K Muffler Acoustics

with M.L. Munjal

The performance of acoustic mufflers relies heavily on reflections at duct discontinuities, such as steps of the cross section or of the duct lining, following each other in short distances. Such transitions, being mostly neglected in ➤ *Ch. J*, about long, homogeneous silencers (often called "industrial silencers"), together with the always necessary consideration of mean flow and often of high temperatures, give muffler acoustics a special character.

Conventions in the Present Chapter

Muffler acoustics preferably is formulated with the field quantities pressure p and volume flow velocity u in the duct. Therefore, mostly the volume flow impedance (or flow impedance), defined by the ratio $p/u = p/(v \cdot S)$, is used, where v is the particle velocity (as usual in this book) and S is the duct cross section. The flow impedance is indicated by underlining:

$$\underline{Z}_x = \frac{p_x}{u_x} = \frac{p_x}{v_x S_x},$$

where p_x, u_x, v_x, S_x respectively are the sound pressure, volume flow, particle velocity, and duct cross section at a position x of the duct. An exception to this rule may be the symbol $\underline{Z}_0 = \rho_0 c_0/S$, if the cross section S is unspecified, and $\underline{Z}_r$, which stands for a radiation impedance in the dimensions of a flow impedance.

It is convenient to formulate expressions for sound fields in a steady flow with *convected quantities* (➤ *Sect. K.1*). Such convected quantities will be indicated by an index c.

Matrix formulations play an important role in muffler acoustics. The conventions for writing a vector are $\{v\} = \{p, u\}$, and [M] for a matrix in the running text. A matrix equation may either be written as $\{u\} = [M] \cdot \{v\}$ or with the elements as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Most graphs in this chapter will indicate by points • u and • d duct cross sections just above the *upsound* and just below the *downsound* cross sections, respectively, between which a transformation matrix will be developed.

K.1 Acoustic Power in a Flow Duct

In the intake and exhaust systems of reciprocating internal combustion engines and compressors, and ducts of the heating, ventilation and air-conditioning (HVAC) systems, the meanflow velocity V is generally small enough for the flow to be assumed as incompressible. Thus, the meanflow Mach number M is limited to the range $M = V/c_0 < 0.2$. For linear waves in a duct with incompressible meanflow, thermodynamic variables p_0, ρ_0, T, and hence sound speed c_0 may be assumed to be constant. For plane waves in such a flow duct, the acoustic power is given by [Morfey (1971)]:

$$\Pi(M) = \langle pu \rangle + \frac{M}{\underline{Z}_0} \langle p^2 \rangle + M\underline{Z}_0 \langle u^2 \rangle + M^2 \langle pu \rangle . \tag{1}$$

With *convected field quantities* this can be written as follows:

$$\begin{aligned} \Pi(M) &= \langle p_c\, u_c \rangle , \\ p_c &= p + M\underline{Z}_0 u, \\ u_c &= u + Mp/\underline{Z}_0, \\ \underline{Z}_0 &= \rho_0\, c_0/S. \end{aligned} \tag{2}$$

Substitution of M = 0 yields the correponding relationships for a stationary medium.

p	= sound pressure;
u	= volume flow velocity;
v	= particle velocity;
V	= meanflow velocity;
M	= V/c_0 = meanflow Mach number;
S	= area of the duct cross-section;
$\underline{Z}_0$	= $\rho_0 c_0/S$ = characteristic flow impedance;
R	= reflection factor;

- subscript c denotes convected quantities;
- $\langle \dots \rangle$ denotes time average over a cycle;
- underlining denotes flow impedances

In terms of the amplitudes p(0), u(0) of the sound pressure p and the volume flow u at a position x = 0, and of the sound pressure amplitudes A, B of the forward wave (which is in the direction of V) and reflected/rearward wave components of a plane standing wave in the duct at the same position x = 0 in the duct:

$$\begin{aligned} &p(0) = A + B ; \quad u(0) = (A - B)/\underline{Z}_0, \\ &R = B/A; \quad R_c \equiv B_c/A_c = \frac{B(1-M)}{A(1+M)} = R\frac{1-M}{1+M}, \\ &A_c = A(1+M) ; \quad B_c = B(1-M), \\ &p_c(0) = A(1+M) + B(1-M) = A_c + B_c, \\ &u_c(0) = \frac{A(1+M) - B(1-M)}{\underline{Z}_0} = (A_c - B_c)/\underline{Z}_0. \end{aligned} \tag{3}$$

The acoustic power is:

$$\begin{aligned} \Pi(M) &= \frac{1}{2}\frac{|A|^2}{\underline{Z}_0}\left((1+M)^2 - |R|^2(1-M)^2\right) = \frac{1}{2}\frac{|A_c|^2}{\underline{Z}_0}\left(1 - |R_c|^2\right); \\ \Pi(0) &= \frac{1}{2}\frac{|A|^2}{\underline{Z}_0}\left(1 - |R|^2\right). \end{aligned} \tag{4}$$

For given sound pressure amplitudes the convective effect of meanflow is to strengthen the forward wave and weaken the reflected wave. The overall effect is to increase the net acoustic power, i.e. $\Pi(M) \geq \Pi(0)$. The reflection factor R, defined with the non-convective, *classical* field quantities, may exceed unity if R_c approaches unity [Mechel/Schiltz/Dietz (1965)].

K.2 Radiation from the Open End of a Flow Duct

The object is an unflanged open end of a duct of radius r_0 with a flow Mach number M in the duct. The radiation flow impedance of the orifice is given by [Levine/Schwinger (1948); Davies et al. (1980)]:

$$\begin{aligned} &\underline{Z}_r(M) = \underline{R}_r(M) + j\underline{X}_r(M), \\ &\underline{R}_r(M) \approx \underline{R}_r(0) - 1.1 \cdot M \cdot \underline{Z}_0, \\ &\underline{X}_r(M) \approx \underline{X}_r(0), \\ &\underline{Z}_r(0) = \underline{R}_r(0) + j\underline{X}_r(0) = \underline{Z}_0 \frac{1+R}{1-R}. \end{aligned} \tag{1}$$

The inside reflection factor R at the orifice is:

$$\begin{aligned} &R \;= |R|\, e^{j(\pi - 2k_0\Delta\ell)}, \\ &|R| \approx 1 + 0.01336\, k_0r_0 - 0.59079(k_0r_0)^2 + 0.33756(k_0r_0)^3 - 0.06432(k_0r_0)^4, \\ &\qquad 0 < k_0r_0 < 1.5. \end{aligned} \tag{2}$$

The relative end correction $\Delta\ell/r_0$ is approximately:

$$\begin{aligned} &\Delta\ell/r_0 = 0.6133 - 0.1168(k_0r_0)^2\,; \quad k_0r_0 \leq 0.5, \\ &\Delta\ell/r_0 = 0.6393 - 0.1104\, k_0r_0\,; \quad 0.5 < k_0r_0 < 2. \end{aligned} \tag{3}$$

At sufficiently low frequencies, such that $k_0r_0 < 0.5$, the stationary impedance radiation flow resistance can be approximated as:

$$\underline{Z}_r(0) \approx \underline{Z}_0 \left\{ \frac{k_0^2r_0^2}{4} + 0.6k_0r_0 \right\}; \quad k_0r_0 < 0.5. \tag{4}$$

This approximate expression is generally good enough to cover the entire range of plane wave propagation with the cut-off frequency given by $k_0r_{sh} = 1.84$, where r_{sh}, the radius of the muffler shell, is about three to four times r_0, the radius of the radiating tail pipe.

Working with the convective state variables, the convective radiation flow impedance is given by the formula:

$$\underline{Z}_{c,r} = \underline{Z}_0 \frac{\underline{Z}_r(M)/\underline{Z}_0 + M}{M\underline{Z}_r(M)/\underline{Z}_0 + 1}. \tag{5}$$

$\underline{Z}_r(M)$ differs from the corresponding stationary medium $\underline{Z}_r(0)$ not only because of the convective effect, but also the interaction of the outgoing (radiated) wave with an unstable cylindrical vortex layer of the meanflow jet [Munt (1990)]. In fact, the total acoustic power in the farfield Π_F is less than the acoustic power Π_T transmitted out [Howe (1980)], particularly at low frequencies:

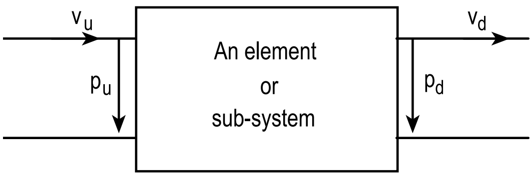
$$\Pi_F \approx \Pi_T \frac{(k_0 r_0)^2}{2M + (k_0 r_0)^2} .^{*)} \tag{6}$$

K.3 Transfer Matrix Representation

Transfer matrix representation is ideally suited for the analysis of cascaded one-dimensional systems like acoustic filters or mufflers. The performance of a muffler may be obtained readily in terms of the fourpole parameters or transfer matrix of the entire system, which in turn may be computed by means of successive multiplication of the transfer matrices of the constituent elements.

Transfer matrices of different elements constituting commercial mufflers are given in the subsequent sections of this chapter. Transformation from classical state variables p and u to the convective ones p_c and u_c may be obtained as follows.

Let $\{S\} = \{p, u\}^T$ and $\{S_c\} = \{p_c, u_c\}^T$. Below, vectors are denoted by braces { } and matrices by brackets [ ].



Subscripts u and d denote the *upstream* end and *downstream* end respectively of a muffler element, and

$$\begin{aligned} \{S\}_u &= [T]\{S\}_d , \\ \{S_c\}_u &= [T_c]\{S_c\}_d , \end{aligned} \tag{1}$$

where $[T]$, $[T_c]$ are transfer matrices, then $\quad [T_c] = [C]_u [T][C]_d^{-1}$. (2)

Conversely: $\quad [T] = [C]_u^{-1} [T_c][C]_d$, (3)

where $[C]$ is the transformation matrix: $\quad [C] = \begin{bmatrix} 1 & M\underline{Z}_0 \\ M/\underline{Z}_0 & 1 \end{bmatrix}$. (4)

Thus, one can work with classical state variables or convective state variables as per personal preference and skip from one system to the other at the end as necessary.

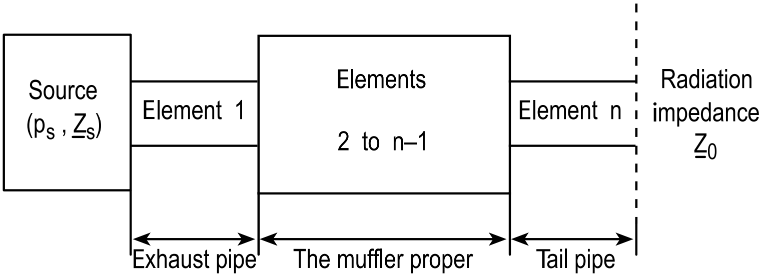
K.4 Muffler Performance Parameters

The performance of a muffler is measured in terms of one of the following parameters:

- insertion loss (IL),
- transmission loss (TL),
- level difference (LD) or noise reduction (NR).

*) See Preface to the 2nd edition.

Insertion loss (IL) is defined as the difference between the acoustic power radiated without any muffler and that with the muffler (inserted, as it were, between the source and the radiation load impedance), as shown in the figure.



Writing $\Pi(M)$ in the form which defines $\underline{R}_e$, [Prasad/Crocker (1983)]:

$$\Pi(M) = \frac{|u|^2}{2}\left(\underline{R}_r + M\underline{Z}_0 + M\left|\underline{Z}_r\right|^2/\underline{Z}_0\right) = \frac{|u|^2}{2}\underline{R}_e, \tag{1}$$

the *insertion loss* may be expressed in terms of the product transfer matrix parameters as:

$$IL = 20\ \log\left[\left(\frac{R_{e1}}{R_{en}}\right)^{1/2}\left|\frac{Z_{rn}T_{11} + T_{12} + Z_{rn}Z_sT_{21} + Z_sT_{22}}{Z_{r1} + Z_s}\right|\right] \tag{2}$$

with

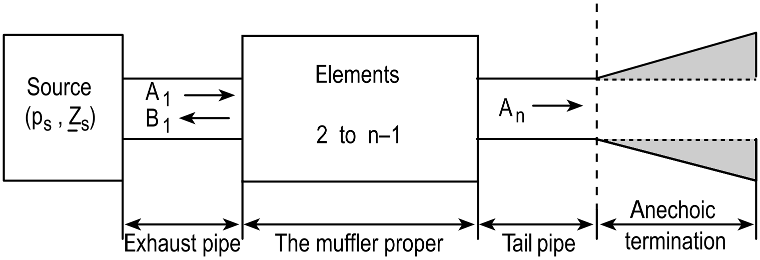
$$\begin{aligned}\underline{R}_{e1} &= \underline{R}_{r1} + M_1\underline{Z}_1 + M_1\,|Z_{r1}|^2/\underline{Z}_1,\\ \underline{R}_{en} &= \underline{R}_{rn} + M_n\underline{Z}_n + M_n\,|Z_{rn}|^2/\underline{Z}_n,\end{aligned} \tag{3}$$

where $\underline{Z}_{rn} = \underline{R}_{rn} + j\cdot\underline{X}_{rn}$ is the (flow) radiation impedance of the nth element, especially $\underline{Z}_{r1}$ that of the exhaust pipe without any muffler and $\underline{Z}_{rn}$ that of the tailpipe, and $\underline{Z}_n$ is the characteristic flow impedance in the nth element. Further, $\underline{Z}_s$ is the source (flow) impedance defined with respect to classical field variables. T_{11}, T_{12}, T_{21} and T_{22} are the fourpole parameters of the product transfer matrix of the entire muffler, obtained by successive multiplication of the transfer matrices of exhaust pipe (element number 1, next to the source), elements of the muffler proper, ending with the tailpipe (element n), as shown in the figure.

The *transmission loss* (TL) is independent of the source and presumes (or requires) an anechoic termination at the downstream end. It is defined as the difference between the power incident on the muffler proper and that transmitted downstream into the anechoic termination, as shown in the figure.

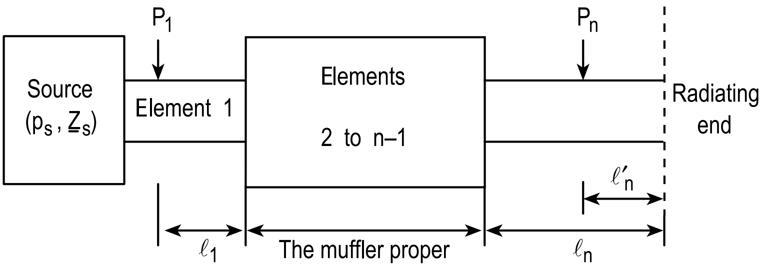
It is given by [Munjal (1987)]:

$$TL = 20\ \log\left[\left(\frac{\underline{Z}_n}{\underline{Z}_1}\right)^{1/2}\frac{(1+M_1)}{2(1+M_n)}\left|T_{11} + T_{12}/\underline{Z}_n + T_{21}\underline{Z}_1 + T_{22}\underline{Z}_1/\underline{Z}_n\right|\right], \tag{4}$$



where M_1 is the meanflow Mach number in the exhaust pipe, M_n is the meanflow Mach number in the tailpipe, and all other symbols carry the same connotations as in the expression for IL above.

The *level difference* (LD) (or noise reduction, NR) is the difference in sound pressure levels at two arbitrarily selected points in the exhaust pipe and tailpipe, as shown in the figure.



It is given by [Munjal (1987)]:

$$\mathrm{LD} = 20\ \log\left|\frac{T_{11} + T_{12}/Z_{rn}}{\cos(k_0\ell'_n) + j\underline{Z}_n \sin(k_0\ell'_n)/Z_{rn}}\right|, \tag{5}$$

where ℓ'_n is the distance of the downstream microphone from the radiation end of the tailpipe and $k_0 = \omega/c_0$; all other symbols carry the same connotations as in the expression for IL above.

K.5 Uniform Tube with Flow and Viscous Losses

For plane linear waves in a uniform-area tube with meanflow Mach number M and viscothermal and turbulent friction loss, the acoustic pressure and particle velocity distribution are given by [Panicker/Munjal, J. Acoust. Soc. India 9 (1981); Munjal (1987)]:

$$\begin{aligned} p(z) &= e^{jMk_cz}\left\{Ae^{-jk_cz} + Be^{jk_cz}\right\}, \\ u(z) &= \frac{e^{jMk_cz}}{\underline{Z}}\left\{Ae^{-jk_cz} - Be^{jk_cz}\right\}, \end{aligned} \tag{1}$$

where

$$k_c = \frac{k_0 + \alpha - j(\alpha + \xi M)}{1 - M^2} = \frac{k_0}{1 - M^2}\left\{1 + \frac{\alpha}{k_0} - j\frac{\alpha + \xi M}{k_0}\right\}, \tag{2}$$

η	=	dynqamic vicosity;
κ	=	adiabatic exponent;
K	=	heat conductivity;
ξ	=	F/2d;
F	=	Froude's friction factor;
d	=	$2r_0$ = hydraulic diameter;
Re	=	$Vd\rho_0/\eta$ = Reynold's number

$$\underline{Z} = \underline{Z}_0 \left\{ 1 + \frac{\alpha + \xi M}{k_0} - j\frac{\alpha + \xi M}{k_0} \right\},$$

$$\alpha = \frac{1}{r_0 c_0} \left(\frac{\omega \eta_e}{2\rho_0} \right)^{1/2},$$

$$\eta_e = \eta \left\{ 1 + \left(\kappa^{1/2} - \frac{1}{\kappa^{1/2}} \right) \left(\frac{K}{\eta C_p} \right)^{1/2} \right\}^2, \tag{3}$$

$$F = 0.0072 + \frac{0.612}{R_e^{0.35}}; \quad R_e < 4 \cdot 10^5.$$

On elimination of A and B from the field equations (1) for $z = 0$ (entrance, indicated by subscript u) and $z = l$ (exit of the duct with length ℓ, indicated by subscript d), one gets the desired transfer matrix relation:

$$\begin{bmatrix} p \\ u \end{bmatrix}_u = e^{-jMk_c\ell} \begin{bmatrix} \cos(k_c\ell) & j\underline{Z}\sin(k_c\ell) \\ (j/\underline{Z})\sin(k_c\ell) & \cos(k_c\ell) \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_d. \tag{4}$$

Incidentally, the same transfer matrix would hold for the convective state variables p_c and u_c at both ends.

