D Reflection of Sound

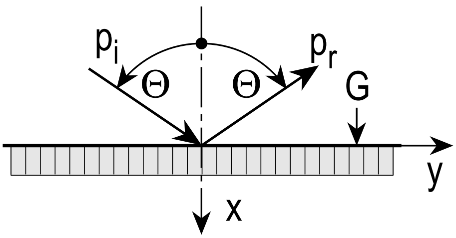
The limit between *reflection* and *scattering* of sound is not sharp. A generally applicable distinction could be that reflection sends sound back only into the half-space of incidence, whereas scattering sends sound also in the forward direction. This is the guideline for placing topics either in this chapter about reflection of sound or in the later chapter about scattering of sound.

The general reference in this chapter is Mechel, "Schallabsorber" (Vol. I – III). Formulas for the input admittance and/or absorption coefficient can also be found in the chapter "Compound Absorbers".

D.1 Plane Wave Reflection at a Locally Reacting Plane

An absorber is said to be *locally reacting* if there is no sound propagation inside the absorber parallel to the absorber surface.

A plane wave with amplitude A is incident in the plane (x,y) on an absorbent plane; the y axis is in the absorber surface; the x axis is normal to the absorber and directed into the absorber; the wave vector of the incident wave $p_i(x,y)$ forms a polar angle Θ with the normal to the surface.



The acoustic quality of the absorber is defined

by the wall admittance:

$$G = \frac{v_x(0,y)}{p(0,y)} = \frac{j}{k_0 Z_0}\,\frac{\partial p(0,y)/\partial x}{p(0,y)} \,. \tag{1}$$

Field formulation:

$$\begin{aligned} p(x,y) &= p_i(x,y) + p_r(x,y) \,, \\ p_i(x,y) &= A \cdot e^{-j\,k_0 (x \cdot \cos\Theta + y \cdot \sin\Theta)} \,, \\ p_r(x,y) &= r \cdot A \cdot e^{-j\,k_0 (-x \cdot \cos\Theta + y \cdot \sin\Theta)} \,. \end{aligned} \tag{2}$$

Reflection factor:

$$r = \frac{p_r(0,y)}{p_i(0,y)} = \frac{\cos\Theta - Z_0 G}{\cos\Theta + Z_0 G}\ . \qquad (3)^{*)}$$

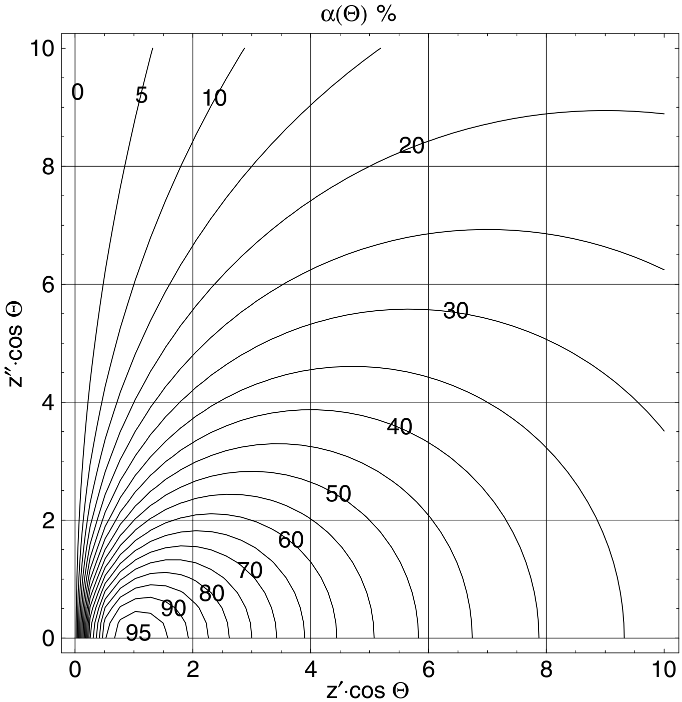
$$\xrightarrow[G\to 0]{} 1 \quad ; \quad \xrightarrow[|G|\to\infty]{} -1\ ,$$

Absorption coefficient with the normalised absorber input impedance $z = z' + j\cdot z'' = Z/Z_0$ or input admittance $g = g' + j\cdot g'' = Z_0 G$.

$$\alpha(\Theta) = 1 - |r(\Theta)|^2$$

$$= \frac{4z'\cos\Theta}{(1 + z'\cos\Theta)^2 + (z''\cos\Theta)^2} \qquad (4)$$

$$= \frac{4g'\cos\Theta}{(g' + \cos\Theta)^2 + g''^2}\ .$$



Contour lines of $\alpha(\Theta)$ in per cent over $z'\cdot\cos\Theta$, $z''\cdot\cos\Theta$

The straight connecting line between the starting point at z', z'', for $\Theta = 0$, with the origin, for $\Theta = \pi/2$, mostly passes through higher absorption values at some finite angle Θ.

*) See Preface to the 2nd Edition.

Sometimes the derivatives

$r^{(n)}(\Theta) = \partial^n r(\Theta)/\partial\Theta^n$ are needed (substitute $Z_0 G \rightarrow g$): (5)*)

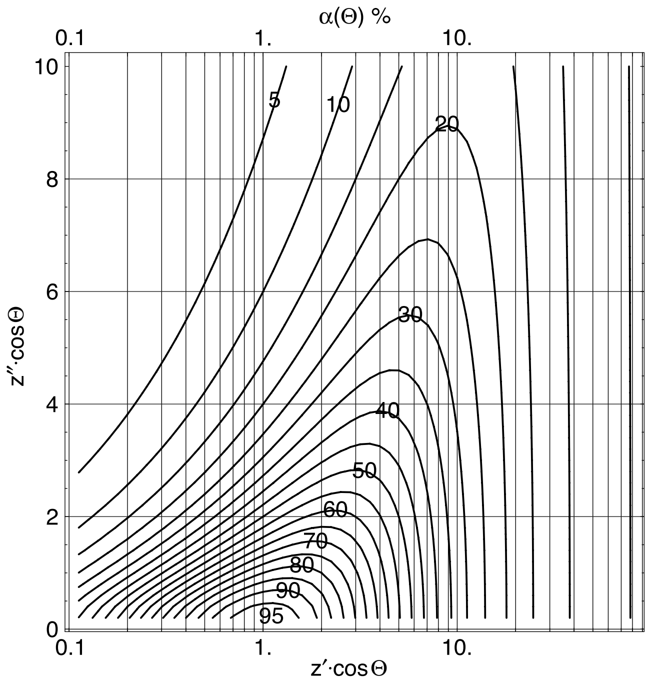
$$r'(\Theta) = \frac{-2g \sin\Theta}{(g+\cos\Theta)^2},$$

$$r''(\Theta) = \frac{-g\left(3 + 2g\cos\Theta - \cos(2\Theta)\right)}{(g+\cos\Theta)^3},$$

$$r^{(3)}(\Theta) = \frac{-g\left(11 - 2g^2 + 8g\cos\Theta - \sin\Theta\cdot\cos(2\Theta)\right)}{(g+\cos\Theta)^4},$$

$$r^{(4)}(\Theta) =$$

$$\frac{-g\left(115 - 20g^2 + 2g(47-4g^2)\cos\Theta - 4(19-11g^2)\cos(2\Theta) - 22g\cos(3\Theta) + \cos(4\Theta)\right)}{4(g+\cos\Theta)^5}.$$



Contour lines of $\alpha(\Theta)$ in per cent over logarithmic $z' \cdot \cos\Theta$ and linear $z'' \cdot \cos\Theta$

D.2 Plane Wave Reflection at an Infinitely Thick Porous Layer

"Porous layer" here stands for any homogeneous, isotropic material with characteristic propagation constant Γ_a and wave impedance Z_a. If the material is air, then $\Gamma_a \rightarrow jk_0$;

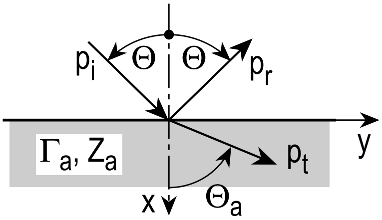
*) See Preface to the 2nd Edition.

$Z_a \to Z_0$.

Sound incidence is as in Sect. D.1.

Sound field above absorber: $p_1 = p_i + p_r$ (as in Sect. D.1) ,

sound field in absorber: $$p_2(x,y) = p_t(x,y) = B \cdot e^{-\Gamma_a (x \cos\Theta_a + y \sin\Theta_a)} \; . \qquad (1)$$



The boundary conditions are:

- Equal propagation constant in y direction on both sides;
- Equal normal admittance component on both sides (is equivalent to matching sound pressure and normal particle velocity).

Refracted angle Θ_a (complex !): $$\frac{\sin\Theta_a}{\sin\Theta} = \frac{j\,k_0}{\Gamma_a} \; . \qquad (2)$$

Reflection factor r: $$r = \frac{Z_a/\cos\Theta_a - Z_0/\cos\Theta}{Z_a/\cos\Theta_a + Z_0/\cos\Theta} = \frac{Z_{an} - Z_{0n}}{Z_{an} + Z_{0n}} \; . \qquad (3)$$

(in the second form Z_{an},Z_{0n} indicate normal components of the impedances). Absorption coefficient again is $\alpha = 1 - |r|^2$.

D.3 Plane Wave Reflection at a Porous Layer of Finite Thickness

The absorber layer of thickness d is backed by a rigid wall.

The input impedance of the layer is: $$Z_2 = \frac{Z_a}{\cos\Theta_a} \cdot \coth(\Gamma_a d \cdot \cos\Theta_a). \qquad (1)$$

Reflection factor: $$r = \frac{Z_2/\cos\Theta_a - Z_0/\cos\Theta}{Z_2/\cos\Theta_a + Z_0/\cos\Theta}. \qquad (2)$$

With normalised characteristic values $\Gamma_{an} = \Gamma_a/k_0$, $Z_{an} = Z_a/Z_0$ it is convenient to evaluate

$$\Gamma_{an} \cdot \cos\Theta_a = \sqrt{\Gamma_{an}^2 + \sin^2\Theta_a} = \sqrt{1 + \Gamma_{an}^2 + \cos^2\Theta_a}. \qquad (3)$$

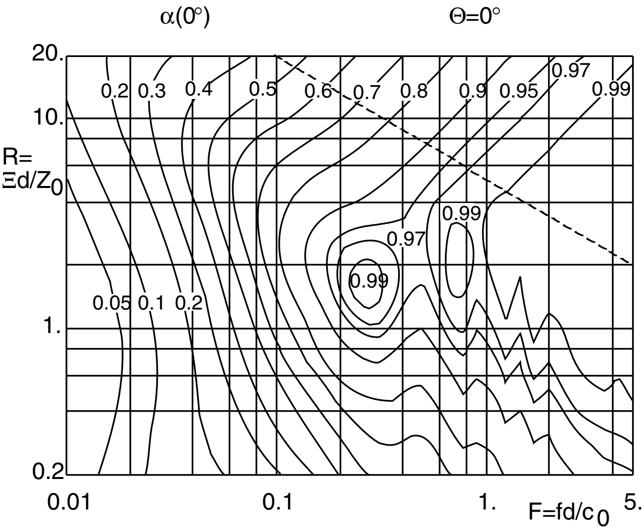
Θ_a follows from the law of refraction in Sect. D.2. $$\cos\Theta_a = \sqrt{1 + (\sin\Theta/\Gamma_{an})^2} \qquad (4)$$

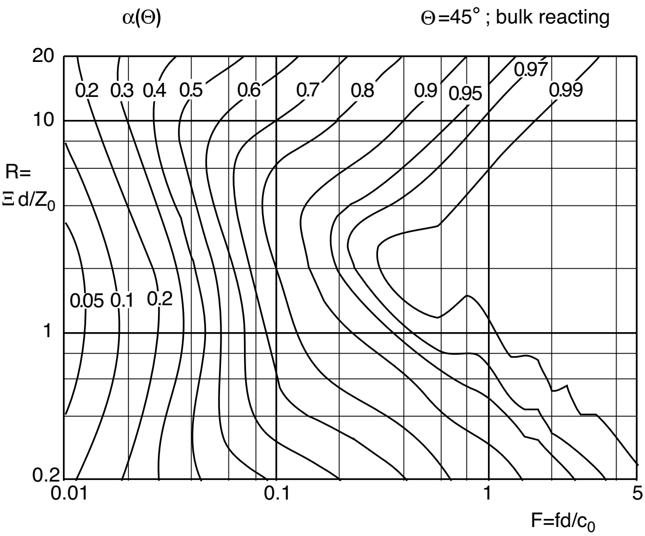
Limit of layer thickness d above which the layer effectively behaves like an infinitely thick layer for normal sound incidence:

The layer is *locally reacting* (either due to large R or by internal partitions):

The limit follows	$R \geq 5.158/F^{0.5886}$;	
from one of the relations	$F \geq 16.233/R^{1.699}$; $F \geq 2.81 \cdot E^{0.629}$;	(5)
for locally reacting layers:	$f_{[\mathrm{Hz}]} \cdot d_{[\mathrm{m}]}^{2.699} \cdot \Xi_{[\mathrm{Pa \cdot s/m^2}]}^{1.699} \geq 2.274 \cdot 10^6$.	

Contour diagrams of $\alpha(\Theta)$ of a porous absorber layer with hard back, for $\Theta = 0$ and $\Theta = 45°$.





The layer is *bulk reacting*:

The limit follows $R \geq 3.209/F^{0.7245}$;

from one of the relations $F \geq 5.00/R^{1.380}$; $F \geq 1.966 \cdot E^{0.580}$; (6)

for bulk reacting layers: $f_{[Hz]} \cdot d_{[m]}^{2.380} \cdot \Xi_{[Pa \cdot s/m^2]}^{1.380} \geq 0.70 \cdot 10^6$;

with the non-dimensional quantities:

$$R = \Xi \cdot d/Z_0 \quad ; \quad F = f \cdot d/c_0 = d/\lambda_0 ; \qquad E = \rho_0 f/\Xi . \tag{7}$$

D.4 Plane Wave Reflection at a Multilayer Absorber

The absorber consists of M layers of homogeneous porous material (or air); the layers are numbered $m = 1, 2, \ldots, M$. The space in front of the layer (with characteristic values k_0, Z_0) takes the index $m = 0$.

Layer thicknesses: d_m; $m = 1, 2, \ldots, M$

Characteristic values: Γ_{am}, Z_{am}; $m = 0, 1, \ldots, M$ (with $\Gamma_{a0} = j$, $Z_{a0} = 1$)

Incidence and refracted angles: Θ_m; $m = 0, 1, \ldots, M$

Reflection factors r_m; $m = 0, 1, \ldots, M$
(r_0= reflection factor of the arrangement)

Layer input admittances: G_m; $m = 0, 1, \ldots, M$
(G_0= input admittance of the arrangement).

Acoustic layer thicknesses: D_m; $m = 1, 2, \ldots, M$.

One can apply the chain circuit algorithm of Sect. C.5 for the evaluation of the input admittance and therewith of the reflection factor using the equivalent four poles of Sect. C.2.

Here will be given a more explicit scheme of iteration with the iteration of the reflection factors (r_m is the reflection factor at the back side of layer $m = 0, 1, 2, \ldots, M$):

$$r_{m-1} = \frac{\dfrac{W_m}{W_{m-1}}\left(1 + r_m \cdot e^{-2D_m}\right) - \left(1 - r_m \cdot e^{-2D_m}\right)}{\dfrac{W_m}{W_{m-1}}\left(1 + r_m \cdot e^{-2D_m}\right) + \left(1 - r_m \cdot e^{-2D_m}\right)} . \tag{1}$$

Auxiliary quantities:

$$W_m = Z_{am}/\cos\Theta_m \quad ; \quad W_0 = Z_0/\cos\Theta_0 \ ,$$

$$\cos\Theta_m = \sqrt{1 + (k_0/\Gamma_{am})^2 \cdot (1 - \cos^2\Theta_0)} \ , \tag{2}$$

$$D_m = \Gamma_{am} d_m \cdot \cos\Theta_m = k_0 d_m \sqrt{1 + (\Gamma_{am}/k_0)^2 - \cos^2\Theta_0} \ .$$

If the arrangement has a rigid backing, start the iteration with $r_M=1$. If the back side of the arrangement is in contact with free space (without a back cover of the last layer), start with

$$r_M = \frac{Z_0/\cos\Theta_0 - Z_{aM}/\cos\Theta_M}{Z_0/\cos\Theta_0 + Z_{aM}/\cos\Theta_M}. \tag{3}$$

Input admittance G_m of the *m*th layer ($m = 1, \ldots, M$):

$$G_m = \frac{1}{W_m}\frac{1 - r_m}{1 + r_m}. \tag{4}$$

D.5 Diffuse Sound Reflection at a Locally Reacting Plane

Generally, the absorption coefficient α_{dif} follows from the absorption coefficient $\alpha(\Theta)$ for oblique incidence by integration over the polar angle Θ. The integrals are

$$\text{in 2-dimensional space:} \qquad \alpha_{2-dif} = \int_0^{\pi/2} \alpha(\Theta) \cdot \cos\Theta \; d\Theta, \tag{1}$$

$$\text{in 3-dimensional space:} \qquad \alpha_{3-dif} = 2\int_0^{\pi/2} \alpha(\Theta) \cdot \cos\Theta \cdot \sin\Theta \; d\Theta. \tag{2}$$

The integral in three dimensions has an analytical solution for a locally reacting plane with normalised input admittance $Z_0 G = g' + j \cdot g''$:

$$\begin{aligned} \alpha_{3-dif} &= 8\,g'\left[1 + \frac{g'^2 - g''^2}{g''} \cdot \arctan\frac{g''}{g' + g'^2 + g''^2} - g' \cdot \ln\left(1 + \frac{1 + 2\,g'}{g'^2 + g''^2}\right)\right] \\ &\xrightarrow[g' \neq 0\,;\,g''=0]{} 8\,g'\left[1 + \frac{g'^2}{g' + g'^2} - g' \cdot \ln\left(1 + \frac{1 + 2\,g'}{g'^2}\right)\right] \\ &\xrightarrow[g'=0\,;\,g'' \neq 0]{} 0 \end{aligned} \tag{3}$$

or with the normalised input impedance $Z/Z_0 = z' + j \cdot z''$:

$$\begin{aligned} \alpha_{3-dif} &= 8\,\frac{z'}{z'^2 + z''^2} \\ &\left[1 + \frac{1}{z''}\frac{z'^2 - z''^2}{z'^2 + z''^2} \cdot \arctan\frac{z''}{1 + z'} - \frac{z'}{z'^2 + z''^2} \cdot \ln\left(1 + 2\,z' + z'^2 + z''^2\right)\right] . \end{aligned} \tag{4}$$

Under condition $z''^2 \ll (1+z')^2$, the absorption coefficient α_{3-dif} can be evaluated from the (measured or computed) absorption coefficient α_0 for normal sound incidence:

$$\alpha_{3-dif} = 8\left[\frac{2}{1-\sqrt{1-\alpha_0}} - \frac{1-\sqrt{1-\alpha_0}}{2} + 2\ln\frac{1-\sqrt{1-\alpha_0}}{2}\right]\cdot\left[\frac{1-\sqrt{1-\alpha_0}}{1+\sqrt{1-\alpha_0}}\right]^2 . \tag{5}$$

The maximum possible value of α_{3-dif} for locally reacting absorbent planes is $\alpha_{3-dif} = 0.951$.

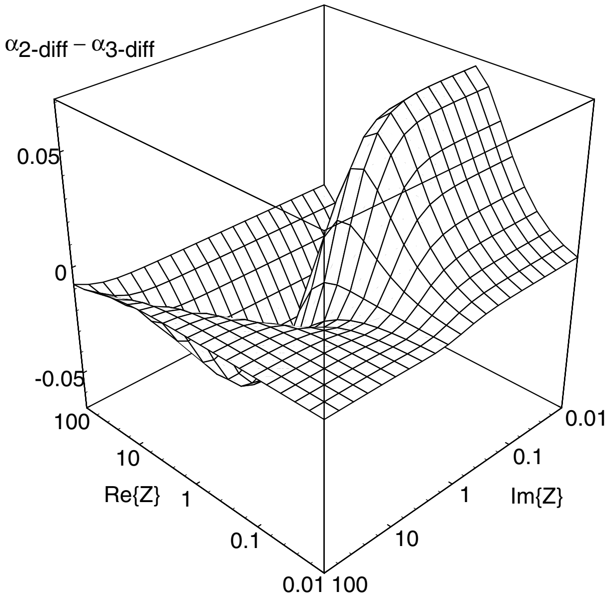
The analytical solution for the integral of α_{2-dif} follows from:

$$\begin{aligned}\alpha_{2-dif} &= \int_0^{\pi/2} \alpha(\Theta)\cdot\cos\Theta\, d\Theta = \int_0^{\pi/2} \frac{4\,z'\cos^2\Theta}{(1+z'\cos\Theta)^2+(z''\cos\Theta)^2}\, d\Theta \\ &= \int_0^1 \frac{4\,z'\,x^2}{(1+z'\,x)^2+z''^2\,x^2}\,\frac{dx}{\sqrt{1-x^2}} \\ &= \int_0^1 \frac{4\,z'\,x^2}{1+2\,z'\,x+(z'^2+z''^2)\,x^2}\,\frac{dx}{\sqrt{1-x^2}}\end{aligned} \tag{6}$$

and is

$$\alpha_{2-dif} = 2\,z'\left[\frac{\pi}{z'^2+z''^2} + 2\,\mathrm{Im}\left\{\frac{\ln\left(z+\sqrt{z^2-1}\right)}{z''\cdot z\,\sqrt{z^2-1}}\right\}\right] . \tag{7}$$

The difference $\alpha_{2-dif} - \alpha_{3-dif}$ is small, in general, so that one absorption coefficient can be approximated by the other. This can be seen from the following diagram showing the difference over the complex plane of the normalised input impedance Z of a locally reacting plane.



