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On the determination of phase and group velocities of dispersive waves in solids

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A new technique is developed to determine the dispersion relation and the propagational speeds of waves in dispersive solids. The phase spectrum of a broadband pulse is linearly related to the dispersion relation of the dispersive medium. The method is simpler than the continuous-wave phase comparison technique. Application is made to measure the phase and group velocities of waves in fiber-reinforced composite materials and in thin wires. This technique is expected to be applicable to measurements of acoustic or electromagnetic wave speeds in other dispersive media.

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I. INTRODUCTION

The velocity of a wave is usually associated with the phase difference between the vibrations observed at two different points during the passage of the wave. A plane harmonic wave of amplitude A, angular frequency ω $(=2\pi f)$, and wavelength $\lambda(=2\pi/k)$ propagating in a nonattenuating medium may be represented by $A\cos(\omega t-kx+\xi)$ $=A\cos\omega(t-kx/\omega+\epsilon)$, where $\xi=\omega\epsilon$ is a constant. The phase of the wave is then defined by the argument of the cosine function, and the wave velocity is $v=\omega/k$ which is usually called the *phase velocity*.

When the phase velocity v of a plane harmonic wave is a function of its frequency or wavelength, then the propagating medium, according to Havelock,[1] is *dispersive*. Such dispersion may be caused by (i) the presence of specimen boundaries (geometric dispersion), (ii) the frequency dependence of material constants, such as mass density, elastic moduli, dielectric constants, etc. (material dispersion), (iii) the scattering of waves by densely distributed fine inhomogeneities in a material (scattering dispersion), (iv) the absorption or dissipation of wave energy into heat or other forms of energy in an irreversible process (dissipative dispersion), and (v) the dependence of the wave speed on the wave amplitude (nonlinear dispersion). We shall omit the last-mentioned source of dispersion in the analysis to be presented. The theory of the propagation of pulses in dispersive media is well known.[1-3] The principal feature is that the pulse does not retain its initial shape as it propagates through the dispersive medium. Thus, a short-duration pulse may be dispersed into wave trains in time.

Measurements of the phase versus frequency of continuous waves in dispersive media determines the *group velocity* of the wave. By definition, the group velocity U equals $d\omega/dk=v+k\,dv/dk$. From the data of U versus ω, it seems that one could evaluate v as a function of k by integrating the preceding equation. However, in order to complete the integration, it would require precise values of ω and k at the lower limit of integrations, and these can not be easily obtained experimentally.

For unidirectional waves in a solid, the dispersion characteristics are similar to those of the propagation of electromagnetic signals in transmission lines or those of ultrasonic signals in delay lines. The former is discussed in many standard texts,[4] and the latter was addressed by Young.[5] Young clearly defined the concept of "phase delay" of continuous waves and "group delay" of a narrow-bandwidth pulse, and he showed that the latter is related to the derivatives of the phase characteristics of the pulse with respect to frequency. From the group delay, the group velocity of a pulse can be determined. However, Young[5] considered only waves without attenuation and pulses with narrow bandwidths.

Previous experimental work in solids has dealt with both continuous waves and pulses. As mentioned above, the phase velocity may be frequency dependent. Thus, errors in its determination are minimized by using monochromatic waves. In the so-called π-point phase comparison technique, which appears to have been first used by Balamuth[6] and is recently reviewed by Papadakis,[7] the half-wavelength of the wave is determined directly by varying the propagation distance between source and receiver and observing the corresponding π-phase change of the received signal. Although such measurements are easily carried out in gaseous and fluid media, this is not the case in solids.

The main difficulty of the π-point phase technique is that the phase must be measured to a frequency low enough so that the phases between the transmitted and received signals differ exactly by π or a known multiple of π.[7] To overcome this difficulty, Lynnworth *et al.*[8] used a sliding-wedge technique to vary the propagation length in three-dimensional carbon-phenolic composite specimens to determine the wavelength and, hence, the phase velocity directly. Unfortunately, the wedge interface and the pronounced material microstructure can give rise to anomalous effects. Other researchers used specimens of fixed thickness and varied the frequency of excitation. The group velocity is thus determined as a function of frequency, as will be discussed in more detail in Sec. II. Recently, however, Ting and Sachse[9] have combined the variable frequency method and the variable path length method. They developed a constraint equation from which the number of wavelengths in a specimen can be established unambiguously.

Pulse measurements of the dispersion have been made with broadband pulses as well as narrowband bursts. In a broadband-pulse technique, the frequency-dependent reflection coefficient of a pulse from a buffer-specimen interface is measured; from this, the frequency-dependent phase velocity of the specimen is inferred. The technique has been ap-

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plied to phase velocity measurements in Teflon[8] and boron-epoxy specimens.[10] There is serious question, however, whether the measured phase velocity is that which characterizes the bulk material, rather than the interfacial layer at the buffer and specimen interface.