For inviscid stationary medium, the transfer matrix for a uniform tube reduces to

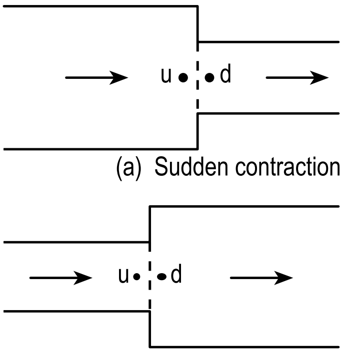
$$\begin{bmatrix} \cos(k_0\ell) & j\underline{Z}_0\sin(k_0\ell) \\ (j/\underline{Z}_0)\sin(k_0\ell) & \cos(k_0\ell) \end{bmatrix}. \tag{5}$$

K.6 Sudden Area Changes

Subscripts u and d indicate points just upstream and just downstream of the sudden area discontinuity. Typically, the meanflow Mach number M in the smaller diameter tube is $M < 0.2$.

For sudden expansion and sudden contraction, the equations of mass continuity, momentum balance, stagnation pressure drop and entropy fluctuations [Alfredson/Davies (1970); Panicker/Munjal, J. Indian Inst. Sc. 63, pp. 1–19 (1981)], for plane waves and incompressible mean flow, yield the following transfer matrix relation [Munjal (1987)]:

$$\begin{bmatrix} p_{c,u} \\ u_{c,u} \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{K_d M_d^2}{1 - M_d^2} & K_d M_d \underline{Z}_d \\ \dfrac{(\kappa - 1)K_d M_d^3}{(1 - M_d^2)\underline{Z}_d} & 1 - \dfrac{(\kappa - 1)K_d M_d^2}{1 - M_d^2} \end{bmatrix} \begin{bmatrix} p_{c,d} \\ u_{c,d} \end{bmatrix}, \tag{1}$$



(a) Sudden contraction

(b) Sudden expansion

where K_d is the K-factor indicating the drop in stagnation pressure of the incompressible mean flow in terms of the dynamic head $\rho_0 V_d^2/2$:

$$\begin{aligned} K_d &= (1 - S_d/S_u)/2 \quad \text{for sudden contraction,} \\ K_d &= [(S_d/S_u) - 1]^2 \quad \text{for sudden expansion.} \end{aligned} \tag{2}$$

If $M < 0.2$ in the smaller diameter tube, the foregoing transfer matrix may be approximated as [Munjal (1987)]:

$$\begin{bmatrix} p_{c,u} \\ u_{c,u} \end{bmatrix} = \begin{bmatrix} 1 & K_d M_d \underline{Z}_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{c,d} \\ u_{c,d} \end{bmatrix}. \tag{3}$$

This relation, on transformation to the classical state variables and on incorporating the low Mach number simplification, becomes:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} 1 & (1 + K_d) M_d \underline{Z}_d - M_u \underline{Z}_u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}. \tag{4}$$

In the simplified, approximate, form, the end-correction effect or evanescent higher-order mode effect can be incorporated readily as an inline lumped inertance [Sahasrabudhe et al. (1995)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} 1 & (1 + K_d) M_d \underline{Z}_d - M_u \underline{Z}_u + j\omega H(\alpha) \cdot 0.85 r_p / S_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{5}$$

where r_p is the radius of the narrower pipe,
$\Delta\ell = 0.85 r_p$ is the end correction of the orifice in a baffle wall,
$H(\alpha) = 1 - 1.25 r_p / r_{ch}$ for co-axial tubes,
r_{ch} is radius of the chamber (or the larger diameter tube).

If the junction of the pipe (the smaller diameter tube) and chamber (the larger diameter tube) is not co-axial, then the factor $H(\alpha)$ is given by the polynomial [Panicker, Munjal (1981)]:

$$\begin{aligned} H(\alpha) = {} & 1.442 + 3.516\,\delta - 5.403\,\alpha - 0.068\,kr - 11.067\,\delta^2 + 10.462\,\delta^2 - 0.099(kr)^2 \\ & + 2.517\,\delta\alpha - 0.197\,\delta \cdot kr + 1.024\,\alpha \cdot kr + 7.774\,\delta^3 - 8.15\,\alpha^3 - 0.05(kr)^3 \\ & - 0.841\,\delta^2\alpha + 0.131\,\delta^2 \cdot kr - 3.378\,\delta\alpha^2 - 1.311\,\alpha^2 \cdot kr \\ & + 0.141(kr)^2 \cdot \delta - 0.067(kr)^2\alpha - 0.031\,\delta \cdot \alpha \cdot kr, \end{aligned} \tag{6}$$

where $kr = k_0 r_{ch}$; $\alpha = r_p / r_{ch}$; $\delta = \text{offset distance}/(r_{ch} - r_p)$.

For the case of a stationary medium, the transfer matrix would reduce to

$$\begin{bmatrix} 1 & j\omega H(\alpha)0.85r_p/S_p \\ 0 & 1 \end{bmatrix}. \tag{7}$$

The description of sudden area changes would become simple if one neglected the evanescent higher-order mode effects, because then the transfer matrix would reduce to a unity matrix, implying $p_u = p_d$ and $u_u = u_d$.

Decomposing the standing wave on the entrance side into the forward moving and reflected progressive waves, for anechoic termination, one gets, [Munjal (1987)]:

$$\text{Reflection factor:} \quad R = \frac{\underline{Z}_d - \underline{Z}_u}{\underline{Z}_d + \underline{Z}_u} \approx \frac{S_u - S_d}{S_u + S_d}; \tag{8}$$

$$\text{Transmission loss:} \quad TL = 10\ \lg\left[\frac{(\underline{Z}_u + \underline{Z}_d)^2}{4\underline{Z}_u\underline{Z}_d}\right] \approx 10 \cdot \log\left[\frac{(S_d + S_u)^2}{4S_dS_u}\right]. \tag{9}$$

These relationships represent the principle of impedance mismatch, which is the underlying principle of reflective (or reactive, or non-dissipative) mufflers. When the characteristic impedance undergoes a sudden change or jump, a significant portion of the incident acoustic power is reflected back to the source. This impedance jump may be obtained in several ingenious ways, one of which is sudden area changes. The symmetry of the expression for TL shows that what matters is a sudden jump or change in characteristic impedance, not whether it increases (as in sudden contraction) or decreases (as in sudden expansion).

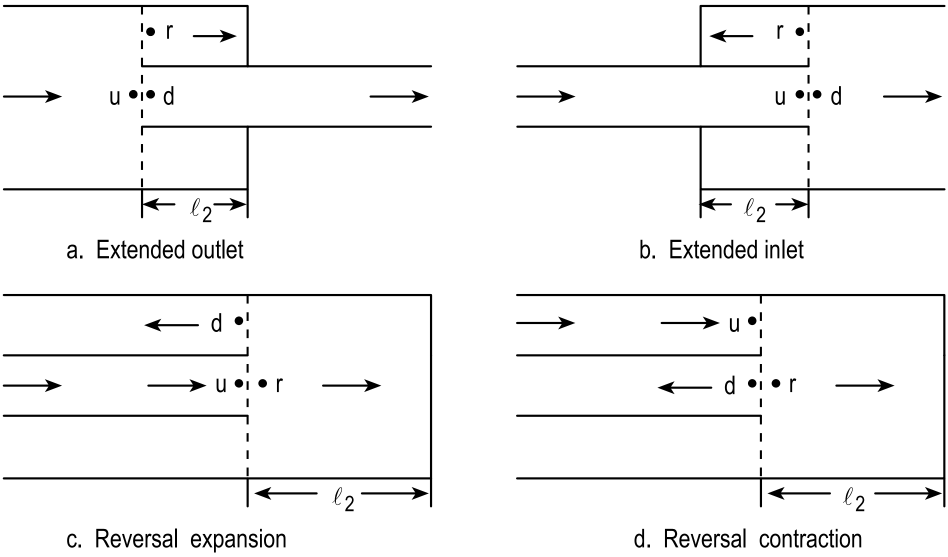
K.7 Extended Inlet/Outlet

▸ *See also:* *Sect. J.18* for simple expansions or contractions of lined ducts, *Sect. J.19* for sequences of duct sections without feedback (all those sections without flow).

Four types of extended-tube sudden area discontinuities are shown below. Equations of mass continuity, momentum balance, stagnation pressure drop and entropy fluctuations for plane waves and incompressible meanflow [Alfredson/Davies (1970); Panicker/Munjal, J. Indian Inst. Sc. 63 (A) (1981), pp. 1–19, 21–38] yield a transfer matrix relation that is a little too involved.

However, for typical cases where the meanflow Mach number M in the smaller diameter tube is $M < 0.2$, the transfer matrix relation between convective state variables p_c, u_c is given by [Munjal (1987)]:

$$\begin{bmatrix} p_{c,u} \\ u_{c,u} \end{bmatrix} = \begin{bmatrix} 1 & K_dM_d\underline{Z}_d \\ \dfrac{C_rS_r}{C_rS_rZ_r + S_uM_u\underline{Z}_u} & \dfrac{C_rS_rZ_r - M_d\underline{Z}_d(C_dS_d + K_dS_u)}{C_rS_rZ_r + S_uM_u\underline{Z}_u} \end{bmatrix} \begin{bmatrix} p_{c,d} \\ u_{c,d} \end{bmatrix}. \tag{1}$$



Therein

- subscripts u and d indicate points just upstream and just downstream of the sudden area discontinuity as shown in the figures;
- K_d is the K-factor indicating the drop in stagnation pressure of incompressible mean flow in terms of the dynamic head $1/2\,\rho_0 V_d^2$;
- S_x are cross-section areas at $x = u, d, r$;
- M_x are Mach numbers at these positions;
- $\underline{Z}_d, \underline{Z}_u, \underline{Z}_2$ are characteristic flow impedances;
- $\underline{Z}_r = -j\,\underline{Z}_2 \cot(k_0\ell_2)$ is the input flow impedance of the annular cavity resonator. (2)

$$\text{Thus,}\quad \begin{aligned} K_d &= (1 - S_d/S_u)/2 && \text{for extended outlet,}\\ &= (S_d/S_u - 1)^2 && \text{for extended inlet,}\\ &= (S_d/S_u)^2 && \text{for reversal expansion,}\\ &= 0.5 && \text{for reversal contraction.}\end{aligned} \tag{3}$$

The constants C_r and C_d are given by the area compatibility equation:

$$S_u + C_d S_d + C_r S_r = 0. \tag{4}$$

$$\text{Thus,}\quad \begin{aligned} C_d &= -1\,; & C_r &= -1 && \text{for extended outlet,}\\ C_d &= -1\,; & C_r &= +1 && \text{for extended inlet,}\\ C_d &= +1\,; & C_r &= -1 && \text{for reversal expansion,}\\ C_d &= +1\,; & C_r &= -1 && \text{for reversal contraction.}\end{aligned} \tag{5}$$

The foregoing transfer matrix relationship between the convective state variables $[T_c]$ can be transformed into the one between classical state variables $[T]$ making use of the relationship given in *Sect. K.3*:

$$[T] = [C_u]^{-1}[T_c][C_d], \tag{6}$$

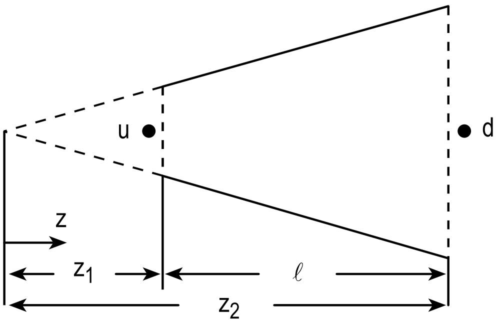
where $[C_x]$ are the transformation matrices.

K.8 Conical Tube

With tube length ℓ and tube radii r_u, r_d, the following relations are used:

$$z_1 = \frac{r_u}{r_d - r_u}\ell, \quad z_2 = z_1 + \ell,$$

$$\underline{Z}_d = \frac{\rho_0 c_0}{\pi r_d^2}.$$



(a) a conical tube

For a stationary medium, the transfer matrix relationship for a conical tube is given by [Munjal (1987)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} \frac{z_2}{z_1}\cos(k_0\ell) - \frac{\sin(k_0\ell)}{k_0 z_1} & j\underline{Z}_d \frac{z_2}{z_1}\sin(k_0\ell) \\ \frac{j}{\underline{Z}_d}\left\{\frac{z_1}{z_2}\left(1 + \frac{1}{k_0^2 z_1 z_2}\right)\sin(k_0\ell) - \left(1 - \frac{z_1}{z_2}\right)\frac{\cos(k_0\ell)}{k_0 z_2}\right\} & \frac{\sin(k_0\ell)}{k_0 z_2} + \frac{z_1}{z_2}\cos(k_0\ell) \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}. \quad (1)$$

These expression hold for a convergent tube as well as a divergent tube.

For a moving medium, assuming that the flare is small enough to avoid separation of boundary layer, Easwaran and Munjal (1992) have solved the wave equation with variable coefficients analytically to obtain the transfer matrix parameters for a conic tube. Dokumaci has obtained these parameters numerically by means of matrizants [Dokumaci (1998)]. The resultant expressions are rather complicated. Fortunately, however, the convective effect of incompressible meanflow (for $M < 0.2$) is negligible in the case of conical tubes as well as uniform tubes.

K.9 Exponential Horn

The medium is supposed to be stationary. The horn (or tube section) is characterised by the relations (with r(z) the radius at position z; m is the flare constant):

$$r(z) = r(0)e^{mz}\,; \quad S(z) = S(0)e^{2mz};$$

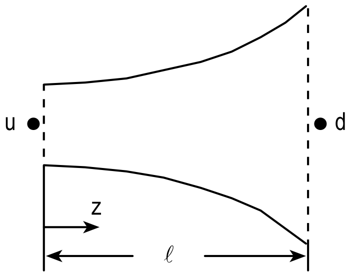
$$\underline{Z}_u = \frac{\rho_0 c_0}{\pi r_u^2}\,; \quad r_u = r(0); \tag{1}$$

$$\begin{bmatrix} p_u \\ \\ u_u \end{bmatrix} = \begin{bmatrix} e^{m\ell}\left(\cos(k'\ell) - \frac{m}{k'}\sin(k'\ell)\right) & je^{-m\ell}\frac{k_0}{k'}\underline{Z}_u \sin(k'\ell) \\ \\ \frac{j}{\underline{Z}_u}e^{m\ell}\frac{k_0}{k'}\sin(k'\ell) & e^{-m\ell}\left(\cos(k'\ell) + \frac{m}{k'}\sin(k'\ell)\right) \end{bmatrix} \begin{bmatrix} p_d \\ \\ u_d \end{bmatrix}, \tag{2}$$

where

$$k' = (k_0^2 - m^2)^{1/2}; \quad k_0 = \omega/c_0. \tag{3}$$

These expressions hold for a convergent horn as well as a divergent horn.



(b) a hyperbolic tube

For a moving medium, assuming that the flare is small enough to avoid separation of boundary layer, [Easwaran/Munjal (1992)] have solved the wave equation with variable coefficients analytically to obtain the transfer matrix parameters for an exponential tube. Dokumaci has obtained the same numerically by means of matrizants [Dokumaci (1998)]. The resultant expressions are rather too complicated. Fortunately, however, the convective effect of incompressible meanflow (for $M < 0.2$) is negligible in the case of exponential tubes as well as conical tubes.

K.10 Hose

A hose is a uniform tube of interior radius r_i with compliant walls. Incorporating the local wall impedance in the mass continuity equation, and the losses due to visco-thermal friction and turbulent eddies in the momentum equation, and neglecting entropy fluctuations for the linear plane waves, yields the following transfer matrix relationship [Munjal/Thawani (1996)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \frac{e^{j(k^+ - k^-)\ell}}{\underline{Z}^+ + \underline{Z}^-} \begin{bmatrix} \underline{Z}^- e^{-jk^+\ell} + \underline{Z}^+ e^{jk^-\ell} & \underline{Z}^+\underline{Z}^-(e^{+jk^-\ell} - e^{-jk^+\ell}) \\ e^{jk^-\ell} - e^{-jk^+\ell} & \underline{Z}^+ e^{-jk^+\ell} + \underline{Z}^- e^{jk^-\ell} \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}. \tag{1}$$

Here, the wave numbers are given by:

$$k^{\pm} = \frac{k_i}{1-M^2}\left[\left\{\left(1-j\frac{\alpha_M r_i + G_i}{k_i r_i}\right)^2 + (1-M^2)\left(\frac{\alpha_M r_i + G_i}{k_i r_i}\right)^2\right\}^{1/2} \mp M\left(1-j\frac{\alpha_M r_i + G_i}{k_i r_i}\right)\right]. \tag{2}$$

The subscript i refers to the medium inside the hose; superscripts + and − refer to the forward wave and reflected wave, respectively. The factor within braces is an empirical factor representing the effect of curvature on the inertance of the hose. The other terms are as follows:

$$k = \omega/c_i\,; \quad M = V/c_i\,;$$

$$\alpha_M = \alpha + \frac{MF}{4r_i}\,; \quad \alpha = \frac{1}{r_i c_0}\left(\frac{\omega\eta}{2\rho_0}\right)^{1/2}; \qquad G_i = \frac{\rho_i c_i}{Z_w}\,; \tag{3}$$

$$F = 0.0072 + \frac{0.612}{R_e^{0.35}}\,; \quad R_e = \frac{V d \rho_i}{\eta}\,; \quad d = 2r_i\,.$$

G_i is the normalised wall admittance, Z_w the wall impedance. The axial volume flow impedances are given by:

$$\underline{Z}^{\pm} = \underline{Z}_0 \frac{k_i \mp Mk^{\pm} - 2j\alpha_M}{k^{\pm}}\,; \quad \underline{Z}_0 = \frac{\rho_i c_i}{\pi r_i^2}\,. \tag{4}$$

The hose-wall impedance is given by $Z_w = Z_c + Z_m + Z_o$ with the component Z_c due to compliance (for thin hoses):

$$Z_c = \frac{E}{j\omega r_i} \Big/ \left(\frac{r_0^2 + r_i^2}{r_0^2 - r_i^2} + \mu\right) \approx \frac{E\,t}{j\,\omega r_i^2}\,, \tag{5}$$

and the component Z_m due to mass reactance:

$$Z_m = j\omega\rho_w t\left[1 + \left(\frac{0.025}{r_i}\right)^2\right]. \tag{6}$$

Here E is the complex elastic modulus, $E_r(\omega)\left(1 + j\eta(\omega)\right)$; $\eta(\omega)$ is the loss factor and $E_r(\omega)$ is the storage modulus of the hose wall material; μ is the Poisson ratio of the hose-wall material; ρ_w and t are the density and thickness of the wall.

