N Flow Acoustics

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Sound propagation in a flowing medium is treated also in *Ch. J*, "Duct Acoustics", and *K*, "Acoustic Mufflers", but there mostly with simplifying assumptions.

This chapter uses throughout the "double subscript summation rule", i.e. terms in an expression in which a subscript (for example i) appears twice represent a sum over that term with the multiple subscript as summation index. Terms containing x_i^2, for example, are also cases of the summation rule.

The general convention to symbolise the density of air with ρ_0 and sound velocity with c_0 must be suspended in this chapter because these quantities may be used in other than standard conditions. These conditions will always be defined in the context.

N.1 Concepts and Notations in Fluid Mechanics, in Connection with the Field of Aeroacoustics

▸ *See also:* Morfey (2001); Lauchle (1996); Douglas (1986); Roger (1996)

N.1.1 Types of Fluids

Ideal fluids:	$\mu = 0\,, \quad \lambda = 0$	μ: dynamic viscosity λ: thermal conductivity
Newtonian fluid:	$\mu = \text{constant}$	
Non-Newtonian fluid:	$\mu \neq \text{constant}$ The relationship between shear stress τ and velocity gradient $\partial v / \partial n$ is non-linear.	

N.1.2 Properties of Fluids

Density	ρ, mass per volume, $[\rho] = \dfrac{\text{kg}}{\text{m}^3}$
Pressure	p, normal force pushing against a plane area divided by the area, $[p] = \dfrac{\text{N}}{\text{m}^2} = \text{Pa}$
Viscosity	dynamic viscosity $\quad \mu, [\mu] = \dfrac{\text{N}\cdot\text{s}}{\text{m}^2} = \text{Pa}\cdot\text{s}$
	kinematic viscosity $\quad \nu, [\nu] = \dfrac{\text{m}^2}{\text{s}}$
Gas constant	$R, [R] = \dfrac{\text{J}}{\text{kg}\cdot\text{K}}$

Specific heats — at constant volume $$c_v = T\left(\frac{\partial s}{\partial T}\right)_\rho = \left(\frac{\partial u}{\partial T}\right)_\rho \quad (1)$$

at constant pressure $$c_p = T\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial h}{\partial T}\right)_p \quad (2)$$

$$[c_p, c_v] = \frac{J}{kg \cdot K}$$

with: s specific entropy
u specific internal energy
h specific enthalpy

Specific heat ratio — $\kappa = c_p/c_v$, ratio of the specific heat at constant pressure to that at constant volume (3)

Speed of sound — $c, [c] = \frac{m}{s}$

Bulk modulus — K, expresses the compressibility of a fluid, [K] = Pa,

adiabatic or isentropic bulk modulus: $$K_s = \rho\left(\frac{\partial p}{\partial \rho}\right)_s \quad (4)$$

isothermal bulk modulus: $$K_T = \rho\left(\frac{\partial p}{\partial \rho}\right)_T \quad (5)$$

The reciprocal $1/K_s$ or $1/K_T$ is the adiabatic or isothermal compressibility.

Coefficient of expansion $$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \qquad [\beta] = \frac{1}{K} \quad (6)$$

Thermal conductivity — $\lambda, [\lambda] = \frac{W}{m \cdot K}$

Shear stress — $\tau, [\tau] = Pa$

N.1.3 Models of Fluid Flows

Real flow:	flow without any assumptions
Ideal flow:	flow without viscosity and thermal conductivity
Inviscid flow:	flow without viscosity
Viscous flow:	$\mu \neq 0$
Incompressible flow:	ρ = constant
Compressible flow:	$\rho \neq$ constant
Adiabatic flow:	flow without heat transfer
Isentropic flow:	$\frac{Ds}{Dt} = 0$, the specific entropy of each fluid particle along its path is constant, but may vary from one particle to another (Roger); inviscid and non-heat-conducting gas flow, also frictionless adiabatic flow

Homentropic flow: s = constant throughout the flow, uniform specific entropy

Isothermal flow: T = constant

Steady flow: no time dependence for v, p, ρ, T, . . .; thus $\frac{\partial \ldots}{\partial t} = 0$

Stationary flow:
$$\frac{\partial \bar{A}}{\partial t} = 0, \text{ with } \bar{A} = \bar{v}, \bar{p}, \bar{\rho}, \bar{T}, \ldots \text{ and } \bar{A} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A\,dt \qquad (7)$$

Unsteady flow:
$$\frac{\partial A}{\partial t} \neq 0, \text{ possibly also } \frac{\partial \bar{A}}{\partial t} \neq 0$$

Uniform flow:
$$\frac{\partial \bar{v}}{\partial s} = 0$$

Non-uniform flow:
$$\frac{\partial \bar{v}}{\partial s} \neq 0$$

Rotational flow:
$$\vec{\omega} = \mathrm{rot}\vec{v} = \mathrm{curl}\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \neq 0 \qquad (8)$$

with: $v_x = u,\ v_y = v,\ v_z = w$

Vorticity: $\vec{\omega} = \mathrm{rot}\vec{v} = \nabla \times \vec{v}$ is a measure of local fluid rotation.

Irrotational flow:
$$\vec{\omega} = \mathrm{rot}\vec{v} = \nabla \times \vec{v} = 0 \qquad (9)$$

Comment:

From Crocco's form of the momentum equations it follows that (stationary flow with constant stagnation enthalpy) $\vec{\omega} \times \vec{v} = T \cdot \mathrm{grad}\, s$.

Consequences:

a rotational flow cannot exist with uniform entropy;
a homentropic flow must be irrotational (except when vorticity field and velocity field are parallel).

Laminar flow: viscous or streamline flow, without turbulence; the particles of the fluid moving in an orderly manner and retaining the same relative positions in successive flow cross sections.

Turbulent flow: a random, non-deterministic motion of eddying fluid flow;

Turbulence: characterised by (Morfey):

- three-dimensional velocity fluctuations field;
- unsteady flow;

- viscous flow;
- rotational flow;
- flow with viscous dissipation of energy;
- viscous dissipation takes place at the smallest length scales of eddies, far removed from the larger scales eddies contain most of the kinetic energy; the smallest scales ≫ molecular scales;
- fluctuations cover a wide frequency range and a wide range of eddy sizes or length scales;
- occurring at high Reynolds numbers.

Turbulence level: based on the averaging of the specific kinetic energy:

$$\frac{1}{2}\overline{v_i^2} = \frac{1}{2}\overline{\left(\bar{v}_i + v_i'\right)\left(\bar{v}_i + v_i'\right)} = \frac{1}{2}\bar{v}_i^2 + \frac{1}{2}\overline{v_i'^2} \tag{10}$$

three-dimensional:

$$\frac{1}{2}\overline{v_i^2} = \frac{1}{2}\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right) + \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \tag{11}$$

turbulence level:

$$\mathrm{Tu} = \sqrt{\frac{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}{\left(\bar{u}^2 + \bar{v}^2 + \bar{w}^2\right)}} \tag{12}$$

in the special case of isotropic turbulence and unidirectional flow
$\bar{v}_i = \{\bar{v}_x = U; 0; 0\}$:

$$\mathrm{Tu} = \frac{\sqrt{\overline{u'^2}}}{U} = \frac{u'_{rms}}{U} \tag{13}$$

Transition: the fluid flow change from laminar to turbulent flow

Boundary layer flow:

- in the mean flow sense (Morfey): flow next to a solid surfaces within which the mean flow $\bar{u}(y)$ varies with distance y from the wall, from zero at the wall (at $y = 0$) to 99% of its free-stream value at $y = \delta$, δ is the boundary layer thickness;

- in the acoustic sense (Morfey):
  a thin region produced by a sound field next to a solid boundary, within which the oscillatory velocity parallel to the wall drops to zero as the wall is approached, as a result of viscosity. The acoustic boundary layer thickness is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \ll \lambda \tag{14}$$

Reynolds stress:

- $\rho v_i v_j$ in unsteady fluid flow;
  v_i, v_j are fluid velocity components in any of the three orthogonal Cartesian coordinate directions;
  $\rho v_i v_j$ represents the transfer rate of j-component fluid momentum per unit area;
  the double divergence of $\rho v_i v_j$ represents a source term in Lighthill's inhomogeneous wave equation (acoustic analogy for aerodynamic sound generation);
- in turbulent flows:
  the time-average Reynolds stress $\overline{\rho v_i' v_j'}$ is a term in the time-averaged momentum equation, as the negative of an effective stress;
  $\overline{\rho v_i' v_j'}$ represents the mean momentum flux due to turbulent eddies;
  the Reynolds stress tensor is $\tau_{ij} = \rho\overline{v_i v_j}$ with normal stress if i = j and shear stress if i ≠ j (Morfey)

N.2 Some Tools in Fluid Mechanics and Aeroacoustics

► *See also:* Telionis (1981); Johnson (1998); Schlichting (1997); Lauchle (1996); Liu (1988); Hussain (1970); Reynolds (1972)

N.2.1 Averaging

General quantity: $f(\vec{x}, t)$

Spatial average:

$$\bar{f}_{\text{spatial}} = \frac{1}{V}\int_V f(\vec{x}, t)\, dV = \bar{f}^s(t) \tag{1}$$

Time average:

$$\bar{f}_{\text{time}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\vec{x}, t)\, dt = \bar{f}^t(\vec{x}) = \bar{f} \tag{2}$$

with: abbreviation: $\bar{f}$

Root mean square:

$$f_{\text{rms}} = \sqrt{\overline{f^2}} = \sqrt{\lim_{T\to\infty} \frac{1}{T}\int_0^\infty f^2(\vec{x}, t)\, dt} = \bar{f}^{\text{rms}}(\vec{x}) \tag{3}$$

the square root of the mean square value

Ensemble average: over N repeated experiments

$$\bar{f}_{\text{ensemble}} = \langle f(\vec{x}, t)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} f^{(\alpha)}(\vec{x}, t) = \bar{f}^{en}(\vec{x}, t) \tag{4}$$

Phase average: for periodic flows

$$\bar{f}_{\text{phase}} = \langle f(\vec{x}, t)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} f(\vec{x}, t + n\tau) = \bar{f}^{ph}(\vec{x}, t) \tag{5}$$

with: τ period of an externally imposed fluctuation

$$\bar{f}_{\text{phase}} = \langle f(\vec{x}, t)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} f\left(\vec{x}, \varphi_0 + n\omega\tau\right) = \bar{f}^{ph}\left(\vec{x}, \varphi_0\right) \tag{6}$$

with: $0 < \varphi_0 < 2\pi$; $\varphi = \varphi_0 + n\omega\tau$ phase of the periodic flow

Reynolds averaging: decomposition of a general quantity in the flow in the following form:

$$f = \bar{f} + f'$$

with: $$\bar{f} = f_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\, dt \quad \text{mean quantity} \tag{7}$$

$$\overline{f'} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f'\, dt = 0 \quad \text{fluctuating quantity} \tag{8}$$

Mass-weighted or Favre averaging:

decomposition of a general quantity in the flow in the following form:

$$f = \tilde{f}^f + f''^f$$

with: $\tilde{f}^f = \dfrac{\overline{\rho f}}{\bar{\rho}}$ filtered part of f;

f''^f unresolved or subgrid part of f:

$\widetilde{f''^f} = 0.$

N.2.2 Decomposition (in General)

Decomposition of a general flow quantity in three (or four) parts:

$$f(\vec{x}, t) = \bar{f}(\vec{x}) + \tilde{f}(\vec{x}, t) + f'(\vec{x}, t) \tag{9}$$

with:

$\bar{f}(\vec{x})$ time-averaged or mean component, obtained by Reynolds averaging: $\bar{f}(\vec{x}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\vec{x}, t)\, dt$

$\bar{\tilde{f}} = 0;$

$\overline{f'} = 0;$

$\tilde{f}(\vec{x}, t)$ organised fluctuation: periodicities are in time, periodic mean component of flow can be split into odd modes $\tilde{f}^{odd}$ and even modes $\tilde{f}^{even}$ (see Liu):

$$\tilde{f}(\vec{x}, t) = \tilde{f}^{odd}(\vec{x}, t) + \tilde{f}^{even}(\vec{x}, t); \tag{10}$$

$f'(\vec{x}, t)$ random fluctuations, incoherent fluctuating flow quantities, e.g. small-scale stochastic fluctuations of fine-grained turbulence;

$$\bar{f}_{phase} = \langle f(\vec{x}, t)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} f(\vec{x}, t + n\tau) = \bar{f} + \tilde{f}$$
$$\text{with:} \quad \langle f'(\vec{x}, t)\rangle = 0 \tag{11}$$

$$\tilde{f} = \langle f(\vec{x}, t)\rangle - \bar{f}. \tag{12}$$

Phase averaging with period 2τ is denoted by $\langle\langle\rangle\rangle$ so that $\langle\langle \tilde{f}^{odd}\rangle\rangle = 0$.

Therefore the even modes are obtained from $\langle\langle \tilde{f}^{odd} + \tilde{f}^{even}\rangle\rangle = \tilde{f}^{even}$, (13)

and the odd modes from subtracting $\tilde{f}^{odd} = \tilde{f} - \langle\langle \tilde{f}^{odd} + \tilde{f}^{even}\rangle\rangle$. (14)

N.2.3 Decomposition of the Physical Quantities in the Basic Equations

Decomposition:
$$\rho = \bar{\rho} + \rho'$$
$$p = \bar{p} + p'$$
$$v_i = \bar{v}_i + v_i'$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0 \tag{15}$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}\left[\bar{\rho}\bar{v}_i + \bar{\rho}v_i' + \bar{v}_i\rho' + v_i'\rho'\right] = 0 \tag{16}$$

with assumptions: $\rho', v_i' \ll \bar{\rho}, \bar{v}_i$ and $\bar{\rho} \neq f(\vec{x})$

Continuity equation in the case of mean flow: $$\frac{\partial \bar{v}_i}{\partial x_i} = 0 \tag{17}$$

and in the case of fluctuating flow: $$\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial v_i'}{\partial x_i} + \bar{v}_i\frac{\partial \rho'}{\partial x_i} = 0 \tag{18}$$

and with the equation of state $\rho' = \dfrac{p'}{c_0^2}$:

$$\frac{\partial p'}{\partial t} + \bar{v}_i\frac{\partial p'}{\partial x_i} + \bar{\rho}c_0^2\frac{\partial v_i'}{\partial x_i} = 0 \tag{19}$$

$$\frac{\bar{D}p'}{Dt} + \bar{\rho}c_0^2\frac{\partial v_i'}{\partial x_i} = 0 \tag{20}$$

$$\text{with:} \quad \frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \bar{v}_i\frac{\partial}{\partial x_i} \tag{21}$$

Momentum equation (without viscosity)

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p\delta_{ij}) = 0 \tag{22}$$

Mean flow: $$\bar{\rho}\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} = 0 \tag{23}$$

Fluctuating flow (with linearisation): $$\bar{\rho}\frac{\partial v_i'}{\partial t} + \overline{\rho v}_j \frac{\partial v_i'}{\partial x_j} + \rho' \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \bar{\rho} v_j' \frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial p'}{\partial x_i} = 0 \tag{24}$$

With constant mean flow
(assumption: $\bar{v}_i$ is uniform): $$\bar{\rho}\frac{\partial v_i'}{\partial t} + \overline{\rho v}_j \frac{\partial v_i'}{\partial x_j} + \frac{\partial p'}{\partial x_i} = 0 \tag{25}$$

Wave equation

Following from the continuity equation $$\frac{\partial p'}{\partial t} + \bar{v}_i \frac{\partial p'}{\partial x_i} + \bar{\rho} c_0^2 \frac{\partial v_i'}{\partial x_i} = 0 \tag{26}$$

and the momentum equations $$\bar{\rho}\frac{\partial v_i'}{\partial t} + \overline{\rho v}_j \frac{\partial v_i'}{\partial x_j} + \frac{\partial p'}{\partial x_i} = 0 \tag{27}$$

Result: convective wave equation: $$\frac{\partial^2 p'}{\partial x_i^2} - \frac{1}{c_0^2}\left[\frac{\partial}{\partial t} + \bar{v}_i \frac{\partial}{\partial x_i}\right]^2 p' = 0 \tag{28}$$

with: $$\left[\frac{\partial}{\partial t} + \bar{v}_i \frac{\partial}{\partial x_i}\right]^2 = \left(\frac{\partial^2}{\partial t^2} + 2\bar{v}_i \frac{\partial^2}{\partial x_i \partial t} + \bar{v}_i \bar{v}_j \frac{\partial^2}{\partial x_i \partial x_j}\right) \tag{29}$$

Navier–Stokes equation:
Double decomposition of quantities and time averaging: Reynolds averaging

Assumptions: incompressible flow

for the mean flow: $$\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j^2} - \frac{\partial \left(\overline{v_i' v_j'}\right)}{\partial x_j} \quad \text{Reynolds equation} \tag{30}$$

with stress tensor of fluctuating flow: $$\begin{pmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{pmatrix} = -\begin{pmatrix} \rho\overline{u'^2} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{u'v'} & \rho\overline{v'^2} & \rho\overline{v'w'} \\ \rho\overline{u'w'} & \rho\overline{v'w'} & \rho\overline{w'^2} \end{pmatrix} \tag{31}$$

Triple decomposition of quantities and time averaging (Telionis):

$$f(\vec{x}, t) = \bar{f}(\vec{x}) + \tilde{f}(\vec{x}, t) + f'(\vec{x}, t)$$

for the mean flow:
$$\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j^2} - \frac{\partial \left(\overline{v_i' v_j'}\right)}{\partial x_j} - \frac{\partial \left(\overline{\tilde{v}_i \tilde{v}_j}\right)}{\partial x_j} \tag{32}$$

with two Reynolds stress terms on the right-hand side of the equation:
non-linear contributions due to the random fluctuations and due to organised fluctuations;

for the organised fluctuations:

$$\frac{\partial \tilde{v}_i}{\partial t} + \bar{v}_j \frac{\partial \tilde{v}_i}{\partial x_j} + \tilde{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{v}_i}{\partial x_j^2} + \frac{\partial \left(\overline{\tilde{v}_i \tilde{v}_j}\right)}{\partial x_j} + \frac{\partial \left(\overline{v_i' v_j'}\right)}{\partial x_j} - \frac{\partial \left(\left\langle v_i' v_j' \right\rangle\right)}{\partial x_j} \tag{33}$$

N.2.4 Correlations

$$R_p(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T p(t)p(t+\tau)dt \qquad \text{autocorrelation function} \tag{34}$$

$$R_p(0) = \overline{p^2(t)} \tag{35}$$

$$R_p(\xi) = \frac{1}{L} \int_0^L p(x)p(x+\xi)dx \tag{36}$$

$$R = \frac{\overline{v_1' v_2'}}{\sqrt{\overline{v_1'^2}\ \overline{v_2'^2}}} \qquad \text{correlation function, fluctuations of velocity} \tag{37}$$

N.2.5 Scales

$$\tau_c = \frac{1}{R_p(0)} \int_0^\infty R_p(\tau)d\tau \qquad \text{integral time scale} \tag{38}$$

$$\Lambda = \frac{1}{R_p(0)} \int_0^\infty R_p(\xi)d\xi \qquad \text{integral length scale} \tag{39}$$

$$\Lambda_x = \frac{E_{11}\,(k_1)_{k_1\to 0}}{4 v_{rms}'^2} \tag{40}$$

integral length scale, limiting value of the power spectrum as k_1 approaches zero

$$E(k) = \alpha \frac{\overline{v'^2_{rms}}}{k_e} \frac{\left(k/k_e\right)^4}{\left(1+\left(k/k_e\right)^2\right)^{17/6}} e^{-2\,(k/k_\nu)^2}$$ power spectral density for isotropic turbulence (von Kármán spectrum) (41)

with (Longatte):

$\alpha \approx 1.453$

$k_e \approx 0.747/\Lambda$

$\Lambda = (\overline{v^2})^{3/2} \Big/ \varepsilon$

ε = dissipation rate of turbulent kinetic energy

$k_\nu = (\varepsilon/\nu^3)^{1/4}$

$$l_\tau^2 = \frac{\overline{p^2(t)}}{\overline{\left(\frac{\partial p}{\partial t}\right)^2}}$$ differential time scale (42)

$$l_\tau^2 = -\frac{R_p(0)}{\frac{\partial^2 R_p(0)}{\partial \tau^2}} \tag{43}$$

N.3 The Basic Equations of Fluid Motion

▸ *See also:* Bangalore/Morris (1996); Bailly/Lafon/Candel (1996)

N.3.1 Continuity Equation, Momentum Equation, Energy Equation

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho v_i\right)}{\partial x_i} = 0 \quad \text{respectively} \quad \frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \vec{v}\right) = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} + v_j \frac{\partial \rho}{\partial x_j} = 0 \tag{2}$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} \tag{3}$$

Momentum equation:

Euler equations (viscous terms are neglected):

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \quad \text{respectively} \quad \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} + \frac{1}{\rho}\operatorname{grad} p = 0 \tag{4}$$

$$\frac{Dv_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \tag{5}$$

Reformulation with help of continuity equation:

$$\frac{\partial\left(\rho v_i\right)}{\partial t} + \frac{\partial\left(\rho v_i v_j\right)}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0 \tag{6}$$

Energy equation:

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_i}\left[v_i\left(e+p\right)\right] = 0 \quad \text{respectively} \quad \frac{\partial e}{\partial t} + \operatorname{div}\left[\vec{v}\left(e+p\right)\right] = 0 \tag{7}$$

with: $e = \rho u + \frac{1}{2}\rho\,|v_i|^2$ fluid energy density (per unit volume) (8)

$$u = \frac{p}{\rho\,(\kappa - 1)} \quad \text{specific internal energy} \tag{9}$$

with: $u = c_v T = c_v \frac{p}{\rho R} = c_v \frac{p}{\rho} \frac{1}{c_v\,(\kappa-1)} = \frac{p}{\rho\,(\kappa-1)}$ (10)

$$e = \frac{p}{\kappa - 1} + \frac{1}{2}\rho\,|v_i|^2 \tag{11}$$

Reformulation with help of continuity and momentum equation:

$$\rho\frac{Du}{Dt} + p\frac{\partial v_i}{\partial x_i} = 0 \quad \text{respectively} \quad \frac{Dp}{Dt} + \kappa p\frac{\partial v_i}{\partial x_i} = 0 \tag{12}$$

N.3.2 Thermodynamic Relationships

The law of energy conservation $dq = du + pdv - dr$ (13)

with:
- dq supplied heat
- du internal energy
- pdv mechanical work
- dr friction loss

Internal energy:

u specific internal energy per unit volume

$$du = Tds - pd\left(\frac{1}{\rho}\right) = c_v dT \tag{14}$$

$$u = \frac{p}{\rho\,(\kappa - 1)} \tag{15}$$

Fluid energy:

e fluid energy density per unit volume

$$e = \rho u + \frac{1}{2}\rho\,|\vec{v}|^2 = \frac{p}{\kappa - 1} + \frac{1}{2}\rho\,|\vec{v}|^2 \tag{16}$$

Enthalpy:

$B = h + \frac{1}{2}v^2 (= h_s)$ total enthalpy, stagnation enthalpy, per unit volume (17)

$h = u + \frac{p}{\rho}$ specific enthalpy per unit volume (18)

$$dh = Tds + \frac{dp}{\rho} = c_p dT \tag{19}$$

Entropy:
s specific entropy per unit volume

$$ds = \frac{dq + dr}{T} \quad (20)$$

Further relationships:

$$p = \rho RT \quad \text{equation of state of an ideal gas} \quad (21)$$

$$R = c_p - c_v \quad \text{gas constant, equal to the difference in the specific heats} \quad (22)$$

$$\kappa = \frac{c_p}{c_v} \quad \text{ratio of the specific heats} \quad (23)$$

$$c^2 = \kappa \frac{p_0}{\rho_0} \quad \text{isentropic speed of sound} \quad (24)$$

Pressure-density relation:

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds = \frac{dp}{c^2} + \left(\frac{\partial \rho}{\partial T}\right)_p \left(\frac{\partial T}{\partial s}\right)_p ds = \frac{dp}{c^2} - \frac{\beta \rho T}{c_p} ds = \frac{dp}{c^2} - \frac{\rho}{c_p} ds \quad (25)$$

with:

$$\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{1}{c^2} \quad \text{isentropic sound speed c} \quad (26)$$

$$c_p = T \left(\frac{\partial s}{\partial T}\right)_p \quad \text{specific heat at constant pressure} \quad (27)$$

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_p = \frac{1}{T} \quad \text{coefficient of expansion} \quad (28)$$

N.3.3 Non-linear Perturbation Equations, non-linear Euler Equations

General form of the three-dimensional fluid flow equations:

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0 \quad (29)$$

The general flow quantity A is split into a mean quantity A_0 and a perturbation quantity A', so that the non-linear disturbance equations follow:

$$\frac{\partial E'}{\partial t} + \frac{\partial F'}{\partial x} + \frac{\partial G'}{\partial y} + \frac{\partial H'}{\partial z} + \frac{\partial F'_n}{\partial x} + \frac{\partial G'_n}{\partial y} + \frac{\partial H'_n}{\partial z} = Q = -\left(\frac{\partial F_0}{\partial x} + \frac{\partial G_0}{\partial y} + \frac{\partial H_0}{\partial z}\right) \quad (30)$$

with:
F', G', H' linear perturbation terms
F'_n, G'_n, H'_n non-linear perturbation terms
F_0, G_0, H_0 Q: sum of the divergence of mean convective fluxes.

