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Elastic waves in layered media: Two-scale homogenization approach

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Using the two-scale convergence approach, we derive equations which govern transversal time-harmonic waves through a layered medium taking the form of a poroelastic composite saturated with a viscous fluid. To improve convergence, we construct a corrector. We study how wave speed and attenuation time depend on porosity and frequency. We prove that the Darcy permeability and the acoustic permeability in the Biot equations do not coincide.

Key words: 74Q10; 74Q15; 76S05; 35B27.

1 Introduction

We consider acoustics of a fluid-saturated porous medium. A number of publications have been devoted to the question. Equations of poroelasticity were first formulated by Biot [5]. His theory predicted a second (slower) compressional wave in a poroelastic medium [6]. Later, such a wave was confirmed experimentally in [7, 17]. To determine the empirical coefficients of the Biot equations, one should use experimental data. On the other hand, these macro-coefficients can be calculated numerically by the two-scale homogenization approach starting from the micro-coefficients of the elastic and fluid components. The potential of such a calculation was proved first by the two-scale expansion technique [4, 9, 14, 18] and then by the method of two-scale convergence [16].

We restrict ourselves to a medium with a periodic layered structure when it consists of alternating elastic and fluid layers. We study time-harmonic transversal waves that are sufficiently long. Using the two-scale convergence approach, we investigate the limit problem when the total thickness of both the solid and fluid layers goes to zero. We prove that the limit waves are elastic or viscoelastic depending on whether the fluid viscosities are low or high. It is a novelty of the present study that we not only find governing equations for waves but we determine both the wave speed and its attenuation for different acoustic and geometrical data of the constitutive components of the composite medium. We construct a corrector which improves weak convergence into strong.

As far as the general solid–fluid geometry of the composite is concerned, it is well-known that, to determine homogenized macro-equations explicitly, one should solve numerically micro-equations reduced to a representative periodicity cell [14]. In our simplified layered composite, the micro-equations are one-dimensional and, consequently, they can be solved in a straightforward manner. This is why we are able to perform an exhaustive acoustic analysis of the homogenized media. Particularly, when the solid and fluid components are acoustically semi-equivalent, i.e. they have the same compressional sound speed, we prove that the sound wave propagates through the homogenized layered media in the transversal direction slower than in the homogeneous components.

We also establish a frequency dispersion effect both for the sound speed and for the attenuation time. Moreover, we study how these parameters depend on porosity. Particularly, we show that sound speed depends on porosity non-monotonically.

We comment also on recent attempts to identify formation permeability by acoustic methods (see for example US Patent No. 2814017). We demonstrate some pitfalls in the approach by showing that permeability in the Biot theory does not coincide with those in the Darcy equation.

Waves in layered media have been studied in a number of publications. We refer the reader to several textbooks [1, 8, 13, 20] and recent articles [3, 19].

2 Problem formulation

We consider small oscillations of a composite material consisting of an elastic porous solid saturated with a viscous fluid. The solid phase is governed by the elasticity equations

$$\rho_s \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i, \tag{2.1}$$

$$\sigma_{ij} = \lambda_s \epsilon^u_{kk} \delta_{ij} + 2\mu_s \epsilon^u_{ij}, \quad \epsilon^u_{ij} \equiv \epsilon_{ij}(u) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \rho^s = \text{const}.$$

Here, $\mathbf{u}$ is the displacement vector, ϵ^u is the strain tensor of small deformations, σ is the stress tensor, ρ is the density, λ_s and μ_s are the Lamé moduli, $\mathbf{f}(t)$ is the mass force density. The equations of a fluid phase are

$$\rho_f \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i, \quad \sigma_{ij} = \left(-p + \lambda_f \frac{\partial \epsilon^u_{kk}}{\partial t}\right)\delta_{ij} + 2\mu_f \frac{\partial \epsilon^u_{ij}}{\partial t}, \tag{2.2}$$

$$p = -c_f^2 \rho_f \epsilon^u_{kk}, \quad \rho_f = \text{const}. \tag{2.3}$$

Here, c_f is the sound speed corresponding to the fluid density ρ_f, p is the pressure deviation from the equilibrium pressure $p_f = c_f^2 \rho_f$, the constants λ_f and μ_f are the viscosities. One can derive equation (2.3) integrating the linearized mass conservation law $\partial_t(\rho - \rho_f) = -\rho_f \operatorname{div} \partial_t \mathbf{u}$ written for density deviation from the initial constant density. Indeed, (2.3) results from the equalities $\rho - \rho_f = -\rho_f \operatorname{div} \mathbf{u}$ and $p = c_f^2(\rho - \rho_f)$. Both the displacement vector and the stress vector are continuous at the surface which separates the solid phase from the fluid one:

$$[u_i] = 0, \quad [\sigma_{ij} n_j] = 0. \tag{2.4}$$

Here, $\mathbf{n}$ is the unit normal to the surface; given a function F, we denote its jump at a point of discontinuity lying on the surface by $[F]$.

In what follows, we assume that the composite has a layered structure. The solid horizontal layers $a_n < z < b_n$ have the same thickness and alternate with the fluid horizontal layers $b_n < z < a_{n+1}$ which are also all of the same thickness, $n = 0, 1, 2, \ldots, k$. With $h_s = b_n - a_n$ and $h_f = a_{n+1} - b_n$ standing for sizes of solid and fluid layers, we introduce the total thickness $L = kh$ and $h = h_f + h_s$.

