# F Radiation of Sound

Radiation of sound takes place, not only if a surface is driven by an internal force, but also if the surface is set in vibration by an incident sound wave. Then radiation is the back reaction of the surface to the incident sound in the process of reflection and/or scattering. Part of the power which the vibrating surface produces with the exciting sound pressure is radiated as effective power to infinity; this gives rise to the *radiation loss* of the surface. Part of the reaction is contained in non-radiating near fields; they will influence the tuning of resonating surfaces by the inertia of their *oscillating mass*. This oscillating mass can be represented as the mass contained in a prism with the cross section of the vibrating surface (e.g. an orifice) and the length of an *end correction*. The advantage of the concept of the oscillating mass and of the end correction is the possibility to include them as members in equivalent networks (they are determined just so that this is possible).

Recall the distinction between "mechanical impedance", "impedance", and "flow impedance" from *Sect. A.3* conventions.

## F.1 Definition of Radiation Impedance and End Corrections

▸ *See also:* Mechel, Vol. I, Ch. 9 (1989)

Let $v_n(s)$ be the oscillating velocity in a surface A with the co-ordinate s in A, and directed normal to the surface towards the side, on which a sound pressure p(s) exists. The time average sound power produced is:

$$\Pi = \Pi' + j \cdot \Pi'' = \frac{1}{2} \iint_A p(s) \cdot v_n^*(s)\ dA = \iint_A I_n(s)\ dA \tag{1}$$

with the normal time average sound intensity $I_n(s)$. The *radiation impedance* $Z_r = Z_r' + j \cdot Z_r''$ is defined by:

$$\Pi := \frac{1}{2} Z_r \cdot \iint_A |v_n(s)|^2\ dA. \tag{2}$$

The *mechanical radiation impedance* $Z_{mr}$ (which is suitable for a small surface A and/or conphase excitation) is defined by:

$$\Pi := \frac{1}{2} Z_{mr} \cdot \langle |v_n(s)|^2 \rangle_A , \tag{3}$$

where $\langle \ldots \rangle_A$ stands for the average over A. It is evident that: $Z_{mr} = A \cdot Z_r$.

A normal component $Z_{Fn}(s)$ of a *field impedance* can be defined by: $p(s) = Z_{Fn}(s) \cdot v_n(s)$ on A. Then

$$\Pi = \frac{1}{2} \iint_A Z_{Fn}(s) \cdot |v_n(s)|^2 \, dA. \tag{4}$$

*Special case:* $Z_{Fn}(s) = \text{const}(s)$: $Z_r = Z_{Fn}$. (5)

*Special case:* $|v_n(s)| = \text{const}(s)$: $Z_r = \frac{1}{A} \iint_A Z_{Fn}(s) \, dA = \frac{1}{A} \iint_A \frac{dA}{G_n}$, (6)

where the field admittance component $G_n = 1/Z_{Fn}$.

*Special case:* $v_n(s) = \text{const}(s)$ in magnitude and phase: $Z_r = \frac{\langle p(s) \rangle_A}{v_n}$. (7)

*Special case:* $p(s) = \text{const}(s)$: $\Pi = \frac{1}{2} p \iint_A v_n^*(s) \, dA = \frac{1}{2} p \cdot q^*$, (8)

where q = volume flow of the surface A.

**Related quantities:**

The *radiation efficiency* σ is defined as the ratio of the real (effective) power radiated by A to the effective power, which a section of size A of an infinite surface with constant surface velocity $v_n$ would radiate:

$$\sigma = \Pi' \Big/ \frac{1}{2} \iint_A |v_n(s)|^2 \, dA = \frac{Z'_r}{Z_0}. \tag{9}$$

The *oscillating mass* $M_r$ is given by $Z''_{mr} = j\,\omega \cdot M_r$ or a mass surface density $m_r$ given by

$Z''_r = j\,\omega \cdot m_r$ with $M_r = A \cdot m_r$.

The *end correction* $\Delta\ell$ is the height of a prism of cross section A containing the oscillating mass $M_r$:

$$\Delta\ell = \frac{M_r}{\rho_0 A} = \frac{m_r}{\rho_0} = \frac{Z''_{mr}}{\omega\rho_0 A} = \frac{Z''_r}{\omega\rho_0} = \frac{Z''_r}{k_0 Z_0} \quad ; \quad \frac{\Delta\ell}{a} = \frac{Z''_r}{k_0 a \cdot Z_0}. \tag{10}$$

The non-dimensional form $\Delta\ell/a$ may contain any meaningful length a, mostly the radius of surface A.

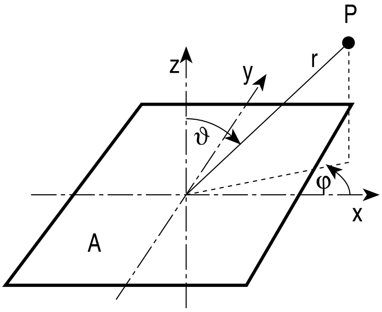
Also used is the *radiation factor* S, which is the ratio of the power Π of A to the power $\Pi_0$ of a small spherical radiator with the same square average of the volume flow density $\langle |q|^2 \rangle_A$ as the considered surface A:

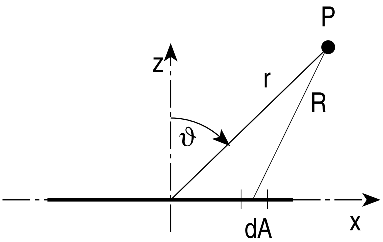
$$\Pi = S \cdot \Pi_0 \quad ; \quad \Pi_0 = \frac{Z_0}{2} \frac{k_0^2}{4\pi} \langle |q|^2 \rangle_A \quad ; \quad Z_r = Z_0 \frac{k_0^2 A}{4\pi} \cdot S. \tag{11}$$

## F.2 Some Methods to Evaluate the Radiation Impedance

The simplest radiators are piston radiators and "breathing" radiators with constant normal particle velocity over the radiator surface A: $v_n(s) = \text{const}$. According to *Sect. F.1*, only the average sound pressure $\langle p(s) \rangle_A$ at the surface must be evaluated.

Also simple are radiators with a surface A which is on a co-ordinate surface of a co-ordinate system in which the wave equation is separable (e.g. spheres, cylinders, ellipsoids, etc.) and if the vibration pattern agrees with an eigenfunction (mode) in that system, because then the modal field impedance of the vibration is constant over A, so it agrees with the radiation impedance (*Sect. F.1*).




An important family of radiators are plane surfaces A in a surrounding plane baffle wall. Let the normal particle velocity at points $(x_0, y_0)$ of A be $v(x_0, y_0)$. The sound pressure at a field point P(x, y, z) is then:

$$\begin{aligned} p(x,y,z) &= \frac{j\,k_0 Z_0}{2\pi} \iint\limits_A v(x_0,y_0)\,\frac{e^{-j\,k_0 R}}{R}\,dx_0\,dy_0 \\ &= \frac{j\,k_0 Z_0}{2\pi} \iint\limits_A v(x_0,y_0)\cdot G(x,y,z|x_0,y_0,0)\,dx_0\,dy_0 \end{aligned} \tag{1}$$

with Green's function $G(x, y, z|x_0, y_0, 0)$.

One gets with the Fourier transform of $v(x_0, y_0)$ (in the hard baffle wall $z = 0$):

$$V(k_1, k_2) = \iint_{-\infty}^{+\infty} v(x_0, y_0) \cdot e^{-j\,(k_1 x_0 + k_2 y_0)}\, dx_0\, dy_0, \tag{2}$$

for the complex power

$$\Pi = \Pi' + j \cdot \Pi'' = \frac{k_0 Z_0}{8\,\pi^2} \iint_{-\infty}^{+\infty} \frac{|V(k_1, k_2)|^2}{\sqrt{k_0^2 - k_1^2 - k_2^2}}\, dk_1\, dk_2, \tag{3}$$

and therefore for the radiation impedance:

$$Z_r = k_0 Z_0 \iint_{-\infty}^{+\infty} \frac{|V(k_1, k_2)|^2}{\sqrt{k_0^2 - k_1^2 - k_2^2}}\, dk_1\, dk_2 \Bigg/ \iint_{-\infty}^{+\infty} |V(k_1, k_2)|^2\, dk_1\, dk_2. \tag{4}$$

The sound pressure in the far field is given by:

$$p(x, y, z) = \frac{k_0 Z_0}{4\,\pi^2} \iint_{-\infty}^{+\infty} \frac{V(k_1, k_2)}{\sqrt{k_0^2 - k_1^2 - k_2^2}} \cdot e^{-j\,(k_1 x + k_2 y + z\sqrt{k_0^2 - k_1^2 - k_2^2})}\, dk_1\, dk_2. \tag{5}$$

*Special case*:

Surface A is a strip with the strip axis on the y axis and $v(x_0, y_0) = \text{const}(y)$:

$$p(x, z) = \frac{k_0 Z_0}{2\,\pi} \int_{-\infty}^{+\infty} \frac{V(k_1)}{\sqrt{k_0^2 - k_1^2}} \cdot e^{-j\,(k_1 x + z\sqrt{k_0^2 - k_1^2})}\, dk_1, \tag{6}$$

$$\Pi' = \frac{k_0 Z_0}{4\,\pi} \int_{-k_0}^{+k_0} \frac{|V(k_1)|^2}{\sqrt{k_0^2 - k_1^2}}\, dk_1, \tag{7}$$

$$Z_r = \frac{k_0 Z_0}{2\,\pi A \langle |v_n|^2 \rangle_A} \int_{-\infty}^{+\infty} \frac{|V(k_1)|^2}{\sqrt{k_0^2 - k_1^2}}\, dk_1 \tag{8}$$

($\Pi'$ and $Z_r$ per unit strip length; A = strip width).

*Special case*:

Plane surface A and the velocity $v(r)$ have a radial symmetry.

The role of the Fourier transform of v(r) is taken over by a Hankel transform:

$$V(k_r) = 2\pi \int_0^\infty v(r_0) \cdot J_0(k_r r_0) \cdot r_0 \, dr_0. \tag{9}$$

One gets for the sound pressure far field:

$$p(r, \vartheta) = \frac{j\, k_0 Z_0}{2\,\pi} \frac{e^{-j\, k_0 r}}{r} \cdot V(k_0 \sin \vartheta), \tag{10}$$

and for the effective sound power $\Pi'$ and the radiation impedance $Z_r$:

$$\Pi' = \frac{k_0^2 Z_0}{4\pi} \int_0^{\pi/2} |V(k_0 \sin \vartheta)|^2 \cdot \sin \vartheta \, d\vartheta = \frac{k_0 Z_0}{4\pi} \int_0^{k_0} \frac{|V(k_r)|^2}{\sqrt{k_0^2 - k_r^2}} \cdot k_r \, dk_r, \tag{11}$$

$$Z_r = \frac{k_0 Z_0}{2\,\pi A \langle |v_n|^2 \rangle_A} \int_{-\infty}^{+\infty} \frac{|V(k_r)|^2}{\sqrt{k_0^2 - k_r^2}} \cdot k_r \, dk_r. \tag{12}$$

Bouwkamp (1945/46), evaluates the radiation impedance of a plane piston radiator with particle velocity distribution $v(x, y) = \text{const}$ as:

$$Z_r = \frac{Z_0 k_0^2 A}{4\pi^2} \int_0^{2\pi} d\varphi \int_0^{\pi/2+j\infty} |D(\vartheta, \varphi)|^2 \cdot \sin \vartheta \, d\vartheta, \tag{13}$$

where $D(\vartheta, \varphi)$ is the far field directivity function of the radiated sound (directivity pattern with unit value in the maximum). The integration over $\vartheta = 0 \to \vartheta = \pi/2$ returns the real part of $Z_r$; the integration $\vartheta = \pi/2 + j \cdot 0 \to \vartheta = \pi/2 + j \cdot \infty$ returns the imaginary part of $Z_r$.

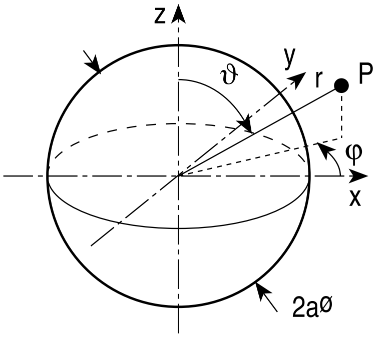
## F.3 Spherical Radiators

▸ *See also:* Mechel, Vol. I, Ch. 9 (1989)

Let $v(\vartheta, \varphi)$ be the pattern of the normal (outward) particle velocity on the sphere with radius a.

The pattern is synthesised with spherical modes:

$$v(\vartheta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} V_{m,n} \cdot P_n^m(\cos \vartheta) \cdot \cos(m\varphi) \tag{1}$$




with associate Legendre functions

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \; ; \; m \geq 1 \quad ; \quad P_n(x) = P_n^0(x) = \frac{1}{2^n \, n!} \frac{d^n}{dx^n} (x^2 - 1)^n \tag{2}$$

defined via the Legendre polynomials $P_n(x)$. Some special values:

$$\begin{aligned} &P_0(x) = 1 \quad &; \quad &P_1(x) = x\,; \\ &P_2(x) = (3x^2 - 1)/2 \quad &; \quad &P_3(x) = (5x^3 - 3x)/2. \end{aligned} \tag{3}$$

The modal velocity amplitudes are:

$$V_{m,n} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\varphi \int_0^{\pi} v(\vartheta, \varphi) \cdot P_n^m(\cos\vartheta) \cdot \cos(m\varphi) \cdot \sin\vartheta \, d\vartheta \tag{4}$$

with the mode norms

$$N_{m,n} = \int_0^{2\pi} \cos^2(m\varphi) \, d\varphi \int_{-1}^{1} \left(P_n^m(x)\right)^2 dx = \frac{2\pi}{\delta_m} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}; \; \delta_m = \begin{cases} 1; \; m = 0, \\ 2; \; m > 0. \end{cases} \tag{5}$$

The sound pressure at the surface of the sphere is:

$$p(a, \vartheta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} Z_n \cdot V_{m,n} \cdot P_n^m(\cos\vartheta) \cdot \cos(m\varphi), \tag{6}$$

where $Z_n$ are the modal impedances at the sphere surface (directed inward):

$$Z_n = -j \, \rho_0 c_0 \frac{h_n^{(2)}(k_0 a)}{h_n'^{(2)}(k_0 a)} \tag{7}$$

with the spherical Hankel functions of the second kind $h_n^{(2)}(z)$.

If the sphere oscillates in a single mode, the modal impedance is the radiation impedance (Sect. F.1).

*Special case*:

The vibration pattern $v(\vartheta, \varphi) = \text{const}(\varphi)$, i.e. the oscillation is symmetrical around the z axis.

$$v(\vartheta) = \sum_{n=0}^{\infty} V_n \cdot P_n(\cos\vartheta),$$
$$V_n = \left(n + \frac{1}{2}\right) \int_0^{\pi} v(\vartheta) \cdot P_n(\cos\vartheta) \cdot \sin\vartheta \, d\vartheta, \tag{8}$$

$$p(r, \vartheta) = -j\,\rho_0 c_0 \sum_{n=0}^{\infty} V_n \cdot P_n(\cos\vartheta) \frac{h_n^{(2)}(k_0 r)}{h_n'^{(2)}(k_0 a)} \xrightarrow[r \to a]{} \sum_{n=0}^{\infty} Z_n V_n \cdot P_n(\cos\vartheta). \tag{9}$$

*Special case:*

Breathing sphere: $V_{n>0} = 0;\ V_0 = v(\vartheta, \varphi) = \text{const}(\vartheta, \varphi).$ (10)

Radiation impedance (= zero mode impedance):

$$Z_{r0} = \rho_0 c_0 \frac{j\,k_0 a}{1 + j\,k_0 a} = \rho_0 c_0 \frac{(k_0 a)^2 + j\,k_0 a}{1 + (k_0 a)^2}. \tag{11}$$

Oscillating mass:

$$M_{r0} = A \cdot \frac{Z''_{r0}}{\omega} = \frac{\rho_0 \cdot 4\pi a^3}{1 + (k_0 a)^2} \xrightarrow[k_0 a \ll 1]{} \rho_0 \cdot 4\pi a^3 = \rho_0 \cdot 3\,\text{Vol}. \tag{12}$$

*Special case:*

Oscillating rigid sphere: $V_{n \neq 1} = 0;\ v(\vartheta) = V_1 \cdot \cos\vartheta.$ (13)

Radiation impedance (= first-order mode impedance):

$$Z_{r1} = \frac{j\,\rho_0 c_0}{\dfrac{2}{k_0 a} - \dfrac{h_0^{(2)}(k_0 a)}{h_1^{(2)}(k_0 a)}} = \rho_0 c_0 \frac{(k_0 a)^4 + j\,k_0 a\,\left(2 + (k_0 a)^2\right)}{4 + (k_0 a)^4}$$
$$\xrightarrow[k_0 a \ll 1]{} \rho_0 c_0 \frac{(k_0 a)^4}{4} + j\,\omega \rho_0\, a/2, \tag{14}$$

$$M_{r1} \xrightarrow[k_0 a \ll 1]{} \rho_0 \cdot \tfrac{3}{2}\,\text{Vol}.$$

In general, for the n-th mode oscillation ($n > 0$):

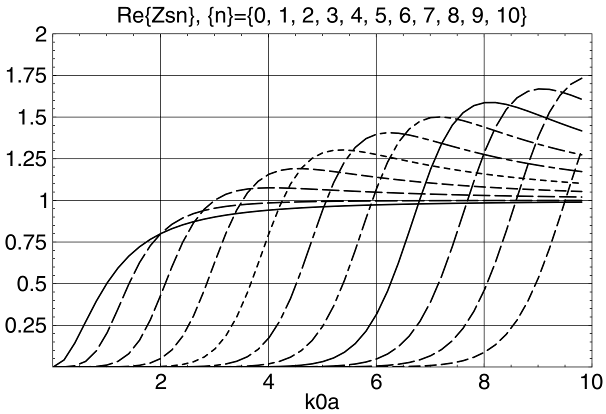
$$Z'_{rn} = \rho_0 c_0 \frac{(k_0 a)^{2n+2}}{(n+1)^2[1 \cdot 3 \cdot \ldots \cdot (2n-1)]^2} \quad ; \quad (k_0 a)^2 \ll |2n-1|$$
$$= \rho_0 c_0 \quad ; \quad k_0 a \gg n^2 + 1, \tag{15}$$

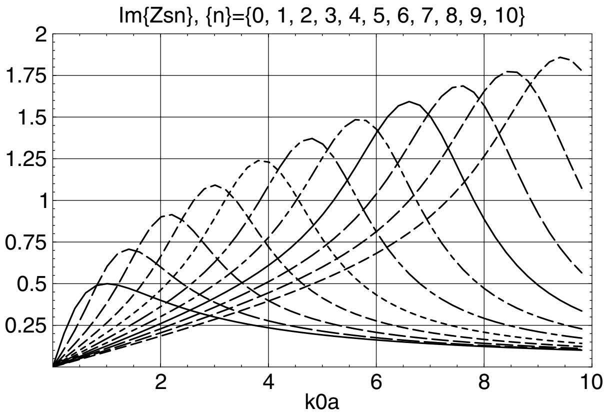
$$Z''_{rn} = \rho_0 c_0 \frac{k_0 a}{n+1} \quad ; \quad (k_0 a)^2 \ll |2n-1|$$
$$= \rho_0 c_0 / k_0 a \quad ; \quad k_0 a \gg n^2 + 1, \tag{16}$$

$$M_{r\,n} \xrightarrow[k_0 a << 1]{} \rho_0 \cdot \frac{3}{n+1} \,\mathrm{Vol.}$$

Correspondence in the graphs below: "Zsn" $\rightarrow$ $Z_{rn}/Z_0$; "k0a" $\rightarrow$ $k_0 a$; order of curves from left to right as in the parameter list $\{n\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

The maximum of $\mathrm{Im}\{Z_{rn}/Z_0\}$ is at about $k_0 a = n$.



## F.4 Cylindrical Radiators

► *See also:* Mechel, Vol. I, Ch. 9 (1989)

Let $v(\vartheta, \varphi)$ be the pattern of the normal (outward) particle velocity on the cylinder with radius a.