The *limit of polar angle of incidence* in the above formulas is assumed to be $\Theta = \pi/2$. In some of the literature it is recommended to integrate up to an angle $\Theta < \pi/2$ (values between 75° and 87° were proposed). Writing the normalised surface admittance as $Z_0G = g_1 + j \cdot g_2$ and using the reflection factor $R(\theta)$ for the absorption coefficient $\alpha(\theta)$ for oblique incidence under a polar angle θ:

$$\alpha(\theta) = 1 - |R(\theta)|^2 = \frac{4g_1 \cos\theta}{(\cos\theta + g_1)^2 + g_2^2}\,,$$

$$R(\theta) = \frac{\cos\theta - Z_0G}{\cos\theta + Z_0G}\,, \tag{8}$$

$$Z_0G = g_1 + j \cdot g_2\,,$$

one gets for the sound absorption coefficient $\alpha_{\mathrm{dif}} = d\Pi_a/d\Pi_e$ of a plane wave with intensity I, with the absorbed sound power $d\Pi_a$ and the incident sound power $d\Pi_e$ on a surface element dS of the absorber:

$$d\Pi_e = I\,dS \int_0^{2\pi} d\varphi \int_0^{\Theta} \cos\theta \sin\theta\, d\theta = 2\pi I\,dS \int_0^{\Theta} \cos\theta \sin\theta\, d\theta = \pi I\,dS \sin^2\Theta\,;$$

$$d\Pi_e = I\,dS \int_0^{2\pi} d\varphi \int_0^{\Theta} \alpha(\theta) \cos\theta \sin\theta\, d\theta = 2\pi I\,dS \int_0^{\Theta} \alpha(\theta) \cos\theta \sin\theta\, d\theta\,. \tag{9}$$

The integral and its special values:

$$\alpha_{\mathrm{dif}} = \frac{d\Pi_a}{d\Pi_e} = \frac{2}{\sin^2\Theta} \int_0^{\Theta} \alpha(\theta) \cos\theta \sin\theta\, d\theta = \frac{8g_1}{\sin^2\Theta} \int_0^{\Theta} \frac{\cos^2\theta}{(\cos\theta + g_1)^2 + g_2^2} \sin\theta\, d\theta$$

$$\xrightarrow[-\sin\theta\, d\theta = dt]{\cos\theta \to t} = \frac{8g_1}{\sin^2\Theta} \int_{\cos\Theta}^{1} \frac{t^2}{(t + g_1)^2 + g_2^2}\, dt = \frac{8g_1}{\sin^2\Theta} \left[1 - \cos\Theta + \frac{g_1^2 - g_2^2}{g_2} \right.$$

$$\left. \left(\arctan\frac{1 + g_1}{g_2} - \arctan\frac{g_1 + \cos\Theta}{g_2}\right) + g_1 \ln\frac{g_1^2 + g_2^2 + 2g_1\cos\Theta + \cos^2\Theta}{1 + g_1^2 + g_2^2 + 2g_1}\right] \tag{10}$$

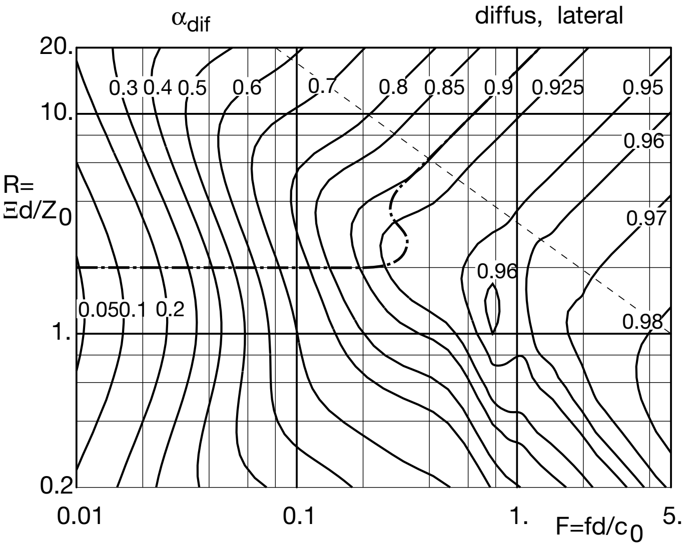
$$\xrightarrow[\Theta = \pi/2]{} 8g_1 \left[1 + \frac{g_1^2 - g_2^2}{g_2}\left(\arctan\frac{1 + g_1}{g_2} - \arctan\frac{g_1}{g_2}\right) + g_1 \ln\frac{g_1^2 + g_2^2}{1 + g_1^2 + g_2^2 + 2g_1}\right]$$

$$\xrightarrow[g_2 = 0]{} \frac{8g_1}{\sin^2\Theta}\left[1 - \cos\Theta + \frac{g_1^2}{g_1 + \cos\Theta} - \frac{g_1^2}{1 + g_1} + 2g_1 \ln\frac{g_1 + \cos\Theta}{1 + g_1}\right]$$

$$\xrightarrow[g_2 = 0]{\Theta = \pi/2} 8g_1 \left[1 + g_1 - \frac{g_1^2}{1 + g_1} + 2g_1 \ln\frac{g_1}{1 + g_1}\right].$$

D.6 Diffuse Sound Reflection at a Bulk Reacting Porous Layer

The integral for α_{dif} from Sect. D.5 of bulk reacting absorbers generally must be evaluated numerically. The diagram below shows a contour plot of α_{dif} for a porous layer of thickness d with hard back over the non-dimensional parameters $F = f \cdot d/c_0$ and $R = \Xi \cdot d/Z_0$. In the parameter range above and on the left-hand side of the dash-dotted curve, the absorption coefficients of a locally reacting and a bulk reacting porous layer agree with each other. In the range above and on the right-hand side of the dashed straight line, the bulk reacting layer effectively has an infinite thickness.

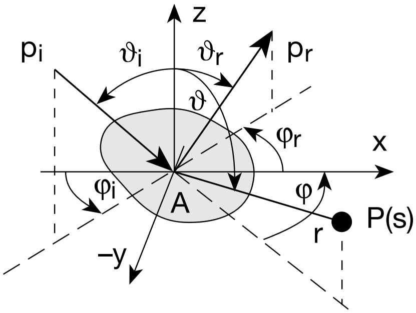


D.7 Sound Reflection and Scattering at Finite-Size Local Absorbers

The wording "local" in this heading (and at other places in this book) is a shorter form of "locally reacting"; the corresponding abbreviation for "bulk reacting" will be "lateral".

If the side dimensions of the plane absorber are finite, scattering takes place at the borders between the absorber and the baffle wall. In fact, some theories determine the sound absorption of finite-size absorbers from the solution of the scattering problem.

Let the absorber with area A be in the plane (x, y); the z co-ordinate shows into the space above the absorber. The sketch shows co-ordinates and angles used, as well as the incident plane wave p_i and the specularly reflected wave p_r. The field point is in P. Let s be a general co-ordinate.



Field composition: $p(s) = p_i(s) + p_r(s) + p_s(s)$

$$p_i(s) = P_i \cdot e^{-j\left(k_x x + k_y y - k_z z\right)} \ ,$$
$$p_r(s) = r_u \cdot P_i \cdot e^{-j\left(k_x x + k_y y + k_z z\right)} \ , \tag{1}$$

with:

$p_i(s)$	=	incident plane wave;
$p_r s)$	=	specularly reflected plane wave;
$p_s(s)$	=	scattered wave

with:

r_u	=	reflection factor of the baffle wall;
F_0	=	normalised admittance of the baffle wall.

Wave number components:

$$k_x = k_0 \cdot \sin\vartheta_i \cdot \cos\varphi_i \quad ; \quad k_y = k_0 \cdot \sin\vartheta_i \cdot \sin\varphi_i \quad ; \quad k_z = k_0 \cdot \cos\vartheta_i \, ;$$
$$k_x^2 + k_y^2 + k_z^2 = k_0^2 \, . \tag{2}$$

Scattered wave:

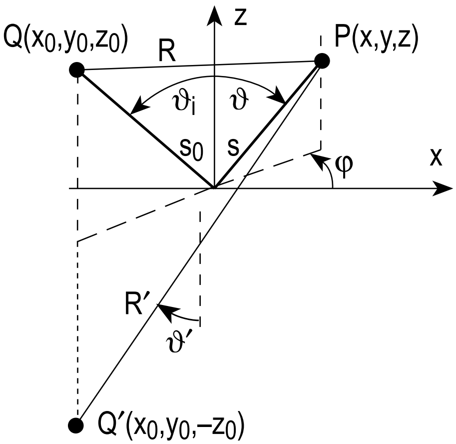
$$p_s(s) = \iint\limits_A \left[G(s|s_0) \cdot \frac{\partial}{\partial n_0} p(s_0) - p(s_0) \cdot \frac{\partial}{\partial n_0} G(s|s_0) \right] ds_0 \ , \tag{3}$$

with Green's function (in which s is a radius, see sketch) for field points at a large distance:

$$G(s|s_0) = \frac{e^{-j k_0 s}}{4\pi\, s} \left[e^{j k_0 z_0 \cos\vartheta} + r_u(\vartheta) \cdot e^{-j k_0 z_0 \cos\vartheta} \right] \cdot e^{j k_0 \sin\vartheta \cdot (x_0 \cos\varphi + y_0 \sin\varphi)} \, . \tag{4}$$

The Green's function corresponds to a superposition of the fields of point sources at Q and at the mirror-reflected point Q'. It satisfies the boundary condition at the baffle wall.

Source point Q, mirror source point Q' and field point P in the construction of Green's function:



The integral equation above for p_s holds also on the absorber surface A if the integral is multiplied with 1/2.

It may be solved by iteration n = 0, 1, . . . for p_{sn}:

- Start with a suitable p_{so} on A (e.g. with the value of $p_i + p_r$ on A);
- Insert $p = p_i + p_r + p_s$ in the integrand;
- Evaluate the first approximation p_{s1}, and so on.

The *n*th iteration gives $$p_n(s) = p_i(s) + p_r(s) + p_{s1}(s) + \ldots + p_{sn}(s) \quad (5)$$

and for $p_s(s)$: $$p_s(s) = p_{s1}(s) + \ldots + p_{sn}(s)$$

Scattering cross section of A: $$Q_s = \frac{\Pi_s}{I_i} = \oiint \mathrm{Re}\left\{\frac{p_s^*}{P_i^*} \cdot \frac{v_{ns}}{P_i/Z_0}\right\} dS. \quad (6)$$

Absorption cross section of A: $$Q_a = \frac{\Pi_a}{I_i} = -\oiint \mathrm{Re}\left\{\frac{p^*}{P_i^*} \cdot \frac{v_n}{P_i/Z_0}\right\} dS. \quad (7)$$

Extinction cross section of A: $$Q_e = Q_a + Q_s \quad (8)$$

with:

Π_s = scattered effective power;
Π_a = absorbed effective power;
I_i = incident effective intensity,
and the integrals over large hemispheres surrounding A.

The absorption coefficient is: $$\alpha(\vartheta_i, \varphi_i) = \frac{\Pi_a}{\Pi_i} = \frac{\Pi_a}{I_i \cdot A \cdot \cos\vartheta_i} = \frac{Q_a(\vartheta_i, \varphi_i)}{A \cdot \cos\vartheta_i}. \tag{9}$$

Q_a can be expressed with the far field angular distribution of $p_s(s)$ with the help of the *extinction theorem*:

The far field p_s can be separated into an angular and a radial function: $$p_s(s) \xrightarrow[s\to\infty]{} P_i \cdot \Phi_s\left(\vartheta_i, \varphi_i | \vartheta, \varphi\right) \cdot \frac{e^{-j\,k_0 s}}{s}. \tag{10}$$

The extinction theorem: $$Q_e = -\frac{4\pi}{k_0} \cdot \mathrm{Im}\{\Phi_s\left(\vartheta_i, \varphi_i | \vartheta_r, \varphi_r\right)\}\,, \tag{11}$$

(with ϑ_r, φ_r in the direction of the mirror-reflected wave, i.e. in our case $\vartheta_r = \vartheta_i$, $\varphi_r = \varphi_i$).

Finally, with $Q_a = Q_e - Q_s$:

$$Q_a = -\frac{4\pi}{k_0} \cdot \mathrm{Im}\{\Phi_s\left(\vartheta_i, \varphi_i | \vartheta_r, \varphi_r\right)\} \int_0^{2\pi} d\varphi \int_0^{\pi/2} |\Phi_s\left(\vartheta_i, \varphi_i | \vartheta, \varphi\right)|^2 \cdot \sin\vartheta\, d\vartheta. \tag{12}$$

Thus one needs the angular distribution of the scattered far field.

Example 1

The absorber area has the normalised admittance G, which possibly is a function of surface co-ordinates, $G(x_0, y_0)$; then $\langle G\rangle$ is its average over A. The baffle wall has the constant normalised admittance F_0. An approximation to the angular far field distribution of the scattered field is

$$\begin{aligned} \Phi_s &= \frac{-j\,k_0 \cos\vartheta \cdot \cos\vartheta_i}{2\pi(\cos\vartheta + F_0)\,(\cos\vartheta_i + \langle G\rangle)} \iint_A (G - F_0)\, e^{-j\,(\mu_x x_0 + \mu_y y_0)}\, dx_0\, dy_0\,, \\ \mu_x &= k_0\left(\sin\vartheta_i \cdot \cos\varphi_i - \sin\vartheta \cdot \cos\varphi\right)\,, \\ \mu_y &= k_0\left(\sin\vartheta_i \cdot \sin\varphi_i - \sin\vartheta \cdot \sin\varphi\right)\,. \end{aligned} \tag{13}$$

Because $G - F_0 = 0$ outside A, the integral can be extended over the whole plane $z = 0$; then it just represents the two-dimensional Fourier integral of the admittance difference.

In the special case $F_0 = 0$, i.e. a hard baffle wall:

$$\Phi_s = \frac{-j\,k_0 \cos\vartheta_i}{2\pi(\cos\vartheta_i + \langle G\rangle)} \iint_A G\, e^{-j\,(\mu_x x_0 + \mu_y y_0)}\, dx_0 dy_0\,. \tag{14}$$

Example 2

The absorber surface $A = a \cdot b$ is a rectangle, centred at the origin, with side length a in the x direction, side length b in the y direction, and $G = \text{const}$. The Fourier transform gives

$$\Phi_s = \frac{-j\,k_0 ab\,(G - F_0) \cos\vartheta_i \cdot \cos\vartheta}{2\pi(\cos\vartheta + F_0)(\cos\vartheta_i + G)} \frac{\sin(a\mu_x/2)}{a\,\mu_x/2} \frac{\sin(b\,\mu_y/2)}{b\,\mu_y/2}\,, \tag{15}$$

and for $F_0 = 0$

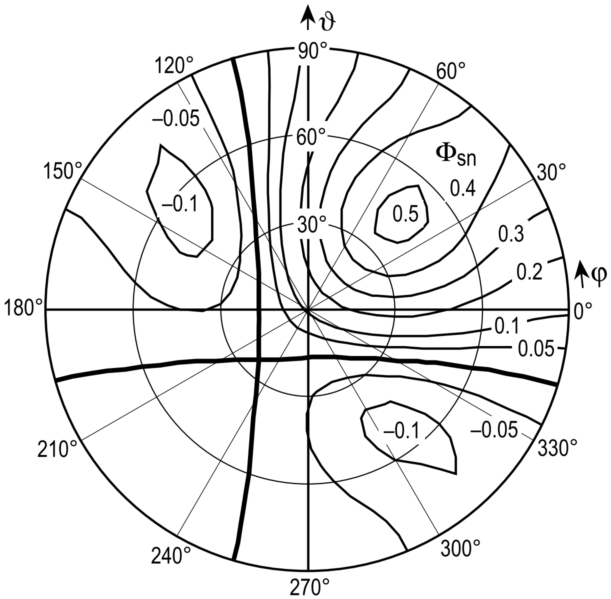
$$\Phi_s = \frac{-j\,k_0 ab \cdot G\cos\vartheta_i}{2\pi(\cos\vartheta_i + G)}\;\frac{\sin(a\,\mu_x/2)}{a\,\mu_x/2}\;\frac{\sin(b\,\mu_y/2)}{b\,\mu_y/2}\,. \tag{16}$$

Example 3

A circular absorber with radius a, centred at the origin, with a constant normalised admittance G [with $J_1(z)$ the Bessel function of the first order]:

$$\Phi_s = \frac{-j\,k_0\pi a^2(G - F_0)\cos\vartheta_i \cdot \cos\vartheta}{2\pi(\cos\vartheta + F_0)(\cos\vartheta_i + G)}\;\frac{2\,J_1(\gamma\,k_0 a)}{\gamma k_0 a}\,, \qquad (\gamma k_0)^2 = \mu_x^2 + \mu_y^2\,. \tag{17}$$

The diagram shows a directivity diagram of Φ_{sn} over φ, ϑ of a square with $k_0 a = 8$; $G = 1$; $\varphi_i = \vartheta_i = 45°$. Contour lines of Φ_{sn} are displayed; the thick lines separate ranges with different signs of Φ_{sn}.



The method of this section can also be applied for diffuse sound incidence. The scattered sound field for diffuse incidence is

$$p_{s,dif}(s,\vartheta,\varphi) = P_i\,\frac{e^{-j\,k_0 s}}{s}\int_0^{2\pi} d\,\varphi_i \int_0^{\pi/2} \Phi_s(\vartheta_i,\varphi_i|\vartheta,\varphi)\cdot\sin\vartheta_i\;d\vartheta_i\,. \tag{18}$$

D.8 Uneven, Local Absorber Surface

This section uses the method described in the previous Sect. D.7. The "unevenness" may be modelled either with a variation of the normalised absorber admittance G(x,y) or with a variation of the co-ordinates of the surface. The first method can be applied to grooves and narrow valleys, for example; the second method is applicable for slow or random variations.

If the absorber surface can be represented by a reference plane with a variable admittance G(x, y), then the admittance first is described by its Fourier series. The following example assumes in A a one-dimensional variation of the admittance F_0 of the surrounding baffle wall

in the form of a cosine modulation $\quad G(x) = F_0 + B \cdot \cos(2\pi x/\ell) \quad (1)$

with the period length ℓ, and the average admittance over the absorber side length a in the x direction $\quad \langle G\rangle = F_0 + B \cdot \mathrm{si}\,(\pi a/\ell), \quad (2)$

of a rectangular absorber with $\quad A = a \cdot b, \quad (3)$

with the function $\quad \mathrm{si}\,(z) = \sin(z)/z\,. \quad (4)$

The angular far field function of the scattered field is:

$$\Phi_s = \frac{-j\,k_0 ab \cdot B \cdot \cos\vartheta_i \cos\vartheta}{4\pi(\cos\vartheta + F_0)\,(\cos\vartheta_i + \langle G\rangle)} \cdot \left[\mathrm{si}\left(a\,\mu_x/2 - \pi a/\ell\right) + \mathrm{si}\left(a\,\mu_x/2 + \pi a/\ell\right)\right] \cdot \mathrm{si}\left(b\mu_y/2\right), \tag{5}$$

$$\begin{aligned} \mu_x &= k_0\left(\sin\vartheta_i \cdot \cos\varphi_i - \sin\vartheta \cdot \cos\varphi\right), \\ \mu_y &= k_0\left(\sin\vartheta_i \cdot \sin\varphi_i - \sin\vartheta \cdot \sin\varphi\right). \end{aligned} \tag{6}$$

Next, the geometrical profile of the absorber surface can be represented by $z = \zeta(x, y)$, and the normalised admittance G(x,y) is given at this surface. The co-ordinate s_0 of the absorber surface in section D.7 now has a non-zero z component $s_0 = (x_0, y_0, \zeta(x_0, y_0))$. The derivative normal to the surface becomes

$$\frac{\partial}{\partial n_0} = -\frac{\partial}{\partial z_0} + \frac{\partial\zeta}{\partial x_0} \cdot \frac{\partial}{\partial x_0} + \frac{\partial\zeta}{\partial y_0} \cdot \frac{\partial}{\partial y_0}\,.$$

If the variation of height is smaller than about half a wavelength, the angular far field distribution of the scattered field is

$$\Phi_s = \frac{-j\,\cos\vartheta\cos\vartheta_i}{2\pi(\cos\vartheta + F_0)(\cos\vartheta_i + \langle G\rangle)} \iint_A \left(k_0(G - F_0) + \mu_x\frac{\partial\zeta}{\partial x_0} + \mu_y\frac{\partial\zeta}{\partial y_0}\right) \cdot e^{-j\,(\mu_x x_0 + \mu_y y_0)}\,dx_0 dy_0\,. \tag{7}$$

In a special case (often encountered in reverberant room measurements) the profile $\zeta(x, y)$ is a constant height h of A over the surrounding baffle wall, with

$$\begin{aligned} \frac{\partial \zeta}{\partial x_0} &= h \cdot [\delta(x_0 + a/2) - \delta(x_0 - a/2)] \quad \text{for} \quad -b/2 < y_0 < b/2 \,, \\ \frac{\partial \zeta}{\partial y_0} &= h \cdot \left[\delta(y_0 + b/2) - \delta(y_0 - b/2)\right] \quad \text{for} \quad -a/2 < x_0 < a/2 \,, \end{aligned} \tag{8}$$

with the Dirac delta function $\delta(z)$. The contribution of the height step h to the far field angular distribution of the scattered field is

$$\Phi_{s\zeta} = \frac{-j\,k_0 ab \cdot k_0 h \cdot \cos\vartheta_i \cos\vartheta}{\pi(\cos\vartheta + F_0)\,(\cos\vartheta_i + \langle G\rangle)}\gamma^2 \cdot \mathrm{si}\left(a\mu_x/2\right) \mathrm{si}\left(b\mu_y/2\right) \,, \tag{9}$$

with $(\gamma k_0)^2 = \mu_x^2 + \mu_y^2$. The ratio of the contribution $\Phi_{s\zeta}$ to the contribution Φ_{sG} which describes the difference of the absorber admittance G from the baffle wall admittance F_0 is

$$\frac{\Phi_{s\zeta}}{\Phi_{sG}} = 2\frac{k_0 h}{G - F_0}\gamma^2 \,. \tag{10}$$

Next, the normalised absorber admittance $G(s_0)$ and/or the absorber surface contour $\zeta(s_0)$ has random variations with correlation distances d_G, d_ζ, respectively, and correlation functions

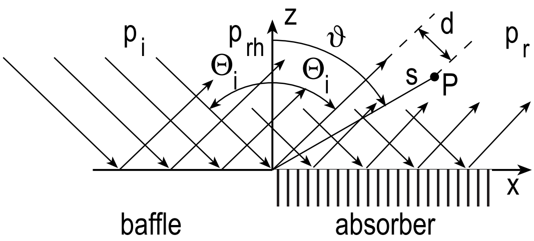
$$\begin{aligned} K_G(d) &= \langle (G - \langle G\rangle)^2\rangle_A \cdot e^{-(d/d_G)^2/2} \,, \\ K_\zeta(d) &= \langle \zeta^2\rangle_A \cdot e^{-(d/d_\zeta)^2/2} \,. \end{aligned} \tag{11}$$

With $G_t(k)$ and $\zeta_t(k)$ the Fourier transforms of $G(s_0) - \langle G\rangle$ and $\zeta(s_0)$, respectively, and using the relation between the far field effective intensity I_s and angular distribution Φ_s of the scattered field $I_s = |p_s|^2/(2Z_0) = I_i \cdot |\Phi_s|^2/s^2$, one gets for the far field contribution of the variations in G and/or ζ to the effective intensity:

$$\begin{aligned} I_{s,G,\zeta} &= 4\pi^2 \cdot I_i \cdot \frac{A}{s^2} \left| \frac{\cos\vartheta_i \cos\vartheta}{(\cos\vartheta + F_0)\,(\cos\vartheta_i + \langle G\rangle)} \right|^2 \\ &\quad \cdot \left(k_0^2 \cdot |G_t(k_0\gamma)|^2 + k_0^4\gamma^4 \cdot |\zeta_t(k_0\gamma)|^2\right) \\ &= I_i \cdot \frac{A}{2\pi s^2} \left| \frac{\cos\vartheta_i \cos\vartheta}{(\cos\vartheta + F_0)\,(\cos\vartheta_i + \langle G\rangle)} \right|^2 \\ &\quad \cdot \left((k_0 d_G)^2 \cdot \langle (G - \langle G\rangle)^2\rangle_A \cdot e^{-(k_0\gamma\, d_G)^2/2} + k_0^4\gamma^4 d_\zeta^2 \cdot \langle \zeta^2\rangle_A \cdot e^{-(k_0\gamma\, d_\zeta)^2/2}\right) \,. \end{aligned} \tag{12}$$

D.9 Scattering at the Border of an Absorbent Half-Plane

A hard half-plane and a locally reacting absorbent half-plane with the normalised surface admittance G have the y axis as common border line. A plane wave p_i is incident from the side of the hard half-plane under the polar angle ϑ_i and azimuthal angle φ_i (measured in the x,y plane relative to the x axis).



