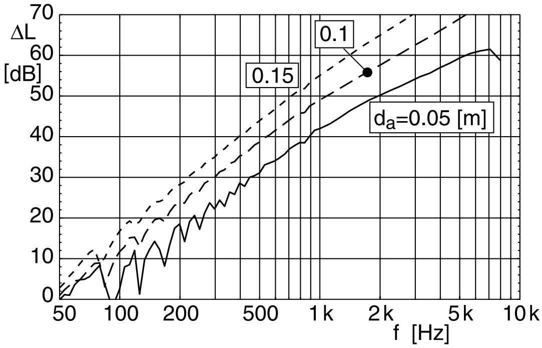
In the examples shown below, the walls on both sides of the cabin are equal, for simplicity.



The two sound pressure measures ΔL and ΔL_0 in a 2-dimensional cabin. Parameters: $\Theta_0 = 0°$; $t = 4$ [m]; $d_a = 0.1$ [m]; $\Xi = 10$ [kPa $\cdot$ s/m^2]; $d_p = 0.0015$ [m]; $f_{cr} d_p = 12.3$ [Hz $\cdot$ m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



Influence of the absorber layer thickness d_a on the sound protection measure ΔL for normal sound incidence on the cabin wall. Parameters: $\Theta_0 = 0°$; $t = 4$ [m]; d_a = var.; $\Xi = 10$ [kPa·s/m^2]; $d_p = 0.0015$ [m]; $f_{cr} d_p = 12.3$ [Hz·m]; $\rho_p = 7850$ [kg/m^3]; $\eta = 0.02$



As above, but for diffuse sound incidence

L.7 Cabin with Rectangular Cross Section

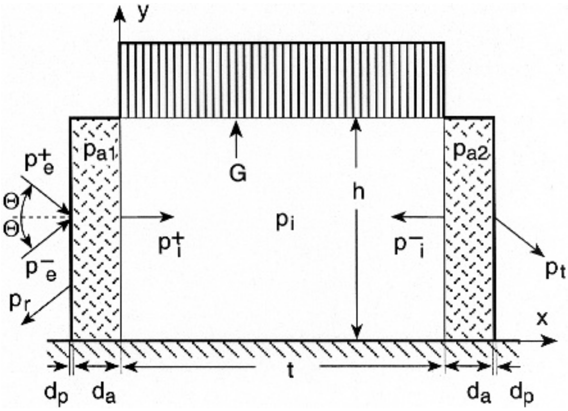
► *See also:* Mechel, Vol. III, Ch. 20 (1998)

The cabin is still two-dimensional, but with a ceiling which (for simplicity) is locally reacting with a surface admittance G.

The incident wave p_e with $p_e = p_e^+ + p_e^-$ considers also the ground reflection.

The field p_i inside the cabin is formulated as a mode sum of a locally lined flat duct with the cabin ceiling as lining. The lateral wave numbers ε_m are solutions of the characteristic equation of the duct.

With the usual dimensions of cabins it makes no significant difference whether the back wall radiates a wave p_t into the free space or if the back wall has a hard termination. Therefore this possibility of simplification will be used below.



Field formulations:

$$\begin{aligned}
p_e &= P_e \cdot e^{-jk_x x} \cdot \cos(k_y y)\,; \quad k_x = k_0 \cdot \cos\Theta; \quad k_y = k_0 \cdot \sin\Theta\,, \\
p_r &= P_r \cdot e^{+jk_x x} \cdot \cos(k_y y)\,, \\
p_i &= \sum_m \left[A_m \cdot e^{-\gamma_m x} + B_m \cdot e^{+\gamma_m x}\right] \cdot \cos(\varepsilon_m y)\,; \quad \gamma_m = \sqrt{\varepsilon_m^2 - k_0^2}\,, \\
p_{a1} &= \sum_m \left[P_{am} \cdot e^{-\gamma_{am} x} + Q_{am} \cdot e^{+\gamma_{am} x}\right] \cdot \cos(\varepsilon_m y)\,; \quad \gamma_{am} = \sqrt{\varepsilon_m^2 - \Gamma_a^2}\,, \\
p_{a2} &= \sum_m R_{am} \cosh\left(\gamma_{am}(x - t - d_a)\right) \cdot \cos(\varepsilon_m y)\,.
\end{aligned} \tag{1}$$