Finally, Z_0 is the radiation impedance (subscript o refers to the outer medium):

$$Z_o = j\omega\frac{\rho_o}{k_{ro}}\frac{H_0^{(2)}(k_{ro} r_o)}{H_1^{(2)}(k_{ro} r_o)}, \tag{7}$$

where

$$H_i^{(2)}(z) = J_i(z) - jY_i(z); \quad i = 0 \text{ or } 1$$

are Hankel functions, and the wave numbers are:

$$k_{ro} = \left(\frac{4j\rho_i c_i k_i}{dZ_w} + k_0^2 - k_i^2\right)^{1/2}; \quad k_0 = \omega/c_0. \tag{8}$$

For the limiting case of rigid walls, $Z_w \to \infty$, the transfer matrix would reduce to that for a uniform tube with rigid walls (*Sect. K.5*) if one neglected α_M^2 with respect to k_i^2.

For the limiting case of a hose with inviscid stationary medium inside ($M \to 0$, $\alpha_M \to 0$), we get:

$$k^{\pm} = k_i \left\{1 - j\frac{G_i}{k_i r_i}\right\} = k_i \left\{1 - j\frac{\rho_i c_i}{Z_w k_i r_i}\right\}. \tag{9}$$

At low frequencies, Z_w would be dominated by its compliance component Z_c; then, for thin hoses, the wave number in either direction would be given by:

$$k = k_i \left(1 + \frac{r_i \rho_i c_i^2}{Et}\right). \tag{10}$$

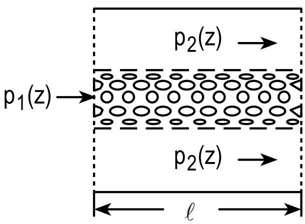
Writing $k = \omega/c_{eq}$, the equivalent sound speed in the fluid inside a hose is given by:

$$c_{eq} = \frac{c_i}{1 + \frac{r_i \rho_i c_i^2}{Et}} = \frac{c_i}{1 + \frac{r_i}{t} \cdot \frac{\rho_i c_i^2}{\rho_w c_w^2}}. \tag{11}$$

Thus, the effect of wall compliance is to match the characteristic impedance and then to reduce the effective velocity of wave propagation inside a hose pipe.

K.11 Two-Duct Perforated Elements

Concentric-tube resonators as well as concentric cross-flow elements have an inner region 1 and outer region 2 which are coupled with each other on a length ℓ via a perforated tube.



The common two-duct perforated section

The mass continuity and momentum equations in the inner region 1 and outer region 2 are combined with the isentropicity equation. These are solved simultaneously for harmonic time dependence, $\exp(j\omega t)$, to obtain coupled quadratic equations in $p_1(z)$ and

$p_2(z)$ [Munjal (1987)]. These are then reduced to four linear differential equations so as to apply the standard eigenvalue program [Peat (1988)]. Thence we obtain the following 4×4 transfer matrix relation [Munjal (1987)]:

$$\begin{bmatrix} p_1(0) \\ p_2(0) \\ \underline{Z}_1 u_1(0) \\ \underline{Z}_2 u_2(0) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} p_1(\ell) \\ p_2(\ell) \\ \underline{Z}_1 u_1(\ell) \\ \underline{Z}_2 u_2(\ell) \end{bmatrix}, \tag{1}$$

where $[T] = [A(0)]\,[A(1)]^{-1}$. Elements of the matrix $[A(z)]$ are given by $(i = 1, 2, 3, 4)$:

$$\begin{aligned} &A_{1,i} = \psi_{3,i} e^{\beta_i z}\,; \quad A_{2i} = \psi_{4i} e^{\beta_i z}\,; \\ &A_{3,i} = -\frac{e^{\beta_i z}}{jk_0 + M_1 \beta_i}\,; \quad A_{4i} = -\frac{\psi_{2,i} e^{\beta_i z}}{jk_0 + M_2 \beta_i}\,. \end{aligned} \tag{2}$$

$[\psi]$ and $\{\beta\}$ are respectively the eigenmatrix (or modal matrix) and eigenvector of the matrix:

$$\begin{bmatrix} -\alpha_1 & -\alpha_3 & -\alpha_2 & -\alpha_4 \\ -\alpha_5 & -\alpha_7 & -\alpha_6 & -\alpha_8 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} &\alpha_1 = -\frac{jM_1}{1 - M_1^2}\left(\frac{k_a^2 + k_0^2}{k_0}\right); \quad \alpha_2 = \frac{k_a^2}{1 - M_1^2}; \\ &\alpha_3 = \frac{jM_1}{1 - M_1^2}\left(\frac{k_a^2 - k_0^2}{k_0}\right); \quad \alpha_4 = -\left(\frac{k_a^2 - k_0^2}{1 - M_1^2}\right); \end{aligned} \tag{3a}$$

$$\begin{aligned} &\alpha_5 = \frac{jM_2}{1 - M_2^2}\left(\frac{k_b^2 - k_0^2}{k_0}\right); \quad \alpha_6 = -\left(\frac{k_b^2 - k_0^2}{1 - M_2^2}\right); \\ &\alpha_7 = -\frac{jM_2}{1 - M_2^2}\left(\frac{k_b^2 + k_0^2}{k_o}\right); \quad \alpha_8 = \frac{k_b^2}{1 - M_2^2} \end{aligned} \tag{3b}$$

with

$$\begin{aligned} &k_0 = \omega/c_0\,; \quad M_1 = V_1/c_0; \quad M_2 = V_2/c_0; \\ &k_a^2 = k_0^2 - \frac{4jk_0}{d_i \zeta}; \quad k_b^2 = k_0^2 - \frac{4jk_0 d_1}{(d_2^2 - d_1^2)\,\zeta}. \end{aligned} \tag{4}$$

ζ is the normalised partition impedance of the perforate. For different flow conditions, ζ is given by the following empirical expressions:

Perforates with cross flow [Sullivan (1979)]:

$$\zeta = \frac{p}{\rho_0 c_0 v} = \left[0.514\frac{d_1 M}{\ell \sigma} + j0.95 k_0 (t + 0.75 d_h)\right]/\sigma, \tag{5}$$

where d_1 is diameter of the perforated tube,
M is the mean-flow Mach number in the tube,
ℓ is the length of perforate,
σ is porosity,
f is frequency,
t is the thickness of the perforated tube,
d_h is the hole diameter,
v is the (average) radial particle velocity at the perforate.

For perforates in stationary media [Sullivan/Crocker (1978)]:

$$\zeta = [0.006 + jk_0(t + 0.75d_h)]/\sigma. \tag{6}$$

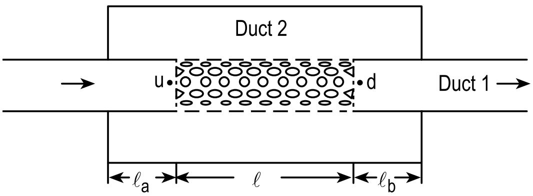
Perforates with grazing flow [Rao/Munjal (1986)]:

$$\zeta = [7.337 \times 10^{-3}(1 + 72.23M) + j2.2245 \times 10^{-5}(1 + 51t)(1 + 204d_h)f]/\sigma, \tag{7}$$

where wall thickness t and hole diameter d_h are in metres.

The desired 2×2 transfer matrix for a particular two-duct element may be obtained from the 4×4 matrix [T] by making use of the appropriate upstream and downstream variables and two boundary conditions characteristic of the element. The final results [Munjal (1987)] are given below for various two-duct elements shown in figures.

(a) Concentric-tube resonator:



(a) Concentric resonator

$$\begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_1(\ell) \\ \underline{Z}_1 u_1(\ell) \end{bmatrix} {}^{*)} \tag{8}$$

with

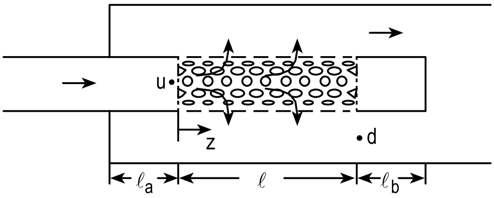
$$\begin{aligned} T_a &= T_{11} + A_1 A_2\,; \quad T_b = T_{13} + B_1 A_2\,; \\ T_c &= T_{31} + A_1 B_2\,; \quad T_d = T_{33} + B_1 B_2\,; \end{aligned} \tag{9}$$

$$\begin{aligned} A_1 &= (X_1 T_{21} - T_{41})/F_1\,; \quad B_1 = (X_1 T_{23} - T_{43})/F_1\,; \\ A_2 &= T_{12} + X_2 T_{14}\,; \quad B_2 = T_{32} + X_2 T_{34}\,; \end{aligned} \tag{10}$$

$$\begin{aligned} F_1 &= T_{42} + X_2 T_{44} - X_1(T_{22} + X_2 T_{24})\,; \\ X_1 &= -j\tan(k_0\ell_a)\,; \quad X_2 = +j\tan(k_0\ell_b)\,. \end{aligned} \tag{11}$$

*) See Preface to the 2nd edition.

(b) Cross-flow expansion element:



(b) Cross-flow expansion element

$$\begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_2(\ell) \\ \underline{Z}_2 u_2(\ell) \end{bmatrix} \tag{12}$$

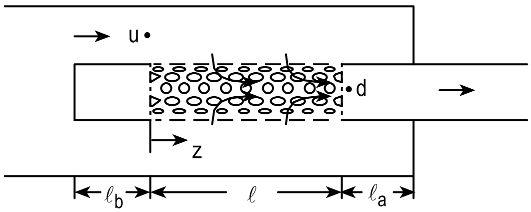
with

$$\begin{aligned} T_a &= T_{12} + A_1 A_2 \; ; \quad T_b = T_{14} + B_1 A_2 \; ; \\ T_c &= T_{32} + A_1 B_2 \; ; \quad T_d = T_{34} + B_1 B_2 \; ; \end{aligned} \tag{13}$$

$$\begin{aligned} A_1 &= (X_1 T_{22} - T_{42})/F_1 \; ; \quad B_1 = (X_1 T_{24} - T_{44})/F_1 \; ; \\ A_2 &= T_{11} + X_2 T_{13} \; ; \quad B_2 = T_{31} + X_2 T_{33} \; ; \end{aligned} \tag{14}$$

$$\begin{aligned} F_1 &= T_{41} + X_2 T_{43} - X_1(T_{21} + X_2 T_{23}) \; ; \\ X_1 &= -j \tan(k_0 \ell_a) \; ; \quad X_2 = j \tan(k_0 \ell_b) \; . \end{aligned} \tag{15}$$

(c) Cross-flow contraction element:



(c) Cross-flow contraction element

$$\begin{bmatrix} p_2(0) \\ \underline{Z}_2 u_2(0) \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_1(\ell) \\ \underline{Z}_1 u_1(\ell) \end{bmatrix} \tag{16}$$

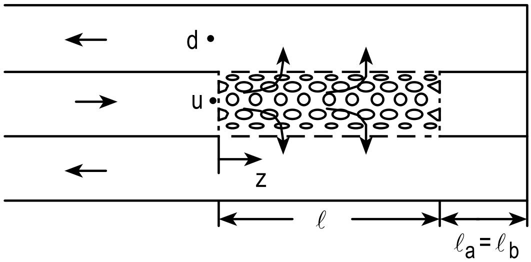
with

$$\begin{aligned} T_a &= T_{21} + A_1 A_2 \; ; \quad T_b = T_{23} + B_1 A_2 \; ; \\ T_c &= T_{41} + A_1 B_2 \; ; \quad T_d = T_{43} + B_1 B_2 \; ; \end{aligned} \tag{17}$$

$$\begin{aligned} A_1 &= (X_1 T_{11} - T_{31})/F_1 \; ; \quad B_1 = (X_1 T_{13} - T_{33})/F_1 \; ; \\ A_2 &= T_{22} + X_2 T_{24} \; ; \quad B_2 = T_{42} + X_2 T_{44} \; ; \end{aligned} \tag{18}$$

$$\begin{aligned} F_1 &= T_{32} + X_2 T_{34} - X_1 (T_{12} + X_2 T_{14}) \; ; \\ X_1 &= -j \tan(k_0 \ell_b) \; ; \quad X_2 = j \tan(k_0 \ell_b) \; . \end{aligned} \tag{19}$$

(d) Reverse-flow expansion element:



(d) Reverse-flow expansion element

$$\begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} = \begin{bmatrix} T_a & -T_b \\ T_c & -T_d \end{bmatrix} \begin{bmatrix} p_2(0) \\ \underline{Z}_2 u_2(0) \end{bmatrix} . \tag{20}$$

The minus sign with T_b and T_d is due to reversal in the direction of $u_2(0)$, which is needed for making the foregoing relation adaptable to similar relations for other downstream elements.

Therein:

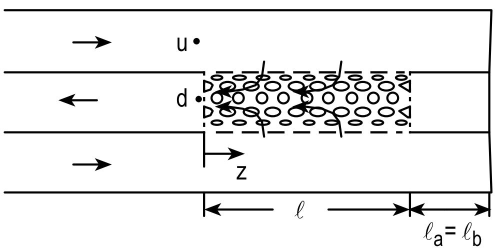
$$\begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}^{-1} ; \tag{21}$$

$$\begin{aligned} A_1 &= T_{11} + X_2 T_{13} \; ; \quad A_2 = T_{12} + X_2 T_{14} \; ; \\ A_3 &= T_{31} + X_2 T_{33} \; ; \quad A_4 = T_{32} + X_2 T_{34} \; ; \end{aligned} \tag{22}$$

$$\begin{aligned} B_1 &= T_{21} + X_2 T_{23} \; ; \quad B_2 = T_{22} + X_2 T_{24} \; ; \\ B_3 &= T_{41} + X_2 T_{43} \; ; \quad B_4 = T_{42} + X_2 T_{44} \; ; \end{aligned} \tag{23}$$

$$X_2 = j \tan(k_0 \ell_b) \; . \tag{24}$$

(e) Reverse-flow contraction element:



(e) Reverse-flow contraction element

$$\begin{bmatrix} p_2(0) \\ \underline{Z}_2 u_2(0) \end{bmatrix} = \begin{bmatrix} -T_a & -T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} \tag{25}$$

with

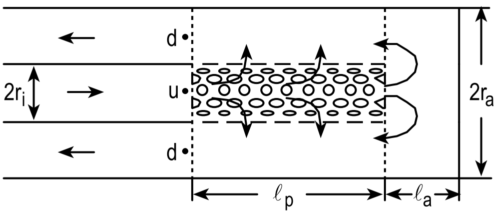
$$\begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}; \tag{26}$$

$$\begin{aligned} A_1 &= T_{32} + X_a T_{12}\ ; \quad A_2 = T_{34} + X_a T_{14}\ ; \\ A_3 &= T_{42} + X_a T_{22}\ ; \quad A_4 = T_{44} + X_a T_{24}\ ; \end{aligned} \tag{27}$$

$$\begin{aligned} B_1 &= T_{31} + X_a T_{11}\ ; \quad B_2 = T_{33} + X_a T_{21}\ ; \\ B_3 &= T_{41} + X_a T_{21}\ ; \quad B_4 = T_{43} + X_a T_{23}\ ; \end{aligned} \tag{28}$$

$$X_a = j \tan(k_0 \ell_a)\ . \tag{29}$$

(f) Reversal expansion, two-duct, open-end, perforated element:



(f) Reversal expansion, 2-duct, open-end perforated element

$$\begin{bmatrix} p_u \\ \underline{Z}_u u_u \end{bmatrix} = \begin{bmatrix} T_a & -T_b \\ T_c & -T_d \end{bmatrix} \begin{bmatrix} p_d \\ \underline{Z}_d u_d \end{bmatrix} \tag{30}$$

with

$$\begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{-1}; \tag{31}$$

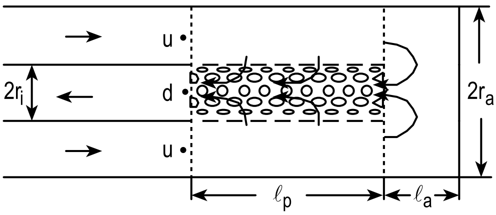
$$\begin{aligned} A_{11} &= T_{11} F_{11} + T_{12} + T_{13} F_{21}\ ; \\ A_{12} &= T_{11} F_{12} + T_{14} + T_{13} F_{22}\ ; \\ A_{21} &= T_{31} F_{11} + T_{32} + T_{33} F_{21}\ ; \\ A_{22} &= T_{31} F_{12} + T_{34} + T_{33} F_{22}\ ; \end{aligned} \tag{32}$$

$$\begin{aligned} B_{11} &= T_{21} F_{11} + T_{22} + T_{23} F_{21}\ ; \\ B_{12} &= T_{21} F_{12} + T_{24} + T_{23} F_{22}\ ; \\ B_{21} &= T_{41} F_{11} + T_{42} + T_{43} F_{21}\ ; \\ B_{22} &= T_{41} F_{12} + T_{44} + T_{43} F_{22}\ ; \end{aligned} \tag{33}$$

$$\begin{aligned} F_{11} &= E_{11}\ ; \quad F_{12} = -E_{12}/\underline{Z}_u\ ; \\ F_{21} &= E_{21} \underline{Z}_d\ ; \quad F_{22} = -E_{22} \underline{Z}_d / \underline{Z}_u\ . \end{aligned} \tag{34}$$

Matrix [E] is the 2 × 2 transfer matrix of the reversal expansion element of Sect. K.7(c). [T] is the 4 × 4 matrix for the perforated section of the two interacting ducts derived earlier [Eq. (1)].