The pattern is synthesised with cylindrical modes:

$$v(\varphi, z) = \sum_{m,n\geq 0} V_{m,n} \cdot \cos(n\varphi) \cdot \cos(k_m z). \tag{1}$$

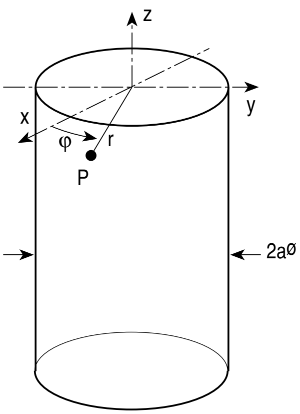
The sound pressure field then is:

$$p(r, \varphi, z) = \sum_{m,n\geq 0} V_{m,n} \cdot Z_{m,n}(r) \cdot \cos(n\varphi) \cdot \cos(k_m z) \tag{2}$$

with the modal field impedances (in radial direction):

$$Z_{m,n}(r) = \frac{-j\, k_0 Z_0}{k_{rm}} \frac{H_n^{(2)}(k_{rm} r)}{H_n^{\prime(2)}(k_{rm} r)} \quad ; \quad k_{rm}^2 = k_0^2 - k_m^2 \tag{3}$$

[$H_n^{(2)}(z)$ Hankel functions of the second kind].




The modal velocity amplitudes $V_{m,n}$ are obtained from the integral transformation of the given pattern $v(\vartheta, \varphi)$:

$$V_{m,n} = \frac{\delta_m \delta_n}{4\pi} \lim_{L\to\infty} \int_{-L}^{L} dz \int_0^{2\pi} v(\varphi, z) \cdot \cos(n\varphi) \cdot \cos(k_m z)\, d\varphi \; ; \; \delta_m = \begin{cases} 1; \; m = 0\,, \\ 2; \; m > 0\,. \end{cases} \tag{4}$$

*Special case*:

The cylinder surface oscillates with only one azimuthal mode n and one axial wave number $k_m$. Then (according to Sect. F.1) the modal wave impedance $Z_{m,n}(a)$ is the radiation impedance $Z_r$:

$$Z_r = Z_{m,n}(a) = \frac{-j\,k_0 Z_0}{k_{rm}} \frac{H_n^{(2)}(k_{rm}a)}{H_n^{\prime(2)}(k_{rm}a)}. \tag{5}$$

For an axially conphase oscillation ($k_m = 0$): $Z_{r\,n} = Z_{0,n}(a) = -j\,\rho_0 c_0 \dfrac{H_n^{(2)}(k_0 a)}{H_n^{\prime(2)}(k_0 a)}$. (6)

For thin cylinders ($k_0 a \ll 1$) and $n > 0$:

$$\frac{Z'_{r\,n}}{\rho_0 c_0} \approx \pi\,k_0 a \frac{(k_0 a)^{2n}}{(n\,!)^2 \cdot 2^{2n-1}}; \qquad \frac{Z''_{r\,n}}{\rho_0 c_0} \approx \frac{k_0 a}{n}. \tag{7}$$

For $n = 0$:

$$\frac{Z_{r\,0}}{\rho_0 c_0} \approx \frac{\pi\,k_0 a}{2} - j\,k_0 a \cdot \ln(k_0 a). \tag{8}$$

*Special case*:

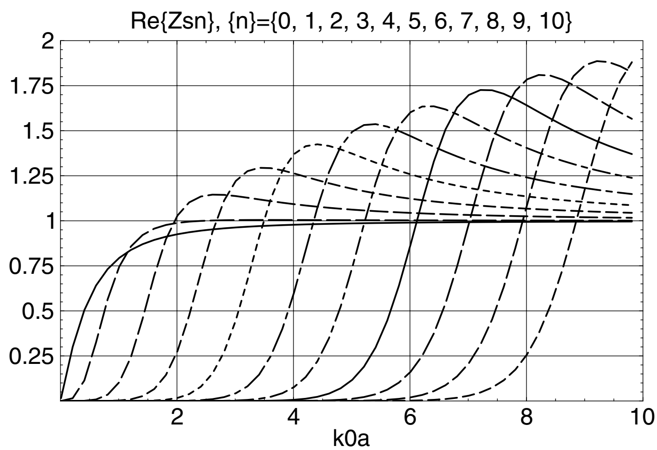
A slow mode in the axial direction: $k_m^2 > k_0^2$.

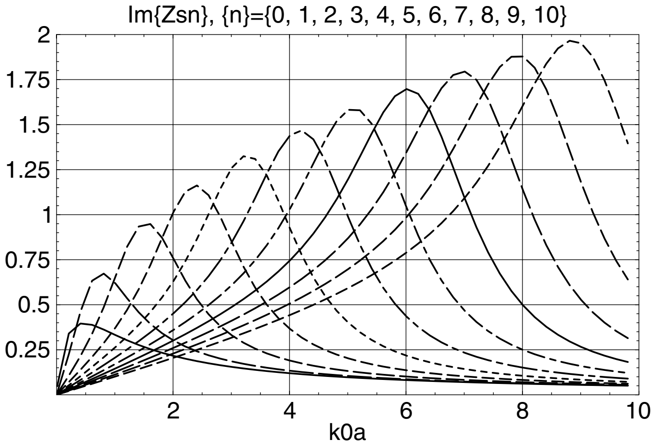
The modal radiation impedances then are:

$$Z_{m,n}(a) = j\,\rho_0 c_0 \frac{k_0}{\sqrt{k_m^2 - k_0^2}} \Bigg/ \left[ \frac{K_{n+1}\left(a\sqrt{k_m^2 - k_0^2}\right)}{K_n\left(a\sqrt{k_m^2 - k_0^2}\right)} - \frac{n}{a\sqrt{k_m^2 - k_0^2}} \right] \tag{9}$$

with $K_n(z)$ modified Bessel functions of the second kind.

Correspondence in the graphs below: "Zsn" $\longrightarrow Z_{rn}/Z_0$ ; "k0a" $\longrightarrow k_0 a$. They are for $k_m = 0$. The curves are ordered from left to right as in the parameter list $\{n\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.



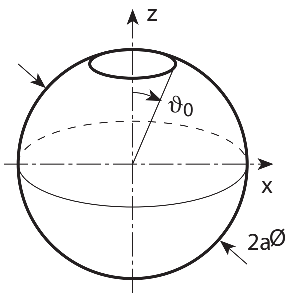



## F.5 Piston Radiator on a Sphere

► *See also:* Mechel, Vol. I, Ch. 9 (1989)

This case corresponds to the classical Helmholtz resonator.

A hollow hard sphere with radius a has a circular hole which subtends an angle $\vartheta_0$ with the z axis.




Let the particle velocity be constant in the hole:

$$v(\vartheta) = \begin{cases} v_0 & ; \quad 0 \le \vartheta < \vartheta_0 \\ 0 & ; \quad \vartheta_0 < \vartheta \le \pi. \end{cases}$$

Modal velocity amplitudes at r = a:

$$V_n = (n + 1/2) \cdot v_0 \int_{\cos\vartheta_0}^{1} P_n(x)\, dx = \frac{v_0}{2}\left[P_{n-1}(\cos\vartheta_0) - P_{n+1}(\cos\vartheta_0)\right] \qquad (1)$$

with $P_n(z)$ being Legendre polynomials and $P_{-1}(z) = 1$.

Radial particle velocity and sound pressure at r = a:

$$v(a,\vartheta)=\sum_{n=0}^{\infty} V_n\cdot P_n(\cos\vartheta)\quad;\quad p(a,\vartheta)=\sum_{n=0}^{\infty} Z_n(a)\cdot V_n\cdot P_n(\cos\vartheta) \tag{2}$$

using the modal (radial) impedances:

$$Z_n(a)=-j\,\rho_0 c_0\frac{h_n^{(2)}(k_0a)}{h_n^{\prime(2)}(k_0a)} \tag{3}$$

with the spherical Hankel functions of the second kind $h_n^{(2)}(z)$.

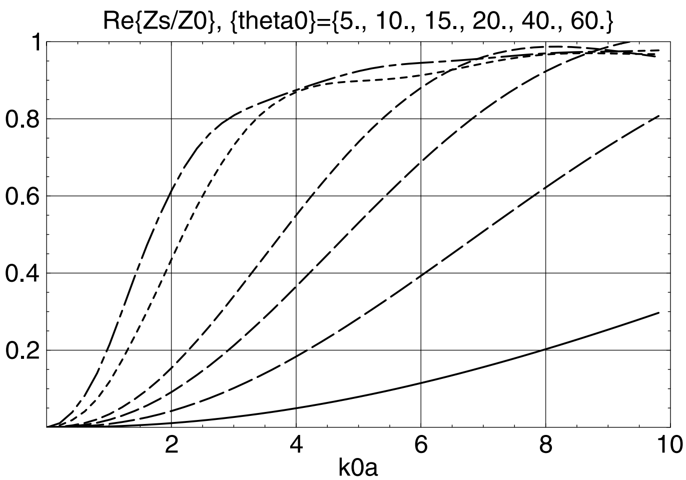
Because $v(\vartheta)$ = const over the hole, its radiation impedance is given by the average sound pressure and the particle velocity (*Sect. F.1*) with the radiator surface:

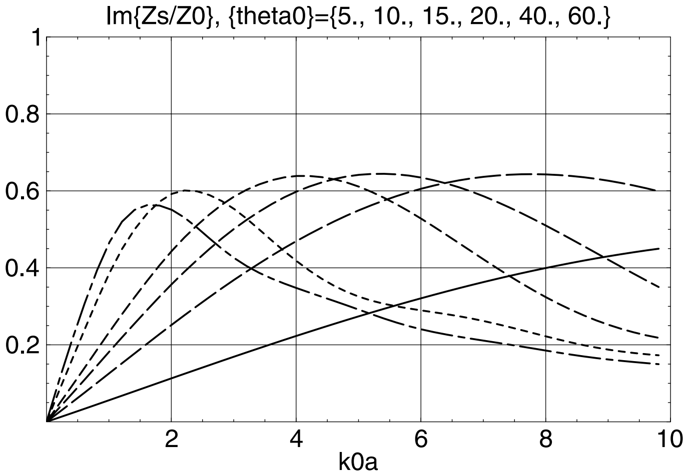
$$A=2\pi a^2\int_0^{\vartheta_0}\sin\vartheta\,d\vartheta=2\pi a^2(1-\cos\vartheta_0), \tag{4}$$

$$\langle p(a,\vartheta)\rangle_A=v_0\frac{\pi a^2}{A}\sum_{n=0}^{\infty}\frac{Z_n(a)}{2n+1}\,[P_{n-1}(\cos\vartheta_0)-P_{n+1}(\cos\vartheta_0)]^2. \tag{5}$$

This gives the radiation impedance:

$$Z_r=\frac{1}{2\,(1-\cos\vartheta_0)}\sum_{n=0}^{\infty}\frac{Z_n(a)}{2n+1}\,[P_{n-1}(\cos\vartheta_0)-P_{n+1}(\cos\vartheta_0)]^2. \tag{6}$$






In the limit of low frequencies:

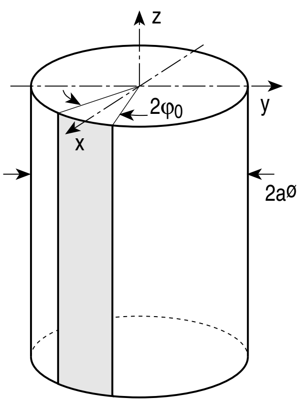
$$\frac{Z_r}{\rho_0 c_0} \approx \frac{1+\cos\vartheta_0}{2}\,\frac{Z_{r0}}{\rho_0 c_0} = \frac{1+\cos\vartheta_0}{2}\;\frac{j\,k_0 a}{1+j\,k_0 a}. \tag{7}$$

Correspondence in the diagrams above: "Zs/Z0" $\to$ $Z_r/\rho_0 c_0$ ; "theta0" $\to$ $\vartheta_0$ ; "k0a" $\to$ $k_0 a$. The dashes become shorter for higher list entries of $\vartheta_0$; The curves are ordered from right to left in the sequence of parameter values in the parameter list $\{\vartheta_0\}$.

## F.6 Strip-Shaped Radiator on Cylinder

► *See also:* Mechel, Vol. I, Ch. 9 (1989)

A hard cylinder with radius a has a vibrating strip on its surface, which subtends an angle $\varphi_0$ with the x axis.



The radial particle velocity be constant in the azimuthal direction and may have a propagating or standing wave pattern in the axial direction:

$$v(a,\varphi,z)=\begin{cases} v_0\cdot g(k_m z) & ;\quad -\varphi_0\le\varphi\le\varphi_0 \\ 0 & ;\quad \varphi_0<\varphi<2\pi-\varphi_0. \end{cases} \tag{1}$$

The modal particle velocity amplitudes are:

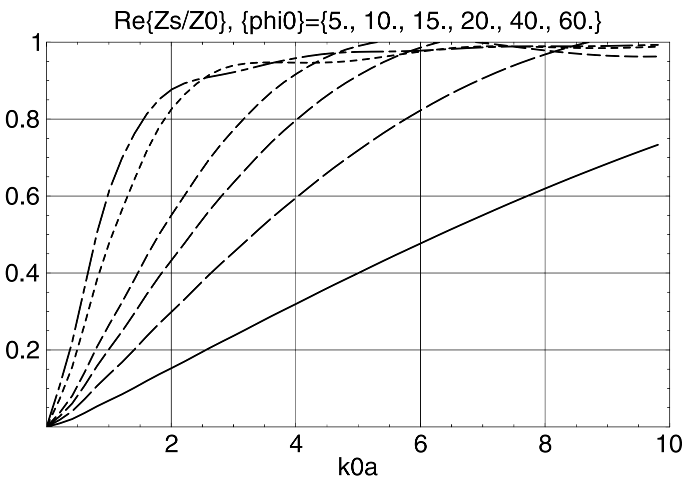
$$V_{m,n}=\frac{\delta_m}{\pi}v_0\varphi_0\frac{\sin(n\varphi_0)}{n\varphi_0}\quad ;\quad \delta_m=\begin{cases}1\,;\ m=0\\ 2\,;\ m>0\end{cases}\quad ;\quad k_r^2=k_0^2-k_m^2. \tag{2}$$

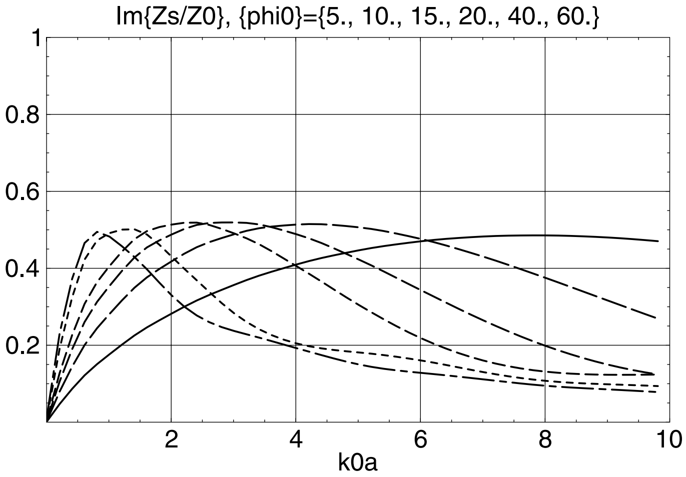
The radiation impedance is evaluated as:

$$\begin{aligned} Z_r&=\iint\limits_A p\cdot v^*\,dA\Big/\iint\limits_A |v|^2 dA=\frac{\varphi_0}{\pi}\sum_{n=0}^{\infty}\delta_n Z_{m,n}(a)\left(\frac{\sin(n\varphi_0)}{n\varphi_0}\right)^2\\ &=\rho_0c_0\frac{-j\,\varphi_0}{\pi}\frac{k_0a}{k_ra}\sum_{n=0}^{\infty}\delta_n\frac{H_n^{(2)}(k_ra)}{H_n'^{(2)}(k_ra)}\left(\frac{\sin(n\varphi_0)}{n\varphi_0}\right)^2. \end{aligned} \tag{3}$$

At high frequencies, $Z_r\to\rho_0c_0\cdot k_0/k_r$. (4)

Correspondence in the diagrams below: "Zs/Z0" $\to Z_r/\rho_0c_0$ ; "phi0" $\to \varphi_0$ ; "k0a" $\to k_0a$. The dashes become shorter for higher list entries of $\varphi_0$; the curves are arranged from left to right (at low $k_0a$) in the order of these entries. The axial wave number there is $k_m=0$.



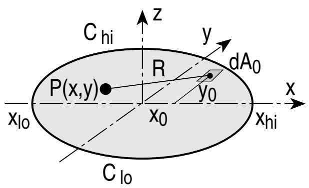



## F.7 Plane Piston Radiators

► *See also:* Mechel, Vol. I, Ch. 9 (1989)

A plane surface A, surrounded by a plane, hard baffle wall, oscillates with a constant velocity v.

A general scheme of evaluation for the radiation impedance $Z_r$ can be designed for surfaces A with convex border lines.




The evaluation applies the field impedance $Z_F(x, y)$ on the radiating surface:

$$\begin{aligned}\frac{Z_F}{Z_0} &= \frac{j}{2\pi} \iint\limits_{k_0^2 A} \frac{e^{-j\,k_0 R}}{R}\, d(k_0^2 A) \\ &= \frac{1}{2\pi} \int\limits_{k_0 x_{lo}}^{k_0 x_{hi}} d(k_0 x_0) \int\limits_{k_0 C_{lo}(x_0)}^{k_0 C_{hi}(x_0)} \left[ \frac{\sin\, k_0 R}{k_0 R} + j \frac{\cos\, k_0 R}{k_0 R} \right] d(k_0 y_0),\end{aligned} \tag{1}$$

$$\frac{Z_r}{Z_0} = \frac{1}{2\pi\,k_0^2 A} \int_{k_0 x_{lo}}^{k_0 x_{hi}} d(k_0 x) \int_{k_0 C_{lo}(x)}^{k_0 C_{hi}(x)} \frac{Z_F(x,y)}{Z_0}\, d(k_0 y). \qquad (2)$$

**Circular piston radiator** with radius a:

$$\frac{Z_r}{Z_0} = 1 - \frac{J_1(2k_0 a)}{k_0 a} + j\,\frac{S_1(2k_0 a)}{k_0 a}, \qquad (3)$$

where $J_1(z)$ is a Bessel function and $S_1(z)$ a Struve function.

Approximation for low $k_0 a$ (with $x = 2k_0 a$; for about $x < 4$; range depends on number of terms):

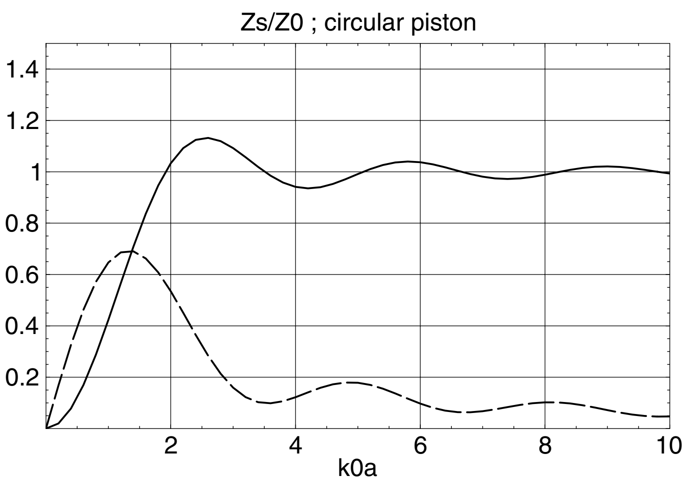
$$\begin{aligned} \frac{Z_r'}{Z_0} &= \frac{x^2}{2\cdot 4} - \frac{x^4}{2\cdot 4^2\cdot 6} + \frac{x^6}{2\cdot 4^2\cdot 6^2\cdot 8} - + \ldots; \\ \frac{Z_r''}{Z_0} &= \frac{4}{\pi}\left[\frac{x}{3} - \frac{x^3}{3^2\cdot 5} + \frac{x^5}{3^2\cdot 5^2\cdot 7} - + \ldots\right]. \end{aligned} \qquad (4)$$

Approximation for high $k_0 a$ (with $x = 2k_0 a$, for about $x > 4$ ):

$$\frac{Z_r'}{Z_0} = 1 - \frac{2}{x}\sqrt{\frac{2}{\pi x}}\cdot \sin(x - \pi/4) \quad ; \quad \frac{Z_r''}{Z_0} = \frac{4}{\pi x}\left[1 - \sqrt{\frac{2}{x}}\cdot \sin(x + \pi/4)\right]. \qquad (5)$$

Correspondence in the diagram below: "Zs/Z0" $\rightarrow Z_r/\rho_0 c_0$ ; "k0a" $\rightarrow k_0 a$.

Solid line: $\mathrm{Re}\{Z_r/\rho_0 c_0\}$, dashed line: $\mathrm{Im}\{Z_r/\rho_0 c_0\}$.