In burst measurements, one measures the delay of a burst with a certain center frequency through the thickness of a specimen. Tauchert,[11] utilizing pairs of transducers with differing center frequencies, measured the group velocity in this way in boron-epoxy composite materials. Using a single pair of broadband transducers and varying the frequency of the electrical excitation burst, Sutherland and Lingle[12] measured dispersive effects in boron-aluminum specimens. With burst-delay measurements in very dispersive materials, it is impossible to unambiguously identify equivalent points in the undelayed burst and the specimen-delayed burst; therefore, it is not clear what wave speeed is actually measured.[7] Furthermore, as emphasized by Young,[5] the envelope of a received signal may differ considerably from the input signal, and this makes it difficult to determine its delay time through a specimen. This problem can be minimized by amplitude modulating the rf carrier with a particular smooth envelope such as a Gaussian contour.[5]

Other examples of elastic-wave dispersion measurements are those with a shock-tube technique by Whittier and Peck[13] in carbon-phenolic composites, by Yew and Yogi[14] in steel-epoxy specimens with an optical interferometry technique, and by Kaarsberg[15] in geological materials with ultrasonic techniques.

Our investigation was motivated primarily by the determination of wave speeds in eutectic and fiber-reinforced composite materials and porous solids. In Sec. III of this paper, we review the principles of the π-point phase comparison method and its limitations in uniquely determining the dispersion relation of a material.

In Sec. IV, we propose a different technique—based on the phase spectral analysis of a broadband pulse—for the determination of phase and group velocities of waves in dispersive media. In this technique, the phase function of a Fourier-analyzed pulse is evaluated. It is shown how the phase and group velocities as well as the dispersion relation for the material can be determined directly from it. The experimental technique and its implementation are described in Sec. V. In Sec. VI, we show the results obtained with this technique to characterize dispersive and nondispersive materials and to compare them with the results obtained with the continuous-wave π-point phase comparison technique.

Although the examples we cite in this paper are limited to ultrasonic stress waves and pulses propagating in dispersive solids, the principle and the experimental technique described in Secs. III and IV are perfectly general such that similar observations can be made with linear acoustic, electromagnetic, or optical waves propagating in dispersive media. A general discussion of the dispersion of waves and the concept of group velocity can be found in Refs. 16 and 17.

II. DISPERSION OF A LINEAR CAUSAL SYSTEM

A solid specimen sustaining the propagation of a plane wave along a length or thickness dimension, comprises a one-dimensional linear system. Such systems admit plane-wave solutions of the form

$$u(x,t)=A\exp[i\omega(t\pm x/v-\xi)]$$
$$=A\exp[(i\omega t\pm kx-\epsilon)]. \tag{1}$$

As noted earlier, the phase velocity v $(=\omega/k)$ is a constant for nondispersive media. More generally, v is a function of ω. Also, in real systems, the wave amplitude gradually attenuated through beam spreading, scattering, mode conversion, coupling losses between transducer and specimen, and the absorption and dissipation of energy in the specimen material. As long as the total energy is conserved, ω must be real, and the attenuation of the wave amplitude can be accounted for by letting k be complex. That is

$$k=\beta+i\alpha. \tag{2}$$

For waves propagating in the region $x\geqslant 0$, Eq. (1) is then written as

$$u(x,t)=A\exp(-\alpha x)\exp[i(\omega t\pm\beta x-\omega\xi)]. \tag{3}$$

In general, both α and β in Eqs. (2) and (3) are functions of ω.

Although the function in Eq. (3) is not periodic, we can still define a phase velocity v and group velocity U of the wave as

$$v=\frac{\omega}{\mathrm{Re}(k)}=\frac{\omega}{\beta},\qquad U=\frac{d\omega}{d\beta}. \tag{4}$$

The wavelength of the decaying sinusoidal wave is $2\pi/\beta$, and α is the attenuation factor. We caution that the group velocity U as customarily defined only has physical significance when $\alpha(\omega)$ is small.[16] By small $\alpha(\omega)$, we mean that the attenuation in Np for one wavelength is small.

The dispersive property of a physical system is characterized by specifying the α and β as functions of ω. For a causal system, in which the response of the system to a driving force cannot precede the first arrival of the signal generated by the driving force, the $\alpha(\omega)$ and $\beta(\omega)$ are related by Kramers-Kronig relation. Mathematically, for a causal system, the $\alpha(\omega)$ are related to the Hilbert transform of $\beta(\omega)$ and vice versa.[3] Thus, we shall regard that $\alpha(\omega)$ can be calculated from $\beta(\omega)$, and the dispersion property of the linear causal system is defined by the relation

$$\beta=\hat{\beta}(\omega)\quad\text{or}\quad\omega=\hat{\omega}(\beta). \tag{5}$$

In Eq. (5), $\hat{\beta}(\omega)$ is a real function of the real variable ω, and $\hat{\omega}(\beta)$ is the inverse function of $\hat{\beta}(\omega)$. The objective of the experiments is to determine the *dispersion relation*, Eq. (5), for a solid medium and then to calculate $v(\omega)$ and $U(\omega)$ from Eq. (4) or to measure either v or U and then determine the dispersion relation, Eq. (5).

If there exists a dissipative mechanism in the system, the frequency ω may also be complex valued. One example is the thermoelastic wave in solids.[18,19] In such a case, there are

two choices: one is to restrict the answer to real ω but complex k, the other is to limit the answer to real k but complex ω. The corresponding phase velocities are defined, respectively, as $v=\omega/\mathrm{Re}(k)$ and $v=\mathrm{Re}(\omega)/k$, and the two results do not always agree. The meaning of group velocity also becomes uncertain. The situation is further complicated by possible anisotropy in the material. The concept of phase and group velocity for waves in a dissipative and anisotropic medium has been discussed in Ref. 19.