The problem becomes a two-dimensional one (in the x,z plane) by the substitutions

$$k_0^2 \;\rightarrow\; k^2 = k_0^2\left(1-\sin^2\vartheta_i\cdot\sin^2\varphi_i\right) \quad;\quad k_x = k\cdot\sin\Theta_i \quad;\quad k_z = k\cdot\cos\Theta_i\,;$$

$$\sin\Theta_i = \frac{\sin\vartheta_i\cdot\cos\varphi_i}{\sqrt{1-\sin^2\vartheta_i\cdot\sin^2\varphi_i}} \qquad;\quad \cos\Theta_i = \frac{\cos\vartheta_i}{\sqrt{1-\sin^2\vartheta_i\cdot\sin^2\varphi_i}}\,; \tag{1}$$

and a common factor $e^{-j\,k_y y}$ to each field quantity.

The sound field is composed of $p(x,z) = p_i(x,z) + p_{rh}(x,z) + p_s(x,z)$ with

- p_i = incident plane wave,
- p_{rh} = reflected wave with "hard reflection",
- p_s = scattered wave.

Combining $p_i + p_{rh}$, the sound field is

$$p(x,z) = 2\,P_i e^{-jkx\cdot\sin\Theta_i}\cdot\cos(kz\cdot\cos\Theta_i) - jkG\int_0^\infty p(x_0,0)\cdot G(x,y|x_0,y_0)dx_0\,, \tag{2}$$

with the Green's function

$$G(x,y|x_0,y_0) = -\frac{j}{4}\left[H_0^{(2)}(kR)+H_0^{(2)}(kR')\right]\,,$$
$$R^2 = (x-x_0)^2+(z-z_0)^2 \quad;\quad R'^2 = (x-x_0)^2+(z+z_0)^2\,, \tag{3}$$

containing Hankel functions of the second kind $H_0^{(2)}(z)$.

The far field of the sound pressure components $p_{rh}(s) + p_s(s)$ is

$$p_{rh+s}(s) \xrightarrow[ks\to\infty]{}$$

$$P_i\begin{cases} 1-\dfrac{G}{\cos\Theta_i+G}\left[(1-C(u)-S(u))+j\,(C(u)-S(u))\right] & ;\quad d<0 \\ \dfrac{\cos\Theta_i-G}{\cos\Theta_i+G}+\dfrac{G}{\cos\Theta_i+G}\left[(1-C(u)-S(u))+j\,(C(u)-S(u))\right] & ;\quad d>0 \end{cases}\,, \tag{4}$$

with u from

$$u^2 = \frac{1}{2}\,kd\frac{d}{s} = \frac{1}{2}kd \cdot \sin(\Theta - \Theta_i) = \frac{1}{2}ks \cdot \sin^2(\Theta - \Theta_i) \tag{5}$$

and C(u), S(u) the Fresnel's integrals:

$$C(u) = \sqrt{\frac{2}{\pi}} \int_0^u \cos(t^2)dt \qquad ; \qquad S(u) = \sqrt{\frac{2}{\pi}} \int_0^u \sin(t^2)dt\,. \tag{6}$$

The sound pressure in the surface at $z = 0$ is

$$p(x,0) = P_i e^{-jkx \cdot \sin\Theta_i} \left[2 - \frac{G}{G + \cos\Theta_i} \left(2 - U(-\Theta_i, kx) + jV(-\Theta_i, kx)\right) \right], \tag{7}$$

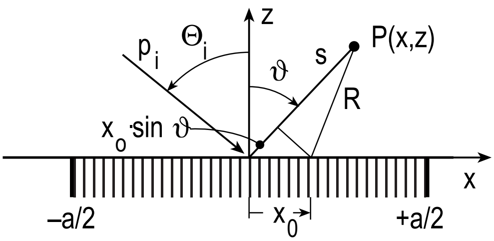
with the functions $U(\Theta, u)$, $V(\Theta, u)$ defined as the real and imaginary parts of

$$\begin{aligned} U(\Theta,u) - jV(\Theta,u) = 1 - \cos\Theta \int_0^u &\left(J_0(w) - jY_0(w)\right) \\ &\cdot \left[\cos(w \cdot \sin\Theta) - j\sin(w \cdot \sin\Theta)\right]\, dw\,, \end{aligned} \tag{8}$$

($J_0(z)$ and $Y_0(z)$ are Bessel and Neumann functions, respectively). The evaluation of these integrals is described in Mechel, Vol. I, Ch. 8 (1989).

D.10 Absorbent Strip in a Hard Baffle Wall, with Far Field Distribution

A locally reacting strip with normalised admittance G and width a, axial direction along the y axis, is placed in the x,y plane at $(-a/2, +a/2)$. A plane sound wave p_i is incident from the $-x$ direction under the polar angle Θ_i.



See Sect. D.9 for a possible component k_y of the wave vector in the y direction.

The sound field is composed of $p(x,z) = p_i(x,z) + p_{rh}(x,z) + p_s(x,z)$ with

- p_i = incident plane wave,
- p_{rh} = reflected wave with "hard reflection",
- p_s = scattered wave.

Combining $p_i + p_{rh}$, the sound field is

$$p(x, z) = 2\,P_i e^{-jkx\cdot\sin\Theta_i}\cdot\cos(kz\cdot\cos\Theta_i) - \frac{kG}{2}\int_0^{\infty} p(x_0, 0)\cdot H_0^{(2)}(kR)dx_0\ , \tag{1}$$

with $R^2 = (x - x_0)^2 + z^2$.

In the far field,

$$p(x, z) = 2P_i e^{-jkx\cdot\sin\Theta_i}\cdot\cos(kz\cdot\cos\Theta_i) + \sqrt{\frac{j}{2\pi k}}\frac{e^{-jks}}{\sqrt{s}}\cdot V_z(k\sin\vartheta)\ , \tag{2}$$

where $V_z(k\sin\vartheta)$ is the Fourier transform of the particle velocity distribution (in the z direction) at the plane $z = 0$. From the equivalent form of the scattered field p_s in the far field,

$$p_s(s) = -P_i\sqrt{\frac{j}{2\pi k}}\cdot\Phi_s(\Theta_i|\vartheta)\cdot\frac{e^{-jks}}{\sqrt{s}}\ , \tag{3}$$

follows

$$\Phi_s(\Theta_i|\vartheta) = \int_{-ka/2}^{+ka/2} G\cdot\frac{p(x_0, 0)}{P_i}\cdot e^{+jkx_0\cdot\sin\vartheta}d(kx_0) = \int_{-\infty}^{+\infty}\frac{v_z(x_0, 0)}{v_i}\cdot e^{+jkx_0\cdot\sin\vartheta}d(kx_0)\ , \tag{4}$$

and the absorption cross section Q_a of the strip

$$Q_a = \frac{2}{k}\mathrm{Re}\{\Phi_s(\Theta_i|\Theta_i)\} - \frac{1}{2\pi k}\int_{-\pi/2}^{+\pi/2}|\Phi_s(\Theta_i|\vartheta)|^2 d\vartheta\ . \tag{5}$$

The needed sound pressure distribution $p(x_0, 0)$ has different possible approximations. For small ka and low values of G, $p(x_0, 0) \approx 2p_i(x_0, 0) = 2P_i e^{-jkx_0\cdot\sin\Theta_i}$

leading to $\Phi_s(\Theta_i|\vartheta)$,

$$\Phi_s(\Theta_i|\vartheta) = 2kaG\cdot\mathrm{si}\,(ka(\sin\vartheta - \sin\Theta_i)/2)\ , \tag{6}$$

with $\mathrm{si}(z) = \sin(z)/z$,

and to Q_a

$$\frac{Q_a}{a} = 4\mathrm{Re}\{G\} - \frac{2}{\pi}ka\cdot|G|^2\int_{-\pi/2}^{+\pi/2}\mathrm{si}^2\,(ka(\sin\vartheta - \sin\Theta_i)/2)\,d\vartheta$$
$$\xrightarrow[ka\ll 1]{} 4\mathrm{Re}\{G\} - 2ka\cdot|G|^2 \rightarrow 4\mathrm{Re}\{G\} \tag{7}$$

Approximations for large ka

$$p(x_0, 0) \approx P_i(1 + r)\cdot e^{-jkx_0\cdot\sin\Theta_i} = 2P_i\frac{\cos\Theta_i}{G + \cos\Theta_i}\cdot e^{-jkx_0\cdot\sin\Theta_i}$$

and the corresponding $\Phi_s(\Theta_i|\vartheta)$ are:

$$\Phi_s(\Theta_i|\vartheta) = 2\frac{G\cos\Theta_i}{G + \cos\Theta_i}ka \cdot\mathrm{si}\,(ka(\sin\vartheta - \sin\Theta_i)/2)\ . \tag{8}$$

The resulting Q_a is

$$\frac{Q_a}{a} = 4\mathrm{Re}\left\{\frac{G\cos\Theta_i}{G+\cos\Theta_i}\right\} - \frac{2ka}{\pi}\left|\frac{G\cos\Theta_i}{G+\cos\Theta_i}\right|^2 \int_{-\pi/2}^{+\pi/2} \mathrm{si}^2\,(ka(\sin\vartheta - \sin\Theta_i)/2)\,d\vartheta\,. \tag{9}$$

D.11 Absorbent Strip in a Hard Baffle Wall, as a Variational Problem

The geometry and field composition are as in *Sect. D.10*.

The variational principle is based on Helmholtz's theorem of superposition, which requires that the power of the "cross intensity" $p(x_0, 0) \cdot v_z^a(x_0, 0)$ of the desired field $p(x, z)$ and the particle velocity $v_z^a(x, z)$ of the adjoint field be zero at the plane $z = 0$. The adjoint field is the solution for exchanged emission and immission points. The cross power is minimised by variation of the amplitude P of the estimate $P \cdot \exp(-jkx \cdot \sin\Theta_i)$. The expression to be minimised is

$$\begin{aligned} &4P_i ka\,G \cdot P - kaG \cdot P^2 - \frac{1}{2}k^2G^2P^2 \int_{-a/2}^{+a/2} e^{jkx\cdot\sin\Theta_i}\,dx \\ &\quad\cdot \int_{-a/2}^{a/2} H_0^{(2)}(k|x-x_0|) \cdot e^{-j\,kx_0\cdot\sin\Theta_i}\,dx_0 = \mathrm{Min}(P)\,. \end{aligned} \tag{1}$$

The partial derivative $\partial/\partial P$ is zero for

$$P = 2P_i \Bigg/ \left\{1 + \frac{kG}{2a}\int_{-a/2}^{+a/2} e^{j\,kx\cdot\sin\Theta_i}\left(\int_{-a/2}^{a/2} H_0^{(2)}(k\,|x-x_0|)\cdot e^{-j\,kx_0\cdot\sin\Theta_i}\,dx_0\right)dx\right\}. \tag{2}$$

Definition of auxiliary functions:

$$\begin{aligned} Z(\Theta, u) &= \cos\Theta \int_0^u \big(J_0(|w|) - jY_0(|w|)\big)\cdot e^{-jw\cdot\sin\Theta}\,dw \\ &= \begin{cases} -1 + U(\Theta,u) - jV(\Theta,u) & ; \quad u < 0 \\ 1 - U(\Theta,u) + jV(\Theta,u) & ; \quad u < 0 \end{cases}, \\ W(\Theta, ka) &= \frac{1}{ka}\int_0^{ka} Z(\Theta,u)\,du \end{aligned} \tag{3}$$

[the functions $U(\Theta, u)$, $V(\Theta, u)$ are defined in *Sect. D.9*].

The amplitude factor P becomes

$$P = \frac{2P_i \cdot \cos\Theta_i}{\cos\Theta_i + G/2 \cdot (W(\Theta_i, ka) + W(-\Theta_i, ka))}\,, \tag{4}$$

and the sound pressure at z = 0 is

$$p(x,0) = 2P_i \left[1 - \frac{G}{2} \frac{Z(\Theta_i, ka/2 - kx) + Z(-\Theta_i, ka/2 + kx)}{\cos\Theta_i + G/2 \cdot (W(\Theta_i, ka) + W(-\Theta_i, ka))} \right] \cdot e^{-jkx \cdot \sin\Theta_i}$$

$$\to \begin{cases} 2P_i \cdot e^{-jkx \cdot \sin\Theta_i} & ; \quad |x| \gg a \\ 2P_i \cdot e^{-j\,kx \cdot \sin\Theta_i} \dfrac{\cos\Theta_i}{G + \cos\Theta_i} & ; \quad |kx| \ll |ka| \gg 1\,. \end{cases} \tag{5}$$

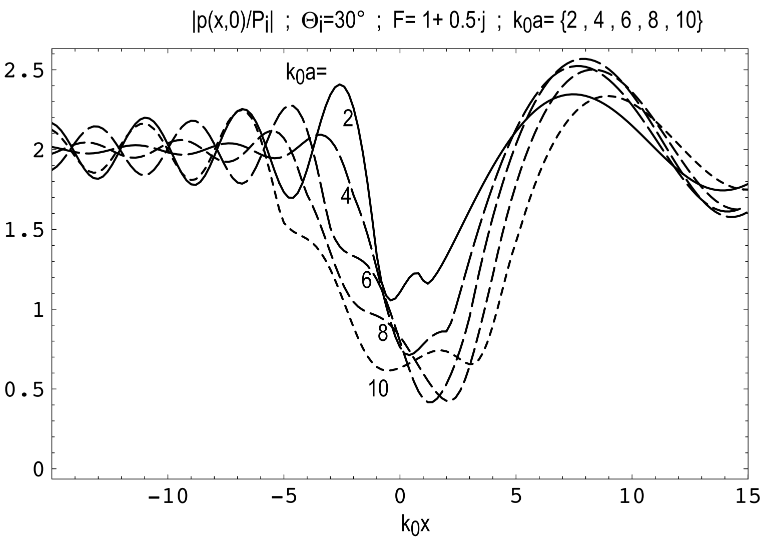
The angular far field distribution of the scattered field is

$$\Phi_s(\Theta_i|\vartheta) = \frac{2kaG\cos\Theta_i}{\cos\Theta_i + \frac{G}{2}\,(W(\Theta_i, ka) + W(-\Theta_i, ka))} \cdot \mathrm{si}\;(ka\,(\sin\Theta - \sin\Theta_i)/2)\,, \tag{6}$$

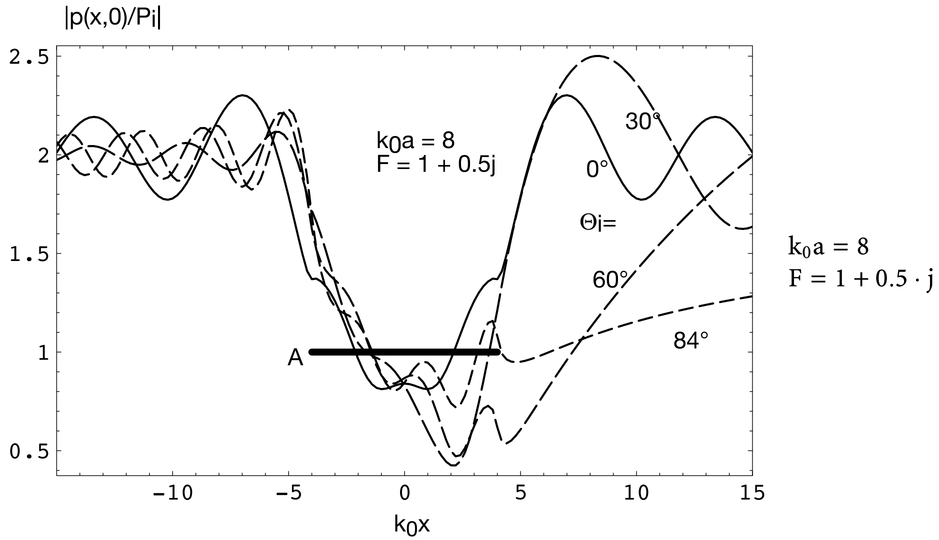
with $\mathrm{si}(z) = \sin(z)/z$.

The absorption cross section Q_a of the strip follows with this from the previous Sections.

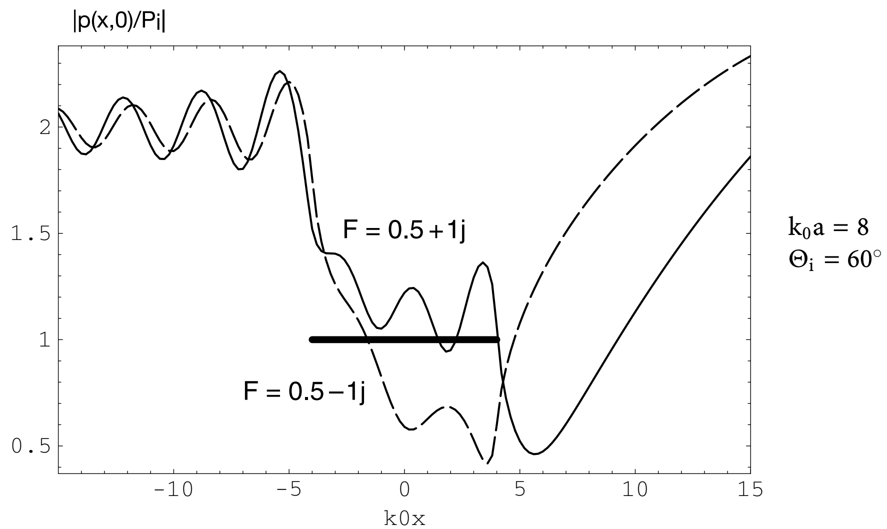
Numerical examples for sound pressure distributions in the plane z = 0: (equivalencies for the parameters in the plot labels: "Theta" $\sim \Theta_i$; "F" $\sim$ G; "k0a" $\sim k_0a$). The curve dashes become shorter for later entries in the parameter lists {...}. Sound incidence is normal to the strip axis ($k_y = 0$).



Sound pressure distributions for different k_0a



Sound pressure distributions for different angles of incidence Θ_i



Sound pressure distributions for two normalised admittances $F \sim G$

D.12 Absorbent Strip in a Hard Baffle Wall, with Mathieu Functions

► *See also:* Mechel (1997), for notations and relations of Mathieu functions.

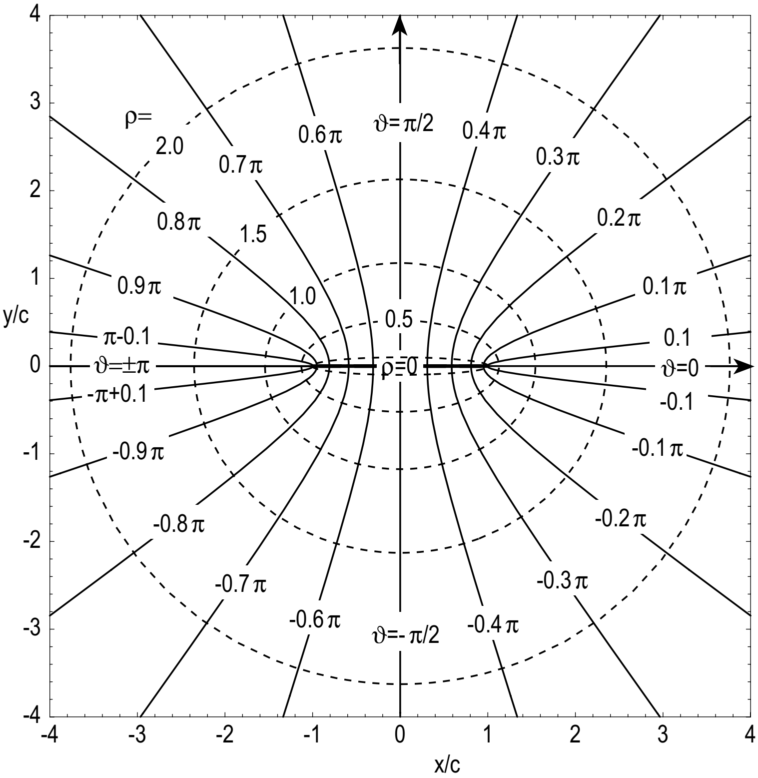
The sound field around a locally reacting strip with normalised admittance G in a hard baffle wall can be formulated as a boundary value problem with exact solutions in elliptic-hyperbolic cylinder co-ordinates (ρ, ϑ). The co-ordinate curves are confocal ellipses and orthogonal confocal hyperbolic branches. The radial and azimuthal eigen-functions in these co-ordinates are Mathieu functions.