Let u be the displacement along the vertical direction. We impose the following boundary conditions

$$u(0,t) = u_0 e^{-i\omega t}, \qquad u(L,t) = 0, \tag{2.5}$$

which prescribe a vibration with the frequency ω. Assuming $\mathbf{f} = \mathbf{0}$, we look for a solution as a one-dimensional time-harmonic wave:

$$u = e^{-i\omega t} U(z), \quad p = e^{-i\omega t} P(z).$$

Then system (2.1)–(2.3) can be reduced to the equations

$$-\rho_s \omega^2 U = \frac{\partial}{\partial z}(\lambda_s U_z + 2\mu_s U_z), \quad a_n < z < b_n, \tag{2.6}$$

$$-\rho_f \omega^2 U = -\frac{\partial}{\partial z}(P + i\omega\lambda_f U_z + 2i\omega\mu_f U_z), \quad P = -c_f^2 \rho_f U_z, \quad b_n < z < a_{n+1}. \tag{2.7}$$

We pass to the dimensionless variables

$$z' = \frac{z}{L}, \quad U' = \frac{U}{L}, \quad P' = \frac{P}{p_0}, \quad \omega' = \frac{\omega}{\omega_0},$$

where the denominators are characteristic reference values of the dimension variables. The system (2.6)–(2.7) for the functions U', P' (we drop the primes in what follows) takes the following dimensionless form:

$$-\alpha_1 \omega^2 U = \frac{\partial}{\partial z}\left(\alpha_2 \frac{\partial U}{\partial z}\right), \quad \frac{a_n}{L} < z < \frac{b_n}{L}, \tag{2.8}$$

$$-\alpha_3 \omega^2 U = -\frac{\partial}{\partial z}\left(P + i\alpha_4 \omega U_z\right), \quad P = -\alpha_5 U_z, \quad \frac{b_n}{L} < z < \frac{a_{n+1}}{L}, \tag{2.9}$$

where the dimensionless constants are defined by the formulae

$$\alpha_1 = \frac{\rho_s \omega_0^2 L^2}{p_0}, \quad \alpha_2 = \frac{\lambda_s + 2\mu_s}{p_0}, \quad \alpha_3 = \frac{\rho_f \omega_0^2 L^2}{p_0},$$

$$\alpha_4 = \frac{\omega_0(\lambda_f + 2\mu_f)}{p_0}, \quad \alpha_5 = \frac{c_f^2 \rho_f}{p_0}.$$

An exact solution of the system (2.8)–(2.9) is given by the formula

$$U(z) = \begin{cases} c_{1,n}^s e^{\lambda_1^s z} + c_{2,n}^s e^{\lambda_2^s z} \equiv U_n^s(z), \ a_n/L < z < b_n/L, \\ c_{1,n}^f e^{\lambda_1^f z} + c_{2,n}^f e^{\lambda_2^f z} \equiv U_n^f(z), \ b_n/L < z < a_{n+1}/L, \end{cases} \tag{2.10}$$

where λ_j^s and λ_j^f are roots of the equations

$$\alpha_2 \lambda^2 + \omega^2 \alpha_1 = 0 \quad \text{and} \quad (\alpha_5 - i\omega\alpha_4)\lambda^2 + \alpha_3 \omega^2 = 0$$

respectively. The constants $c^s_{j,n}, c^f_{j,n}$, $(j = 1,2; n = 1,2,\dots,k)$, can be determined from the boundary conditions (2.5) and the continuity conditions (2.4). The latter imply that the solid displacements U^s_n coincide with the fluid displacements U^f_n, and the solid fluxes $\alpha_2 dU^s_n/dz$ coincide with the fluid fluxes $(\alpha_5 - i\omega\alpha_4)dU^f_n/dz$ at the points b_n/L, $0 \leqslant n < k$, and the points a_n/L, $1 \leqslant n < k$.

These continuity conditions and the boundary conditions can be written as a linear system $A^{(k)}c = F$ for the $4k$-dimensional vector c consisting of the coefficients $c^s_{j,n}, c^f_{j,n}$, $(j = 1,2; n = 1,2,\dots,k)$. Hence, the frequency ω obeys the restriction

$$\det A^{(k)}(\omega) \neq 0, \quad k = 1,2,\dots\ . \tag{2.11}$$

The resonance equation $\det A^{(k)}(\omega_*) = 0$ implies that

$$\max_z |U(z)| \to \infty \quad \text{as} \quad \omega \to \omega_*.$$

Assuming that $\delta = h/L = 1/k$ is a small number, we perform an asymptotic analysis for $\delta \to 0$. The formula (2.10) is of no help in such an analysis, since the representation formulae for the vector c are not manageable. Observe that one should verify an infinite number of the conditions (2.11) with $k = 1,2,\dots$. We apply the homogenization theory to study the limit as $\delta \to 0$. We consider a viscous and a 'weakly viscous fluid'. For the viscous fluid, none of the constants α_i depends on δ. For the weakly viscous fluid, none of the constants α_i depends on δ except for α_4, which obeys the law $\alpha_4 = \bar\alpha_4\delta^m$, $m > 1$, $\bar\alpha_4 > 0$. For the sake of convenience, we introduce the function $\alpha_4(\delta)$ which is equal to the constant α_4 in the viscous case and is given by the formula $\alpha_4(\delta) = \bar\alpha_4\delta^m$ in the weakly viscous case. In the limit, as $\delta \to 0$, we have $\alpha_4(0) = \alpha_4$ in the viscous case and $\alpha_4(0) = 0$ in the weakly viscous case.

We introduce the micro-variable $y = z/\delta$ and the characteristic function of the fluid phase

$$1_f(y) = \begin{cases} 1, & 1-\phi < y < 1, \\ 0, & 0 < y < 1-\phi, \end{cases} \qquad y \in Y = \{y : 0 < y < 1\},$$

where $\phi = h_f/h$ is the porosity. The function $1_s(y) = 1 - 1_f(y)$ stands for the characteristic function of the solid phase. We extend these characteristic functions periodically onto the real line $\mathbb{R}$. Clearly, $1^\delta_f(z) = 1_f(z/\delta)$ is a periodic function with the period δ. We introduce the dimensionless density

$$\rho^\delta(z) = \tilde\rho(y)|_{y=z/\delta}, \quad \tilde\rho(y) \equiv \alpha_3 1_f(y) + \alpha_1 1_s(y).$$

Let us pass to a reduced displacement with homogeneous boundary conditions. We introduce the following notation:

$$u_1(z) = (1-z)u_0, \quad w = U - u_1, \quad z \in \Omega \equiv \{z : 0 < z < 1\}.$$

The system (2.8)–(2.9) can be written as an equation

$$-\omega^2\rho^\delta(w + u_1) = \sigma_z, \quad w|_{z\in\partial\Omega} = 0, \tag{2.12}$$

where

$$\sigma = -P - i\omega\alpha_4(\delta)(w_z - u_0)1_f^\delta + \alpha_2(w_z - u_0)1_s^\delta, \quad P(z) = -\alpha_5(w_z - u_0)1_f^\delta. \tag{2.13}$$