The exciting and the reflected field at $x = 0$ are synthesised with duct modes:

$$\begin{aligned}
p_e(0, y) &= P_e \cdot \cos(k_y y) = \sum_m P_{em} \cdot \cos(\varepsilon_m y)\,; \\
p_r(0, y) &= P_r \cdot \cos(k_y y) = \sum_m P_{rm} \cdot \cos(\varepsilon_m y)
\end{aligned} \tag{2}$$

with mode amplitudes:

$$P_{em} = \frac{P_e}{N_m} \cdot \frac{1}{h} \int_0^h \cos(k_y y) \cdot \cos(\varepsilon_m y)\, dy = \frac{S_m}{N_m} \cdot P_e \quad ; \qquad P_{rm} = \frac{S_m}{N_m} \cdot P_r \tag{3}$$

using mode norms: $$N_m = \frac{1}{h} \int_0^h \cos^2(\varepsilon_m y)\, dy = \frac{1}{2} \left(1 + \frac{\sin(2\varepsilon_m h)}{(2\varepsilon_m h)}\right) \tag{4}$$

and mode coupling coefficients: $$S_m = \frac{1}{2} \left(\frac{\sin\left((\varepsilon_m - k_y)h\right)}{(\varepsilon_m - k_y)h} + \frac{\sin\left((\varepsilon_m + k_y)h\right)}{(\varepsilon_m + k_y)h} \right). \tag{5}$$

The boundary conditions (which hold term-wise) at the front and back side walls

$$\begin{aligned}
&p_e(-d_a) + p_r(-d_a) - p_{a1}(-d_a) \overset{!}{=} Z_T \cdot v_{a1x}(-d_a)\,,\\
&v_{ex}(-d_a) + v_{rx}(-d_a) \overset{!}{=} v_{a1x}(-d_a)\,,\\
&p_{a1}(0) \overset{!}{=} p_i(0); \quad v_{a1x}(0) \overset{!}{=} v_{ix}(0)\,,\\
&p_i(t) \overset{!}{=} p_{a2}(t); \quad v_{ix}(t) \overset{!}{=} v_{a2x}(t)\,,
\end{aligned} \tag{6}$$

(Z_T is the partition impedance of an outer cover plate of the walls; see previous Sect. L.6) lead to the following system of equations:

$$\begin{pmatrix} \ddots & \cdots & \iddots \\ \vdots & a_{i,k} & \vdots \\ \iddots & \cdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} P_{rm} \\ A_m \\ B_m \\ P_{am} \\ Q_{am} \\ R_{am} \end{pmatrix} = e^{+jk_x d_a} \cdot P_{em} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

using the matrix coefficients:

$$\begin{aligned}
&a_{1,1} = -e^{-jk_x d_a}; \quad a_{1,2} = a_{1,3} = a_{1,6} = 0;\\
&a_{1,4} = e^{+\gamma_{am} d_a}\left(1 + \frac{\gamma_{am} d_a Z_T/Z_0}{k_0 d_a \cdot \Gamma_{an} Z_{an}}\right); \quad a_{1,5} = e^{-\gamma_{am} d_a}\left(1 - \frac{\gamma_{am} d_a Z_T/Z_0}{k_0 d_a \cdot \Gamma_{an} Z_{an}}\right),\\
&a_{2,1} = +e^{-jk_x d_a}; \quad a_{2,2} = a_{2,3} = a_{2,6} = 0;\\
&a_{2,4} = e^{+\gamma_{am} d_a}\frac{\gamma_{am} d_a}{k_x d_a \cdot \Gamma_{an} Z_{an}}; \quad a_{2,5} = -e^{-\gamma_{am} d_a}\frac{\gamma_{am} d_a}{k_x d_a \cdot \Gamma_{an} Z_{an}},\\
&a_{3,1} = a_{3,6} = 0; \quad a_{3,2} = 1; \quad a_{3,3} = e^{-\gamma_m t}; \quad a_{3,4} = a_{3,5} = -1\,,\\
&a_{4,1} = a_{4,6} = 0; \quad a_{4,2} = 1; \quad a_{4,3} = -e^{-\gamma_m t};\\
&a_{4,4} = -\frac{j\gamma_{am} d_a/\gamma_m t}{\Gamma_{an} Z_{an}}\frac{t}{d_a}; \quad a_{4,5} = +\frac{j\gamma_{am} d_a/\gamma_m t}{\Gamma_{an} Z_{an}}\frac{t}{d_a},\\
&a_{5,1} = a_{5,4} = a_{5,5} = 0; \quad a_{5,2} = e^{-\gamma_m t}; \quad a_{5,3} = 1;\\
&a_{5,6} = \frac{-1}{2}\left(e^{+\gamma_{am} d_a} + e^{-\gamma_{am} d_a}\right),\\
&a_{6,1} = a_{6,4} = a_{6,5} = 0; \quad a_{6,2} = e^{-\gamma_m t}; \quad a_{6,3} = -1;\\
&a_{6,6} = \frac{-j\gamma_{am} d_a/\gamma_m t}{2\Gamma_{an} Z_{an}}\frac{t}{d_a}\left(e^{+\gamma_{am} d_a} - e^{-\gamma_{am} d_a}\right).
\end{aligned} \tag{8}$$

With the abbreviations

$$X_m := j\Gamma_{an} Z_{an} \cdot \gamma_m t + \gamma_{am} t; \quad Y_m := j\Gamma_{an} Z_{an} \cdot \gamma_m t - \gamma_{am} t\,, \tag{9}$$

the required amplitudes A_m, B_m will be:

$$\begin{aligned}
A_m = {} & 4\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x t \cdot \gamma_{am} d_a \left(X_m \cdot e^{+\gamma_{am} d_a} + Y_m \cdot e^{-\gamma_{am} d_a}\right) \cdot e^{+jk_x d_a} \cdot P_{em} \\
& \cdot \Big\{\Big[\left(e^{-2\gamma_{am} d_a} + e^{-2\gamma_m t}\right) \cdot Y_m + \left(1 + e^{-2(\gamma_{am} d_a + \gamma_m t)}\right) \cdot X_m\Big] \\
& \cdot \Big[\left(\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x d_a - \gamma_{am} d_a \cdot (k_0 d_a + k_x d_a \cdot Z_T/Z_0)\right) \cdot X_m \\
& - \left(\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x d_a + \gamma_{am} d_a \cdot (k_0 d_a + k_x d_a \cdot Z_T/Z_0)\right) \cdot e^{+2\gamma_{am} d_a} \cdot Y_m\Big]\Big\}^{-1} ,
\end{aligned} \tag{10}$$

$$\begin{aligned}
B_m = {} & 4\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x t \cdot \gamma_{am} d_a \left(X_m \cdot e^{-\gamma_{am} d_a} + Y_m \cdot e^{+\gamma_{am} d_a}\right) \cdot e^{-\gamma_m t} \cdot e^{+jk_x d_a} \cdot P_{em} \\
& \cdot \Big\{\Big[\left(e^{-2\gamma_{am} d_a} + e^{-2\gamma_m t}\right) \cdot Y_m + \left(1 + e^{-2(\gamma_{am} d_a + \gamma_m t)}\right) \cdot X_m\Big] \\
& \cdot \Big[\left(\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x d_a - \gamma_{am} d_a \cdot (k_0 d_a + k_x d_a \cdot Z_T/Z_0)\right) \cdot X_m \\
& - \left(\Gamma_{an} Z_{an} \cdot k_0 d_a \cdot k_x d_a + \gamma_{am} d_a \cdot (k_0 d_a + k_x d_a \cdot Z_T/Z_0)\right) \cdot e^{+2\gamma_{am} d_a} \cdot Y_m\Big]\Big\}^{-1} .
\end{aligned} \tag{11}$$

References

Mechel, F.P.: Schallabsorber, Vol. III, Ch. 20: Capsules and cabins. Hirzel, Stuttgart (1998)