Oscillating free circular disk with radius a; oscillation normal to disk:

The sound field is described in oblate spheroidal co-ordinates ($\rho$, $\vartheta$, $\varphi$) [generated by rotation of the elliptic-hyperbolic cylinder co-ordinates ($\rho$, $\vartheta$) around the short axis of the ellipses], in relation to the Cartesian co-ordinates:

$$z = a \cdot \sinh\rho \cdot \cos\vartheta \quad ; \quad \begin{matrix} x \\ y \end{matrix} = a \cdot \cosh\rho \cdot \sin\vartheta \cdot \begin{matrix} \cos \\ \sin \end{matrix}\,\varphi. \tag{6}$$

The co-ordinate value $\rho = 0$ describes a circular disk with radius a normal to the z axis:

$$\frac{Z_r}{Z_0} = \frac{-8j\,k_0 a}{9} \sum_{n=1,3,\ldots}^{\infty} \left[\frac{he_{0n}(-j\,k_0 a\,,\,j\,\sinh\,\rho)}{d\,he_{0n}(-j\,k_0 a\,,\,j\,\sinh\,\rho)/d\rho}\right]_{\rho=0}$$

$$\cdot \frac{d_1(-j\,k_0 a|\,0,\,n)}{\Delta_{0n}}\,S_{0n}(-j\,k_0 a\,,\,\cos\,\vartheta), \tag{7}$$

where $S_{0n}(\rho, \vartheta)$ is an azimuthal spheroidal function; $he_{0n}(\rho, z)$ is an even radial spheroidal function of the third kind; the term $d_1(-jk_0a|0, n)$ comes from the expansion of $S_{0n}(\rho, \vartheta)$ in associated

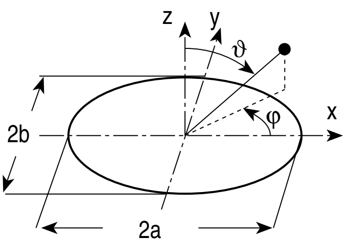
Legendre functions $$S_{0n}(\rho, \vartheta) = \sum_{m=1,3,\ldots}^{\infty} d_m(\rho|\,0,\,1) \cdot T_m^0(\vartheta) \tag{8}$$

and $\Delta_{0n}$ from $$\Delta_{0n} = \int_{-1}^{+1} S_{0n}^2(\rho, \vartheta)\,d\vartheta. \tag{9}$$

Approximation for low $k_0a$: $$\frac{Z_r}{Z_0} \approx \frac{16}{27\,\pi^2}(k_0 a)^4 + j\frac{8}{3\pi}k_0 a. \tag{10}$$

**Elliptic piston in a baffle wall:**

The ellipse has a long axis 2a and a short axis 2b; the ratio of the axes is $\beta = b/a$.




Some evaluations in the literature start from the Bouwkamp integral (Sect. F.2) with the following far field directivity function of the radiated sound:

$$D(\vartheta, \varphi) = 2\,\frac{J_1(k_0\sin\,\vartheta\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi})}{k_0\sin\,\vartheta\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}. \tag{11}$$

One solution for the real part of the radiation impedance $Z_r = Z'_r + j \cdot Z''_r$ is:

$$\frac{Z'_r}{Z_0} = k_0 a \cdot k_0 b \sum_{m=0}^{\infty} \frac{(k_0 a)^{2m}}{(m+1)!\,(m+2)!} \cdot {}_2F_1\left(-m\,;\,\tfrac{1}{2}\,;\,1\,;\,\mu^2\right) \quad ; \quad \mu^2 = 1-\beta^2, \tag{12}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the hypergeometric function. The numerical errors become large for $k_0 a \gg 1$.

A solution suited for numerical integration is:

$$\begin{aligned} \frac{Z'_r}{Z_0} &= 1 - \frac{2}{\pi} k_0^2 ab \int_0^{\pi/2} \frac{J_1(2B)}{B^3}\, d\varphi \quad ; \quad B = k_0 a \sqrt{\cos^2\varphi + \beta^2 \sin^2\varphi}, \\ \frac{Z''_r}{Z_0} &= \frac{2}{\pi} k_0^2 ab \int_0^{\pi/2} \frac{S_1(2B)}{B^3}\, d\varphi, \end{aligned} \tag{13}$$

where $J_1(z)$ is a Bessel function and $S_1(z)$ a Struve function. The numerical integration can be avoided by an expansion of the integrands. This leads to the following iterative evaluation:

$$\begin{aligned} \frac{Z'_r}{Z_0} &= \beta \left[ (k_0 a)^2/2 + \sum_{n=2}^{n_{hi}} c'_n \cdot I'_n \right], \\ c'_1 &= (k_0 a)^2/2 \quad ; \quad c'_n = \frac{-(k_0 a)^2}{n \cdot (n+1)} \cdot c'_{n-1} \quad ; \quad I'_0 = 1/\beta \quad ; \quad I'_1 = 1 \quad ; \quad I'_n = 2\, I''_n/\pi, {}^{*)} \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{Z''_r}{Z_0} &= \frac{4 k_0 b}{\pi^2} \left[ \frac{4}{3} I''_0 - \frac{16}{45} (k_0 a)^2 \cdot I''_1 + \sum_{n=2}^{n_{hi}} c''_n \cdot I''_n \right], \\ c''_1 &= -16\,(k_0 a)^2/45 \quad ; \quad c''_n = -4\,(k_0 a)^2 / \left((2n+1)(2n+3)\right), \\ I''_0 &= K(\mu^2) \quad ; \quad I''_1 = E(\mu^2) \quad ; \quad I''_n = \frac{2n-2}{2n-1}(1+\beta^2) \cdot I''_{n-1} - \frac{2n-3}{2n-1}\beta^2 \cdot I''_{n-2} \end{aligned} \tag{15}$$

with $K(z)$, $E(z)$ the complete elliptic integrals of the first and second kind. The upper summation limit should be $n_{hi} \geq 2(k_0 a + 1)$.

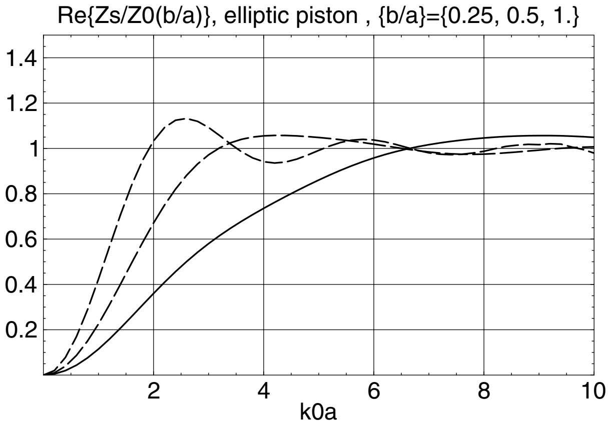
A further solution for the real component of $Z_r$ is:

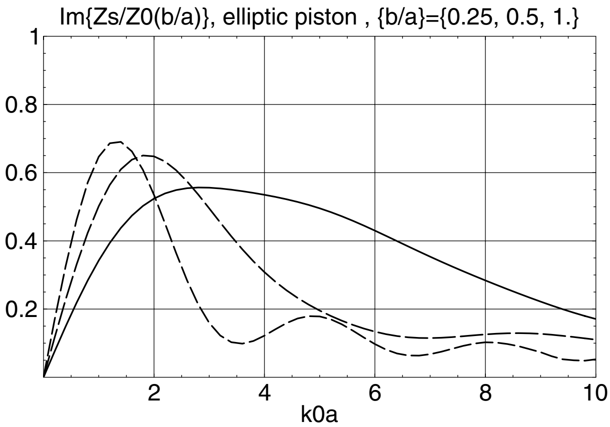
$$\begin{aligned} \frac{Z'_r}{Z_0} &= 1 - \frac{J_1(2k_0 a)}{k_0 a} - (1-\beta) \cdot J_2(2k_0 a) - \frac{\beta}{k_0 a} \sum_{n=2}^{n_{hi}} \hat{c}_n \cdot \hat{I}_n \cdot J_{1+n}(2k_0 a), \\ \hat{c}_1 &= (1-\beta^2) \cdot k_0 a \quad ; \quad \hat{c}_n = \frac{(1-\beta^2) k_0 a}{n} \cdot \hat{c}_{n-1}, \\ \hat{I}_0 &= 1/\beta\,;\ \hat{I}_1 = 1/\left(\beta(1+\beta)\right)\,;\ \hat{I}_n = \left( \frac{1}{1-\beta^2} + \frac{2n-3}{2n-2} \right) \cdot \hat{I}_{n-1} - \frac{1}{1-\beta^2} \frac{2n-3}{2n-2} \cdot \hat{I}_{n-2}, \end{aligned} \tag{16}$$

where $J_n(z)$ are Bessel functions.

*) See Preface to the 2nd Edition.

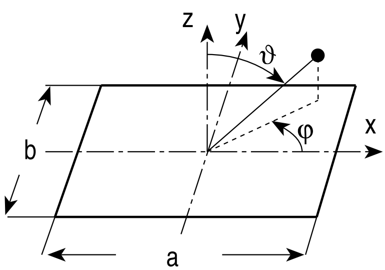
Correspondence in the diagrams below: "Zs/Z0" → $Z_r/Z_0$; "k0a" → $k_0a$; dashes become shorter for higher positions in the parameter list $\{\beta\} = \{b/a\} = \{0.25, 0.5, 1\}$.




**Rectangular piston in a baffle wall:**

The rectangle has a long side a and a short side b; the side length ratio is $\beta = b/a$.



Some evaluations in the literature start from the Bouwkamp integral (Sect. F.2) with the far field directivity function of the radiated sound [with si(z) = (sinz)/z]:

$$D(\vartheta,\varphi) = \mathrm{si}\left(k_0a/2\cdot\sin\vartheta\cdot\cos\varphi\right)\cdot\mathrm{si}\left(k_0b/2\cdot\sin\vartheta\cdot\sin\varphi\right). \tag{17}$$

A first form of the radiation impedance $Z_r = Z'_r + j\cdot Z''_r$ is:

$$\begin{aligned}\frac{Z'_r}{Z_0} &= 1+\frac{\beta}{\pi}\left[\mathrm{Ci}(k_0a)-\frac{\sin k_0a}{k_0a}+\frac{\cos k_0a-1}{(k_0a)^2}\right]\\&\quad+\frac{1}{\pi\beta}\left[\mathrm{Ci}(k_0b)-\frac{\sin k_0b}{k_0b}+\frac{\cos k_0b-1}{(k_0b)^2}\right]-\frac{2\beta}{\pi}I_1(k_0a,\beta),\\\frac{Z''_r}{Z_0} &= -\frac{\beta}{\pi}\left[\mathrm{Si}(k_0a)+\frac{\sin k_0a}{(k_0a)^2}+\frac{\cos k_0a-2}{k_0a}\right]\\&\quad-\frac{1}{\pi\beta}\left[\mathrm{Si}(k_0b)+\frac{\sin k_0b}{(k_0b)^2}+\frac{\cos k_0b-2}{k_0b}\right]-\frac{2\beta}{\pi}I_2(k_0a,\beta)\end{aligned} \tag{18}$$

with Ci(z), Si(z) the integral cosine and sine functions and the integrals:

$$\begin{aligned}I_1(k_0a,\beta) &= \int_0^1\left[\mathrm{Ci}\left(k_0a\sqrt{x^2+1/\beta^2}\right)+\frac{1}{\beta^2}\mathrm{Ci}\left(k_0b\sqrt{x^2+\beta^2}\right)\right]\cdot(1-x)\,dx,\\I_2(k_0a,\beta) &= \int_0^1\left[\mathrm{Si}\left(k_0a\sqrt{x^2+1/\beta^2}\right)+\frac{1}{\beta^2}\mathrm{Si}\left(k_0b\sqrt{x^2+\beta^2}\right)\right]\cdot(1-x)\,dx.\end{aligned} \tag{19}$$

A second form of the radiation impedance $Z_r = Z'_r + j\cdot Z''_r$ is:

$$\begin{aligned}\frac{Z'_r}{Z_0} &= 1-\frac{2}{\pi\beta(k_0a)^2}\cdot\Big[1+\cos\left(k_0a\sqrt{1+\beta^2}\right)+k_0a\sqrt{1+\beta^2}\cdot\sin\left(k_0a\sqrt{1+\beta^2}\right)\\&\quad-\cos(k_0a)-\cos(k_0b)\Big]+\frac{2}{\pi\sqrt{\beta}}\cdot I_a(k_0a,\beta),\\\frac{Z''_r}{Z_0} &= \frac{2}{\pi\beta(k_0a)^2}\cdot\Big[\sin\left(k_0a\sqrt{1+\beta^2}\right)-k_0a\sqrt{1+\beta^2}\cdot\cos\left(k_0a\sqrt{1+\beta^2}\right)\\&\quad+k_0a(1+1/\beta)-\sin(k_0a)-\sin(k_0b)\Big]-\frac{2}{\pi\sqrt{\beta}}\cdot I_b(k_0a,\beta),\end{aligned} \tag{20}$$

with the integrals

$$\begin{aligned}I_a(k_0a,\beta) &= \int_{\sqrt{\beta}}^{\sqrt{\beta+1/\beta}}\sqrt{1-\beta/x^2}\cdot\cos\left(x\,k_0a\sqrt{\beta}\right)dx+\beta\int_{1/\sqrt{\beta}}^{\sqrt{\beta+1/\beta}}\sqrt{1-1/(\beta x)^2}\\&\quad\cdot\cos\left(x\,k_0a\sqrt{\beta}\right)dx,\end{aligned} \tag{21}$$

$$I_b(k_0a\,,\ \beta) = \int\limits_{\sqrt{\beta}}^{\sqrt{\beta+1/\beta}} \sqrt{1-\beta/x^2}\cdot\sin(x\,k_0a\sqrt{\beta})\,dx + \beta\int\limits_{1/\sqrt{\beta}}^{\sqrt{\beta+1/\beta}} \sqrt{1-1/(\beta x)^2}\cdot\sin(x\,k_0a\sqrt{\beta})\,dx.$$

A modification of these formulas leads to a fast numerical evaluation:

$$\begin{aligned}\frac{Z_r'}{Z_0} &= 1-\frac{2}{\pi\,k_0^2ab}\cdot\Big[1+\cos(k_0\sqrt{a^2+b^2})+k_0\sqrt{a^2+b^2}\cdot\\ &\quad\cdot\sin(k_0\sqrt{a^2+b^2})-\cos(k_0a)-\cos(k_0b)\Big]+\frac{2}{\pi}\cdot\hat{I}_a,\\ \frac{Z_r''}{Z_0} &= \frac{2}{\pi\,k_0^2ab}\cdot\Big[k_0(a+b)+\sin(k_0\sqrt{a^2+b^2})-k_0\sqrt{a^2+b^2}\\ &\quad\cdot\cos(k_0\sqrt{a^2+b^2})-\sin(k_0a)-\sin(k_0b)\Big]-\frac{2}{\pi}\cdot\hat{I}_b\end{aligned}\tag{22}$$

with the integrals

$$\begin{aligned}\hat{I}_a &= \int\limits_1^{\sqrt{1+(b/a)^2}}\sqrt{1-1/x^2}\cdot\cos(x\,k_0a)\,dx+\int\limits_1^{\sqrt{1+(a/b)^2}}\sqrt{1-1/x^2}\cdot\cos(x\,k_0b)\,dx,\\ \hat{I}_b &= \int\limits_1^{\sqrt{1+(b/a)^2}}\sqrt{1-1/x^2}\cdot\sin(x\,k_0a)\,dx+\int\limits_1^{\sqrt{1+(a/b)^2}}\sqrt{1-1/x^2}\cdot\sin(x\,k_0b)\,dx.\end{aligned}\tag{23}$$

The component integrals are of the forms:

$$\tilde{I}_a(A,B)=\int\limits_1^B\sqrt{1-1/x^2}\cdot\cos(Ax)\,dx\,;\ \tilde{I}_b(A,B)=\int\limits_1^B\sqrt{1-1/x^2}\cdot\sin(Ax)\,dx.\tag{24}$$

They can be evaluated iteratively:

$$\tilde{I}_a(A,B)=I_{-1}+\sum_{n=1}^{\infty}(-1)^n\frac{A^{2n}}{(2n)!}\cdot I_{2n-1}\,;\ \tilde{I}_b(A,B)=A\cdot I_0+\sum_{n=1}^{\infty}(-1)^n\frac{A^{2n+1}}{(2n+1)!}\cdot I_{2n}\tag{25}$$

with start values and recursion for the $I_m$:

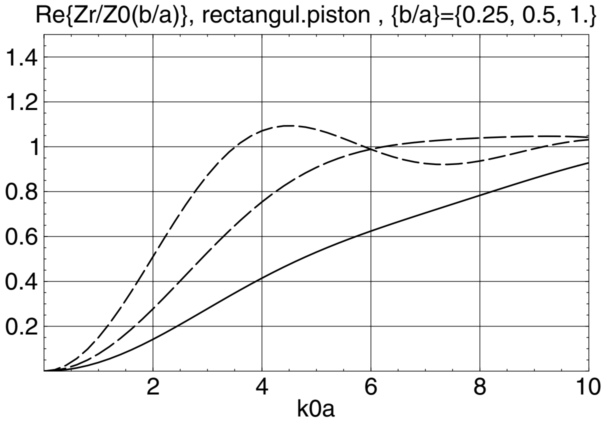
$$\begin{aligned}I_m &= \frac{B^{m-1}}{m+2}(B^2-1)^{3/2}+\frac{m-1}{m+2}\cdot I_{m-2},\\ I_{-1} &= \sqrt{B^2-1}-\arccos(1/B)\quad;\quad I_0=\frac{B}{2}\sqrt{B^2-1}-\frac{1}{2}\ln(B+\sqrt{B^2-1}).\end{aligned}\tag{26}$$

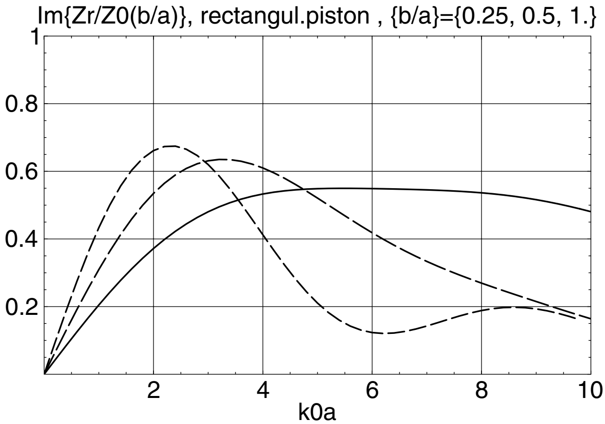
An approximation for large $k_0a$ ($>$ 5) and not too small b/a is:

$$\begin{aligned}\frac{Z_r'}{Z_0} = 1 - \sqrt{\frac{2}{\pi}}\left[\frac{\cos(k_0a-\pi/4)}{(k_0a)^{3/2}} + \frac{\cos(k_0b-\pi/4)}{(k_0b)^{3/2}}\right] - \frac{2}{\pi k_0^2 ab}[1-\cos k_0a - \cos k_0b] \\ - \frac{9}{8}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(k_0a-\pi/4)}{(k_0a)^{5/2}} + \frac{\sin(k_0b-\pi/4)}{(k_0b)^{5/2}}\right] + \frac{2(a^2+b^2)^{3/2}}{\pi(k_0ab)^3}\sin(k_0\sqrt{a^2+b^2}),\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{Z_r''}{Z_0} = \frac{2(a+b)}{\pi k_0ab} + \sqrt{\frac{2}{\pi}}\left[\frac{\sin(k_0a-\pi/4)}{(k_0a)^{3/2}} + \frac{\sin(k_0b-\pi/4)}{(k_0b)^{3/2}}\right] - \frac{2}{\pi k_0^2 ab}[\sin k_0a + \sin k_0b] \\ - \frac{9}{8}\sqrt{\frac{2}{\pi}}\left[\frac{\cos(k_0a-\pi/4)}{(k_0a)^{5/2}} + \frac{\cos(k_0b-\pi/4)}{(k_0b)^{5/2}}\right] + \frac{2(a^2+b^2)^{3/2}}{\pi(k_0ab)^3}\cos(k_0\sqrt{a^2+b^2}).\end{aligned} \tag{28}$$

Correspondence in the diagrams below: "Zr/Z0" $\to Z_r/Z_0$; "k0a" $\to k_0a$; dashes become shorter for higher positions in the parameter list $\{\beta\} = \{b/a\} = \{0.25, 0.5, 1\}$.



## F.8 Uniform End Correction of Plane Piston Radiators

► *See also:* Mechel, Vol. I, Ch. 9 (1989)

The normalised end correction of a radiator is defined from its radiation reactance $Z_r''$ by:

$$\frac{\Delta\ell}{a} = \frac{Z_r''}{k_0 a \cdot Z_0}, \tag{1}$$

where a is any side length. Thus $\Delta\ell/a$ equals the tangent of the curve of $Z_r''/Z_0$ over $k_0 a$ at the origin $k_0 a = 0$.