Due to the solid–fluid boundary conditions (2.4), equation (2.12) admits the following weak formulation. We look for functions $w \in W_0^{1,2}(\Omega)$ and $P \in L^2(\Omega)$ such that equality (2.13) holds jointly with the condition

$$\int_0^1 \omega^2\rho^\delta(w + u_1)\varphi\, dz = \int_0^1 \sigma\varphi_z\, dz \qquad \forall\varphi \in W_0^{1,2}(\Omega). \tag{2.14}$$

3 Asymptotic analysis

Given $\delta > 0$, the problem (2.12)–(2.13) has a unique weak solution. It follows from the Lax–Milgram theorem. Indeed, let us denote

$$R^\delta(z) = [\alpha_5 - i\omega\alpha_4(\delta)]\, 1_f^\delta(z) + \alpha_2 1_s^\delta(z).$$

We eliminate pressure by formula $(2.13)_2$ and find equation (2.14) is reduced to the following problem: one should find a complex-valued function $w^\delta(z)$ such that $w^\delta \in W_0^{1,2}(\Omega)$ and

$$A(w^\delta,\varphi) \equiv \int\limits_\Omega w_z^\delta\bar{\varphi}_z R^\delta - \omega^2\rho^\delta w^\delta\bar{\varphi}\, dz = \int\limits_\Omega \omega^2\rho^\delta u_1\bar{\varphi} + u_0\bar{\varphi}_z R^\delta\, dz \quad \forall\varphi \in W_0^{1,2}(\Omega). \tag{3.1}$$

The form A is sesquilinear and continuous on the Hilbert space $W_0^{1,2}(\Omega)$ equipped with the scalar product $(u,v) = \int_\Omega u_z\bar{v}_z\, dz$. For small ω, the form is coercive. It follows from the inequalities

$$|w(z)|^2 \leqslant z\int\limits_\Omega |w_z|^2\, dz, \quad |A(w,w)| \geqslant \int\limits_\Omega \left(\alpha_5 1_f^\delta + \alpha_2 1_s^\delta\right)|w_z|^2 - \omega^2\rho^\delta|w|^2\, dz \geqslant$$

$$\geqslant \left[\min(\alpha_2,\alpha_5) - \frac{\omega^2\max(\alpha_1,\alpha_3)}{2}\right]\int\limits_\Omega |w_z|^2\, dz.$$

Since the right-hand side of (3.1) is a linear continuous functional on the space $W_0^{1,2}(\Omega)$, the above properties of the form A enable us to apply the Lax–Milgram theorem [18] and conclude that a solution of (3.1) does exist and it is unique provided the frequencies are small enough.

To describe the homogenized layered media, one should find equations for the principal terms (w^0 and P^0) of the expansion series

$$w^\delta(z) = w^0(z) + O(\delta), \quad P^\delta(z) = P^0(z) + O(\delta), \quad z \in \Omega.$$

While calculating w^0 and P^0, we apply the notion of two-scale convergence [2, 15]. We recall that a sequence $u^\delta \subset L^2(\Omega)$ is said to be two-scale convergent to a limit $u \in L^2(\Omega\times Y)$ if for any test function $\psi \in C(\Omega; C_{per}(Y))$ one has

$$\lim_{\delta\to 0}\int_\Omega u^\delta(z)\psi\left(z,\frac{z}{\delta}\right)dz \to \int_\Omega\int_Y u(z,y)\psi(z,y)\,dz\,dy. \tag{3.2}$$

For brevity, we write the property (3.2) as $u^\delta(z) \xrightarrow{t.s.} u(z,y)$.

The two-scale limit has the following property [15]. From each bounded sequence in $L^2(\Omega)$ one can extract a subsequence which two-scale converges to a function from $L^2(\Omega \times Y)$. Moreover, if $u^\delta(z) \xrightarrow{t.s.} u(z,y)$ we have $u^\delta \to \int_Y u(z,y)\,dy \equiv \tilde{u}(z)$ weakly in $L^2(\Omega)$.

As for derivatives, we will use the following assertion [15]. Let $u^\delta(z)$ and $u^\delta_z(z)$ be bounded sequences in $L^2(\Omega)$. Then there exist a subsequence, still denoted by $u^\delta(z)$, and functions $u \in W^{1,2}(\Omega)$, $v \in L^2(\Omega; W^{1,2}_{per}(Y))$ such that both $u^\delta(z)$ and $u^\delta_z(z)$ two-scale converge to $u(z)$ and $u_z(z) + v_y(z,y)$ respectively. Moreover, $u^\delta \to u$ weakly in $W^{1,2}(\Omega)$.

Assume that frequencies satisfy the following restriction:

$$\min(\alpha_2,\alpha_5) - \frac{3\omega^2\max(\alpha_1,\alpha_3)}{4} > 0. \tag{3.3}$$

We show that, under this constraint, solutions of (3.1) are uniformly bounded in δ in the norm of the space $W^{1,2}_0(\Omega)$. Indeed, setting $\varphi = w^\delta$ in (3.1) we derive that

$$\operatorname{Re}\int_\Omega R^\delta|w^\delta_z|^2\,dz = \omega^2\int_\Omega \rho^\delta|w^\delta|^2\,dz + \operatorname{Re}\int_\Omega\left(\omega^2\rho^\delta u_1\bar{w}^\delta + u_0\bar{w}^\delta_z R^\delta\right)dz. \tag{3.4}$$

Due to the inequalities

$$2\int_\Omega |w|^2\,dz \leqslant \int_\Omega |w_z|^2\,dz, \quad |xy| \leqslant \frac{\varepsilon x^2}{2} + \frac{y^2}{2\varepsilon}\ \forall\varepsilon,$$

we find that the right-hand side of (3.4) is bounded from above by

$$\left(\frac{3\omega^2\max(\alpha_1,\alpha_3)}{4} + \frac{\varepsilon}{2}\right)\int_\Omega |w^\delta_z|^2\,dz + \frac{u_0^2}{2} + \int_\Omega \frac{u_0^2|R^\delta|^2}{2\varepsilon}\,dz.$$