The abbreviations are calculated from the following three-dimensional equations

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{31}$$

Momentum equations:

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho uw)}{\partial z} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho vv)}{\partial y} + \frac{\partial (\rho wv)}{\partial z} + \frac{\partial p}{\partial y} = 0 \tag{32}$$

$$\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho uw)}{\partial x} + \frac{\partial (\rho vw)}{\partial y} + \frac{\partial (\rho ww)}{\partial z} + \frac{\partial p}{\partial z} = 0$$

Energy equation:

$$\frac{\partial e}{\partial t} + \frac{\partial \left[u\left(e+p\right)\right]}{\partial x} + \frac{\partial \left[v\left(e+p\right)\right]}{\partial y} + \frac{\partial \left[w\left(e+p\right)\right]}{\partial z} = 0 \tag{33}$$

with: $$e = \frac{p}{\kappa - 1} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) \tag{34}$$

The abbreviations in the general form of the non-linear disturbance equations mean in the sequence of the equations below:

- Continuity equation
- Euler equation in x direction
- Euler equation in y direction
- Euler equation in z direction
- Energy equation

$$E' \left\{ \begin{array}{l} \rho' \\ \rho_0 u' + \rho' u_0 + \rho' u' \\ \rho_0 v' + \rho' v_0 + \rho' v' \\ \rho_0 w' + \rho' w_0 + \rho' w' \\ e' \end{array} \right. \qquad F' \left\{ \begin{array}{l} \rho_0 u' + u_0 \rho' \\ u_0^2 \rho' + 2\rho_0 u_0 u' + p' \\ \rho_0 u_0 v' + \rho_0 v_0 u' + u_0 v_0 \rho' \\ \rho_0 u_0 w' + \rho_0 w_0 u' + u_0 w_0 \rho' \\ u'\left(e_0 + p_0\right) + u_0\left(e'_{lin} + p'\right) \end{array} \right. \tag{35,36}$$

$$G' \left\{ \begin{array}{l} \rho_0 v' + v_0 \rho' \\ \rho_0 u_0 u' + \rho_0 u_0 v' + u_0 v_0 \rho' \\ v_0^2 \rho' + 2\rho_0 v_0 v' + p' \\ \rho_0 v_0 w' + \rho_0 w_0 v' + v_0 w_0 \rho' \\ v'\left(e_0 + p_0\right) + v_0\left(e'_{lin} + p'\right) \end{array} \right. \qquad H' \left\{ \begin{array}{l} \rho_0 w' + w_0 \rho' \\ \rho_0 w_0 u' + \rho_0 u_0 w' + u_0 w_0 \rho' \\ \rho_0 w_0 v' + \rho_0 v_0 w' + v_0 w_0 \rho' \\ w_0^2 \rho' + 2\rho_0 w_0 w' + p' \\ w'\left(e_0 + p_0\right) + w_0\left(e'_{lin} + p'\right) \end{array} \right. \tag{37,38}$$

$$F'_n \left\{ \begin{array}{l} \rho' u' \\ 2u_0 \rho' u' + \rho_0 u'^2 + \rho' u'^2 \\ \rho_0 u' v' + u_0 \rho' v' + v_0 \rho' u' + \rho' u' v' \\ \rho_0 u' w' + u_0 \rho' w' + w_0 \rho' u' + \rho' u' w' \\ u'\left(e' + p'\right) + u_0 e'_{nonlin} \end{array} \right. \tag{39}$$

$$G'_n \begin{cases} \rho'v' \\ \rho_0 u'v' + u_0\rho'v' + v_0\rho'u' + \rho'u'v' \\ 2v_0\rho'v' + \rho_0 v'^2 + \rho'v'^2 \\ \rho_0 v'w' + v_0\rho'w' + w_0\rho'v' + \rho'v'w' \\ v'\left(e' + p'\right) + v_0 e'_{\mathrm{nonlin}} \end{cases} \tag{40}$$

$$H'_n \begin{cases} \rho'w' \\ \rho_0 u'w' + u_0\rho'w' + w_0\rho'u' + \rho'u'w' \\ \rho_0 v'w' + v_0\rho'w' + w_0\rho'v' + \rho'v'w' \\ 2w_0\rho'w' + \rho_0 w'^2 + \rho'w'^2 \\ w'\left(e' + p'\right) + w_0 e'_{\mathrm{nonlin}} \end{cases} \tag{41}$$

$$F_0 \begin{cases} \rho_0 u_0 \\ \rho_0 u_0^2 + p_0 \\ \rho_0 u_0 v_0 \\ \rho_0 u_0 w_0 \\ u_0\left(e_0 + p_0\right) \end{cases} \quad G_0 \begin{cases} \rho_0 v_0 \\ \rho_0 u_0 v_0 \\ \rho_0 v_0^2 + p_0 \\ \rho_0 v_0 w_0 \\ v_0\left(e_0 + p_0\right) \end{cases} \quad H_0 \begin{cases} \rho_0 w_0 \\ \rho_0 u_0 w_0 \\ \rho_0 v_0 w_0 \\ \rho_0 w_0^2 + p_0 \\ w_0\left(e_0 + p_0\right) \end{cases} \tag{42,43,44}$$

Fluctuating part of energy density:

$$\begin{aligned} e' = &\frac{p'}{\kappa - 1} + \frac{1}{2}\rho'\left(u_0^2 + v_0^2 + w_0^2\right) + \frac{1}{2}\left(\rho_0 + \rho'\right)\left(u'^2 + v'^2 + w'^2\right) \\ &+ \left(\rho_0 + \rho'\right)\left(u'u_0 + v'v_0 + w'w_0\right) \end{aligned} \tag{45}$$

subdivided in two terms:

- linear term

$$e'_{\mathrm{lin}} = \frac{p'}{\kappa - 1} + \frac{1}{2}\rho'\left(u_0^2 + v_0^2 + w_0^2\right) + \rho_0\left(u'u_0 + v'v_0 + w'w_0\right) \tag{46}$$

- non-linear term

$$e'_{\mathrm{nonlin}} = \frac{1}{2}\left(\rho_0 + \rho'\right)\left(u'^2 + v'^2 + w'^2\right) \tag{47}$$

Stationary part of energy density:

$$e_0 = \frac{p_0}{\kappa - 1} + \frac{1}{2}\rho_0\left(u_0^2 + v_0^2 + w_0^2\right) \tag{48}$$

N.3.4 Formulation of Euler Equations to Use in Computational Aeroacoustics (CAA)

Basic equations:

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \tag{49}$$

$$\frac{Dv_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \tag{50}$$

Decomposition: $\vec{v}\,(\vec{x}, t) = \vec{v}_0\,(\vec{x}, t) + \vec{v}'\,(\vec{x}, t)$ (51)

with: $$\vec{v}_0(\vec{x},t) = \frac{1}{T}\int_t^{t+T} \vec{v}(\vec{x},t)\,dt \tag{52}$$

$$T_1 \ll T \underset{\approx}{\overset{\ll}{-}} T_2 \tag{53}$$

T_1 time scale of turbulent fluctuations

T_2 time scale of large variations in mean flow

$$\frac{1}{T}\int_t^{t+T} \vec{v}'(\vec{x},t)\,dt = \overline{\vec{v}'} = 0 \tag{54}$$

The short forms $v = v_0 + v'$; $p = p_0 + p'$; $\rho = \rho_0 + \rho'$ are introduced: (55)

- in the continuity equation:

$$\frac{\partial\rho'}{\partial t} + v_{0j}\frac{\partial\rho'}{\partial x_j} + v_j'\frac{\partial\rho_0}{\partial x_j} + \rho_0\frac{\partial v_i'}{\partial x_i} + \rho'\frac{\partial v_{0i}}{\partial x_i} = -\frac{\partial\rho_0}{\partial t} - v_{0j}\frac{\partial\rho_0}{\partial x_j} - v_j'\frac{\partial\rho'}{\partial x_j} - \rho_0\frac{\partial v_{0i}}{\partial x_i} - \rho'\frac{\partial v_i'}{\partial x_i} \tag{56}$$

with condition of the incompressibility: $\rho_0 = \text{const.}$ and $\dfrac{\partial v_{0i}}{\partial x_i} = 0$:

$$\frac{\partial\rho'}{\partial t} + v_{0j}\frac{\partial\rho'}{\partial x_j} + \rho_0\frac{\partial v_i'}{\partial x_i} = -v_j'\frac{\partial\rho'}{\partial x_j} - \rho'\frac{\partial v_i'}{\partial x_i} \tag{57}$$

- in the momentum equation:

$$\frac{\partial v_i'}{\partial t} + v_{0j}\frac{\partial v_i'}{\partial x_j} + v_j'\frac{\partial v_{0i}}{\partial x_j} - \frac{\rho'}{\rho_0^2}\frac{\partial p_0}{\partial x_i} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} = -\frac{\partial v_{0i}}{\partial t} - v_{0j}\frac{\partial v_{0i}}{\partial x_j} - v_j'\frac{\partial v_i'}{\partial x_j} + \frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x_i} - \frac{1}{\rho_0}\frac{\partial p_0}{\partial x_i} \tag{58}$$

with: $$\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{\frac{1}{\rho_0}}{1 + \frac{\rho'}{\rho_0}} \approx \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right) = \frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2} \tag{59}$$

$(\rho' \ll \rho_0)$

Averaging over time T_1:

$$\frac{\partial v_{0i}}{\partial t} + v_{0j}\frac{\partial v_{0i}}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial p_0}{\partial x_i} = \overline{-v_j'\frac{\partial v_i'}{\partial x_j}} + \overline{\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x_i}} \tag{60}$$

$$\frac{\partial v_i'}{\partial t} + v_{0j}\frac{\partial v_i'}{\partial x_j} + v_j'\frac{\partial v_{0i}}{\partial x_j} - \frac{\rho'}{\rho_0^2}\frac{\partial p_0}{\partial x_i} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} = -v_j'\frac{\partial v_i'}{\partial x_j} + \frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x_i} + \overline{v_j'\frac{\partial v_i'}{\partial x_j}} - \overline{\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x_i}} \tag{61}$$

Introducing decomposition in turbulent and acoustic fluctuating quantities:

$$\rho' = \rho_t + \rho_a; \quad \vec{v}' = \vec{v}_t + \vec{v}_a; \quad p' = p_t + p_a \tag{62}$$

furthermore introducing the approximate condition of incompressibility of turbulent flow:
$\rho_t \approx 0$ and $\partial v_{ti}/\partial x_i = 0$ results:

- for the continuity equation:

$$\frac{\partial \rho_a}{\partial t} + v_{0j}\frac{\partial \rho_a}{\partial x_j} + \rho_0 \frac{\partial v_{ai}}{\partial x_i} = Q_{at}^{\text{cont.eq.}} + Q_{aa}^{\text{cont.eq.}} \tag{63}$$

with: $$Q_{at}^{\text{cont.eq.}} = -v_{tj}\frac{\partial \rho_a}{\partial x_j}\,^{*)} \tag{64}$$

$$Q_{aa}^{\text{cont.eq.}} = -v_{aj}\frac{\partial \rho_a}{\partial x_j} - \rho_a \frac{\partial v_{ai}}{\partial x_i} \tag{65}$$

- for the momentum equation:

$$\frac{\partial v_{ai}}{\partial t} + v_{0j}\frac{\partial v_{ai}}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial p_a}{\partial x_i} + v_{aj}\frac{\partial v_{0i}}{\partial x_j} - \frac{\rho_a}{\rho_0^2}\frac{\partial p_0}{\partial x_i} = Q_{ot}^{\text{mom.eq.}} + Q_{tt}^{\text{mom.eq.}} + Q_{at}^{\text{mom.eq.}} + Q_{aa}^{\text{mom.eq.}} \tag{66}$$

with: $$Q_{0t}^{\text{mom.eq.}} = -\frac{\partial v_{ti}}{\partial t} - v_{0j}\frac{\partial v_{ti}}{\partial x_j} - v_{tj}\frac{\partial v_{0i}}{\partial x_j} - \frac{1}{\rho_0}\frac{\partial p_t}{\partial x_i} \tag{67}$$

sound source due to the interaction between the mean flow and turbulent flow: partly shear noise;

$$Q_{tt}^{\text{mom.eq.}} = -v_{tj}\frac{\partial v_{ti}}{\partial x_j} + \overline{v_{tj}\frac{\partial v_{ti}}{\partial x_j}} \tag{68}$$

sound generated by turbulent interaction: self-noise;

$$Q_{at}^{\text{mom.eq.}} = -v_{tj}\frac{\partial v_{ai}}{\partial x_j} - v_{aj}\frac{\partial v_{ti}}{\partial x_j} + \frac{\rho_a}{\rho_0^2}\frac{\partial p_t}{\partial x_i} + \overline{v_{tj}\frac{\partial v_{ai}}{\partial x_j}} + \overline{v_{aj}\frac{\partial v_{ti}}{\partial x_j}} - \overline{\frac{\rho_a}{\rho_0^2}\frac{\partial p_t}{\partial x_i}} \tag{69}$$

sound generated by interaction between turbulence and sound;

$$Q_{aa}^{\text{mom.eq.}} = -v_{aj}\frac{\partial v_{ai}}{\partial x_j} + \frac{\rho_a}{\rho_0^2}\frac{\partial p_a}{\partial x_i} + \overline{v_{aj}\frac{\partial v_{ai}}{\partial x_j}} - \overline{\frac{\rho_a}{\rho_0^2}\frac{\partial p_a}{\partial x_i}} \tag{70}$$

sound generated by sound, e.g. scattering of sound.

N.4 The Equations of Linear Acoustics

The basic equations of fluid mechanics to use in the acoustics are as follows:

- equation of continuity, law of conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = \dot{M} \quad \text{(in tensor notation, double suffix summation convention)} \tag{1}$$

with: $\dot{M}$ mass flux per unit volume

- equation of motion, law of conservation of momentum, Newton's law of motion, momentum equation, Navier–Stokes equations (for a viscous fluid):

$$\rho\frac{\partial v_i}{\partial t} + \rho\, v_j\frac{\partial v_i}{\partial x_j} = \rho F_i - \frac{\partial}{\partial x_j}\left(p\delta_{ij} - \tau_{ij}\right) \tag{2}$$

*) See Preface to the 2nd edition.

and in the form due to Reynolds:

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p_{ij}\right) = \rho F_i + \dot{M} v_i \tag{3}$$

with: $\quad p_{ij} = p\delta_{ij} - \tau_{ij}$ (4)

- equation of state (the pressure-density):

$$d\rho = \frac{dp}{c^2} - \frac{\rho}{c_p} ds \tag{5}$$

and with the isentropic condition $ds = 0$:

$$d\rho = \frac{dp}{c^2} \tag{6}$$

Premises of linear acoustics:
Assumptions, applied to the basic equations of fluid mechanics:

- without mass sources: $\dot{M} = 0$;
- without external forces: $F_i = 0$;
- inviscid flow: $p_{ij} = p \quad (\tau_{ij} = 0)$;
- decomposition of all physical quantities in mean values and fluctuating components:
  $p = \bar{p} + p' \quad \rho = \bar{\rho} + \rho' \quad v_i = \bar{v}_i + v_i'$

  with the following assumptions and definitions:
  $\bar{p}$, $\bar{\rho}$ constant in time and space;
  $p' \ll \bar{p}$ (with: p' sound pressure);
  $\rho' \ll \bar{\rho}$ (with: ρ' acoustic fluctuation of mass density);
  $\bar{v}_i = 0$ without mean flow;
  v_i' acoustic part of fluid velocity, particle velocity;
- linearisation of equations of fluid mechanics;
- irrotational flow: $\vec{\omega} = \mathrm{rot}\vec{v} = \nabla \times \vec{v} = 0$.

The basic equations of linear acoustics:

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial v_i'}{\partial x_i} = 0 \quad \text{linearised continuity equation;} \tag{7}$$

$$\bar{\rho}\frac{\partial v_i'}{\partial t} + \frac{\partial p'}{\partial x_i} = 0 \quad \text{linearised Euler equation (momentum equation);} \tag{8}$$

$$\frac{p'}{\rho'} = c^2 \quad \text{linearised equation of state} \tag{9}$$

with: p', ρ', v_i' sound pressure, acoustic density fluctuation, particle velocity.

The homogeneous wave equation of linear acoustics;
there are the following homogeneous wave equations:

- pressure fluctuations (sound pressure): $$\frac{\partial^2 p}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad (10)$$
- density fluctuations: $$\frac{\partial^2 \rho}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 \rho}{\partial t^2} = 0 \quad (11)$$
- velocity fluctuations: $$\frac{\partial^2 v_i}{\partial x_j^2} - \frac{1}{c_0^2}\frac{\partial^2 v_i}{\partial t^2} = 0 \quad \text{resp.} \quad \Delta\vec{v} - \frac{1}{c_0^2}\frac{\partial^2 \vec{v}}{\partial t^2} = 0 \quad (12)$$

  with: $$\Delta = \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{Laplace operator} \quad (13)$$
- velocity potential: $$\frac{\partial^2 \Phi}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \quad \text{resp.} \quad \Delta\Phi - \frac{1}{c_0^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \quad (14)$$

Wave equation for uniform flow:

$$\Delta\Phi - \left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{U_\infty}{c_0}\frac{\partial}{\partial x}\right)^2 \Phi = 0 \quad \text{convective wave equation} \quad (15)$$

$$\Delta\Phi - \frac{1}{c_0^2}\frac{D^2\Phi}{Dt^2} = 0 \quad (16)$$

with: $$\frac{D^2}{Dt^2} = \left(\frac{\partial}{\partial t} + U_\infty\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial t^2} + 2U_\infty\frac{\partial^2}{\partial x\partial t} + U_\infty^2\frac{\partial^2}{\partial x^2} \quad (17)$$

U_∞ uniform time-averaged velocity in the x direction;

$$\left(1 - \frac{U_\infty^2}{c_0^2}\right)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - 2\frac{U_\infty}{c_0^2}\frac{\partial^2\Phi}{\partial x\partial t} - \frac{1}{c_0^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \quad (18)$$

$$\left[\Delta - \left(M\frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)^2\right]\Phi = 0 \quad (19)$$

with: $M = \dfrac{U_\infty}{c_0}$ Mach number.

With harmonic components, separating the time factor $e^{j\omega t}$:

$$\left\{\Delta - M^2\frac{\partial^2}{\partial x^2} - 2jkM\frac{\partial}{\partial x} + k^2\right\}\Phi = 0 \quad (20)$$

with: Φ complex amplitude of the velocity potential,
$k = \omega/c$ wave number.

Cylindrical co-ordinate system r, x, Θ:
(with Laplace operator in cylindrical co-ordinates)

$$\Delta\Phi - \frac{1}{c_0^2}\frac{D^2\Phi}{Dt^2} = 0 \quad \text{convective wave equation of velocity potential} \quad (21)$$

becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\Theta^2}-\frac{1}{c_0^2}\frac{D^2}{Dt^2}\right)\Phi=0. \tag{22}$$

Spherical co-ordinate system r, ϑ, φ
(with Laplace operator in spherical co-ordinates):

$$\left\{\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)-\frac{1}{c_0^2}\frac{D^2}{Dt^2}\right\}\Phi=0. \tag{23}$$

N.5 Inhomogeneous Wave Equation, Lighthill's Acoustic Analogy

▶ *See also:* Lighthill (1952); Howe (1998); Crighton et al. (1992); Goldstein (1976); Curle (1955)

N.5.1 Lighthill's Inhomogeneous Wave Equation

Equation of continuity, the law of conservation of mass:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_i)}{\partial x_i}=0 \tag{1}$$

Equation of motion, in fact in the form due to Reynolds:

$$\frac{\partial}{\partial t}\left(\rho v_i\right)+\frac{\partial}{\partial x_j}\left(\rho v_i v_j+p_{ij}\right)=0 \tag{2}$$

with: $p_{ij}=p\delta_{ij}-\tau_{ij}$ compressive stress tensor (3)

$\tau_{ij}=\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}$ viscous stresses (4)

Lighthill's equation is obtained by taking the time derivative of the continuity equation and subtracting the divergence of the momentum equation:

$$\frac{\partial}{\partial t}\left(\text{continuity equation}\right)-\frac{\partial}{\partial x_i}\left(\text{momentum equation}\right)$$

eliminating the term ρv_i, that is the mass density flux in the continuity equation but the momentum density in the momentum equation:

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x_i\partial x_j}\left(\rho v_i v_j+p_{ij}\right) \tag{5}$$

addition of the term $-c_0^2\frac{\partial^2\rho}{\partial x_i^2}$ (with c_0 characteristic speed of sound in the medium surrounding the flow region), gives

$$\frac{\partial^2\rho}{\partial t^2}-c_0^2\frac{\partial^2\rho}{\partial x_i^2}=\frac{\partial^2}{\partial x_i\partial x_j}\left(\rho v_i v_j+p_{ij}-c_0^2\rho\delta_{ij}\right)=q \tag{6}$$

Lighthill's inhomogeneous wave equation

with: $T_{ij} = \rho v_i v_j + p_{ij} - c_0^2 \rho \delta_{ij} = \rho v_i v_j + \left(p - c_0^2 \rho\right) - \tau_{ij}$ (7)

T_{ij} Lighthill's stress tensor
q source term.