Transformation between Cartesian and elliptic-hyperbolic co-ordinates: The common foci are at $x = \pm c$.

$$\begin{aligned} x &= x_1 = c \cdot \cosh\rho \cdot \cos\vartheta\,, \\ y &= x_2 = c \cdot \sinh\rho \cdot \sin\vartheta\,, \\ z &= x_3 = z\,. \end{aligned} \tag{1}$$

The boundary surface of the absorbent strip is at $\rho = 0$, the focus distance is $c = a/2$, and the boundaries of the baffle wall are at $\vartheta = 0$ and $\vartheta = \pi$. The Helmholtz differential equation (wave equation) $(\Delta + k_0^2)\,u = 0$ in elliptic-hyperbolic co-ordinates is:

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{\partial^2 u}{\partial\vartheta^2} + \left(\cosh^2\rho - \cos^2\vartheta\right) \cdot \left(c^2 \frac{\partial^2 u}{\partial z^2} + (k_0 c)^2 u\right) = 0\,. \tag{2}$$



Co-ordinate lines in elliptic-hyperbolic cylinder co-ordinates

For separated field functions $u(\rho,\vartheta) = T(\vartheta)\cdot R(\rho)$. This is equivalent to the pair of Mathieu differential equations:

$$\begin{aligned}&\frac{d^2R(\rho)}{d\rho^2} - \big(\lambda - 2q\cdot\cosh(2\rho)\big)\cdot R(\rho) = 0\,,\\&\frac{d^2T(\vartheta)}{d\vartheta^2} + \big(\lambda - 2q\cdot\cos(2\vartheta)\big)\cdot T(\vartheta) = 0\,.\end{aligned} \tag{3}$$

The parameter q is determined by $q = (k_0c)^2/4$.

The parameter λ stands for *characteristic values* of the Mathieu functions, for which the Mathieu differential equations have finite and periodic (in ϑ) solutions; they will be called $\lambda = ac_m$ for cos-like (symmetrical in ϑ) azimuthal Mathieu functions $T(\vartheta) = ce_m(\vartheta, q)$ and $\lambda = bc_m$ for sin-like (antisymmetrical in ϑ) azimuthal Mathieu functions $T(\vartheta) = se_m(\vartheta, q)$. The azimuthal functions are associated, respectively, with radial Mathieu functions of the Bessel type, $Jc_m(\rho)$, $Js_m(\rho)$, of the Neumann type, $Yc_m(\rho)$, $Ys_m(\rho)$, and of the Hankel-type for outward propagating waves

$$Hc_m^{(2)}(\rho) = Jc_m(\rho) - j\cdot Yc_m(\rho) \quad \text{or}$$

$$Hs_m^{(2)}(\rho) = Js_m(\rho) - j\cdot Ys_m(\rho).$$

A plane wave incident at an angle Θ against the major axis of the ellipses, i.e. against the plane of the strip and the baffle wall, is in Cartesian co-ordinates

$$\begin{aligned}&p_i(x,y) = e^{-jk_0(x\cos\Theta + y\sin\Theta)} = e^{-2jw\sqrt{q}} = p_i(\rho,\vartheta)\,,\\&w = \cos\rho\cos\vartheta\cos\Theta + \sin\rho\sin\vartheta\sin\Theta\,.\end{aligned} \tag{4}$$

Its expansion in Mathieu functions is

$$\begin{aligned}p_i(\rho,\vartheta) = {}&2\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot Jc_m(\rho;q)\cdot ce_m(\vartheta;q)\\&+2\sum_{m=1}^{\infty}(-j)^m se_m(\Theta;q)\cdot Js_m(\rho;q)\cdot se_m(\vartheta;q)\,.\end{aligned} \tag{5}$$

The sum of the incident wave p_i and of the reflected wave p_r with reflection at a hard plane containing the major axis of the ellipses (i.e. the baffle wall) is

$$p_i(\rho,\vartheta) + p_r(\rho,\vartheta) = 4\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot Jc_m(\rho;q)\cdot ce_m(\vartheta;q)\,. \tag{6}$$

A scattered field $p_s(\rho,\vartheta)$ is added to these field components; it is formulated as a sum of terms as in $p_i + p_r$, but with yet undetermined term amplitudes a_m, and the Mathieu-Bessel functions $Jc_m(\rho;q)$ (which represent radial standing waves) replaced with Mathieu-Hankel functions $Hc_m^{(2)}(\rho;q)$; so it satisfies the boundary condition at the baffle wall. Thus:

$$\begin{aligned}p(\rho,\vartheta) &= p_i(\rho,\vartheta) + p_r(\rho,\vartheta) + p_s(\rho,\vartheta)\\&= 4\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot ce_m(\vartheta;q)\cdot\big(Jc_m(\rho;q) + a_m\cdot Hc_m^{(2)}(\rho;q)\big)\,.\end{aligned} \tag{7}$$

The gradient in elliptic-hyperbolic co-ordinates is

$$\operatorname{grad} u = \frac{\vec{e}_\rho}{c\sqrt{\sinh^2\rho + \sin^2\vartheta}}\frac{\partial u}{\partial\rho} + \frac{\vec{e}_\vartheta}{c\sqrt{\sinh^2\rho + \sin^2\vartheta}}\frac{\partial u}{\partial\vartheta} + \vec{e}_z \cdot \frac{\partial u}{\partial z}\,, \tag{8}$$

and therefore the particle velocity v_ρ in the direction of the hyperbolic ρ-lines:

This is used for the boundary condition at the strip: $Z_0 v_\rho(0,\vartheta) = -G \cdot p(0,\vartheta)$.

$$\begin{aligned} v_\rho &= \frac{j}{k_0 Z_0}\operatorname{grad}_\rho p = \frac{j}{k_0 c Z_0}\frac{\partial p/\partial\rho}{\sqrt{\sinh^2\rho + \sin^2\vartheta}}\,. \\ &\xrightarrow[\rho=0]{} \frac{j}{k_0 c\, Z_0}\frac{\partial p/\partial\rho}{|\sin\vartheta|}\,, \\ &\xrightarrow[\substack{\vartheta=0\\ \vartheta=\pi}]{} \frac{j}{k_0 c Z_0}\frac{\partial p/\partial\rho}{\sinh\rho}\,. \end{aligned} \tag{9}$$

The boundary condition gives (a prime at the Mathieu functions indicates the derivative in ρ)

$$\begin{aligned} &\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot ce_m(\vartheta;q)\cdot\left(Jc'_m(0;q) + a_m\cdot Hc'^{(2)}_m(0;q)\right) \\ &= jk_0cG\cdot|\sin\vartheta|\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot ce_m(\vartheta;q)\cdot\left(Jc_m(0;q) + a_m\cdot Hc^{(2)}_m(0;q)\right)\,. \end{aligned} \tag{10}$$

The functions $ce_m(\vartheta;q)$ are orthogonal in ϑ over $(0,\pi)$ with the norms N_m:

$$\int_0^\pi ce_m(\vartheta;q)\cdot ce_n(\vartheta;q)d\vartheta = \delta_{m,n}\cdot N_m\,. \tag{11}$$

Application of the orthogonality integral on both sides of the boundary condition gives, with the mode-coupling coefficients:

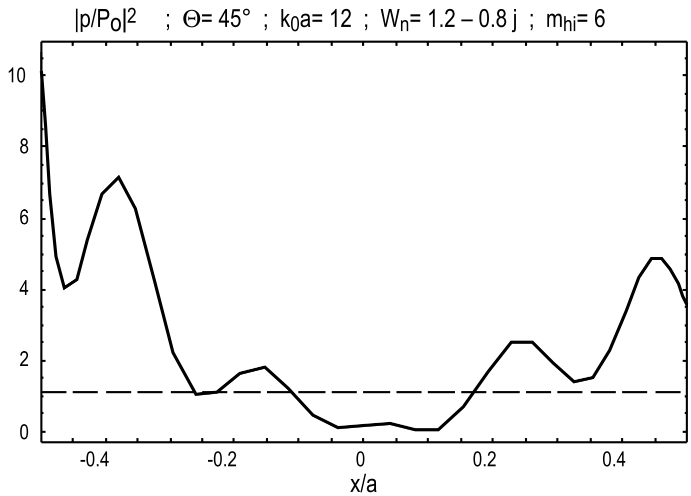
$$T_{m,n} = \int_0^\pi ce_m(\vartheta;q)\cdot ce_n(\vartheta;q)\cdot|\sin\vartheta|d\vartheta. \tag{12}$$

the linear, inhomogeneous system of equations for the amplitudes a_m:

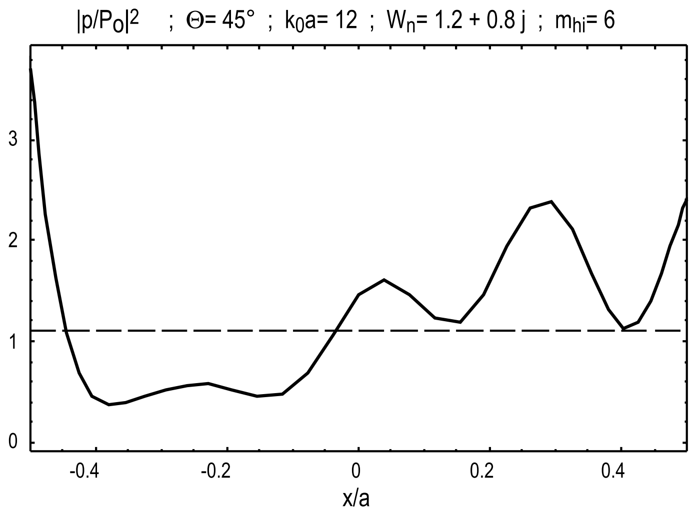
$$\begin{aligned} &\sum_{m=0}^{\infty} a_m\cdot(-j)^m ce_m(\Theta;q)\cdot\left(j\,k_0c\,G\cdot T_{m,n}\cdot Hc^{(2)}_m(0;q) - \delta_{m,n}N_n\cdot Hc'^{(2)}_n(0;q)\right) \\ &= -\sum_{m=0}^{\infty}(-j)^m ce_m(\Theta;q)\cdot\left(j\,k_0c\,G\cdot T_{m,n}\cdot Jc_m(0;q) - \delta_{m,n}N_n\cdot Jc'_n(0;q)\right)\,. \end{aligned} \tag{13}$$

The mode norms are $N_m = \pi/2$; the coupling coefficients $T_{m,n}$ can be expressed in terms of the Fourier coefficients of the Fourier series representation of $ce_m(\vartheta)$ (see Mechel (1997), *Sect. 19.5*). After the solution of this (truncated) system of equations, the sound field is known.

Numerical examples:
The horizontal dashed lines in the plots below are the squared sound pressure magnitudes at an absorber of infinite extend.

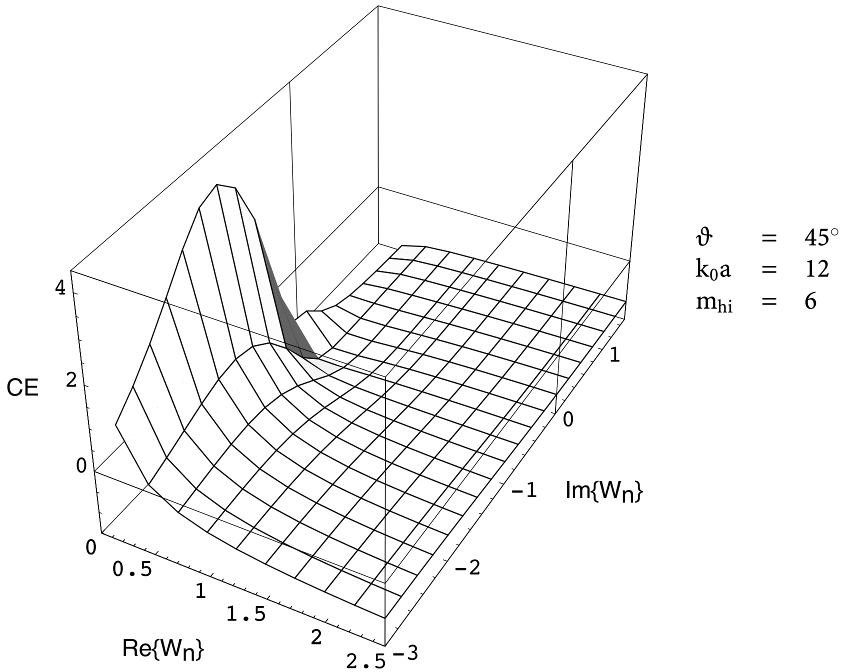


Distribution of sound pressure magnitude squared at the surface of an absorbent strip with mass-type reactance



Distribution of sound pressure magnitude squared at the surface of an absorbent strip with spring-type reactance

The *acoustic corner effect*:
Due to scattering at the borders of a finite-size absorber, its absorption in general is different from the absorption of an infinite, but otherwise equal, absorber. The *quantitative corner effect* is defined as the ratio of the effective power Π absorbed by the strip



The corner effect may be positive or negative, depending on the sign of the reactance

to the power Π_∞ absorbed by an area of the same size of an infinite, but otherwise equal, absorber:

$$\begin{aligned} CE(\vartheta) = \frac{\Pi}{\Pi_\infty} = \frac{1}{4\,k_0a \cdot \mathrm{Re}\,\{G\}} \cdot \left| 1 + \frac{G}{\sin\vartheta} \right|^2 \\ \cdot \sum_{m \geqslant 0} \left[4ce_m(\vartheta; q) \cdot \mathrm{Re}\left\{ (-j)^m\, a_m^* \right\} + |\,a_m|^2 \right]. \end{aligned} \tag{14}$$

D.13 Absorption of Finite-Size Absorbers, as a Problem of Radiation

► *See also:* Mechel, Vol. I, Ch. 8 (1989)

The surface impedance Z_A of an infinite absorber is generally easily evaluated. The problem with finite-size absorbers in a baffle wall is the influence of border scattering. This influence can be taken into account by a simple equivalent network.

The absorbed effective power is:

$$\Pi_a = A \cdot \frac{|P_i|^2}{2\,Z_0}\,\frac{4\,\mathrm{Re}\{Z_A\}}{|Z_A + Z_s|^2}\,. \tag{1}$$

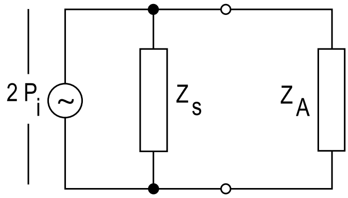
The normalised absorption cross section is:

$$\frac{Q_a}{A} = \frac{4\,\mathrm{Re}\{Z_A\}}{|Z_A + Z_s|^2}\,. \tag{2}$$

A	=	area of the absorber;
P_i	=	amplitude of the incident plane wave;
Z_A	=	surface impedance of the infinite absorber;
Z_s	=	radiation impedance of a radiator with the size and shape of the absorber, when its surface oscillation pattern agrees with that of the exciting wave at the absorber surface

This can be represented in an equivalent network:

- The pressure source has an amplitude $2P_i$;
- The radiation impedance Z_s is the internal source impedance;
- The impedance Z_A of the infinite absorber is the load impedance;
- The power in Z_A is the absorbed power Π_a.

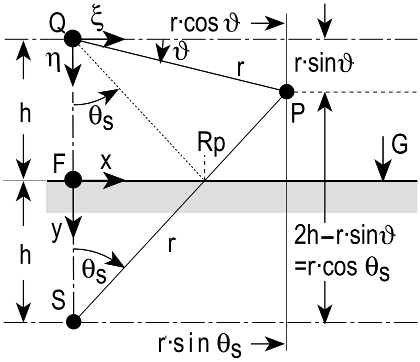


Thus the determination of the absorption by finite-size absorbers is reduced to the determination of their radiation impedance.

D.14 A Monopole Line Source Above an Infinite, Plane Absorber; Integration Method

► *See also:* Mechel, A line source above a plane absorber (2000)

A monopole line source placed at Q is parallel to the absorber, with a normalised surface admittance G. S is the mirror-reflected point to Q. P is a field point.



The absorber may be locally or bulk reacting; it will be mentioned if results are valid for locally reacting absorbers only.

In what follows, k is the wave vector component in the plane containing Q, S, P.

The field is set up as $p = p_Q + p_r$; p_Q = source free field; p_r = reflected field.

Source free field (with unit amplitude) $$p_Q(r) = H_0^{(2)}(kr). \tag{1}$$

Field of plane wave incident under polar angle θ [with reflection factor R(θ) for this angle of incidence, and $\Theta = \theta + \vartheta$]:

$$p_e + p_{er} = e^{-j\,kr\cdot\sin\Theta} + R(\Theta - \vartheta)\cdot e^{-2j\,kh\cdot\cos(\Theta-\vartheta)}\cdot e^{-jkr\cdot\sin(\Theta-2\vartheta)}. \tag{2}$$

After application of the integral operation:

$$\frac{1}{\pi}\int\limits_{C(\Theta)} \left(p_e + p_{er}\right) d\Theta \tag{3}$$

the first term is the integral representation of the Hankel function

$$H_0^{(2)}(kr) = \frac{1}{\pi}\int\limits_{C(\Theta)} e^{-j\,kr\cdot\sin\Theta}\,d\Theta \quad ; \quad \mathrm{Re}\{kr\} > 0\,. \tag{4}$$

whith path $C(\Theta)$: $-j\infty \rightarrow 0 \rightarrow \pi \rightarrow \pi + j\infty$.

Thus the second term yields

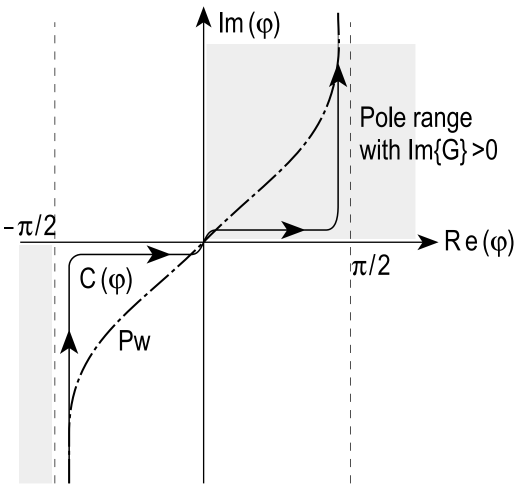
$$p_r = \frac{1}{\pi}\int\limits_{C(\Theta)} R(\Theta - \vartheta)\cdot e^{-j\,kr\cdot(\sin(\Theta-2\vartheta)+2h/r\cdot\cos(\Theta-\vartheta))}\,d\Theta, \tag{5}$$

and after a horizontal shift of the path, with $\varphi = \theta - \theta_s$ (see sketch for θ_s and r'):

$$p_r = \frac{1}{\pi}\int\limits_{C(\varphi)} R(\varphi + \theta_s)\cdot e^{-j\,kr'\cdot\cos\varphi}\,d\varphi \quad ; \quad R(\varphi + \theta_s) = \frac{\cos(\varphi + \theta_s) - G}{\cos(\varphi + \theta_s) + G}. \tag{6}$$

This is an exact representation for p_r; the path $C(\varphi)$ is shown in the diagram, together with the shaded range, where poles of $R(\varphi + \theta_s)$ are possible (only for $\mathrm{Im}\{G\} > 0$), and the "path of steepest descent" (pass way) Pw.

If during the deformation $C(\varphi) \rightarrow$ Pw a pole is crossed, a "pole contribution" must be added to the integral of steepest descent; it has the form of a surface wave.



For direct numerical integration, use

$$p_r = \frac{1}{\pi} \int_{(-\pi/2)}^{(-\pi/2)} \frac{\cos(\varphi + \theta_s) - G}{\cos(\varphi + \theta_s) + G} \cdot e^{-j\,kr' \cdot \cos\varphi}\, d\varphi$$
$$+ \frac{2j}{\pi} \int_0^{\infty} \frac{1 + G^2 - \cos^2\theta_s + \sinh^2\varphi''}{1 - G^2 - \cos^2\theta_s + \sinh^2\varphi'' + 2jG\cos\theta_s \cdot \sinh\varphi''} \cdot e^{-kr' \cdot \sinh\varphi''}\, d\varphi''. \tag{7}$$

The first integrand oscillates strongly for $kr' \gg 1$. Therefore use the method of integration along the steepest descent (also "saddle point integration" or "pass integration"). Some cases must be distinguished.

Saddle point at $\varphi_s = 0$; on the pass way is $\varphi(s) = \pm\arccos(1 - j \cdot s^2)$; $s \gtrless 0$, s being a running parameter on the pass way from $-\infty$ to $+\infty$; the saddle point is at $s = 0$; the slope of the pass way in the saddle point is $d\varphi(0)/ds = 1 + j$.