III. THE METHOD OF π-POINT PHASE COMPARISON

By transmitting continuous waves of varying frequencies through a specimen of length L, a phase-vs-frequency relation can be measured for the specimen. By adjusting the frequency to, say, f_1, the signal at the receiver is first brought into in-phase or π rad out of phase with that from the transmitter. At this frequency, let the wavelength be λ_1 and the number of complete sine waves be N_1; thus $f_1=v_1N_1/L$, where v_1 is the phase velocity. An increment of the phase by a known multiple of π rad is made by changing the frequency to f_2 with N_2 the number of sine waves of length λ_2 and $f_2=v_2N_2/L$. For nondispersive waves, $v_1=v_2=v$. By setting $N_2=N_1+1$ in the experiment,[7] we find

$$v=L(f_2-f_1). \tag{6}$$

In this situation, it is not necessary to know the exact value of N_1 at the starting point.

For dispersive media, v_1 differs from v_2 because of the change of frequency, and so one cannot determine either, unless N_1 is known exactly. However, as illustrated by the hypothetical experimental data shown in Fig. 1, it is still possible to determine the group velocity U by measuring $f_j(j=1,2,3,\cdots)$ and N_j, where $N_{j+1}=N_j+1$. This follows from the following relations:

$$U=\lim_{\Delta\to 0}\left(L\frac{\Delta f}{\Delta N}\right)=L\frac{df}{dN}=L\left(\frac{dN}{df}\right)^{-1}. \tag{7}$$

In this case, U is only determined in a range of frequencies higher than f_1.

Once $U(N)$ is determined for a range of $N\geqslant N_1$, the phase velocity can be calculated by the following integration:

$$v=(1/N)\int^N U(N)\,dN+v_1(N_1/N). \tag{8}$$

Thus, we are facing an uncertainty unless the value for N_1 is known. This can easily be seen from Fig. 1 in which the location of N_1 is uncertain. The slope of the tangent to the curve f versus N which is related to the group velocity can be calculated for $N > N_1$, but the slope of the secant (a line from the origin to a point on the curve) cannot be determined. The slope of the secant is precisely v/L.

IV. THE METHOD OF PHASE SPECTRAL ANALYSIS

A. Fourier synthesis of the transmitted pulse

It was mentioned that a linear system admits plane harmonic waves described by Eq. (1) and, if attenuation exists, by Eq. (3). Based on the principle of superposition, a pulse $u(x,t)$ propagating in the linear medium $x\geqslant 0$ may be expressed as an addition of all plane harmonic waves.

$$u(x,t)=(1/2\pi)\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}A(\xi)\times\exp(-\alpha x)\exp[i\omega(t-x/v-\xi)]\,d\xi, \tag{9}$$

where $A(\xi)$ is an unspecified amplitude function, and both v and α are functions of ω.

Assuming that the medium is at rest initially, and a disturbance $F(t)$ is generated at the location $x=0$, for $t\geqslant 0$, that is,

$$u(0,t)=F(t), \tag{10}$$

then, from Eq. (9), we find

$$F(t)=(1/2\pi)\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}A(\xi)\exp[i\omega(t-\xi)]\,d\xi. \tag{11}$$

This expression is exactly the Fourier integral representation of the input time function $F(t)$. Hence, $A(\xi)=F(\xi)$, and the amplitudes of the superposed plane waves are fixed by the end condition at $x=0$. We thus write Eq. (9) as

$$u(x,t)=(1/2\pi)\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}F(\xi)\times\exp(-\alpha x)\exp[i\omega(t-x/v-\xi)]\,d\xi. \tag{12}$$

If α and v are constant, we find from Eq. (12) and the theorem of Fourier integral the well-known result

$$u(x,t)=F(t-x/v)\exp(-\alpha x). \tag{13}$$

The input pulse $F(t)$ at $x=0$ propagates through the nondispersive medium without a change of shape, except for an exponential decrease in amplitude.

When both α and v are functions of ω, the double integral in Eq. (12) gives rise to the dispersed pulse at any location. To determine the signal at $x=L$ in a specimen, we

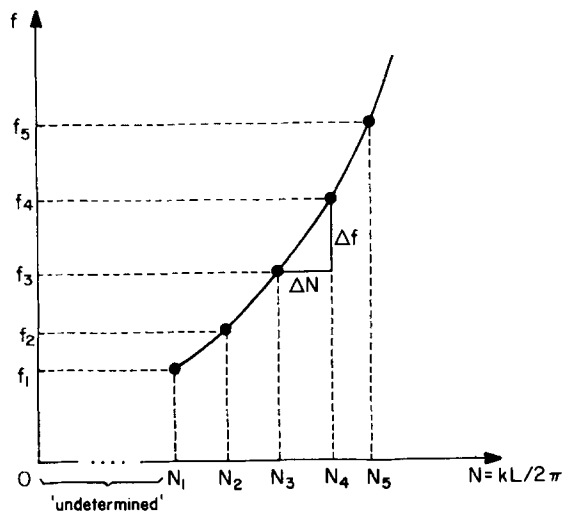

FIG. 1. Hypothetical dispersion curve frequency f versus number of waves N in a length L.

evaluate, first, the integration with respect to ξ,

$$u(L,t)=(1/2\pi)\int_{-\infty}^{\infty}\bar{F}(\omega)\exp(-\alpha L)\times\exp[i\omega(t-L/v)]\,d\omega, \quad (14)$$

where

$$\bar{F}(\omega)=\int_{-\infty}^{\infty}F(\xi)\exp(-i\omega\xi)\,d\xi. \quad (15)$$