The fluid mechanical problem of calculating the aerodynamic sound is formally equivalent to solving this equation for radiation into a stationary ideal fluid produced by a distribution of quadrupole sources whose strength per unit volume is the Lighthill stress tensor T_{ij} (Howe).

Incompressible approximation: $T_{ij} \approx \rho_0 v_i v_j$ (8)

with assumptions:

- low Mach number, velocity fluctuations are of order $\rho_0 Ma^2$,
- isentropic flow,
- high Reynolds number, viscous effects are much smaller than inertial effects, the viscous stress tensor is neglected compared with the Reynolds stresses $\rho v_i v_j$,
- furthermore: viscous terms in T_{ij} are

$$\tau_{ij} = \mu \frac{\partial v_i}{\partial x_j}, \quad \text{so that} \quad \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = \mu \frac{\partial^3 v_i}{\partial x_j \partial x_i \partial x_j}, \tag{9}$$

corresponding to a octupole source (a very ineffective sound radiator).

$T_{ij} \approx \rho_0 v_i v_j$ can be used as a source term, generating the acoustic field.

Lighthill's development can be expanded as an inhomogeneous wave equation in general form, starting from:

- equation of continuity with external sources:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = \dot{M} \tag{10}$$

with: $\dot{M}$ external source flux of mass (per unit volume);

- equation of motion with external forces:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = F_i - \frac{\partial}{\partial x_j}\left(p\delta_{ij} - \tau_{ij}\right) \tag{11}$$

with: F_i external forces (per unit volume),

or in the form due to Reynolds:

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p_{ij}\right) = F_i + \dot{M} v_i. \tag{12}$$

Inhomogeneous wave equation in general form:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial \dot{M}}{\partial t} - \frac{\partial}{\partial x_i}\left(F_i + \dot{M} v_i\right) + \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j + p_{ij} - c_0^2 \rho \delta_{ij}\right) = q \tag{13}$$

Source terms:

$$q = \frac{\partial \dot{M}}{\partial t} - \frac{\partial}{\partial x_i}\left(\dot{M} v_i + F_i\right) + \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} \tag{14}$$

with: $\dfrac{\partial \dot{M}}{\partial t}$ monopole source (15)

$-\dfrac{\partial}{\partial x_i}\left(F_i + \dot{M}v_i\right)$ dipole source (16)

$\dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ quadrupole source (17)

N.5.2 Solutions of Inhomogeneous Wave Equation

Using the generalisation of Kirchhoff's equation:

$$p(x_i, t) = \frac{1}{4\pi}\int_S \left(\frac{1}{c_0 r}\frac{\partial r}{\partial n}\frac{\partial p}{\partial t} + \frac{1}{r^2}\frac{\partial r}{\partial n}p + \frac{1}{r}\frac{\partial p}{\partial n}\right)_\tau dS + \frac{1}{4\pi}\int_V \left(\frac{q}{r}\right)_\tau dV, \tag{18}$$

a formal solution of the inhomogeneous wave equation follows:

$$\begin{aligned} p(x_i, t) = &\int_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV - \int_S \frac{1}{4\pi r}\left(\frac{\partial(\rho v_i)}{\partial t}\right)_\tau n_i dS \\ &- \frac{\partial}{\partial x_i}\int_V \frac{1}{4\pi r}\left(F_i + \dot{M}v_i\right)_\tau dV + \frac{\partial}{\partial x_i}\int_S \frac{1}{4\pi r}\left(\rho v_i v_j + p_{ij}\right)_\tau n_j dS \\ &+ \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{1}{4\pi r}\left(T_{ij}\right)_\tau dV \end{aligned} \tag{19}$$

with: $(....)_\tau$ retarded source strength,

$\tau = t - \dfrac{r}{c_0}$ retardation time,

$r = |x_i - y_i|$ distance between source point and observer point,

x_i vector from origin of co-ordinates to the observer point,

y_i vector from origin of co-ordinates to the source point.

Free-space solution:

$$p(x_i, t) = \int_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV - \frac{\partial}{\partial x_i}\int_V \frac{1}{4\pi r}\left(F_i + \dot{M}v_i\right)_\tau dV + \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{1}{4\pi r}\left(T_{ij}\right)_\tau dV \tag{20}$$

Decomposition into the near-field and the far-field solution:

$$\begin{aligned} p(x_i, t) = &\int_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV \\ &+ \frac{1}{4\pi}\int_V \frac{1}{r^2}\left(F_i + \dot{M}v_i\right)_\tau \frac{(x_i - y_i)}{r} dV + \frac{1}{4\pi c_0}\int_V \frac{1}{r}\frac{\partial}{\partial t}\left(F_i + \dot{M}v_i\right)_\tau \frac{(x_i - y_i)}{r} dV \\ &+ \frac{1}{4\pi}\int_V \frac{1}{r^2}\left(T_{ij}\right)_\tau \left[\frac{3}{r^2}(x_i - y_i)(x_j - y_j) - \delta_{ij}\right] dV \\ &+ \frac{1}{4\pi c_0^2}\int_V \frac{1}{r}\frac{\partial^2 \left(T_{ij}\right)_\tau}{\partial t^2}\left[\frac{1}{r^2}(x_i - y_i)(x_j - y_j)\right] dV \end{aligned} \tag{21}$$

with the far-field solution:

$$p(x_i,t)=\int\limits_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV+\frac{1}{4\pi c_0}\int\limits_V \frac{1}{r}\frac{\partial}{\partial t}\left(F_i+\dot{M}v_i\right)_\tau \frac{(x_i-y_i)}{r}dV$$
$$+\frac{1}{4\pi c_0^2}\int\limits_V \frac{1}{r}\frac{\partial^2\left(T_{ij}\right)_\tau}{\partial t^2}\left[\frac{1}{r^2}(x_i-y_i)(x_j-y_j)\right]dV \tag{22}$$

Solution with solid boundaries in flow (Curle):

$$p(x_i,t)=\int\limits_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV-\int\limits_S \frac{1}{4\pi r}\left(\frac{\partial(\rho v_i)}{\partial t}\right)_\tau n_i dS$$
$$-\frac{\partial}{\partial x_i}\int\limits_V \frac{1}{4\pi r}\left(F_i+\dot{M}v_i\right)_\tau dV+\frac{\partial}{\partial x_i}\int\limits_S \frac{1}{4\pi r}\left(\rho v_i v_j+p_{ij}\right)_\tau n_j dS \tag{23}$$
$$+\frac{\partial^2}{\partial x_i\partial x_j}\int\limits_V \frac{1}{4\pi r}\left(T_{ij}\right)_\tau dV$$

with: surface S is stationary,
the body can have mass injection or suction: $v_i n_i = v_n \neq 0$,
body vibrations: $\dfrac{\partial v_n}{\partial t}\neq 0$.

Solution when the boundaries S are rigid and impermeable:

$$p(x_i,t)=\int\limits_V \frac{1}{4\pi r}\left(\frac{\partial \dot{M}}{\partial t}\right)_\tau dV$$
$$-\frac{\partial}{\partial x_i}\int\limits_V \frac{1}{4\pi r}\left(F_i+\dot{M}v_i\right)_\tau dV+\frac{\partial}{\partial x_i}\int\limits_S \frac{1}{4\pi r}\left(p_{ij}\right)_\tau n_j dS \tag{24}$$
$$+\frac{\partial^2}{\partial x_i\partial x_j}\int\limits_V \frac{1}{4\pi r}\left(T_{ij}\right)_\tau dV$$

Sound from free turbulence:

$$p(x_i,t)=\frac{1}{4\pi}\int\limits_V \frac{1}{r}\left(\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\right)_\tau dV \tag{25}$$

$$p(x_i,t)=\frac{1}{4\pi}\frac{\partial^2}{\partial x_i\partial x_j}\int\limits_V \frac{1}{r}\left(T_{ij}\right)_\tau dV \tag{26}$$

$$p(x_i,t)=\frac{1}{4\pi c_0^2}\int\limits_V \frac{1}{r}\left\{\frac{\partial^2\left(T_{ij}\right)}{\partial t^2}\right\}_\tau\left[\frac{1}{r^2}(x_i-y_i)(x_j-y_j)\right]dV \tag{27}$$

in the far field $x_i \gg y_i$:

$$p(x_i,t)=\frac{1}{4\pi r}\int\limits_V \left(\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\right)_\tau dV \tag{28}$$

$$p(x_i, t) = \frac{1}{4\pi r} \frac{\partial^2}{\partial x_i \partial x_j} \int_V (T_{ij})_\tau \, dV \tag{29}$$

$$p(x_i, t) = \frac{1}{4\pi c_0^2 r} \frac{x_i x_j}{r^2} \int_V \left[\frac{\partial^2 (T_{ij})}{\partial t^2}\right]_\tau dV \tag{30}$$

$$p(x_i, t) = \frac{1}{4\pi c_0^2 r} \int_V \left[\frac{\partial^2 (T_{rr})}{\partial t^2}\right]_\tau dV \tag{31}$$

$$\text{with:} \quad T_{rr} = \frac{x_i x_j}{r^2} T_{ij}$$

Solutions using a Fourier transformation:

$$p(x_i, \omega) = -\frac{\omega^2}{4\pi c_0^2} \frac{e^{-j\omega x/c_0}}{r} \int_0^T \int_V e^{j\vec{k}\cdot\vec{y}} e^{-j\omega t} T_{rr}(y_i, t) \, dV dt \tag{32}$$

Solutions neglecting viscous stresses: $T_{ij} = \rho v_i v_j$:

$$p(x_i, t) = \frac{x_i x_j}{4\pi c_0^2 r^3} \int_V \frac{\partial^2 (\rho v_i v_j)_\tau}{\partial t^2} dV \tag{33}$$

N.6 Acoustic Analogy with Source Terms Using Pressure

► *See also:* Ribner (1959); Meecham (1958, 1981)

N.6.1 Lighthill's Representation of the Source Term with Use of Pressure

Reformulation of the source strength using equations of continuity and momentum to introduce the pressure as a source (see Lighthill):

$$\frac{\partial}{\partial t}(\rho v_i v_j) = p_{ik} \frac{\partial v_j}{\partial x_k} + p_{jk} \frac{\partial v_i}{\partial x_k} - \frac{\partial}{\partial x_k}(\rho v_i v_j v_k + p_{ik} v_j + p_{jk} v_i), \tag{1}$$

neglecting the viscous stresses and the octupole source:

$$\frac{\partial}{\partial t}(\rho v_i v_j) \approx p \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2}$$

calculating the source term:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} T_{ij} &\approx \frac{\partial^2}{\partial t^2}(\rho v_i v_j) \\ &\approx \frac{\partial}{\partial t}\left[p\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] = \frac{\partial p}{\partial t}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{\partial}{\partial t}\left[p\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)\right] \end{aligned} \tag{3}$$

first term: shear noise,
second term: self-noise.

N.6.2 Pressure-Source theory (Ribner)

Decomposition of pressure fluctuations inside flow $p - p_0 = p' + p_a$

with: $p - p_0$ pressure fluctuations,

p_0 constant pressure,

p' pseudo-sound:
pressure fluctuations in a nearly incompressible flow, that is, inside the flow pressure fluctuations are dominated by inertial effects rather than compressibility,

p_a acoustic pressure.

Pseudo-sound pressure is a solution of Poisson's equation:
application of divergence operator to the momentum equation

$$\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial t}\left(\rho v_i\right)\right] + \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j + p_{ij}\right) = 0 \tag{4}$$

Incompressible flow $\dfrac{\partial v_i}{\partial x_i} = 0$ (without viscosity) leads to the Poisson equations:

$$\frac{\partial^2 p}{\partial x_i^2} = -\frac{\partial^2\left(\rho v_i v_j\right)}{\partial x_i \partial x_j} \tag{5}$$

$$\frac{\partial^2 p'}{\partial x_i^2} = -\frac{\partial^2\left(\rho v_i v_j\right)}{\partial x_i \partial x_j} \tag{6}$$

$$\frac{\partial^2 p'}{\partial x_i^2} = -\rho_0 \frac{\partial^2\left(v_i v_j\right)}{\partial x_i \partial x_j} = -\rho_0 \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \tag{7}$$

Inhomogeneous wave equation:

$$\frac{\partial^2 p}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p'}{\partial x_i^2} \tag{8}$$

and with $p - p_0 = p' + p_a$:

$$\frac{\partial^2 p_a}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} = \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} \quad \text{Ribner's inhomogeneous wave equation} \tag{9}$$

The far-field solution:

$$p\left(x_i, t\right) = -\frac{1}{4\pi c_0^2}\int_V \frac{1}{r}\left(\frac{\partial^2 p'}{\partial t^2}\right)_\tau dV \tag{10}$$

in comparison with Lighthill:

$$p\left(x_i, t\right) = \frac{1}{4\pi c_0^2}\int_V \frac{1}{r}\frac{\partial^2\left(T_{ij}\right)_\tau}{\partial t^2}\left[\frac{1}{r^2}(x_i - y_i)(x_j - y_j)\right] dV \tag{11}$$

(In far field both equations are identical!)

N.6.3 Pressure-Source Theory (Meecham)

Expanding the field about an incompressible flow, low Mach number fluctuating flow

$$p = p_0 + p_a \tag{12}$$

$$\rho = \rho_0 + \rho_1 + \rho_a \tag{13}$$

$$v = v_0 + v_a \tag{14}$$

with: subscript 0 for incompressible flow quantities
p_a, ρ_a, v_a acoustic field

Definition of ρ_1:
The change in density ρ_1 is caused by the nearly incompressible pressure change p_0:

$$\frac{\bar{D}\rho_1}{Dt} = \frac{1}{c^2}\frac{\bar{D}p_0}{Dt} \tag{15}$$

with: $\dfrac{\bar{D}}{Dt} = \dfrac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla$ substantial derivative, following the incompressible flow v_0.

Wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} - \nabla^2 p_a = -\frac{\partial^2 \rho_1}{\partial t^2} + \nabla \cdot \left(\rho_1 \frac{\partial \vec{v}_0}{\partial t}\right) \tag{16}$$

The second source term on the right side may be neglected.

Reformulation with $$\frac{\partial p_0}{\partial t} = c_0^2 \frac{\partial \rho_1}{\partial t} \tag{17}$$

leads to:

$$\frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} - \nabla^2 p_a = -\frac{1}{c_0^2}\frac{\partial^2 p_0}{\partial t^2} \quad \text{Meecham's inhomogeneous wave equation.} \tag{18}$$

Solution:

$$p(x_i, t) = -\frac{1}{4\pi c_0^2}\int\limits_V \frac{1}{r}\left(\frac{\partial^2 p_0}{\partial t^2}\right)_\tau dV \tag{19}$$

(see Ribner's solution mentioned above).

The incompressible pressure fluctuations are the solution to Poisson's differential equation:

$$\frac{\partial^2 p_0}{\partial x_i^2} = -\rho_0 \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j}. \tag{20}$$

N.7 Acoustic Analogy with Mean Flow Effects, in the Form of Convective Inhomogeneous Wave Equation

► *See also:* Phillips (1960); Pao (1972); Lilley (1958, 1993, 1999); Legendre (1981); Morfey (2000); Goldstein/Howes (1973); Ribner (1996); Albring (1981); Detsch (1976); Dittmar (1983)

N.7.1 Phillips's Convective Inhomogeneous Wave Equation

$$\frac{D^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[c^2\frac{\partial\Pi}{\partial x_i}\right] = \kappa\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \frac{D}{Dt}\left(\frac{\kappa}{c_p}\frac{Ds}{Dt}\right) - \frac{\partial}{\partial x_i}\left\{\frac{\kappa}{\rho}\frac{\partial\tau_{ij}}{\partial x_j}\right\} + \frac{D}{Dt}\left(\frac{\kappa}{\rho}\dot{M}\right) - \frac{\partial}{\partial x_i}\left\{\frac{\kappa}{\rho}F_i\right\} \tag{1}$$

with: $\Pi = \ln\frac{p}{p_0}$ definition of a dimensionless logarithmic pressure ratio

p_0 constant reference pressure

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij} \quad \text{viscous stresses} \tag{2}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} \quad \text{substantial derivative.}$$

Left-hand side: corresponding to a wave equation in a moving medium with variable speed of sound (Phillips);

Right-hand side: contains propagation terms (in the first member) and source terms, generation of pressure fluctuations by velocity fluctuations in the fluid, by effects of entropy fluctuations, of fluid viscosity and by external mass and force sources (Goldstein).

Neglecting the effects of viscosity and thermal conductivity, furthermore the external mass and force sources:

$$\frac{D^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[c^2\frac{\partial\Pi}{\partial x_i}\right] = \kappa\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \tag{3}$$

in comparison with Lighthill's equation:

$$\frac{\partial^2\rho}{\partial t^2} - c_0^2\frac{\partial^2\rho}{\partial x_i^2} = \frac{\partial^2\left(\rho v_i v_j\right)}{\partial x_i\partial x_j} \tag{4}$$

(Both equations are identical with assumptions: low Mach number and constant speed of sound.)

Example: shear flow $v_i = \bar{v}_i + v_i'$ $\bar{v}_i = \left\{\bar{v}_x(y); 0; 0\right\}$

$$\frac{\bar{D}^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[\overline{c^2}\frac{\partial\Pi}{\partial x_i}\right] = 2\kappa\frac{\partial\bar{v}_x}{\partial y}\frac{\partial v_y'}{\partial x} + \kappa\frac{\partial v_j'}{\partial x_i}\frac{\partial v_i'}{\partial x_j} \tag{5}$$

with: $\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \bar{v}_i\frac{\partial}{\partial x_i}$

$\overline{c^2}$ mean value the square of the speed of sound

right-hand side: first term: shear noise
second term: self noise

In comparison with the equation of sound propagation in non-uniform flow:

$$\frac{\bar{D}^2\Pi'}{Dt^2} - \frac{\partial}{\partial x_i}\left[c_0^2\frac{\partial \Pi'}{\partial x_i}\right] = 2\kappa\frac{\partial \bar{v}_x}{\partial y}\frac{\partial v'_y}{\partial x} + \kappa\left[\frac{\partial v'_j}{\partial x_i}\frac{\partial v'_i}{\partial x_j} - \overline{\frac{\partial v'_j}{\partial x_i}\frac{\partial v'_i}{\partial x_j}}\right] \tag{6}$$

with: $\Pi' = \ln\frac{p'}{p_0}$; $p' = p_a$

N.7.2 Lilley's Convective Inhomogeneous Wave Equation

Continuity equation: $$\frac{D\Pi}{Dt} + \kappa \operatorname{div}\vec{v} = \frac{\kappa}{c_p}\frac{Ds}{Dt} \tag{7}$$

with: $\Pi = \ln\frac{p}{p_0}$,

following from continuity equation in the form

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\vec{v} = 0$$

with: $$\frac{d\rho}{\rho} = \frac{1}{\kappa}\frac{dp}{p} - \frac{ds}{c_p}$$

Momentum equation: $$\kappa\frac{Dv_i}{Dt} = -c^2\frac{\partial \Pi}{\partial x_i} + \frac{\kappa}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \tag{8}$$

Lilley's wave equation:

$$\begin{aligned}\frac{D}{Dt}\left(\frac{D^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[c^2\frac{\partial \Pi}{\partial x_i}\right]\right) + 2\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j}\left[c^2\frac{\partial \Pi}{\partial x_i}\right] &= -2\kappa\frac{\partial v_j}{\partial x_i}\frac{\partial v_k}{\partial x_j}\frac{\partial v_i}{\partial x_k}\\ &+ 2\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j}\left[\frac{\kappa}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right] - \frac{D}{D\tau}\left\{\frac{\partial}{\partial x_i}\left[\frac{\kappa}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right]\right\} + \frac{D^2}{Dt^2}\left[\frac{\kappa}{c_p}\frac{Ds}{Dt}\right]\end{aligned} \tag{9}$$

Interpretation: third-order equation,
left-hand side contains all propagation effects,
right-hand side includes all source terms.

Neglecting the effects of viscosity and thermal conductivity, furthermore introducing the mean values in the propagation terms on the left-hand sides, replacing v_i by $\bar{v}_i$ and c^2 by $\overline{c^2}$:

$$\frac{\bar{D}}{Dt}\left(\frac{\bar{D}^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[\overline{c^2}\frac{\partial \Pi}{\partial x_i}\right]\right) + 2\frac{\partial \bar{v}_j}{\partial x_i}\frac{\partial}{\partial x_j}\left[\overline{c^2}\frac{\partial \Pi}{\partial x_i}\right] = -2\kappa\frac{\partial v_j}{\partial x_i}\frac{\partial v_k}{\partial x_j}\frac{\partial v_i}{\partial x_k} \tag{10}$$

Example: shear flow: $v_i = \bar{v}_i + v'_i \quad \bar{v}_i = \{\bar{v}_x(y); 0; 0\}$

$$\frac{\bar{D}}{Dt}\left(\frac{\bar{D}^2\Pi}{Dt^2} - \frac{\partial}{\partial x_i}\left[c^2\frac{\partial \Pi}{\partial x_i}\right]\right) + 2\frac{\partial \bar{v}_x}{\partial y}\frac{\partial}{\partial x}\left[c^2\frac{\partial \Pi}{\partial y}\right] = q \tag{11}$$

with: q — different source terms but no terms which are linear in fluctuating velocities.

In comparison with equation applicable to the sound propagation in a shear flow:

$$\frac{\bar{D}}{Dt}\left(\frac{1}{c_0^2}\frac{\bar{D}^2\Pi'}{Dt^2}-\frac{\partial^2\Pi'}{\partial x_i^2}\right)+2\frac{\partial\bar{v}_x}{\partial y}\frac{\partial^2\Pi'}{\partial x\partial y}=0 \tag{12}$$

$$\frac{\bar{D}}{Dt}\left(\frac{1}{c_0^2}\frac{\bar{D}^2 p_a}{Dt^2}-\frac{\partial^2 p_a}{\partial x_i^2}\right)+2\frac{\partial\bar{v}_x}{\partial y}\frac{\partial^2 p_a}{\partial x\partial y}=0 \tag{13}$$

(This equation follows as a result of Lilley's equation with assumptions:
$q=0 \qquad \Pi=\ln\frac{p}{p_0}=\ln\frac{p_0+p_a}{p_0}\approx\frac{p_a}{p_0}=\Pi' \quad \text{with } p_a\ll p_0 \qquad c^2=c_0^2$)

N.7.3 Lilley's Wave Equation with a New Lighthill Stress Tensor

Definition of a new Lighthill stress tensor:

$$T'_{ij}=\rho v'_i v'_j-\tau_{ij}+\left(p-c_\infty^2\rho\right)\delta_{ij} \tag{14}$$

with: v' — velocity fluctuations; that is: the Lighthill tensor involves only quadratic fluctuations of the velocity field.