Thus,

$$\|w^\delta\|_{W^{1,2}_0(\Omega)} \leqslant c, \quad \|P^\delta\|_{L^2(\Omega)} \leqslant c,$$

uniformly in δ. In view of these estimates, there are functions $w^0(z)$, $w^1(z,y)$, $P^0(z,y)$ and subsequences, still denoted by $w^\delta(z)$ and $P^\delta(z)$, such that $w^0 \in W^{1,2}_0(\Omega)$, $w^1 \in L^2(\Omega; W^{1,2}_{per}(Y))$, $P^0 \in L^2(\Omega \times Y)$, and

$$w^\delta \to w^0 \quad \text{weakly in} \quad W^{1,2}_0(\Omega),$$

$$P^\delta \to \tilde{P} \equiv \int_Y P^0(z,y)\,dy \quad \text{weakly in} \quad L^2(\Omega),$$

$$w^\delta(z) \xrightarrow{t.s.} w^0(z), \quad P^\delta(z) \xrightarrow{t.s.} P^0(z,y), \quad w_z^\delta(z) \xrightarrow{t.s.} w_z^0(z) + w_y^1(z,y).$$

We denote

$$\phi_s = 1 - \phi, \quad \rho^h = \int_Y \tilde{\rho}(y)\,dy = \alpha_3\phi + \alpha_1\phi_s.$$

Let us show that the limit functions satisfy the equations

$$-\omega^2\rho^h\left(w^0 + u_1\right) = \tilde{P}_z\left(1 - \frac{\alpha_2}{\alpha_5} - i\omega\frac{\alpha_4(0)}{\alpha_5}\right) + \alpha_2 w_{zz}^0,$$
$$\tilde{P}\left(\frac{\phi_s}{\alpha_2} + \frac{\phi}{\alpha_5} - i\omega\frac{\alpha_4(0)\phi_s}{\alpha_2\alpha_5}\right) = -\phi\left(w_z^0 - u_0\right).$$

We recall that $\alpha_4(0) = \alpha_4$ and $\alpha_4(0) = 0$ in the viscous and weakly viscous cases, respectively.

First, we prove an auxiliary result.

Lemma *Let a sequence $u^\delta(z)$ be bounded in $L^2(\Omega)$ and $u^\delta(z) \xrightarrow{t.s.} u(z,y)$. Then $1_f^\delta(z)u^\delta(z) \xrightarrow{t.s.} 1_f(y)u(z,y)$.*

Proof Let $g_m(y)$ be a continuous periodic function on Y such that $\|1_f - g_m\|_{L^2(Y)} < 1/m$, $m > 0$. From the definition (3.2) of the two-scale convergence, it follows that $g_m(z/\delta)u^\delta(z) \xrightarrow{t.s.} g_m(y)u(z,y)$.

With $\varphi(x,y)$ being a test function, $\varphi \in C(\Omega; C_{per}(Y))$, we have

$$\int_\Omega 1_f^\delta(z)u^\delta\varphi(z,z/\delta)\,dz = \int_\Omega \left(1_f^\delta - g_m(z/\delta)\right)u^\delta\varphi(z,z/\delta)\,dz$$
$$+ \int_\Omega g_m(z/\delta)u^\delta\varphi(z,z/\delta)\,dz = I_1^{\delta,m} + I_2^{\delta,m}.$$

Clearly,

$$\lim_{\delta\to 0} I_1^{\delta,m} = O(1/m), \quad \lim_{\delta\to 0} I_2 = \int_\Omega\int_Y g_m(y)u(z,y)\varphi(z,y)\,dzdy = I_2^m.$$

Since

$$I_2^m = \int_\Omega\int_Y 1_f(y)u(z,y)\varphi(z,y)\,dzdy + O(1/m),$$

the lemma is proved. □

The equality (2.13) is equivalent to

$$\int_\Omega P^\delta(z)v(z)dz = -\int_\Omega \alpha_5 1_f^\delta(w_z^\delta(z) - u_0)v(z)\,dz, \quad \forall v \in D(\Omega).$$

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By passing to the limit, as $\delta \to 0$, we obtain that

$$\int_\Omega \int_Y P^0(z,y)v(z)\,dzdy = -\alpha_5 \int_\Omega \int_Y 1_f(y)\left(w_z^0(z) + w_y^1(z,y) - u_0\right) v(z)\,dzdy,$$

for any $v \in D(\Omega)$. Since the function $v(z)$ is arbitrary, we have

$$\tilde{P}(z) = -\alpha_5\phi(w_z^0(z) - u_0) - \alpha_5 \int_Y 1_f(y)w_y^1(z,y)\,dy. \tag{3.5}$$

Eliminating pressure with the help of $(2.13)_2$, we write the equality (2.14) as

$$-\int_\Omega \omega^2\rho^\delta(w^\delta + u_1)\varphi\,dz = -\int_\Omega \varphi_z(w_z^0 - u_0)\left[(\alpha_5 - i\omega\alpha_4(\delta))1_f^\delta + \alpha_2 1_s^\delta\right]\,dz, \tag{3.6}$$

where $\varphi(z)$ is an arbitrary function from $D(\Omega)$. Let us choose $\varphi(z) = \psi(z) + \delta\psi_1(z, z/\delta)$, where $\psi \in D(\Omega)$ and $\psi_1 \in D(\Omega, C_{per}^\infty(Y))$. Taking φ as a test function in (3.6), we find that

$$\begin{aligned}
&-\int_\Omega \omega^2\rho^\delta(w^\delta + u_1)\left[\psi(z) + \delta\psi_1(z, z/\delta)\right]\,dz \\
&= -\int_\Omega \left[\psi_z(z) + \psi_{1y}(z, z/\delta)\right]\left[\alpha_5 1_f^\delta - i\omega\alpha_4(\delta)1_f^\delta + \alpha_2 1_s^\delta\right](w_z^\delta - u_0)\,dz \\
&\quad -\delta\int_\Omega \psi_{1z}(z, z/\delta)\left[\alpha_5 1_f^\delta - i\omega\alpha_4(\delta)1_f^\delta + \alpha_2 1_s^\delta\right](w_z^\delta - u_0)\,dz.
\end{aligned}$$

In the limit, as $\delta \to 0$, we obtain

$$\begin{aligned}
&-\int_\Omega \int_Y \omega^2\tilde{\rho}(y)(w^0 + u_1)\psi\,dzdy \\
&\quad = -\int_\Omega \int_Y \left(\psi_z + \psi_{1y}\right)\{[\alpha_5 - i\omega\alpha_4(0)]1_f(y) + \alpha_2 1_s(y)\}(w_z^0 + w_y^1 - u_0)\,dzdy.
\end{aligned} \tag{3.7}$$