Note that $\bar{F}(\omega)$ is the Fourier transform of the input $F(t)$ which is supposed to be known. Since $\beta=\omega/v$, Eq. (14) can be rewritten as

$$u(L,t)=(1/2\pi)\int_{-\infty}^{\infty}[\bar{F}(\omega)\exp(-\alpha L)\times\exp(-i\beta L)]\exp(i\omega t)\,d\omega. \quad (16)$$

The exact shape $u(L,t)$ can be calculated from integral (16) if the dispersion relation [Eq. (5)] is known, but the calculation will, in general, be very difficult. However, the spectrum of the transmitted pulse at $x=L$ is given by the Fourier transform of $u(L,t)$, which is precisely the quantity inside the brackets of Eq. (16),

$$\bar{u}(L,\omega)=\bar{F}(\omega)\exp(-\alpha L)\exp(-i\beta L). \quad (17)$$

In general, the Fourier transform of a causal time function $F(t)$ may be complex. Thus,

$$\bar{F}(\omega)=|\bar{F}(\omega)|\exp(-i\omega\xi)=|\bar{F}(\omega)|\exp(-i\phi_0), \quad (18)$$

where $\phi_0=\omega\xi$. Equation (17) can then be written as

$$\bar{u}(L,\omega)=|\bar{F}(\omega)\exp(-\alpha L)|\exp[-i(\beta L+\omega\xi)]. \quad (19)$$

This means that the phase spectrum of $u(L,t)$ is given by

$$\phi(\omega)=\beta L+\phi_0=L\hat{\beta}(\omega)+\omega\xi. \quad (20)$$

It is seen from Eq. (20) that the phase spectrum of the transmitted pulse is linearly related to the dispersion relation $\hat{\beta}(\omega)$ of the medium.

B. Phase spectra and dispersion relations

In Sec. V, we show experimentally that an accurate spectrum of the transmitted pulse $u(L,t)$ can be determined by digital analysis. The spectrum is composed of two parts: the amplitude spectrum and the phase spectrum. The latter is precisely the $\phi(\omega)$ of Eq. (20). Thus, the dispersion relation of a medium is determined uniquely by the phase spectrum of a propagating pulse,

$$\hat{\beta}(\omega)=(1/L)[\phi(\omega)-\phi_0]. \quad (21)$$

The phase velocity is obtainable from the experimental data,

$$v(\omega)=\omega/\beta=\omega L/[\phi(\omega)-\phi_0]. \quad (22)$$

By differentiating the phase function, we obtain

$$\frac{d\phi(\omega)}{d\omega}=L\frac{d\hat{\beta}}{d\omega}+\xi=L\left(\frac{d\omega}{d\beta}\right)^{-1}+\xi=\frac{L}{U}+\xi. \quad (23)$$

The group velocity U can then be calculated from the derivatives of the phase spectrum

$$U=L\left(\frac{d\phi}{d\omega}-\xi\right)^{-1}. \quad (24)$$

When ω is complex valued, as in a dissipative medium, the Fourier synthesis in Eq. (9) is meaningless unless the path of integration for the complex variable ω is specified. An alternative would be to assume k real valued and to write

$$u(x,t)=(1/2\pi)\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}A(\zeta)\exp[ik(vt-x-\zeta)]\,d\zeta. \quad (25)$$

The ζ is a phase constant in the space axis, and $v=\omega/k$ must be complex valued for a medium with damping. As shown by Havelock,[1] an integral representation such as Eq. (25) is suitable for analyzing initial value problems (u is given at $t=0$). For waves in a medium without dissipation, these two representations [Eqs. (9) and (25)] are equivalent.

C. A simple example

That the phase spectrum of a propagating pulse is related to the group velocity may be illustrated by the following simple example. Consider a square pulse of duration $2T_0$ generated at $x=0$.

$$u(0,t)=u_0, \quad 0<t<2T_0, \quad (26)$$

and $u(0,t)=0$ for $t<0$ and for $t>2T_0$. At time $t=T$, the pulse travels through a nondispersive and nonattenuating medium a distance L. Since the pulse shape is preserved [Eq. (13)], the disturbance at $x=L$ is

$$u(L,t)=u_0, \quad T<t<T+2T_0, \quad (27)$$

and $u(L,t)=0$ for $t<T$ and $t>T+2T_0$. The Fourier transforms of $u(0,t)$ and $u(L,t)$ are, respectively,

$$\bar{u}(0,\omega)=2T_0(\sin\omega T_0/\omega T_0)\exp(-i\omega T_0), \quad (28)$$

$$\bar{u}(L,\omega)=2T_0(\sin\omega T_0/\omega T_0)\exp[-\omega(T+T_0)]. \quad (29)$$

The phase spectrum of $\bar{u}(L,\omega)$ and its derivative are

$$\phi(\omega)=\omega(T+T_0), \quad \frac{d\phi}{d\omega}=T+T_0. \quad (30)$$

Comparing Eq. (29) with Eq. (19), with $\alpha=0$, and Eq. (30) with Eq. (23) we identify $\xi=T_0$ and find $T=L/U$. This last result states that the square pulse traverses the distance L with the group velocity U which, in this case, is also equal to the phase velocity.