(g) Reversal contraction, two-duct, open-end perforated element:



(g) Reversal contraction, 2-duct, open-end perforated element

$$\begin{bmatrix} p_u \\ \underline{Z}_u u_u \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ -T_c & -T_d \end{bmatrix} \begin{bmatrix} p_d \\ \underline{Z}_d u_d \end{bmatrix} \tag{35}$$

with

$$\begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{-1} ; \tag{36}$$

$$\begin{aligned} [B] &= [P] + [Q][F] ; \\ [C] &= [R] + [U][F] ; \end{aligned} \tag{37}$$

$$\begin{aligned} &P_{11} = A_{11} ; \quad P_{12} = A_{13} ; \quad P_{21} = A_{31} ; \quad P_{22} = A_{33} ; \\ &Q_{11} = A_{12} ; \quad Q_{12} = A_{14} ; \quad Q_{21} = A_{32}; \quad Q_{22} = A_{34} ; \\ &R_{11} = A_{12} ; \quad R_{12} = A_{23} ; \quad R_{21} = A_{41} ; \quad R_{22} = A_{43}; \\ &U_{11} = A_{22} ; \quad U_{12} = A_{24} ; \quad U_{21} = A_{42} ; \quad U_{22} = A_{44} ; \end{aligned} \tag{38}$$

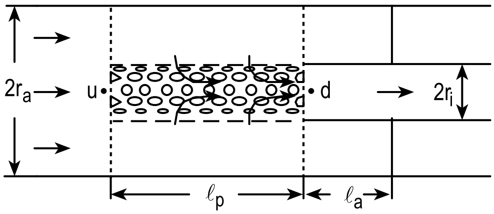
$$[A] = [T]^{-1}. \tag{39}$$

[T] is the 4 × 4 matrix for the perforated section of the two interacting ducts, derived above.

$$\begin{aligned} &F_{11} = D_{11} ; \qquad F_{12} = D_{12}/\underline{Z}_d ; \\ &F_{21} = D_{21}\underline{Z}_u ; \quad F_{22} = -D_{22}\underline{Z}_u/\underline{Z}_d . \end{aligned} \tag{40}$$

[D] is the 2 × 2 transfer matrix of the reversal contraction element of Sect. K.7(d).

(h) Perforated extended outlet:



(h) Perforated extended outlet

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12}\underline{Z}_d \\ C_{21}/\underline{Z}_u & C_{22}\underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix} \tag{41}$$

with

$$[C] = [A]^{-1}[B] \; ; \tag{42}$$

$$\begin{aligned} A_{11} &= 1 - f_3\underline{Z}_{2u} \; ; \quad A_{12} = M_u(1 - 2f_3\underline{Z}_{2u}) \; ; \\ A_{21} &= M_u - f_4\underline{Z}_{2u} \; ; \quad A_{22} = 1 - 2f_4\underline{Z}_{2u}M_u \; ; \end{aligned} \tag{43}$$

$$\begin{aligned} B_{11} &= T_{11} + M_dK_{p1}T_{31} - f_3\left[T_{21} + \underline{Z}_{21}\{T_{11} + 2M_uT_{31}\}\right]; \\ B_{12} &= T_{13} + M_dK_{p1}T_{33} - f_3\left[T_{23} + \underline{Z}_{21}\{T_{13} + 2M_uT_{33}\}\right]; \\ B_{21} &= \underline{Z}_{u2}T_{u1} + \underline{Z}_{u1}\{T_{31} + M_dT_{11}\} - \mathrm{num1}\cdot f_4 \; ; \\ B_{22} &= \underline{Z}_{u2}T_{43} + \underline{Z}_{u1}\{T_{33} + M_dT_{13}\} - \mathrm{num2}\cdot f_4 \; ; \end{aligned} \tag{44}$$

$$f_3 = \mathrm{num3}/\mathrm{den}; \quad f_4 = \mathrm{num4}/\mathrm{den}; \tag{45}$$

$$\begin{aligned} \mathrm{num1} &= T_{21} + \underline{Z}_{21}\{T_{11} + 2M_uT_{31}\}; \\ \mathrm{num2} &= T_{23} + \underline{Z}_{21}\{T_{13} + 2M_uT_{33}\}; \\ \mathrm{num3} &= T_{12} + X_a\,T_{14} + M_dK_{p1}\{T_{32} + X_a\,T_{34}\}; \\ \mathrm{num4} &= \underline{Z}_{u2}\{T_{42} + X_a\,T_{44}\} + \underline{Z}_{u1}[T_{32} + M_dT_{12} + X_a\{T_{34} + M_dT_{14}\}]; \end{aligned} \tag{46}$$

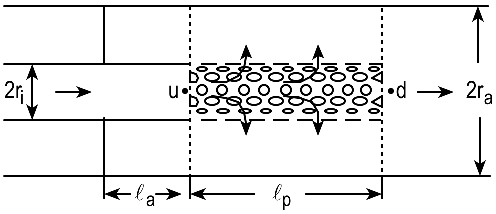
$$\mathrm{den} = T_{22} + X_a\,T_{24} + \underline{Z}_{21}[T_{12} + 2M_uT_{32} + X_a\{T_{14} + 2M_uT_{34}\}]; \tag{47}$$

$$\begin{aligned} \underline{Z}_{21} &= \underline{Z}_2/\underline{Z}_1 \; ; \quad \underline{Z}_{2u} = \underline{Z}_2/\underline{Z}_u \; ; \\ \underline{Z}_{u2} &= \underline{Z}_u/\underline{Z}_2 \; ; \quad \underline{Z}_{u1} = \underline{Z}_u/\underline{Z}_1 \; ; \\ \underline{Z}_d &= \rho_0c_0/S_d \; ; \quad \underline{Z}_u = \rho_0c_0/S_u \; ; \\ \underline{Z}_2 &= \rho_0c_0/S_a \; ; \quad \underline{Z}_1 = \underline{Z}_d \; ; \end{aligned} \tag{48}$$

$$\begin{aligned} & S_2 = S_u - S_d \; ; \quad X_a = j\tan(k\ell_a) \; ; \\ & K_c = \frac{1}{2}\left\{1 - \left(\frac{r_i}{r_{sh}}\right)^2\right\}; \quad K_{p1} = K_c + 1. \end{aligned} \tag{49}$$

[T] is the 4 × 4 matrix for the perforated section of the two interacting ducts, derived above.

(i) Perforated extended inlet:



(i) Perforated extended inlet

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12}\underline{Z}_d \\ D_{31}/\underline{Z}_u & D_{32}Z_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{50}$$

where [D] is a 4 × 2 matrix given by:

$$[D] = [T][A][B]. \tag{51}$$

[A] is a 4 × 4 matrix and [B] is a 4 × 2 matrix, elements of which are as follows:

$$\begin{aligned}
&A_{11} = 1\,; \quad A_{12} = 0\,; \quad A_{13} = M_u\,; \quad A_{14} = 0\,; \\
&A_{21} = M_1/\underline{Z}_w\,; \quad A_{22} = 0\,; \quad A_{23} = 1/\underline{Z}_u\,; \quad A_{24} = 1/\underline{Z}_d\,; \\
&A_{31} = 1/\underline{Z}_u\,; \quad A_{32} = 1/\underline{Z}_d\,; \quad A_{33} = 2M_u/\underline{Z}_u\,; \quad A_{34} = 0\,; \\
&A_{41} = T_{41} - X_a\,T_{21}\,; \quad A_{42} = T_{42} - X_a\,T_{22}\,; \\
&A_{43} = T_{43} - X_a\,T_{23}\,; \quad A_{44} = T_{44} - X_a\,T_{24}\,;
\end{aligned} \tag{52}$$

$$\begin{aligned}
&B_{11} = 1\,; \quad B_{12} = M_d(1 + K_e)\,; \quad B_{21} = M_d/\underline{Z}_d\,; \quad B_{22} = 1/\underline{Z}_d\,; \\
&B_{31} = 1/\underline{Z}_d\,; \quad B_{32} = 2M_d/\underline{Z}_d\,; \quad B_{41} = 0\,; \quad B_{42} = 0\,;
\end{aligned} \tag{53}$$

$$\underline{Z}_u = \rho_0 c_0/S_u; \quad \underline{Z}_d = \rho_0 c_0/S_d; \quad \underline{Z}_2 = \rho_0 c_0/(S_d - S_u); \tag{54}$$

$$X_a = -j\tan(k\ell_a); \quad K_e = \left(1 - (r_u/r_d)^2\right)^2. \tag{55}$$

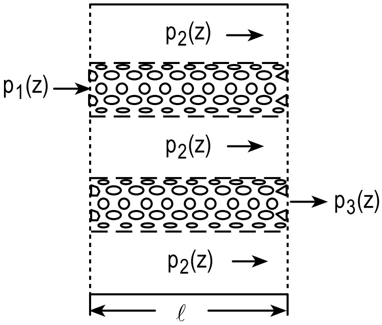
[T] is the 4 × 4 matrix for the perforated section of the two interactive ducts, derived above.

K.12 Three-Duct Perforated Elements

The two-duct perforated section shown in *Sect. K.11* is common to the two-duct perforated muffler elements, and it was treated in *Sect. K.11*.

Similarly, the three-duct perforated section shown below is common to the three-duct muffler elements discussed below.
This common three-duct perforated section is described by the transfer matrix relation [Munjal (1987)]:

$$\begin{bmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ \underline{Z}_1 u_1(0) \\ \underline{Z}_2 u_2(0) \\ \underline{Z}_3 u_3(0) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{bmatrix} p_1(\ell) \\ p_2(\ell) \\ p_3(\ell) \\ \underline{Z}_1 u_1(\ell) \\ \underline{Z}_2 u_2(\ell) \\ \underline{Z}_3 u_3(\ell) \end{bmatrix}, \tag{1}$$



The common two-duct perforation

where

$$[T] = [A(0)]\,[A(\ell)]^{-1}\,. \tag{2}$$

The elements of the matrix [A(z)] are given by (i = 1, 2, . . ., 6):

$$\begin{aligned}
&A_{1,i} = \psi_{4,i}e^{\beta_{i,z}}; \quad A_{2,i} = \psi_{5,i}e^{\beta_{i,z}}; \quad A_{3,i} = \psi_{6,i}e^{\beta_{i,z}};\\
&A_{4,i} = -e^{\beta_{i,z}} / \left(jk_0 + M_1\beta_i\right); \quad A_{5,i} = -\psi_{2,i}e^{\beta_{i,z}} / \left(jk_0 + M_2\beta_i\right);\\
&A_{6,i} = -\psi_{3,i}e^{\beta_{i,z}} / \left(jk_0 + M_3\beta_i\right).
\end{aligned} \tag{3}$$

[ψ] and {β} are, respectively, the eigenmatrix (or modal matrix) and eigenvector of the following matrix [Peat (1988)]:

$$\begin{bmatrix}
-\alpha_1 & \alpha_3 & 0 & -\alpha_2 & -\alpha_4 & 0\\
-\alpha_5 & -\alpha_7 & -\alpha_9 & -\alpha_6 & -\alpha_8 & -\alpha_{10}\\
0 & -\alpha_{11} & -\alpha_{13} & 0 & -\alpha_{12} & -\alpha_{14}\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}, \tag{4}$$

where

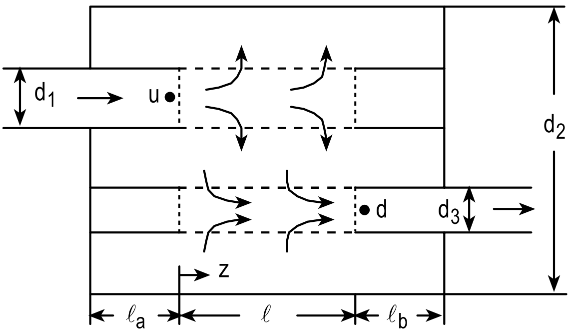
$$\begin{aligned}
&\alpha_1 = \frac{-jM_1}{1-M_1^2}\left(\frac{k_a^2+k_0^2}{k_0}\right); \quad \alpha_2 = \frac{k_a^2}{1-M_1^2}; \quad \alpha_3 = \frac{jM_1}{1-M_1^2}\left(\frac{k_a^2-k_0^2}{k_0}\right);\\
&\alpha_4 = \frac{k_a^2-k_0^2}{1-M_1^2}; \quad \alpha_5 = \frac{jM_2}{1-M_2^2}\left(\frac{k_b^2-k_0^2}{k_0}\right); \quad \alpha_6 = \frac{k_b^2-k_0^2}{1-M_2^2};\\
&\alpha_7 = \frac{-jM_2}{1-M_2^2}\left(\frac{k_b^2-k_c^2}{k_0}\right); \quad \alpha_8 = \frac{k_b^2+k_c^2-k_0^2}{1-M_2^2}; \quad \alpha_9 = \frac{jM_2}{1-M_2^2}\,\frac{k_c^2-k_0^2}{k_0};\\
&\alpha_{10} = \frac{k_c^2-k_0^2}{1-M_2^2}; \quad \alpha_{11} = \frac{jM_3}{1-M_3^2}\left(\frac{k_b^2-k_0^2}{k_0}\right); \quad \alpha_{12} = \frac{k_d^2-k_0^2}{1-M_3^2};\\
&\alpha_{13} = \frac{-jM_3}{1-M_3^2}\left(\frac{k_d^2+k_0^2}{k_0}\right); \quad \alpha_{14} = \frac{k_d^2}{1-M_3^2},
\end{aligned} \tag{5}$$

and

$$M_1 = V_1/c_0\ ; \quad M_2 = V_2/c_0\ ; \quad M_3 = V_3/c_0\ ;$$
$$k_a^2 = k_0^2 - \frac{4jk_0}{d_1\zeta_1}\ ; \quad k_b^2 = k_0^2 - \frac{4jk_0 d_1}{(d_2^2 - d_1^2 - d_3^2)\zeta_1}\ ; \quad k_c^2 = k_0^2 - \frac{4jk_0 d_3}{(d_2^2 - d_1^2 - d_3^2)\zeta_2}\ ; \tag{6}$$
$$k_d^2 = k_0^2 - \frac{4jk_0}{d_3\zeta_2}\ .$$

The desired 2 × 2 transfer matrix for a particular three-duct perforated element may be obtained from the 6 × 6 transfer matrix [T] above, making use of the appropriate upstream and downstream variables and four boundary conditions characteristic of the element. The final results are given below for different three-duct perforated elements shown in graphs.

(a) Cross-flow, three-duct, closed-end element



(a) Cross-flow, three-duct, closed-end element

$$\begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_3(\ell) \\ \underline{Z}_3 u_3(\ell) \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} T_a &= TT_{1,2} + A_3C_3\ ; \quad T_b = TT_{1,4} + B_3C_3\ ; \\ T_c &= TT_{3,2} + A_3D_3\ ; \quad T_d = TT_{3,4} + B_3D_3\ ; \end{aligned} \tag{8}$$

$$\begin{aligned} A_3 &= (TT_{2,2}X_2 - TT_{4,2})/F_2\ ; \quad B_3 = (TT_{2,4}X_2 - TT_{4,4})/F_2\ ; \\ C_3 &= TT_{1,1} + X_1TT_{1,3}\ ; \quad D_3 = TT_{3,1} + X_1TT_{3,3}\ ; \end{aligned} \tag{9}$$

$$F_2 = TT_{4,1} + X_1TT_{4,3} - X_2(TT_{2,1} + X_1TT_{2,3}) \tag{10}$$

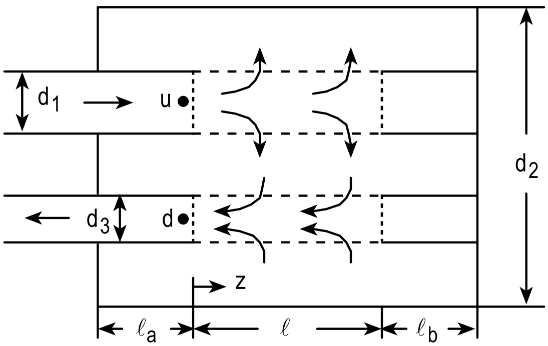
with

$$
\begin{aligned}
&TT_{1,1} = A_1A_2 + T_{1,2}\ ;\ TT_{1,2} = B_1A_2 + T_{1,3}\ ;\ TT_{1,3} = C_1A_2 + T_{1,5}\ ;\\
&TT_{1,4} = D_1A_2 + T_{1,6};\\
&TT_{2,1} = A_1B_2 + T_{2,2}\ ;\ TT_{2,2} = B_1B_2 + T_{2,3}\ ;\ TT_{2,3} = C_1B_2 + T_{2,5}\ ;\\
&TT_{2,4} = D_1B_2 + T_{2,6};\\
&TT_{3,1} = A_1C_2 + T_{4,2}\ ;\ TT_{3,2} = B_1C_2 + T_{4,3}\ ;\ TT_{3,3} = C_1C_2 + T_{4,5}\ ;\\
&TT_{3,4} = D_1C_2 + T_{4,6};\\
&TT_{4,1} = A_1D_2 + T_{5,2}\ ;\ TT_{4,2} = B_1D_2 + T_{5,3}\ ;\ TT_{4,3} = C_1D_2 + T_{5,5}\ ;\\
&TT_{4,4} = D_1D_2 + T_{5,6};
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&A_1 = (T_{3,2}X_2 - T_{6,2})/F_1\ ; && B_1 = (T_{3,3}X_2 - T_{6,3})/F_1\ ;\\
&C_1 = (T_{3,5}X_2 - T_{6,5})/F_1\ ; && D_1 = (T_{3,6}X_2 - T_{6,6})/F_1\ ;\\
&A_2 = T_{1,1} + T_{1,4}X_1\ ; && B_2 = T_{2,1} + T_{2,4}X_1\ ;\\
&C_2 = T_{4,1} + T_{4,4}X_1\ ; && D_2 = T_{5,1} + T_{5,4}X_1\ ;
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&F_1 = T_{6,1} + X_1T_{6,4} - X_2(T_{3,1} + X_1T_{3,4})\ ;\\
&X_1 = j\tan(k_0\ell_b)\ ; \quad X_2 = -j\tan(k_0\ell_a)\ .
\end{aligned}
\tag{13}
$$