Setting $\psi_1 \equiv 0$ and keeping in mind that the function $\psi \in D(\Omega)$ is arbitrary, we arrive at the equality

$$-\omega^2\rho^h(w^0(z) + u_1(z)) = \frac{d}{dz}\int_Y \{[\alpha_5 - i\omega\alpha_4(0)]1_f(y) + \alpha_2 1_s(y)\}(w_z^0 + w_y^1 - u_0)\,dy.$$

Thus,

$$-\omega^2\rho^h(w^0 + u_1) = \{[\alpha_5 - i\omega\alpha_4(0)]\phi + \alpha_2\phi_s\}w_{zz}^0 + (\alpha_5 - i\omega\alpha_4(0) - \alpha_2)\frac{d}{dz}\int_Y 1_f(y)w_y^1(z,y)\,dy.$$

By applying (3.5), we arrive at the first macro-equation for $w^0(z)$ and $P^0(z)$:

$$-\omega^2\rho^h(w^0 + u_1) = -\tilde{P}_z\left(1 - \frac{\alpha_2}{\alpha_5} - i\omega\frac{\alpha_4(0)}{\alpha_5}\right) + \alpha_2 w_{zz}^0.$$

Let us address (3.7), setting $\psi \equiv 0$, $\psi_1(z,y) = \varphi(z)\theta(y)$, where $\phi \in D(\Omega)$ and $\theta \in C^\infty_{per}(Y)$. By virtue of the fact that the function $\varphi(z)$ is arbitrary, we have

$$0 = -\int_Y \theta_y(y)\left\{[\alpha_5 - i\omega\alpha_4(0)]1_f(y) + \alpha_2 1_s(y)\right\}(w^0_z + w^1_y(z,y) - u_0)\,dy.$$

We look for w^1 by the method of separation of variables. Starting from the assumption that $w^1 = (w^0_z - u_0)W(y)$, we obtain that a periodic function $W(y)$ satisfies the equation

$$0 = -\int_Y \theta_y(y)\left\{[\alpha_5 - i\omega\alpha_4(0)]1_f(y) + \alpha_2 1_s(y)\right\}(1 + W_y)\,dy.$$

In terms of distributions, it implies that

$$\frac{d}{dy}(\left\{[\alpha_5 - i\omega\alpha_4(0)]1_f(y) + \alpha_2 1_s(y)\right\}(1 + W_y)) = 0.$$

Any constant satisfies this equation. In the interests of uniqueness, we impose the restriction $\int_y W\,dy = 0$.

By integrating, we find that in the fluid zone the function W_y is a constant:

$$W_y(y) = \frac{[\alpha_2 - \alpha_5 + i\omega\alpha_4(0)]\phi_s}{\alpha_2\phi + [\alpha_5 - i\omega\alpha_4(0)]\phi_s}, \quad 1 - \Phi < y < 1.$$

By setting $w^1 = (w^0_z - u_0)W$ into (3.5), we obtain the second macro-equation

$$\tilde{P} = -\frac{\alpha_2\alpha_5\phi(w^0_z - u_0)}{\alpha_2\phi + [\alpha_5 - i\omega\alpha_4(0)]\phi_s}.$$

The two-scale limit $w^0(z)$ approximates the function $w^\delta(z)$ for small values of δ only weakly in $W^{1,2}_0(\Omega)$. Here, we improve the approximation by finding a corrector to the function $w^0(z)$.

Let us denote

$$r^\delta(z) = w^\delta(z) - w^0(z) - \delta w^1(z, z/\delta),$$

$$h^\delta(z) = -\delta(1-z)\left[w^0_z(0) - u_0\right]W(0) - \delta z\left[w^0_z(1) - u_0\right]W(1).$$

We prove that the sequence $w^\delta(z) - \delta w^1(z, z/\delta)$ converges to $w^0(z)$ strongly in $W^{1,2}_0(\Omega)$. The term $\delta w^1(z, z/\delta)$ is called the corrector. Such an approach has been pursued by many authors. Let us mention just a reference [10]. One can verify that the function

$$r^\delta_1(z) = r^\delta(z) - h^\delta(z)$$

vanishes at $\partial\Omega$. It is the crucial property of r^δ_1 that it solves weakly the equation

$$\frac{d}{dz}\left(R^\delta(z)\frac{d}{dz}r^\delta_1\right) + \omega^2\rho^\delta r^\delta_1 = -\omega^2\rho^\delta h^\delta + f^\delta_1 + \frac{d}{dz}f^\delta_2, \tag{3.8}$$

where

$$f_1^\delta(z) = -\omega^2\left[(\rho^\delta(z)-\rho^h)(w^0(z)+u_1)+\delta\rho^\delta(z)(w_z^0(z)-u_0)W(z/\delta)\right],$$
$$f_2^\delta(z) = -R^\delta(z)\frac{d}{dz}h^\delta(z)+i\omega 1_f^\delta(z)(w_z^0(z)-u_0)\left[1+W_y\left(\frac{z}{\delta}\right)\right][\alpha_4(\delta)-\alpha_4(0)]$$
$$+\delta w_{0zz}^0 R^\delta W(z/\delta).$$

Observe that $h^\delta \to 0$ strongly in $L^2(\Omega)$, $f_1^\delta \to 0$ weakly in $L^2(\Omega)$ and $f_2^\delta \to 0$ strongly in $L^2(\Omega)$ as $\delta \to 0$.

It follows from (3.8) that

$$\mathrm{Re}\int_\Omega R^\delta\left|r_{1z}^\delta\right|^2 dz = \int_\Omega \omega^2\rho^\delta\left|r_1^\delta\right|^2 + \mathrm{Re}\left[(\omega^2\rho^\delta h^\delta - f_1^\delta)\overline{r_1^\delta} + f_2^\delta\overline{r_{1z}^\delta}\right] dz.$$

Hence, by the Young inequality

$$\left[\min(\alpha_2,\alpha_5)-\frac{\varepsilon}{2}\right]\int_\Omega\left|r_{1z}^\delta\right|^2 dz \leqslant \int_\Omega \omega^2\rho^\delta\left|r_1^\delta\right|^2+\frac{|f_2^\delta|^2}{2\varepsilon}\,dz+\left|\mathrm{Re}\int_\Omega(\omega^2\rho^\delta h^\delta - f_1^\delta)\overline{r_1^\delta}\,dz\right|. \quad (3.9)$$

Assume that the sequence r_1^δ does not converge strongly to zero. Hence, there is subsequence still denoted by r_1^δ such that

$$\|r_1^\delta\|_{W_0^{1,2}(\Omega)} \geqslant \varepsilon_0 > 0 \quad (3.10)$$

for some ε_0 and any δ. By compact imbedding of $W_0^{1,2}(\Omega)$ into $L^2(\Omega)$, there is a subsequence still denoted by r_1^δ such that $r_1^\delta \to 0$ in $L^2(\Omega)$ strongly. Thus, the right-hand side of (3.9) converges to zero in contradiction with the assumption (3.10).