No pole crossing, and no pole near the saddle point:

$$p_r = \sqrt{\frac{1}{\pi kr'}}\, e^{-j\,kr'} \left[\Phi(0) + \frac{1}{4\,kr'}\Phi''(0) + \ldots + \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)}{(2n)!(2\,kr')^n}\Phi^{(2n)}(0) \right], \tag{8}$$

(primes at Φ indicate derivatives with respect to s), with:

$$\Phi(s) = R(\varphi(s) + \theta_s) \cdot \frac{2\,j}{\sqrt{2\,j + s^2}} = \frac{\cos(\varphi(s) + \theta_s) - G}{\cos(\varphi(s) + \theta_s) + G} \cdot \frac{2\,j}{\sqrt{2\,j + s^2}}. \tag{9}$$

With some derivatives performed [leaving $R^{(n)}(\theta_s)$ unevaluated] we have

$$p_r = \sqrt{\frac{2j}{\pi\,kr'}}\, e^{-j\,kr'} \cdot \left[\left(1 + \frac{j}{8\,kr'} - \frac{9}{128(kr')^2} - \frac{75\,j}{1024(kr')^3} + \frac{3675}{32768(kr')^4} \right) \cdot R(\theta_s) \right.$$
$$+ \left(\frac{j}{2} - \frac{5}{16\,kr'} - \frac{259\,j}{768(kr')^2} + \frac{3229}{6144(kr')^3} \right) \cdot \frac{R^{(2)}(\theta_s)}{kr'}$$
$$- \left(\frac{1}{8} + \frac{35\,j}{192\,kr'} - \frac{329}{1024(kr')^2} \right) \cdot \frac{R^{(4)}(\theta_s)}{(kr')^2}$$
$$\left. - \left(\frac{j}{48} - \frac{7}{128\,kr'} \right) \cdot \frac{R^{(6)}(\theta_s)}{(kr')^3} + \frac{1}{384} \cdot \frac{R^{(8)}(\theta_s)}{(kr')^3} \right]. \tag{10}$$

This form is valid for both locally and bulk reacting absorbers. The first terms in the parentheses give the *geometrical acoustic approximation* (or *mirror source approximation*):

$$p_r \xrightarrow[kr' \gg 1]{} R(\theta_s) \cdot H_0^{(2)}(kr'). \tag{11}$$

For locally reacting absorbers the derivatives $R^{(n)}(\theta_s)$ can be evaluated in advance (for bulk reacting absorbers they depend on the internal structure of the absorber); G = normalised admittance:

$$R(\theta_s) = \frac{\cos\theta_s - G}{\cos\theta_s + G}\,,$$

$$R^{(2)}(\theta_s) = -2G\frac{2-\cos^2\theta_s + G\cdot\cos\theta_s}{(\cos\theta_s + G)^3}\,,$$

$$R^{(4)}(\theta_s) =$$

$$2G\frac{(G-5\cos\theta_s)(\cos\theta_s+G)^2\cos\theta_s + 4(\cos\theta_s+G)(2G-7\cos\theta_s)\sin^2\theta_s - 24\sin^4\theta_s}{(\cos\theta_s+G)^5}\,,$$

$$R^{(6)}(\theta_s) = 2\,G\frac{(\cos\theta_s+G)^3(28\,G\cos\theta_s - G^2 - 61\cos^2\theta_s)\cos\theta_s + \ldots}{(\cos\theta_s+G)^7} \qquad (12)$$

$$\frac{\ldots + 2(\cos\theta_s+G)^2(193\,G\cos\theta_s - 16\,G^2 - 331\cos^2\theta_s)\sin\theta_s + \ldots}{\ldots}$$

$$\frac{\ldots + 120(\cos\theta_s+G)(4\,G - 11\cos\theta_s)\sin^4\theta_s - 720\sin^6\theta_s}{\ldots}\,,$$

$$R^{(8)}(\theta_s) = 2\,G\frac{(\cos\theta_s+G)^4(G^3 - 123\,G^2\cos\theta_s + 1011\,G\cos^2\theta_s - 1385\cos^3\theta_s)\times\ldots}{(\cos\theta_s+G)^9}$$

$$\frac{\ldots\times\cos\theta_s + 8(\cos\theta_s+G)^3(16\,G^3 - 519\,G^2\cos\theta_s + 2694\,G\cos^2\theta_s - \ldots}{\ldots}$$

$$\frac{\ldots - 3071\cos^3\theta_s)\sin^2\theta_s + 1008(\cos\theta_s+G)^2(-8\,G^2 + 59\,G\cos\theta_s - \ldots}{\ldots}$$

$$\frac{\ldots - 83\cos^2\theta_s)\sin^4\theta_s + 20160(\cos\theta_s+G)(2\,G - 5\cos\theta_s)\sin^6\theta_s - \ldots}{\ldots}$$

$$\frac{\ldots - 40320\sin^8\theta_s}{\ldots}\,.$$

Special case $\theta_s = 0$, i.e. field point P on line through S and Q:

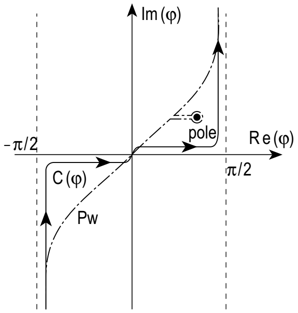
$$\begin{aligned} p_r(r',0) = \sqrt{\frac{2\,j}{\pi\,kr'}}e^{-j\,kr'}\cdot\Bigg[&\left(1+\frac{j}{8\,kr'} - \frac{9}{128\,(kr')^2} - \frac{75\,j}{1024\,(kr')^3} + \frac{3675}{32768\,(kr')^4}\right)\\ &\times\frac{1-G}{1+G} - \left(\frac{j}{2} - \frac{5}{16\,kr'} - \frac{259\,j}{768\,(kr')^2} + \frac{3229}{6144\,(kr')^3}\right)\cdot\frac{2\,G}{kr'(1+G)^2}\\ &+\left(\frac{1}{8} + \frac{35\,j}{192\,kr'} - \frac{329}{1024\,(kr')^2}\right)\cdot\frac{2\,G\,(5-G)}{(kr')^2(1+G)^3} + \left(\frac{j}{48} - \frac{7}{128\,kr'}\right)\\ &\times\frac{2\,G(61-28\,G+G^2)}{(kr')^3(1+G)^4} - \frac{1}{384}\cdot\frac{2\,G(1385-1011\,G+123\,G^2-G^3)}{(kr')^4(1+G)^5}\Bigg]\,. \end{aligned} \qquad (13)$$

Special case $\theta_s = \pi/2$, i.e. P and Q are on the absorber:

$$
\begin{aligned}
p_r(r', \pi/2) = {} & \sqrt{\frac{2j}{\pi\, kr'}}\, e^{-j\,kr'} \\
& \cdot \Bigg[-\left(1 + \frac{j}{8\,kr'} - \frac{9}{128\,(kr')^2} - \frac{75\,j}{1024\,(kr')^3} + \frac{3675}{32768\,(kr')^4}\right) \\
& - \left(\frac{j}{2} - \frac{5}{16\,kr'} - \frac{259\,j}{768\,(kr')^2} + \frac{3229}{6144\,(kr')^3}\right) \cdot \frac{4}{kr' G^2} + \\
& + \left(\frac{1}{8} + \frac{35\,j}{192\,kr'} - \frac{329}{1024\,(kr')^2}\right) \cdot \frac{48 - 16\,G^2}{(kr')^2 G^4} + \left(\frac{j}{48} - \frac{7}{128\,kr'}\right) \\
& \times \frac{2(720 - 480\,G^2 + 32\,G^4)}{(kr')^3 G^6} - \frac{1}{384} \cdot \frac{2\left(40320(1-G^2) + 8064\,G^4 - 128\,G^6\right)}{(kr')^4 G^8} \Bigg] .
\end{aligned}
\tag{14}
$$

A pole is crossed which is not near the saddle point:

The pole is circumvented by an indentation of Pw; the first-order pole of R gives a “pole contribution”, which must be added to the above result.



The pole contribution is

$$
\begin{aligned}
p_{rp}(r', \theta_s) &= \frac{1}{\pi} \oint_{\varphi_p} R(\varphi + \theta_s) \cdot e^{-j\,kr' \cdot \cos\varphi}\, d\varphi \\
&= 2j \cdot \operatorname{Res}\left(R(\varphi + \theta_s) \cdot e^{-j\,kr' \cdot \cos\varphi}\right)_{\varphi = \varphi_p} ,
\end{aligned}
\tag{15}
$$

where φ_p is the position of the pole in the complex φ and $\operatorname{Res}(f(z))$ is the residue of $f(z)$ at a pole of $f(z)$.

For locally reacting absorbers

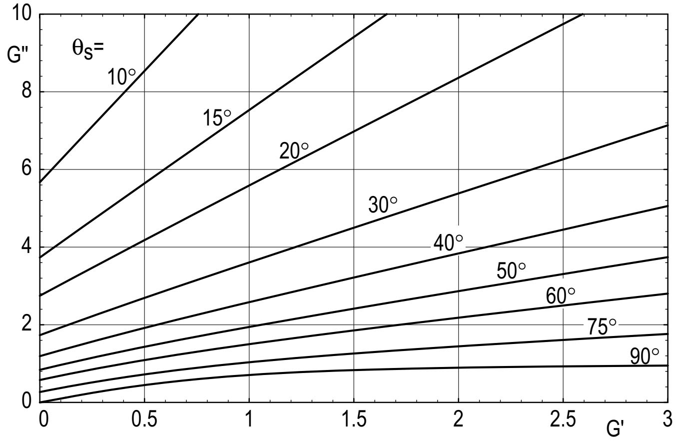
$$
\begin{aligned}
p_{rp}(r', \theta_s) &= \frac{4j\,G}{\sqrt{1-G^2}} \cdot e^{-j\,kr' \cdot \cos\varphi_p} \\
&= \frac{4j\,G}{\sqrt{1-G^2}} \cdot e^{+j\,kG \cdot r' \cdot \cos\theta_s} \cdot e^{-j\,k\sqrt{1-G^2} \cdot r' \cdot \sin\theta_s} \\
&= \frac{4j\,G}{\sqrt{1-G^2}} \cdot e^{+j\,kG \cdot h} \cdot e^{+j\,kG \cdot |y|} \cdot e^{-j\,k\sqrt{1-G^2} \cdot x} .
\end{aligned}
\tag{16}
$$

Condition for pole crossing (if $\leq$ holds):

$$G'' \leq \frac{1}{\sin\theta_s}\frac{(G'+\cos\theta_s)(G'\cdot\cos\theta_s+1)}{\sqrt{(G'+\cos\theta_s)^2+\sin^2\theta_s}} = \frac{1}{\sin\theta_s}\frac{(G'+\cos\theta_s)(G'\cdot\cos\theta_s+1)}{\sqrt{1+2G'\cos\theta_s+G'^2}}\ ,$$

$$\xrightarrow[\theta_s\to\pi/2]{} \leq \frac{G'}{\sqrt{1+G'^2}} \xrightarrow[G'\to 0]{} 0\ , \tag{17}$$

$$\xrightarrow[\theta_s\to 0]{} \leq \frac{1}{\sin\theta_s} \to \infty\ .$$



The diagram shows limits for pole crossing in the complex plane of $G = G' + j\cdot G''$ with θ_s as parameter for a locally reacting absorber. Pole contributions are below the curves (their magnitude, however, may make them negligible)

A uniform pass integration:
It can be used also if the pole is near the saddle point; however, the simple pass integration above is preferable if the pole is not near the saddle point.

Let the integral to be computed along the pass way Pw be of the following form, with real $x > 1$:

$$I = \int_{Pw} e^{x\cdot f(\varphi)}\cdot F(\varphi)d\varphi\ . \tag{18}$$

It can be evaluated by

$$\begin{aligned} I &= e^{x\cdot f(\varphi_s)}\cdot\left[\pm j\pi a\cdot W\left(\pm b\sqrt{x}\right)+\sqrt{\frac{\pi}{x}}\cdot T\right] \quad ; \quad \mathrm{Im}\{b\}\gtrless 0\ ,\\ &\quad e^{x\cdot f(\varphi_s)}\cdot\left[j\pi a\cdot W\left(b\sqrt{x}\right)+\sqrt{\frac{\pi}{x}}\cdot T - j\pi a\cdot e^{-x\cdot b^2}\right] \quad ; \quad \left\{\begin{array}{l}\mathrm{Im}\{b\} = \ \ 0\\ \mathrm{Re}\{b\}\neq\ \ 0\end{array}\right.\ ,\\ &\quad e^{x\cdot f(\varphi_s)}\cdot\sqrt{\frac{\pi}{x}}\cdot T \quad ; \quad \mathrm{Im}\{b\}=0\ , \end{aligned} \tag{19}$$

with the following definitions (φ_s = value of φ at saddle point):

$$
\begin{aligned}
&a =: \lim_{\varphi\to\varphi_p} \left[(\varphi-\varphi_p)\cdot F(\varphi)\right] = N(\varphi_p) \bigg/ \left.\frac{dD}{d\varphi}\right|_{\varphi=\varphi_p} , \\
&b =: \sqrt{f(\varphi_s)-f(\varphi_p)} \quad ; \quad b \xrightarrow[\varphi_p\to\varphi_s]{} \frac{\varphi_p-\varphi_s}{\gamma} , \\
&\gamma =: \sqrt{-2/f''(\varphi_s)} \quad ; \quad \arg(\gamma) = \left(\arg(d\varphi)\right)_{\varphi_s} \quad ; \quad \varphi \quad \text{along} \quad Pw , \qquad (20) \\
&T =: \gamma\cdot F(\varphi_s) + \frac{a}{b} , \\
&W(u) =: e^{-u^2}\cdot \operatorname{erfc}(-j\,u) \quad ; \quad \operatorname{erfc}(z) =: \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-y^2}\,dy .
\end{aligned}
$$

It is supposed that $F(\varphi) = N(\varphi)/D(\varphi)$ can be written as the quotient of a numerator and denominator; thus a is the residue of $F(\varphi)$. The quantity b distinguishes cases of the relative position φ_p of the pole to the pass way Pw or to the saddle point φ_s. For $\mathrm{Im}\{b\} > 0$ the pole is still outside Pw; for $\mathrm{Im}\{b\} < 0$ it has been crossed by $C(\varphi) \to$ Pw, and $\mathrm{Im}\{b\} = 0$ describes the situation where φ_p is on Pw. The addendum in the definition of b defines the sign of the root in b. The addendum in the definition of γ also serves to select the sign of the root; it demands that the argument of γ should agree with the argument of a step $d\varphi$ from the saddle point φ_s in the direction of the pass way. The function $W(u)$ is based on the complementary error function erfc(z).

The correspondences to the present integral along the pass way Pw as defined above are

$$
x \Rightarrow kr' \quad ; \quad f(\varphi) \Rightarrow -j\cos\varphi \quad ; \quad F(\varphi) \Rightarrow R(\varphi+\theta_s) = \frac{\cos(\varphi+\theta_s)-G}{\cos(\varphi+\theta_s)+G} , \qquad (21)
$$

and the required quantities are

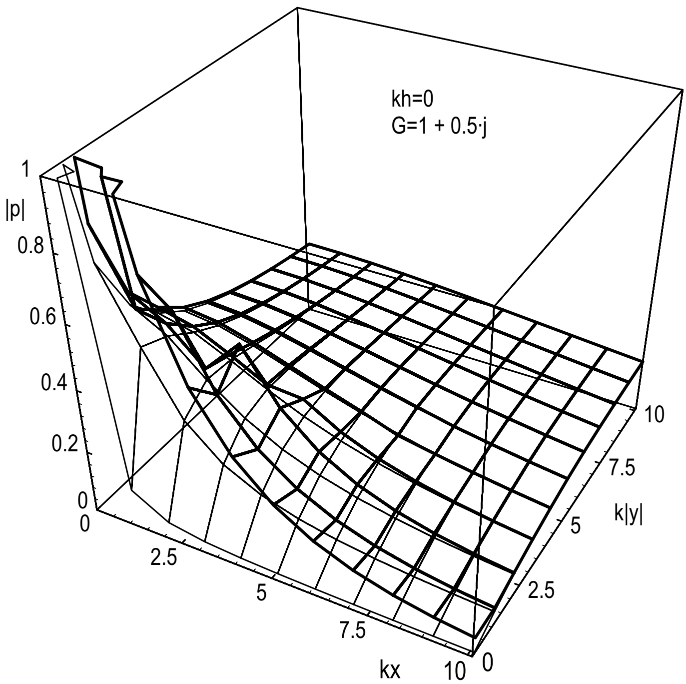
$$
\begin{aligned}
&\varphi_s = 0 \quad ; \quad \varphi_p = \arccos(-G) - \theta_s , \\
&\cos(\varphi_p+\theta_s) = -G , \qquad (22) \\
&\cos\varphi_p = -G\cdot\cos\theta_s + \sqrt{1-G^2}\cdot\sin\theta_s \quad ; \quad \mathrm{Im}\left\{\sqrt{1-G^2}\right\} \le 0 .
\end{aligned}
$$

Other quantities in the above definitions are for a locally reacting absorber:

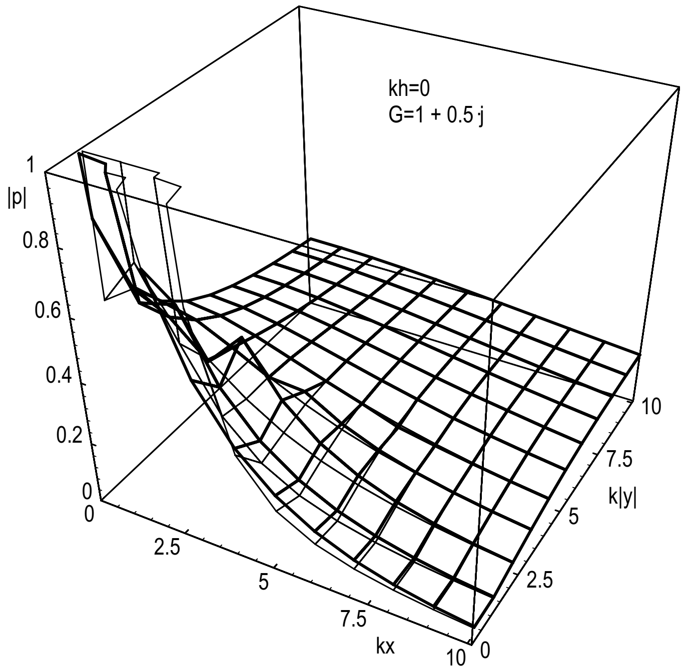
$$
\begin{aligned}
a &= -\frac{\cos(\varphi_p + \theta_s) - G}{\sin(\varphi_p + \theta_s) - G} = \frac{2\,G}{\sqrt{1-G^2}}\,, \\
b &= \pm\sqrt{j}\sqrt{\cos\varphi_p - 1} \xrightarrow[\varphi_p \to \varphi_s = 0]{} \mp\frac{1-j}{2}\varphi_s \\
&= \pm(j)^{3/2}\sqrt{1 + G\cdot\cos\theta_s - \sqrt{1-G^2}\cdot\sin\theta_s}\quad, \\
\gamma &= \sqrt{2\,j}\,, \\
T &= \sqrt{2\,j}\cdot R(\theta_s) \mp \frac{2\sqrt{j}\,G}{\sqrt{1-G^2}}\frac{1}{\sqrt{1 + G\cdot\cos\theta_s - \sqrt{1-G^2}\cdot\sin\theta_s}}\,.
\end{aligned}
\tag{23}
$$

The sign convention for γ is satisfied; the sign convention in b requires the lower signs in b and T if the last root in b, T is evaluated with a positive real part. The desired field $p_r = I/\pi$ can be evaluated by insertion.

The diagrams below compare in 3D plots (as “wire graphics”) the magnitude of the sound pressure |p| over kx, ky from numerical integration of the exact integral (thick lines) with results from approximate methods (to which the pass integration belongs).



Comparison of numerical integration of exact integral (thick lines)with the mirror source approximation (thin lines)

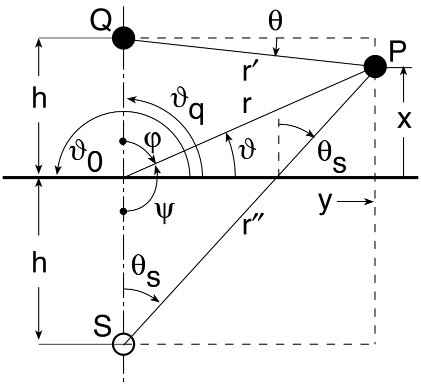


Comparison of numerical integration of exact integral (thick lines) with the simple pass integration (thin lines)

D.15 A Monopole Line Source Above an Infinite, Plane Absorber; with Principle of Superposition

► *See also:* Mechel, Modified Mirror and Corner Sources with a Principle of Superposition (2000)

A monopole line source with volume flow q (per unit length) is placed at Q with a height h above a locally reacting plane with normalised surface admittance G. S is the mirror-reflected point to Q, and P is a field point.



The sound field is formulated as $p_a(\mathbf{r}) = A \cdot p_h(\mathbf{r}) + B \cdot p_s(\mathbf{r})$, with p_h the field above a hard plane and p_s the field above a soft plane, both satisfying individually the source condition.