V. EXPERIMENTS AND DIGITAL SPECTRAL ANALYSIS

The two methods described in Secs. III and IV were implemented to measure the dispersion relation and phase and group velocities for nondispersive and dispersive materials. The experimental setup for the continuous-wave π-point phase difference measurements is simiar to that in Ref. 7 (Fig. 22). The block diagram for the equipment used in the

pulse transmission experiments is shown in Fig. 2. Two nearly identical broadband transducers were used in both experiments.

In the continuous-wave experiments, the source transducer was excited with a 10-V peak-to-peak harmonic wave ranging in frequency from 50 kHz to 20 MHz. The signals from the receiver transducer were amplified and displayed on an oscilloscope. The amplitude and phase measurements of the continuous waves were made directly from the oscilloscope and with a vector voltmeter for frequencies above 1 MHz.

In the pulse experiments, the source transducer was excited with a -250-V pulse of 35-nsec duration. A sampling oscilloscope was used to overcome the bandwidth limitation of the A/D converter attached to a minicomputer. Any selected portion of a time record could be sampled by means of a digital delay unit used in conjunction with the sampling plug-in unit of the oscilloscope.

A fast-Fourier transform algorithm was used to compute the Fourier amplitudes of the input time function $F(t)$. The actual time record was associated with 1024 time domain data points which were padded with 1024 zeros. The real and imaginary parts of of the Fourier transform, $\bar{F}_1(\omega)$ and $\bar{F}_2(\omega)$, respectively, were first calculated. The absolute value of the transform is given by $|\bar{F}(\omega)| = (\bar{F}_1^2+\bar{F}_2^2)^{1/2}$ and the phase function by $\phi(\omega)=\tan^{-1}[\bar{F}_2(\omega)/\bar{F}_1(\omega)]$.

The computer algorithm limits the values of ϕ in the reduced range from π to $-\pi$. Thus, discontinuities of π for the phase spectra were found whenever the magnitude of $\phi(\omega)$ becomes less than or greater than zero and 2π. This was corrected by adding or subtracting the amount of jump to render the phase spectrum continuous. The resultant data was then used to compute the wave number, phase velocity, and group velocity associated with the pulses $u(0,t)$ and $u(L,t)$ according to Eqs. (21), (22), and (24).

To initialize the pulse analysis system and thus set ϕ_0, the source transducer was placed in intimate contact with the receiving transducer, and the maximum of the received excitation pulse was made to coincide with the start of the sampled time record by means of a vernier-controlled digital delay unit. In this way, ϕ_0 was set equal to 0. Alternatively, the phase spectrum of the received excitation pulse could be computed and stored in the data analysis system. This spectrum is exactly ϕ_0, and it could be subtracted from all subsequent phase spectra calculations.

The sampled sweep was digitally delayed in steps of 100 nsec until the pulse to be analyzed was displayed on the sampling oscilloscope. The time shift τ_s in increments of 100 nsec, relative to $\tau_s=0.0$, was read into the computer as was the propagation distance L. Equations (21), (22), and (24) for the dispersion relation, phase velocity, and group velocity, respectively, were modified to include the time-shift factor. They are, in terms of frequency f,

$$\hat{\beta}(f)=L^{-1}[\phi_0-\phi(f)]+2\pi f\tau_s, \tag{31}$$

$$v(f)=\frac{2\pi fL}{[\phi_0-\phi(f)]+2\pi f\tau_s}, \tag{32}$$

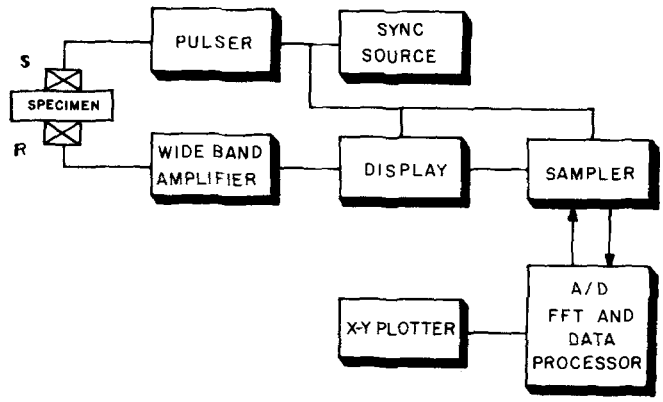

FIG. 2. Block diagram of the apparatus for measuring the phase spectrum.

$$U(f)=2\pi L\left(\frac{d\phi}{df}-\tau_s\right)^{-1}. \tag{33}$$

The operation of the system was checked by entering a simulated square pulse, and the phase spectrum $\phi(\omega)$ was indeed a straight line as given by Eq. (30).

VI. EXPERIMENTAL RESULTS

To experiment with waves in a nondispersive medium, a broadband longitudinal ultrasonic pulse, comprised mainly of frequency components from 3 to 12 MHz was propagated through a 1.900-cm-thick specimen of 6061-T6 aluminum. In Fig. 3(a) is shown the excitation pulse $u(L,t)$ at $L=1.900$ cm. In Figs. 3(c) and 3(d) are shown the Fourier amplitude spectrum and phase spectrum of $u(L,t)$. Figure 3(d) also yields the dispersion relation as $\beta=\phi/L$ [Eq. (21)]. The calculated phase velocity and group velocity of this specimen are shown in Figs. 3(e) and 3(f). It is clear from Fig. 3(d) that there is little dispersion in this material in the frequency range 0–20 MHz.