Assuming a uniform mean flow: $\bar{v}_i=\left\{\bar{v}_x(y);0;0\right\};$

decomposition: $v_i=\bar{v}_i+v'_i;$

convective operator for mean flow $\dfrac{\bar{D}}{Dt}=\dfrac{\partial}{\partial t}+\bar{v}_x(y)\dfrac{\partial}{\partial x}.$

This leads to the generalised linear convective wave equation, a third-order equation:

$$\frac{\bar{D}}{Dt}\left(\frac{\bar{D}^2}{Dt^2}-c_0^2\Delta\right)\rho+2c_0^2\frac{d\bar{v}_x}{dy}\frac{\partial^2\rho}{\partial x\partial y}=\frac{\bar{D}}{Dt}\frac{\partial^2 T'_{ij}}{\partial x_i\partial x_j}-2\frac{d\bar{v}_x}{dy}\frac{\partial^2 T'_{yi}}{\partial x\partial x_j}, \tag{15}$$

left-side hand: all linear fluctuating terms;
right-hand side: generation terms, all quadratic in the fluctuations.

N.7.4 Convected Wave Equation for the Dilatation (Legendre)

$$\begin{aligned}\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D\Theta}{Dt}\right)-\Delta\Theta &= -\frac{D}{Dt}\left(\frac{1}{c^2}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}-\frac{\nabla\ln h}{c^2}\frac{Dv_i}{Dt}\right)\\ &\quad-\frac{\Delta v_i}{c^2}\frac{Dv_i}{Dt}-2\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{1}{c^2}\frac{Dv_i}{Dt}\right)\end{aligned} \tag{16}$$

with: $\Theta=\nabla\cdot\vec{v}=-\dfrac{1}{\rho}\dfrac{D\rho}{Dt}=-\dfrac{D}{Dt}\left(\ln\dfrac{\rho}{\rho_0}\right)$ dilatation

h specific enthalpy

N.7.5 Goldstein's Third-Order Inhomogeneous Wave Equation

Assumptions: parallel shear flow in the x direction, mean velocity U, density $\bar{\rho}$, sound speed $\bar{c}$: all independent of x and t:

$$\begin{aligned}\frac{\bar{D}}{Dt}\left[\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - \bar{\rho}\,\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla p\right)\right] + 2\frac{\partial}{\partial x}\left(\nabla U\cdot\nabla p\right) \\ = \bar{\rho}\left\{\frac{\bar{D}^2 q}{Dt^2} - \frac{\bar{D}}{Dt}\left(\nabla\cdot\vec{f}\right) + 2\frac{\partial}{\partial x}\left(\vec{f}\cdot\nabla U\right)\right\}\end{aligned} \tag{17}$$

with: $\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ material derivative

$\vec{f}$ force per unit mass

q mass flux per unit mass

$\nabla\cdot\vec{v} = q - \frac{1}{\rho}\frac{D\rho}{Dt}$ source term in the continuity equation

N.7.6 Goldstein-Howes Inhomogeneous Wave Equation

Assumptions: high Reynolds number, no entropy fluctuations; wave operator is linearised by assuming that only the first-rder interaction between the mean flow and the fluctuating field is retained; the mean flow is decomposed into a mean part and a fluctuating part:

$v_i = U\left(y\right)\delta_{xi} + v_i'$.

Introducing the convective derivative: $\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$,

use of Phillips wave equation: $\frac{D^2\Pi}{Dt^2} - c_0^2\nabla^2\Pi - 2\kappa\frac{\partial U}{\partial y}\frac{\partial v_y'}{\partial x} = \kappa\frac{\partial v_j'}{\partial x_i}\frac{\partial v_i'}{\partial x_j}$,

convective derivative and introduction the momentum equation (no viscosity) leads to:

$$\frac{D}{Dt}\left\{\frac{D^2\Pi}{Dt^2} - c_0^2\nabla^2\Pi\right\} + 2c_0^2\frac{dU}{dy}\frac{\partial^2\Pi}{\partial x\partial y} = \kappa\frac{D}{Dt}\left(\frac{\partial v_i'}{\partial x_j}\frac{\partial v_j'}{\partial x_i}\right) - 2\kappa\frac{\partial U}{\partial y}\frac{\partial}{\partial x}\left(v_i'\frac{\partial v_y'}{\partial x_i}\right). \tag{18}$$

This is an acoustic analogy, like Lighthill's equation.

Further assumptions: acoustic fluctuations are negligible compared to the turbulent fluctuations in the source volume;turbulent velocity is incompressible, these lead to another formulation of the wave equation:

$$\frac{D}{Dt}\left\{\frac{D^2\Pi}{Dt^2} - c_0^2\nabla^2\Pi\right\} + 2c_0^2\frac{dU}{dy}\frac{\partial^2\Pi}{\partial x\partial y} = \kappa\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{Dv_i'v_j'}{Dt}\right) - 4\kappa\frac{\partial U}{\partial y}\frac{\partial}{\partial x}\left(v_i'\frac{\partial v_y'}{\partial x_i}\right) - \kappa\frac{\partial}{\partial x}\left(\frac{d^2U}{dy^2}v_y'^2\right). \tag{19}$$

New form of inhomogeneous convective wave equation:

$$\frac{D^2\Gamma}{Dt^2} - c_0^2\nabla^2\Gamma = \kappa\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{Dv_i'v_j'}{Dt}\right) - 4\kappa\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\frac{\partial}{\partial x_i}\left(v_i'v_y' + \frac{c_0^2}{\kappa}\pi\delta_{yi}\right)\right] - \kappa\frac{\partial}{\partial x}\left[\frac{d^2U}{dy^2}\left(v_y'^2 + \frac{c_0^2}{\kappa}\pi\right)\right] \tag{20}$$

with new variable: $$\Gamma = \frac{D\Pi}{Dt} = \frac{\partial\Pi}{\partial t} + U\frac{\partial\Pi}{\partial x} \tag{21}$$

Alternatively:

$$\frac{D^2\Gamma}{Dt^2} - c_0^2\nabla^2\Gamma = \kappa\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{Dv_i'v_j'}{Dt}\right) + 4\kappa\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\frac{Dv_y'}{Dt}\right] - \kappa\frac{\partial}{\partial x}\left[\frac{d^2U}{dy^2}\left(v_y'^2 + \frac{c_0^2}{\kappa}\pi\right)\right] \tag{22}$$

Simplification of source term:

In mixing layer of a jet: gradient of mean velocity is constant: $\frac{d}{dy}\left(\frac{dU}{dy}\right) = 0$ from which follows the source term q (right-hand side of wave equation):

$$q = \kappa\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{Dv_i'v_j'}{Dt}\right) + 4\kappa\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\frac{Dv_y'}{Dt}\right]. \tag{23}$$

N.7.7 Ribner's Recent Reformulation of Lighthill's Source Term

Lighthill's wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial^2\rho v_i v_j}{\partial x_i\partial x_j} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2\rho}{\partial t^2} \tag{24}$$

Assumptions:

- instantaneous local velocity: $v_i = \bar{v}_i + v_i'$
- unidirectional, transversely sheared, mean flow: $\bar{v}_i = [U(y), 0, 0]$

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial^2\rho v_i'v_j'}{\partial x_i\partial x_j} + 2\frac{\partial U}{\partial y}\frac{\partial\rho v_y'}{\partial x} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\bar{D}^2\rho}{Dt^2} \tag{25}$$

with: $\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ convective derivative following mean flow.

With approximation: $\dfrac{\bar{D}^2\rho}{Dt^2} = \dfrac{1}{\bar{c}^2}\dfrac{\bar{D}^2 p}{Dt^2}$ ($\bar{c} = \bar{c}\,(\vec{x})$ local time-averaged sound speed):

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - \Delta p = \frac{\partial^2 \rho v_i' v_j'}{\partial x_i \partial x_j} + 2\frac{\partial U}{\partial y}\frac{\partial \rho v_y'}{\partial x} \tag{26}$$

In the case of an exact wave equation with the following notations:
$v_i = \bar{v}_i + v_i'$, $\langle v_i \rangle_{av} = \bar{v}_i\,(\vec{x})$, $\rho = \bar{\rho}\,(\vec{x}) + \rho'$, $\langle \rho \rangle_{av} = \bar{\rho}\,(\vec{x})$,

there follows the wave equation:

$$\begin{aligned}\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = {} & 2\rho\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial v_j'}{\partial x_i} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\bar{D}^2\rho}{Dt^2} + \frac{\partial^2 v_i' v_j'}{\partial x_i \partial x_j} + 2v_j'\frac{\partial \bar{v}_i}{\partial x_j}\frac{\partial \rho}{\partial x_i}\\ & + 2\frac{\partial v_i' v_j'}{\partial x_j}\frac{\partial \rho}{\partial x_i} + v_i' v_j'\frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2\frac{\partial}{\partial x_j}\left(\rho v_j'\frac{\partial \bar{v}_i}{\partial x_i}\right)\\ & + \rho\left(\frac{\partial \bar{v}_j}{\partial x_i}\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_i}{\partial x_i}\frac{\partial \bar{v}_j}{\partial x_j}\right)\end{aligned} \tag{27}$$

with: $$\frac{\bar{D}^2}{Dt^2} = \left(\frac{\partial}{\partial t} + \bar{v}_i\frac{\partial}{\partial x_i}\right)^2 = \frac{\partial^2}{\partial t^2} + 2\bar{v}_i\frac{\partial^2}{\partial x_i \partial t} + \bar{v}_i\bar{v}_j\frac{\partial^2}{\partial x_i \partial x_j} \tag{28}$$

N.7.8 Inhomogeneous Wave Equation Including Stream Function (Albring/Detsch)

Inhomogeneous wave equation in approximated form of Lighthill:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_i^2} = \rho_0\frac{\partial^2 \left(v_i v_j\right)}{\partial x_i \partial x_j} = q \tag{29}$$

Introducing the stream function in a two-dimensional flow

$$v_x = \frac{\partial \Psi}{\partial y}; \qquad v_y = -\frac{\partial \Psi}{\partial x}$$

leads to:

$$q = 2\rho_0\left[\left(\frac{\partial^2 \Psi}{\partial x \partial y}\right)^2 - \frac{\partial^2 \Psi}{\partial x^2}\frac{\partial^2 \Psi}{\partial y^2}\right] \tag{30}$$

Decomposition $\Psi = \bar{\Psi} + \Psi'$ gives for the source term:

$$q = 2\rho_0\left\{2\left(\frac{\partial^2 \bar{\Psi}}{\partial x \partial y}\frac{\partial^2 \Psi'}{\partial x \partial y}\right) + \left(\frac{\partial^2 \Psi'}{\partial x \partial y}\right)^2 - \frac{\partial^2 \bar{\Psi}}{\partial x^2}\frac{\partial^2 \Psi'}{\partial y^2} - \frac{\partial^2 \bar{\Psi}}{\partial y^2}\frac{\partial^2 \Psi'}{\partial x^2} - \frac{\partial^2 \Psi'}{\partial x^2}\frac{\partial^2 \Psi'}{\partial y^2}\right\} \tag{31}$$

with: term number 1, 3, 4: shear noise
term number 2, 5: self-noise.

N.8 Acoustic Analogy in Terms of Vorticity, Wave Operators for Enthalpy

▸ *See also:* Powell (1963, 1964); Howe (1975, 1998); Crighton et al. (1992); Möhring (1978, 1999); Möhring/Obermeier (1980); Doak (1995, 1998)

N.8.1 Powell's Theory of Vortex Sound

Assumption: fluid motion is isentropic.

Use of vector identities:

$$\nabla\left(\frac{1}{2}v^2\right) = (\vec{v}\cdot\nabla)\,\vec{v} - (\nabla\times\vec{v})\times\vec{v} = (\vec{v}\cdot\nabla)\,\vec{v} - \omega\times\vec{v} \tag{1}$$

with: $\vec{\omega} = \nabla\times\vec{v}$ vorticity vector,
or:

$$\nabla\left(\frac{1}{2}v^2\right) = v_j\frac{\partial v_i}{\partial x_j} - \left[\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right)v_k - \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)v_j\right] \tag{2}$$

of the continuity equation:

$$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\,\rho + \rho\nabla\cdot\vec{v} = 0, \tag{3}$$

and of the momentum equation (neglecting the viscous stress tensor τ_{ij}):

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\,(\vec{\omega}\times\vec{v}) + \rho\nabla\left(\frac{1}{2}\,|\vec{v}|^2\right) = -\nabla p \tag{4}$$

leads to an inhomogeneous wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} - \Delta p_a = \nabla\left\{\rho\,(\vec{\omega}\times\vec{v}) + \nabla\left(\rho\frac{v^2}{2}\right) - \frac{v^2}{2}\nabla\rho - \vec{v}\frac{\partial\rho}{\partial t} + \nabla\left(p - c_0^2\rho\right)\right\}.$$

With the assumption: $|\vec{v}| \ll c_0$ Powell's equation follows in the theory of vortex sound:

$$\frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} - \Delta p_a = \nabla\left\{\rho\,(\vec{\omega}\times\vec{v}) + \nabla\left(\rho\frac{v^2}{2}\right)\right\} \tag{5}$$

Solution in free space:

$$p\,(\vec{x},t) = \frac{\partial}{\partial x_i}\int\limits_V \frac{1}{4\pi r}\left[\rho\,(\vec{\omega}\times\vec{v})\right]_\tau dV + \frac{\partial^2}{\partial x_i^2}\int\limits_V \frac{1}{4\pi r}\left[\frac{1}{2}\rho\,|\vec{v}|^2\right]_\tau dV \tag{6}$$

$$p\,(\vec{x},t) = \frac{\rho_0 x_i}{4\pi c_0 r^2}\frac{\partial}{\partial t}\int\limits_V \left[(\vec{\omega}\times\vec{v})_i\right]_\tau dV + \frac{\rho_0}{4\pi c_0^2 r}\frac{\partial^2}{\partial t^2}\int\limits_V \left[\frac{1}{2}\,|\vec{v}|^2\right]_\tau dV \tag{7}$$

Source terms:

first term: dipole source, vorticity distribution, incompressible approximation of fluctuating flow, rate of change of vortex stretching by fluid flow → principal source of sound at low Mach number

second term: quadrupole source, isotropic quadrupole (three longitudinal quadrupoles with undirected radiation characteristic), rate of change of kinetic energy of source flow

N.8.2 Howe's Formulation of Acoustic Analogy Equation for Total Enthalpy

Momentum equation (Crocco's formulation, neglecting effects of external forces):

$$\frac{\partial v_i}{\partial t} + \frac{\partial B}{\partial x_i} = -(\vec{\omega} \times \vec{v}) + T\frac{\partial s}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \tag{8}$$

with: thermodynamic relations:

$B = h + \frac{1}{2}v^2$ total enthalpy, stagnation enthalpy, per unit volume

$h = u + \frac{p}{\rho}$ specific enthalpy (per unit volume)

u specific internal energy (per unit volume)

vector relations:

$\vec{\omega} = \mathrm{rot}\vec{v} = \mathrm{curl}\vec{v} = \nabla \times \vec{v}$ vorticity

$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = \frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v} + \nabla\left(\frac{1}{2}v^2\right)$ vector identity

$\vec{L} = (\vec{\omega} \times \vec{v})$ Lamb vector, unsteady vortical lifting force (per mass unit)

or $$\vec{L} = (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\omega} \times \vec{v} + \mathrm{grad}\frac{1}{2}v^2 \tag{9}$$

Continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \mathrm{div}\,\vec{v} = 0 \quad \text{or} \quad \frac{1}{\rho c^2}\frac{Dp}{Dt} + \mathrm{div}\,\vec{v} = \frac{\beta T}{c_p}\frac{Ds}{Dt} \tag{10}$$

with: elimination of $\frac{D\rho}{Dt}$ by means of relations:

$$d\rho = \frac{dp}{c^2} + \left(\frac{\partial \rho}{\partial s}\right)_p ds = \frac{dp}{c^2} + \left(\frac{\partial \rho}{\partial T}\right)_p \left(\frac{\partial T}{\partial s}\right)_p ds = \frac{dp}{c^2} - \frac{\beta \rho T}{c_p}ds \tag{11}$$

Subtracting the divergence of the momentum equation from the time derivative of the continuity equation gives:

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{Dp}{Dt}\right) - \Delta B = \mathrm{div}\,(\vec{\omega} \times \vec{v} - T\nabla s - \vec{\sigma}) + \frac{\partial}{\partial t}\left(\frac{\beta T}{c_p}\frac{Ds}{Dt}\right) \tag{12}$$

with: $\sigma_i = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$,

τ_{ij} = viscous stress tensor.

Howe's inhomogeneous wave equation in terms of total enthalpy:

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)-\frac{\nabla p\cdot\nabla}{\rho c^2}-\Delta\right\}B=\left(\operatorname{div}+\frac{\nabla p}{\rho c^2}\right)\cdot(\vec{\omega}\times\vec{v}-T\nabla s-\vec{\sigma})$$
$$+\frac{\partial}{\partial t}\left(\frac{\beta T}{c_p}\frac{Ds}{Dt}\right)+\frac{D}{Dt}\left[\frac{1}{c^2}\left(T\frac{Ds}{Dt}+\vec{v}\cdot\vec{\sigma}\right)\right] \quad (13)$$

Special cases:

- Absence of viscous dissipation and heat transfer, with momentum equation $\rho\frac{D\vec{v}}{Dt}=-\nabla p$:

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)+\frac{1}{c^2}\frac{D\vec{v}}{Dt}\cdot\nabla-\Delta\right\}B=\left[\operatorname{div}-\frac{1}{c^2}\frac{D\vec{v}}{Dt}\right](\vec{\omega}\times\vec{v}-T\nabla s) \quad (14)$$

- High Reynolds number, homentropic flow (dissipation and heat transfer are neglected, s = const.):

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)-\frac{1}{\rho}\nabla\cdot(\rho\nabla)\right\}B=\frac{1}{\rho}\operatorname{div}(\rho\vec{\omega}\times\vec{v}) \quad (15)$$

- Low Mach number, $\rho=\rho_0$ and $c=c_0$, neglecting non-linear effects of propagation and scattering of sound by vorticity:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\Delta\right)B=\operatorname{div}(\vec{\omega}\times\vec{v}) \quad (16)$$

- Non-homentropic source flow, fluid is temporarily incompressible, dissipation is ignored, mean flow is irrotational, mean velocity $\vec{v}(\vec{x})$, density $\rho(\vec{x})$, sound velocity $c(\vec{x})$:

$$\left\{\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\right]-\frac{1}{\rho}\nabla\cdot(\rho\nabla)\right\}B$$
$$=\operatorname{div}(\vec{\omega}\times\vec{v}-T\nabla s)+\frac{\partial}{\partial t}\left(\frac{\beta T}{c_p}\frac{Ds}{Dt}\right) \quad (17)$$

- At very low Mach number:

$$\left\{\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\Delta\right\}B=\operatorname{div}(\vec{\omega}\times\vec{v}-T\nabla s)+\frac{\partial}{\partial t}\left(\frac{\beta T}{c_p}\frac{Ds}{Dt}\right) \quad (18)$$

right-hand side:
sources of noise are vorticity, entropy gradients and unsteady heating of the fluid; the last source is equivalent to a volume monopole of strength $q(x_i,t)=\frac{\beta T}{c_p}\frac{Ds}{Dt}$.

- At low Mach number, mean density, entropy and sound velocity are constant, isentropic flow:

$$\left\{\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)^2-\Delta\right\}B=\operatorname{div}(\vec{\omega}\times\vec{v}) \tag{19}$$

The acoustic pressure p can be calculated from the fluctuations in total enthalpy by:

$$\frac{p}{\rho_0}=B. \tag{20}$$

This is a linearised relation of acoustic pressure p in the far field, to the first order in Mach number, furthermore considering a very low mean flow Mach number.

Formulation of the wave equation in terms of the pressure:

$$\left\{\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\Delta\right\}p=\rho_0\operatorname{div}(\vec{\omega}\times\vec{v}) \tag{21}$$

The right-hand side represents an aerodynamic source for incompressible flow; the generated sound is dipole in nature; $\vec{\omega}\times\vec{v}$ is a force distribution.
Lamb vector: $\vec{L}=(\vec{\omega}\times\vec{v})$, quantity ρL vortex force (per unit volume);
$\vec{v}$ can be calculated directly from $\vec{\omega}$:

$$\vec{v}(\vec{x},t)=\operatorname{curl}\left[\int\frac{\vec{\omega}(\vec{y},t)}{4\pi\left|\vec{x}-\vec{y}\right|}dV\right]\quad\text{(Biot-Savart law).} \tag{22}$$

Therefore the source term depends only on the vorticity. See also the Helmholtz vorticity equation:

$$\frac{\partial\vec{\omega}}{\partial t}+\operatorname{curl}(\vec{\omega}\times\vec{v})=0. \tag{23}$$

It is a direct relationship between the non-linear term $\vec{\omega}\times\vec{v}$ and the linear term in the vorticity.

Solution:

$$p(x_i,t)=-\rho_0\int\frac{1}{r}\frac{\partial}{\partial y_i}(\vec{\omega}\times\vec{v})_i\left(y_i,t-\frac{r}{c_0}\right)dV \tag{24}$$

$$\text{or:}\quad p(x_i,t)=-\rho_0\frac{\partial}{\partial x_i}\int\frac{1}{r}(\vec{\omega}\times\vec{v})_i\left(y_i,t-\frac{r}{c_0}\right)dV \tag{25}$$

In the far field (with the approximations of low Mach number flow, compact turbulent eddies):

$$p(x_i,t)=-\frac{\rho_0 x_i}{4\pi c_0 x^2}\frac{\partial}{\partial t}\int(\vec{\omega}\times\vec{v})_i\left(y_i,t-\frac{r}{c_0}\right)dV \tag{26}$$

$$p(x_i,t)=-\frac{\rho_0 x_i}{4\pi c_0^2 x^3}\frac{\partial^2}{\partial t^2}\int(\vec{x}\cdot\vec{y})\,\vec{x}\cdot(\vec{\omega}\times\vec{v})\left(y_i,t-\frac{r}{c_0}\right)dV \tag{27}$$

N.8.3 Möhring's Equation with Source Term Linearly Dependent on Vorticity Field

Definition of a vector Green's function: $\mathrm{curl}\,\vec{G} = \mathrm{grad}\,G$ (with: G a scalar Green's function).
Solution of wave equation:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right) B = \mathrm{div}\,(\vec{\omega} \times \vec{v}) \tag{28}$$

Introduce a Green's function $G\left(\vec{y}, \tau\,|\vec{x}, t\right)$ which satisfies

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial y_i^2}\right\} G = \delta\left(\vec{x} - \vec{y}, t - \tau\right). \tag{29}$$

Use the combination $\quad B(\vec{x}, t) = \int\limits_V \mathrm{div}\,(\vec{\omega} \times \vec{v})\, G\, dV d\tau \quad (30)$

in the far field $\quad p_a(\vec{x}, t) = -\rho_0 \int\limits_V (\vec{\omega} \times \vec{v})_i \frac{\partial G}{\partial y_i} dV d\tau \quad (31)$

with the vector Green's function $\quad p_a(\vec{x}, t) = -\rho_0 \int\limits_V (\vec{\omega} \times \vec{v})\ \mathrm{curl}\vec{G}\ dV d\tau \quad (32)$

with the Helmholtz vorticity equation $\quad p_a(\vec{x}, t) = \rho_0 \int\limits_V \frac{\partial \vec{\omega}}{\partial \tau} \vec{G} dV d\tau \quad (33)$

to obtain $\quad p_a(\vec{x}, t) = -\rho_0 \int \vec{\omega} \frac{\partial \vec{G}}{\partial \tau} dV d\tau = \rho_0 \frac{\partial}{\partial t} \int \vec{\omega} \vec{G} dV d\tau. \quad (34)$

This equation does not contain the flow velocity. It depends linearly on the vorticity field, that is, the contributions from several vortices add linearly.