If one takes

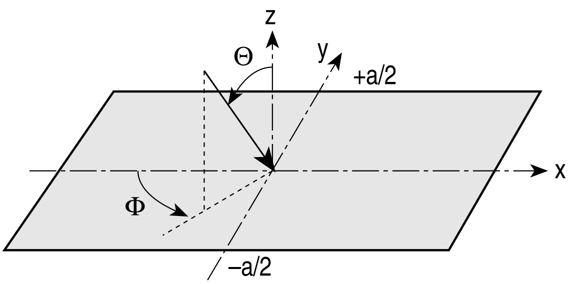
$$a = A^{3/4} \cdot U^{1/2}, \tag{2}$$

where A is the area and U is the periphery of the piston surface, then the curves of $Z_r''/Z_0$ over $k_0 a$ coincide at the origin $k_0 a = 0$ for different shapes of the surface, assuming its border line is convex. So one can deduce end corrections for piston shapes with unknown solutions for $Z_r$ from end corrections of shapes with known solutions.

## F.9 Narrow Strip-Shaped, Field-Excited Radiator

► *See also:* Mechel, Vol. I, Ch. 10 (1989)

A plane radiator is called "field excited" if its vibration pattern agrees with that of an obliquely incident plane wave at the surface.




The object here is an infinitely long strip of width a in a hard baffle wall, the strip by a plane wave with polar angle Θ of incidence and azimuthal angle Φ with the strip axis. If either $\Phi = 0$ (then a is unlimited), or $\Phi \neq 0$, and $a \ll \lambda_0$, the oscillation velocity of the strip surface can be assumed to be constant across the strip:

$$v(x, y) = V_0 \cdot e^{-j k_x x} \quad ; \quad k_x = k_0 \cdot \sin\Theta \cdot \cos\Phi. \tag{1}$$

According to Sect. F.1, because of

$$|v| = \text{const}, \quad Z_r = \frac{1}{A}\iint\limits_A Z_F \, dA = \frac{1}{V_0 a}\int\limits_{-a/2}^{+a/2} p(y,0)\, dy \tag{2}$$

with the field impedance $Z_F = p(x,y,0)/v(x,y)$ and $p(x,y,z) = p(y,z)\cdot e^{-j k_x x}$. The lateral sound pressure distribution is:

$$p(y,z) = \frac{k_0 Z_0 V_0 a}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{\sin(k_y a/2)}{k_y a/2}\, \frac{e^{-j\left(k_y y + z\sqrt{k_0^2-k_x^2-k_y^2}\right)}}{\sqrt{k_0^2-k_x^2-k_y^2}}\, dk_y, \tag{3}$$

and therewith the radiation impedance:

$$\frac{Z_r}{Z_0} = \frac{k_0 a}{2\pi}\int\limits_{-\infty}^{+\infty} \left(\frac{\sin(k_y a/2)}{k_y a/2}\right)^2 \frac{dk_y}{\sqrt{k_0^2-k_x^2-k_y^2}}. \tag{4}$$

In a different form:

$$\frac{Z_r}{Z_0} = \frac{k_0}{k^2 a}\int\limits_0^{ka} (ka-|u|)\cdot H_0^{(2)}(|u|)\, du \quad ; \quad k^2 = k_0^2 - k_x^2 = k_0^2(1-\sin^2\Theta\cdot\cos^2\Phi) \tag{5}$$

with the Hankel function of the second kind $H_0^{(2)}(z)$. After analytical evaluation of the integral:

$$\begin{aligned}\frac{Z_r}{Z_0} = k_0 a \Bigg\{ & H_0^{(2)}(ka) - \frac{H_1^{(2)}(ka)}{ka} + \frac{2j}{\pi (ka)^2} \\ & + \frac{\pi}{2}\left[H_1^{(2)}(ka)\cdot S_0(ka) - H_0^{(2)}(ka)\cdot S_1(ka)\right]\Bigg\}\end{aligned} \tag{6}$$

or as real and imaginary parts

$$\begin{aligned}\frac{Z_r'}{Z_0} &= k_0 a\left\{J_0(ka) - \frac{J_1(ka)}{ka} + \frac{\pi}{2}\left[J_1(ka)\cdot S_0(ka) - J_0(ka)\cdot S_1(ka)\right]\right\}, \\ \frac{Z_r''}{Z_0} &= -k_0 a\left\{Y_0(ka) - \frac{Y_1(ka)}{ka} - \frac{2}{\pi (ka)^2} + \frac{\pi}{2}\left[Y_1(ka)\cdot S_0(ka) - Y_0(ka)\cdot S_1(ka)\right]\right\},\end{aligned} \tag{7}$$

where $J_n(z)$ is a Bessel function, $Y_n(z)$ a Neumann function and $S_n(z)$ a Struve function.

Approximation for small ka (with c = 0.57721, Euler's constant):

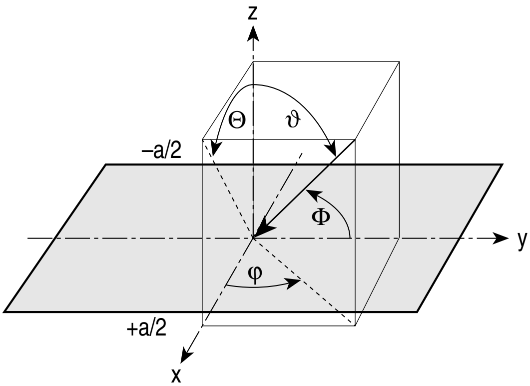
$$\frac{Z_r'}{Z_0} = k_0 a \left\{ \left(1 - \frac{(ka)^2}{6} + \frac{(ka)^4}{64}\right) \left(1 - \frac{(ka)^2}{3} + \frac{(ka)^4}{45}\right) - \left(\frac{1}{2} - \frac{(ka)^2}{16} + \frac{(ka)^4}{192}\right) \left(1 - (ka)^2 + \frac{(ka)^4}{9}\right) \right\}, \tag{8}$$

$$\begin{aligned}\frac{Z_r''}{Z_0} = \frac{2\,k_0 a}{\pi} \Bigg\{ & 1 - \frac{(ka)^2}{9} + \frac{(ka)^4}{225} - \left(1 - \frac{(ka)^2}{3} + \frac{(ka)^4}{45}\right) \left[ \left(\ln \frac{ka}{2} + c\right) \right. \\ & \left. \cdot \left(1 - \frac{(ka)^2}{4} + \frac{(ka)^4}{64}\right) + (ka)^2 \left(\frac{1}{4} - \frac{3(ka)^2}{128}\right) \right] + \left(1 - (ka)^2 + \frac{(ka)^4}{9}\right) \\ & \cdot \left[ \left(\ln \frac{ka}{2} + c\right) \left(\frac{1}{2} - \frac{(ka)^2}{16} + \frac{(ka)^4}{192}\right) - \frac{1}{4} + \frac{5(ka)^2}{64} - \frac{10(ka)^4}{2304} \right] \Bigg\}. \end{aligned} \tag{9}$$

## F.10 Wide Strip-Shaped, Field-Excited Radiator

► *See also:* Mechel, Vol. I, Ch. 10 (1989)

A plane radiator is called "field excited" if its vibration pattern agrees with that of an obliquely incident plane wave at the surface.




The object here is an infinitely long strip of width a in a hard baffle wall, the strip is excited by a plane wave with polar angle ϑ of incidence and azimuthal angle φ with the normal to the strip axis.

Notice the different co-ordinates and angles as compared to *Sect. F.9*:

$$\begin{aligned}
\cos\,\Phi &= \sin\varphi\cdot\sin\vartheta,\\
\cos\,\Theta &= \cos\,\vartheta/\sqrt{1-\sin^2\varphi\cdot\sin^2\vartheta},\\
\cos\,\vartheta &= \sin\Phi\cdot\cos\Theta,\\
\sin\varphi &= \cos\Phi/\sqrt{1-\sin^2\Phi\cdot\cos^2\,\Theta}.
\end{aligned}\tag{1}$$

Radiation impedance:

$$\frac{Z_r}{Z_0}=\frac{C}{\sin\,\Phi}\quad;\quad b=k_0a\cdot\sin\,\Phi$$

$$C=A+j\,B=\int_0^b\left(1-\frac{x}{b}\right)\cdot\cos\,(x\cdot\sin\,\Theta)\cdot H_0^{(2)}(x)\,dx,\tag{2}$$

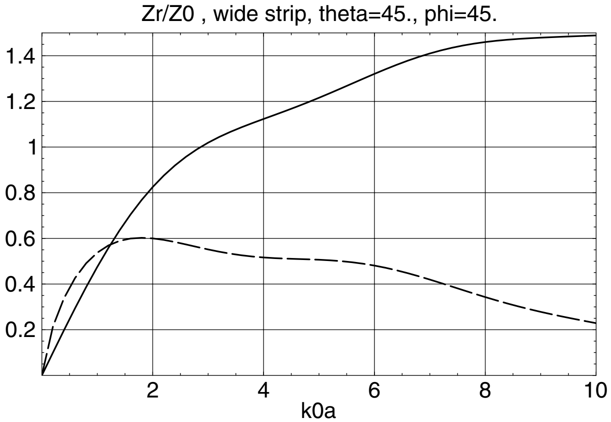
where $H_0^{(2)}(x)$ is a Hankel function of the second kind. After power series expansion of the factor to the Hankel function in the integrand:

$$\begin{aligned}
A=\sum_{n=0}^{\infty}(-1)^n\frac{\sin^{2n}\Theta\cdot b^{2n+1}}{(2n)\,!}\cdot&\left[\frac{1}{2n+1}\,{}_1F_2\left(1/2+n\,;\,1\,,\,3/2+n\,;\,-b^2/4\right)\right.\\
&\left.-\frac{1}{2n+2}\,{}_1F_2\left(1+n\,;\,1\,,\,2+n\,;\,-b^2/4\right)\right],
\end{aligned}\tag{3}$$

$$\begin{aligned}
B=\frac{-1}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\sin^{2n}\Theta\cdot b^{2n+1}}{(2n)\,!}\cdot&\left[\ln\left(\frac{4}{b^2}\right)\cdot\left(\frac{1}{2n+2}\,{}_1F_2\left(1+n\,;\,1\,,\,2+n\,;\,-b^2/4\right)\right.\right.\\
&\left.-\frac{1}{2n+1}\,{}_1F_2\left(1/2+n\,;\,1\,,\,3/2+n\,;\,-b^2/4\right)\right)\\
&+\frac{2}{(2n+2)^2}\cdot{}_2F_3\left(1+n\,,\,1+n\,;\,1\,,\,2+n\,,\,2+n\,;\,-b^2/4\right)\\
&\left.-\frac{2}{(2n+1)^2}\cdot{}_2F_3\left(1/2+n\,,\,1/2+n\,;\,1\,,\,3/2+n\,,\,3/2+n\,;\,-b^2/4\right)\right]
\end{aligned}$$

with hypergeometric functions ${}_1F_2(a_1;b_1,b_2;z)$ and ${}_2F_3(a_1,a_2;b_1,b_2,b_3;z)$.

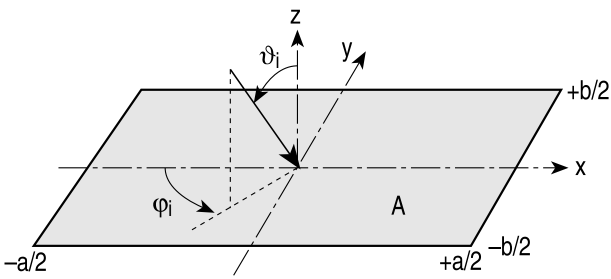
Correspondence in the diagram below: "Zr/Z0" $\to Z_r/Z_0$; "theta" $\to\vartheta$; "phi" $\to\varphi$; "k0a" $\to k_0a$; solid line: real part; dashed line: imaginary part.




## F.11 Wide Rectangular, Field-Excited Radiator

► *See also:* Mechel, Vol. I, Ch. 10 (1989)

A plane radiator is called "field excited" if its vibration pattern agrees with that of an obliquely incident plane wave at the surface.




The object here is a rectangle A with side lengths a,b in a hard baffle wall, the rectangle is excited by a plane wave with polar angle $\vartheta_i$ of incidence and azimuthal angle $\varphi_i$ with the axis parallel to side a.

Velocity pattern on A:

$$\begin{aligned} v(x, y) &= V_0 \cdot e^{-j\,(k_x x + k_y y)}, \\ k_x &= k_0 \cdot \sin\vartheta_i \cdot \cos\varphi_i = k_0 \cdot \mu_x, \\ k_y &= k_0 \cdot \sin\vartheta_i \cdot \sin\varphi_i = k_0 \cdot \mu_y. \end{aligned} \tag{1}$$

The sound pressure field is:

$$p(x,y,z)=\frac{j\,k_0Z_0}{2\pi}\iint\limits_A v(x_0,y_0)\,\frac{e^{-j\,k_0R}}{R}\,dA_0 \quad ; \quad R=\sqrt{(x-x_0)^2+(y-y_0)^2}. \tag{2}$$

The definition of the radiation impedance $Z_r$ with the radiated power gives a first form:

$$Z_r=\frac{j\,k_0Z_0}{2\pi A}\iint\limits_A dA\iint\limits_A \frac{e^{-j\,k_0R}}{R}\,e^{-j\,(k_x(x_0-x)+k_y(y_0-y))}\,dA_0\,;\; R^2=(x_0-x)^2+(y_0-y)^2. \tag{3}$$

The fact that $|v(x,y)|=\text{const}$ on A and that, therefore, the radiation impedance follows from the average field impedance with the Fourier transform of the velocity distribution leads to a form with fewer integrations:

$$\begin{aligned} V(k_1,k_2)&=\iint\limits_A v(x_0,y_0)\,e^{-j\,(k_1x_0+k_2y_0)}\,dx_0\,dy_0\\ &=V_0ab\,\frac{\sin\left((k_1+k_x)\,a/2\right)}{(k_1+k_x)\,a/2}\,\frac{\sin\left((k_2+k_y)\,b/2\right)}{(k_2+k_y)\,b/2}\end{aligned} \tag{4}$$

using $\alpha_1=k_1/k_0$; $\alpha_2=k_2/k_0$:

$$\begin{aligned}\frac{Z_r}{Z_0}&=\frac{k_0a\cdot k_0b}{4\pi^2}\iint\limits_{-\infty}^{+\infty}\left(\frac{\sin\left((\mu_x-\alpha_1)k_0a/2\right)}{(\mu_x-\alpha_1)k_0a/2}\right)^2\\ &\quad\cdot\left(\frac{\sin\left((\mu_y-\alpha_2)k_0b/2\right)}{(\mu_y-\alpha_2)k_0b/2}\right)^2\frac{d\alpha_1\,d\alpha_2}{\sqrt{1-\alpha_1^2-\alpha_2^2}}.\end{aligned} \tag{5}$$

The third form starts from the Bouwkamp integral (*Sect. F.2*):

$$\frac{Z_r}{Z_0}=\frac{k_0a\cdot k_0b}{4\,\pi^2}\int_0^{2\pi}d\varphi\int_0^{\pi/2+j\,\infty}|D(\vartheta,\varphi)|^2\cdot\sin\vartheta\,d\vartheta \tag{6}$$

with the far field directivity function

$$\begin{aligned}D(\vartheta,\varphi)&=\frac{\sin\left(\dfrac{k_0a}{2}(\sin\vartheta_i\cos\varphi_i-\sin\vartheta\cos\varphi)\right)}{\dfrac{k_0a}{2}(\sin\vartheta_i\cos\varphi_i-\sin\vartheta\cos\varphi)}\\ &\quad\cdot\frac{\sin\left(\dfrac{k_0b}{2}(\sin\vartheta_i\sin\varphi_i-\sin\vartheta\sin\varphi)\right)}{\dfrac{k_0b}{2}(\sin\vartheta_i\sin\varphi_i-\sin\vartheta\sin\varphi)}.\end{aligned} \tag{7}$$

The second form can be transformed into:

$$\frac{Z_r}{Z_0} = \frac{2j}{\pi\, k_0a \cdot k_0b} \int_{x=0}^{k_0a} dx \int_0^{k_0b} (k_0a - x)\,(k_0b - y)\; \cos(\mu_x x)\; \cos(\mu_y y) \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dy. \tag{8}$$

This becomes, for normal sound incidence with $\vartheta_i = 0;\ \mu_x = \mu_y = 0$

$$\frac{Z_r}{Z_0} = \frac{2j}{\pi\, k_0a \cdot k_0b} \int_{x=0}^{k_0a} dx \int_0^{k_0b} (k_0a - x)\,(k_0b - y) \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dy. \tag{9}$$

The double integral can be transformed by substitution of variables into:

$$\frac{Z_r}{Z_0} = \frac{2j}{\pi\, k_0a \cdot k_0b} \left[ \int_0^{\mathrm{arctg}(b/a)} I(k_0a/\cos\varphi)\, d\varphi + \int_{\mathrm{arctg}(b/a)}^{\pi/2} I(k_0b/\sin\varphi)\, d\varphi \right] \tag{10}$$

with the intermediate integrals:

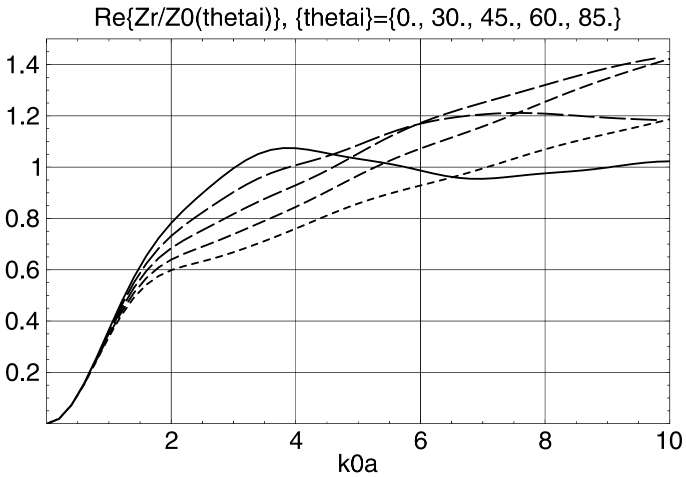
$$I(R) = \int_0^R \left(U + V \cdot r + W \cdot r^2\right) \cos(\alpha r) \cdot \cos(\beta r) \cdot e^{-j\,r}\, dr, \tag{11}$$

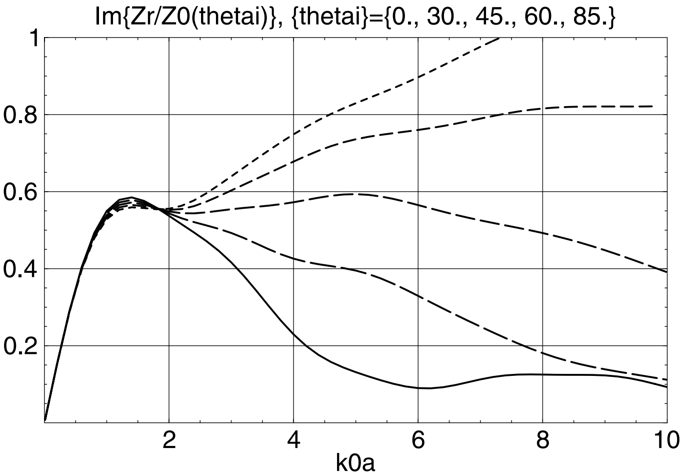
$U = k_0a \cdot k_0b \quad ; \quad V = -(k_0a \cdot \sin\varphi + k_0b \cdot \cos\varphi) \quad ; \quad W = \sin\varphi \cdot \cos\varphi,$

$\alpha = \mu_x \cdot \cos\varphi \quad ; \quad \beta = \mu_y \cdot \sin\varphi.$

See the reference for an analytical procedure to solve the integrals contained in I(R).

Correspondence in the following diagrams: “Zr/Z0” $\to$ $Z_r/Z_0$; “thetai” $\to$ $\vartheta_i$; “k0a” $\to$ $k_0a$; parameters: $b/a = 3$; $\varphi_i = 0$; the dashes become shorter with increasing position of the parameter value of $\vartheta_i$ in the list $\{\vartheta_i\}$.