Let us write down the macro-equations for $U = U^0(z)$ and $P = \tilde{P}(z)$ in terms of dimensional variables. We introduce the notation

$$c_s = \sqrt{\frac{\lambda_s+2\mu_s}{\rho_s}}, \quad \bar{\rho} = \phi\rho_f + \phi_s\rho_s.$$

Clearly, they stand for compressional velocity in the homogeneous elastic medium and average density, respectively. The equations that govern propagation of time-harmonic waves in a homogenized layered medium, with a porous fluid having low viscosities, are of the form

$$-\omega^2\bar{\rho}U = -P_z\left(1-\frac{c_s^2\rho_s}{c_f^2\rho_f}\right)+c_s^2\rho_s U_{zz}, \quad -P\left(\frac{\phi}{c_f^2\rho_f}+\frac{\phi_s}{c_s^2\rho_s}\right)=\phi U_z. \quad (3.11)$$

As for the fluid with high viscosities, the equations are

$$-\omega^2\bar{\rho}U = -P_z\left(1-\frac{c_s^2\rho_s}{c_f^2\rho_f}-i\omega\frac{\lambda_f+2\mu_f}{c_f^2\rho_f}\right)+c_s^2\rho_s U_{zz}, \quad (3.12)$$

$$P(\phi c_s^2\rho_s+\phi_s c_f^2\rho_f)-i\omega\phi_s(\lambda_f+2\mu_f)P = -c_f^2\rho_f c_s^2\rho_s\phi U_z. \quad (3.13)$$

Let c_0 be a reference sound speed, say c_s or c_f. The frequency restriction condition (3.3) can be formulated in terms of the wave length $L_w = c_0/\omega$ as follows

$$\frac{L_w^2}{L^2} > \frac{3c_0^2 \max(\rho_s, \rho_f)}{4 \min(c_s^2\rho_s, c_f^2\rho_f)}. \tag{3.14}$$

4 Sound waves in a homogenized medium

First, we consider the case of a weakly viscous fluid. The harmonic-wave equations (3.11) suggest that the general wave equations are of the form

$$\bar{\rho}u_{tt} = c_s^2\rho_s u_{zz} - \left(1 - \frac{c_s^2\rho_s}{c^2\rho_f}\right)p_z, \quad -p\left(\frac{\phi}{c_f^2\rho_f} + \frac{\phi_s}{c_s^2\rho_s}\right) = \phi u_z. \tag{4.1}$$

Though these equations do not result from above calculations we present them to see that the effective homogenized medium is an elastic solid. Rigorously, equations (4.1) can be derived by the technique developed in [11, 15, 18]. In the special case that a solution of (4.1) depends on the variable $e^{i(kz-\omega t)}$ only, one can determine the wave number k. We find the wave number k directly from equations (3.11) looking for the solution in the form $U = U_0e^{ikz}$, $P = P_0e^{ikz}$. Calculations reveal that

$$k^2 = \bar{\rho}\omega^2\left(\frac{\phi}{c_f^2\rho_f} + \frac{\phi_s}{c_s^2\rho_s}\right).$$

Hence, the sound speed $c_h = \omega/k$ in the homogenized layered medium is defined by the formula

$$c_h = \frac{1}{\sqrt{\bar{\rho}\left[\phi/(c_f^2\rho_f) + \phi_s/(c_s^2\rho_s)\right]}}. \tag{4.2}$$

Let us restrict ourselves to the case $c_f = c_s = c$. By analysing (4.2), we find that the composite sound speed c_h is lower than the sound speed in the components, with a minimal value achieved at $\phi = 1/2$ and being equal to

$$\min_{\phi} c_h = c\frac{\sqrt{\rho_f\rho_s}}{(\rho_f + \rho_s)/2}.$$

Generally, the dependence of c_h on ϕ is not monotone (Figure 1).

Now we pass to the case when the fluid viscosities are not small. As equations (3.12) and (3.13) suggest, the homogenized medium is a viscoelastic medium which is governed, generally, by the equations

$$\bar{\rho}u_{tt} = c_s^2\rho_s u_{zz} - \left(1 - \frac{c_s^2\rho_s}{c_f^2\rho_f}\right)p_z - \frac{\lambda_f + 2\mu_f}{c_f^2\rho_f}p_{zt},$$

$$p(\phi c_s^2\rho_s + \phi_s c_f^2\rho_f) + \phi_s(\lambda_f + 2\mu_f)p_t = -c_f^2\rho_f c_s^2\rho_s\phi u_z.$$

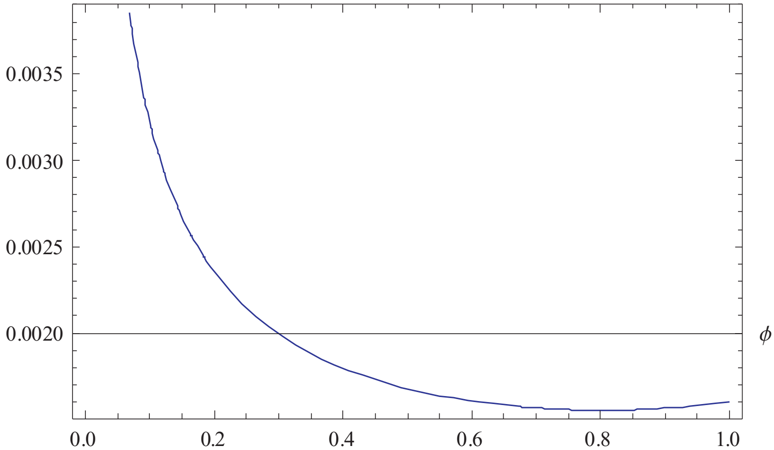

FIGURE 1. (Colour online) The dimensionless sound speed c_h/c_s versus the porosity ϕ for the composite consisting of Berea sand stone and water. The data are $\rho_f = 1$ g/cm^3, $\rho_s = 2.65$ g/cm^3, $c_f = 1.5323 \times 10^2$ cm/s, $c_s = 0.9558 \times 10^5$ cm/s.