Vorticity sound in a low Mach number flow in unbounded space (see Dowling):

$$p(\vec{x}, t) = \frac{\rho_0}{12\pi c_0^2 r^3} \frac{\partial^3}{\partial t^3} \int (\vec{x} \cdot \vec{y})\vec{y} \cdot [\vec{\omega} \times \vec{x}]_{t - \frac{x}{c_0}}\ dV \tag{35}$$

right-hand side:
only components of vorticity perpendicular to the observer's position vector $\vec{x}$ contribute to the sound far field.

N.8.4 Convected Wave Operators for Total Enthalpy in Comparison

Möhring:

$$L_{Möhring} B = \nabla \cdot \left(\rho \nabla B - \frac{\rho \vec{v}}{c^2}\frac{DB}{Dt}\right) - \frac{\partial}{\partial t}\frac{\rho}{c^2}\frac{DB}{Dt} = -\mathrm{div}\rho\vec{L} + \frac{\partial}{\partial t}\frac{\partial \rho}{\partial s}\frac{\partial s}{\partial t} + \mathrm{div}\frac{\partial \rho}{\partial s}\vec{v}\frac{\partial s}{\partial t} \tag{36}$$

with: $\quad \vec{L} = \vec{\omega} \times \vec{v} - T\nabla s$

Möhring/Obermeier:

$$\boldsymbol{L}_{\textit{Möhring/Obermeier}}\mathrm{B} = \nabla \cdot \left(\rho\nabla\mathrm{B}\right) - \rho\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{1}{\mathrm{c}^2}\frac{\mathrm{DB}}{\mathrm{Dt}}\right) = -\mathrm{div}\rho\vec{\mathrm{L}} - \frac{\partial\vec{\mathrm{v}}}{\partial\mathrm{t}}\frac{\partial\rho}{\partial\mathrm{s}}\nabla\mathrm{s} \tag{37}$$

with: $\vec{\mathrm{L}} = \vec{\omega} \times \vec{\mathrm{v}} = \mathrm{curl}\vec{\mathrm{v}} \times \vec{\mathrm{v}}$

Howe:

$$\boldsymbol{L}_{\textit{Howe}}\mathrm{B} = \Delta\mathrm{B} - \frac{1}{\mathrm{c}^2}\frac{\mathrm{D}\vec{\mathrm{v}}}{\mathrm{Dt}} \cdot \nabla\mathrm{B} - \frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{1}{\mathrm{c}^2}\frac{\mathrm{DB}}{\mathrm{Dt}}\right) \tag{38}$$

Doak:

$$\boldsymbol{L}_{\textit{Doak}}\mathrm{B} = \Delta\mathrm{B} - \frac{1}{\mathrm{c}^2}\left[\frac{\partial^2\mathrm{B}}{\partial\mathrm{t}^2} + \left(2\vec{\mathrm{v}}\frac{\partial}{\partial\mathrm{t}} + \vec{\omega} \times \vec{\mathrm{v}} + \mathrm{T}\nabla\mathrm{s} - 2\nabla\mathrm{h}\right)\nabla\mathrm{B} + \vec{\mathrm{v}}\vec{\mathrm{v}} \cdot \nabla \cdot \nabla\mathrm{B}\right] \tag{39}$$

Those terms of these three convected wave operators which contain second derivatives of B agree. This means that they agree in the high frequency limit of geometrical acoustics (Möhring/Obermeier).

N.8.5 Doak's Theory of Aerodynamic Sound Including the Fluctuating Total Enthalpy as a Basic Generalised Acoustic Field for a Fluid

Continuity equation:

$$\frac{\partial\rho}{\partial\mathrm{t}} + \frac{\partial\left(\rho\mathrm{v_i}\right)}{\partial\mathrm{x_i}} = \dot{\mathrm{M}} \tag{40}$$

with: $\dot{\mathrm{M}}$ rate of mass creation per unit volume

Momentum equation:

$$\frac{\partial\left(\rho\mathrm{v_i}\right)}{\partial\mathrm{t}} + \frac{\partial\left(\rho\mathrm{v_i}\mathrm{v_j}\right)}{\partial\mathrm{x_j}} + \frac{\partial\mathrm{p_{ij}}}{\partial\mathrm{x_j}} = \dot{\mathrm{M}}\mathrm{v_i} + \dot{\mathrm{I}}_\mathrm{i} + \mathrm{F_i} \tag{41}$$

with: $\dot{\mathrm{I}}_\mathrm{i}$ rate of production of momentum per unit volume by internal processes such as chemical reactions,

$\mathrm{F_i}$ rate of production of momentum per unit volume due to external forces such as gravitational and electromagnetic fields.

Energy equation:

$$\frac{\partial\left(\rho\mathrm{u} + \frac{1}{2}\rho\mathrm{v_i^2}\right)}{\partial\mathrm{t}} + \frac{\partial\left(\rho\mathrm{u}\mathrm{v_j} + \frac{1}{2}\rho\mathrm{v_i^2}\mathrm{v_j} + \mathrm{p_{ij}}\mathrm{v_i} - \lambda\frac{\partial\mathrm{T}}{\partial\mathrm{x_j}}\right)}{\partial\mathrm{x_j}} = \dot{\mathrm{M}}\mathrm{v_i^2} + \dot{\mathrm{I}}_\mathrm{i}\mathrm{v_i} + \mathrm{F_i}\mathrm{v_i} + \rho\mathrm{q} + \rho\mathrm{Q} \tag{42}$$

with: u internal energy per unit volume

λ coefficient of thermal conductivity

Q rate of external heat addition per unit mass

ρq rate of energy production per unit volume due to internal processes

ρQ rate of energy production per unit volume due to external processes

Inhomogeneous convected scalar wave equation for fluctuating total enthalpy:

$$
\begin{aligned}
&\frac{\partial^2 B'}{\partial x_i^2} - \left\{ \frac{1}{c^2} \left[\frac{\partial^2 B'}{\partial t^2} + \left(2v_i \frac{\partial}{\partial t} - (\vec{v} \times \vec{\omega})_i + \boldsymbol{V}_i - 2\frac{\partial h}{\partial x_i} \right) \frac{\partial B'}{\partial x_i} + v_i v_j \frac{\partial^2 B'}{\partial x_i \partial x_j} \right] \right\}' \\
&\quad = \frac{\partial}{\partial x_i} \left[(\vec{v} \times \vec{\omega})'_j + \boldsymbol{V}'_j \right] - \left\{ \frac{1}{c^2} \left[-(\vec{v} \times \vec{\omega})_i + \boldsymbol{V}_i - 2\frac{\partial h}{\partial x_i} + v_i v_j \frac{\partial}{\partial x_j} \right] \right. \\
&\quad \left. \cdot \left[\left((\vec{v} \times \vec{\omega})'_j + \boldsymbol{V}'_j \right) + \overline{\left[\frac{\partial v'_i}{\partial t} \right]} \right] \right\}' \\
&\quad + \left[\frac{\partial}{\partial t} \left(\frac{1}{c^2} \right)' \frac{Dh}{Dt} \right]' - \left[\frac{\partial}{\partial t} \left(\frac{1}{R} \frac{Ds}{Dt} + \frac{\dot{M}}{\rho} \right)' \right]' - \left[\frac{1}{c^2} v_i \frac{\partial \boldsymbol{V}'_i}{\partial t} \right]'
\end{aligned}
\tag{43}
$$

with: fluctuating part of the quantity is denoted by the prime, all quantities without primes are the sum of the respective mean and fluctuating parts;

$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$ material derivative;

B' fluctuating total enthalpy, basic generalised acoustic field for a fluid (Doak);

$\boldsymbol{V}_i = T\frac{\partial s}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i} + \frac{\dot{I}_i}{\rho} + \frac{F_i}{\rho}$ sum of accelerations (“forces” per unit mass)

with: first term vanishes if the motion is homentropic; second term vanishes if the fluid is inviscid; third term vanishes if there is no production of momentum by internal processes such as chemical processes; fourth term vanishes if there are no external forces.

Rewriting in a more compact form:

$$
\begin{aligned}
&\frac{\partial^2 B'}{\partial x_i^2} - \left\{ \frac{1}{c^2} \left[\frac{\partial^2}{\partial t^2} + \left(2v_i \frac{\partial}{\partial t} - (\vec{v} \times \vec{\omega})_i + \boldsymbol{V}_i - 2\frac{\partial h}{\partial x_i} \right) \frac{\partial}{\partial x_i} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j} \right] B' \right\}' \\
&\quad = \left\{ \left[\frac{\partial}{\partial x_i} - \frac{1}{c^2} \left(-(\vec{v} \times \vec{\omega})_i + \boldsymbol{V}_i - 2\frac{\partial h}{\partial x_i} + v_i v_j \frac{\partial}{\partial x_i} \right) \right] \right. \\
&\quad \left. \cdot \left(\left[(\vec{v} \times \vec{\omega})'_j + \boldsymbol{V}'_j \right] + \overline{\left[\frac{\partial v'_i}{\partial t} \right]} \right) \right\}' \\
&\quad + \left[\frac{\partial}{\partial t} \left(\frac{1}{c^2} \right)' \frac{Dh}{Dt} \right]' - \left[\frac{\partial}{\partial t} \left(\frac{1}{R} \frac{Ds}{Dt} + \frac{\dot{M}}{\rho} \right)' \right]' - \left[\frac{1}{c^2} v_i \frac{\partial \boldsymbol{V}'_i}{\partial t} \right]'
\end{aligned}
\tag{44}
$$

Reformulation:

The idealised case of the homentropic flow of the lossless fluid, under no external force, mass creation or heat addition:

$$
\begin{aligned}
&\frac{\partial^2 B'}{\partial x_i^2} - \left\{ \frac{1}{c^2} \left[\frac{\partial^2}{\partial t^2} + \left(2v_i \frac{\partial}{\partial t} - (\vec{v} \times \vec{\omega})_i - 2\frac{\partial h}{\partial x_i} \right) \frac{\partial}{\partial x_i} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j} \right] B' \right\}' \\
&\quad = \left\{ \left[\frac{\partial}{\partial x_i} - \frac{1}{c^2} \left(-(\vec{v} \times \vec{\omega})_i - 2\frac{\partial h}{\partial x_i} + v_i v_j \frac{\partial}{\partial x_i} \right) \right] \right. \\
&\quad \left. \cdot \left((\vec{v} \times \vec{\omega})'_j + \overline{\left[\frac{\partial v'_i}{\partial t} \right]} \right) \right\}' + \left[\frac{\partial}{\partial t} \left(\frac{1}{c^2} \right)' \frac{Dh}{Dt} \right]'
\end{aligned}
\tag{45}
$$

If the flow is also irrotational:

$$\begin{aligned}&\frac{\partial^2 B'}{\partial x_i^2} - \left\{\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + \left(2v_i\frac{\partial}{\partial t} - 2\frac{\partial h}{\partial x_i}\right)\frac{\partial}{\partial x_i} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j}\right]B'\right\}' \\ &\quad = \left\{\left[\frac{\partial}{\partial x_i} - \frac{1}{c^2}\left(-2\frac{\partial h}{\partial x_i} + v_i v_j \frac{\partial}{\partial x_i}\right)\right]\left(\overline{\left[\frac{\partial v_i'}{\partial t}\right]}\right)\right\}' + \left[\frac{\partial}{\partial t}\left(\frac{1}{c^2}\right)'\frac{Dh}{Dt}\right]'\end{aligned} \tag{46}$$

If the flow is also time-stationary:

$$\frac{\partial^2 B'}{\partial x_i^2} - \left\{\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + \left(2v_i\frac{\partial}{\partial t} - 2\frac{\partial h}{\partial x_i}\right)\frac{\partial}{\partial x_i} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j}\right]B'\right\}' = \left[\frac{\partial}{\partial t}\left(\frac{1}{c^2}\right)'\frac{Dh}{Dt}\right]' \tag{47}$$

For flows in which temperature variations are rather negligible:

$$\begin{aligned}&\frac{\partial^2 B'}{\partial x_i^2} - \left\{\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + \left(2v_i\frac{\partial}{\partial t} - (\vec{v}\times\vec{\omega})_i\right)\frac{\partial}{\partial x_i} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j}\right]B'\right\}' \\ &\quad = \left\{\left[\frac{\partial}{\partial x_i} - \frac{1}{c^2}\left(-(\vec{v}\times\vec{\omega})_i + v_i v_j \frac{\partial}{\partial x_i}\right)\right]\left((\vec{v}\times\vec{\omega})_j' + \overline{\left[\frac{\partial v_i'}{\partial t}\right]}\right)\right\}'\end{aligned} \tag{48}$$

In the field of aeroacoustics:

- External forces and heat addition are of relatively less significance,
- like the effects of temperature and entropy fluctuations, mean entropy gradients and viscous diffusion on propagation.
- Also the dependence of the inhomogeneous source terms on these quantities is of less significance.
- Therefore one can suggest neglecting them and linearise the equation mentioned above in the fluctuations:

$$\begin{aligned}&\frac{\partial^2 B'}{\partial x_i^2} - \frac{1}{\bar{c}^2}\left[\frac{\partial^2}{\partial t^2} + \left(2\bar{v}_i\frac{\partial}{\partial t} - \left(\vec{v}\times\bar{\vec{\omega}}\right)_i - \frac{2}{\kappa-1}\frac{\partial \bar{c}^2}{\partial x_i}\right)\frac{\partial}{\partial x_i} + \bar{v}_i\bar{v}_j\frac{\partial^2}{\partial x_i\partial x_j}\right]B' = \left\{\frac{\partial}{\partial x_i} - \frac{1}{\bar{c}^2}\right. \\ &\quad \cdot\left.\left[-\left(\bar{\vec{v}}\times\bar{\vec{\omega}}\right)_i - \frac{2}{\kappa-1}\frac{\partial\bar{c}^2}{\partial x_i} + \bar{v}_i\bar{v}_j\frac{\partial}{\partial x_j}\right]\left[\left(\vec{v}'\times\bar{\vec{\omega}}\right)_i + \left(\bar{\vec{v}}\times\vec{\omega}'\right)_i\right]\right\} \\ &\qquad - \frac{1}{\bar{c}^2}\left[-\left(\vec{v}'\times\bar{\vec{\omega}}\right)_i - \left(\bar{\vec{v}}\times\vec{\omega}'\right)_i - \frac{2}{\kappa-1}\frac{\partial c^{2\prime}}{\partial x_i} + \left(v_i'\bar{v}_j + \bar{v}_i v_j'\right)\frac{\partial}{\partial x_j}\right]\left[\overline{\frac{\partial v_i'}{\partial t}}\right]\end{aligned} \tag{49}$$

with the approximation $$\bar{h} = \frac{\bar{c}^2}{\kappa-1}\left(= \frac{\kappa R\bar{T}}{\kappa-1}\right). \tag{50}$$

If the flow is time-stationary, then the last line of the last equation is zero.

N.9 Acoustic Analogy with Effects of Solid Boundaries

▸ *See also:* Ffowcs Williams/Hawkings, Sound generation by turbulence (1969); Howe (1998); Crighton et al. (1992); Farassat (1986); Prieur/Rahier (1998); Long/Watts (1987); Pilon/Lyrintzis (1998); Farassat/Myers (1988); Brentner/Farassat (1998); Brentner (1997); Farassat (2001)

N.9.1 Ffowcs Williams–Hawkings (FW-H) Inhomogeneous Wave Equation, FW-H Equation in Differential and Integral Form

The aeroacoustic analogy in the representation of Ffowcs Williams–Hawkings is a very general Lighthill acoustic analogy. Primarily it is used for problems of sound generation by flow with moving boundaries and by moving sources interacting with such boundaries.

Notations in connection with moving surface S:
$f = f(x_i, t) = 0$,

$u_i(x_i, t)$	velocity of the surface,
$\vec{n} = \nabla f$	subscript n indicates projection of a vector quantity in surface normal direction,
$u_n = u_i n_i, \quad v_n = v_i n_i,$	
$\lvert\nabla f\rvert = 1$	on surface,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} = 0.$$

Heaviside function $H(f)$:

$H(f) = 1$	for	$f(x_i, t) > 0$,	that is in the fluid region, exterior to S
$H(f) = 0$	for	$f(x_i, t) < 0$,	that is in the volume enclosed by the surface S interior of S

$$\frac{\partial H(f)}{\partial t} = \frac{\partial H(f)}{\partial f}\frac{\partial f}{\partial t} = \delta(f)\frac{\partial f}{\partial t}$$

$$\frac{\partial H(f)}{\partial x_i} = \frac{\partial H(f)}{\partial f}\frac{\partial f}{\partial x_i} = \delta(f)\frac{\partial f}{\partial x_i} = \delta(f) n_i \tag{1}$$

In what follows the function $(\rho - \rho_0) H(f)$ is defined in the continuity and momentum equations. That is, the derived equations are valid for all space. The term $(\rho - \rho_0) H(f)$ is determined only in the region of space that is of interest, i.e. space occupied by fluid.

Continuity equation:

$$\frac{\partial}{\partial t}(\rho - \rho_0) + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{2}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left[(\rho - \rho_0) H(f)\right] + \frac{\partial}{\partial x_i}\left[\rho v_i H(f)\right] &= \left[\rho_0 u_i + \rho(v_i - u_i)\right]\delta(f)\frac{\partial f}{\partial x_i} \\ &= \left[\rho_0 u_i + \rho(v_i - u_i)\right]\frac{\partial H(f)}{\partial x_i} = Q\delta(f) \end{aligned} \tag{3}$$

$$
\begin{aligned}
Q &= \left[\rho_0 u_i + \rho\left(v_i - u_i\right)\right]\frac{\partial f}{\partial x_i} = \left[\rho_0 u_n + \rho\left(v_n - u_n\right)\right] \\
&= \rho_0\left[\left(1 - \frac{\rho}{\rho_0}\right)u_n + \frac{\rho}{\rho_0}v_n\right] = \rho_0 U_n
\end{aligned}
\tag{4}
$$

with: Q mass flux per unit volume
ρ_0 density of undisturbed medium
u_n local normal velocity of the surface

Momentum equation:

$$
\frac{\partial}{\partial t}\left(\rho v_i\right) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + p_{ij}\right) = 0 \tag{5}
$$

with: $p_{ij} = p\delta_{ij} - \tau_{ij}$

$$
\frac{\partial}{\partial t}\left[\rho v_i H(f)\right] + \frac{\partial}{\partial x_j}\left[\left(\rho v_i v_j + p_{ij}\right)H(f)\right] = F_i\delta(f) \tag{6}
$$

with: $F_i = \left[p_{ij} + \rho v_i\left(v_j - u_j\right)\right]\frac{\partial f}{\partial x_j} = \left[p_{ij}n_j + \rho v_i\left(v_n - u_n\right)\right]$ (7)

F_i force per unit area acting on medium

Eliminate $\rho v_i H(f)$ by cross differentiation:

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\left[\left(\rho - \rho_0\right)H(f)\right] - c_0^2\frac{\partial^2\left[\left(\rho - \rho_0\right)H(f)\right]}{\partial x_i^2} &= \frac{\partial}{\partial t}\left[Q\delta(f)\right] - \frac{\partial}{\partial x_i}\left[F_i\delta(f)\right] \\
&+ \frac{\partial^2}{\partial x_i\partial x_j}\left[\left(\rho v_i v_j + p_{ij}\right)H(f)\right] - c_0^2\frac{\partial^2\left[\left(\rho - \rho_0\right)H(f)\right]}{\partial x_i^2}
\end{aligned}
\tag{8}
$$

and with Lighthill tensor $T_{ij} = \rho v_i v_j + p_{ij} - c_0^2\left(\rho - \rho_0\right)\delta_{ij}$ (9)

follows:

Ffowcs Williams–Hawkings equation in differential form

$$
\left\{\frac{\partial^2}{\partial t^2} - c_0^2\frac{\partial^2}{\partial x_i^2}\right\}\left[\left(\rho - \rho_0\right)H(f)\right] = \frac{\partial}{\partial t}\left[Q\delta(f)\right] - \frac{\partial}{\partial x_i}\left[F_i\delta(f)\right] + \frac{\partial^2}{\partial x_i\partial x_j}\left[T_{ij}H(f)\right]. \tag{10}
$$

This equation is valid throughout the whole of space. Inside the fluid, $H = 1$, are Lighthill's quadrupole sources. In addition to these volume sources there exist surface sources in the form of dipoles and monopoles with:

dipole strength density: surface stress
monopole strength density: rate at which mass is transmitted across unit area of surface.

Source contributions caused by surface permeability:

- monopole source: $\frac{\partial}{\partial t}\left[\rho\left(v_n - u_n\right)\delta(f)\right]$
- dipole source: $-\frac{\partial}{\partial x_i}\left[\rho v_i\left(v_n - u_n\right)\delta(f)\right]$

Ffowcs Williams–Hawkings equation in integral form:

$$Hc_0^2\left(\rho-\rho_0\right)=\frac{\partial^2}{\partial x_i\partial x_j}\int\limits_{V(\tau)}\left[T_{ij}\right]_\tau\frac{dV}{4\pi\left|x_i-y_i\right|}-\frac{\partial}{\partial x_i}\int\limits_{S(\tau)}\left[\rho v_i\left(v_j-u_j\right)+p_{ij}\right]_\tau \cdot\frac{dS_j(y_i)}{4\pi\left|x_i-y_i\right|}+\frac{\partial}{\partial t}\int\limits_{S(\tau)}\left[\rho\left(v_j-u_j\right)+\rho_0 u_j\right]_\tau\frac{dS_j(y_i)}{4\pi\left|x_i-y_i\right|} \tag{11}$$

with: $\tau=t-\frac{\left|x_i-y_i\right|}{c_0}=t-\frac{\left|\vec{x}-\vec{y}\right|}{c_0}$ retarded time, surface integrals over the retarded surface $S(\tau)$ defined by $f(y_i,t)=0$, surface element dS_i directed into the region $V(\tau)$ where $f>0$.