## F.12 End Corrections

▸ *See also:* Mechel, Vol. II, Ch. 22 (1995)

See *Sect. F.1* for the definition of end corrections. End corrections represent the inertial near fields at expansions (orifices) of the cross section available for the sound wave. End corrections are mostly of interest for small $k_0a$, where a is a characteristic lateral dimension of the orifice. End corrections are influenced by the shape of the orifice and of the space which is available for the sound wave behind the orifice. Therefore in general the orifices on both sides of a "neck" must be distinguished (exterior and interior end correction). The relations of the end correction $\Delta\ell$ of an orifice with area A to the radiation impedance $Z_r = Z'_r + j \cdot Z''_r$ and the oscillating mass $M_r$ are

$$\Delta\ell = \frac{M_r}{\rho_0 A} = \frac{m_r}{\rho_0} = \frac{Z''_{mr}}{\omega\rho_0 A} = \frac{Z''_r}{\omega\rho_0} = \frac{Z''_r}{k_0 Z_0} \quad ; \quad \frac{\Delta\ell}{a} = \frac{Z''_r}{k_0 a \cdot Z_0}. \tag{1}$$

**Table 1** Oscillating mass $M_r$ of simple oscillating bodies

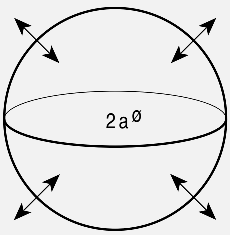
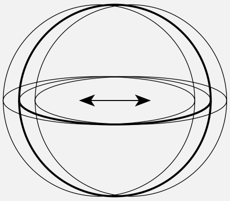
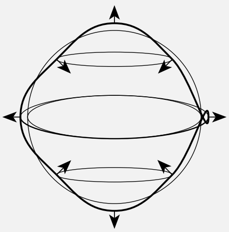
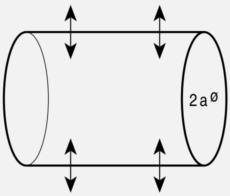
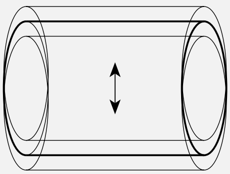
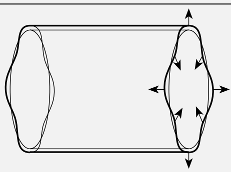
| Object | | $M_r$ | Remarks |
|---|---|---|---|
| Monopole sphere |   | $M_r = 3\,\rho_0 V \dfrac{1}{1+(k_0a)^2}$ $\xrightarrow[k_0a\ll 1]{} 3\,\rho_0 V$ | $A = 4\pi a^2$ $V = 4\pi a^3/3$ |
| Oscillating sphere |  | $M_r = 3\,\rho_0 V \dfrac{2+(k_0a)^2}{4+(k_0a)^4}$ $\xrightarrow[k_0a\ll 1]{} \frac{3}{2}\,\rho_0 V$ | |
| Sphere in *n*th mode |  | $M_r \xrightarrow[k_0a\ll \lvert 2n-1\rvert]{} 3\,\rho_0 V \dfrac{1}{n+1}$ $\xrightarrow[k_0a \gg n^2+1]{} 3\,\rho_0 V \dfrac{1}{(k_0a)^2}$ | |
| Monopole cylinder |   | $M_r \xrightarrow[k_0a\ll 1]{} -2\,\rho_0 V\ \ln(k_0a)$ | $A = 2\pi a$ $V = \pi a^2$ |
| Oscillating cylinder |  | $M_r \xrightarrow[k_0a\ll 1]{} 2\,\rho_0 V$ | |
| Cylinder in n-th mode |  | $M_r \xrightarrow[k_0a\ll 1]{} 2\,\rho_0 V\ \dfrac{1}{n}$ | |

**Table 2** End corrections $\Delta l/a$ of orifices

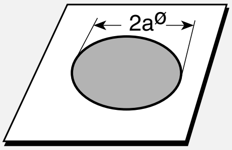
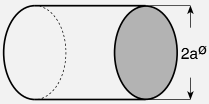
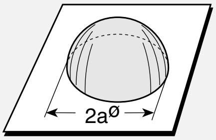
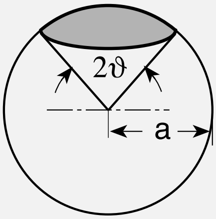
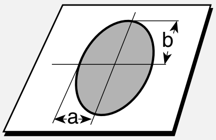
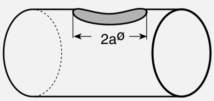
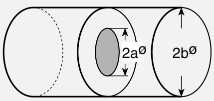
| Object | | $\frac{\Delta \ell}{a}$ | Remarks |
|---|---|---|---|
| Circle in baffle wall |  | $0.785 = \pi/4 < \Delta\ell/a \leq 8/3\pi = 0.85$<br>$\frac{\Delta\ell}{a} = \frac{8}{3\pi}\left[1 - \frac{2}{15}(k_0a)^2 + \frac{8}{525}(k_0a)^4\right]$ | a = radius |
| Tube orifice in free space |  | $(0.65 \text{ to } 0.69) \cdot a^2/\lambda_0$ | a = radius |
| Half monopole sphere in baffle wall |  | $\Delta\ell/a = 2/[1 + (k_0a)^2] \rightarrow 2$ | a = radius |
| Orifice on sphere |  | $\frac{\Delta\ell}{a} = [1 + \cos\vartheta] \,/\, [2\,(1 + (k_0a)^2)]$ | a = radius<br>a · sin ϑ = orifice radius |
| Elliptical orifice in baffle wall |  | $\frac{\Delta\ell}{a} = \frac{16}{3\pi^2} K(1-\beta^2)\,;\ \frac{\Delta\ell}{b} = \beta \cdot \frac{\Delta\ell}{a}$<br>$K(1-\beta^2) \approx \frac{4+\beta^2}{8} \ln\frac{16}{\beta^2} - \frac{\beta^2}{4}$<br>$0 < \beta \leq 0.641;$<br>$K(1-\beta^2) \approx \frac{\pi}{2}\,\frac{11+5\beta^2}{7+9\beta^2}$<br>$0.641 \leq \beta \leq 1$ | a = small<br>b = large<br>half axis<br>ß = a/b < 1 |
| Orifice in tube wall |  | $\Delta\ell \approx U/8 + (\lambda_0/2\pi) \cdot \chi_0(2k_0\sqrt{S/\pi})$<br>$\chi_0(x) = \frac{4}{\pi}\int_0^{\pi/2} \sin(x\,\cos\alpha)\,\sin^2\alpha\,d\alpha$<br>$\approx x^2/8\,;\ x \ll 1$ | U = 2πa = periphery<br>S = πa² = area of orifice |
| Circular fence in tube |  | $\Delta l/a \approx -0.0445\,728 - 0.728\,326\,x$<br>$-\,0.177\,078\,x^2 + 0.0339\,531\,y +$<br>$+\,0.00810\,471\,y^2 - 0.00100\,762\,xy$<br>$\sigma = (a/b)^2; x = \lg\sigma\,; y = \lg(b/\lambda_0)$ | a = fence radius<br>b = tube radius |

**Table 2** continued

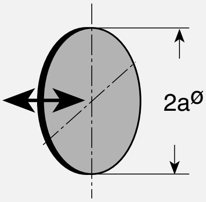
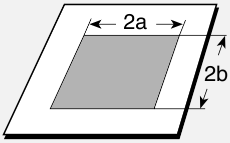
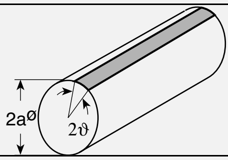
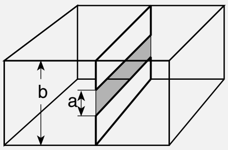
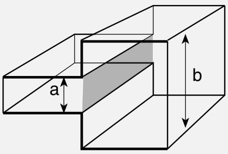
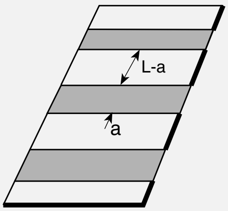
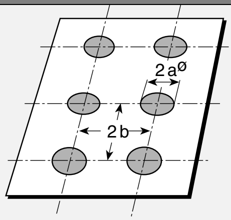
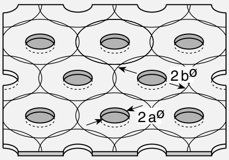
| Object | | $\frac{\Delta\ell}{a}$ | Remarks |
|---|---|---|---|
| Free circular disk |  | $\Delta l/a \approx 8/3\pi$ | a = radius |
| Rectangular orifice in baffle |  | $\frac{\Delta l}{a} = \frac{2}{3\pi}\left[\beta + \frac{1-(1+\beta^2)^{3/2}}{\beta^2}\right] + \frac{2}{\pi}\left[\frac{1}{\beta}\ln\left(\beta+\sqrt{1+\beta^2}\right) + \ln\left(\frac{1}{\beta}\left(1+\sqrt{1+\beta^2}\right)\right)\right]$ | 2a, 2b = sides<br>$\beta = a/b \leq 1$ |
| Slit on a cylinder |  | $\frac{\Delta\ell}{2a\vartheta} = \frac{1}{\pi}\sum_{n\geq 1}\frac{1}{n}\left(\frac{\sin\ n\vartheta}{n\vartheta}\right)^2 - \frac{\ln(k_0 a)}{2\pi}$ | a = cylinder radius<br>$\vartheta$ = angle of slit |
| Rectangular orifice in baffle wall |  | $\Delta\ell/a = \frac{1}{\pi}\ln\left[\frac{1}{2}\mathrm{tg}\left(\frac{\pi\sigma}{4}\right) + \frac{1}{2}\cot\left(\frac{\pi\sigma}{4}\right)\right]$<br>$\approx \frac{1}{\pi}\ln\left[\sin\left(\frac{\pi\sigma}{2}\right)\right];\ \sigma < 1$<br>$\approx \frac{\pi}{8}(1-\sigma)^2;\ b-a \ll b$<br>$\approx -0.395\,450\,x + 0.346\,161\,x^2$<br>$+ 0.141\,928\,x^3 + 0.0200\,128\,x^4$ | a = slit width<br>b = duct width<br>$\sigma = a/b$<br>$x = \lg\sigma$ |
| Expansion of a flat duct |  | $\Delta\ell/a = \frac{1}{\pi}\left[\frac{(1-\beta)^2}{2\beta}\ln\frac{1+\beta}{1-\beta} + \ln\frac{(1+\beta)^2}{4\beta}\right]$<br>$\approx \frac{1}{\pi}[1-\ln(4\beta)];\ a \ll b$<br>$\approx \frac{1}{4\pi}(1-\beta)^2[1-\ln((1-\beta)/2)];$<br>$a \approx b$ | a = narrow duct height<br>b = wide duct height<br>$\beta = a/b$ |
| Grid of slits |  | $\Delta\ell/a = \sigma\sum_{n=1}^{\infty}\frac{\sin^2(n\pi\sigma)}{(n\pi\sigma)^3}$<br>$\approx -\sqrt{2}/\pi\cdot\ln[\sin(\pi\sigma/2)]\ ;$<br>$0,1 < \sigma < 0,7$ | a = slit width<br>L = slit distance<br>$\sigma = a/L$ |

**Table 2** continued

| Object | | $\frac{\Delta\ell}{a}$ | Remarks |
|---|---|---|---|
| Hole grid; square arrangement |  | $\Delta\ell/a = 0.79\,(1 - 1.47\sqrt{\sigma} + 0.47\,\sigma^{3/2})$ | a = hole radius<br>2b = hole distance<br>$\sigma = \pi a^2/(2b)^2$ |
| Hole grid; hexagonal arrangement |  | $\Delta\ell/a \approx -0.0454\,728 - 0.728\,326\,x$ $- 0.177\,078\,x^2 + 0.0339\,531\,y$ $+ 0.00810\,471\,y^2 - 0.00100\,762\,xy$ ; $\sigma = (a/b)^2$ ; $x = \lg\sigma$ ; $y = \lg(b/\lambda_0)$ | a = hole radius<br>b = hole distance |

*End correction of a slit in a grid of parallel slits:*

Alternative representations; a = width of slit; L = a + b = slit centre distance:

$$\frac{\Delta\ell}{a} = \frac{a}{L}\cdot\sum_{n=1}^{\infty}\frac{\sin^2(n\pi a/L)}{(n\pi a/L)^3}, \tag{2}$$

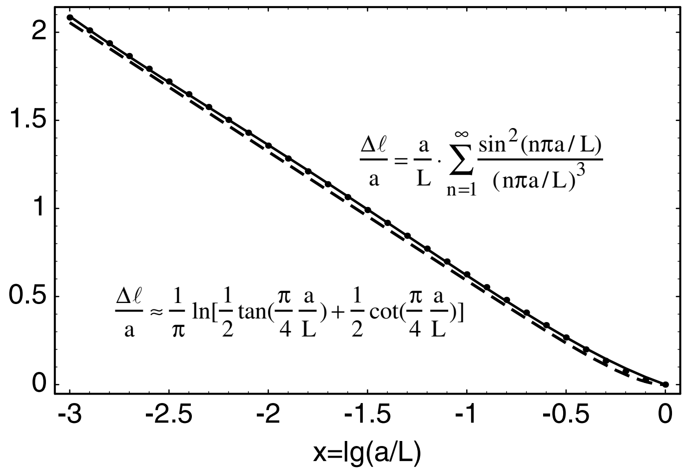
$$\frac{\Delta\ell}{a} \approx \frac{1}{\pi}\,\ln\left[\frac{1}{2}\tan\left(\frac{\pi}{4}\,\frac{a}{L}\right) + \frac{1}{2}\cot\left(\frac{\pi}{4}\,\frac{a}{L}\right)\right] \tag{3}$$

from regression:

$$\frac{\Delta\ell}{a} = -0.395\,450\cdot x + 0.346\,161\cdot x^2 + 0.141\,928\cdot x^3 + 0.0200\,128\cdot x^4. \tag{4}$$

Radiation reactance (used below for reference):

$$\frac{Z''_{r0}}{Z_0} = \frac{2a}{L}\sum_{n>0}\frac{1}{\sqrt{\left(n\,\frac{\lambda_0}{L}\right)^2 - 1}}\left(\frac{\sin(n\pi\,a/L)}{n\pi\,a/L}\right)^2. \tag{5}$$




*Influence of higher modes in the neck of a slit grid plate:*

Width and distance of slits as above: the slits are in a plate of thickness d; radiation reactance of a back orifice:

$$Z''_{rb} = Z''_{r0} \cdot \left(1 + \frac{\Delta Z''_{rb}}{Z''_{r0}}\right) = Z''_{rb} \cdot (1 - 10^{F(x,y)}),$$

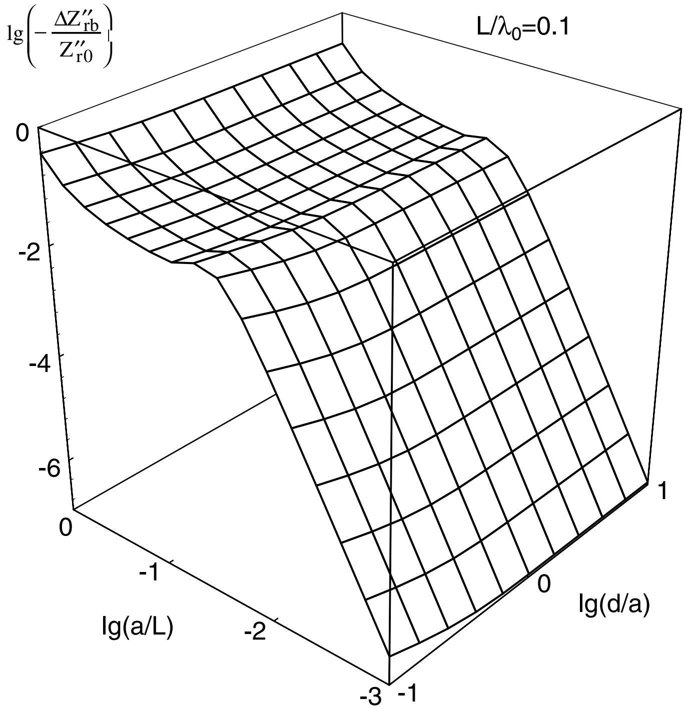
$$F(x, y) = \lg\left(-\frac{\Delta Z''_{sh}}{Z''_{sh0}}\right) = f(x) \cdot (1 + g(y)), \qquad (6)$$

$$x = \lg(a/L) \quad ; \quad y = \lg(d/a)$$

with (in $-3 \le x < 0$ and $-1 \le y \le 1$):

$$\begin{aligned} f(x) \quad &= -1.739\,68 + 1.484\,35\,(x + 1.5) - 1.842\,30\,(x + 1.5)^2 \\ &\quad + 0.292\,538\,(x + 1.5)^3 + 0.428\,402\,(x + 1.5)^4, \\ g(y) \quad &= H(-y) \cdot [0.00\,259\,355\,y - 0.0758\,181\,y^2 \\ &\quad + 0.330\,845\,y^3 + 0.226\,933\,y^4], \end{aligned} \qquad (7)$$

$$H(-y) = \begin{cases} 1\,; & y \le 0 \\ 0\,; & y > 0. \end{cases}$$




Relative change of radiation reactance of a slit in a slit grid due to higher modes in the neck of the slit plate.

*Interior end correction of the slit orifice in a slit resonator array:*

No losses and only a plane wave in the slit (i.e. narrow slit).

The resonators repeat in the y direction with a period length $L = a + b$.

Lateral wave numbers in the volume:

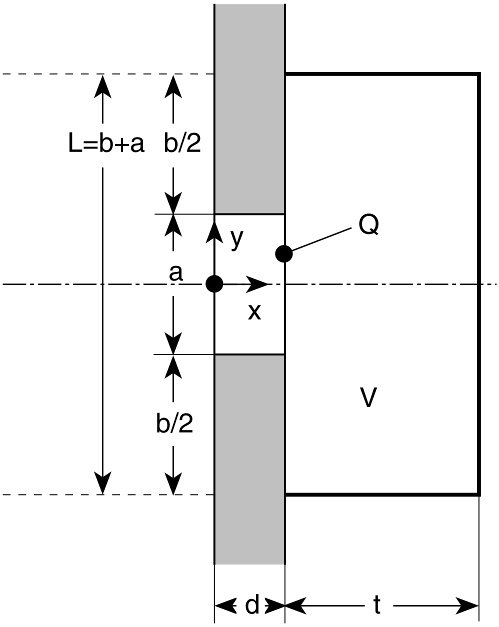
$$\gamma_0 = jk_0\,; \quad \gamma_n = k_0\sqrt{\left(n\frac{\lambda_0}{L}\right)^2 - 1}, \tag{8}$$

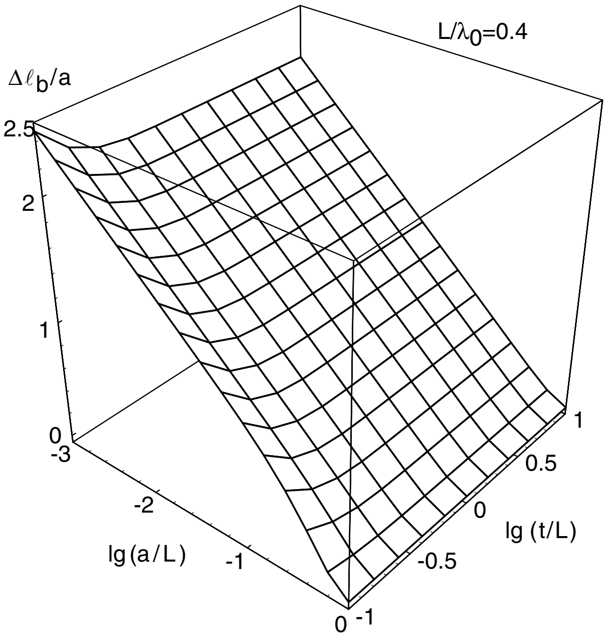
$$\mathrm{Re}\{\gamma_n\} \geq 0 \quad \text{or} \quad \mathrm{Im}\{\gamma_n\} \geq 0.$$

Impedance of the back orifice:

$$\frac{Z_{sh}}{Z_0} = -j\,\frac{a/L}{\tan(k_0 t)} + j\,2\frac{a}{L}\sum_{i>0}\frac{k_0}{\gamma_i}\,\frac{s_i^2}{\tanh(\gamma_i t)}, \tag{9}$$

$$s_0 = 1 \quad ; \quad s_i = \frac{\sin(i\pi\, a/L)}{i\pi\, a/L}. \tag{10}$$




Influence of the shape parameter t/L on the interior end correction of the slit in a slit resonator

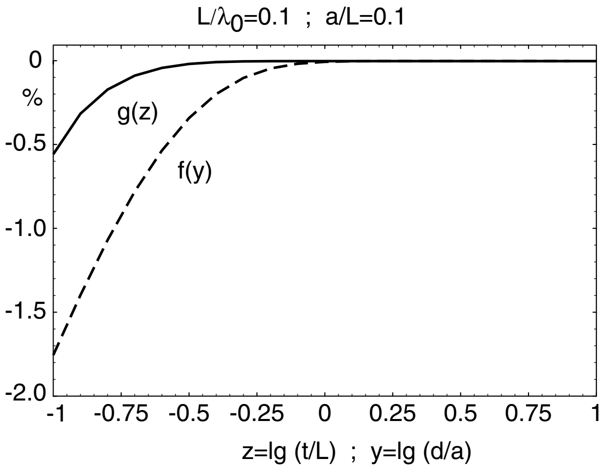
The first term (outside the sum) is the spring reactance of the volume; thus the sum term is the mass reactance at the interior orifice. The back side end correction therefore is:

$$\frac{\Delta\ell_b}{a} = \frac{2}{L}\sum_{i>0}\frac{1}{\gamma_i}\,\frac{s_i^2}{\tanh(\gamma_i t)}. \tag{11}$$

*Interior end correction of the slit orifice in a slit resonator array with higher modes in the neck:*

Geometrical parameters as above. $\Delta\ell_{b0}/a$ interior end correction from above with only plane wave in the neck:

$$\begin{aligned}
&\frac{\Delta\ell_b}{a} \approx \frac{\Delta\ell_{b0}}{a}(x)\cdot[1+f(y)]\cdot[1+g(z)],\\
&x = \lg\frac{a}{L}\quad;\quad y=\lg\frac{d}{a}\quad;\quad z=\lg\frac{t}{L},\\
&f(y) = 0.001\,448\,29\cdot y + 0.002\,555\,10\cdot y^2 + 0.034\,305\,10\cdot y^3 + 0.015\,682\,99\cdot y^4,\\
&g(z) = -0.000\,932\,290\cdot z - 0.007\,672\,04\cdot z^2 - 0.019\,259\,72\cdot z^3 - 0.018\,048\,39\cdot z^4.
\end{aligned} \tag{12}$$




Influence (in per cent) of shape factors $d/a$ and $t/L$ on the interior end correction, with higher neck modes taken into account

*Interior orifice impedance* of a slit in a slit array, with viscous and caloric losses in the neck taken into account:

Let $Z_{b0}$ be the back orifice impedance without losses. The back orifice impedance $Z_b = Z'_b + j \cdot Z''_b$ can be approximated with:

$$\frac{Z'_b}{Z_0} = \frac{Z'_{b0}}{Z_0}\left(1 + \frac{10^{F'(x)}}{\sqrt{a_{[m]}} \cdot \sqrt[3]{a/L}}\right);\ \frac{Z''_b}{Z_0} = \frac{Z''_{b0}}{Z_0}\left(1 + \frac{10^{F''(x)}}{\sqrt{a_{[m]}} \cdot \sqrt[3]{a/L}}\right);\ x = \lg \frac{f_{[Hz]}a_{[m]}}{(a/L)^{3/2}}, \tag{13}$$

$$F'(x) = -4.641\,06 \ + 0.435\,993\ x + 0.0142\,851\ x^2 + 0.000\,461\,347\ x^3,$$

$$F''(x) = -2.266\,65 - 0.492\,331\ x - 0.000\,719\,182\ x^2 - 0.001\,0208\ x^3.$$

*Interior orifice impedance of a slit in a slit array in contact with a porous absorber layer* (i.e. t = 0 in the sketch):

Let the characteristic propagation constant and wave impedance of the porous material be $\Gamma_a, Z_a$. Air gap thickness $t = 0$. $\Xi$ = flow resistivity of the porous material.