Equations (3.12) and (3.13) admit the solution $U = U_0 e^{ikz}$, $P = P_0 e^{ikz}$ provided the dispersion relation

$$\bar{\rho} V^2 = \chi, \quad \omega/k \equiv V = V_1 + iV_2,$$

holds, where

$$\chi = \frac{1 - i\omega \frac{\lambda_f + 2\mu^f}{c_f^2 \rho_f}}{\frac{\phi_s}{\lambda_s + 2\mu_s} + \frac{\Phi}{c_f^2 \rho_f} - i\omega \frac{\phi_s(\lambda_f + 2\mu_f)}{c_f^2 \rho_f (\lambda_s + 2\mu_s)}}.$$

We pass to the dimensionless variables

$$v = \frac{V}{c_s}, \quad \mathrm{f} = \frac{\omega}{\omega_0}, \quad \omega_0 \equiv \frac{c_f^2 \rho_f}{\lambda_f + 2\mu_f}.$$

Now, the dispersion relation becomes

$$v^2 = \frac{1 - i\mathrm{f}}{d_1(d_2 + d_3(1 - i\mathrm{f}))},$$

where

$$d_1 = \phi_s + \frac{\rho_f}{\rho_s}\phi, \quad d_2 = \phi \frac{c_s^2}{c_f^2}\frac{\rho_s}{\rho_f}, \quad d_3 = \phi_s.$$

Writing down v in the complex form $v = v_1 + iv_2$, we find the representation formulae

$$v_1(\mathrm{f}) = \sqrt{\frac{d_2 + d_3 + d_3\mathrm{f}^2 + \sqrt{(d_2 + d_3 + d_3\mathrm{f}^2)^2 + d_2^2\mathrm{f}^2}}{2d_1\left[(d_2 + d_3)^2 + d_3^2\mathrm{f}^2\right]}},$$

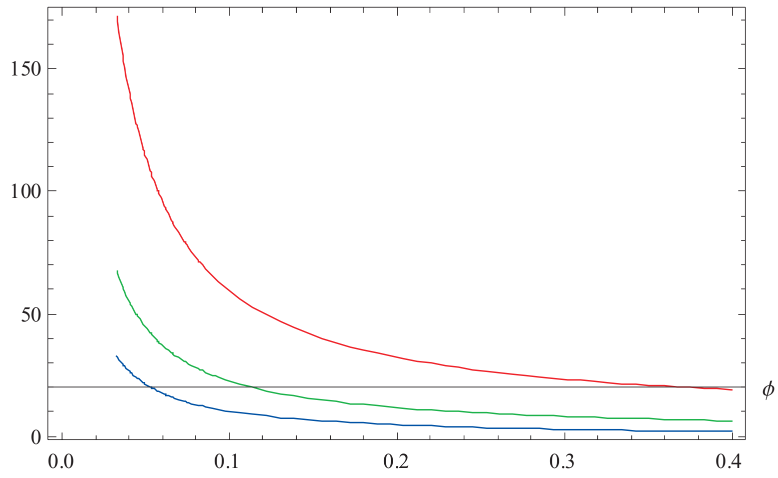

FIGURE 2. (Colour online) The dimensionless attenuation time $T\omega_0$ versus the porosity ϕ for resin particles dispersed in liquid paraffin for different values of dimensionless frequency $\mathrm{f} = \omega/\omega_0$. The curves from the bottom upwards correspond to f taking values $\mathrm{f} = 0.5$, 1 and 300, respectively.

$$v_2(\mathrm{f}) = -\frac{d_2\mathrm{f}}{2v_1(\mathrm{f})d_1\left[(d_2+d_3)^2 + d_3^2\mathrm{f}^2\right]}.$$

These formulae enable us to calculate the sound speed and find the attenuation time for different frequencies. We introduce the variables

$$c_0 = \frac{|V|^2}{V_1}, \quad \xi = z - c_0 t,$$

which stand for the sound speed and the phase variable, respectively. Simple calculations give

$$e^{i(kz-\omega t)} = e^{i\omega V_1\xi/|V|^2} e^{\omega V_2\xi/V^2} e^{\omega V_2 c_0 t/|V|^2}.$$

Since in our case $V_2 < 0$, the wave dies out. The time T satisfying the relation $e^{\omega V_2 c_0 T/|V|^2} = e^{-1}$ is called the attenuation time. One can see that both the attenuation time and the wave speed can be calculated by the formulae

$$T = -\frac{v_1(\mathrm{f})}{\mathrm{f}v_2(\mathrm{f})\omega_0}, \quad c_0 = \frac{c_s\left[v_1^2(\mathrm{f}) + v_2^2(\mathrm{f})\right]}{v_1(\mathrm{f})}.$$

As an example, we show how these formulae can be applied to tell the concentration of resin particles that are dispersed in a fluid. The drug industry calls for concentration measurement while developing encapsulation technology [12]. Encapsulation is a process of enclosing micron-sized particles of solids or droplets of liquids or gasses in an inert shell, which in turn isolates and protects them from the external environment. Microencapsulation can be done to protect the sensitive substances from the external environment, to mask the organoleptic properties like colour, taste and odour of the substance, to obtain controlled release of the drug substance and so on. Figures 2 and