(b) Reverse-flow, three-duct, closed-end element



(b) Reverse-flow, three-duct, closed-end element

$$
\begin{bmatrix} p_1(0) \\ \underline{Z}_1 u_1(0) \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_3(\ell) \\ \underline{Z}_3 u_3(\ell) \end{bmatrix},
\tag{14}
$$

where

$$
\begin{aligned}
&T_a = B_{1,1}D_{1,1} + B_{1,2}D_{2,1} + B_{1,3}D_{3,1}\ ; && T_b = B_{1,1}D_{1,2} + B_{1,2}D_{2,2} + B_{1,3}D_{3,2}\ ;\\
&T_c = B_{4,1}D_{1,1} + B_{4,2}D_{2,1} + B_{4,3}D_{3,1}\ ; && T_d = B_{4,1}D_{1,2} + B_{4,2}D_{2,2} + B_{4,3}D_{3,2}\ ;
\end{aligned}
\tag{15}
$$

$$
B_{i1,i2} = T_{i1,i2} + X_1T_{i1,i2+3}\ ; \quad \begin{cases} i1 = 1, 2, \ldots, 6, \\ i2 = 1, 2, 3\ ; \end{cases}
\tag{16}
$$

$$\begin{aligned}&D_{1,1} = C_{1,1}D_{2,1} + C_{1,2}D_{3,1} \quad ; \quad D_{1,2} = C_{1,1}D_{2,2} + C_{1,2}D_{3,2};\\ &D_{2,1} = C_{3,2}/F_4 \quad ; \quad D_{2,2} = -C_{2,2}/F_4\ ;\\ &D_{3,1} = -C_{3,1}/F_4 \quad ; \quad D_{3,2} = -C_{2,1}/F_4\ ;\end{aligned} \tag{17}$$

$$\begin{aligned}&C_{1,1} = \left(B_{5,2} - X_2B_{2,2}\right)/F_3\ ; \quad C_{1,2} = \left(B_{5,3} - X_2B_{2,3}\right)/F_3\ ;\\ &C_{2,1} = B_{3,2} - C_{1,1}B_{3,1}\ ; \quad C_{2,2} = B_{3,3} - C_{1,2}B_{3,1}\ ;\\ &C_{3,1} = B_{6,2} - C_{1,1}B_{6,1}\ ; \quad C_{3,2} = B_{6,3} - C_{1,2}B_{6,1}\ ;\end{aligned} \tag{18}$$

$$\begin{aligned}&F_3 = X_2B_{2,1} - B_{5,1}\ ; \quad F_4 = C_{2,1}C_{3,2} - C_{2,2}C_{3,1}\ ;\\ &X_1 = j\tan(k_0\ell_b)\ ; \quad X_2 = -j\tan(k_0\ell_a)\ .\end{aligned} \tag{19}$$

The transfer matrices in Sects. *K.11 and K.12* for perforated elements have been derived in the form:

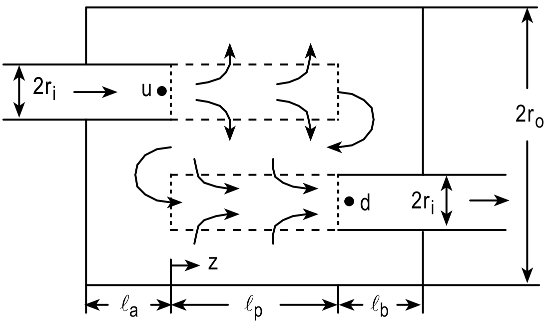
$$\begin{bmatrix} p_u \\ \underline{Z}_u u_u \end{bmatrix} = \begin{bmatrix} T_a & T_b \\ T_c & T_d \end{bmatrix} \begin{bmatrix} p_d \\ \underline{Z}_d u_d \end{bmatrix}. \tag{20}$$

This can be rewritten in the standard dimensional form as:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} T_a & \underline{Z}_d T_b \\ T_c/\underline{Z}_u & \underline{Z}_d T_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}. \tag{21}$$

(c) Cross-flow, three-duct open-end element

For the open-end elements shown below, the boundary conditions are not available as simple end-impedance expressions, but in the form of reversed-flow elements, the transfer matrices for which were given earlier in *Sect. K.11*.



(c) Cross-flow, three-duct, open-end element

Another characteristic of these elements is that the ends of the perforated inlet and outlet ducts being open, the gases moving through the ducts escape (or enter) partially through the perforations and partially through the end. This is expected to result in a much lower pressure drop compared to the closed-end elements. It has been observed that most of the meanflow moves straight grazing the perforates, and therefore the perforation impedance is given by the grazing flow impedance. The transfer matrix relationship between the upstream point u and downstream point d is given by [Gogate/Munjal (1995)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{22}$$

where

$$[P] = [[TT_{11}][X] + [TT_{12}]]\,[MAT] + [TT_{13}]\,;$$
$$[MAT] = [[[TT_{21}][X] + [TT_{22}]] - [Y]\,[TT_{31}][X] + [TT_{32}]]]^{-1}\,[[Y][TT_{33}] - [TT_{23}]]\,. \tag{23}$$

[X] is the transfer matrix of the reverse-flow expansion element at the right-hand junction, given in Sect. K.11; and

[Y] is the transfer matrix of the reverse-flow contraction element at the left-hand junction, given in Sect. K.11.

$$\left[TT_{ij}\right] = \begin{bmatrix} TM_{ij} & TM_{i,j+3} \\ TM_{i+3,j} & TM_{i+3,j+3} \end{bmatrix}; \quad i, j = 1, 2, 3. \tag{24}$$

[TM] is the 6 × 6 transfer matrix for the common perforated section of the three interacting ducts, with respect to state variables p and v. Thus:

$$TM_{i,4} = T_{i,4} \cdot \underline{Z}_1\,; \quad TM_{i,5} = T_{i,5} \cdot \underline{Z}_2\,; \quad TM_{i,6} = T_{i,6} \cdot \underline{Z}_1\,;$$
$$TM_{4,i} = T_{4,i}/\underline{Z}_1\,; \quad TM_{5,i} = T_{5,i}/\underline{Z}_2\,; \quad TM_{6,i} = T_{6,i}/\underline{Z}_1\,. \tag{25}$$

$\underline{Z}_1$ is the volume-flow impedance of the upstream/downstream or inlet/outlet ducts

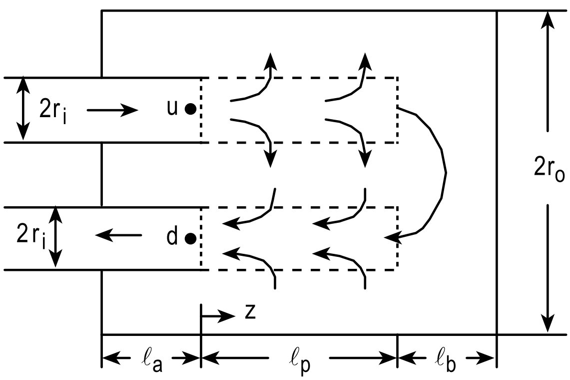
$\underline{Z}_1 = \rho_0 c_0 / S_u$;

$\underline{Z}_2$ is the volume-flow impedance of the annular duct

$\underline{Z}_2 = \rho_0 c_0 / (S_{shell} - 2S_u)$ neglecting the duct wall thickness;

[T] is the 6 × 6 transfer matrix of the common perforated section of the three interacting ducts, [Eq. (1)].

(d) Reverse-flow, open-end, three-duct element



(d) Reverse-flow, three-duct, open-end element

This element is a combination of elements (b) and (c) inasmuch as on the left-hand end it is like the closed-end element (b) and on the right-hand end it is like the open-end of element (c). The transfer matrix relationship between the upstream point u and the downstream point d is given by [Gogate/Munjal (1995)]

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} T_a & -T_b \underline{Z}_i \\ T_c/Z_i & -T_d \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{26}$$

where

$$\begin{aligned} T_a &= B_{11}C_{11} + (B_{12} + B_{15}X_a)C_{21} + B_{13}C_{31} + B_{14}C_{41} + B_{16}C_{51} \ ; \\ T_b &= B_{11}C_{12} + (B_{12} + B_{15}X_a)C_{22} + B_{13}C_{32} + B_{14}C_{42} + B_{16}C_{52} \ ; \\ T_c &= B_{41}C_{11} + (B_{42} + B_{45}X_a)C_{21} + B_{43}C_{31} + B_{44}C_{41} + B_{46}C_{51} \ ; \\ T_d &= B_{41}C_{12} + (B_{42} + B_{45}X_a)C_{22} + B_{43}C_{32} + B_{44}C_{42} + B_{46}C_{52} \ ; \end{aligned} \tag{27}$$

$$X_a = j\tan(k\ell_a) \quad ; \quad X_b = -j\tan(k\ell_b). \tag{28}$$

Here $[C] = [A]^{-1}$, and $[A]$ is a 5×5 matrix whose elements are related to the 6×6 matrix $[T]$ as follows:

$$\begin{aligned} &A_{11} = T_{31} \ ; \quad A_{12} = T_{32} + X_a\, T_{35} \ ; \quad A_{13} = T_{33} \ ; \quad A_{14} = T_{34} \ ; \\ &A_{15} = T_{36} \ ; \quad A_{21} = T_{61} \ ; \quad A_{22} = T_{62} + X_a\, T_{65} \ ; \quad A_{23} = T_{63} \ ; \\ &A_{24} = T_{64} \ ; \quad A_{25} = T_{66} \ ; \quad A_{31} = T_{51} - X_b T_{21} \ ; \quad A_{32} = T_{52} - X_b T_{22} + X_a\, T_{55} - X_b T_{25} \ ; \end{aligned} \tag{29}$$

$$\begin{aligned} &A_{33} = T_{53} - X_b T_{23} \, ; \quad A_{34} = T_{54} - X_b T_{24} \, ; \quad A_{35} = T_{56} - X_b T_{26} \, ; \\ &A_{41} = 1 \, ; \quad A_{42} = 0 \, ; \quad A_{43} = -1 \, ; \quad A_{44} = M_i \, ; \quad A_{45} = M_i(1 + K_{re} + K_{rc}) \, ; \\ &A_{51} = M_i/\underline{Z}_i \, ; \quad A_{52} = X_a/\underline{Z}_6 \, ; \quad A_{53} = -M_i/\underline{Z}_i \, ; \\ &A_{54} = 1/\underline{Z}_i \, ; \quad A_{55} = 1/\underline{Z}_i . \end{aligned} \tag{30}$$

K_{re}, K_{rc}	are the pressure-loss factors for reversal-expansion and reversal-contraction;
M_i	is the meanflow Mach number in the inner pipes of radius r_i;
$\underline{Z}_i$	is the volume-flow impedance of the inner pipe, $\underline{Z}_i = \rho_0 c_0 / S_i$; $S_i = \pi r_i^2$;
$[T]$	is the 6×6 transfer matrix for the common perforated section of the three interacting ducts, given above in Eq. (1).

K.13 Three-Duct Perforated Elements with Extended Perforations

Three possible configurations in this class of elements are shown below. The derivation of the transfer matrix between the upstream point u and downstream point d calls for simultaneous solution of equations representing

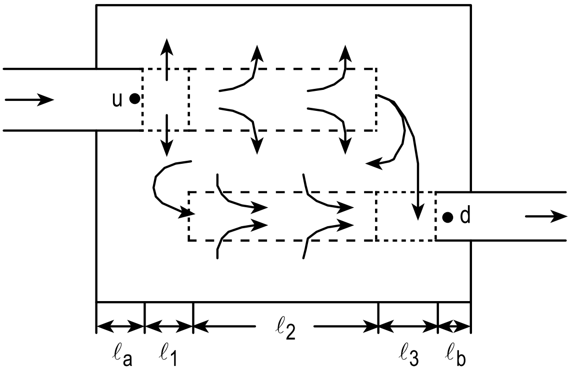
(1) the common three-duct perforated section of length ℓ_2;
(2) two-duct extended perforated sections on either end, of lengths ℓ_1 and ℓ_3;
(3) the closed-end cavities of lengths ℓ_a and ℓ_b, and l_1 and l_3 in the closed-end configuration (b).

The algebraic equations for item (1) are in the form of a 6×6 transfer matrix, discussed earlier in Sect. K.12. Equations for item (2) are in the form of two 4×4 transfer matrices discussed in Sect. K.11; and equations for the end-cavities of item (3) are in the form of an impedance expression.

In the open-end elements (a) and (c) below, as indicated for similar elements earlier, most of the meanflow grazes the perforations; very little flows across the perforations. So, for convenience, the entire flow may be assumed to be of the grazing type for selection of the appropriate expression for the perforated impedance. In the closed-end configuration (b) below, however, the entire flow has to get across the perforations, calling for the cross-flow or through-flow expression for perforate impedance.

While combining different sets of equations, care has to be taken to account for change of directions of flow and acoustic particle velocities, and also for the transformation of normalised volume velocity and the standard dimensional volume velocity.

(a) Cross-flow, open-end, extended-perforation element



(a) Cross-flow, open-end, extended-perforation element

The final transfer matrix relation for the element shown is given by:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} T_a & T_b \underline{Z}_d \\ T_c/\underline{Z}_u & T_d \underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} T_a &= R_{11} \cdot (R_{12} + R_{14} X_b) \cdot NP/DN\,; \\ T_b &= R_{13} \cdot (R_{12} + R_{14} X_b) \cdot NV/DN\,; \\ T_c &= R_{31} \cdot (R_{32} + R_{34} X_b) \cdot NP/DN\,; \\ T_d &= R_{33} \cdot (R_{32} + R_{34} X_b) \cdot NV/DN \end{aligned} \tag{2}$$

with

$$\begin{aligned} &NP = X_a\, R_{21} - R_{41}\,; \quad NV = X_a\, R_{23} - R_{43}\,; \\ &DN = R_{42} - X_a\, R_{22} + X_b (R_{44} - X_a\, R_{24})\,; \end{aligned} \tag{3}$$

$$X_a = -j \tan(k_0 \ell_a)\,; \quad X_b = j \tan(k_0 \ell_b)\,; \tag{4}$$

$$[R] = [A][Q][C]. \tag{5}$$

[A] and [C] are 4 × 4 transfer matrices for the extended perforated pipes of lengths ℓ_1 and ℓ_3, respectively (see Figure (c) below).

$$
\begin{aligned}
&Q_{11} = T_{11}DE_1 + T_{12}DE_2 + T_{14}DE_6 + T_{15}DE_7 + T_{13}\,;\\
&Q_{12} = T_{11}DF_1 + T_{12}DF_2 + T_{14}DF_6 + T_{15}DF_7\,;\\
&Q_{13} = T_{11}DG_1 + T_{12}DG_2 + T_{14}DG_6 + T_{15}DG_7 + T_{16}\,;\\
&Q_{14} = T_{11}DH_1 + T_{12}DH_2 + T_{14}DH_6 + T_{15}DH_7\,;\\
&Q_{21} = DE_5\,;\quad Q_{22} = DF_5\,;\quad Q_{23} = DG_5\,;\quad Q_{24} = DH_5\,;\\
&Q_{31} = T_{41}DE_1 + T_{42}DE_2 + T_{44}DE_6 + T_{45}DE_7 + T_{43}\,;\\
&Q_{32} = T_{41}DF_1 + T_{42}DF_2 + T_{44}DF_6 + T_{45}DF_7\,;\\
&Q_{33} = T_{41}DG_1 + T_{42}DG_2 + T_{44}DG_6 + T_{45}DG_7 + T_{46}\,;\\
&Q_{34} = T_{41}DH_1 + T_{42}DH_2 + T_{44}DH_6 + T_{45}DH_7\,;\\
&Q_{41} = DE_{10}\,;\quad Q_{42} = DF_{10}\,;\quad Q_{43} = DG_{10}\,;\quad Q_{44} = DH_{10}\,;
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\{DE\} = [D]^{-1}\{E\}\,;\quad \{DF\} = [D]^{-1}\{F\}\,;\\
&\{DG\} = [D]^{-1}\{G\}\,;\quad \{DH\} = [D]^{-1}\{H\}\,.
\end{aligned}
\tag{7}
$$

{E}, {F}, {G} and {H} are 10 × 1 column matrices with all elements zero, except the following:

$$
\begin{aligned}
&E_4 = T_{23}\,;\quad E_5 = T_{33}\,;\quad E_6 = T_{53}\,;\quad E_7 = T_{63}\,;\quad F_3 = 1\,;\\
&G_4 = T_{26}\,;\quad G_5 = T_{36}\,;\quad G_6 = T_{56}\,;\quad G_7 = T_{66}\,;\quad H_2 = 1\,.
\end{aligned}
\tag{8}
$$

[D] is a 10 × 10 matrix whose non-zero elements are:

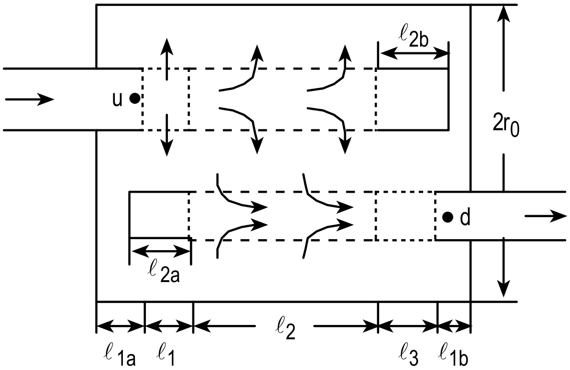
$$
\begin{aligned}
&D_{11} = 1\,;\quad D_{12} = -1\,;\quad D_{16} = M_u\,;\quad D_{17} = -(1+K_{re})M_6\,;\\
&D_{21} = M_u\underline{Z}_4/\underline{Z}_u\,;\quad D_{22} = M_6\underline{Z}_4/\underline{Z}_6\,;\quad D_{26} = \underline{Z}_4/\underline{Z}_u\,;\\
&D_{27} = \underline{Z}_4/\underline{Z}_6\,;\quad D_{31} = Z_4/\underline{Z}_u\,;\quad D_{32} = \underline{Z}_4/\underline{Z}_6\,;\\
&D_{36} = 2M_u\underline{Z}_4/\underline{Z}_u\,;\quad D_{37} = 2M_6\underline{Z}_4/\underline{Z}_6\,;\\
&D_{41} = -T_{21}\,;\quad D_{42} = -T_{22}\,;\quad D_{43} = -1\,;\quad D_{46} = -T_{24}\,;\quad D_{47} = -T_{25}\,;\\
&D_{51} = -T_{31}\,;\quad D_{52} = -T_{32}\,;\quad D_{54} = 1\,;\quad T_{56} = -T_{34}\,;\quad D_{57} = -T_{35}\,;\\
&D_{61} = -T_{51}\,;\quad D_{62} = -T_{52}\,;\quad D_{66} = -T_{54}\,;\quad D_{67} = -T_{55}\,;\quad D_{68} = 1\,;\\
&D_{71} = -T_{61}\,;\quad D_{72} = -T_{62}\,;\quad D_{76} = -T_{64}\,;\quad D_{77} = -T_{65}\,;\quad D_{79} = 1\,;\\
&D_{83} = 1\,;\quad D_{84} = -1\,;\quad D_{88} = M_8\,;\quad D_{89} = -(1+k_{rc})M_d\,;\\
&D_{93} = M_6\underline{Z}_{10}/\underline{Z}_8\,;\quad D_{94} = M_d\underline{Z}_{10}/\underline{Z}_d\,;\\
&D_{98} = \underline{Z}_{10}/\underline{Z}_8\,;\quad D_{99} = \underline{Z}_{10}/\underline{Z}_d\,;\quad D_{9,10} = -1\,;\\
&D_{10,3} = \underline{Z}_{10}/\underline{Z}_8\,;\quad D_{10,4} = \underline{Z}_{10}/\underline{Z}_d\,;\\
&D_{10,8} = 2M_8\underline{Z}_{10}/\underline{Z}_8\,;\quad D_{10,9} = 2M_d\underline{Z}_{10}/\underline{Z}_d\,;\quad D_{10,5} = -1\,;
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\underline{Z}_u = \rho_0 c_0/S_u\,;\quad \underline{Z}_d = \rho_0 c_0/S_d\,;\quad \underline{Z}_4 = \rho_0 c_0/(S_{sh}-S_d)\,;\\
&\underline{Z}_6 = \rho_0 c_0/(S_{sh}-S_u-S_d)\,;\quad \underline{Z}_8 = \underline{Z}_6\,;\quad \underline{Z}_{10} = \rho_0 c_0/(S_{sh}-S_d)\,;
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&M_6 = -M_u S_u/(S_{sh} - S_u - S_d)\,;\quad M_8 = M_6\,;\\
&k_{rc} = 0.5\,;\quad k_{re} = \{(S_{sh} - S_u - S_d)/S_u\}^2\,.
\end{aligned}
\tag{11}
$$

[T] is the 6 × 6 transfer matrix for the common perforated section of the three interacting ducts, given in Eq. (1).