The control surface is a non-porous surface (impenetrable): $\rho\left(v_i-u_i\right)=0$

Monopole term: $$Q=\rho_0 u_i\frac{\partial f}{\partial x_i}=\rho_0 u_n \tag{12}$$

Dipole term: $$F_i=p_{ij}\frac{\partial f}{\partial x_j}=p_{ij}n_j \tag{13}$$

with: $p_{ij}=p\delta_{ij}-\tau_{ij}$

$F_i=p\frac{\partial f}{\partial x_i}=pn_i$, if viscous stresses are neglected;

p local surface pressure,

n_i local unit outward normal to surface.

Sources can be interpreted

- as a volume distribution of quadrupoles $\frac{\partial^2 T_{ij}}{\partial x_i\partial x_j}$ in the outer region of the surfaces, due to the turbulent flow,
- as a surface distribution of dipoles $-\frac{\partial}{\partial x_i}\left[F_i\delta(f)\right]$ due to the interaction of the flow with moving bodies, especially due to surface pressure and stress fluctuations on the bodies in the flow,
- as a surface distribution of monopoles $\frac{\partial}{\partial t}\left[Q\delta(f)\right]$, due to the kinematics of the bodies, especially from normal accelerations of the body surfaces.

Detailed representation of the Ffowcs Williams–Hawkings equation in differential form:

$$\Box^2\left[\phi H(f)\right]:=\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\left[\phi H(f)\right]-\frac{\partial^2}{\partial x_i^2}\left[\phi H(f)\right]$$

$$=\frac{\partial^2}{\partial x_i\partial x_j}\left[T_{ij}H(f)\right]-\frac{\partial}{\partial x_i}\left\{\left[\left(p-p_0\right)\delta_{ij}-\tau_{ij}\right]n_j\delta(f)\right\}-\frac{\partial}{\partial x_i}\left[\rho v_i\left(v_n-u_n\right)\delta(f)\right] +\frac{\partial}{\partial t}\left[\rho_0 u_n\delta(f)\right]+\frac{\partial}{\partial t}\left[\rho\left(v_n-u_n\right)\delta(f)\right] \tag{14}$$

with: $\phi=c_0^2\left(\rho-\rho_0\right)$ wave variable in linear acoustics corresponding to sound pressure.

Reformulation (Pilon/Lyrintzis):

$$\Box^2\left[\phi H(f)\right] = -\left(\frac{\partial\phi}{\partial n} + \frac{M_n}{c_0}\frac{\partial\phi}{\partial t_x}\right)\delta(f) - \frac{1}{c_0}\frac{\partial}{\partial t}\left[M_n\phi\delta(f)\right] - \frac{\partial}{\partial x_i}\left[\phi n_i\delta(f)\right] + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}H(f) \tag{15}$$

with: subscript x in the time derivative denoting differentiation with respect to time, holding the observer co-ordinates fixed.

In comparison to the generalised wave equation, which is the governing equation for the Kirchhoff formulation:

- the domain is considered in terms of wave propagation,
- the generalised pressure perturbation: $p' = \begin{cases} p' & f > 0 \\ 0 & f < 0 \end{cases}$.

The generalised wave equation:

$$\Box^2 p' := \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = -\left(\frac{\partial p'}{\partial t}\frac{M_n}{c} + \frac{\partial p'}{\partial n}\right)\delta(f) - \frac{\partial}{\partial t}\left[p'\frac{M_n}{c}\delta(f)\right] - \frac{\partial}{\partial x_i}\left[p' n_i\delta(f)\right] \tag{16}$$

with: $M_n = \frac{u_n}{c}$ local normal Mach number on S

$$\Box^2 p_a = q_{\mathrm{Kirchhoff}} \tag{17}$$

Manipulation the FW–H source terms into the form of Kirchhoff source terms (inviscid fluid):

$$\begin{aligned}\Box^2 p' &= q_{\mathrm{FW-H}} \\ &= q_{\mathrm{Kirchhoff}} + \frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij}H(f)\right] - \frac{\partial}{\partial x_j}\left[\rho v_i v_j\right]n_i\delta(f) - \frac{\partial}{\partial x_j}\left[\rho v_i v_n\delta(f)\right] \\ &\quad + \frac{\partial}{\partial t}\left[p' - c^2\rho'\right]\frac{M_n}{c}\delta(f) + \frac{\partial}{\partial t}\left[\left(p' - c^2\rho'\right)\frac{M_n}{c}\delta(f)\right]\end{aligned} \tag{18}$$

Extra source terms are of second order in perturbation quantities:

- $q_{\mathrm{FW-H}} \approx q_{\mathrm{Kirchhoff}}$ equivalent in linear region $p' \approx c_0^2\rho'$; $v_i \ll 1$,
- $q_{\mathrm{FW-H}} \neq q_{\mathrm{Kirchhoff}}$ not equivalent in non-linear flow region.

There are four types of source terms in the inhomogeneous wave equation in the form of the FW–H equation and Kirchhoff equation:

$$\Box^2 p_a = \frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij}H(f)\right]; \quad \Box^2 p_a = Q(x_i, t)\,\delta(f); \tag{19,20}$$

$$\Box^2 p_a = \frac{\partial}{\partial t}\left[Q\delta(f)\right]; \quad \Box^2 p_a = \frac{\partial}{\partial x_i}\left[Q_i\delta(f)\right]. \tag{21,22}$$

N.9.2 Curle's Equation

Curle's equation is a special case of the FW-H equation in integral form, with control surface stationary and rigid: $u_i = 0$:

$$Hc_0^2\left(\rho-\rho_0\right)=\frac{\partial^2}{\partial x_i \partial x_j}\int\limits_{V(\tau)}\left[T_{ij}\right]_\tau \frac{dV}{4\pi\left|x_i-y_i\right|}-\frac{\partial}{\partial x_i}\int\limits_{S(\tau)}\left[p_{ij}\right]_\tau \frac{dS_j(y_i)}{4\pi\left|x_i-y_i\right|}. \tag{23}$$

N.10 Acoustic Analogy in Terms of Entropy, Heat Sources as Sound Sources, Sound Generation by Turbulent Two-Phase Flow

► *See also:* Howe (1998); Crighton/Dowling et al. (1992); Morfey (1973); Strahle (1971, 1972, 1975); Boineau/Gervais (1998); Perrey-Debain et al. (1980); Crighton/Ffowcs Williams (1969)

N.10.1 Acoustic Analogy in Terms of Entropy, Sound Generation by Fluctuating Heat Sources (Dowling, Howe)

Lighthill's inhomogeneous wave equation:

$$\frac{\partial^2\rho}{\partial t^2}-c_0^2\Delta\rho=\frac{\partial^2}{\partial x_i\partial x_j}\left\{\rho v_i v_j+\left[\left(p-p_0\right)-c_0^2\left(\rho-\rho_0\right)\right]\delta_{ij}-\tau_{ij}\right\} \tag{1}$$

with: τ_{ij} viscous stress tensor

p_0, ρ_0, c_0 mean value in acoustic field

Reformulation by Dowling (1992):

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\Delta p=\frac{\partial^2}{\partial x_i\partial x_j}\left\{\rho v_i v_j-\tau_{ij}\right\}-\frac{\partial^2\rho_e}{\partial t^2} \tag{2}$$

with: $\rho_e=\rho-\rho_0-\dfrac{(p-p_0)}{c_0^2}$

"excess" density ρ_e vanishing in the far field but is non-zero in regions where the entropy is different from ambient

$$\begin{aligned}\frac{\partial\rho_e}{\partial t}=&\frac{\alpha\rho_0}{c_p\rho}\left[\sum_{n=1}^{N}\left.\frac{\partial h}{\partial Y_n}\right|_{\rho,p,Y_m}\rho\frac{DY_n}{DT}+\nabla\cdot q_i-\frac{\partial v_i}{\partial x_j}\tau_{ij}\right]-\nabla\cdot\left(v_i\rho_e\right)\\&-\frac{1}{c_0^2}\left[\left(1-\frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{Dp}{Dt}-\frac{(p-p_0)}{\rho}\frac{D\rho}{Dt}\right]\end{aligned} \tag{3}$$

α volumetric expansion coefficient, for an ideal gas equal to $\beta = T^{-1}$

h enthalpy

Y_n mass fraction of *n*th species

N number of N (possible reacting) species

$\vec{q}$ heat flux

$$\rho\frac{DY_n}{Dt} = w_n - \nabla \cdot J_{n,i}$$

w_n production rate per unit volume of species n by reaction

$J_{n,i}$ flux of species n by diffusion

Inhomogeneous wave equation:

$$\begin{aligned}\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = &-\frac{\partial}{\partial t}\left\{\frac{\alpha\rho_0}{c_p\rho}\left[\sum_{n=1}^{N}\frac{\partial h}{\partial Y_n}\bigg|_{\rho,p,Y_m}\rho\frac{DY_n}{DT} + \nabla\cdot q_i - \frac{\partial v_i}{\partial x_j}\tau_{ij}\right]\right\}\\ &+\frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j - \tau_{ij}\right) + \frac{1}{c_0^2}\frac{\partial}{\partial t}\left[\left(1-\frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{Dp}{Dt} - \frac{(p-p_0)}{\rho}\frac{D\rho}{Dt}\right]\\ &+\frac{\partial^2}{\partial x_i \partial t}\left(v_i \rho_e\right)\end{aligned} \tag{4}$$

Solution, with the help of the Green's function in unbounded space and with the far-field approximations:

$$\begin{aligned}(p-p_0)(x_i,t) = &-\frac{1}{4\pi r}\frac{\partial}{\partial t}\int\left\{\frac{\alpha\rho_0}{c_p\rho}\left[\sum_{n=1}^{N}\frac{\partial h}{\partial Y_n}\bigg|_{\rho,p,Y_m}\rho\frac{DY_n}{DT} + \nabla\cdot q_i - \frac{\partial v_i}{\partial x_j}\tau_{ij}\right]\right\}_\tau dV\\ &+\frac{x_i x_j}{4\pi r^3 c_0^2}\frac{\partial^2}{\partial t^2}\int\left(\rho v_i v_j - \tau_{ij}\right)_\tau dV\\ &+\frac{1}{4\pi r c_0^2}\frac{\partial}{\partial t}\int\left[\left(1-\frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{Dp}{Dt} - \frac{(p-p_0)}{\rho}\frac{D\rho}{Dt}\right]_\tau dV\\ &-\frac{x_i}{4\pi r^2 c_0}\frac{\partial^2}{\partial t^2}\int\left(v_i\rho_e\right)_\tau dV.\end{aligned} \tag{5}$$

Right-hand side: thermoacoustic source mechanisms

- first term: sound generated by irreversible flow processes, including diffusion of mass and heat; → monopole sources;
- second term: sound generated by momentum flux density fluctuations and by fluctuations of viscous stress (Lighthill's jet noise theory); → quadrupole and octupole sources;
- third term: sound generated by unsteady flow regions with different mean density and sound speed from ambient fluid; → dipole sources;
- fourth term: sound generated by effects of momentum changes of density inhomogeneities; → dipole sources.

Other formulation by Howe (1998):

$$\begin{aligned}\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = {} & \frac{\partial}{\partial t}\left(\frac{\rho_0}{c_p}\frac{Ds}{Dt}\right) + \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_0 v_i v_j\right) \\ & + \frac{1}{c_0^2}\frac{\partial}{\partial t}\left[\left(1 - \frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{\partial p}{\partial t}\right] - \rho_0 \operatorname{div}\left[\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\nabla p\right]\end{aligned} \tag{6}$$

Solution:

$$\begin{aligned}p(x_i, t) = {} & \frac{\rho_0}{4\pi r}\frac{\partial}{\partial t}\int\left[\frac{q(y_i, t)}{c_p \rho T}\right]_\tau dV + \frac{x_i x_j}{4\pi r^3 c_0^2}\frac{\partial^2}{\partial t^2}\int\left(\rho_0 v_i v_j\right)_\tau \\ & + \frac{\rho_0 x_j}{4\pi r^2 c_0}\frac{\partial}{\partial t}\int\left[\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)\frac{\partial p}{\partial y_j}\right]_\tau dV + \frac{\rho_0}{4\pi r}\frac{\partial^2}{\partial t^2}\int\left[\left(\frac{1}{\rho_0 c_0^2} - \frac{1}{\rho c^2}\right)\left(p - p_0\right)\right]_\tau dV\end{aligned} \tag{7}$$

Right-hand side: thermoacoustic source mechanisms

- first term: direct combustion noise → monopole sources;
- second term: jet noise (Lighthill) → quadrupole sources;
- third term: indirect combustion noise, "entropy" noise → dipole sources;
  dipole source strength is proportional to
  $-\left[\frac{1}{\rho} - \frac{1}{\rho_0}\right]\nabla p,$
  that is, to the difference between the acceleration of fluid of density ρ in the jet and which fluid of ambient mean density ρ_0 would experience in the same pressure gradient;
  other interpretation:
  $\rho_0\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) = \frac{\rho - \rho_0}{\rho} \approx \frac{\Delta T}{T}$
  that is, the dipole source strength is proportional to the fractional temperature difference between the hot spot and its environment;
- fourth term: indirect combustion noise → monopole sources;
  monopole source strength is proportional to the difference between the adiabatic compressibility $(1/\rho c^2)$ in the hot source region and in the ambient medium.

Re: third and fourth terms:
The "hot spots" or "entropy inhomogeneities" behave as scattering centres at which dynamic pressure fluctuations are converted directly into sound.

N.10.2 Acoustic Analogy in Terms of Heat Release, Turbulent Density Fluctuations and Turbulent Velocity Fluctuations on Outer Flame Surface (Strahle)

Solution of inhomogeneous wave equation:

$$p(\vec{x}, t) = \frac{(\kappa - 1)}{4\pi r c_0^2} \int_V \left[\frac{\partial q}{\partial t}\right]_\tau dV \tag{8}$$

with: q rate of heat release per unit volume

$$\rho(\vec{x}, t) = \frac{1}{4\pi r c_0^2} \frac{\partial^2}{\partial t^2} \int_V [\rho_t]_\tau dV \tag{9}$$

with: ρ_t turbulent fluctuating components of density, is considered negligible outside of reacting region of flame, even if non-reacting portions of flow field are turbulent, if Mach number is low

ρ acoustic fluctuating components of density

$$\rho(\vec{x}, t) = \frac{\bar{\rho}_1}{4\pi r c_0} \frac{\partial}{\partial t} \int_{S_r} v_{i,t} n_i dS \tag{10}$$

with: $\bar{\rho}_1$ mean density behind flame

S_r surface of direct combustion region, of turbulent reaction zone

$v_{i,t}$ turbulent velocity fluctuations on outer flame surface, produced by interior density fluctuations

N.10.3 Sound Power Radiated by a Turbulent Flame

Far-field acoustic pressure:

$$p(x_i, t) = \frac{(\kappa - 1)}{4\pi r c_0^2} \frac{\partial}{\partial t} \int_V [Q]_\tau \, dV \tag{11}$$

with: q heat release rate per unit volume

$q(y_i, t) = \frac{\partial}{\partial t} S(y_i, t) = \frac{\partial}{\partial t} \left[\bar{\rho}(y_i)\, \bar{c}_p(y_i)\, T'(y_i, t)\right]$ acoustic source strength

$T'(y_i, t)$ temperature fluctuations in combustion zone

Sound power spectrum radiated by a turbulent flame:

$$W(\omega) = \frac{(\kappa - 1)^2}{4\pi \rho_0 c_0} k^4 \int_{V_y} \int_{V_\eta} S_{rms}(y_i') S_{rms}(y_i'') \Gamma(y_i', \eta_i, \omega) \, dy_i' d\eta_i \tag{12}$$

with: k acoustic wave number

Γ Fourier transform of acoustic sources, space–time correlation coefficient

η_i separation vector between two acoustic sources

S_{rms} rms amplitude of acoustic sources

Respectively in the case of axisymmetrical flow:

$$W(\omega) = \frac{(\kappa-1)^2}{4\pi\rho_0 c_0} k^4 \int\limits_{V_{y'}} \int\limits_{V_{y''}} S_{rms}(y_i')S_{rms}(y_i'')\Gamma\left(\xi_z, \xi_{r,}, \omega\right) \times \sqrt{P\left(y_i', \omega\right)}\sqrt{P\left(y_i'', \omega\right)} J_0^2\left(\frac{k_{ac} r_i}{2}\right) J_0^2\left(\frac{k_{ac} r_j}{2}\right) dy_i' dy_i'' \tag{13}$$

with: $$\Gamma\left(\xi_z, \xi_r, \omega\right) = \exp\left(\frac{-\pi\xi_z^2}{L_{cz}^2}\right)\exp\left(\frac{-\pi\xi_r^2}{L_{cr}^2}\right)\frac{\sqrt{\pi}}{\omega_t}\exp\left(\frac{-\omega^2}{4\omega_t^2}\right) \tag{14}$$

$\Gamma\left(\xi_z, \xi_r, \omega\right)$	spatial and frequency coherence function
ξ_z, ξ_r	longitudinal and radial separation distances respectively between acoustic sources
L_{cz}, L_{cr}	longitudinal and radial coherence scale respectively
$\omega_t = 2\pi C_\omega \frac{\varepsilon}{k}$	
ω_t	characteristic angular frequency of turbulence
$C_\omega = 1{,}5$	
$P(\omega) = \frac{2a}{\pi\left(1 + a^2\omega^2\right)}$	
$P(\omega)$	normalised temperature fluctuation spectrum, containing characteristic time for temperature fluctuations, which is a function of the turbulence level and the characteristic turbulence angular frequency
$a = \frac{k}{C_S \varepsilon}$	characteristic time of temperature fluctuations
k	turbulent kinetic energy
ε	dissipation rate of turbulent kinetic energy
$C_S = 6{,}4$	see Sanders, J.P.H. and P.G.G. Lammers (“Combustion and Flame”, 1994)

N.10.4 Sound Generation by Turbulent Two-Phase Flow

Mass continuity equation for α-phase:

$$\frac{\partial}{\partial t}\left(1-\beta\right)\rho^\alpha + \frac{\partial}{\partial x_j}\left(1-\beta\right)\rho^\alpha v_j^\alpha = 0 \tag{15}$$

with:

α	α-phase, e.g. water
β	β-phase, e.g. gas bubbles
ρ^α, ρ^β	mass of α-phase (resp. β-phase) per volume occupied by α-phase (resp.β-phase)
$\left(1-\beta\right)\rho^\alpha$	mass of α-phase in unit volume of mixture
$\left(1-\beta\right)\rho^\alpha + \beta\rho^\beta$	total mass per unit volume

Reformulation:

$$\frac{\partial}{\partial t}\rho^\alpha + \frac{\partial}{\partial x_j}\rho^\alpha v_j^\alpha = Q \tag{16}$$

with: $$Q = -\rho^{\alpha}\left(\frac{\partial}{\partial t} + v_j^{\alpha}\frac{\partial}{\partial x_j}\right)\ln\left(1-\beta\right) \tag{17}$$

Momentum equation for α-phase:

$$\frac{\partial}{\partial t}\left(1-\beta\right)\rho^{\alpha}v_i^{\alpha} + \frac{\partial}{\partial x_j}\left[\left(1-\beta\right)\rho^{\alpha}v_i^{\alpha}v_j^{\alpha} + p_{ij}\right] = F_i \tag{18}$$

with: p_{ij} stress tensor

F_i interphase force

Reformulation:

$$\frac{\partial}{\partial t}\rho^{\alpha}v_i^{\alpha} + \frac{\partial}{\partial x_j}\left[\left(1-\beta\right)\rho^{\alpha}v_i^{\alpha}v_j^{\alpha} + p_{ij}\right] = G_i \tag{19}$$

with: $$G_i = F_i + G_i' = F_i + \frac{\partial}{\partial t}\left(\beta\rho^{\alpha}v_i^{\alpha}\right) \tag{20}$$

Inhomogeneous wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_{\alpha}^2\nabla^2\right)\rho^{\alpha} = \frac{\partial Q}{\partial t} - \frac{\partial G_i}{\partial x_i} + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{21}$$

with: $T_{ij} = \left(1-\beta\right)\rho^{\alpha}v_i^{\alpha}v_j^{\alpha} + p_{ij} - c_{\alpha}^2\rho^{\alpha}\delta_{ij}$
c_{α} sound speed in pure water

Sound is generated in turbulent two-phase flow by three physical mechanisms:

- first term: distribution of monopoles, of strength Q, equal to rate of mass injection into α-phase;
- second term: distribution of dipoles, of strength G_i, equal to effective force on α-fluid, composed in part of interphase force F_i and in part of term G_i', which represents momentum defect arising from the fact that a fraction β of the total volume is not occupied by α-phase;
- third term: distribution of quadrupoles, of strength T_{ij} (Lighthill), is dominated by the Reynolds stress (viscous contributions are neglected).