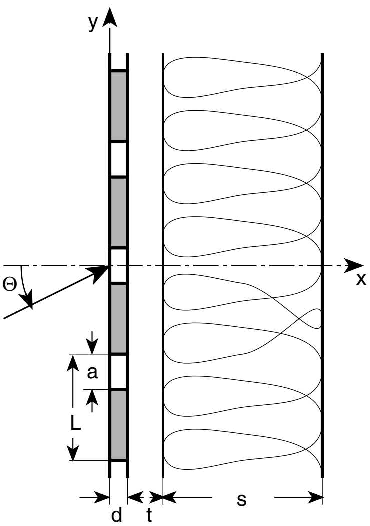
$$\varepsilon_n = k_0\sqrt{(\sin\,\Theta + n\,\lambda_0/L)^2 + (\Gamma_a/k_0)^2}. \tag{14}$$

Impedance $Z_b$ of the back slit orifice:

$$\frac{Z_b}{Z_0} = 2\frac{a}{L}\,\frac{\Gamma_a Z_a}{k_0 Z_0}\sum_{n>0}\frac{k_0}{\varepsilon_n}\left(\frac{\sin(n\pi\,a/L)}{n\pi\,a/L}\right)^2 \coth(\varepsilon_n s)\,. \tag{15}$$

Back orifice end correction:

$$\frac{\Delta\ell_b}{a} = -2j\,\frac{s}{L}\,\frac{\Gamma_a Z_a}{k_0 Z_0}\sum_{n>0}\left(\frac{\sin(n\pi\,a/L)}{n\pi\,a/L}\right)^2 \frac{\coth(\varepsilon_n s)}{\varepsilon_n s}. \tag{16}$$



Interior end correction of a slit in a slit array in contact with a porous absorber layer:

$$\begin{aligned}\frac{\Delta\ell_b}{a} &= j\,\frac{\Gamma_a Z_a}{k_0 Z_0}\\ &\cdot (0.0389998 + 0.454066\cdot x - 0.345328\cdot x^2 - 0.125386\cdot x^3\\ &- 0.0143782\cdot y + 0.00418541\cdot y^2 + 0.0170766\cdot y^3\\ &- 0.0142094\cdot z - 0.0715597\cdot z^2 + 0.0915584\cdot z^3\\ &- 0.0115326\cdot x\cdot y - 0.0195509\cdot x\cdot z - 0.0595634\cdot y\cdot z),\end{aligned} \tag{17}$$

$x = \lg(a/L);\ y = \lg(\Xi\, s/Z_0);\ z = \lg(s/L).$

*Interior orifice impedance of a slit in a slit array with an air gap* t *between the slit plate and a porous absorber layer:*

Geometrical and material parameters as well as $\varepsilon_n$ as above:

$$\gamma_0 = j\,k_0 \cos\Theta \quad ; \quad \gamma_n = k_0\sqrt{(\sin\theta + n\,\lambda_0/L)^2 - 1}. \tag{18}$$

Impedance $Z_b$ of back side orifice:

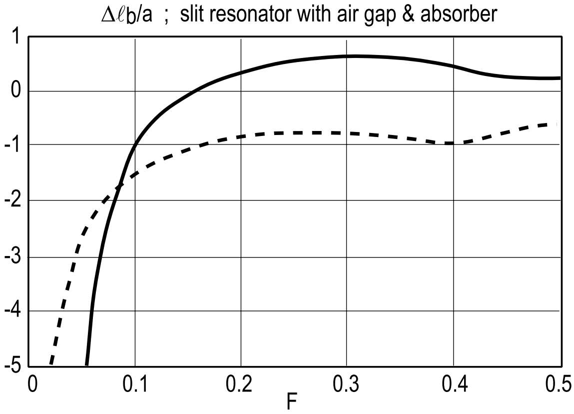
$$\frac{Z_b}{Z_0} = \frac{a}{L}\left[\frac{1 + r_0\, e^{-2j\,k_0 t}}{1 - r_0\, e^{-2j\,k_0 t}} + 2j\sum_{n>0}\frac{k_0}{\gamma_n}\left(\frac{\sin(n\pi\, a/L)}{n\pi\, a/L}\right)^2 \frac{1 + r_n\, e^{-2\gamma_n t}}{1 - r_n\, e^{-2\gamma_n t}}\right], \tag{19}$$

$$r_n = \frac{1 - j\,\dfrac{k_0 Z_0}{\Gamma_a Z_a}\,\dfrac{\varepsilon_n}{\gamma_n}\tanh(\varepsilon_n s)}{1 + j\,\dfrac{k_0 Z_0}{\Gamma_a Z_a}\,\dfrac{\varepsilon_n}{\gamma_n}\tanh(\varepsilon_n s)}. \tag{20}$$

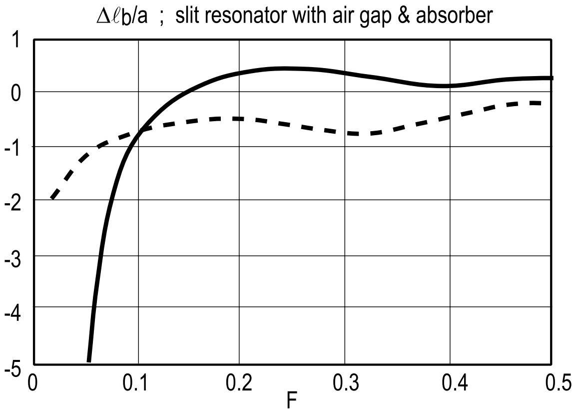
The first term in the brackets is the front side impedance of the porous layer transformed to the plane of the back side orifices of the slit plate. Therefore the second term (sum term) is the mass impedance $Z_{bm}$ of the oscillating mass of the back side orifice. The $r_n$ are the modal reflection factors at the front side of the porous layer. The end correction of the back slit orifice is:

$$\frac{\Delta\ell_b}{a} = \frac{-j}{k_0 a}\,\frac{Z_{bm}}{Z_0} = 2\sum_{n>0}\frac{1}{\gamma_n L}\left(\frac{\sin(n\pi\, a/L)}{n\pi\, a/L}\right)^2 \frac{1 + r_n\, e^{-2\gamma_n t}}{1 - r_n\, e^{-2\gamma_n t}}. \tag{21}$$

Correspondence and parameters in the following diagrams: “F” $\to L/\lambda_0$; parameters: $a/L = 0.25;\ d/a = 1;\ s/L = 1;\ R = \Xi\cdot s/Z_0 = 1$; porous layer of glass fibres.




Real (solid line) and imaginary (dashed line) part of interior end correction $\Delta\ell_b/a$ for $t/a = 0.01$ (other parameters given above). The real part represents a mass reactance if it is positive; at negative values it represents the influence of the porous material on the spring reactance of the volume. The negative imaginary part represents a flow resistance

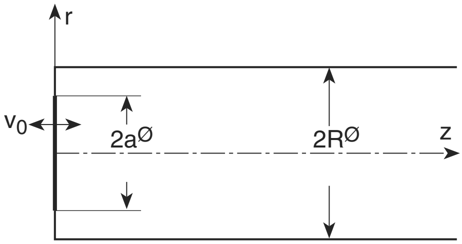



As above, but with a larger distance $t/a = 1$ between plane of orifices and absorber layer

## F.13 Piston Radiating Into a Hard Tube

► *See also:* Mawardi (1951)

A circular piston with diameter 2a oscillates with a velocity amplitude $v_0$ in a hard end surface of a hard, circular tube with diameter 2R. $S = \pi a^2$ = piston area.




Sound pressure on the piston surface $z = 0$:

$$p(r, z=0) = v_0 \left[ Z_0 + j\,\frac{\omega\,\rho_0}{S} \sum_{n\geq 1} \frac{J_0(k_{0n}r)\cdot J_1(k_{0n}a)\cdot a}{2\,k_{0n}\sqrt{k_{0n}^2 - k_0^2}\cdot J_0^2(k_{0n}a)} \right] \tag{1}$$

with $k_0 = \omega/c_0$; $k_{mn}$ = n-th root of $J'_m(k_{mn}R) = 0$.

The second term vanishes in the special case $a = R$ with $J_1(k_{0n}a) = 0$.

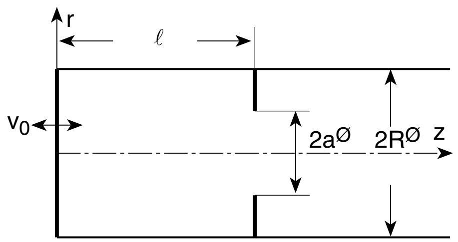
The radiation impedance $Z_s$ is:

$$Z_s = Z_0 + j\,\omega\,\rho_0 \sum_{n\geq 1} \frac{J_1^2(k_{0n}a)}{k_{0n}^2\sqrt{k_{0n}^2 - k_0^2}\cdot J_0^2(k_{0n}a)}. \tag{2}$$

## F.14 Oscillating Mass of a Fence in a Hard Tube

► *See also:* Iwanov-Schitz/Rscherkin (1963)

A hard tube with diameter 2R is driven by a plane wave with velocity amplitude $v_0$ from a piston in a distance $\ell$ to a thin fence with aperture diameter 2a; $S = \pi a^2$.




$M_I, M_{II}$ are the oscillating masses of the fence orifice towards the piston and towards the tube, respectively:

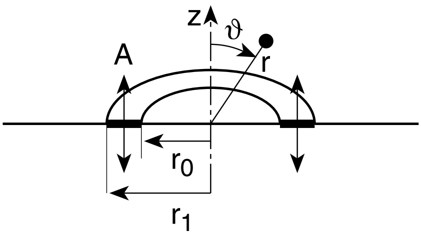
$$M_I = 4S\rho_0 R \sum_{m\geq 1} \frac{J_1^2(x_m a)\cdot \coth(x_m \ell)}{(x_m R)^3\cdot J_0^2(x_m R)} \quad ; \quad M_{II} = 4S\rho_0 R \sum_{m\geq 1} \frac{J_1^2(x_m a)}{(x_m R)^3\cdot J_0^2(x_m R)} \tag{1}$$

with $x_m$ the roots of $J'_0(x_m R) = 0$. In the limit $M_I \xrightarrow[\ell\to\infty]{} M_{II}$.

## F.15 A Ring-Shaped Piston in a Baffle Wall

► *See also:* Antonov/Putyrev (1984)

A ring with interior radius $r_0$ and exterior radius $r_1$ oscillates in a baffle wall. The ring surface area is $A_R = \pi(r_1^2 - r_0^2)$; the circle areas are $A_0 = \pi\, r_0^2$; $A_1 = \pi\, r_1^2$; the radius ratio $\alpha = r_0/r_1$ with $0 \le \alpha < 1$; the area ratio $\beta = A_R/A_0 = (r_1/r_0)^2 - 1$.




The mechanical radiation impedance $Z_s = Z_s' + j \cdot Z_s''$ (force/velocity) is evaluated by:

$$\begin{aligned} Z_s &= \frac{j\, k_0 Z_0}{2\pi} \iint_{A_R} dA_1 \iint_{A_R} \frac{e^{-j\, k_0 r}}{r}\, dA \\ &= 2\pi Z_0 \left[ \int_{r_0}^{r_1} \left( 1 - J_0(2k_0 r) - \frac{4}{\pi} I_s \right) \cdot r\, dr + j \int_{r_0}^{r_1} \left( S_0(2k_0 r) - \frac{4}{\pi} I_c \right) \cdot r\, dr \right] \end{aligned} \tag{1}$$

with $J_0(z)$ the Bessel function, $S_0(z)$ the Struve function of zero order, and the integrals:

$$\left.\begin{matrix} I_s \\ I_c \end{matrix}\right\} = \int_0^{\arcsin(r_0/r)} \left.\begin{matrix} \sin\,(k_0 r \cdot \cos\vartheta) \\ \cos\,(k_0 r \cdot \cos\vartheta) \end{matrix}\right\} \cdot \sin\left( k_0 \sqrt{r_0^2 - r^2 \sin^2\vartheta} \right) d\vartheta. \tag{2}$$

Approximation for low frequencies $k_0 r_0 \ll 1$ and $k_0 r_1 \ll 1$:

$$\frac{Z_s}{Z_0} \approx A_0 \left[ \frac{\beta^2}{2} (k_0 r_0)^2 + j \frac{8}{3\pi} \frac{k_0 r_0}{\alpha^2} \left( (1+\alpha^2)(1 - E(\alpha)) + (1-\alpha^2) K(\alpha) \right) \right] \tag{3}$$

with $E(\alpha)$, $K(\alpha)$ the complete elliptic integrals of the first and second kinds, respectively.

For low frequencies $k_0 r_0 \ll 1$ and $k_0 r_1 \ll 1$ and a slender ring $0 < \beta < 0.6$:

$$\frac{Z_s''}{Z_0} \approx \frac{A_0 \beta^2}{2\pi} k_0 r_0 \left[ (1 - 0.25\,\beta)\, \ln \frac{16}{\beta} + \frac{3}{2} \right]. \tag{4}$$

Special case of a small circular piston radiator, i.e. $r_0 \to 0$ and $k_0 r_1 \ll 1$:

$$\frac{Z_s}{Z_0} \approx A_1 \left[ \frac{(k_0 r_1)^2}{2} + j\,\frac{8}{3\,\pi} k_0 r_1 \right]. \tag{5}$$

Far field of a ring-shaped piston radiator with elongation amplitude a:

$$p(r, \vartheta) \approx -\frac{1}{2}\, a\, Z_0 \frac{r_1}{r}\, k_0 r_1 \left[ 2\, \frac{J_1(k_0 r_1 \sin\vartheta)}{k_0 r_1 \sin\vartheta} - 2\,\alpha^2 \frac{J_1(k_0 r_0 \sin\vartheta)}{k_0 r \sin\vartheta} \right] \cdot e^{-j k_0 r}. \tag{6}$$

## F.16 Measures of Radiation Directivity

Let $p(r, \vartheta, \varphi)$ be the sound pressure generated by a radiator in the far field, $k_0 r \gg 1$.

*Directivity factor*: $$D_0(\vartheta, \varphi) = \frac{p(r, \vartheta, \varphi)}{p(r, \vartheta_0, \varphi_0)} \quad \left( \text{or} \quad = \frac{|p(r, \vartheta, \varphi)|}{|p(r, \vartheta_0, \varphi_0)|} \right) \tag{1}$$

where $p(r, \vartheta_0, \varphi_0)$ is the sound pressure in a reference direction (mostly the direction of some axis of symmetry of the radiator).

*Directivity coefficient*: $$D_0^2(\vartheta, \varphi) = \frac{|p(r, \vartheta, \varphi)|^2}{|p(r, \vartheta_0, \varphi_0)|^2} \tag{2}$$

*Directivity value*: $$D_m(\vartheta, \varphi) = \frac{|p(r, \vartheta, \varphi)|^2}{\langle |p(r, \vartheta, \varphi)|^2 \rangle_{\vartheta,\varphi}} \tag{3}$$

*Directivity index*: $$D_{L0}(\vartheta, \varphi) = 10 \cdot \lg \frac{|p(r, \vartheta, \varphi)|^2}{|p(r, \vartheta_0, \varphi_0)|^2} \tag{4}$$

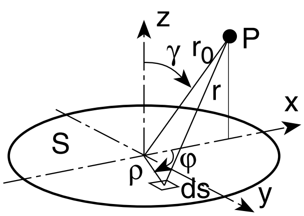
*Directivity*: $$D_{Lm}(\vartheta, \varphi) = 10 \cdot \lg \frac{|p(r, \vartheta, \varphi)|^2}{\langle |p(r, \vartheta, \varphi)|^2 \rangle_{\vartheta,\varphi}} \tag{5}$$

With $\langle \ldots \rangle_{\vartheta,\phi}$ the average over the directions $\vartheta$ and $\phi$.

*Sharpness of directivity pattern* $\beta$ is given as the angle between the normal to the radiator and the direction for which the intensity decreases to 1/2 of the maximum value.

## F.17 Directivity of Radiator Arrays

► *See also:* Skudrzyk, Ch. 26 (1971)



The far field p of a plane radiator with area S and normal velocity distribution V(x, y) in an infinite baffle wall can be evaluated with the Huygens-Rayleigh integral:

$$\begin{aligned} p &= \frac{j\,k_0 Z_0}{2\pi} \int_S \frac{V(x,y) \cdot e^{-j\,k_0 r}}{r}\,ds \\ &= \frac{j\,k_0 Z_0}{2\pi} \frac{e^{-j\,k_0 r_0}}{r_0} \int_S V(x,y) \cdot e^{+j\,k_0 (x\,\cos(r_0,x) + y\,\cos(r_0,y))}\,ds, \end{aligned} \tag{1}$$

where $r_0$ is the radius from a reference point on the radiator to the field point P. If the velocity V has a constant phase on the radiator, the sound pressure attains its maximum $P_0$ in the direction normal to the radiator:

$$P_0 = \frac{j\,k_0 Z_0}{2\pi} \frac{e^{-j\,k_0 r_0}}{r_0} \cdot Q \quad ; \quad Q = \int_S V\,ds. \tag{2}$$

Describe the sound pressure in other directions with the directivity factor D: $p = D \cdot P_0$ with:

$$\begin{aligned} D &= \frac{1}{Q} \int_S e^{+j\,k_0 (x\,\cos(r_0,x) + y\,\cos(r_0,y))}\,dQ \quad ; \quad dQ = V \cdot ds \\ &= \frac{1}{Q} \int_S e^{+j\,k_0 \rho\,\cos\varphi\,\sin\gamma}\,dQ \xrightarrow[\text{symm.}]{} \frac{1}{Q} \int_S \cos(k_0 \rho\,\cos\varphi\,\sin\gamma)\,dQ \end{aligned} \tag{3}$$

(see the graph for $\gamma$, $\varphi$). The last relation holds for a radiator with a central axis of symmetry.