(b) Cross-flow, closed-end, extended peroration element



(b) Cross-flow, closed-end, extended-perforation element

The overall transfer matrix between the upstream point u and downstream point d of the configuration of the figure is given by:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} T_a & T_b\underline{Z}_d \\ T_c/\underline{Z}_u & T_d\underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} T_a &= (E_{12} + X_{1b}E_{14})NP/DN + E_{11} ; \\ T_b &= (E_{12} + X_{1b}E_{14})NV/DN + E_{13} ; \\ T_c &= (E_{32} + X_{1b}E_{34})NP/DN + E_{31} ; \\ T_d &= (E_{32} + X_{1b}E_{34})NV/DN + E_{33} ; \end{aligned} \tag{13}$$

$$\begin{aligned} &NP = X_{1a}E_{21} - E_{41} ; \quad NV = X_{1a}E_{23} - E_{43} ; \\ &DN = E_{42} + X_{1b}E_{44} - (E_{22} + X_{1b}E_{24})\,X_{1a} ; \end{aligned} \tag{14}$$

$$\begin{aligned} X_{1a} &= -j\tan(k_0\ell_{a1}) ; \quad X_{1b} = j\tan(k_0\ell_{b1}) ; \\ X_{2a} &= -j\tan(k_0\ell_{a2}) ; \quad X_{2b} = j\tan(k_0\ell_{b2}) ; \end{aligned} \tag{15}$$

$$[E] = [A][D][C]. \tag{16}$$

[A] and [C] are 4 × 4 transfer matrices for the extended perforated pipes of lengths ℓ_1 and ℓ_3, respectively (see figure).

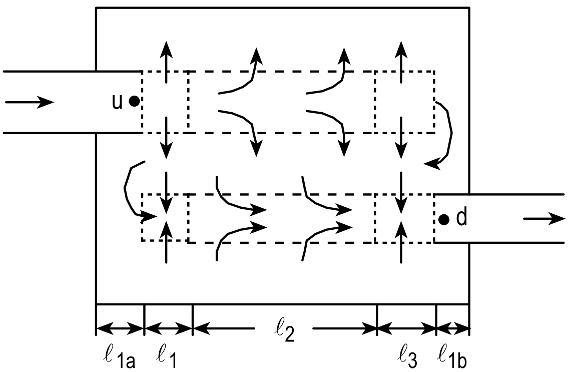
$$\begin{aligned} D_{11} &= T_{11}G_2 + T_{13} + T_{14}G_2X_{2b} ; & D_{12} &= T_{11}G_1 + T_{12} + T_{14}G_1X_{2b} ; \\ D_{13} &= T_{11}G_4 + T_{16} + T_{14}G_4X_{2b} ; & D_{14} &= T_{11}G_3 + T_{15} + T_{14}G_3X_{2b} ; \\ D_{21} &= T_{21}G_2 + T_{23} + T_{24}G_2X_{2b} ; & D_{22} &= T_{21}G_1 + T_{22} + T_{24}G_1X_{2b} ; \\ D_{23} &= T_{21}G_4 + T_{26} + T_{24}G_4X_{2b} ; & D_{24} &= T_{21}G_3 + T_{25} + T_{24}G_3X_{2b} ; \\ D_{31} &= T_{41}G_2 + T_{43} + T_{44}G_2X_{2b} ; & D_{32} &= T_{41}G_1 + T_{42} + T_{44}G_1X_{2b} ; \\ D_{33} &= T_{41}G_4 + T_{46} + T_{44}G_4X_{2b} ; & D_{34} &= T_{41}G_3 + T_{45} + T_{44}G_3X_{2b} ; \\ D_{41} &= T_{51}G_2 + T_{53} + T_{54}G_2X_{2b} ; & D_{42} &= T_{51}G_1 + T_{52} + T_{54}G_1X_{2b} ; \\ D_{43} &= T_{51}G_4 + T_{56} + T_{54}G_4X_{2b} ; & D_{44} &= T_{51}G_3 + T_{55} + T_{54}G_3X_{2b} . \end{aligned} \tag{17}$$

[T] is the 6 × 6 transfer matrix for the common perforated section of the three interacting ducts, given in Eq. (1).

$$G_1 = (X_{2a}T_{32} - T_{62})/h; \quad G_2 = (X_{2a}T_{33} - T_{63})/h; \quad G_3 = (X_{2a}T_{35} - T_{65})/h; \quad G_4 = (X_{2a}T_{36} - T_{66})/h; \tag{18}$$

$$h = T_{61} - X_{2a}T_{31} + X_{2b}(T_{64} - X_{2a}T_{34}); \quad \underline{Z}_u = \rho_0 c_0/S_u; \quad \underline{Z}_d = \rho_0 c_0/S_d. \tag{19}$$

(c) Reverse-flow, open-end, extended-perforated element:



(c) Reverse-flow, open-end, extended-perforation element

For the configuration shown in the figure, the final transfer matrix relationship is given by [Munjal/Behera/Thawani (1997)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} T_a & T_b\underline{Z}_d \\ T_c/\underline{Z}_u & T_d\underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{20}$$

where

$$\begin{aligned} T_a &= B_{11}W_{11} + B_{12}U_{11} + B_{13} + B_{14}W_{21} + B_{15}U_{21}; \\ T_b &= B_{11}W_{12} + B_{12}U_{12} + B_{16} + B_{14}W_{22} + B_{15}U_{22}; \\ T_c &= B_{41}W_{11} + B_{42}U_{11} + B_{43} + B_{44}W_{21} + B_{45}U_{21}; \\ T_d &= B_{41}W_{12} + B_{42}U_{12} + B_{46} + B_{44}W_{22} + B_{45}U_{22}; \end{aligned} \tag{21}$$

$$[W] = [G][U]; \quad [U] = [Q][R]; \quad [Q] = [X]^{-1}; \tag{22}$$

$$\begin{aligned} R_{11} &= H_{11}B_{33} + H_{12}B_{63} - B_{23}; \quad R_{12} = H_{11}B_{36} + H_{12}B_{66} - B_{26}; \\ R_{21} &= H_{21}B_{33} + H_{22}B_{63} - B_{53}; \quad R_{22} = H_{21}B_{36} + H_{22}B_{66} - B_{56}; \end{aligned} \tag{23}$$

$$\begin{aligned} X_{11} &= B_{21}G_{11} + B_{22} + B_{24}G_{21} - H_{11}(B_{31}G_{11} + B_{32} \\ &\quad + B_{34}G_{21}) - H_{12}(B_{61}G_{11} + B_{62} + B_{64}G_{21}); \\ X_{12} &= B_{21}G_{12} + B_{25} + B_{24}G_{22} - H_{11}(B_{31}G_{12} + B_{35} \\ &\quad + B_{34}G_{22}) - H_{12}(B_{61}G_{12} + B_{65} + B_{64}G_{22}); \\ X_{21} &= B_{51}G_{11} + B_{52} + B_{54}G_{21} - H_{21}(B_{31}G_{11} + B_{32} \\ &\quad + B_{34}G_{21}) - H_{22}(B_{61}G_{11} + B_{62} + B_{64}G_{21}); \\ X_{22} &= B_{51}G_{12} + B_{55} + B_{54}G_{22} - H_{21}(B_{31}G_{12} + B_{35} \\ &\quad + B_{34}G_{22}) - H_{22}(B_{61}G_{12} + B_{65} + B_{64}G_{22}); \end{aligned} \tag{24}$$

$$[G] = [GA][GI]\,; \quad [GI] = [GB]^{-1}; \tag{25}$$

$$\begin{aligned} GB_{11} &= C_{21}F_{11} + C_{22} + C_{23}F_{21}\,; \quad GB_{12} = C_{21}F_{12} + C_{24} + C_{23}F_{22}\,; \\ GB_{21} &= C_{41}F_{11} + C_{42} + C_{43}F_{21}\,; \quad GB_{22} = C_{41}F_{12} + C_{44} + C_{43}F_{22}\,; \end{aligned} \tag{26}$$

$$\begin{aligned} GA_{11} &= C_{11}F_{11} + C_{12} + C_{13}F_{21}\,; \quad GA_{12} = C_{11}F_{12} + C_{14} + C_{13}F_{22}\,; \\ GA_{21} &= C_{31}F_{11} + C_{32} + C_{33}F_{21}\,; \quad GA_{22} = C_{31}F_{12} + C_{34} + C_{33}F_{22}\,; \end{aligned} \tag{27}$$

$$\begin{aligned} &[H] = [HC][HI]; \quad [HI] = [HB]^{-1}\,; \\ &[HC] = [HR] + [UF]; \quad [HB] = [HP] + [QF]\,; \\ &[UF] = [HU][F]; \quad [QF] = [HQ][F]\,; \end{aligned} \tag{28}$$

$$\begin{aligned} &\begin{bmatrix} HU_{11} & HU_{12} \\ HU_{21} & HU_{22} \end{bmatrix} = \begin{bmatrix} AI_{22} & AI_{24} \\ AI_{42} & AI_{44} \end{bmatrix}; \quad \begin{bmatrix} HR_{11} & HR_{12} \\ HR_{21} & HR_{22} \end{bmatrix} = \begin{bmatrix} AI_{21} & AI_{23} \\ AI_{41} & AI_{43} \end{bmatrix}; \\ &\begin{bmatrix} HQ_{11} & HQ_{12} \\ HQ_{21} & HQ_{22} \end{bmatrix} = \begin{bmatrix} AI_{12} & AI_{14} \\ AI_{32} & AI_{34} \end{bmatrix}; \quad \begin{bmatrix} HP_{11} & HP_{12} \\ HP_{21} & HP_{22} \end{bmatrix} = \begin{bmatrix} AI_{11} & AI_{13} \\ AI_{31} & AI_{33} \end{bmatrix}; \\ &[AI] = [A]^{-1}. \end{aligned} \tag{29}$$

[T] [see (Eq. K.12.1)] is the 6×6 transfer matrix for the common perforated section of the three interacting ducts, and [A] and [C] are 4×4 transfer matrices for the extended perforated pipes of lengths ℓ_a and ℓ_3, respectively [see Eq. (K.11.1)].

K.14 Three-Pass (or Four-Duct) Perforated Elements

The figures below show typical three-pass element mufflers where waves in four ducts interact with each other. These comprise three perforated ducts of radius r_1, r_2 and r_3 and the annular space of equivalent radius r_4. These mufflers have the advantage of good acoustic performance coupled with low back pressure.

Following the same procedure as for the two-duct elements in *Sect. K.11* or three-duct elements in *Sect. K.12*, we obtain first an 8×8 transfer matrix for the common interaction section. Then, making use of the closed-end boundary conditions for the annular duct, we get a 6×6 transfer matrix relationship.

The common 8×8 matrix relationship is given by:

$$\{S(0)\} = [TM]\,\{S(\ell_p)\}, \tag{1}$$

where {S(z)} is the state vector:

$$\left[p_1(z), p_2(z), p_3(z), p_4(z), V_1(z), V_2(z), V_3(z), V_4(z)\right]^T \tag{2}$$

and

$$V \equiv \rho_0 c_0 v = \frac{\rho_0 c_0}{S} u = \underline{Z}_0 u, \tag{3}$$

so that $\underline{Z}_0$ is the characteristic impedance with respect to volume velocity u and S is the area of cross section of the appropriate duct.

$$[TM] = [A(0)][A(\ell_1)]^{-1}, \tag{4}$$

where $[A(z)]$ is an 8×8 matrix with constituent elements given by, $i = 1, 2, \ldots, 8$:

$$\begin{aligned}
&A_{1,i} = \psi_{1,i}\, e^{\beta_i z}; \quad A_{2,i} = -\psi_{5,i}\, e^{\beta_i z}/\left(jk_0 + M_1\beta_i\right);\\
&A_{3,i} = \psi_{2,i}\, e^{\beta_i z}; \quad A_{4,i} = -\psi_{6,i}\, e^{\beta_i z}/\left(jk_0 + M_2\beta_i\right);\\
&A_{5,i} = \psi_{3,i}\, e^{\beta_i z}; \quad A_{6,i} = -\psi_{7,i}\, e^{\beta_i z}/\left(jk_0 + M_3\beta_i\right);\\
&A_{7,i} = \psi_{4,i}\, e^{\beta_i z}; \quad A_{8,i} = -\psi_{8,i}\, e^{\beta_i z}/\left(jk_0\right).
\end{aligned} \tag{5}$$

$[\psi]$ is the modal matrix and $\{\beta\}$ is the eigenvector of the following coefficient matrix [Munjal, Int. J. Acoust. and Vib. 2, pp. 63–68 (1997)]:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
-\alpha_2 & 0 & 0 & -\alpha_4 & -\alpha_1 & 0 & 0 & -\alpha_3\\
0 & -\alpha_6 & 0 & -\alpha_8 & 0 & -\alpha_5 & 0 & -\alpha_7\\
0 & 0 & -\alpha_{10} & -\alpha_{12} & 0 & 0 & -\alpha_9 & -\alpha_{11}\\
0 & -\alpha_{14} & -\alpha_{15} & -\alpha_{16} & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}
&\alpha_1 = \frac{2jk_0M_1}{1-M_1^2}\left(1 - \frac{2j}{k_0 d_1 \zeta_1}\right); \quad \alpha_2 = \frac{k_0^2}{1-M_1^2}\left(1 - \frac{4j}{k_0 d_1 \zeta_1}\right);\\
&\alpha_3 = \frac{4M_1}{d_1\zeta_1(1-M_1^2)}; \quad \alpha_4 = \frac{j4k_0}{d_1\zeta_1(1-M_1^2)};\\
&\alpha_5 = \alpha_1, \quad \alpha_6 = \alpha_2, \quad \alpha_7 = \alpha_3, \quad \alpha_8 = \alpha_4 \text{ with subscript 1 replaced by 2};\\
&\alpha_9 = \alpha_1, \quad \alpha_{10} = \alpha_2, \quad \alpha_{11} = \alpha_3, \quad \alpha_{12} = \alpha_4 \text{ with subscript 1 replaced by 3};\\
&\alpha_{13} = \frac{j4k_0 d_1}{\zeta_1\left\{d_4^2 - (d_1^2 + d_2^2 + d_3^2)\right\}};\\
&\alpha_{14} = \alpha_{13} \quad \text{with} \quad d_1/\zeta_1 \quad \text{replaced by} \quad d_2/\zeta_2;\\
&\alpha_{15} = \alpha_{13} \quad \text{with} \quad d_1/\zeta_1 \quad \text{replaced by} \quad d_3/\zeta_3;\\
&\alpha_{16} = k_0^2 - (\alpha_{13} + \alpha_{14} + \alpha_{15});
\end{aligned} \tag{7}$$

$$M_1 = U_1/c_0; \quad M_2 = U_2/c_0; \quad M_3 = U_3/c_0; \quad k_0 = \omega/c_0. \tag{8}$$

Use of the boundary conditions of the annular duct, and rearranging yields a reduced, and more useful, form of the transfer matrix relationship:

$$\begin{bmatrix} p_1(0)\\ V_1(0)\\ p_2(0)\\ V_2(0)\\ p_3(0)\\ V_3(0) \end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16}\\
T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26}\\
T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36}\\
T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46}\\
T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56}\\
T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66}
\end{bmatrix} \begin{bmatrix} p_1(\ell)\\ V_1(\ell)\\ p_2(\ell)\\ V_2(\ell)\\ p_3(\ell)\\ V_3(\ell) \end{bmatrix}; \tag{9}$$

$$T_{ij} = TM_{ij} + \frac{(TM_{i7} + X_{pb}TM_{i8})(X_{pa}TM_{7j} - TM_{8j})}{TM_{87} + X_{pb}TM_{88} - X_{pa}(TM_{77} + X_{pb}TM_{78})}; \quad i,j = 1, 2, \ldots, 6; \tag{10}$$

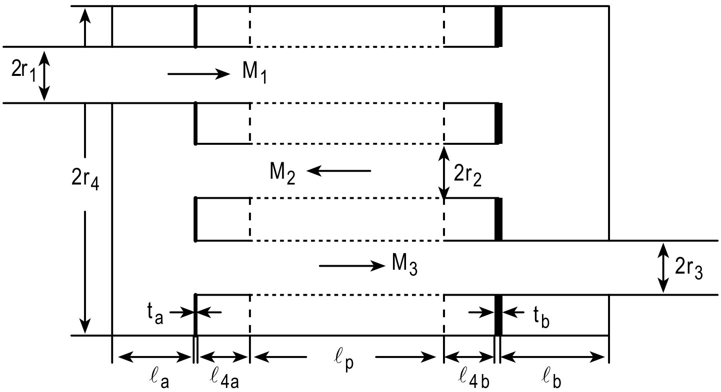
$$X_{pa} = -j\tan(k_0\ell_a); \quad X_{pb} = -j\tan(k_0\ell_b). \tag{11}$$

Partitioning the foregoing matrix equation as:

$$\begin{bmatrix} S_1(0) \\ S_2(0) \\ S_3(0) \end{bmatrix} = \begin{bmatrix} D & E & F \\ G & H & K \\ P & Q & R \end{bmatrix} \begin{bmatrix} S_1(\ell_p) \\ S_2(\ell_p) \\ S_3(\ell_p) \end{bmatrix}, \tag{12}$$

where $\{S_i\} = [p_i\ V_i]^T$ and D, E, F, G, H, K, P, Q and R are 2×2 submatrices as becomes clear from comparison of the two corresponding matrix equations.