N.11 Acoustics of Moving Sources

► *See also:* Lowson (1965); Tanna/Morfey, J. Sound Vibr. 15 and J. Sound Vibr. 16 (1971); Roger (1996); Ianniello (1999); Lyrintzis (1997)

N.11.1 Sound Field of Moving Point Sources

Monopole point source:

Starting point:

$$p(x_i, t) = \frac{1}{4\pi} \int_V \left[\frac{1}{r} \frac{\partial (\dot{M}\delta)}{\partial t} \right]_\tau dV \qquad (1)$$

with:

$\dot{M}$	mass flux, point source
$\delta = \delta\left[y_i - y_{qi}(t)\right]$	three-dimensional delta function
y_i	coordinates of the source point
$y_{qi}(t)$	time-variable place of the mass flux source
$v_{qi} = \frac{\partial y_{qi}}{\partial t} = c_0 M_{qi}$	velocity of the source point
$M_{qi} = \frac{v_{qi}}{c_0}$	Mach number, based on the source velocity
$\tau = t - \frac{r_\tau}{c_0}$	retarded time, time of radiation
t	time of observation
r_τ	observer – source separation, dependent on τ

Sound pressure (general):

$$p(x_i, t) = \frac{1}{4\pi\left(1 - M_{qr}\right)_\tau} \left\{ \frac{1}{r}\frac{\partial \dot{M}}{\partial t} + \frac{\partial}{\partial y_i}\left[\frac{\dot{M} c_0 M_{qi}}{r\left(1 - M_{qr}\right)}\right] + \frac{\left(x_i - y_i\right)}{r^2}\frac{\partial}{\partial t}\left[\frac{\dot{M} M_{qi}}{\left(1 - M_{qr}\right)}\right]\right\}_\tau \qquad (2)$$

Sound pressure in the far field:

$$p(x_i, t) = \left\{ \frac{1}{4\pi r\left(1 - M_{qr}\right)^2} \left[\frac{\partial \cdot M}{\partial t} + \frac{\dot{M}}{\left(1 - M_{qr}\right)} \frac{\partial M_{qr}}{\partial t} \right] \right\}_\tau \qquad (3)$$

with:

$M_{qr} = \frac{\left(x_i - y_i\right)}{\left\lvert x_i - y_i \right\rvert} M_{qi}$	projection of Mach number vector M_{qi} in the direction of sound radiation $\vec{r} = \vec{x} - \vec{y}$
$\dot{M}$	mass flux

Other formulation:

$$p(x_i, t) = \left\{ \frac{1}{4\pi r\left(1 - M_{qr}\right)} \frac{\partial \dot{M}}{\partial t} + \frac{\left(x_i - y_i\right)}{4\pi r^2\left(1 - M_{qr}\right)^2} \left[\frac{\partial\left(\dot{M} M_{qi}\right)}{\partial t} + \frac{\dot{M} M_{qi}}{\left(1 - M_{qr}\right)} \frac{\partial M_{qr}}{\partial t} \right] \right\}_\tau \qquad (4)$$

Proportionalities:

- first term: temporal change of the mass flux

$$p \sim \rho_0 \frac{L}{r} \frac{1}{\left(1 - M_{qr}\right)} U^2 \quad \cong \text{monopole source} \tag{5}$$

- second term: temporal change of momentum flux

$$p \sim \frac{\rho_0}{c_0} \frac{L}{r} \frac{1}{\left(1 - M_{qr}\right)^2} U^3 \quad \cong \text{dipole source} \tag{6}$$

- third term: accelerated motion of momentum flux

$$p \sim \frac{\rho_0}{c_0^2} \frac{L}{r} \frac{1}{\left(1 - M_{qr}\right)^3} U^4 \quad \cong \text{quadrupole source} \tag{7}$$

Sound pressure in far field, uniform motion of monopole point source:

$$p\left(x_i, t\right) = \left\{ \frac{1}{4\pi r \left(1 - M_{qr}\right)} \frac{\partial \dot{M}}{\partial t} \right\}_\tau \tag{8}$$

Dipole point source:

Starting point:

$$p\left(x_i, t\right) = -\frac{1}{4\pi} \int_V \left[\frac{1}{r} \frac{\partial \left(F_i \delta\right)}{\partial y_i} \right]_\tau dV \tag{9}$$

with: F_i force, point source

Sound pressure (general):

$$\begin{aligned} p\left(x_i, t\right) = & \frac{1}{4\pi \left(1 - M_{qr}\right)_\tau} \left\{ F_{i\tau} \frac{\partial \left(\frac{1}{r}\right)_\tau}{\partial y_i} + \left[\frac{\left(x_i - y_i\right)}{c_0 r^2} \frac{\partial F_i}{\partial t} \right]_\tau \right. \\ & + \left[\frac{\partial}{\partial y_j} \left(\frac{\left(x_i - y_i\right)}{r^2} \frac{F_i M_{qj}}{\left(1 - M_{qr}\right)} \right) \right]_\tau \\ & \left. + \left[\frac{\left(x_i - y_i\right)\left(x_j - y_j\right)}{c_0 r^3} \frac{\partial}{\partial t} \left(\frac{F_i M_{qj}}{\left(1 - M_{qr}\right)} \right) \right]_\tau \right\} \end{aligned} \tag{10}$$

Sound pressure in far field:

$$p\left(x_i, t\right) = \left\{ \frac{\left(x_i - y_i\right)}{4\pi r^2 c_0 \left(1 - M_{qr}\right)^2} \left[\frac{\partial F_i}{\partial t} + \frac{F_i}{\left(1 - M_{qr}\right)} \frac{\partial M_{qr}}{\partial t} \right] \right\}_\tau \tag{11}$$

Proportionalities:

- first term: temporal change of force

$$p \sim \frac{\rho_0}{c_0}\frac{L}{r}\frac{1}{\left(1-M_{qr}\right)^2}U^3 \qquad \cong \text{dipole source} \tag{12}$$

- second term: accelerated motion of force

$$p \sim \frac{\rho_0}{c_0^2}\frac{L}{r}\frac{1}{\left(1-M_{qr}\right)^3}U^4 \qquad \cong \text{quadrupole source} \tag{13}$$

Other formulation (Lighthill):

$$p\left(x_i, t\right) = \left\{\frac{\left(x_i - y_i\right)}{rc_0\left(1-M_{qr}\right)}\frac{\partial}{\partial t}\left[\frac{F_i}{4\pi r\left(1-M_{qr}\right)}\right]\right\}_\tau \tag{14}$$

Sound pressure in far field, uniform motion of dipole point source:

$$p\left(x_i, t\right) = \left\{\frac{\left(x_i - y_i\right)}{4\pi r^2 c_0\left(1-M_{qr}\right)^2}\frac{\partial F_i}{\partial t}\right\}_\tau \tag{15}$$

Quadrupole point source:

Starting point:

$$p\left(x_i, t\right) = \frac{1}{4\pi}\int_V \left[\frac{1}{r}\frac{\partial^2\left(T_{ij}\delta\right)}{\partial y_i \partial y_j}\right]_\tau dV \tag{16}$$

with: T_{ij} pressure-stress tensor, Lighthill tensor, point source

Sound pressure:

$$\begin{aligned} p\left(x_i, t\right) = & \left\{\frac{\left(x_i - y_i\right)\left(x_j - y_j\right)}{4\pi r^3 c_0^2\left(1-M_{qr}\right)^3}\left[\frac{\partial^2 T_{ij}}{\partial t^2} + \frac{3}{\left(1-M_{qr}\right)}\frac{\partial T_{ij}}{\partial t}\frac{\partial M_{qr}}{\partial t}\right.\right. \\ & \left.\left. + \frac{T_{ij}}{\left(1-M_{qr}\right)}\frac{\partial^2 M_{qr}}{\partial t^2} + \frac{3T_{ij}}{\left(1-M_{qr}\right)^2}\left(\frac{\partial M_{qr}}{\partial t}\right)^2\right]\right\}_\tau \end{aligned} \tag{17}$$

Proportionalities:

- first term: second time derivative of momentum flux density

$$p \sim \frac{\rho_0}{c_0^2}\frac{L}{r}\frac{1}{\left(1-M_{qr}\right)^3}U^4 \qquad \cong \text{quadrupole source} \tag{18}$$

- second/third term: accelerated motion of momentum flux density, respectively, and time derivative of this quantity

$$p \sim \frac{\rho_0}{c_0^3}\frac{L}{r}\frac{1}{\left(1-M_{qr}\right)^4}U^5 \qquad \cong \text{octupole source} \tag{19}$$

- fourth term: strong accelerated motion of momentum flux density

$$p \sim \frac{\rho_0}{c_0^4}\frac{L}{r}\frac{1}{\left(1-M_{qr}\right)^5}U^6 \quad \cong \text{sexdecupole source} \tag{20}$$

Other formulation (Lighthill):

$$p\left(x_i, t\right) = \left\{\frac{\left(x_i - y_i\right)}{rc_0\left(1-M_{qr}\right)}\frac{\partial}{\partial t}\left[\frac{\left(x_j - y_j\right)}{rc_0\left(1-M_{qr}\right)}\frac{\partial}{\partial t}\left(\frac{T_{ij}}{4\pi r\left(1-M_{qr}\right)}\right)\right]\right\}_\tau \tag{21}$$

Sound pressure in far field, uniform motion of quadrupole point source:

$$p\left(x_i, t\right) = \left\{\frac{\left(x_i - y_i\right)\left(x_j - y_j\right)}{4\pi r^3 c_0^2\left(1-M_{qr}\right)^3}\frac{\partial^2 T_{ij}}{\partial t^2}\right\}_\tau \tag{22}$$

N.11.2 Formulation of Equation of Sound Sources in Motion Based on Ffowcs Williams–Hawkings Equation

Far-field solution:

$$\begin{aligned} p\left(x_i, t\right) = {} & \frac{\partial^2}{\partial x_i \partial x_j}\int\limits_V\left[\frac{T_{ij}}{4\pi r\left(1-M_r\right)}\right]_\tau dV - \frac{\partial}{\partial x_i}\int\limits_S\left[\frac{p_{ij}n_j}{4\pi r\left(1-M_r\right)}\right]_\tau dS \\ & + \frac{\partial}{\partial t}\int\limits_S\left[\frac{\rho_0 u_n}{4\pi r\left(1-M_r\right)}\right]_\tau dS \end{aligned} \tag{23}$$

with:

$p_{ij}n_j dS$	force on fluid from each surface element
u_n	normal velocity field on surfaces
$M_r = \dfrac{x_i u_i}{rc_0}$	Mach number of sources
$1 - M_r$	Doppler amplification factor related to projected motion on line between source and observer point

Quadrupole source:

$$p\left(\vec{x}, t\right) = \frac{\partial^2}{\partial x_i \partial x_j}\int\limits_V\left[\frac{T_{ij}}{4\pi r\left|1-M_r\right|}\right]_\tau dV \tag{24}$$

respectively:

$$\begin{aligned} p\left(\vec{x}, t\right) = {} & \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int\limits_V\left[\frac{T_{rr}}{4\pi r\left|1-M_r\right|}\right]_\tau dV + \frac{1}{c_0}\frac{\partial}{\partial t}\int\limits_V\left[\frac{3T_{rr} - T_{ii}}{4\pi r^2\left|1-M_r\right|}\right]_\tau dV \\ & + \int\limits_V\left[\frac{3T_{rr} - T_{ii}}{4\pi r^3\left|1-M_r\right|}\right]_\tau dV \end{aligned} \tag{25}$$

with:

M_r	projection of the rotational Mach number in source-observer direction
$T_{rr} = T_{ij}\hat{r}_i\hat{r}_j$	Lighthill stress tensor in radiation direction
$\hat{\vec{r}}$	unit vector in radiation direction, with components $\hat{r}_i$

N.11.3 Moving Kirchhoff Surfaces

Kirchhoff's surface S:
- encloses all non-linear effects and sound sources;
- outside Kirchhoff's surface the acoustic field is linear.

Wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = 0 \tag{26}$$

with: Φ acoustic quantity satisfying wave equation in exterior of surface S

Classical Kirchhoff formulation for stationary control surface:

$$4\pi\Phi(\vec{x},t) = \int_S \left[\frac{1}{r^2}\Phi\frac{\partial r}{\partial n} - \frac{1}{r}\frac{\partial\Phi}{\partial n} + \frac{1}{rc_0}\frac{\partial r}{\partial n}\frac{\partial\Phi}{\partial\tau}\right]_\tau dS \tag{27}$$

with: τ retarded (emission) time
r distance between observer and source
$\frac{\partial r}{\partial n} = \cos\Theta$
Θ angle between the normal vector $\vec{n}$ on the surface and the radiation direction $\vec{r} = \vec{x} - \vec{y}$

Interpretation:
- integral representation of Φ at points exterior to S in terms of quantities on control surface S,
- computation of noise at an arbitrary point, if the solution is known on surface S,
- first term: not significant in far field.

Uniformly moving surface:

Convective wave equation:

$$\Delta\Phi - \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + U_\infty\frac{\partial}{\partial x}\right)^2\Phi = 0 \tag{28}$$

with: U_∞ uniform velocity of control surface S

Distance between observer and surface point:

$$r_0 = \sqrt{(x-x')^2 + \beta^2\left[(y-y')^2 + (z-z')^2\right]} \tag{29}$$

with: $\vec{x} = (x, y, z, t)$ observer location
$\vec{y} = (x', y', z', \tau')$ source location
$\tau' = t - \tau = t - r/c_0$ source time, retarded (emission) time
$\tau = [r_0 - M_\infty(x-x')]/(c\beta^2)$ time delay between emission and detection

$$x_0 = x, \quad y_0 = \beta y, \quad z_0 = \beta z \qquad \text{Prandtl–Glauert transformation}$$

$$\beta = \sqrt{1 - M_\infty^2}$$

Solution for case of subsonically moving surface:

$$4\pi\Phi(\vec{x}, t) = \int_{S_0} \left[\frac{1}{r_0^2} \Phi \frac{\partial r_0}{\partial n_0} - \frac{1}{r_0} \frac{\partial \Phi}{\partial n_0} + \frac{1}{r_0 c_0 \beta^2} \frac{\partial \Phi}{\partial \tau} \left(\frac{\partial r_0}{\partial n_0} - M_\infty \frac{\partial x_0}{\partial n_0} \right) \right]_\tau dS_0 \tag{30}$$

with: subscript 0: transformed variable, e.g. $\vec{n}_0$ is outward pointing vector normal to surface S_0

Solution for case of supersonically moving surface:

$$4\pi\Phi(\vec{x}, t) = \int_{S_0} \left[\frac{1}{r_0^2} \Phi \frac{\partial r_0}{\partial n_0} - \frac{1}{r_0} \frac{\partial \Phi}{\partial n_0} + \frac{1}{r_0 c_0 \beta^{*2}} \frac{\partial \Phi}{\partial \tau} \left(\pm \frac{\partial r_0}{\partial n_0} - M_\infty \frac{\partial x_0}{\partial n_0} \right) \right]_{\tau^\pm} dS_0 \tag{31}$$

with: $\beta^* = \sqrt{M_\infty^2 - 1}$

$\tau^\pm = \dfrac{[\pm r_0 - M_\infty (x - x')]}{c\beta^{*2}}$ time delay

$x_0 = x, \quad y_0 = \beta^* y, \quad z_0 = \beta^* z$ Prandtl–Glauert transformation

$r_0 = \sqrt{(x - x')^2 + \beta^{*2}\left[(y - y')^2 + (z - z')^2\right]}$ distance between observer and surface point in Prandtl–Glauert co-ordinates

sign $\pm$ evaluation at both retarded times τ^+ and τ^-

Arbitrarily moving surface (subsonically moving, rigid surface):

$$4\pi\Phi(\vec{x}, t) = \int_S \left\{ \frac{\frac{\partial r}{\partial n}}{r^2(1 - M_r)} \Phi - \frac{1}{r(1 - M_r)} \left[\frac{\partial \Phi}{\partial n} + \frac{M_n}{c_0} \frac{\partial \Phi}{\partial \tau} \right] + \frac{1}{c_0(1 - M_r)} \frac{\partial}{\partial \tau} \left[\frac{\frac{\partial r}{\partial n} - M_n}{r(1 - M_r)} \Phi \right] \right\}_\tau dS \tag{32}$$

with: $\vec{y}, \vec{r}$ functions of time: $\vec{y}(\tau'), \ \vec{r}(\tau')$

observer point is stationary

all quantities are evaluated for retarded (emission) time, which is root of equation $\tau - t + r(\tau)/c_0 = 0$

$M_r = \vec{u} \cdot \vec{r}/(rc_0)$ Mach number in direction of wave propagation from source to observer

$\vec{u} = \partial\vec{y}/\partial\tau$ local source-surface velocity

$M_n = u_n/c_0$ local normal Mach number

u_n local normal velocity of S with respect to undisturbed medium

$\dfrac{\partial \Phi}{\partial \tau}, \dfrac{\partial \Phi}{\partial n}$ taken with respect to source co-ordinates and time

N.12 Aerodynamic Sound Sources in Practice

► *See also:* Bailly (1996); Goldstein (1976); Fuchs/Michalke (1973); Ribner (1969); Roger (1996); Goldstein/Howes (1973); Béchara et al. (1995); Goldstein/Rosenbaum (1973); Heckl (1969); Ffowcs Williams/Hawkings, J. Sound Vibr. 10 (1969); Lowson (1970); Farassat/Brentner (1998); Singer/Brentner (1999); Brentner (1997); Ianiello (1999); Költzsch (1974, 1986, 1994)

N.12.1 Jet Noise

The acoustic intensity and power radiated by a jet are evaluated.

Starting point: Lighthill's inhomogeneous wave equation:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial \dot{M}}{\partial t} - \frac{\partial}{\partial x_i} \left(F_i + \dot{M}\, \nu_i \right) + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = q \tag{1}$$

with: $T_{ij} = \rho v_i v_j + p_{ij} - c_0^2 \rho \delta_{ij}$ Lighthill tensor

Solution:

$$p\,(x_i, t) = \frac{1}{4\pi} \int\limits_V \frac{1}{r} \left(\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \right)_\tau dV \tag{2}$$

$$p\,(x_i, t) = \frac{1}{4\pi c_0^2} \int\limits_V \frac{1}{r} \left\{ \frac{\partial^2 \left(T_{ij}\right)}{\partial t^2} \right\}_\tau \left[\frac{1}{r^2} (x_i - y_i)(x_j - y_j) \right] dV \tag{3}$$

In far field:

$$p\,(x_i, t) = \frac{1}{4\pi c_0^2 r} \frac{x_i x_j}{r^2} \int\limits_V \left[\frac{\partial^2 \left(T_{ij}\right)}{\partial t^2} \right]_\tau dV \tag{4}$$

Calculation of the acoustic intensity I with the help of the two-point correlation function R_a of the acoustic pressure fluctuations:

$$R_a\,(x_i, \tau) = R_{pp}\,(x_i, \tau) = \frac{\overline{p\,(x_i, t)\, p\,(x_i, t + \tau)}}{\rho_0 c_0} = \lim_{T \to \infty} \frac{1}{T} \int\limits_0^T \frac{p\,(x_i, t)\, p\,(x_i, t + \tau)}{\rho_0 c_0} dt \tag{5}$$

$I\,(x_i) = R_a\,(x_i, \tau = 0)$ overall intensity at observation point

Power spectral density of acoustic pressure at observer point:

$$S_a\,(x_i, \omega) = S_{pp}\,(x_i, \omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} R_a\,(x_i, \tau)\, e^{i\omega\tau} d\tau \tag{6}$$

Introducing Lighthill's solution:

$$R_a(x_i,\tau) = \frac{1}{16\pi^2 c_0^5 \rho_0 x^2}\frac{x_i x_j x_k x_l}{x^4}\int\limits_{V'}\int\limits_{V''}\overline{\frac{\partial^2 T_{ij}}{\partial t^2}(y_i',t')\frac{\partial^2 T_{kl}}{\partial t^2}(y_i'',t'')}\,dV'dV'' \tag{7}$$

with: retarded times

$$t' = t - \frac{|x_i - y_i'|}{c_0} \quad \text{and} \quad t'' = t + \tau - \frac{|x_i - y_i''|}{c_0}$$

Formulation with coherent source regions:
Volume correlation:

$$\int\limits_{V'}\int\limits_{V''}\overline{\left[\frac{\partial^2 T_{ij}}{\partial t^2}(y_i',t')\right]\left[\frac{\partial^2 T_{kl}}{\partial t^2}(y_i'',t'')\right]}\,dV'dV'' = \int\limits_{V'}\overline{\left[\frac{\partial^2 T_{ij}}{\partial t^2}(y_i',t')\right]^2}V_{corr}dV' \tag{8}$$

with: $$V_{corr} = \int\limits_{V''}\frac{\overline{\left[\frac{\partial^2 T_{ij}}{\partial t^2}(y_i',t')\right]\left[\frac{\partial^2 T_{kl}}{\partial t^2}(y_i'',t'')\right]}}{\overline{\left[\frac{\partial^2 T_{ij}}{\partial t^2}(y_i',t')\right]^2}}dV'' \tag{9}$$

Assumption of stationary properties of turbulence:

$$R_a(x_i,\tau) = \frac{1}{16\pi^2 \rho_0 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\frac{\partial^4}{\partial\tau^4}\int\limits_{V'}\int\limits_{V''}\overline{T_{ij}(y_i',t')\,T_{kl}(y_i'',t'')}\,dV'dV'' \tag{10}$$

Far-field approximation:

$$R_a(x_i,\tau) = \frac{1}{16\pi^2 \rho_0 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\frac{\partial^4}{\partial\tau^4}\int\limits_{V'}\int\limits_{V''}R_{ijkl}\left(y_i',\eta_i,\tau+\frac{x_i\cdot\eta_i}{xc_0}\right)dV'dV_\eta \tag{11}$$

with: $$R_{ijkl}(y_i',\eta_i,\tau) = \overline{T_{ij}(y_i',t)\,T_{kl}(y_i'+\eta_i,t+\tau)} \tag{12}$$

R_{ijkl}	two-point fourth-order correlation tensor of turbulent velocity
$\eta_i = y_i'' - y_i'$	distance vector between two points y_i' and y_i'' in source volume
$\eta \ll x$	far-field assumption

$$S_a(x_i,\omega) = S_{pp}(x_i,\omega)$$

$$= \frac{\omega^4}{32\pi^3\rho_0 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\int\limits_{-\infty}^{+\infty}\iint e^{i\omega\left(\tau-\frac{\vec{x}}{x}\frac{\vec{\eta}}{c_0}\right)}R_{ijkl}(\vec{y}',\vec{\eta},\tau)\,dV'dV_\eta d\tau \tag{13}$$

Reformulation with following assumptions:

- isotropic turbulence in a frame moving with mean convection velocity U_c;
- introducing new coordinate (Ffowcs Williams) $\vec{\xi} = \vec{\eta} - U_c\tau\vec{y}_1$ ($\vec{y}_1$ direction of mean flow) and variable $\lambda = \alpha U_c\tau$, with $u' = \alpha U_c$ (typical turbulence velocity, α small parameter, corresponding to the turbulence level);
- definition of new correlation tensor with change of variables $(\vec{\eta}, \tau) \rightarrow (\vec{\xi}, \lambda)$.
  $R^*_{ijkl}(\vec{y}, \vec{\eta}, \tau) = R_{ijkl}(\vec{y}, \vec{\xi} = \vec{\eta} - U_c\tau\vec{y}_1, \lambda)\,.$ (14)

This leads to an expression for the far-field correlation for noise generated by convected isotropic turbulence:

$$R_a(\vec{x}, \tau) = \frac{1}{16\pi^2\rho_0 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\iint\limits_V \frac{1}{C^5}\frac{\partial^4}{\partial t^4}R_{ijkl}\left(\vec{y}', \vec{\xi}, t = \frac{\tau}{C}\right)dV' dV_\xi \tag{15}$$

with: $C = \sqrt{(1 - M_c\cos\Theta)^2 + \alpha^2 M_c^2}$ convection factor, Doppler factor;

$M_c = U_c/c_0$ convection Mach number;

Θ angle between the mean flow direction $\vec{y}_1$ and the observer point $\vec{x}$;

$\cos\Theta = \dfrac{\vec{x}\cdot\vec{y}_1}{x}$;

$\alpha^2 = \dfrac{\omega^2 L^2}{\pi c_0^2}$ (usually α = const., e.g. $\alpha \approx 0,55$ (Ribner), $\alpha \approx 0,3$ (Davies), often approximately $\alpha \approx 0$, i.e. using the simplified convection factor $C = 1 - M_c\cos\Theta$).