The principle of hard-soft superposition (third principle of superposition in *Sect. B.10*) gives

$$p_a(\mathbf{r}) = \frac{1}{1 + G \cdot X(\mathbf{s})} \cdot \left[p_h(\mathbf{r}) + G \cdot X(\mathbf{s}) \cdot p_s(\mathbf{r})\right] \,, \tag{1}$$

with $\mathbf{s}$ the projection of the field point P (along co-ordinate lines) on the plane, and the "cross impedance"

$$X(\mathbf{s}) = \frac{p_h(\mathbf{s})}{Z_0 v_{sn}(\mathbf{s})} = -\frac{jk_0 \cdot p_h(\mathbf{s})}{\mathrm{grad}_n p_s(\mathbf{s})} \,. \tag{2}$$

In the present task is $p_h = p_Q + p_{Sh}$; $p_w = p_Q + p_{Sw}$,

with p_Q the free source field:

$$p_Q = P_0 \cdot H_0^{(2)}(k_0 r') = \frac{k_0 Z_0 \cdot q}{4} \cdot H_0^{(2)}(k_0 r') = \frac{p_Q(k_0 h)}{H_0^{(2)}(k_0 h)} \cdot H_0^{(2)}(k_0 r') \,, \tag{3}$$

(the second form replaces the amplitude P_0 by the source volume flow q; the third form describes the source strength by the free field sound pressure $p_Q(h)$ at the origin), and $p_h = p_Q + p_{Sh}$; $p_s = p_Q + p_{Ss}$, where p_{Sh}, p_{Ss} are the fields from the mirror sources in the case of a hard or soft plane, respectively, which for "ideal" reflection are exact forms of the scattered field:

$$p_{Sh} = P_0 \cdot H_0^{(2)}(k_0 r''); \quad p_{Ss} = -P_0 \cdot H_0^{(2)}(k_0 r'') \,. \tag{4}$$

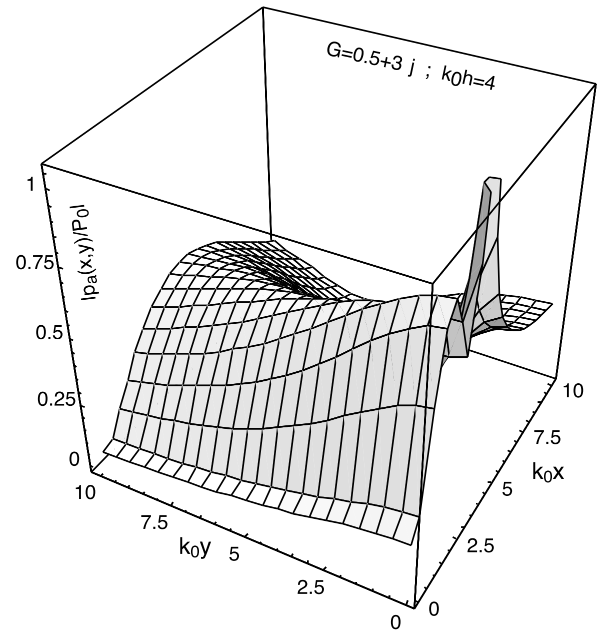
One gets for the cross impedance X(y)

$$\begin{aligned} \frac{1}{X(y)} &= \frac{-Z_0 v_{Qsx}}{p_{Qh}} = -j \cdot \frac{k_0 r_q}{2} \cdot \left(1 + \frac{H_2^{(2)}(k_0 r')}{H_0^{(2)}(k_0 r')}\right) \\ &= -j \cdot \frac{k_0 r_q}{2} \cdot \left(1 + \frac{H_2^{(2)}\left(\sqrt{(k_0 y)^2 + (k_0 r_q)^2}\right)}{H_0^{(2)}\left(\sqrt{(k_0 y)^2 + (k_0 r_q)^2}\right)}\right) \,. \end{aligned} \tag{5}$$

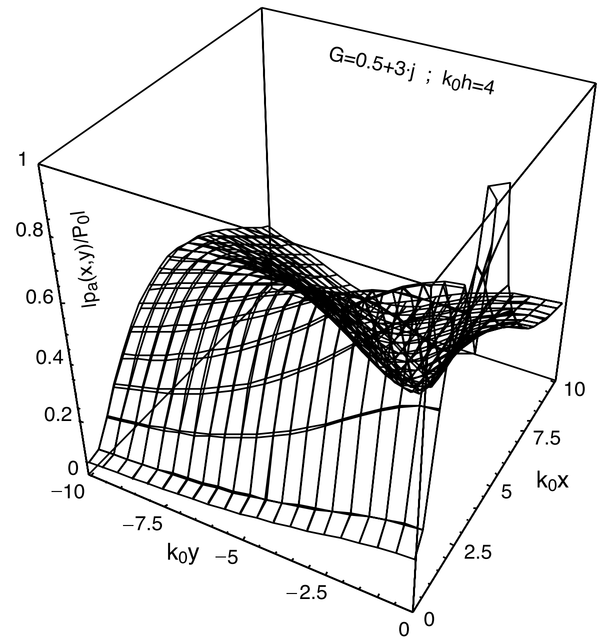
With $p_{Sw} = -p_{Sh}$ one can simplify to

$$\begin{aligned} p_a(x, y) &= p_Q(x, y) + \frac{1 - G \cdot X(y)}{1 + G \cdot X(y)} \cdot p_{Sh}(x, y) \\ &= p_Q(x, y) + \frac{1/X(y) - G}{1/X(y) + G} \cdot p_{Sh}(x, y) \,. \end{aligned} \tag{6}$$

Numerical comparison with saddle point integration (see *Sect. D.14*):



Sound pressure magnitude from a line source above an absorbing plane (on x axis), evaluated using the principle of superposition



This diagram compares the above diagram (in a 3D wire plot) with results from the method of saddle point integration

D.16 A Monopole Point Source Above a Bulk Reacting Plane, Exact Forms

► *See also:* Mechel, Vol. I, Ch. 13 (1989)

A monopole point source is placed at a point Q at height h above an absorbent plane; a field point is at P. The plane may be bulk reacting. See *Sect. D.17* for a locally reacting plane. See Mechel (1989) for references to the extensive literature about this problem. As an exception, the time factor in this section is $e^{-i\,\omega t}$ in order to facilitate the comparison with the literature, where this sign convention mostly is used.

The free field of the point source is $$p_Q(r_1) = P_0 \frac{e^{i\,k_0 r_1}}{k_0 r_1} \,, \tag{1}$$

the field p above the absorber is $$\frac{p}{P_0} = \frac{e^{i\,k_0 r_1}}{k_0 r_1} + \frac{p_r}{P_0} \,, \tag{2}$$

with $r_1 = \mathrm{dist}(Q, P)$ and p_r the reflected field. The task is to find p_r.

An exact integral expression for p_r is

$$\frac{p_r}{P_0} = i \int_0^{\pi/2 - i\infty} J_0(k_0\, r \cdot \sin\Theta) \cdot e^{i\,k_0(z+h)\cdot\cos\Theta} \cdot R(\Theta) \cdot \sin\Theta \; d\Theta \,, \tag{3}$$

where $R(\Theta) = (\cos\Theta - G)/(\cos\Theta + G)$ is the reflection factor of a plane wave incident under a polar angle Θ, z is the co-ordinate normal to the plane directed into the half-space above the plane, r is the radius of P from the foot point of Q on the plane, and $J_0(z)$ is the Bessel function of zero order. The path of integration in the complex Θ plane is

$0 \to \pi/2 \to \pi/2 - i \cdot \infty$.

If the absorber is a half-space (indicated with index $ß = 2$, in contrast to index $ß = 1$ for half-space above the absorber) of a homogeneous, isotropic material, the characteristic wave numbers and wave impedances in both half-spaces are $k_ß$, $Z_ß$, respectively, and the ratios $k = k_2/k_1$, $Z = Z_2/Z_1$. An exact formulation (Sommerfeld) of the field in the upper half-space $ß = 1$ is

$$\frac{p_1(r, z)}{P_0} = (1 + kZ) \int_0^{\infty} \frac{y \cdot J_0(y\,k_1 r) \cdot e^{-k_1 z \sqrt{y^2 - 1}}}{\sqrt{y^2 - k^2} + kZ\sqrt{y^2 - 1}} \, dy \,. \tag{4}$$

This integral is used for numerical integration (as a reference for approximations), for which the interval of integration is subdivided into $(0,\infty) = (0,1)+(1,2)+(2,y_{hi})+(y_{hi},\infty)$, and the precision and convergence are checked separately in each subinterval.

In the case of two half-spaces, the integral above for p_r/p_0 can be transformed into (a form, which is suited for saddle point integration)

$$\frac{p_r(r, \vartheta)}{P_0} = \frac{i}{2} \int_C H_0^{(1)}(k_1 r \cdot \sin\vartheta) \cdot e^{i\,k_1 H\cdot\cos\vartheta} \cdot R(\vartheta) \cdot \sin\vartheta \; d\vartheta \,, \tag{5}$$

with the reflection factor

$$R(\vartheta) = \frac{kZ\cos\vartheta - \sqrt{k^2 - \sin^2\vartheta}}{kZ\ \cos\ \vartheta + \sqrt{k^2 - \sin^2\vartheta}}\,. \tag{6}$$

This form can be applied also for bulk reacting layers of finite thickness if a corresponding reflection factor is used. The path of integration is

$$C = -\pi/2 + i\cdot\infty \quad \to -\pi/2 \to +\pi/2 \to +\pi/2 - i\cdot\infty\,.$$

The cross-over from the positive bank of $\mathrm{Re}\{\vartheta\}$ to the negative bank is at $\mathrm{Re}\{\vartheta\} = 0$.

Further exact forms (Butov) for the reflected field above a homogeneous half-space and the field in the lower half-space are

$$\begin{aligned}
\frac{p_r}{P_0} &= \frac{i}{2\pi k_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k_{2z} - kZ\cdot k_{1z}}{k_{2z} + kZ\cdot k_{1z}}\cdot e^{i\,(k_x x + k_y y)}\,\frac{e^{i\,k_{1z}|z+h|}}{k_{1z}}\ dk_x\ dk_y\,;\\
\frac{p_2}{P_0} &= \frac{i}{2\pi k_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2kZ}{k_{2z} + kZ\cdot k_{1z}}\cdot e^{i\,(k_x x + k_y y)}\,e^{i\,k_{1z}h}\,e^{-i\,k_{2z}z}\ dk_x\ dk_y\,;
\end{aligned} \tag{7}$$

with wave number components $k_ßx$, $k_ßy$, $k_ßz$ of $k_ß$. The first line can be transformed into

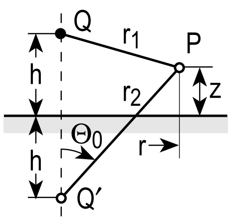
$$\frac{p_r}{P_0} = i\sum_{n=0}^{\infty}(-1)^n(4n+1)\cdot V_{2n}\cdot h_{2n}^{(1)}(k_1 r_2)\cdot P_{2n}(\cos\vartheta)\,, \tag{8}$$

with r_2 = dist (mirror point of Q, P); $P_{2n}(z)$ = Legendre polynomial; $h_{2n}^{(1)}(z)$ = spherical Hankel function of the first kind; and

$$V_{2n} = \frac{1}{2}\int_{-1}^{1} V(x)\cdot P_{2n}(x)\ dx \quad ; \quad V(x) = \frac{kz\cdot x - \sqrt{k^2 - 1 + x^2}}{kz\cdot x + \sqrt{k^2 - 1 + x^2}}\,. \tag{9}$$

Although this form is elegant, it is not suited for numerical evaluations because of problems of convergence caused by the spherical Hankel functions.

Another exact form for p_1 (Brekhovskikh) above a homogeneous absorber half-space is



$$\frac{p_1(r,z;h)}{P_0}=\frac{e^{i\,k_1 r_1}}{k_1 r_1}-\frac{e^{i\,k_1 r_2}}{k_1 r_2}+\frac{p_1(r,z+h;0)}{P_0}\ ,$$
$$\frac{p_1(r,z+h;0)}{P_0}=2\,(1+kZ)\int_0^{\infty} J_0(y\cdot k_1 r)\frac{y\cdot e^{-k_1 H\sqrt{y^2-1}}}{\sqrt{y^2-k^2}+kZ\sqrt{y^2-1}}\,dy\ , \tag{10}$$

or with $\gamma = y \cdot k_1$:

$$\frac{p_1(r,z+h;0)}{P_0}=\frac{2\,(1+kZ)}{k_1}\int_0^{\infty} J_0(\gamma r)\frac{\gamma\cdot e^{-H\sqrt{\gamma^2-k_1^2}}}{\sqrt{\gamma^2-k_2^2}+kZ\sqrt{\gamma^2-k_1^2}}\,d\gamma\ , \tag{11}$$

with $r_1 = \text{dist}(Q,p)$; $r_2 = \text{dist}$ (mirror point of Q, P); $H = h + z$ = sum of heights of P and Q. The inclusion of the source height h in $p_1(r,z;h)$ indicates Brekhovskikh's rule: if one subtracts from the source-free field the mirror source field, the remaining scattering term depends only on the sum of source and receiver heights. The second form can be further modified to a form which is suited for saddle point integration:

$$\frac{p_1(r,H;0)}{P_0}=\frac{2\,(1+kZ)}{k_1}\int_{-\infty}^{\infty} \frac{\gamma\cdot e^{-H\sqrt{\gamma^2-k_1^2}+i\gamma\, r}}{\sqrt{\gamma^2-k_2^2}+kZ\sqrt{\gamma^2-k_1^2}}\cdot H_0^{(1)}(\gamma r)\cdot e^{-\gamma\, r}\,d\gamma\ . \tag{12}$$

The path of integration is parallel to $\text{Re}\{\gamma\}$ with a small distance above this axis for $\text{Re}\{\gamma\} < 0$ and a small distance below it for $\text{Re}\{\gamma\} > 0$.

The exact form of Van der Pol for two half-spaces is

$$\frac{p_1(r,z)}{P_0}=\frac{e^{i\,k_1 r_1}}{k_1 r_1}+\frac{e^{i\,k_1 r_2}}{k_1 r_2}-\frac{1}{\pi k_1}\iiint_{V_2}\frac{\partial^2}{\partial z^2}\left(\frac{e^{i\,k_2 r_1}}{r_1}\right)\frac{e^{i\,k_1 r_2}}{r_2}\cdot r_0\,dr_0\,dz\,d\varphi\ , \tag{13}$$

with integration over the half-space V_2 below the plane which contains the mirror-reflected point to Q, and r_0, φ determined from

$$r_1^2=r_0^2+z^2\quad ;\quad r_2^2=r^2-2\,r\,r_0\cos\varphi+r_0^2+(H+kZ\cdot z)^2\ . \tag{14}$$

D.17 A Monopole Point Source Above a Locally Reacting Plane, Exact Forms

▸ *See also:* Mechel, Vol. I, Ch. 13 (1989); Ochmann (2004)

A monopole point source is placed at a point Q at height h above an absorbent plane; a field point is at P. The plane is locally reacting with a normalised admittance $G = 1/Z$. See *Sect. D.16* for a bulk reacting plane; some of the forms for the field above the absorbent plane in that section can be used also for a locally reacting plane, if such forms apply the reflection factor R of a plane wave at the plane. See Mechel (1989) for references to the extensive literature about this problem. As an exception, the time factor in this section is $e^{-i\omega t}$ in order to facilitate the comparison with the literature, where this sign convention mostly is used.

The free field of the point source is $$p_Q(r_1) = P_0 \frac{e^{i k_0 r_1}}{k_0 r_1} \,, \tag{1}$$

the field p above the absorber is $$\frac{p}{P_0} = \frac{e^{i k_0 r_1}}{k_0 r_1} + \frac{p_r}{P_0} \,, \tag{2}$$

with r_1=dist (Q,P) and p_r the reflected field. The task is to find p_r.

The reflection factor of a plane wave incident under a polar angle Θ is (with Z=1/G)

$$R(\Theta) = (\cos\Theta - G)/(\cos\Theta + G) = (Z \cdot \cos\Theta - 1)/(Z \cdot \cos\Theta + 1) \,. \tag{3}$$

An exact form of p_r is:

$$\frac{p_r}{P_0} = i \int_0^{\pi/2 - i\infty} J_0(k_0 r \cdot \sin\vartheta) \cdot e^{i k_0 H \cdot \cos\vartheta} \cdot R(\vartheta) \cdot \sin\vartheta \, d\vartheta \,. \tag{4}$$

$H = z + h$, z is the co-ordinate normal to the plane directed into the half-space above the plane, r is the radius to P from the foot point of Q on the plane, and $J_0(z)$ is the Bessel function of zero order. The path of integration in the complex ϑ plane is: $0 \to \pi/2 \to \pi/2 - i \cdot \infty$. Decomposition into real and imaginary parts with $\vartheta = \vartheta' + i \cdot \vartheta''$ of

$$\begin{aligned} \sin\vartheta &= \sin\vartheta' \cdot \cosh\vartheta'' + i \cdot \cos\vartheta' \cdot \sinh\vartheta'' \,, \\ \cos\vartheta &= \cos\vartheta' \cdot \cosh\vartheta'' - i \cdot \sin\vartheta' \cdot \sinh\vartheta'' \,, \end{aligned} \tag{5}$$

gives

$$\begin{aligned} \frac{p_r}{P_0} = & \, i \int_0^1 J_0\left(k_0 r \cdot \sqrt{1 - y^2}\right) \cdot e^{i k_0 H \cdot y} \frac{Z \cdot y - i}{Z \cdot y + i} \, dy \\ & + \int_0^\infty J_0\left(k_0 r \cdot \sqrt{1 + y^2}\right) \cdot e^{-k_0 H \cdot y} \frac{Z \cdot y + i}{Z \cdot y - i} \, dy \,. \end{aligned} \tag{6}$$

Replacement of the Bessel function with the Hankel function yields

$$\frac{p_r}{P_0} = \frac{i}{2} \int_C H_0^{(1)}(k_0 r \cdot \sin\vartheta) \cdot e^{i k_0 H \cdot \cos\vartheta} \cdot R(\vartheta) \cdot \sin\vartheta \, d\vartheta \,, \tag{7}$$

with the path of integration $C = -\pi/2 + i \cdot \infty \to -\pi/2 \to +\pi/2 \to +\pi/2 - i \cdot \infty$.

A different exact form is

$$\frac{p_r}{P_0} = \int_0^\infty J_0(k_0 r \cdot y) \cdot e^{-k_0 H \cdot \sqrt{y^2 - 1}} \frac{Z \cdot \sqrt{y^2 - 1} + i}{Z \cdot \sqrt{y^2 - 1} - i} \frac{y}{\sqrt{y^2 - 1}} \, dy \,, \tag{8}$$

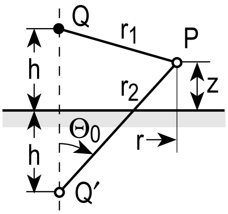
with a negative imaginary root in $0 \leqslant y < 1$.

With a field composition

$$\frac{p(r,z;h)}{P_0}=\frac{e^{i k_0 r_1}}{k_0 r_1}-\frac{e^{i k_0 r_2}}{k_0 r_2}+\frac{p_1(r,z+h;0)}{P_0} \tag{9}$$

one gets ($H = z + h$)

$$\frac{p_1(r,H;0)}{P_0}=2Z\int_0^\infty J_0(y\cdot k_0 r)\frac{y\cdot e^{-k_0 H\sqrt{y^2-1}}}{Z\sqrt{y^2-1}-i}\,dy\,. \tag{10}$$



Butov's form for bulk reacting absorbers can be transformed so that it can be applied to locally reacting absorbers:

$$\frac{p_r}{P_0}=i\sum_{n=0}^{\infty}(-1)^n(4n+1)\cdot V_{2n}\cdot h_{2n}^{(1)}(k_1 r_2)\cdot P_{2n}(\cos\vartheta)\,, \tag{11}$$

with $h_{2n}^{(1)}(z)$ spherical Hankel functions of the first kind, $P_{2n}(z)$ are Legendre polynomials, r_2 = dist (mirror point of Q, P), and

$$V_{2n=0}=\frac{1}{2Z}\int_{-Z}^{Z}\frac{y-1}{y+1}\,dy=1-\frac{1}{Z}\ln\frac{1+Z}{1-Z}\,, \tag{12}$$

$$\begin{aligned}V_{2n>0}&=\frac{-2}{Z}Q_{2n}(1/Z)\\&=\frac{-1}{Z}\left[P_{2n}(1/Z)\cdot\ln\frac{1+Z}{1-Z}-4\sum_{m=0}^{[n-1/2]}\frac{2(n-m)-1}{(2m+1)(2n-m)}P_{2(n-m)-1}(1/Z)\right]\,,\end{aligned} \tag{13}$$

with $Q_{2n}(z)$ Legendre polynomials of the second kind and $[n-1/2]$ the highest integer $\leq (n-1/2)$.

All integrals of the exact forms have oscillating integrands, and the interval of integration extends to infinity. If numerical integration is applied, the convergence must be improved. This is done according to the scheme

$$I=\int_y^\infty f(x)\,dx=\int_y^\infty f_\infty(x)\,dx+\int_y^\infty (f(x)-f_\infty(x))\,dx\,, \tag{14}$$

where $f_\infty(x)$ is an asymptotic approximation to $f(x)$, and the analytical integral over $f_\infty(x)$ is known (it is dangerous to apply an approximation to that integral). In $(f(x) - f_\infty(x))$ the oscillations at large x are reduced.

An integral solution with favourable numerical behaviour of the numerical integral (due to a fast convergence of the integrand) was recently published by Ochmann (2004). The structure of the solution is [cf. (9)]

$$\frac{p(r,z;h)}{P_0} = \frac{e^{i k_0 r_1}}{k_0 r_1} + \frac{e^{i k_0 r_2}}{k_0 r_2} - \frac{p_s(r,z+h;0)}{P_0}\,, \tag{15}$$

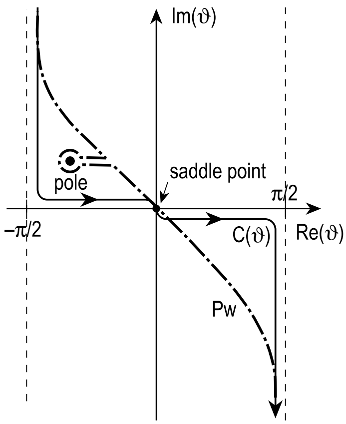
$$\begin{aligned}\frac{p(r,z;h)}{P_0} &= \frac{e^{i\,2\pi\sqrt{(r/\lambda_0)^2+(z/\lambda_0-h/\lambda_0)^2}}}{2\pi\sqrt{(r/\lambda_0)^2+(z/\lambda_0-h/\lambda_0)^2}} + \frac{e^{i\,2\pi\sqrt{(r/\lambda_0)^2+(z/\lambda_0+h/\lambda_0)^2}}}{2\pi\sqrt{(r/\lambda_0)^2+(z/\lambda_0+h/\lambda_0)^2}} \\ &\quad -2G\int_0^\infty e^{-2\pi G\sigma}\,\frac{e^{i\,2\pi\sqrt{(r/\lambda_0)^2+((z/\lambda_0)+(h/\lambda_0)+i\,\sigma)^2}}}{\sqrt{(r/\lambda_0)^2+((z/\lambda_0)+(h/\lambda_0)+i\,\sigma)^2}}\,d\sigma\,.\end{aligned} \tag{16}$$

This solution does not need explicit additional surface wave terms for spring-type reactive surfaces (the contributions of the surface wave are contained implicitly in the integral).