(a) Flush-tube three-pass perforated element



(a) Flush-tube, three-pass perforated element chamber

Now, the desired 2×2 transfer matrix relationship between the upstream point u and the downstream point d in the configuration shown is given by [Munjal, Int. J. Acoust. and Vib. 2, pp. 63–68 (1997)]

$$\begin{bmatrix} p_1(0) \\ u_1(0) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \cdot \underline{Z}_d \\ C_{21}/\underline{Z}_u & C_{22}\underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_3(\ell_p) \\ u_3(\ell_p) \end{bmatrix}, \tag{13}$$

where

$$\begin{aligned} [C] &= [D][A][W] + [E][W] + [F]; \\ [W] &= [[G][A] + [H] - [B][P][A] - [B][Q]]^{-1}\,[[B][R] - [K]]\,. \end{aligned} \tag{14}$$

[A] is the product of the transfer matrices of

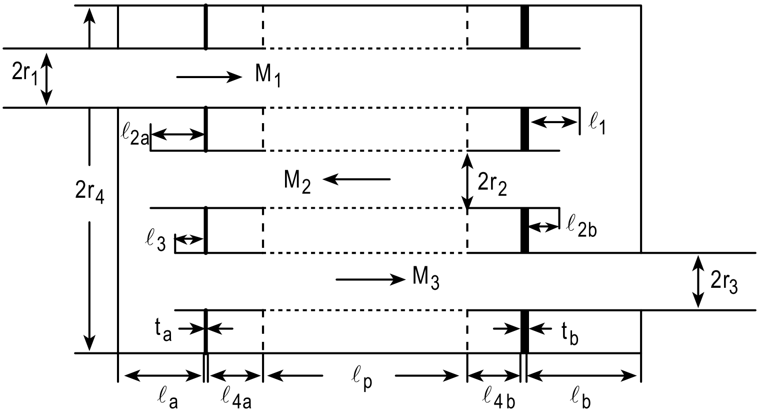
(1) a duct of length $\delta_1 + \ell_{4b} + t_b$,
(2) reversal expansion element,
(3) sudden contraction element,
(4) a duct of length $\delta_2 + t_b + \ell_{4b}$.

[B] is the product of transfer matrices of

(1) a duct of length $\delta_1 + \ell_{4a} + t_a$,
(2) reversal expansion element,
(3) sudden contraction element,
(4) a duct of length $\delta_3 + t_a + \ell_{4a}$.

Here, δ_a and δ_b are end corrections. These as well as other constituent transfer matrices are given in *Sects. K.5 and K.6.*

(b) Extended-tube three-pass perforated element



(b) Extended-tube, three-pass perforated element chamber

The foregoing transfer matrix relationship between the state vectors $[p_1(0)\ u_1(0)]^T$ and $[p_3(\ell_p)\ u_3(\ell_p)]^T$ would hold for the extended-tube three-pass perforated element shown above, with the difference that end matrices [A] and [B] would now be different. For the extended-tube end chambers,

[A] would be the product of the transfer matrices of [Munjal, ICSV-5, Adelaide (1997)]:

(1) a duct of length $\delta_1 + \ell_{4b} + t_b + \ell_1$,
(2) reversal expansion element,
(3) extended outlet element,
(4) a duct of length $\delta_2 + t_b + \ell_{4b} + \ell_{2b}$.

[B] would be the product of the transfer matrices of

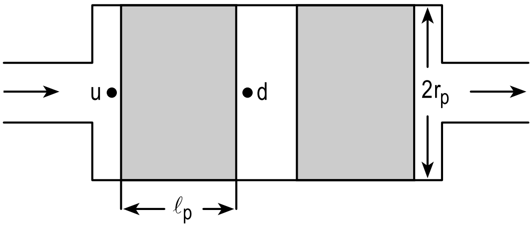
(1) a duct of length $\delta_2 + \ell_{4a} + t_a + \ell_{2a}$,
(2) reversal expansion element,
(3) extended outlet element,
(4) a duct of length $\delta_3 + \ell_3 + t_a + \ell_{4a}$.

Transfer matrices of the extended-tube elements are given in *Sect. K.7*, while the end-correcting $\delta_1, \delta_2, \delta_3$ and other transfer matrices are given in *Sects. K.5 and K.6*.

K.15 Catalytic Converter Elements

Catalytic converters are used often in series with exhaust mufflers for control of air emissions by means of oxidation of unburnt carbon particles and carbon monoxide to carbon dioxide. These converters involve area changes and porous blocks of catalyst pellets, or a bank of capillary tubes as shown below.

(a) Pellet block element



(a) Pellet-block catalytic converter element

While transfer matrices of simple uniform tubes and sudden area changes have been given earlier in *Sects. K.5 and K.6*, the transfer matrix of a pellet block element is given by:

$$\begin{bmatrix} \cos(k\ell) & j\underline{Z}\sin(k\ell) \\ \frac{j}{\underline{Z}}\sin(k\ell) & \cos(k\ell) \end{bmatrix}, \tag{1}$$

where $\underline{Z}$ and k for granular pellets are given by Attenborough's expression [Altenborough (1983)]:

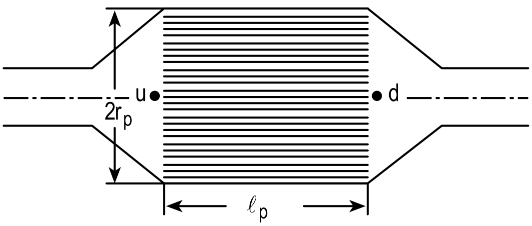
$$\frac{k}{k_0} = q\left(\frac{1+(\kappa-1)T(C)}{1-T(B)}\right)^{1/2}; \quad \frac{\underline{Z}}{\underline{Z}_0} = \frac{q^2}{\sigma}\frac{1}{1-T(B)}\frac{k_0}{k}, \tag{2}$$

where q is a tortuosity factor, κ is the ratio of specific heats for the gaseous medium, and

$$\begin{aligned} &T(x) = \frac{2J_1(x)}{xJ_o(x)}; \quad x = B \text{ or } C; \\ &B = (-j)^{1/2}\lambda_p; \quad C = BN_{pr}^{1/2}; \\ &\lambda_p^2 = 8\rho_0 q^2 S\omega/(n^2\sigma E). \end{aligned} \tag{3}$$

q	is the steady flow factor;
n	is the dynamic shape factor, n = 2 − S;
N_{Pr}	is the Prandtl number

(b) Capillary tube monolith



(b) Capillary-tube monolyth catalytic converter element

The transfer matrix of a monolith or bank of capillary tubes (coated with catalyst), is given by:

$$\begin{bmatrix} \cos(k_m\ell) & j\,\underline{Z}\sin(k_m\ell)/\Phi \\ \frac{j\Phi}{\underline{Z}}\sin(k_m\ell) & \cos(k_m\ell) \end{bmatrix}, \tag{4}$$

where Φ is the open area ratio.

$$\underline{Z} = \rho_m c_m/S\,; \quad k_m = k_0 c_0/c_m\,;$$

$$\frac{c_0}{c_m} = \left[(1+\Phi EG/D)\left(\kappa - \frac{\kappa-1}{1+\Phi EG'/(D\,Pr)}\right)\right]^{1/2} \tag{5}$$

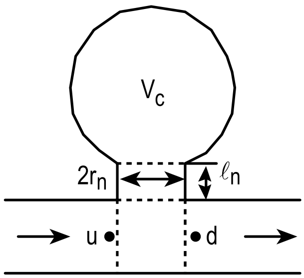
with

$$\begin{aligned} &D = j\omega\rho_0\,; \quad G = \frac{-ab/4}{1-2b/a}\,; \quad G' = \frac{-a'b'/4}{1-2b'/a'}\,; \\ &a = s(-j)^{1/2}\,; \quad b = J_1(a)/J_0(a)\,; \\ &a' = s(-j)^{1/2}\,Pr^{1/2}\,; \quad b' = J_1(a')/J_0(a')\,; \\ &\rho_m = \rho_0 + E\Phi G/(j\omega)\,; \\ &s = \alpha\left(\frac{8\omega\rho_0}{E\Phi}\right)^{1/2}. \end{aligned} \tag{6}$$

E is the specific flow resistance for laminar flow, $E = 32/(\Phi d^2)$. For air, $\kappa = 1.4$, $Pr = 0.7$, $\mu = 1.81 \cdot 10^{-5}$ Pa · s. Typically, $E \approx 500$ Pa · s/m^2 and $\alpha = 1.07$.

K.16 Helmholtz Resonator

See *Sects. H.4–H.16* for a more detailed description of Helmholtz resonators without flow and *Sects. J.38–J.39* for non-linear effects of flow.



The transfer matrix of a Helmholtz resonator as shown is given by:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\underline{Z}_r & 1 \end{bmatrix}\begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{1}$$

where $\underline{Z}_r$ is the flow impedance of the resonator at the junction, made up of a resistance term, an inertance term and a compliance term:

$$\begin{aligned} &\underline{Z}_r = \rho_0\left[\frac{\omega^2}{\pi c_0}\left\{2-\frac{r_n}{r_u}\right\} + 0.425\frac{Mc_0}{S_n} + j\left\{\frac{\omega\ell_{eq}}{S_n} - \frac{c_0^2}{\omega V_c}\right\}\right], \\ &\ell_{eq} = \ell_n + t_w + 0.85r_n\left(2-\frac{r_n}{r_u}\right). \end{aligned} \tag{2}$$

r_u is the radius of the upstream (or downstream) duct; r_n and ℓ_n are, respectively, the radius and length of the resonator neck; V_c is the volume of the resonator cavity; $S_n = \pi r_n^2$.

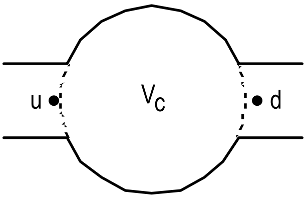
K.17 In-Line Cavity

The transfer matrix of an inline cavity as shown is given by:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\underline{Z} & 1 \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{1}$$

where $\underline{Z}$ is the compliance-type flow impedance and V_c the volume of the cavity:

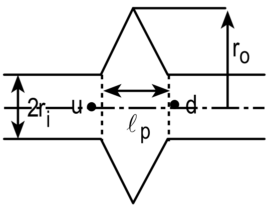
$$\underline{Z} = \frac{\rho_0 c_0^2}{j\omega V_c}. \tag{2}$$



K.18 Bellows

A bellow is characteriszed by wall compliance coupled with a gradual area change. The matrizant approach leads to the following transfer matrix [Singhal/Munjal (1999)] for the divergent conical part of a single step of a bellow:

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{1}$$



$$
\begin{aligned}
T_{11} &= \frac{e^{\beta\ell/2}}{2\delta'}\left(2\delta'\cos(\delta'\ell) - \beta\sin(\delta'\ell)\right);\\
T_{12} &= j\frac{e^{\beta\ell/2}}{2\delta'}\frac{k_0\underline{Z}(\ell)}{\rho_0\delta'}\sin(\delta'\ell);\\
T_{21} &= j\frac{-\rho_0 e^{\beta\ell/2}}{k_0\delta'\underline{Z}(0)}\left(\frac{\beta^2}{4} + \delta'^2\right)\sin(\delta'\ell);\\
T_{22} &= \frac{e^{\beta\ell/2}}{2\delta'}\left(\beta\sin(\delta'\ell) + 2\delta'\cos(\delta'\ell)\right)\frac{\underline{Z}(0)}{\underline{Z}(\ell)}
\end{aligned}
\tag{2}
$$

with

$$
\begin{aligned}
&\delta' = j\delta; \quad \underline{Z}(0) = \frac{\rho_0 c_0}{\pi r_i^2}; \quad \underline{Z}(\ell) = \frac{\rho_0 c_0}{\pi r_o^2};\\
&\delta = \frac{1}{2}\left\{\beta^2 - 4\left(k_0^2 - jk_0\alpha\right)\right\}; \quad \beta = 2\,a\,\alpha/B \quad; \quad \ell = \ell_p/2;\\
&a = (r_o - r_i)/\ell; \quad \alpha = \frac{B\ln(r_o/r_i)}{a\ell}; \quad B = 2\rho_0 c_0/Z_w.
\end{aligned}
\tag{3}
$$

Z_w is the wall impedance given earlier in *Sect. K.10* on hoses.

By interchanging the inlet and outlet radii, we obtain the transfer matrix for the convergent, conical part of the bellows. Successive multiplication of the transfer matrices of the two halves of the step shown above yields the transfer matrix of the full stop (single bellow). Extension of this multiplication process would yield the transfer matrix of multistep bellows.

Evaluation of the transfer matrix of flexible-wall bellows, incorporating the convective effect of mean flow is done by means of the same matrizant approach as given in reference [Singhal/Munjal (1999)]. However, the more important effect of flow separation and the consequent losses have been neglected; only the convective effect of the mean flow is considered.

K.19 Pod Silencer

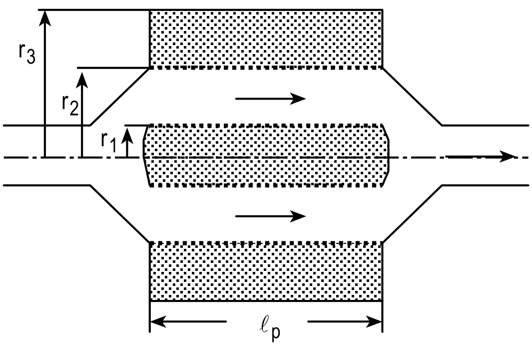
A pod silencer is a combination of a simple acoustically lined duct and a parallel baffle muffler, as shown.

See *Ch. J*, "Duct Acoustics", for lined ducts and especially *Sects. J.23–J.25* for baffle silencers. *Sect. J.28* for conical duct transitions.

Often a pod is inserted to increase the absorptive contact surface and thereby increase the attenuation or transmission loss of the silencer. Pod silencers are often available in prefabricated form for ready use in a noisy duct.

The transfer matrix for a pod silencer of length ℓ_p, and radii r_1, r_2 and r_3, as shown above, is given by:

$$
\begin{bmatrix} \cos(k_z\ell_p) & j\underline{Z}\sin(k_z\ell_p) \\ \frac{j}{\underline{Z}}\sin(k_z\ell_p) & \cos(k_z\ell_p) \end{bmatrix}; \quad \underline{Z} = \underline{Z}_0 k_0/k_z, \tag{1}
$$



where the convective effect of meanflow and the presence of highly perforated plate have been neglected; k_z is a root of the transcendental determinant equation:

$$\begin{vmatrix} \dfrac{-k_{r1}J_1(k_{r1}r_1)}{k_w\underline{Z}_w} & \dfrac{k_{r2}J_1(k_{r2}r_1)}{k_0\underline{Z}_0} & \dfrac{k_{r2}Y_1(k_{r2}r_1)}{k_0\underline{Z}0} & 0 & 0 \\ J_0(k_{r1}r_1) & -J_0(k_{r2}r_1) & -Y_0(k_{r2}r_1) & 0 & 0 \\ 0 & -\dfrac{k_{r2}J_1(k_{r2}r_1)}{k_0\underline{Z}_0} & \dfrac{-k_{r2}Y_1(k_{r2}r_2)}{k_0\underline{Z}_0} & \dfrac{k_{r3}J_1(k_{r3}r_2)}{k_w\underline{Z}_w} & \dfrac{k_{r3}Y_1(k_{r3}r_2)}{k_w\underline{Z}_w} \\ 0 & J_0(k_{r2}r_1) & Y_0(k_{r2}r_2) & -J_0(k_{r3}r_2) & -Y_0(k_{r3}r_2) \\ 0 & 0 & 0 & \dfrac{k_{r3}J_1(k_{r3}r_3)}{k_w\underline{Z}_w} & \dfrac{k_{r3}Y_1(k_{r3}r_3)}{k_w\underline{Z}_w} \end{vmatrix} = 0, \quad (2)$$

where

$$k_{r1} = \left(k_w^2 - k_z^2\right)^{1/2}; \quad k_{r_2} = \left(k_0^2 - k_z^2\right)^{1/2}; \quad k_{r3} = \left(k_w^2 - k_z^2\right)^{1/2} = k_{r1} \quad (3)$$

and $J_n(z), Y_n(z)$ denote Bessel and Neumann functions, respectively. Y_w and k_w are given by Mechel's formulae presented in *Sect. J.25*.

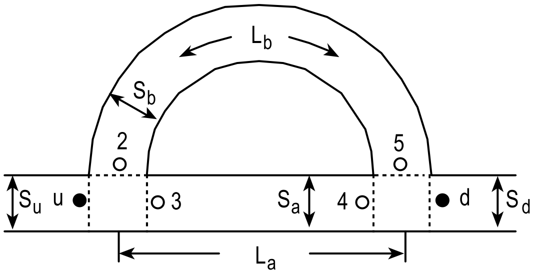
The choice of the starting value of k_z for the Newton-Raphson iteration process may be done as for a circular lined duct (*Sects. J.13 and J.15*).