In the case of an array with small elementary radiators having conphase volume flows $Q_n$ the integral is replaced by a sum:

$$D = \frac{1}{Q} \sum_n Q_n \cdot e^{+j\,k_0 (x_n\,\cos(r_n,x_n) + y_n\,\cos(r_n,y_n))} \quad ; \quad Q = \sum_n Q_n. \tag{4}$$

*Two point sources with equal volume flow* $Q_i$ *at* $x = 0$ *and* $x = d$:

$$D = e^{j\,k_0 d/2 \cdot \cos(r,x)} \cdot \cos\left(k_0 d/2 \cdot \cos(r,x)\right) = e^{j\,k_0 d/2 \cdot \sin\gamma} \cdot \cos\left(k_0 d/2 \cdot \sin\gamma\right).$$

Maxima of $|D|$ are at angles $\gamma$ with $d\,\sin\gamma = 2\nu \cdot \lambda_0/2 \quad ; \quad \nu = 1, 2, 3, \ldots$ ; minima occur at odd multiples of $\lambda_0/2$.

*Point sources equally spaced along a line*:

The n point sources spaced at intervals d again are conphase and of equal strength.

$$D = \frac{1}{n} \sum_{\nu=0}^{n-1} e^{j\,\nu k_0 d\,\sin\gamma} = e^{j\,(n-1)\Delta} \frac{\sin(n\Delta)}{n \cdot \sin\Delta} \quad ; \quad \Delta = \tfrac{1}{2} k_0 d\,\sin\gamma. \tag{5}$$

Zeroes of the directivity are at angles $\gamma$ with $\sin\gamma = \nu\lambda_0/nd$; the principal maximum (with unit value) is at $\gamma = 0$; the angles for the following maxima are at:

$$\sin\gamma = \frac{(2\nu+1)\pi}{2nd} \quad ; \quad \nu = 1,\ 2,\ 3, \ldots \tag{6}$$

with values at the maxima:

$$D_\nu = \frac{1}{n\ \sin\Delta} = \frac{1}{n\ \sin\left((2\nu+1)\pi/(2n)\right)}. \tag{7}$$

*Densely packed linear array*:

With $\ell = n\,d$ the length of the array:

$$D = e^{j\,\frac{1}{2}k_0\ell\ \sin\gamma}\,\frac{\sin\left(\frac{1}{2}k_0\ell\ \sin\gamma\right)}{\frac{1}{2}k_0\ell\ \sin\gamma}. \tag{8}$$

*Densely packed circular array*:

The circle has the radius a; the elementary volume flow $dQ = Q_0 ds = Q_0 \cdot a d\varphi$ is constant along the circle.

$$D = \frac{1}{2\pi}\int_0^{2\pi} e^{j\,k_0 a\ \sin\gamma\,\cos\varphi}\,d\varphi = J_0(\Delta) \quad ; \quad \Delta = k_0 a\ \sin\gamma \tag{9}$$

*Sources at constant intervals along a circle*:

Let n point sources with equal volume flow Q be distributed with equal intervals on a circle with radius a. $r_0$ = radius from circle centre to field point P; $\gamma$ = angle between circle axis and $r_0$; $\varphi$ = angle between the x axis in the plane of the circle and the projection of $r_0$ on the circle plane.

$$\begin{aligned} D = {} & J_0(k_0 a\ \sin\gamma) + 2\,j^n J_n(k_0 a\ \sin\gamma)\cdot\cos(n\varphi) \\ & + 2\,j^{2n} J_{2n}(k_0 a\ \sin\gamma)\cdot\cos(2n\varphi) + \ldots \end{aligned} \tag{10}$$

*Circular piston in a baffle wall*:

The piston radius is a. Elementary volume flow $dQ = Q_0 \cdot r\,dr\,d\varphi$; $x = r\cdot\cos\varphi$; $y = r\cdot\sin\varphi$.

$$\begin{aligned} D &= \frac{1}{\pi a^2}\int_0^a\int_0^{2\pi} \cos(k_0 r\ \cos\varphi\ \sin\gamma)\ r\ dr\ d\varphi \\ &= \frac{2}{a^2}\int_0^a r\ J_0(k_0 r\ \sin\gamma)\ dr = 2\frac{J_1(k_0 a\ \sin\gamma)}{k_0 a\ \sin\gamma} \end{aligned} \tag{11}$$

*Rectangular piston in a baffle wall*:

The side lengths are 2a, 2b; the elementary volume flow $dQ = Q_0 \cdot dx\,dy$.

$$D = D_1 \cdot D_2$$

$$D_1 = \frac{1}{2a}\int_{-a}^{+a} e^{j\,k_0 x\ \cos(r,x)}\,dx = \frac{\sin\left(k_0 a\ \cos(r_0, x)\right)}{k_0 a\ \cos(r_0, x)}$$
$$D_2 = \frac{1}{2b}\int_{-b}^{+b} e^{j\,k_0 y\ \cos(r,y)}\,dy = \frac{\sin\left(k_0 b\ \cos(r_0, y)\right)}{k_0 b\ \cos(r_0, y)} \tag{12}$$

*Rectangular plate, clamped at opposite edges, vibrating in its fundamental mode*:

Let the plate be in a one-dimensional vibration with (approximate) velocity distribution:

$$V(y) = V_0 \cdot \left(1 - y^2/b^2\right), \tag{13}$$

where 2a is the length of the supported edges and 2b that of the other two edges. Average velocities: $\langle V\rangle = \frac{2}{3}V_0 \quad ; \quad \langle V^2\rangle = \frac{8}{15}V_0^2,$ (14)

$$D = \frac{3}{\Delta^2}\left(\frac{\sin\Delta}{\Delta} - \cos\Delta\right) \quad ; \quad \Delta = k_0 b\ \sin\gamma. \tag{15}$$

*Rectangular plate, free at opposite edges, vibrating in its fundamental resonance*:

Let the plate be in a one-dimensional vibration with (approximate) velocity distribution:

$$V(y) = V_0 \cdot \left(1 - 2y^2/b^2\right). \tag{16}$$

The nodal lines ($V = 0$) are at $y = \pm b/\sqrt{2}$. The average velocities are:

$$\langle V\rangle = \frac{1}{3}V_0 \quad ; \quad \langle V^2\rangle = \frac{7}{15}V_0^2, \tag{17}$$

$$D = \frac{12}{\Delta^2}\left(\frac{\sin\Delta}{\Delta} - \cos\Delta\right) - \frac{3\ \sin\Delta}{\Delta} \quad ; \quad \Delta = k_0 b\ \sin\gamma. \tag{18}$$

*Circular membrane and plate*:

Let the radius be a. The velocity distribution of the fundamental mode can be represented by a power series:

$$V(\rho) = V_0 + V_1\left(1-\rho^2/a^2\right) + V_2\left(1-\rho^2/a^2\right)^2 + \ldots, \tag{19}$$

$$\begin{aligned} D = &\left[V_0 + \frac{1}{2}V_1 + \frac{1}{3}V_2 + \ldots + \frac{1}{n+1}V_n\right] \\ &\cdot\left[2V_0\frac{J_1(\Delta)}{\Delta} + 2\cdot 1!\cdot V_1\frac{J_2(\Delta)}{\Delta^2} + \ldots + 2^{n+1}\cdot n!\cdot V_n\frac{J_{n+1}(\Delta)}{\Delta^{n+1}}\right]^{-1} ;\ \Delta = k_0 a\ \sin\gamma . \end{aligned} \tag{20}$$

For a velocity distribution $\qquad V(\rho) = V_0 \cdot J_0(k_B\rho) \qquad (21)$

with the bending wave number $k_B$ on the radiator:

$$\begin{aligned} D = &\frac{1}{a^2}\,\frac{k_B a}{J_1(k_B a)}\,\frac{a}{k_B^2 - k_0^2\ \sin\gamma} \\ &\cdot\left[k_B\, J_0(k_0 a\ \sin\gamma)\, J_1(k_B a) - k_0\ \sin\gamma\ J_1(k_0 a\ \sin\gamma)\, J_0(k_B a)\right]. \end{aligned} \tag{22}$$

If the membrane or plate is supported at its edge, i.e. $J_0(k_B a) = 0$:

$$D = \frac{k_B^2}{k_B^2 - k_0^2\ \sin\gamma}\, J_0(k_0 a\ \sin\gamma). \tag{23}$$

*Circular radiator with radial and azimuthal nodal lines*:

Develop the velocity distribution into a Fourier series:

$$V(\rho,\varphi_0) = \sum_{m\geq 0} V_m(\rho)\cdot\cos(m\varphi_0) \tag{24}$$

with radial nodal lines for integer $m > 0$, and circular nodal lines at $V_m(\rho) = 0$.

Write the far field pressure as: $\qquad p(r,\gamma,\varphi) = \dfrac{e^{-j k_0 r}}{r}\sum_{m\geq 0} K_m(\gamma,\varphi). \qquad (25)$

The directivity factor of a sum term then is:

$$D_m(\gamma,\varphi) = \frac{p_m(r,\gamma,\varphi)}{p_0(r,0,\varphi)} = \frac{K_m(\gamma,\varphi)}{K_0(0,\varphi)} = \frac{2\pi\cdot K_m(\gamma,\varphi)}{Q} = \frac{2\pi\cdot K_m(\gamma,\varphi)}{\langle V\rangle\, S}\ ;\ S = \pi a^2, \tag{26}$$

$$K_m(\gamma,\varphi) = \cos(m\varphi)\cdot e^{j\,m\pi/2}\int_0^a J_m(k_0\rho\sin\gamma)\cdot V_m(\rho)\cdot\rho\, d\rho. \tag{27}$$

Introducing the integral transform (which is tabulated for many $V_{cm}(\rho)$):

$$f_m(\lambda) = \frac{1}{a^2} \int_0^a J_m(\lambda\rho) \cdot V_m(\rho) \cdot \rho \, d\rho, \tag{28}$$

one gets: $$K_m(\gamma, \varphi) = a^2 \cdot j^m \cdot f_m(k_0 \sin\gamma) \cdot \cos(m\varphi). \tag{29}$$

The directivity factor $D(\gamma, \vartheta)$ is the sum of the $D_m(\gamma, \vartheta)$.

*Array of finite size radiators*:

If all radiators have the same directivity factor $D_a$, and the similar array with point sources has the directivity factor $D_0$, then the array with finite size radiators has the directivity factor

$$D = D_a \cdot D_0. \tag{30}$$

## F.18 Radiation of Finite Length Cylinder

► *See also:* Skudrzyk, Ch. 21 (1971)

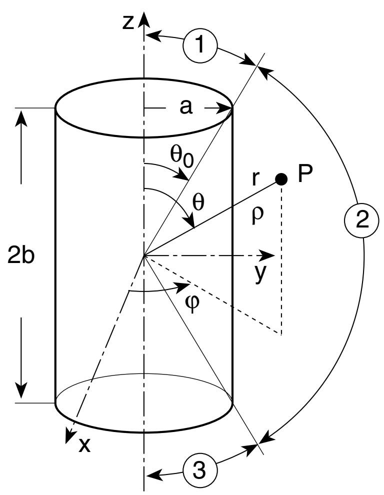
A cylinder with radius a and length 2b oscillates on its circumference with the velocity V and is at rest on its end caps.

The centre of the cylinder is the origin of a cylindrical co-ordinate system $(\rho, z, \varphi)$ and of a spherical system $(r, \theta, \varphi)$.

Three angular sections are distinguished:

$$\begin{aligned} &(1) \quad 0 \le \theta \le \theta_0 = \arctan(a/b) \\ &(2) \quad \theta_0 < \theta < \pi - \theta_0 \\ &(3) \quad \pi - \theta_0 \le \theta \le \pi. \end{aligned} \tag{1}$$

The switch functions are defined for the section i: $H_i = 1$ for $\theta$ in i; $H_i = 0$ else.



Field formulation:

$$p(r,\theta) = -j\,k_0 Z_0 \sum_{n=0,2,4\ldots} a_n \cdot P_n(\cos\theta) \cdot h_n^{(2)}(k_0 r) \tag{2}$$

with $P_n(z)$ = Legendre polynomials; $h_n^{(2)}(z)$= spherical Hankel functions of second kind.
The coefficients $a_n$ are the solutions of the linear system of equations:

$$\sum_{n=0,2,4\ldots} a_n \cdot (\Phi_m, \Phi_n) = V \cdot (\Phi_m, H_2) \quad ; \quad m = 0,\ 2,\ 4,\ \ldots \tag{3}$$

with the integrals:

$$\begin{aligned}
(\Phi_m, \Phi_n) &= b^2 \int_0^{\theta_0} \Phi_m^{*(1)}(\theta) \cdot \Phi_n^{(1)}(\theta) \frac{\sin\theta}{\cos^3\theta}\, d\theta + a^2 \int_{\theta_0}^{\pi/2} \Phi_m^{*(2)}(\theta) \cdot \Phi_n^{(2)}(\theta) \frac{d\theta}{\sin^2\theta}, \\
(\Phi_m, H_2) &= a^2 \int_{\theta_0}^{\pi/2} \Phi_m^{*(2)}(\theta) \frac{d\theta}{\sin^2\theta}
\end{aligned} \tag{4}$$

containing the functions (primes indicate the derivative with respect to the argument):

$$\begin{aligned}
\Phi_n^{(1)}(\theta) &= k_0\ \cos\theta \cdot h_n^{\prime(2)}\ (k_0 b/\cos\theta) \cdot P_n(\cos\theta) \\
&\quad + \frac{\sin^2\theta\cos\theta}{b} \cdot h_n^{(2)}\ (k_0 b/\cos\theta) \cdot P_n'(\cos\theta), \\
\Phi_n^{(2)}(\theta) &= k_0\ \sin\theta \cdot h_n^{\prime(2)}\ (k_0 a/\sin\theta) \cdot P_n(\cos\theta) \\
&\quad - \frac{\sin^2\theta\cos\theta}{a} \cdot h_n^{(2)}\ (k_0 a/\sin\theta) \cdot P_n'(\cos\theta), \\
\Phi_n^{(3)}(\theta) &= -k_0\ \cos\theta \cdot h_n^{\prime(2)}\ (-k_0 b/\cos\theta) \cdot P_n(\cos\theta) \\
&\quad + \frac{\sin^2\theta\cos\theta}{b} \cdot h_n^{(2)}\ (-k_0 b/\cos\theta) \cdot P_n'(\cos\theta).
\end{aligned} \tag{5}$$

In the far field:

$$p(r,\theta) = k_0 Z_0 \frac{e^{-j\,k_0 r}}{k_0 r} \sum_{n=0,2,4\ldots} j^n \cdot a_n \cdot P_n(\cos\theta). \tag{6}$$

The corresponding result for an infinite cylinder which oscillates on a length 2b and is hard outside this band:

$$p(r,\theta) = V \frac{2k_0 b Z_0}{\pi} \frac{e^{-j\,k_0 r}}{k_0 r} \frac{\sin(k_0 b\ \cos\theta)}{k_0 b\ \cos\theta} \frac{1}{\sin\theta \cdot H_0^{\prime(2)}(k_0 a\ \sin\theta)} \tag{7}$$

with $H_0^{(2)}(z)$ the Hankel function of second kind and order zero.

## F.19 Monopole and Multipole Radiators

▶ *See also:* Morse/Ingard, Ch. 7 (1968)

*Monopole*:

A point source is placed at the origin of a spherical co-ordinate system with a volume flow amplitude q (outward).

Sound pressure: $$p(r) = \frac{j\,k_0 Z_0}{4\pi} q \frac{e^{-j\,k_0 r}}{r}. \quad (1)$$

Particle velocity: $$v = v_r = \frac{p(r)}{Z_0}\left(1 - \frac{j}{k_0 r}\right). \quad (2)$$

Energy density: $$w = \frac{\rho_0}{(4\pi r^2)^2}|q|^2\left((k_0 r)^2 + \frac{1}{2}\right). \quad (3)$$

Effective intensity: $$I = I_r = \frac{|p(r)|^2}{2Z_0}. \quad (4)$$

Radiated (effective) power: $$\Pi = 4\pi r^2 \cdot I_r = \frac{\rho_0 \omega^2}{8\pi\, c_0}|q|^2. \quad (5)$$

Radiant energy in a shell of unit thickness: $$E' = \frac{\rho_0 k_0^2}{4\pi}|q|^2. \quad (6)$$

Reactive energy outside the radius r: $$E'' = \frac{\rho_0}{8\pi\, r}|q|^2. \quad (7)$$

If the source has a finite radius $a \ll \lambda_0$:

Surface impedance (outward): $$Z_s = \frac{p(a)}{v_r(a)} = \frac{Z_0}{1 - j/k_0 a} = Z_0 \frac{k_0 a\,(k_0 a + j)}{1 + (k_0 a)^2}. \quad (8)$$

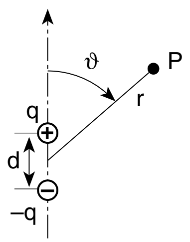
Let a monopole source with volume flow amplitude q be at $\mathbf{r_0} = (x_0, y_0, z_0)$:

Sound pressure in $\mathbf{r} = (x, y, z)$: $$p(\mathbf{r}) = j\,k_0 Z_0 \cdot q \cdot g(\mathbf{r}|\mathbf{r_0}) \quad (9)$$

with $$g(\mathbf{r}|\mathbf{r_0}) = \frac{e^{-j\,k_0 R}}{4\pi R} \quad ; \quad R^2 = |\mathbf{r} - \mathbf{r_0}|^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2. \quad (10)$$

*Dipole*:

Two monopoles with opposite sign of the volume flow q at a mutual distance $d \ll \lambda_0$.



Dipole strength: $$D = q \cdot d. \tag{11}$$

Sound pressure:

$$p(\mathbf{r}) = j\,k_0 Z_0 \cdot q \cdot \left( g(\mathbf{r}|\tfrac{1}{2}\mathbf{d}) - g(\mathbf{r}|-\tfrac{1}{2}\mathbf{d}) \right) = -k_0^2 Z_0 D \frac{e^{-j\,k_0 r}}{4\pi\, r} \left( 1 - \frac{j}{k_0 r} \right) \cdot \cos\vartheta. \tag{12}$$

Velocity components:

$$v_r = -k_0^2 D \frac{e^{-j\,k_0 r}}{4\pi\, r} \left( 1 - \frac{j}{k_0 r} - \frac{2}{(k_0 r)^2} \right) \cdot \cos\vartheta\ ;\ v_\vartheta = -j\,k_0 D \frac{e^{-j\,k_0 r}}{4\pi\, r^2} \left( 1 - \frac{j}{k_0 r} \right) \cdot \sin\vartheta. \tag{13}$$

Effective energy density: $$w = \rho_0 \left( \frac{k_0^2 D}{4\pi\, r} \right)^2 \left[ \cos^2\vartheta + \frac{1}{2(k_0 r)^2} + \frac{1 + 3\cos^2\vartheta}{2(k_0 r)^4} \right]. \tag{14}$$

Effective intensity: $$I_r = \frac{Z_0}{2} \left( \frac{k_0^2 D}{4\pi\, r} \right)^2 \cos^2\vartheta. \tag{15}$$

Effective power: $$\Pi = \frac{Z_0}{2} \frac{4\pi^2}{3\lambda_0^4} |D|^2 = \frac{\rho_0 \omega^4}{24\pi\, c_0^3} |D|^2. \tag{16}$$

Radiant energy in a shell of unit thickness: $$E' = \frac{\rho_0 k_0^2}{12\pi} |D|^2. \tag{17}$$

Reactive energy outside the radius r: $$E'' = \frac{\rho_0}{12\pi\, r^3} |D|^2. \tag{18}$$

A dipole corresponds to a small hard sphere with radius $a \ll \lambda_0$ oscillating back and forth in the direction of the dipole axis with a maximum surface velocity $U_d$ in that direction.