D.18 A Monopole Point Source Above a Locally Reacting Plane, Exact Saddle Point Integration

► *See also:* Mechel, Vol. I, Ch. 13 (1989)

The method of saddle point integration mostly is considered as an approximate method to evaluate an integral which satisfies some criteria. In the present task, the saddle point integration can be applied so that it is an exact transformation of the integral, which makes it suited to numerical integration with high precision. If an integral as it appears in the present problem can be cast exactly as an integral over the path of steepest descent, it has the best possible form for a precise numerical evaluation.



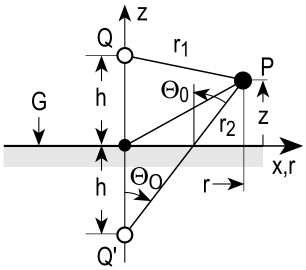
Suppose the integral to be evaluated is of the form

$$I = \int_C e^{a \cdot f(\vartheta)} \cdot F(\vartheta)\, d\vartheta \ . \tag{1}$$

This integral is suited for saddle point integration if $a \gg 1$ is a large real number and the path C goes to infinity on both sides.

The saddle point integration in most of its applications is an approximation because $f(\vartheta)$ is approximated as $f(\vartheta) \approx \alpha + \beta \cdot s^2$ with a real variable s on the pass way and, more seriously, $F(\vartheta)$ is expanded as a power series. If a pole is near the saddle point, the radius of convergence becomes small and the precision goes down.

The start integral I_0 in our problem comes from the third integral of *Sect. D.17*, after multiplication and division of the integrand by $\exp(\pm ik_0 r \cdot \sin\vartheta)$:



$$\frac{p_r}{P_0} = \frac{i}{2} \int_C e^{i\, k_0 r_2 \cdot \cos(\vartheta - \Theta_o)} \cdot \left[H_0^{(1)}(k_0 r \cdot \sin\vartheta) \cdot e^{-i\, k_0 r \cdot \sin\vartheta} \right] \cdot R(\vartheta) \cdot \sin\vartheta\, d\vartheta = \frac{i}{2} \cdot I_0 \ , \tag{2}$$

with the geometrical quantities as in the sketch and making use of

$$k_0\,((h+z)\cos\vartheta + r\sin\vartheta) = k_0 r_2 \cdot \cos(\vartheta - \Theta_0) \ . \tag{3}$$

The integration path $C(\varphi)$ and the path of steepest descent (pass way Pw) are shown above in the sketch in the complex plane of $\varphi = \vartheta - \Theta_0$. If during the deformation $C(\varphi) \to$ Pw a pole of the reflection factor $R(\vartheta)$ is crossed, it is encircled as shown. This extra circle will give a "pole contribution".

The oscillations of the term in brackets in the integral go to zero for large argument values because the Hankel function oscillations are compensated by the exponential factor.

Comparing I_0 with the general integral I, correspondences are $a \to k_0 r_2$; $f(\vartheta) \to i \cdot \cos\varphi$. The saddle point ϑ_s with the maximum exponential factor (outside the brackets) follows from $df(\vartheta)/d\vartheta = 0$, which in our case is $\varphi_s = 0$, i.e. $\vartheta_s = \Theta_0$. The parameter form of the pass way equation is (with $\vartheta = \vartheta' + i \cdot \vartheta''$; values of ϑ on Pw are called ϑ_{Pw})

$$\cos(\vartheta'_{Pw} - \Theta_0) \cdot \cosh\vartheta''_{Pw} = 1 \ ,$$

$$\cos(\vartheta'_{Pw} - \Theta_0) = \frac{1}{\cosh\vartheta''_{Pw}} \ , \tag{4}$$

$$\sin(\vartheta'_{Pw} - \Theta_0) = -\tanh\vartheta''_{Pw} \ ,$$

or, equivalently,

$$\begin{aligned}\sin\vartheta'_{Pw} &= \frac{\sin\Theta_0 - \cos\Theta_0 \cdot \sinh\vartheta''_{Pw}}{\cosh\vartheta''_{Pw}}\ ,\\ \cos\vartheta'_{Pw} &= \frac{\cos\Theta_0 + \sin\Theta_0 \cdot \sinh\vartheta''_{Pw}}{\cosh\vartheta''_{Pw}}\ ,\end{aligned} \tag{5}$$

and therefore the function $f(\vartheta)$ in the exponent can be expressed as follows:

$$f(\vartheta_{Pw}) = -\tanh\vartheta''_{Pw} \cdot \sinh\vartheta''_{Pw} + i \xrightarrow[|\vartheta''_{Pw}|\ll 1]{} -(\vartheta''_{Pw})^2 + i\ . \tag{6}$$

All factors in the integrand of I_0 can be expressed as functions of ϑ''_{Pw}, especially

$$R(\vartheta_{Pw}) = R\langle\vartheta''_{Pw}\rangle = \frac{Z\cdot\cos\langle\vartheta''_{Pw}\rangle - 1}{Z\cdot\cos\langle\vartheta''_{Pw}\rangle + 1}\ , \tag{7}$$

with the definitions:

$$\begin{aligned}\cos\langle\vartheta''_{Pw}\rangle &= \cos\Theta_0 + \sin\Theta_0 \cdot \sinh\vartheta''_{Pw}\\ &\quad -i\cdot\tanh\vartheta''_{Pw}\cdot\left(\sin\Theta_0 - \cos\Theta_0\cdot\sinh\vartheta''_{Pw}\right)\ ,\\ \sin\langle\vartheta''_{Pw}\rangle &= \sin\Theta_0 - \cos\Theta_0 \cdot \sinh\ \vartheta''_{Pw}\\ &\quad +i\cdot\tanh\vartheta''_{Pw}\cdot\left(\cos\Theta_0 + \sin\Theta_0\cdot\sinh\vartheta''_{Pw}\right)\ .\end{aligned} \tag{8}$$

With the transition $\vartheta \to \vartheta''_{Pw}$ the general integral I is transformed into

$$I = \int_{+\infty}^{-\infty} e^{a\cdot g(\vartheta''_{Pw})}\cdot G(\vartheta''_{Pw})\, d\vartheta''_{Pw} \quad ; \quad \begin{cases} g(\vartheta''_{Pw}) = i - \tanh\vartheta''_{Pw}\cdot\sinh\vartheta''_{Pw}\ ,\\ G(\vartheta''_{Pw}) = F\left(\vartheta(\vartheta''_{Pw})\right)\dfrac{d\vartheta}{d\vartheta''_{Pw}}\ ,\\ \qquad\qquad = F\left(\vartheta(\vartheta''_{Pw})\right)\dfrac{dg(\vartheta''_{Pw})/d\vartheta''_{Pw}}{df(\vartheta)/d\vartheta}\ .\end{cases} \tag{9}$$

The last fraction becomes

$$\frac{dg(\vartheta''_{Pw})/d\vartheta''_{Pw}}{df(\vartheta)/d\vartheta} = -\frac{\sinh\vartheta''_{Pw}\cdot(2-\tanh^2\vartheta''_{Pw})}{\tanh\vartheta''_{Pw} + i\cdot\sinh\vartheta''_{Pw}} = -\frac{2-\tanh^2\vartheta''_{Pw}}{i + 1/\cosh\vartheta''_{Pw}}\ . \tag{10}$$

The desired integral I_0 finally is (substitute for ease of writing $\vartheta''_{Pw} \to t$)

$$\begin{aligned} I_0 = e^{i k_0 r_2}\int_0^{\infty} & e^{-k_0 r_2\cdot\tanh\ t\cdot\sinh\ t}\,\frac{2-\tanh^2 t}{i + 1/\cosh\ t}\\ &\cdot\Big[R\langle t\rangle\cdot\sin\langle t\rangle\cdot H_0^{(1)}\left(k_0 r\sin\langle t\rangle\right)\cdot e^{-i k_0 r \sin\langle t\rangle} \qquad {}^{*)}\\ &+ R\langle -t\rangle\cdot\sin\langle -t\rangle\cdot H_0^{(1)}\left(k_0 r\sin\langle -t\rangle\right)\cdot e^{-i k_0 r\sin\langle -t\rangle}\Big]\ dt\ .\end{aligned} \tag{11}$$

*) See Preface to the 2nd Edition.

In the special case $h = z = 0$, i.e. $\Theta_0 = \pi/2$, one gets

$$\begin{aligned}
&\cos\langle\vartheta''_{Pw}\rangle = \sinh\vartheta''_{Pw} - i\cdot\tanh\vartheta''_{Pw} = -\cos\langle-\vartheta''_{Pw}\rangle\ ,\\
&\sin\langle\vartheta''_{Pw}\rangle = 1 + i\cdot\tanh\vartheta''_{Pw}\cdot\sinh\vartheta''_{Pw} = \sin\langle-\vartheta''_{Pw}\rangle\ ,\\
&R\langle\vartheta''_{Pw}\rangle = \frac{Z\cdot\left(\sinh\vartheta''_{Pw} - i\cdot\tanh\vartheta''_{Pw}\right) - 1}{Z\cdot\left(\sinh\vartheta''_{Pw} - i\cdot\tanh\vartheta''_{Pw}\right) + 1} = 1/R\langle-\vartheta''_{Pw}\rangle\ ,
\end{aligned} \tag{12}$$

and therewith

$$\begin{aligned}
I_0 = &\int_0^\infty e^{-k_0 r_2\cdot\tanh\, t\cdot\sinh t}\,\frac{2-\tanh^2 t}{i + 1/\cosh\, t}\cdot(1 + i\cdot\tanh\ t\cdot\sinh\ t)\\
&\cdot H_0^{(1)}\left(k_0 r(1 + i\cdot\tanh\ t\cdot\sinh\ t)\right)\cdot\left(R\langle t\rangle + 1/R\langle t\rangle\right)\ dt\ .
\end{aligned} \tag{13}$$

The integrand in I_0 decreases quickly with increasing t. This is paid for with a complex argument of the Hankel function. The scattered field is $p_r/P_0 = i/2\cdot I_0$.

If during the deformation $C(\varphi) \to$ Pw a pole of the reflection factor $R(\vartheta)$ is crossed, the pole contribution p_{rp} must be added to p_r:

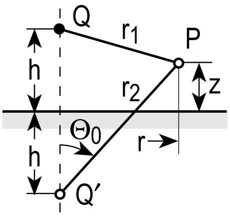
$$\frac{p_{rp}}{P_0} = \frac{-2\pi}{Z}\,H_0^{(1)}(k_0 r\sqrt{1-1/Z^2})\cdot e^{-i\,k_0 H/Z}\quad ;\quad \mathrm{Re}\{\sqrt{1-1/Z^2}\} > 0\ . \tag{14}$$

D.19 A Monopole Point Source Above a Locally Reacting Plane, Approximations

▸ *See also:* Mechel, Vol. I, Ch. 13 (1989)

See ❯ *Sect. D.17* for exact integral formulations of the solution, and see Mechel (1989) for a discussion of the approximations and their precision.

A monopole point source is placed at a point Q at height h above an absorbent plane; a field point is at P with height z above the plane and horizontal distance r from Q. Q′ is the mirror-reflected point to Q. The plane is locally reacting with a normalised admittance $G = 1/Z$.



See Mechel (1989) for a detailed discussion of the approximations. As an exception, the time factor in this section is $e^{-i\omega t}$ in order to facilitate comparison with the literature, where this sign convention mostly is used. Radii used are $r_1 = \mathrm{dist}(Q, P)$ and $r_2 = \mathrm{dist}(Q', P)$, and the angle $\Theta_0 = \angle((Q', Q), (Q', P))$.

The start equation for a first approximation to the reflected field p_r is

$$\frac{p_r}{P_0} = \frac{i}{2} \int_C e^{i\,k_0 r_2 \cdot \cos(\vartheta - \Theta_o)} \cdot \left[H_0^{(1)}(k_0 r \cdot \sin\vartheta) \cdot e^{-i\,k_0 r \cdot \sin\vartheta} \right] \cdot R(\vartheta) \cdot \sin\vartheta\, d\vartheta = \frac{i}{2} \cdot I_0. \quad (1)$$

See Sect. D.17 for definitions of r, r_2, $R\langle t\rangle$. An approximate saddle point integration is applied to I_0 [condition: a pole at $\cos\vartheta_p = -1/Z$ of the reflection factor $R(\vartheta)$ is not near the saddle point $\vartheta_s = \Theta_0$; see sketch in Sect. D.18 for Θ_0]. The first-order approximation is

$$\frac{p_r(r,z)}{P_0} = \frac{e^{i\,k_0 r_2}}{k_0 r_2} \left[R(\Theta_0) - \frac{i}{2\,k_0 r_2} \left(R'(\Theta_0)\cot\Theta_0 + R''(\Theta_0) \right) \right] + \left\langle \frac{p_{rp}}{P_0} \right\rangle. \quad (2)$$

The term $\langle p_{rp}/P_0 \rangle$ indicates a possible pole contribution p_{rp} (for more, see below). In these equations, r = horizontal distance between source Q and field point P; z = height of field point P; $r_1 = \text{dist}(Q, P)$; $r_2 = \text{dist}(Q', P)$; $\Theta_0 = \angle((Q', Q), (Q', P))$; see Sect. D.1 for $R(\Theta_0)$ [there $r(\Theta)$] and derivatives.

A higher approximation (condition: pole not near the saddle point) is

$$\frac{p_r(r,z)}{P_0} = -\frac{e^{i\,k_0 r_2}}{k_0 r_2} \left[\left(\frac{1}{8\,k_0 r_2} + i \right) \cdot F(\Theta_0) + \frac{1}{2\,k_0 r_2} F''(\Theta_0) \right] + \left\langle \frac{p_{rp}}{P_0} \right\rangle, \quad (3)$$

with

$$F(\Theta_0) = R(\Theta_0) \left[1 - \frac{\alpha}{\sin^2\Theta_0} - \frac{i\beta}{\sin\Theta_0} \right] \quad ; \quad \alpha = \frac{9}{128\,(k_0 r)^2} \quad ; \quad \beta = \frac{1}{8\,k_0 r};$$

$$\begin{aligned} F''(\Theta_0) &= \frac{1}{4} R(\Theta_0) \left[\frac{1 - 3\sin^2\Theta_0}{\sin^2\Theta_0} - \alpha \frac{6 + 11\cos^2\Theta_0}{\sin^4\Theta_0} - i\beta \frac{2 + 3\cos^2\Theta_0}{\sin^3\Theta_0} \right] \\ &+ R'(\Theta_0)\cos\Theta_0 \left[\frac{1}{\sin\Theta_0} + \frac{3\alpha}{\sin^3\Theta_0} + \frac{i\beta}{\sin^2\Theta_0} \right] \\ &+ R''(\Theta_0) \left[1 - \frac{\alpha}{\sin^2\Theta_0} - \frac{i\beta}{\sin\Theta_0} \right]. \end{aligned} \quad (4)$$

If the pole of the reflection factor is near the saddle point, the integral to be evaluated, and which can be transformed into the general form

$$I = e^{a \cdot f(\vartheta_s)} \int_{-\infty}^{\infty} e^{-a \cdot f(\vartheta)} \cdot \Phi(s)\, ds\,, \quad (5)$$

is modified further by separation of the simple pole in $\Phi(s)$, i.e. by setting

$$\Phi(s) = \frac{\alpha}{s - \beta} + T(s), \quad (6)$$

with

$$\alpha = \frac{2}{Z} H_0^{(1)}(k_0 r\sqrt{1-1/Z^2}) \cdot e^{-i\,k_0 r\sqrt{1-1/Z^2}} \quad ; \quad \mathrm{Re}\{\sqrt{1-1/Z^2}\} > 0\,;$$

$$\beta = \sqrt{i\left(1+\cos\Theta_0/Z - \sin\,\Theta_0\sqrt{1-1/Z^2}\right)}\,; \tag{7}$$

$$\mathrm{Im}\{\beta\}\begin{cases} > 0\,;\ \text{pole above pass way} \\ = 0\,;\ \text{pole on pass way} \\ < 0\,;\ \text{pole below pass way}\end{cases}$$

The cases for $\mathrm{Im}\{\beta\}$ correspond to

$$\mathrm{Im}\{\beta\} \gtreqless 0 \Leftrightarrow G'' \gtreqless \frac{-1}{\sin\,\Theta_0}\,\frac{(\cos\Theta_0 + G')\ (1+\cos\Theta_0\cdot G')}{\sqrt{1+2\cos\Theta_0\,G' + G'^2}}\,. \tag{8}$$

One gets the approximation

$$\frac{p_r}{P_0} = \frac{i\cdot e^{i\,k_0 r_2}}{2} \cdot \begin{cases} \pm i\,\pi\,\alpha\cdot W(\pm\beta\sqrt{k_0 r_2}) + \sqrt{\pi/(k_0 r_2)}\cdot T(0) \quad ; \quad \mathrm{Im}\{\beta\} \gtrless 0 \\ i\pi\alpha\cdot W(\beta\sqrt{k_0 r_2}) + \sqrt{\pi/(k_0 r_2)}\cdot T(0) - i\pi\alpha\cdot e^{-k_0 r_2\cdot\beta^2} \quad ; \quad \mathrm{Im}\{\beta\} = 0 \end{cases} \tag{9}$$

with

$$T(0) = \frac{\alpha}{\beta} + (1-i)\,H_0^{(1)}(k_0 r\ \ \sin\,\Theta_0)\cdot e^{-i\,k_0 r \sin\Theta_0}\cdot\sin\Theta_0\frac{\cos\Theta_0 - G}{\cos\Theta_0 + G}$$

$$W(u) = e^{-u^2}\cdot \mathrm{erfc}(-iu) \quad ; \quad \mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}\,dx\,. \tag{10}$$

where erfc(z) is the complementary error function.

Van Moorhem's approximation:

$$\frac{p_r}{P_0} = \frac{e^{i\,k_0 r_2}}{k_0 r_2}\left[R(\Theta_0) + (1-R(\Theta_0))\cdot F(\Theta_0, k_0 r_2)\right], \tag{11}$$

with

$$F(\Theta_0, k_0 r_2) = \frac{i}{k_0 r_2}\,\frac{1+G\cdot\cos\Theta_0}{(G+\cos\Theta_0)^2} + \frac{1}{(k_0 r_2)^2}\,\frac{(3\,G^2-1)\cos^2\Theta_0 + 4\,G\cos\Theta_0 - 3 - G^2}{(G+\cos\Theta_0)^4}$$

$$-\frac{3i}{(k_0 r_2)^3}\,\frac{(5\,G^3-3G)\cos^3\Theta_0 + (9\,G^2-3)\cos^2\Theta_0 + (9\,G-3G^2)\cos\Theta_0 + 5 - 3G^2}{(G+\cos\Theta_0)^6}$$

$$-\frac{3}{(k_0 r_2)^4}\,\frac{(3-30\,G^2+35\,G^4)\cos^4\Theta_0 + (80\,G^3-48\,G)\cos^3\Theta_0 - (30-108\,G^2+30\,G^4)}{(G+\cos\Theta_0)^8}$$

$$\frac{\ldots\cdot\ \cos^2\Theta_0\ + (80\,G - 48\,G^3)\cos\Theta_0 + 3G^4 - 30G^2 + 35}{\ldots}\,. \tag{12}$$

Lawhead / Rudnick's approximation (valid for $\mathrm{Im}\{G\} < 0$, i.e. spring-type reactance):

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0 r_2}\left[R(\Theta_0) + (1 - R(\Theta_0)) \cdot F(u)\right], \tag{13}$$

$$\text{with} \quad \begin{aligned} F(u) &= 1 + \sqrt{\pi} \cdot u \cdot e^{u^2} \cdot \mathrm{erfc}(-u) \\ u &= (1 - i)\sqrt{\frac{k_0 r_2}{2}} \, \frac{G + \cos\Theta_0}{\sin\Theta_0} \quad ; \quad \mathrm{Im}\{G\} < 0 \end{aligned} \tag{14}$$

Ingård's approximation is like Rudnick's approximation; however, the function F now is

$$\begin{aligned} F &= 1 - \sqrt{\pi\rho} \cdot e^{\rho} \cdot \left(1 - \Phi(\sqrt{\rho})\right) \\ \rho &= \frac{i k_0 r_2}{2} \, \frac{(G + \cos\Theta_0)^2}{1 + G\cos\Theta_0} \quad ; \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \end{aligned} \tag{15}$$

Approximation by Chien / Soroka: (valid for $|G| \ll 1$ and $k_0 r \gg 1$) with p_r/P_0 as above, but with the function $F(u)$:

$$\begin{aligned} F(u) &= 1 + i\sqrt{\pi} \cdot u \cdot e^{-u^2} \cdot \mathrm{erfc}(-i\,u) \\ u &= \sqrt{i\,k_0 r_2/2} \cdot (G + \cos\Theta_0) \end{aligned} \tag{16}$$

These authors also derived an approximation with a wider range of applicability:

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0 r_2} + \frac{p_s}{P_0} + \frac{p_p}{P_0}, \tag{17}$$

with

$$\begin{aligned} \frac{p_s}{P_0} = \frac{2G}{G + \cos\Theta_0} \, \frac{e^{i k_0 r_2}}{k_0 r_2} &\left\{ -1 + \frac{1}{\sqrt{2}}\sqrt{1 + \frac{1 + G\cos\Theta_0}{\sin\Theta_0\sqrt{1 - G^2}}} + \right. \\ &+ \frac{i}{k_0 r_2 (G + \cos\Theta_0)^2} \left[1 + G\cos\Theta_0 - \frac{\sin\Theta_0}{8\sqrt{2}}\sqrt{1 - G^2} \right. \\ &\left.\left. \cdot \left(1 + \frac{1 + G\cos\Theta_0}{\sin\Theta_0\sqrt{1 - G^2}}\right)^{3/2} \left(3 + \frac{1 + G\cos\Theta_0}{\sin\Theta_0\sqrt{1 - G^2}}\right)\right]\right\} ; \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{p_p}{P_0} &= -\pi\, G\, \mathrm{erfc}(-i\,x_0/\sqrt{2}) \cdot H_0^{(1)}(k_0 r\sqrt{1 - G^2}) \cdot e^{-i k_0 (h+z) \cdot G} ; \\ \frac{x_0^2}{2} &= i\,k_0 r_2 \left(1 + G\cos\Theta_0 - \sin\Theta_0\sqrt{1 - G^2}\right) . \end{aligned} \tag{19}$$

This approximation was also derived by Attenborough et al.