The circular dots shown in Fig. 3(f) represent the group velocity values determined from the data measured by the continuous-wave π-phase technique [Eq. (7)]. For this specimen, no measurements were possible below 1.5-MHz because transducer near-field effects dominated the observations. At frequencies above 7 MHz, data were recorded only at integer frequency points. The average group velocity measured using the continuous-wave technique was 0.618 $\pm$0.009 cm/μsec.

The results for a dispersive medium such as composite materials are shown in Fig. 4. A broadband shear pulse is propagated through a 0.546-cm-thick specimen of 96-ply boron epoxy. In this case, the wave propagates in the direction of the fiber, and the particle displacement direction is perpendicular to the fiber and parallel to the ply layers. Figures 4(a) and 4(b) show the initial pulse and the dispersed pulse, respectively, detected at the receiving transducer. It is apparent that the high-frequency components of the pulse propagate faster than the low-frequency components. The relative amplitudes of the various frequency components in the pulse are shown in Fig. 4(c). As expected, the high-frequency components are markedly lower in amplitude

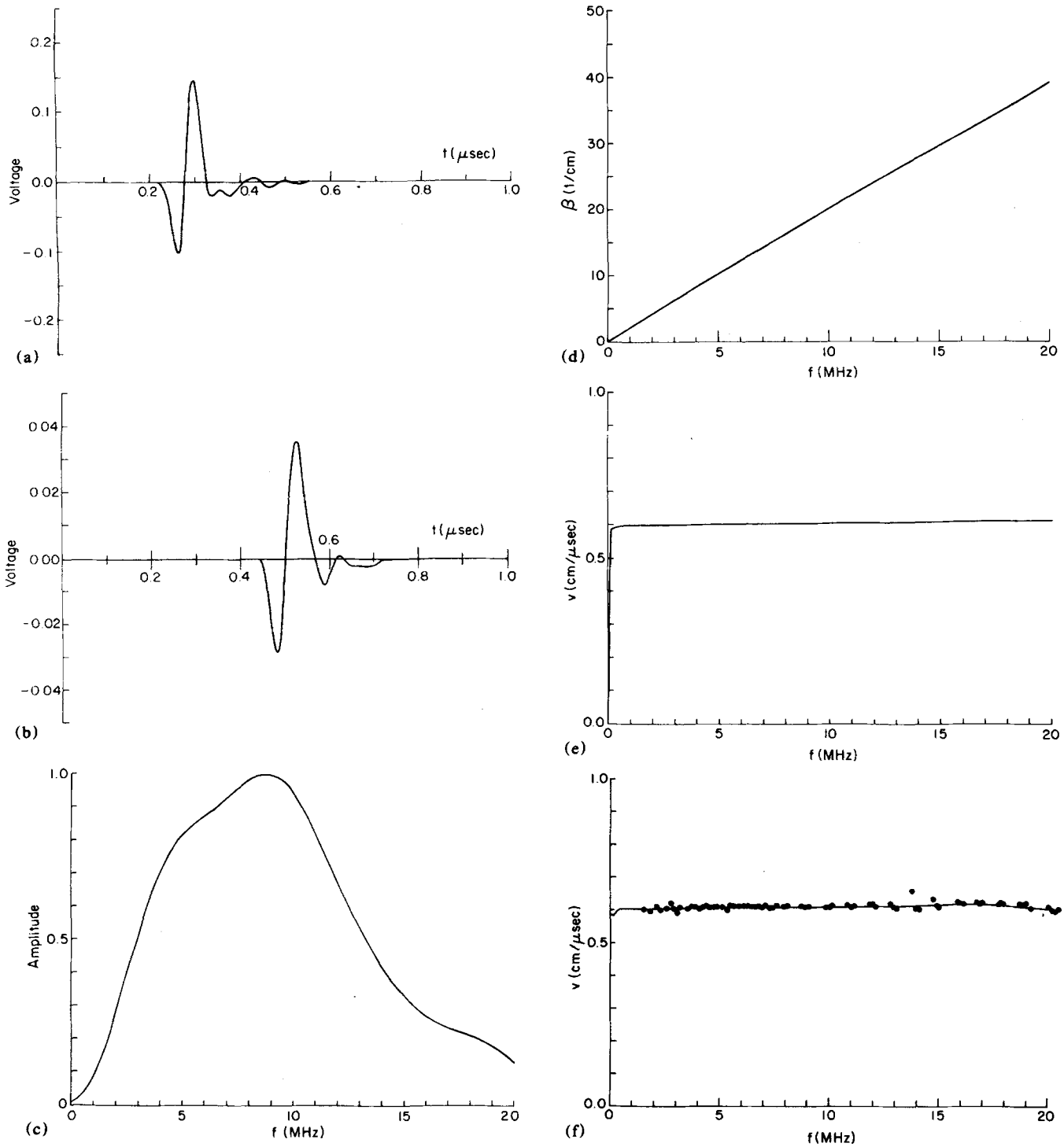

FIG. 3. Pulse propagation, dispersion, phase and group velocities of waves in 6061–T6 aluminum. (a) Input P pulse in voltage ($\tau_s = 0.0\,\mu$sec). (b) Received P pulse at L in voltage ($\tau_s = 2.6\,\mu$sec). (c) The amplitude spectrum of the received pulse (relative amplitude versus frequency f in MHz). (d) The phase spectrum of the received pulse and the dispersion relation (wave number β in cm^{-1} versus frequency f in MHz). (e) Phase velocity v (cm/μsec) versus frequency f (MHz). (f) Group velocity U (cm/μsec) versus frequency f (MHz).

