MATH 473: EULER'S FORMULA

Here is Euler's Formula

Proposition 1. *If x is a real number, and $i = \sqrt{-1}$, then*

$$e^{ix} = \cos x + i \sin x.$$

Idea of proof. The Taylor series of the exponential function e^z is still valid if z is a complex number. We will also need the formula for powers of i:

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad i^7 = -i, \quad i^8 = 1 \dots$$

Note the formula is periodic in the exponent with period 4, so that $i^{n+4} = i^n i^4 = i^n \cdot 1 = i^n$.

So compute the Taylor series of e^{ix}:

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\
&= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \cos x + i \sin x,
\end{aligned}$$

where the last line follows by recognizing the Taylor series for $\cos x$ and $\sin x$. $\square$

Euler's formula is useful in terms of deriving more difficult trigonometric formulas from easier formulas for the exponential function. To proceed, note that $\cos x$ is the real part of the complex number e^{ix} and $\sin x$ is the imaginary part of the complex number e^{ix}. We can write

$$\cos x = \mathrm{Re}(e^{ix}), \qquad \sin x = \mathrm{Im}(e^{ix}).$$

We can use these formulas to derive some of the standard trigonometric formulas.

Lemma 2. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Proof. Compute

$$\begin{aligned}
\cos(\alpha+\beta) &= \mathrm{Re}(e^{i(\alpha+\beta)}) \\
&= \mathrm{Re}(e^{i\alpha+i\beta}) \\
&= \mathrm{Re}(e^{i\alpha}e^{i\beta}) \\
&= \mathrm{Re}(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= \mathrm{Re}(\cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + i^2\sin\alpha\sin\beta) \\
&= \mathrm{Re}(\cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta - \sin\alpha\sin\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta.
\end{aligned}$$

□

Note the same proof shows, by consider the imaginary part of $e^{i(\alpha+\beta)}$, that

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

To prove more trigonometric formulas, it is useful to have more a more explicit characterization of the real and imaginary parts of e^{ix}. First of all, note that

$$e^{-ix} = \cos x - i\sin x.$$

This follows by Euler's Formula and that $\cos x$ is an even function and $\sin x$ is an odd function. Then we can easily compute from Euler's Formula

$$\cos x = \tfrac{1}{2}e^{ix} + \tfrac{1}{2}e^{-ix}, \qquad \sin x = -\tfrac{i}{2}e^{ix} + \tfrac{i}{2}e^{-ix}.$$

These allow us to prove some basic formulas about the integrals and derivatives of trigonometric functions. For example we can verify the formula for the derivative of $\cos x$ by using the formula for the derivative of e^x, Euler's Formula, and the chain rule:

Proposition 3. *$\frac{d}{dx}\cos x = -\sin x$.*

Proof. Compute

$$\begin{aligned}
\frac{d}{dx}\cos x &= \frac{d}{dx}(\tfrac{1}{2}e^{ix} + \tfrac{1}{2}e^{-ix}) \\
&= \tfrac{1}{2}\frac{d}{dx}e^{ix} + \tfrac{1}{2}\frac{d}{dx}e^{-ix} \\
&= \tfrac{1}{2}e^{ix}\cdot i + \tfrac{1}{2}e^{-ix}\cdot(-i) \\
&= -(-\tfrac{i}{2}e^{ix} + \tfrac{i}{2}e^{-ix}) \\
&= -\sin x.
\end{aligned}$$

□

Finally, we compute an example of the integral formulas used in the orthogonality relations of trigonometric polynomials:

Proposition 4. *Let m, n be distinct positive integers. Then*

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0.$$

Proof. Since m and n are distinct and positive, we see that $n-m$ and $n+m$ are both nonzero. Compute

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \int_{-\pi}^{\pi} \tfrac{1}{2}(e^{imx} + e^{-imx})\tfrac{1}{2}(e^{inx} + e^{-inx}) \, dx \\
&= \tfrac{1}{4} \int_{-\pi}^{\pi} (e^{i(m+n)x} + e^{i(m-n)x} + e^{i(n-m)x} + e^{-i(m+n)x}) \, dx \\
&= \tfrac{1}{4} \left(\frac{e^{i(m+n)x}}{i(m+n)} + \frac{e^{i(m-n)x}}{i(m-n)} + \frac{e^{i(n-m)x}}{i(n-m)} + \frac{e^{-i(m+n)x}}{-i(m+n)} \right) \Bigg|_{-\pi}^{\pi}
\end{aligned}$$

Now for each integer k, Euler's formula shows that the function $e^{ikx} = \cos kx + i \sin kx$ is periodic, so that $e^{ikx} = e^{ik(x+2\pi)}$. In particular,

$$e^{ikx}\Big|_{-\pi}^{\pi} = e^{ik\pi} - e^{-ik\pi} = \cos(k\pi) + i\sin(k\pi) - \cos(k\pi) + i\sin(k\pi) = 0.$$

Apply this to the above formula with $k = m+n, m-n, n-m, -(m+n)$ to complete the proof. □

To recap, here are the basic formulas:

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x, \\
\cos x &= \mathrm{Re}(e^{ix}) = \tfrac{1}{2}e^{ix} + \tfrac{1}{2}e^{-ix}, \\
\sin x &= \mathrm{Im}(e^{ix}) = -\tfrac{i}{2}e^{ix} + \tfrac{i}{2}e^{-ix}.
\end{aligned}$$