K.20 Quincke Tube

A Quincke tube is a passive method for generating wave interference in ducts which is more simple, inexpensive and durable than the corresponding active noise control system. The Quincke tube consists of two pipes. The main pipe is connected with a parallel bypass as shown. For plane waves in an incompressible flow, the transfer matrix relationship between the upstream point u = 1 and downstream point d = 6 is given by:

$$\frac{1}{1-M_u^2}\begin{bmatrix} 1 & -M_u\underline{Z}_u \\ -M_u/\underline{Z}_u & 1 \end{bmatrix}\begin{bmatrix} TM_{11} & TM_{12} \\ TM_{21} & TM_{22} \end{bmatrix}\begin{bmatrix} 1 & M_d\underline{Z}_d \\ M_d/\underline{Z}_d & 1 \end{bmatrix}, \quad (1)$$

where M_u and $\underline{Z}_u$ are the meanflow Mach number and characteristic flow impedance in the main pipe upstream of the truncation and $M_d = M_u$; $\underline{Z}_d = \underline{Z}_u$.



$$TM_{11} = (A_{11}B_{12} + B_{11}A_{12})\,/\,(A_{12} + B_{12})\,; \quad TM_{12} = A_{12}B_{12}/\,(A_{12} + B_{12})\,;$$
$$TM_{21} = \frac{A_{11}B_{22} + A_{22}B_{11} - A_{11}A_{22} + A_{21}A_{12} + A_{12}B_{21} + A_{21}B_{12} - B_{11}B_{22} + B_{12}B_{21}}{A_{12} + B_{12}}; \tag{2}$$
$$TM_{22} = (A_{22}B_{12} + B_{22}A_{12})\,/\,(A_{12} + B_{12})\,;$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = e^{-jM_a k_{ca} \ell_a} \begin{bmatrix} \cos(k_{ca}\ell_a) & j\underline{Z}_a \sin(k_{ca}\ell_a) \\ j\sin(k_{ca}\ell_a)/\underline{Z}_a & \cos(k_{ca}\ell_a) \end{bmatrix}; \tag{3}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = e^{-jM_a k_{ca} \ell_a} \begin{bmatrix} \cos(k_{cb}\ell_b) & j\underline{Z}_b \sin(k_{cb}\ell_b) \\ j\sin(k_{cb}\ell_b)/\underline{Z}_b & \cos(k_{cb}\ell_b) \end{bmatrix}; \tag{4}$$

$$k_{ca} = k_0/(1 - M_a^2)\,; \quad k_{cb} = k_0/(1 - M_b^2)\,. \tag{5}$$

ℓ_a (L_a) and ℓ_b (L_b) are lengths of the main duct and bypass duct, respectively, and

$$\underline{Z}_a = \rho_0 c_0/S_a\,; \quad \underline{Z}_b = \rho_0 c_0/S_b\,. \tag{6}$$

S_a and S_b are areas of cross section of the main duct and bypass duct, respectively; M_a and M_b are the meanflow Mach number in the main duct and bypass duct, respectively. These are given by the following expressions:

$$M_a = M_u - \frac{S_b}{S_a} M_b\,; \quad S_a = S_u\,; \quad M_b = M_u/(f_{11} + S_3/S_2)\,;$$
$$f_{11} = \left\{ \frac{d_a}{\ell_a} \left(\frac{\ell_b}{d_b} + 187.5 \right) \right\}^{1/2}. \tag{7}$$

It may be noted that generally M_b will be much less than M_a because of the transverse connection and the requirement of equal pressure drop across the two parallel arms.

K.21 Annular Airgap Lined Duct

Automotive exhaust systems are characterised by hot gas flows, often containing some unburnt carbon particles. In an acoustically lined duct with high velocity grazing flow, there is a strong possibility of some fibres of a fibrous absorptive material like glass wool, ceramic wool or mineral/rock wool being swept away continuously and the perforated protective plate getting progressively clogged with unburnt carbon particles and possibly lubricating oil. One of the alternatives would be to make use of another perforated cylinder between the inner flow pipe and the absorptive layer with an airgap in between, as shown. This element involves acoustic interaction between three ducts, viz. the inner flow duct, the annular airgap duct, and the outer acoustically lined duct. Therefore, the analysis of this element runs on the same lines as those of *Sect. K.12*, with the important difference that the medium in the outer lined duct is different; its wave number k_w and characteristic impedance $\underline{Z}_w$ are given in terms of the specific flow resistance by Mechel's expressions [Mechel (1976)]. The final transfer matrix is given by [Munjal/Venkatesham/Iutam (2000)].

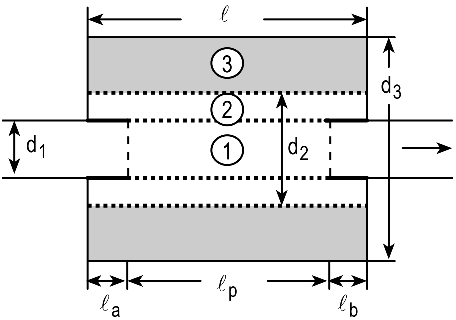
$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} ACP_{11} & ACV_{11}\underline{Z}_d \\ ACP_{61}/\underline{Z}_u & ACV_{61}\underline{Z}_d/\underline{Z}_u \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{1}$$

where [ACP] and [ACV] are 10 × 10 matrices given by:

$$[ACP] = [A]^{-1}\{CP\}; \quad [ACV] = [A]^{-1}\{CV\} \tag{2}$$

with $\underline{Z}_u = \underline{Z}_d = \rho_0 c_0/(\pi d_1^2/4)$ and

$$\begin{aligned} \{CP\} &= \begin{bmatrix} T_{11} & T_{21} & T_{31} & T_{41} & T_{51} & T_{61} & 0 & 0 & 0 & 0 \end{bmatrix}^T; \\ \{CV\} &= \begin{bmatrix} T_{14} & T_{24} & T_{34} & T_{44} & T_{54} & T_{64} & 0 & 0 & 0 & 0 \end{bmatrix}^T; \end{aligned} \tag{3}$$



$$[A] = \begin{bmatrix} 1 & 0 & 0 & -T_{12} & -T_{13} & 0 & 0 & 0 & -T_{15} & -T_{16} \\ 0 & 1 & 0 & -T_{22} & -T_{23} & 0 & 0 & 0 & -T_{25} & -T_{26} \\ 0 & 0 & 1 & -T_{32} & -T_{33} & 0 & 0 & 0 & -T_{35} & -T_{36} \\ 0 & 0 & 0 & -T_{42} & -T_{43} & 1 & 0 & 0 & -T_{45} & -T_{46} \\ 0 & 0 & 0 & -T_{52} & -T_{53} & 0 & 1 & 0 & -T_{55} & -T_{56} \\ 0 & 0 & 0 & -T_{62} & -T_{63} & 0 & 0 & 1 & -T_{65} & -T_{66} \\ 0 & -X_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -X_4 & 1 \\ 0 & 0 & 0 & -X_4 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \tag{4}$$

$$\begin{aligned} X_1 &= -j\tan(k_0\ell_a); \quad X_2 = -j\tan(k_0\ell_b); \\ X_3 &= -j\tan(k_w\ell_a); \quad X_4 = -j\tan(k_w\ell_b). \end{aligned} \tag{5}$$

[T] is the 6 × 6 transfer matrix for the common perforated section where all three duct sections interact. It is evaluated exactly as shown in Sect. K.12; only the values of αs in the coefficient matrix are different. These are as follows:

$$\begin{aligned}&\alpha_1 = -(0.5f_0M_1 + f_1)/\text{den}; \quad \alpha_2 = k_0^2/\text{den}; \quad \alpha_3 = f_1/\text{den}; \quad \alpha_4 = f_2/\text{den};\\ &\alpha_5 = 0; \quad \alpha_6 = f_4; \quad \alpha_7 = 0; \quad \alpha_8 = -k_0^2 - f_4 - f_6; \quad \alpha_9 = 0; \quad \alpha_{10} = f_6;\\ &\alpha_{11} = 0; \quad f_{12} = f_7; \quad \alpha_{13} = 0; \quad \alpha_{14} = k_w^2 - f_7,\end{aligned} \tag{6}$$

$$\text{where: } f_0 = 4jk_0; \quad f_1 = 4M_1\xi_1/d_1; \quad f_2 = f_0\xi_1/d_1. \tag{7}$$

Y_w and k_w are given by Mechel's formulae given in Sect. J.25 and den $= 1 - M_1^2$.

ξ_1 is the reciprocal of the non-dimensional grazing-flow impedance of the perforate given in Sect. K.11, at the interface 1–2 (diameter d_1); ξ_2 is the reciprocal of the non-dimensional stationary-flow impedance of the perforate given in Sect. K.12, at the interface 2–3 (diameter d_2).

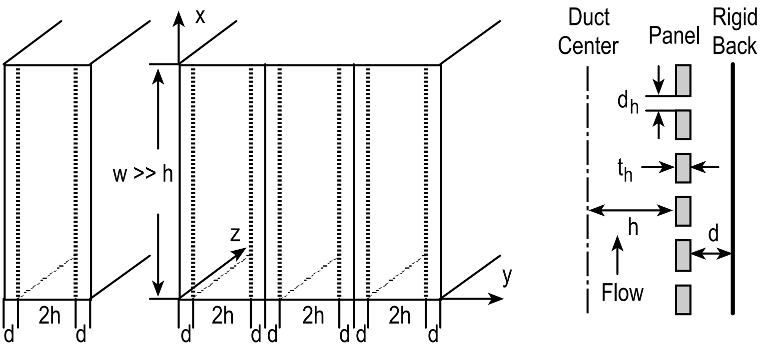
K.22 Micro-Perforated Helmholtz Panel Parallel Baffle Muffler

This element, in a way, is a combination of a concentric tube resonator [Sect. K.11(a)] and a parallel baffle muffler (Sect. J.24) without absorbing material. Following Wu (1997) the transfer matrix of this element across a length ℓ_p is given by:

$$\begin{bmatrix} \cos(k_z\ell_p) & j\underline{Z}\sin(k_z\ell_p) \\ j\sin(k_z\ell_p)/\underline{Z} & \cos(k_z\ell_p) \end{bmatrix}, \tag{1}$$

where

$$\begin{aligned}&\underline{Z} = \underline{Z}_0 k_0/k_z; \quad \underline{Z}_0 = \rho_0 c_0/\left(2n_p hW\right);\\ &k_z = (k_0^2 - k_y^2)^{1/2}.\end{aligned} \tag{2}$$



k_y is a root of the transcendental equation

$$k_y h \cdot \tan(k_y h) - jk_0 h\, \rho_0 c_0/Z = 0. \tag{3}$$

Z is the grazing-flow impedance of the micro-perforated panel:

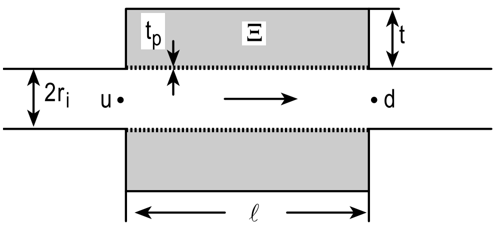
$$Z = (1+M)\left\{\frac{32\nu\rho_0\cdot t_h}{\sigma d_h^2}\left[\left(1+x^2/32\right)^{1/2}+0.177\cdot x\cdot d_h/t_h\right]\right.$$
$$\left.+\frac{j\omega\cdot t_h\cdot\rho_0}{\sigma}\left[1+\frac{1}{(9+x^2/2)^{1/2}}+0.85\frac{d_h}{t_h}\right]-j\frac{\rho_0 c_0}{\tan(k_0 d)}\right\} \tag{4}$$

with $x = d\cdot h\ \left(10^5\cdot f\right)^{1/2}$; M is the meanflow Mach number in the flow passage; σ = panel porosity; d, t_h, d_h, h and W are shown in the figure; $\nu = 1.56\cdot 10^{-5}$ Pa · s for air.

K.23 Acoustically Lined Circular Duct

Circular and annular ducts are extensively treated in Ch. J. Whereas that chapter describes steps in such ducts with a multimode analysis, the present Ch. K uses representations with a fundamental-mode analysis. Thus, in order to get some completeness in the collection of muffler elements in the present chapter, this section will present the transfer matrix method for an element which is also contained in Ch. J, there with higher mode analysis.

A uniform circular duct is often lined on the inside for acoustical absorption or dissipation into heat, as shown. For mechanical protection against the eroding effect of the moving medium, the acoustic layer is covered on the exposed side with a very thin membrane like Mylar or a highly perforated thin metallic plate. This protective cover affects the absorptive properties of the acoustic lining at certain frequencies and therefore needs to be included in the acoustic model.



The protective layer is characterised by a radial partition impedance that would support a pressure difference between the two media. The porous/fibrous acoustic layer is characterised by a complex wave number k_w (= $-j\cdot\Gamma_a$ in earlier chapters) and a characteristic impedance Z_w (= Z_a in earlier chapters). These in turn may be expressed as a function of the flow resistivity Ξ or of an “absorber variable” E, given by:

$$E = \frac{\rho_0 f}{\Xi} = \frac{\rho_0 c_0}{\lambda_0 \Xi}, \tag{1}$$

with λ_0 the free field wave length (see Ch. *G*). For the fibrous materials, the characteristic constants Z_w and k_w are approximately described by the empirical formulae of [Delany/ Bazley (1970)], as modified and improved by [Mechel (1976)]:

$$\frac{Z_w}{\rho_0 c_0} = \begin{cases} 1 + 0.0485E^{-0.754} - j0.087E^{-0.73}\,; & E > 60 \\ \dfrac{0.5/(\pi E) + j1.4}{(-1.466 + j0.212/E)^{1/2}}\,; & E < 60 \end{cases}; \tag{2}$$

$$\frac{k_w}{k_0} = \begin{cases} 1 - j0.189E^{-0.6185} + 0.0978E^{-0.6929}\,; & E > 60 \\ -\left(1.466 - j0.212/E\right)^{1/2}\,; & E < 60 \end{cases}.\ ^{*)} \tag{3}$$

The transfer matrix for a bulk reacting lined duct shown above is given by [Munjal/Thawani (1997)]:

$$\begin{bmatrix} p_u \\ u_u \end{bmatrix} = \begin{bmatrix} \cos(k_z\ell) & j\underline{Z}\sin(k_z\ell) \\ (j/\underline{Z})\sin(k_z\ell) & \cos(k_z\ell) \end{bmatrix} \begin{bmatrix} p_d \\ u_d \end{bmatrix}, \tag{4}$$

where $\underline{Z} = \underline{Z}_0 k_0/k_z\,;\quad \underline{Z}_0 = \rho_0 c_0/S\,;\quad k_0 = \omega/c_0\,.$

The axial wave number for the fundamental mode is the first (lowest) root of the transcendental equation:

$$j\frac{\omega\rho_0 J_0(k_{r,0}r_i)}{k_{r,0}J_1(k_{r,0}r_i)} = j\frac{\omega\rho_w}{k_{r,w}}\frac{J_0(k_{r,w}r_i) + C\cdot Y_0(k_{r,w}r_i)}{J_0(k_{r,w}r_i) + C\cdot Y_1(k_{r,\omega}r_i)} + Z_p(\omega)\,; \tag{5}$$

$$k_{r,0} = (k_0^2 - k_z^2)^{1/2}\,;\quad k_{r,w} = (k_w^2 - k_z^2)^{1/2}\,;\quad C = -\frac{J_1(k_w r_o)}{Y_1(k_w r_o)}\,;\quad r_o = r_i + t\,. \tag{6}$$

Z_p, the partition impedance of a thin impermeable protective foil, is given by $Z_p = j\omega\rho_p t_p$ where t_p are the thickness and material density of the foil, respectively.

For a perforated plate, in the presence of a grazing meanflow, the corresponding expression is [Rao/Munjal (1986)]:

$$Z_p = \rho_0 c_0 \left[7.337\cdot 10^{-3}(1 + 72.23M) + j2.2245\cdot 10^{-5}f(1 + 51t_p)(1 + 204d_h)\right]/\sigma, \tag{7}$$

where d_h is the hole diameter in m, M is the meanflow Mach number, and σ is the porosity of the perforated plate.

A locally reacting lining would not support waves inside the lining in the axial direction. Therefore, for a locally reacting lining, $k_{r,w} = k_w$, and therefore the right-hand side of the foregoing transcendental Eq. (5) would be independent of the variable k_z.

In either case (for either type of lining), the appropriate transcendental equation can be solved for the axial wave number k_z by means of a Newton-Raphson scheme, making use of the start approximation:

$$\left(k_{r,0}r_o\right)^2 \approx \frac{96 + 36jQ \pm \left(9216 + 2304jQ - 912Q^2\right)^{1/2}}{12 + jQ}\,;\quad Q = (k_0 r_o)\frac{\rho_0 c_0}{Z_w}\,, \tag{8}$$

and Z_w is the right-hand side of the transcendental equation above for $k_{r,w} = k_w$, i.e. for the locally reacting lining case.

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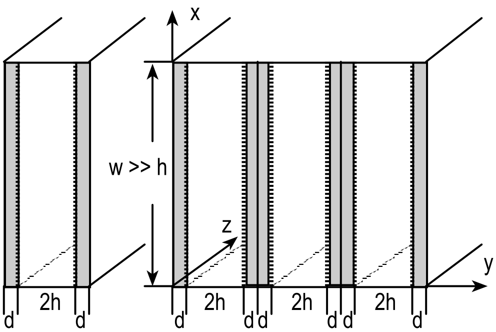
*) See Preface to the 2nd edition.

K.24 Parallel Baffle Muffler (Multipass Lined Duct)

See the introductory comment of the previous Sect. K.23.



The figure shows a typical parallel baffle muffle used in general to increase the contact area and to obtain the required attenuation within a short axial length. For plane-wave propagation, each of the passes will act as a two-dimensional rectangular duct as shown in the separate figure. The transfer matrix for axial wave propagation for the element is the same as for a lined circular duct (Sect. K.23). The difference lies in the value of the axial wave number, k_z. The eigenequation for a two-dimensional rectangular duct with bulk-reacting lining is:

$$jZ_0 \frac{k_0}{k_{y,0}} \cot(k_{y,0}h) = Z_p - jZ_w \cot(k_{y,w}d)\,, \tag{1}$$

where

$$Z_0 = \rho_0 c_0\,; \quad Z_w = \rho_w c_w\,; \quad k_{y,0} = (k_0^2 - k_z^2)^{1/2}\,; \quad k_{y,w} = (k_w^2 - k_z^2)^{1/2}\,. \tag{2}$$

k_w and Z_w are as given in Sect. K.23, and Z_p is the impedance of the porous cover.

For locally reacting linings, the foregoing transcendental equation would hold with the simplification $k_{y,w} = k_w$.

In either case, the transcendental equation or eigenequation may be solved for k_z by means of a Newton-Raphson iteration scheme, with the starting value being given by:

$$(k_{y,0}h/2)^2 \approx \frac{2.47 + Q + \left[(2.47+Q)^2 - 1.87Q\right]^{1/2}}{0.38}\,; \quad Q = jk_0 \frac{h}{2}\frac{\rho_0 c_0}{Z_w}\,. \tag{3}$$

Incidentally, the transmission loss or attenuation of a lined duct of length ℓ may be obtained from the imaginary component of the axial wave number k_z, by means of the equation:

$$TL = -8.68\,\mathrm{Im}\{k_z \ell\}\ [\mathrm{dB}], \tag{4}$$

or else from the transfer matrix of the lined duct or parallel baffle muffler, making use of the expression given in Sect. K.4.

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