Thomasson presented the approximation with good precision:

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0 r_2}\left[R(\Theta_0) + (1 - R(\Theta_0)) \cdot U(\pm i\, x_0/\sqrt{2})\right] \quad ; \quad \mathrm{Im}\{x_0\} \gtrless 0 ;$$
$$U(\pm u) = 1 \mp i\sqrt{\pi} \cdot e^{-x_0^2/2} \cdot \mathrm{erfc}\,(\pm i\, x_0/\sqrt{2}) \,, \tag{20}$$

with x_0 as above. A further approximation by Thomasson is based on

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0\, r_2} - \pi\,(1 - C) \cdot G \cdot H_0^{(1)}(k_0\, r\sqrt{1 - G^2}) \cdot e^{-i\, k_0 (z+h) \cdot G} - 2\, G \cdot e^{i k_0\, r_2} \int_0^\infty \frac{e^{-t}}{\sqrt{V(t)}}\, dt, \tag{21}$$

with

$$V(t) = \left(A^2 + t\right)\left(B^2 - t\right) ;$$
$$A = \sqrt{e^{i k_0 r_2}\,(\gamma_0 - 1)} \quad ; \quad B = \sqrt{e^{i k_0 r_2}\,(1 - \gamma_1)} ; \tag{22}$$
$$\gamma_{0,1} = -G\,\cos\,\Theta_0 \pm \sin\,\Theta_0\sqrt{1 - G^2}$$

and the sign rules $\mathrm{Re}\{\sqrt{V(0)}\} > 0$ and

$$\mathrm{Re}\{\sqrt{V(t)}\} < 0 \quad \text{if} \quad \begin{cases} \pi/4 < \arg(A) < \pi/2 \,, \\ t > \mathrm{Im}\{A^2 B^2\}/\mathrm{Im}\{A^2 - B^2\} \end{cases}$$
$$\mathrm{Re}\{\sqrt{V(0)}\} > 0 \quad \text{else} \tag{23}$$

C is a “switch function”: $$C = \begin{cases} +1 \,;\; -\pi/2 \leq \arg(A) \leq \pi/4 \\ -1 \,;\; \pi/4 < \arg(A) < \pi/2 \,. \end{cases} \tag{24}$$

The signs of other square roots are selected so that their real parts are positive.

This form is well suited for numerical integration; the results coincide with those of numerical integrations of other exact forms. It can be applied also for $h = z = 0$, i.e. source and receiver on the plane.

An approximation which is derived from the last form of p_r/P_0 is for $k_0 r_2 \gg 1$ and $|B|^2 \gg |A|^2$ and $|B|^2 \gg k_0 r_2$:

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0 r_2}\left[1 - 2\, k_0 r_2\, G\, \frac{C}{B} \sum_{m=0}^{\infty} \frac{(2m)\,!}{(m\,!)^2 (4B^2)^m} \cdot I_m\right]$$
$$- \pi\,(1 - C)\, G \cdot H_0^{(1)}\left(k_0 r\sqrt{1 - G^2}\right) \cdot e^{-i\, k_0 (h+z) \cdot G} \,, \tag{25}$$

with iterative evaluation of the I_m:

$$I_0 = \sqrt{\pi} \cdot e^{A^2} \cdot \mathrm{erfc}\,(A) \quad ; \quad I_1 = A + \left(\frac{1}{2} - A^2\right) \cdot I_0 ;$$
$$I_m = \left(m - \frac{1}{2} - A^2\right) \cdot I_{m-1} + (m - 1)\, A^2 \cdot I_{m-2} \,. \tag{26}$$

This approximation computes very precisely in the mentioned range of conditions.

An approximation by Nobile:

$$\frac{p_r}{P_0} = \frac{e^{i k_0 r_2}}{k_0 r_2} - \frac{4\,i\,G \cdot B}{G + \cos\,\Theta_0} \cdot e^{i k_0 r_2} \sum_{n=0}^{\infty} (e_0 \cdot E_n + K_n) \cdot T_n\ , \tag{27}$$

with

$$B\ = -i\sqrt{1 + G\cos\Theta_0 - \sin\Theta_0\sqrt{1-G^2}} \quad ; \quad \mathrm{Re}\{\sqrt{\ldots}\} > 0\ ,$$

$$C\ = 1 + G\cos\Theta_0 - \sin\Theta_0\sqrt{1-G^2} \quad ; \quad \mathrm{Re}\{\sqrt{1-G^2}\} \geq 0\ ,$$

$$T_n = \frac{1}{(2\,B)^n} \sum_{m=0}^{[n/2]} \begin{pmatrix} n-m \\ m \end{pmatrix} \cdot a_{n-m} \cdot \left(\frac{-4\,B^2}{C}\right)^{n-m}, \tag{28}$$

$$a_0\ = 1 \quad ; \quad a_n\ = \frac{\frac{1}{2} - n}{n} \cdot a_{n-1} \quad ; \quad e_0\ = \frac{1}{2}\sqrt{\frac{\pi}{i\,k_0 r_2}} \cdot e^{-\lambda^2} \cdot \mathrm{erfc}\,(-i\,\lambda)\ ,$$

$$\lambda\ = \sqrt{i\,k_0 r_2}\sqrt{1 + G\cos\Theta_0 + \sin\Theta_0\sqrt{1-G^2}} \quad ; \quad \text{all } \mathrm{Re}\{\sqrt{\ldots}\} > 0\ ,$$

and iterative evaluation of

$$\begin{aligned} &E_0\ = 1 \quad ; \quad E_1\ = -B \quad ; \quad E_n\ = -B \cdot E_{n-1} - i\frac{n-1}{2\,k_0 r_2} \cdot E_{n-2}\,; \\ &K_0\ = 0 \quad ; \quad K_1\ = -\frac{i}{2\,k_0 r_2} \quad ; \quad K_n\ = -B \cdot K_{n-1} - i\frac{n-1}{2\,k_0 r_2} \cdot K_{n-2}\,. \end{aligned} \tag{29}$$

Another approximation by Nobile, in which $\mathfrak{R}$ is a "reflection factor for a spherical wave" (see Sect. D.20) reads as follows:

$$\frac{p_r}{P_0}\ = \mathfrak{R}\frac{e^{i k_0 r_2}}{k_0 r_2} \quad ; \quad \mathfrak{R}\ = 1 + \frac{2\,G}{G + \cos\,\Theta_0} \sum_{n=0}^{\infty} (e_1 \cdot \bar{E}_n + \bar{K}_n) \cdot \bar{T}_n\,; \tag{30}$$

with auxiliary quantities from above, except for the newly defined quantities:

$$\bar{T}_n\ = \sum_{m=0}^{[n/2]} \begin{pmatrix} n-m \\ m \end{pmatrix} \cdot a_{n-m} \cdot \left(\frac{-4\,B^2}{C}\right)^{n-m} \quad ; \quad e_1\ = -2\,i\,B\,k_0 r_2 \cdot e_0\,; \tag{31}$$

$$\begin{aligned} &\bar{E}_0\ = 1 \quad ; \quad \bar{E}_1\ = -\frac{1}{2} \quad ; \quad \bar{E}_n\ = -\frac{1}{2} \cdot \bar{E}_{n-1} - i\frac{n-1}{8\,k_0 r_2 B^2} \cdot \bar{E}_{n-2}\,; \\ &\bar{K}_0\ = 0 \quad ; \quad \bar{K}_1\ = -\frac{1}{2} \quad ; \quad \bar{K}_n\ = -\frac{1}{2} \cdot \bar{K}_{n-1} - i\frac{n-1}{8\,k_0 r_2 B^2} \cdot \bar{K}_{n-2}\,. \end{aligned} \tag{32}$$

In the finite sum the upper limits is $[x]$, the highest integer $\leq x$.

Finally, the approximation obtained with the principle of superposition applied in Sect. D.15 is

$$\frac{p_r}{P_0}\ = \mathfrak{R}\frac{e^{i k_0 r_2}}{k_0 r_2} \quad ; \quad \mathfrak{R}\ = \frac{1/X(r) - G}{1/X(r) + G}, \tag{33}$$

with the normalised cross impedance X(r) of the plane z = 0 defined and given by

$$\frac{1}{X(r)}=\frac{-Z_0\,v_{sz}(r,z=0)}{p_h(r,z=0)}=\frac{i}{k_0}\,\frac{\partial\left(p_Q(r,0)-p_{Q'}(r,0)\right)/\partial z}{p_Q(r,0)+p_{Q'}(r,0)}, \tag{34}$$

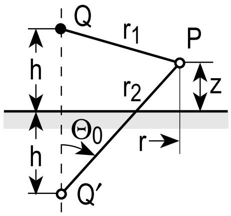
where $p_Q(r_1)$ is the source free field and $p_{Q'}(r_2)$ the free field of a point source (of same strength) in the mirror-reflected point Q′ to Q · p_h is the field, for which the plane z = 0 is hard; p_s is the field for which that plane is soft. It is

$$\begin{aligned} p_Q&=\frac{e^{i\,k_0 r_1}}{k_0 r_1}\quad;\quad p_{Q'}=\frac{e^{i\,k_0 r_2}}{k_0 r_2}\,;\\ r_1&=\sqrt{(h-z)^2+r^2}\quad;\quad r_2=\sqrt{(h+z)^2+r^2}\,; \end{aligned} \tag{35}$$

and therefore

$$\begin{aligned} \frac{1}{X(r)}&=\frac{k_0 h\left(i+k_0\sqrt{h^2+r^2}\right)}{k_0^2\,(h^2+r^2)}\\ \Re&=\frac{i-G\,(k_0 h+r/h\cdot k_0 r)+k_0\sqrt{h^2+r^2}}{i+G\,(k_0 h+r/h\cdot k_0 r)+k_0\sqrt{h^2+r^2}}\;\xrightarrow[r=0]{}\;\frac{i+k_0 h\,(1-G)}{i+k_0 h\,(1+G)} \end{aligned} \tag{36}$$

with the limit $\Re\to -1$ for $r\to\infty$. $\Re$ can be considered as a “reflection factor for spherical waves”.



D.20 A Monopole Point Source Above a Bulk Reacting Plane, Approximations

▸ *See also:* Mechel, Vol. I, Ch. 13 (1989)

See *Sect. D.16* for conventions used, and see Mechel (1989) for a discussion of the approximations and their precision.

The object is a half-space with homogeneous, isotropic material having the characteristic wave number k_2 and wave impedance Z_2. The point source is in the upper half-space with k_1, Z_1 as characteristic wave number and wave impedance; it is at the source point Q with a height h above the plane. The field point P has a horizontal distance r of Q and a height z. The ratios $k = k_2/k_1$ and $Z = Z_2/Z_1$ are used; further $H = h + z$.

The approximation by Delany / Bazley starts from Van der Pol's exact form; it is

$$\frac{p_r}{P_0} = \frac{e^{i\,k_0 r_2}}{k_0 r_2} + 2\,i\,k \int_0^\infty \frac{e^{i\,(k_1 r_3 + ky)}}{k_1 r_3}\,dy\,; \tag{1}$$
$$k_1 r_3 = \sqrt{(k_1 r)^2 + (k_1 H + kZ\cdot y)^2} \quad ; \quad \mathrm{Im}\{\sqrt{\ldots}\} > 0\,.$$

The exponential function in the integrand decays exponentially, therefore this form is suited for numerical integration.

Norton / Rudnick propose a correction to this approximation:

$$kZ \;\rightarrow\; k^2 Z\sqrt{k^2 - \sin^2\Theta_0} \quad ; \quad \mathrm{Re}\{\sqrt{\ldots}\} > 0. \tag{2}$$

Soomerfeld's approximation for the total field in the upper space:

$$\begin{aligned}\frac{p_1}{P_0} &= \frac{e^{i\,k_1 r_1}}{k_1 r_1} + \frac{p_r}{P_0} \\ &= 2\pi C\cdot H_0^{(1)}(\gamma_p r)\cdot e^{-k_1 z\sqrt{(\gamma_p/k_1)^2-1}} + \frac{C_1(z)}{(k_1 r)^2}\cdot e^{i\,k_1 r} + \frac{C_2}{(k_1 r)^2}\cdot e^{i\,k_1 r\cdot k - k_1 z\sqrt{k^2-1}}\,,\end{aligned} \tag{3}$$

with

$$\begin{aligned} C &= \frac{kZ}{1-kZ}\sqrt{\frac{k^2-1}{(kZ)^2-1}} \quad ; \quad C_1(z) = -2\,i\,(1+kZ)\left(\frac{kZ}{1-k^2} + \frac{kz}{\sqrt{1-k^2}}\right) \\ C_2 &= -2\,i\,(1+kZ)\frac{k}{(kZ)^2(k^2-1)} \quad ; \quad \gamma_p/k_1 = k\sqrt{\frac{Z^2-1}{(kZ)^2-1}}\,.\end{aligned} \tag{4}$$

The approximation by Paul uses the notations $\delta = 1/(kZ)$; $k_1 H = k_1(h+z)$:

$$\frac{p_r}{P_0} = -\frac{e^{i\,k_0 r_2}}{k_0 r_2} + V(H,r) \quad ; \quad V(H,r) = V_1(H,r) + V_2(H,r)\,, \tag{5}$$

with

$$\begin{aligned} V_1(H,r) &= -2\,i\,\frac{1+\delta}{\delta^2(1-k^2)}\,\frac{e^{i\,k_1 r}}{(k_1 r)^2}\left[F_0(H) - i\,\frac{F_1(H)}{2\,k_1 r} - \frac{F_2(H)}{8\,(k_1 r)^2}\right] \\ F_0(H) &= 1 + k_1 H\cdot\delta\cdot\sqrt{1-k^2}\end{aligned} \tag{6}$$

$$
\begin{aligned}
F_1(H) = \frac{1}{1-k^2} \Big\{ & 4 - 3(k_1H)^2 + k^2 \left(2 + 3\,(k_1H)^2\right) + k_1H\sqrt{1-k^2} \\
& \cdot \left[\delta \cdot \left(1 - k_1H + k^2(2 + k_1H)\right) - 6/\delta\right] \\
& + k_1H\sqrt{1-k^2} \left[\delta \cdot \left(1 - k_1H + k^2(2 + k_1H)\right) - 6/\delta\right] + 6/\delta^2 \Big\}\ ; \\
F_2(H) = \frac{1}{(1-k^2)^2} \cdot \Big\{ & k_1H\,\delta\sqrt{1-k^2}\left[g + 2(k_1H)^2 + (k_1H)^4 + k^2\right. \\
& \left.\cdot \left(36 - 14(k_1H)^2 - (k_1H)^4\right) + 12\,k^4(k_1H)^2\right] \\
& + 48 - 24(k_1H)^2 + 5(k_1H)^4 + k^2\left(72 - 12(k_1H)^2 - 5(k_1H)^4\right) \\
& + 36\,k^4(k_1H)^2 - \frac{k_1H}{\delta}\sqrt{1-k^2}\left[108 - 20(k_1H)^2 + k^2\left(72 + 20(k_1H)^2\right)\right] \\
& - \frac{1}{\delta}\left[168 - 60(k_1H)^2 + k^2\left(72 + 60(k_1H)^2\right)\right] + 120\frac{k_1H}{\delta^3}\sqrt{1-k^2} + \frac{120}{\delta^4} \Big\}\ ;
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
V_2(H,r) &= -2\,i\,k(1+\delta)\frac{e^{i\,(k_1 r\cdot k + k_1 H\sqrt{1-k^2})}}{(k_1r)^2}\left[G_0(H) - \frac{G_1(H)}{2\,k_1r\cdot k} - \frac{G_2(H)}{8(k_1r)^2\cdot k^2}\right]\ , \\
G_0(H) &= \frac{\delta}{1-k^2} \\
G_1(H) &= -\frac{\delta}{(1-k^2)^2}\left[2 + 4\,k^2 - 6\,\delta^2k^2 + 3\,i\,k_1H\cdot k^2\sqrt{1-k^2}\right] \\
G_2(H) &= -\frac{k^2\delta}{(1-k^2)^2}\left[72 + 48\,k^2 - i\,k_1H\left(36 + 39\,k^2\right)\sqrt{1-k^2} - 15\,(k_1H)^2\right. \\
&\quad \left.\cdot k^2(1-k^2) - \delta^2\left(72 + 168\,k^2 - 60\,i\,k_1H\cdot k^2\sqrt{1-k^2}\right) + 120\,k^2\delta^4\right]\ .
\end{aligned}
\tag{8}
$$

An approximation given by Attenborough/Hayek/Lawther is

$$
\frac{p_r}{P_0} = \frac{e^{i\,k_1r_2}}{k_1r_2}\left[-1 + 2(1+kZ)\frac{\cos\,\Theta_0}{kZ\cdot\cos\,\Theta_0 + \sqrt{k^2 - \sin\,\Theta_0}}\left(1 + \frac{i\,F}{k_1r_2}\right)\right], \tag{9}
$$

with $\mathrm{Im}\{\sqrt{k^2 - \sin\,\Theta_0}\} > 0$ and

$$
\begin{aligned}
F = 1 - & \frac{\sin^2\Theta_0}{k^2 - \sin^2\Theta_0}\left(\frac{\cos\Theta_0 + kZ\sqrt{k^2 - \sin\Theta_0}}{kZ\cdot\cos\Theta_0 + \sqrt{k^2 - \sin\Theta_0}}\right) \\
+ & \frac{kZ\cdot\sin^2\Theta_0}{\cos\Theta_0\cdot(kz\cdot\cos\Theta_0 + \sqrt{k^2 - \sin\Theta_0})} \\
\cdot & \left\{1 - \frac{\cos\Theta_0}{\sin^2\Theta_0}\left[\cos\Theta_0 + \frac{1}{kZ\sqrt{k^2 - \sin\Theta_0}}\left(\cos^2\Theta_0 - \frac{3}{2}\sin^2\Theta_0 + \frac{\cos^2\Theta_0\cdot\sin^2\Theta_0}{2(k^2 - \sin\Theta_0)}\right)\right]\right\}
\end{aligned}
\tag{10}
$$

The denominators create problems when they go to zero, i.e. for large flow resistivity values of a porous material in the lower half-space, and at the same time $\Theta_0 \to \pi/2$, i.e. source and receiver on the plane.

An approximation obtained by saddle point integration is

$$\frac{p_r}{P_0} = \frac{e^{i k_1 r_2}}{k_1 r_2} \cdot \frac{-i}{\sqrt{\sin\Theta_0}} \left[\left(\frac{1}{8 k_1 r_2} + i \right) \cdot F(\Theta_0) + \frac{1}{2 k_1 r_2} F''(\Theta_0) \right], \tag{11}$$

with

$$F(\Theta_0) = R(\Theta_0)\sqrt{\sin\Theta_0}\left[1 - \frac{\alpha}{\sin^2\Theta_0} - \frac{i\,ß}{\sin\Theta_0}\right], \tag{12}$$

$$\begin{aligned} F''(\Theta_0) = {} & \frac{R(\Theta_0)}{4}\sqrt{\sin\Theta_0}\left[\frac{1-3\sin^2\Theta_0}{\sin^2\Theta_0} - \alpha\frac{6+11\cos^2\Theta_0}{\sin^4\Theta_0} - i\,ß\frac{2+3\cos^2\Theta_0}{\sin^3\Theta_0}\right] \\ & + R'(\Theta_0)\sqrt{\sin\Theta_0}\cos\Theta_0\left[\frac{1}{\sin\Theta_0} + \alpha\frac{3}{\sin^3\Theta_0} + i\,ß\frac{1}{\sin^2\Theta_0}\right] \\ & + R''(\Theta_0)\sqrt{\sin\Theta_0}\left[1 - \frac{\alpha}{\sin^2\Theta_0} - \frac{i\,ß}{\sin\Theta_0}\right] \end{aligned}$$

and the reflection factor and its derivatives:

$$\begin{aligned} R(\Theta_0) &= \frac{kZ\cdot\cos\Theta_0 - w}{kZ\cdot\cos\Theta_0 + w}, \\ R'(\Theta_0) &= \frac{kZ}{w\cdot(kZ\cdot\cos\Theta_0 + w)^2}\sin\Theta_0\cdot\left(1+\sin^2\Theta_0 - 2k^2\right), \\ R''(\Theta_0) &= \frac{kZ}{w\cdot(kZ\cdot\cos\Theta_0 + w)^2}\Bigg[\cos\Theta_0\left(1+\sin^2\Theta_0 - 2k^2\right) + 2\sin^2\Theta_0\cdot\cos\Theta_0 \\ & \quad + \left(1+\sin^2\Theta_0 - 2k^2\right)\cdot\left(\frac{\sin^2\Theta_0\cdot\cos\Theta_0}{w^2} + \frac{2\sin^2\Theta_0}{w}\frac{kZ\cdot w + \cos\Theta_0}{kZ\cdot\cos\Theta_0 + w}\right)\Bigg]; \end{aligned} \tag{13}$$

using

$$\alpha = \frac{9}{128\,(k_1 r)^2} \quad ; \quad ß = \frac{1}{8 k_1 r} \quad ; \quad w = \sqrt{k^2 - \sin^2\Theta_0}\ ;\ \mathrm{Im}\{w\} > 0\,. \tag{14}$$

For $k_1 r_2 \gg 1$ this approximation can be simplified to

$$\frac{p_r}{P_0} = \frac{e^{i k_1 r_2}}{k_1 r_2}\left[R(\Theta_0) - \frac{i}{2 k_1 r_2}\left(R'(\Theta_0)\cdot\cot\Theta_0 + R''(\Theta_0)\right)\right]. \tag{15}$$

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