than the low-frequency components. The dispersion relation is shown in Fig. 4(d), which is no longer a linear function of ω. This curve is obtained from the experimental phase spectrum by scaling the ordinate with the length L. Figure 4(e) and 4(f) show the results of the phase and group velocity computation for the frequency range 0–10.0 MHz. Results for the spectral amplitudes beyond 10.0 MHz are not dependable since the amplitudes become negligibly small there.

The circular dots on the group velocity curve are obtained from the π-phase technique. The extent of the dispersion can be readily seen in the dispersion curve in Fig. 4(d).

As another demonstration of the application of the

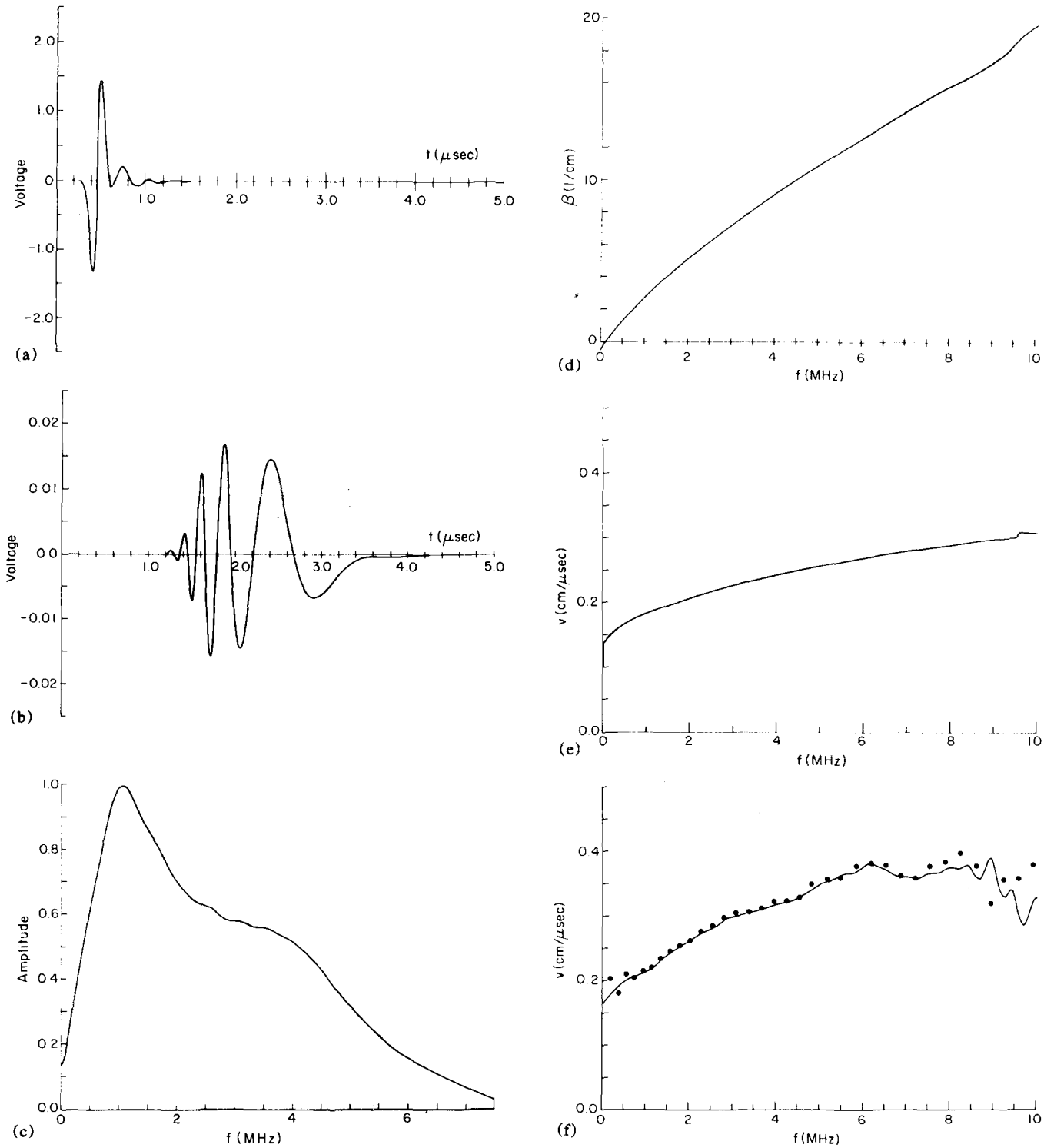

FIG. 4. Pulse propagation, dispersion, phase and group velocities of waves in 96-ply boron-epoxy composite material. (Shear wave propagating in the direction of the fibers). (a) Input S pulse in voltage ($\tau_s = 0.0\,\mu$sec). (b) Received S pulse at L in voltage ($\tau_s = 0.0\,\mu$sec). (c) The amplitude spectrum of the received pulse (relative amplitude versus frequency f in MHz). (d) The phase spectrum of the received pulse and the dispersion relation (wave number β in cm^{-1} versus frequency f in MHz). (e) Phase velocity v (cm/μsec) versus frequency f (MHz). (f) Group velocity U (cm/μsec) versus frequency f (MHz).