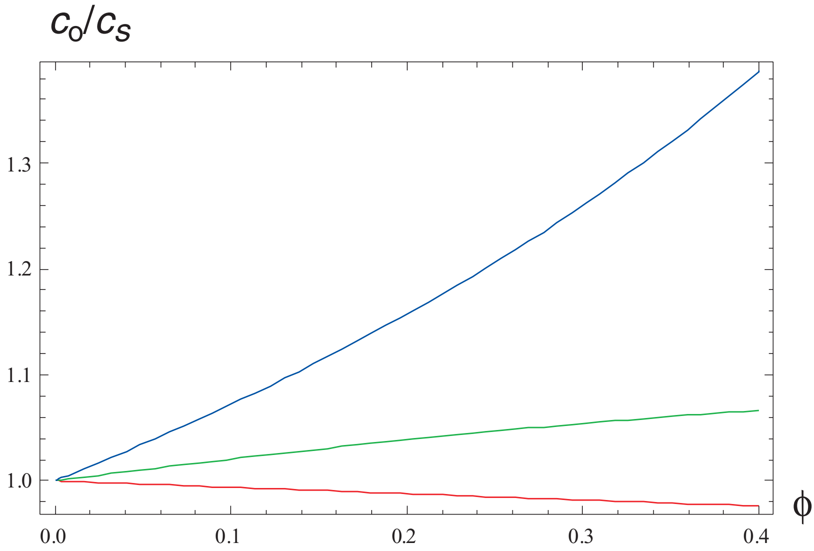

FIGURE 3. (Colour online) The dimensionless sound speed c_0/c_s versus the porosity ϕ for resin particles dispersed in liquid paraffin for different values of dimensionless frequency $\mathrm{f} = \omega/\omega_0$. The curves from the bottom upwards correspond to f taking values $\mathrm{f} = 0.5$, 1 and 300, respectively.

3 depict the attenuation time and wave speed versus porosity for the mixture of liquid paraffin with resin particles. The component data are $\rho_f = 0.8$ g/cm^3, $\rho_s = 1.2$ g/cm^3, $c_f = 14.5 \cdot 10^4$ cm/s, $c_s = 16 \times 10^4$ cm/s, $\lambda_f = 0$ and $2\mu_f/\rho_f = 25$ cm^2/s. One can see that there is a good correlation between the porosity (volume fraction of the fluid phase) and both the attenuation time and the wave speed. Hence, one can measure concentration by acoustic methods. Clearly, there is a limitation on the use of the above calculations to particle concentration measurements since we are restricted in the present study to an idealized layered medium. However, this study suggests that the two-scale homogenization approach to three-dimensional poroelasticity equations can be of use in the topic.

5 Acoustic Darcy's permeability

We introduce the fluid phase velocity $V_f^\delta(z) = -i\omega U^\delta(z) 1_f^\delta(z)$. Let us calculate a two-scale limit of V_f^δ as $\delta \to 0$. Given an arbitrary function $\varphi(z)$ from $D(\Omega)$, we have

$$\int_\Omega V_f^\delta \varphi \, dz \to -i\omega \int_\Omega \int_Y (w^0 + u_1) 1_f(y) \varphi(z) \, dz dy = -i\omega\phi U(z).$$

Therefore, $V_f^\delta(z) \xrightarrow{t.s.} -i\omega\phi U(z) \equiv V_f(z)$. The relative velocity of fluid and solid phases [14]

$$Q = \phi[V_f - (-i\omega U)] = \phi\phi_s i\omega U$$

is called the filtration or Darcy's velocity. Equations (3.11) valid for the case of weakly viscous fluid enable us to represent U via P_z. On this way we arrive at the following formula for Darcy's velocity:

$$Q = -\frac{k}{\mu_f}(P_z - \omega^2\rho_f U), \quad k = i\frac{\mu_f\phi\phi_s}{\omega(\rho_f - \phi\bar{\rho})}. \tag{5.1}$$

The fact that k is pure imaginary implies that, with porosity being small and satisfying the restriction $\rho_f > \phi\bar{\rho}$, the phase shift γ in the complex representations of Q and $p_z - \omega^2\rho_f u$ is equal to $\pi/2$:

$$q = \mathrm{Re}\left(Qe^{-i(\omega t+\beta_1)}\right), \quad p_z - \omega^2\rho_f u = \mathrm{Re}\left([P_z - \omega^2\rho_f U]e^{-i(\omega t+\beta_2)}\right), \quad \gamma = \beta_1 - \beta_2.$$

It results from (5.1) that time-harmonic waves satisfy the equation

$$q = -\frac{k_a}{\mu_f}(p_z - \omega^2\rho_f u), \quad k_a = \frac{\mu_f\phi\phi_s}{\omega(\rho_f - \phi\bar{\rho})}.$$

Observe that Biot derived this equation for a general poroelastic medium, with k_a being an empiric constant [5]. The variable k_a has the same dimension as permeability. This is why one may call it acoustic Darcy's permeability.

If the fluid viscosities are not small, we derive from (3.12) and (3.13) that the Darcy velocity is given by the formula

$$Q = -\frac{k}{\mu_f}\left(P_z - \omega^2\rho_f U\right), \quad k = \frac{i\mu_f\phi\phi_s\left[c_f^2\rho_f - i\omega(\lambda_f + 2\mu_f)\right]}{\omega\left[c_f^2\rho_f(\rho_f - \phi\bar{\rho}) - i\omega\rho_f(\lambda_f + 2\mu_f)\right]}.$$

In this case the acoustic Darcy's law is of the form

$$q = -\frac{k_a}{\mu_f}\left(p_z - \omega^2\rho_f u\right),$$

where

$$k_a = \frac{\mu_f\phi\phi_s\sqrt{\omega^2\phi^2c_f^4\bar{\rho}^2(\lambda_f + 2\mu_f)^2 + \left(c_f^4\rho_f(\rho_f - \phi\bar{\rho}) + \omega^2(\lambda_f + 2\mu_f)^2\right)^2}}{\rho_f\omega\left[c_f^4(\rho_f - \phi\bar{\rho})^2 + \omega^2(\lambda_f + 2\mu_f)^2\right]}.$$

The above study has been performed for the case when fluid layers are isolated from each other. Clearly, the homogenized layered medium has zero permeability in the transversal direction. However, the acoustic permeability does not vanish and enjoys the frequency dispersion effect. It should be noted that some authors do not make distinction between hydrodynamic and acoustic permeabilities (see [14]). The above analysis demonstrates that such a distinction should be recognized especially if one tries to define permeability by an acoustic method.

6 Conclusions

Using the two-scale homogenization approach, we study propagations of time-harmonic transversal waves in a layered medium when elastic layers alternate with compressible viscous fluid layers. The effective homogenized equations describe elastic waves or viscoelastic waves depending on whether the fluid viscosities are low or high. To improve the two-scale convergence, we construct a corrector. We study how the wave speed and the attenuation time depend on the fluid volume fraction and frequencies. We are restricted to the case when the wave length is great enough. When the viscosities are not small, we establish the frequency dispersion effect both for the sound speed and the attenuation time. We find a representation formula for the acoustic Darcy's permeability and show that it does not coincide with the hydraulic Darcy's permeability.

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