Maximum velocity: $$U_d = \frac{D}{2\pi a^3} e^{-j\,k_0 a} \left( 1 + j\,k_0 a - \frac{1}{2}(k_0 a)^2 \right). \tag{19}$$

Driving force: $$F_d = \frac{j\,k_0 Z_0 \cdot D}{3} e^{-j\,k_0 a} \left( 1 - j\,k_0 a \right). \tag{20}$$

Mechanical driving impedance: $$Z_d = \frac{F_d}{U_d} = \frac{2\pi a^3 k_0 Z_0}{3} \frac{(k_0 a + j)}{\left( 1 + j\,k_0 a - \frac{1}{2}(k_0 a)^2 \right)}. \tag{21}$$

A dipole centred at the point $\mathbf{r_0} = (x_0, y_0, z_0)$ with dipole strength vector $\mathbf{D} = (D_x, D_y, D_z)$, with $\mathbf{R} = \mathbf{r} - \mathbf{r_0}$ and $\mathbf{R}$ having the spherical angles $\vartheta_R, \varphi_R$ has the sound pressure field:

$$p(\mathbf{r}) = j\,k_0 Z_0 \cdot \mathbf{D} \cdot \mathbf{g}(\mathbf{r}|\mathbf{r_0}) \quad ; \quad \mathbf{g}(\mathbf{r}|\mathbf{r_0}) = (g_x, g_y, g_z); \tag{22}$$

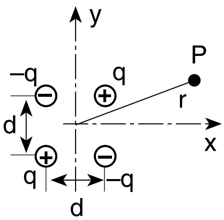
$$g_x = \sin\vartheta_R\ \cos\varphi_R \cdot |g_\omega| \quad ; \quad g_y = \sin\vartheta_R\ \sin\varphi_R \cdot |g_\omega| \quad ; \quad g_z = \cos\vartheta_R \cdot |g_\omega|,$$
$$|g_\omega| = \frac{j\,k_0}{4\pi} \frac{e^{-j\,k_0 R}}{R} \left( 1 - \frac{j}{k_0 R} \right). \tag{23}$$

*Lateral quadrupole*:

For $d \ll \lambda_0$, with $D_{xy} = q \cdot d^2$. (24)

Sound pressure field:

$$p = -j\,k_0^3 Z_0 \cdot D_{xy} \frac{x\,y\,e^{-j\,k_0 r}}{4\pi\,r^3} \left(1 - \frac{3j}{k_0 r} - \frac{3}{(k_0 r)^2}\right). \tag{25}$$



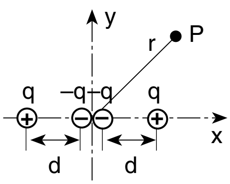

*Linear quadrupole*:

The two central monopoles collapse to a volume flow −2q.

For $d \ll \lambda_0$, with $D_{xx} = q \cdot d^2$. (26)

Sound pressure field:

$$p = -j\,k_0^3 Z_0 \cdot D_{xx} \frac{e^{-j\,k_0 r}}{4\pi\,r} \left[\left(\frac{x}{r}\right)^2 - \frac{3x^2 - r^2}{r^2}\left(\frac{j}{k_0 r} + \frac{1}{(k_0 r)^2}\right)\right]. \tag{27}$$

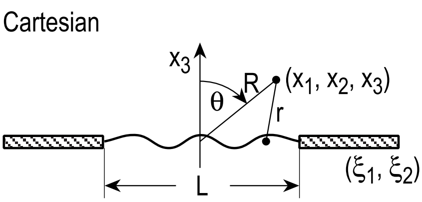


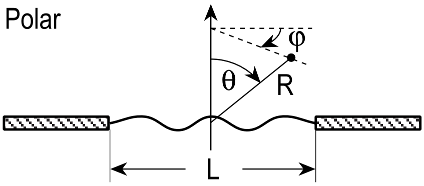

## F.20 Plane Radiator in a Baffle Wall

► *See also:* Heckl (1977)

A plane radiator with either dimensions $L \times B$ in Cartesian co-ordinates $(x_1, x_2, x_3)$ or radius a in polar co-ordinates $(R, \theta, \varphi)$ is contained in a hard baffle wall.

A point on the radiator is at $(\xi_1, \xi_2)$. The radiator area is, respectively, $S = L \cdot B = \pi \cdot a^2$.




Geometrical relations:

$$r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2,$$
$$R^2 = x_1^2 + x_2^2 + x_3^2, \tag{1}$$

$$x_1 = R \cdot \sin\theta \cdot \cos\varphi \quad ; \quad x_2 = R \cdot \sin\theta \cdot \sin\varphi \quad ; \quad x_3 = R \cdot \cos\theta. \tag{2}$$

Quantities:

| | |
|---|---|
| $v(\xi_1, \xi_2)$ | given velocity distribution of the radiator |
| $\hat{v}(k_1, k_2)$ | Fourier transform of $v(\xi_1, \xi_2)$ |
| $p(x_1, x_2, x_3)$ | sound pressure in a field point |
| $\Pi$ | effective sound power radiated towards one side |
| $k_b = 2\pi/\lambda_b$ | wave number of radiator bending wave |
| $k_1, k_2$ | bending wave number components in directions $x_1, x_2$ |
| $r_q, \chi$ | polar co-ordinates r, φ of a point on the source in polar co-ordinates |
| $p_L, \Pi_L$ | sound pressure and effective power radiated by a line source |
| $v_0$ | velocity amplitude of the radiator |
| $Z_s$ | radiation impedance |
| $\sigma$ | radiation efficiency |
| $m_w$ | oscillating medium mass |
| $\eta$ | bending wave loss factor |
| $g(x)$ | envelope of the radiator velocity distribution |

Sound pressure in a far field point, i.e. $k_0L^2/R \ll 1$ or $R \cdot \lambda_0 > L^2$:

$$\text{Cartesian:} \quad p(x_1, x_2, x_3) = \frac{j\,k_0 Z_0}{2\pi} \iint_S v(\xi_1, \xi_2)\,\frac{e^{-j\,k_0 r}}{r}\,d\xi_1\,d\xi_2, \tag{3}$$

polar: $$p(R,\theta,\varphi) = \frac{j\,k_0 Z_0}{2\pi}\,\frac{e^{-j\,k_0 R}}{R}\iint_S v(\xi_1,\xi_2)\cdot e^{j\,k_0\,\sin\theta\,(\xi_1\cos\varphi+\xi_2\sin\varphi)}\,d\xi_1\,d\xi_2. \quad (4)$$

Using the wave number spectrum $\hat{v}(k_1,k_2)$ of the radiator pattern:

$$p(x_1,x_2,x_3) = \frac{k_0 Z_0}{4\pi^2}\iint_{-\infty}^{+\infty}\frac{\hat{v}(k_1,k_2)}{\sqrt{k_0^2-k_1^2-k_2^2}}\cdot e^{j\,(k_1x_1+k_2x_2)}\cdot e^{-j\,x_3\sqrt{k_0^2-k_1^2-k_2^2}}\,dk_1\,dk_2. \quad (5)$$

Long source (in $x_2$ direction):

$$p_L(x_1,x_3) = \frac{k_0 Z_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{v}(k_1)}{\sqrt{k_0^2-k_1^2}}\cdot e^{j\,k_1x_1}\cdot e^{-j\,x_3\sqrt{k_0^2-k_1^2}}\,dk_1. \quad (6)$$

Radiator with radial symmetry (index r):

$$p_r(R,\theta) = \frac{j\,k_0 Z_0}{2\pi R}\cdot\hat{v}_r(k_0\,\sin\,\theta)\cdot e^{-j\,k_0 R}. \quad (7)$$

*Wave number spectrum* of radiator velocity pattern:
rectangular (Fourier transform):

$$\begin{aligned}\hat{v}(k_1,k_2) &= \iint_{-\infty}^{+\infty} v(\xi_1,\xi_2)\cdot e^{-j\,(k_1\xi_1+k_2\xi_2)}\,d\xi_1\,d\xi_2 = \iint_S v(\xi_1,\xi_2)\cdot e^{-j\,(k_1\xi_1+k_2\xi_2)}\,d\xi_1\,d\xi_2,\\ v(\xi_1,\xi_2) &= \frac{1}{4\pi^2}\iint_{-\infty}^{+\infty}\hat{v}(k_1,k_2)\cdot e^{+j\,(k_1\xi_1+k_2\xi_2)}\,dk_1\,dk_2\end{aligned} \quad (8)$$

with radial symmetry (Hankel transform):

$$\begin{aligned}&\hat{v}(k_1,k_2)\rightarrow\hat{v}_r(k_r) = 2\pi\int_0^{\infty} v(r_q)\cdot J_0(k_r r_q)\cdot r_q\,dr_q \quad ;\\ &k_r = \sqrt{k_1^2+k_2^2} \quad ; \quad \begin{aligned}\xi_1 &= r_q\cdot\cos\chi\\ \xi_2 &= r_q\cdot\sin\chi\end{aligned} \; .\end{aligned} \quad (9)$$

*Effective sound power* $\Pi$ radiated towards one side:

$$\Pi = \frac{1}{2Z_0}\int_0^{\pi/2}\int_0^{2\pi}\left|p(R,\theta,\varphi)\right|^2\cdot R^2\sin\,\theta\;d\varphi\,d\theta\,, \quad (10)$$

$$\begin{aligned}\Pi &= \frac{k_0^2 Z_0}{8\pi^2}\int_0^{\pi/2}\int_0^{2\pi}\left|\hat{v}(-k_0\sin\,\theta\cos\varphi\,,\,-k_0\sin\,\theta\,\sin\,\varphi)\right|^2\sin\,\theta\;d\varphi\,d\theta\\ &= \frac{k_0 Z_0}{8\pi^2}\,\mathrm{Re}\left\{\iint_{-\infty}^{+\infty}\left|\hat{v}(k_1\,,\,k_2)\right|^2\frac{dk_1\,dk_2}{\sqrt{k_0^2-k_1^2-k_2^2}}\right\}.\end{aligned} \quad (11)$$

Special case of *line source*:

$$\Pi_L = \frac{k_0 Z_0}{4\pi} \int_{-k_0}^{+k_0} \left| \hat{v}(k_1) \right|^2 \frac{dk_1}{\sqrt{k_0^2 - k_1^2}} = \frac{k_0 Z_0}{4\pi} \int_{-\pi/2}^{+\pi/2} \left| \hat{v}(k_0 \cos \psi) \right|^2 d\psi. \tag{12}$$

Special case of *source with radial symmetry*:

$$\Pi_r = \frac{k_0^2 Z_0}{4\pi} \int_0^{\pi/2} \left| \hat{v}_r(k_0 \sin \theta) \right|^2 \cdot \sin \theta \, d\theta. \tag{13}$$

*Radiation impedance* $Z_s$ ($\Pi$ is complex power; $\langle \ldots \rangle$ indicates average):

Definition: $$\Pi = \frac{S}{2} \cdot Z_s \cdot \langle v^2 \rangle_{\xi_1,\xi_2}, \tag{14}$$

$$Z_s = \frac{k_0 Z_0}{4\pi \langle v^2 \rangle_{\xi_1,\xi_2}} \iint_{-\infty}^{+\infty} \left| \hat{v}(k_1, k_2) \right|^2 \frac{dk_1 \, dk_2}{\sqrt{k_0^2 - k_1^2 - k_2^2}}. \tag{15}$$

*Radiation efficiency* $\sigma$:

Definition: $$\sigma = \frac{\mathrm{Re}\{Z_s\}}{Z_0}, \tag{16}$$

$$\sigma = k_0 \cdot \iint_{k_1^2+k_2^2<k_0^2} \left| \hat{v}(k_1, k_2) \right|^2 \frac{dk_1 \, dk_2}{\sqrt{k_0^2 - k_1^2 - k_2^2}} \Bigg/ \iint_{-\infty}^{+\infty} \left| \hat{v}(k_1, k_2) \right|^2 dk_1 \, dk_2. \tag{17}$$

*Oscillating mass* $m_w$:

Definition: $$m_w = \frac{S \cdot \mathrm{Im}\{Z_s\}}{\omega}, \tag{18}$$

$$m_w = \rho_0 S \iint_{k_1^2+k_2^2>k_0^2} \left| \hat{v}(k_1, k_2) \right|^2 \frac{dk_1 \, dk_2}{\sqrt{-k_0^2 + k_1^2 + k_2^2}} \Bigg/ \iint_{-\infty}^{+\infty} \left| \hat{v}(k_1, k_2) \right|^2 dk_1 \, dk_2. \tag{19}$$

Useful substitutions for evaluation:

Set $k_1 \to k_0 \cosh(z \cos \varphi) \quad ; \quad k_2 \to k_0 \cosh(z \sin \varphi)$

makes $$\frac{dk_1 \, dk_2}{\sqrt{k_1^2 + k_2^2 - k_0^2}} = k_0 \cdot \cosh z \cdot dz \, d\varphi. \tag{20}$$

For line sources, set $k_1 \to k_0 \cosh(z) \quad ; \quad k_2 \to 0 \quad ; \quad S \to B$

makes $$\frac{dk_1}{\sqrt{k_1^2 - k_0^2}} = dz. \tag{21}$$

| Velocity Pattern | Range | Transform |
|---|---|---|
| *Fourier transforms* of some 1-dimensional velocity patterns: $z \equiv k_1 L/2$ | | |
| $v(\xi_1) = v_0$ | $\lvert\xi_1\rvert < L/2$ | $\hat{v}(k_1)/v_0 = L \cdot \sin z/z$ |
| $v(\xi_1) = v_0\left(1 - 2\lvert\xi_1/L\rvert\right)$ | $\lvert\xi_1\rvert < L/2$ | $\hat{v}(k_1)/v_0 = L/2 \cdot (\sin(z/2)/(z/2))^2$ |
| $v(\xi_1) = v_0\left(1 - (2\xi_1/L)^2\right)^2$ | $\lvert\xi_1\rvert < L/2$ | $\frac{\hat{v}(k_1)}{v_0} = L\left[\left(24 z^{-5} - 8 z^{-3}\right)\cdot \sin z - 24 z^{-4}\cdot\cos z\right]$ |
| $v(\xi_1) = v_0\left(2\xi_1/L\right)$ | $\lvert\xi_1\rvert < L/2$ | $\hat{v}(k_1)/v_0 = -j L\left[\sin z/z^2 - \cos z/z\right]$ |
| $v(\xi_1) = v_0\left[3\left(2\xi_1/L\right)^2 - 1\right]/2$ | $\lvert\xi_1\rvert < L/2$ | $\frac{\hat{v}(k_1)}{v_0} = L\left[3\cos z/z^2 + \left(1/z - 3/z^3\right)\sin z\right]$ |
| $v(\xi_1) = v_0\, e^{-\alpha\lvert\xi_1\rvert}$ | $\lvert\xi_1\rvert < \infty$ | $\frac{\hat{v}(k_1)}{v_0} = \frac{2\alpha}{\alpha^2 + \xi_1^2}$ |
| *Hankel transforms* of some velocity patterns with radial symmetry: | | |
| $v(r_q) = v_0$ | $r_q < a$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = 2\pi\frac{J_1(k_r a)}{k_r a}$ |
| $v(r_q) = v_0 J_0(k_b r_q)$ | $r_q < a$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = 2\pi\frac{k_r a\, J_0(k_b a)\, J_1(k_r a) - k_b a\, J_0(k_r a)\, J_1(k_b a)}{(k_r a)^2 - (k_b a)^2}$ |
| $v(r_q) = v_0\, e^{-\alpha r_q}$ | $r_q < \infty$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = 2\pi\frac{\alpha}{(\alpha^2 + k_r^2)^{3/2}}$ |
| $v(r_q) = v_0\, e^{-p^2 r_q^2}$ | $r_q < \infty$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = \frac{2\pi}{2 p^2} e^{-k_r^2/4p^2}$ |
| $v(r_q) = v_0/r_q$ | $r_q < \infty$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = \frac{2\pi}{k_r}$ |
| $v(r_q) = v_0/r_q^\kappa$ | $r_q < \infty$ | $\frac{\hat{v}_r(k_r)}{v_0 a^2} = \pi\, 2^{\kappa+2}\frac{\Gamma(1+\kappa/2)}{\Gamma(-\kappa/2)} k_r^{-(\kappa+2)}$ |

Radiator with *nearly periodic velocity pattern* $v_M(x_1)$:

$$v_M(x_1) = v_0 \cdot g(x_1) \cdot \begin{matrix}\cos(k_b x_1)\\ \sin(k_b x_1)\end{matrix} ,$$

$$\hat{v}_M(k_1) = \frac{v_0}{2}\int \hat{g}(\mu)\left[\delta(k_1 - k_b - \mu) + \delta(k_1 + k_b - \mu)\right] d\mu \qquad (22)$$

$$= \frac{v_0}{2}\left[\hat{g}(k_1 - k_b) \pm \hat{g}(k_1 + k_b)\right] \cdot \begin{cases} 1 \\ j \end{cases} ,$$

with $\delta(x)$ the Dirac delta function and $\hat{g}(k)$ the Fourier transform of the envelope $g(x_1)$.

## F.21 Ratio of Radiation and Excitation Efficiencies of Plates

▸ *See also:* Heckl (1964)

Consider two "experiments":

- Plate excited at a point with a force F radiates a sound power $\Pi$;
- Plate excited by a diffuse sound field with pressure p vibrates with a velocity v.

Define the radiation efficiency a by $\Pi_{eff} = a \cdot F_{eff}^2$; (1)

define the excitation efficiency b by $v_{eff}^2 = b \cdot p_{eff}^2$. (2)

Then: $$\frac{a}{b} = \frac{Z_0 k_0^2}{4\pi}. \quad (3)$$

Consider two "experiments":

- Plate excited by a line source with a force $F_L$ radiates a sound power $\Pi_L$;
- Plate excited by a diffuse sound field with pressure p vibrates with a velocity v.

Define the radiation efficiency $\alpha$ by $\Pi_{L,eff} = \alpha \cdot F_{L,eff}^2$; (4)

define the excitation efficiency $\beta$ by $v_{eff}^2 = \beta \cdot p_{eff}^2$. (5)

Then: $$\frac{\alpha}{\beta} = \frac{Z_0 k_0}{4}. \quad (6)$$

## F.22 Radiation of Plates with Special Excitations

▸ *See also:* Ver, Ch. 11 (1971)

Let S be the area of a plate in a baffle wall, $\langle v^2 \rangle_S$ the average of the squared vibration velocity, $\Pi$ the sound power radiated into one half-space, $\sigma$ with $\Pi = Z_0 S \cdot \sigma \cdot \langle v^2 \rangle_S / 2$ the radiation efficiency, $f_{cr}$ the critical frequenc, $\sigma_P$ the Poisson ratio, $c_D$ the dilatation wave velocity, h the plate thickness, and $m'' = \rho_p h$ the surface mass density.

*Infinite plate with a free bending wave* $k_B$ at $f > f_{cr}$:

$$\sigma = 1 \Big/ \sqrt{1 - (k_B/k_0)^2} = 1 \Big/ \sqrt{1 - (f_{cr}/f)}. \quad (1)$$

*Infinite plate (without losses) excited by a point force* with amplitude F for $f \ll f_{cr}$ (with $m'' = \rho_p h$ surface mass density):

$$\Pi \approx \frac{\rho_0}{2\pi\, c_0} \left( \frac{F}{2\, m''} \right)^2 \left[ 1 - \frac{\rho_0}{k_0 m''} \tan^{-1} \frac{k_0 m''}{\rho_0} \right] \approx \begin{cases} \dfrac{\rho_0}{2\pi\, c_0} \left( \dfrac{F}{2\, m''} \right)^2; & \dfrac{k_0 m''}{\rho_0} \gg 1 \\[2ex] \dfrac{(k_0 F)^2}{24\pi\, Z_0}; & \dfrac{k_0 m''}{\rho_0} \ll 1. \end{cases} \quad (2)$$

*Infinite plate (without losses) with a point velocity source* with amplitude v for $k_0 m''/\rho_o \gg 1$:

$$\Pi \approx \frac{2}{\pi^3}\,\frac{\rho_0 c_0^3}{f_{cr}^2}\,v^2. \tag{3}$$

Radius a of the equivalent ideal piston radiator (with same $\langle v^2\rangle_S$ and unit efficiency):

$$a = \sqrt{8/\pi^3}\cdot\lambda_{cr} = 0.286\cdot\lambda_{cr} \quad ; \quad \lambda_{cr} = c_0/f_{cr}. \tag{4}$$

*Far field for a point force acting on an infinite plate* with force amplitude F
(R = radius from excitation point to field point, $\vartheta$ its polar angle):

For a thin plate without losses ($f < 0.7\cdot f_{cr}$):

$$p(R,\vartheta) = \frac{j\,k_0}{2\,\pi}\cdot F\cdot\frac{e^{-j\,k_0 R}}{R}\cdot\frac{\cos\vartheta}{1+j\dfrac{k_0 m''}{\rho_0}\cos\vartheta\cdot\left(1-(f/f_{cr})^2\sin^4\vartheta\right)}. \tag{5}$$

For a thick plate without losses ($f > 0.7\cdot f_{cr}$):

$$p(R,\vartheta) = \frac{j\,k_0}{2\,\pi}\cdot F\cdot\frac{e^{-j\,k_0 R}}{R}$$
$$\cdot\frac{\left[1+\varphi(\vartheta)\right]\cos\vartheta}{\left[1+\varphi(\vartheta)\right]+j\dfrac{k_0 m''}{\rho_0}\left\{1+\left[1-\dfrac{1-\sigma_P}{24}\,\dfrac{(\pi\,c_D\sin\vartheta)^2}{c_0^2}\right]\cdot\varphi(\vartheta)\right\}}, \tag{6}$$
$$\varphi(\vartheta) = \frac{2(k_0 h)^2}{\pi^2(1-\sigma_P)}\left(\sin^2\vartheta-(c_0/c_D)^2\right).$$

For plates with a loss factor $\eta$ substitute

$$c_D \to c_D\cdot(1+j\cdot\eta/2);\; f_{cr} \to f_{cr}\cdot(1+j\cdot\eta/2). \tag{7}$$

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