Fourier phase analysis technique, we show its use in the analysis of a broadband pulse propagating in a 76.4-cm length of 1.61-mm (outside diameter) Remendur tubing. In this case, both source and receiving transducers consisted of 280-turn coils of No. 32 AWG wire wound on a section of glass tubing and a −50-V pulse having a width of about 1 μsec was used as excitation. The incident pulse contains frequency components from 150 to 500 kHz. The signal, received with a second coil placed over the other end of the tubing, is shown after 40-dB amplification in Fig. 5(a). The pulse has been shifted by 135.7 μsec to bring it on scale. In contrast to the previous example, the low-frequency components of the pulse in this case propagate faster than the high-frequency components. The amplitude spectrum normalized to the

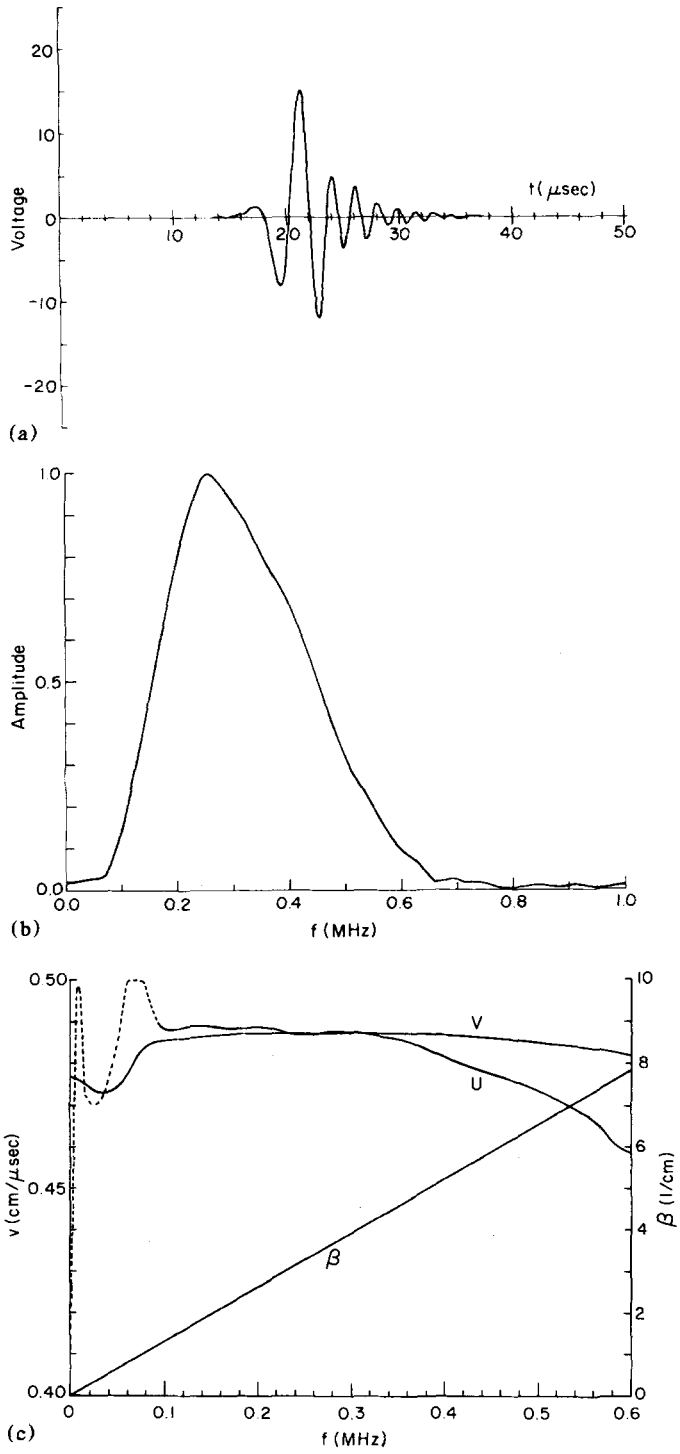

FIG. 5. Dispersion of a pulse in a Remendur tubing of 1.61-mm outside diameter. (a) Received extensional pulse at L in voltage ($\tau_s = 135.7$ μsec). (b) The amplitude spectrum of the received pulse (relative magnitude versus frequency f in MHz). (c) Dispersion relation (wave number β in cm^{-1} versus frequency in MHz), phase velocity v in cm/μsec, and group velocity U in cm/μsec.

maximum value is shown in Fig. 5(b), while in Fig. 5(c) are shown the derived dispersion relation and phase and group velocities for this material.

VIII. CONCLUSIONS

In this paper, it has been shown that the pulse phase spectrum defines the dispersion relation of a linear causal medium, and a phase spectral analysis of a dispersed pulse yields results for both phase and group velocities of waves in dispersive media.

Most continuous-wave techniques require tedious frequency scanning and data reduction. In contrast, the pulse analysis technique is rapid. The technique appears not to be restricted to the analysis of elastic pulses propagating in solids. It could well find application to studies of the dispersion of acoustic or electromagnetic waves in dispersive media.

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