Acoustic intensity:

$$I(\vec{x}) = \frac{1}{16\pi^2\rho_0 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\iint\limits_V \frac{1}{C^5}\frac{\partial^4}{\partial t^4}R_{ijkl}\left(\vec{y}', \vec{\xi}, 0\right)dV' dV_\xi \tag{16}$$

Dimensional development leads to a power law for acoustic intensity (Lighthill):

$$I(\vec{x}) \sim \frac{1}{\rho_0 c_0^5 x^2}\frac{1}{C^5}\left(\frac{U}{D}\right)^4\left(\rho^2 U^4\right)D^6, \tag{17}$$

$$I(\vec{x}) \sim \frac{D^2}{x^2}\frac{\rho^2}{\rho_0}\frac{U_c^3 M_c^5}{\left[(1 - M_c\cos\Theta)^2 + \alpha^2 M_c^2\right]^{5/2}} \tag{18}$$

with: D typical length of the source volume;
ρ mean flow density;
U typical velocity;
$U_c \approx 2/3U$ for a round jet.

$$I(\vec{x}) \sim U^8 \quad \text{(Lighthill)} \tag{19}$$

Acoustic power:

May be obtained by integrating acoustic intensity over sphere of radius r:

$$P = \int_0^{\pi} I(r, \Theta)\, 2\pi r^2 \sin\Theta d\Theta \tag{20}$$

Integration with a simplified convection factor $C = 1 - M_c \cos\Theta$ for subsonic region $M_c \leq 1$:

$$P \sim \frac{\rho^2}{\rho_0} D^2 \frac{U^8}{c_0^5} \frac{1 + M_c^2}{\left(1 - M_c^2\right)^4} \tag{21}$$

(for practical applications: with proportionality factor $K \approx 5 \cdot 10^{-5}$)

Acoustic efficiency η:

$$\eta = \frac{P}{P_{\text{mech}}} \sim \frac{\rho^2}{\rho_0} \frac{U^5}{c_0^5} \frac{1 + M_c^2}{\left(1 - M_c^2\right)^4} \tag{22}$$

$$\text{with:} \quad P_{\text{mech}} = \frac{\pi}{4} D^2 \rho U^3 \tag{23}$$

The convection amplifies the radiated acoustic power by the factor $\frac{1 + M_c^2}{\left(1 - M_c^2\right)^4}$.

For supersonic jet flow $M_c > 1$:

$$P \sim \frac{\rho^2}{\rho_0} D^2 U^3 \tag{24}$$

$$\eta \sim \frac{\rho}{\rho_0} \tag{25}$$

Power spectral density of far-field noise:
(statistical source models in jet noise of Ribner 1969)

Notations:

Density autocorrelation function:

$$C_{\rho\rho}(\vec{x},\tau) = \frac{1}{\rho_0 c_0^3}\overline{\left[\rho(\vec{x},t+\tau)-\rho_0\right]\left[\rho(\vec{x},t)-\rho_0\right]} \tag{26}$$

$$C_{\rho\rho}(\vec{x},\tau) = \frac{\rho_0}{16\pi^2 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\iint_V \overline{\frac{\partial^2}{\partial t^2}v_i' v_j'\left(\vec{y}',t'\right)\frac{\partial^2}{\partial t^2}v_k'' v_l''\left(\vec{y}'',t''\right)}\,dV' dV'' \tag{27}$$

with: $\vec{y}'$, $\vec{y}''$ two running points in source domain

$$t' = t - \frac{\left|\vec{x}-\vec{y}'\right|}{c_0}, \quad t'' = t + \tau - \frac{\left|\vec{x}-\vec{y}''\right|}{c_0}$$

$$C_{\rho\rho}(\vec{x},\tau) = \frac{\rho_0}{16\pi^2 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\iint_V \frac{\partial^4}{\partial\tau^4}R_{ijkl}\left(\vec{y}',\vec{\eta},\tau+\frac{\vec{\eta}\cdot\vec{x}}{c_0 x}\right)dV' dV_\eta \tag{28}$$

with: $R_{ijkl}\left(\vec{y}',\vec{\eta},\tau\right) = \overline{v_i' v_j'\left(\vec{y}',t\right)v_k'' v_l''\left(\vec{y}'',t+\tau\right)}$

two-point time-delayed fourth-order correlation tensor

Directional acoustical intensity spectrum, which is the temporal Fourier transform of the density autocorrelation $C_{\rho\rho}(\vec{x},\tau)$:

$$I_\omega(\vec{x}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} C_{\rho\rho}(\vec{x},\tau)\, e^{j\omega\tau} d\tau \tag{29}$$

Acoustical power spectrum (emitted from a unit volume located at $\vec{y}$):

$$P_\omega\left(\vec{y}\right) = 2\pi r^2 \int_0^\pi I_\omega\left(\vec{x},\Theta\left|\vec{y}\right.\right)\sin\Theta\, d\Theta \tag{30}$$

Total acoustic power:

$$P = \int_V \int_{-\infty}^{+\infty} P_\omega\left(\vec{y}\right) dV d\omega \tag{31}$$

Far-field noise radiated: acoustic intensity from an elementary volume of jet:

$$dI^{\substack{\text{self}\\ \text{noise}}}(\vec{x}) \sim \frac{\rho_0 \overline{u'^2}^2 L^3 \omega_t^4}{c_0^5 r^2 C^5} \tag{32}$$

$$dI^{\text{shear noise}}(\vec{x}) \sim \frac{\rho_0 \overline{u'^2} L^5 \omega_t^4}{c_0^5 r^2 C^5} \left(\frac{\partial U}{\partial y_2}\right)^2 D_\Theta \tag{33}$$

respectively the intensity spectrum:

$$dI_\omega^{\text{self noise}}\left(\vec{x}, \Theta \,|\vec{y}\right) = \frac{\rho_0 \overline{u'^2}^2 L^3}{128\pi^{5/2} c_0^5 r^2} \frac{\omega^4}{\omega_t} \exp\left(-\frac{\omega^2 C^2}{8\omega_t^2}\right) \tag{34}$$

$$dI_\omega^{\text{shear noise}}\left(\vec{x}, \Theta \,|\vec{y}\right) = \frac{\rho_0 \overline{u'^2} L^5}{24\pi^{7/2} c_0^5 r^2} \left(\frac{\partial U}{\partial y_2}\right)^2 \frac{\omega^4}{\omega_t} \exp\left(-\frac{\omega^2 C^2}{4\omega_t^2}\right) D_\Theta \tag{35}$$

with:	$D_\Theta = \frac{1}{2}\left(\cos^2\Theta + \cos^4\Theta\right)$	directivity of shear-noise component (isotropic directivity of self noise is a necessary consequence of isotropy of turbulence)
	C	convection factor (at Goldstein/Howes: C^{-3} instead of C^{-5})
	$\omega_t = 2\pi\frac{\varepsilon}{k}$	
	ω_t	local characteristic frequency of turbulence, related to eddy lifetime
	$L \approx \frac{k^{3/2}}{\varepsilon}$	
	L	integral length scale of turbulence
	u'	turbulent velocity
	$U\left(y_2\right)$	mean velocity in direction of y_1, dependent on co-ordinate y_2

Other developments (Goldstein/Rosenbaum):

Acoustic intensity per unit source volume:

$$dI^{\text{self noise}}(\vec{x}) \sim \frac{\rho_0 \overline{u'^2}^2 L_1 L_2^2 \omega_t^4}{c_0^5 r^2 C^5} D_1 \tag{36}$$

$$dI^{\text{shear noise}}(\vec{x}) \sim \frac{\rho_0 \overline{u_1'^2} L_1 L_2^4 \omega_t^4}{c_0^5 r^2 C^5} \left(\frac{\partial U_1}{\partial y_2}\right)^2 D_2 \tag{37}$$

respectively:

$$dI_\omega^{\text{self noise}}\left(\vec{x}, \Theta \,|\vec{y}\right) = \frac{\rho_0 \overline{u_1'^2}^2 L_1 L_2^2}{40\sqrt{2}\pi^{3/2} c_0^5 r^2} \frac{\omega^4}{\omega_t} D_1 \exp\left(-\frac{\omega^2 C^2}{8\omega_t^2}\right) \tag{38}$$

$$dI_{\omega}^{\substack{\text{shear}\\ \text{noise}}}\left(\vec{x},\Theta\,\middle|\,\vec{y}\right)=\frac{\rho_0\overline{u_1'^2}L_1L_2^4}{\pi^{3/2}c_0^5r^2}\frac{\omega^4}{\omega_t}\left(\frac{\partial U_1}{\partial y_2}\right)^2 D_2\exp\left(-\frac{\omega^2C^2}{4\omega_t^2}\right) \tag{39}$$

with:

$$\begin{aligned}D_1 &= 1+2\left(\frac{M}{9}-N\right)\cos^2\Theta\sin^2\Theta\\ &\quad+\frac{1}{3}\left[\frac{M^2}{7}+M-\frac{3N}{2}\left(3-3N+\frac{3}{2\Delta^2}-\frac{\Delta^2}{2}\right)\right]\sin^4\Theta\end{aligned} \tag{40}$$

$$D_2=\cos^2\Theta\left[\cos^2\Theta+\frac{1}{2}\left(\frac{1}{\Delta^2}-2N\right)\sin^2\Theta\right] \tag{41}$$

with: anisotropic structure of turbulence

$$\Delta=\frac{L_2}{L_1}$$

$$M=\left[\frac{3}{2}\left(\Delta-\frac{1}{\Delta}\right)\right]^2 \tag{42}$$

$$N=1-\frac{\overline{u_2'^2}}{\overline{u_1'^2}} \tag{43}$$

$\overline{u_1'^2}, \overline{u_2'^2}$	axial and transversal turbulent kinetic energy
U_1	axial mean flow velocity
L_1, L_2	integral length scale in direction of flow and in transverse direction

$$L_2\approx\frac{1}{3}L_1 \tag{44}$$

$$L_1\approx\frac{(2k/3)^{3/2}}{\varepsilon} \tag{45}$$

$\omega_t=2\pi\dfrac{\varepsilon}{k}$ angular frequency of turbulence for isotropic turbulence: $$\overline{u_1'^2} = \frac{2}{3}k \tag{46}$$

for anisotropic turbulence: $$\overline{u_1'^2} = \frac{2}{3}k - \nu_t \frac{\partial \bar{U}_1}{\partial x_1} \tag{47}$$

$$\overline{u_2'^2} = \frac{2}{3}k - 2\nu_t \frac{\partial \bar{U}_2}{\partial x_2} \tag{48}$$

$$\text{with: } \nu_t = 0,09 \frac{k^2}{\varepsilon} \tag{49}$$

kinematic turbulent viscosity

N.12.2 Rotor Noise

Computation of rotor noise, based on Ffowcs Williams–Hawkings equation:

$$\Box^2 p_a(\vec{x}, t) = \frac{\partial}{\partial t}[Q\delta(f)] - \frac{\partial}{\partial x_i}[L_i\delta(f)] + \frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij}H(f)\right] \tag{50}$$

Assumptions:

- (at first) neglecting quadrupole sources,
- moving surface is non-porous.

Blade thickness noise:

$$\Box^2 p_T = \frac{\partial}{\partial t}\left[\rho_0 u_n \delta(f)\right] \tag{51}$$

Solution:

$$p_T(x_i, t) = \frac{\partial}{\partial t}\int\limits_S \left[\frac{\rho_0 u_n}{4\pi r(1 - M_r)}\right]_\tau dS = \int\limits_S \left[\frac{1}{1 - M_r}\frac{\partial}{\partial \tau}\frac{\rho_0 u_n}{4\pi r(1 - M_r)}\right]_\tau dS \tag{52}$$

for open, rotating blades with a subsonic tip Mach number:

$$p_T(\vec{x}, t) = \int\limits_{f=0} \left[\frac{\rho_0(\dot{u}_n + u_{\dot{n}})}{4\pi r(1 - M_r)^2}\right]_\tau dS + \int\limits_{f=0} \left\{\frac{\rho_0 u_n\left[r\dot{M}_r + c_0\left(M_r - M^2\right)\right]}{4\pi r^2(1 - M_r)^3}\right\}_\tau dS \tag{53}$$

with: u_n — local velocity of blade surface in directional normal to $f = 0$

$\dot{u}_n = \dot{u}_i n_i$

$u_{\dot{n}} = u_i \dot{n}_i = u_i \dfrac{\partial n_i}{\partial \tau}$

$\vec{n}$ — unit outward normal vector to surface, with components n_i

$M = |\vec{M}|$

$\vec{M}$ — local Mach number vector of source, with components M_i

M_r — component of velocity in radiation direction normalised by c_0

$f = 0$ — function describing rotor blade surface

c_0 — sound speed in quiescent medium

The dot over a symbol implies source-time differentiation of that symbol, e.g. $\dot{M}_r = \left(\dfrac{\partial M_i}{\partial \tau}\right) r_i$.

Loading noise:

$$\Box^2 p_L = -\frac{\partial}{\partial x_i}\left[F_i \delta(f)\right] \tag{54}$$

Solution:

$$\begin{aligned} p_L(x_i, t) &= -\frac{\partial}{\partial x_i} \int\limits_{f=0} \left[\frac{F_i}{4\pi r(1 - M_r)}\right]_\tau dS \\ &= \frac{1}{c}\frac{\partial}{\partial t} \int\limits_{f=0} \left[\frac{F_r}{4\pi r(1 - M_r)}\right]_\tau dS + \int\limits_{f=0} \left[\frac{F_r}{4\pi r^2(1 - M_r)}\right]_\tau dS \end{aligned} \tag{55}$$

for open, rotating blades with a subsonic tip Mach number:

$$\begin{aligned} p_L(\vec{x}, t) = \frac{1}{c_0} \int\limits_{f=0} \left[\frac{\dot{L}_r}{4\pi r(1 - M_r)^2}\right]_\tau dS &+ \int\limits_{f=0} \left[\frac{L_r - L_M}{4\pi r^2(1 - M_r)^2}\right]_\tau dS \\ &+ \frac{1}{c_0} \int\limits_{f=0} \left\{\frac{L_r\left[r\dot{M}_r + c_0\left(M_r - M^2\right)\right]}{4\pi r^2(1 - M_r)^3}\right\}_\tau dS \end{aligned} \tag{56}$$

with: L_i components of local force that acts on fluid (L_i is identical with L_i used above)

$L_i = \left[p_{ij} n_j + \rho v_i (v_n - u_n)\right]$

$L_r = L_i r_i$

L_r component of local force that acts on fluid (due to body) in radiation direction

$L_M = L_i M_i$

M_i velocity of surface $f = 0$ normalised to ambient sound speed

M_r component of velocity in radiation direction normalised to c_0

r distance from source point on surface to observer

The dot over a symbol implies source-time differentiation of that symbol, e.g. $\dot{L}_r = \left(\frac{\partial L_i}{\partial \tau}\right) r_i$.

Other formulation:

Thickness and loading noise together

Integral representation of solution (of FW–H equation):

$$p_a(\vec{x}, t) = \int\limits_{f=0} \left[\frac{\dot{Q} + \dot{L}_r/c_0}{4\pi r (1 - M_r)^2}\right]_\tau dS + \int\limits_{f=0} \left[\frac{L_r - L_M}{4\pi r^2 (1 - M_r)^2}\right]_\tau dS + \int\limits_{f=0} \left\{\frac{(Q + L_r/c_0)\left[r\dot{M}_r + c_0 (M_r - M^2)\right]}{4\pi r^2 (1 - M_r)^3}\right\}_\tau dS \tag{57}$$

Quadrupole noise:

$$\Box^2 p_a(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right] \tag{58}$$

Solution:

$$p_T(x_i, t) = \frac{\partial}{\partial t}\int\limits_S \left[\frac{\rho_0 u_n}{4\pi r (1 - M_r)}\right]_\tau dS = \int\limits_S \left[\frac{1}{1 - M_r}\frac{\partial}{\partial \tau}\frac{\rho_0 u_n}{4\pi r (1 - M_r)}\right]_\tau dS \tag{59}$$

$$p_Q(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j}\int\limits_V \left[\frac{T_{ij}}{4\pi r |1 - M_r|}\right]_\tau dV \tag{60}$$

respectively:

$$p_Q(\vec{x},t) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int\limits_V \left[\frac{T_{rr}}{4\pi r\,|1-M_r|}\right]_\tau dV + \frac{1}{c_0}\frac{\partial}{\partial t}\int\limits_V \left[\frac{3T_{rr}-T_{ii}}{4\pi r^2\,|1-M_r|}\right]_\tau dV + \int\limits_V \left[\frac{3T_{rr}-T_{ii}}{4\pi r^3\,|1-M_r|}\right]_\tau dV \tag{61}$$

with: M_r projection of rotational Mach number in source-observer direction

$T_{rr} = T_{ij}\hat{r}_i\hat{r}_j$ Lighthill stress tensor in radiation direction

$\hat{\vec{r}}$ unit vector in radiation direction, with components $\hat{r}_i$

Rotor noise in practice:

sound far field in terms of source strength spectrum
(rotor and stator noise, propeller noise, helicopter rotor noise, etc.)

Monopole:

$$p(x_i,\omega) = \frac{e^{-2\pi j\omega\, r/c_0}}{4\pi r}\sum_{n=-\infty}^{+\infty} m(\phi,\omega-n\omega_0)\, e^{-jn(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R}{c_0}\sin\Theta\right) \tag{62}$$

with: $m(\phi,\omega-n\omega_0)$ point monopole, spectrum of the source strength

ϕ angular position of the point source at time $t=0$

ω_0 rotational angular velocity

R position vector of the point source

Dipole:

$$p(x_i,\omega) = -\frac{e^{-2\pi j\omega\, r/c_0}}{4\pi r}\frac{2\pi j\omega}{c_0}\widehat{r}_i\sum_{n=-\infty}^{+\infty} f_i(\phi,\omega-n\omega_0)\, e^{-jn(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R}{c_0}\sin\Theta\right) \tag{63}$$

with: $f_i(\phi,\omega-n\omega_0)$ point dipole, spectrum of the source strength

$\widehat{r}_i = \frac{\vec{r}}{|\vec{r}|}$ component of unit vector in direction i

Quadrupole:

$$p(x_i,\omega) = -\frac{e^{-2\pi j\omega\, r/c_0}}{4\pi r}\left(2\pi\frac{\omega}{c_0}\right)^2 \widehat{r}_i\widehat{r}_j\sum_{n=-\infty}^{+\infty} t_{ij}(\phi,\omega-n\omega_0)\, e^{-jn(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R}{c_0}\sin\Theta\right) \tag{64}$$

with: $t_{ij}(\phi, \omega - n\omega_0)$ point quadrupole, spectrum of source strength

$\widehat{r}_i, \widehat{r}_j$ components of unit vector in direction i, j

Rotor monopole sound:

$$P = \frac{\dot{M}_B^2 \omega_0^2}{8\pi\rho_0 c_0} \sum_{m=1}^{\infty} (mB)^2 \int_0^{\pi} J_{mB}^2 \left(mBM_q \sin\Theta\right) \sin\Theta \, d\Theta \tag{65}$$

with:

B	number of rotor blades
$\dot{M}_B = B\dot{M}$	overall fluid mass displaced per unit time by all rotor blades
ω_0	rotor angular frequency
m	integer
J_{mB}	Bessel function (first kind) of order mB
$M_q = \omega_0 R / c_0$	Mach number
R	radius of rotor, equivalent radius from hub point on rotor blade, in which source strength is concentrated
Θ	angle between rotor axis and vector from rotor centre to observer point

Rotor dipole sound due to stationary rotating forces:

$$p_m = \frac{mB\omega_0}{2\pi c_0 r_0} (-j)^{mB+1} e^{-jmB\,\omega_0 r_0/c_0} \left(F_T \cos\Theta - \frac{F_D}{M_q}\right) J_{mB}\left(mBM_q \sin\Theta\right) \tag{66}$$

$$P = \frac{\omega_0^2}{4\pi\rho_0 c_0^3} \sum_{m=1}^{\infty} \int_0^{\pi} \left(F_T \cos\Theta - \frac{F_D}{M_q}\right)^2 (mB)^2 J_{mB}^2 \left(mBM_q \sin\Theta\right) \sin\Theta \, d\Theta \tag{67}$$

with:

p_m	m-th sound pressure harmonic
F_T, F_D	thrust and drag, respectively, which act on air and in site direction, effect of all rotor blades together

Rotor dipole sound due to rotating periodic time-variable forces:

- sound radiation of rotor:

$$p_{m\mu} = \frac{jmB\omega_0}{2\pi c_0 r_0} (-j)^{mB-\mu} \left(F_{T\mu} \cos\Theta - \frac{mB-\mu}{mB} \frac{F_{D\mu}}{M_q}\right) J_{mB-\mu}\left(mBM_q \sin\Theta\right) \tag{68}$$

$$P_{m\mu} = \frac{\omega_0^2}{4\pi\rho_0 c_0^3} \int_0^{\pi} \left(F_{T\mu} \cos\Theta - \frac{mB-\mu}{mB} \frac{F_{D\mu}}{M_q}\right)^2 (mB)^2 J_{mB-\mu}^2 \left(mBM_q \sin\Theta\right) \sin\Theta \, d\Theta \tag{69}$$

with: $p_{m\mu}/P_{m\mu}$ *m*th harmonic of sound pressure / sound power radiated by μth harmonic of fluctuating forces (blade loading harmonic) of rotor blades

$F_{T\mu}, F_{D\mu}$ Fourier components of periodic time-variable blade forces (thrust and drag)

in the case of B rotor blades and V stator vanes: $\mu = kV$

- sound radiation of stator:

$$p_{mk} = -\frac{jmBV\omega_0}{2\pi c_0 r_0} j^{mB-kV}\left[F_{Tm}\cos\Theta - \left(\frac{mB-kV}{mB}\right)\frac{F_{Dm}}{M_q}\right] J_{mB-kV}\left(mBM_q\sin\Theta\right) \quad (70)$$

with: p_{mk} m-th harmonic of sound pressure radiated by *k*th harmonic of fluctuating forces on stator vanes owing to rotor-stator interaction

Rotor dipole sound by rotating random blade forces:

$$p_\omega(x_i,\omega) = -\frac{\omega}{2c_0 r_0} e^{-j\omega r_0/c_0} \sum_{n=-\infty}^{+\infty}\left(F_{T\Omega}\cos\Theta - \frac{n\omega_0}{\omega}\frac{F_{D\Omega}}{M_q}\right) j^{n+1} J_n\left(\frac{\omega}{\omega_0}M_q\sin\Theta\right) \quad (71)$$

$$P_\omega = \frac{z_R\omega^2}{4\pi\rho_0 c_0^3}\int_0^\pi \sum_{n=-\infty}^{+\infty}\left(F_{T\Omega}\cos\Theta - \frac{n\omega_0}{\omega}\frac{F_{D\Omega}}{M_q}\right)^2 J_n^2\left(\frac{\omega}{\omega_0}M_q\sin\Theta\right)\sin\Theta\, d\Theta \quad (72)$$

with: $F_{T\Omega}, F_{D\Omega}$ Fourier transforms of stochastic time-variable blade forces (thrust and drag)

$\omega = \Omega + n\omega_0$

Ω frequency variable of force spectrum

N.13 Power Law of the Aerodynamic Sound Sources

► *See also:* Ffowcs Williams, Annual Review of Fluid Mechanics 1 (1969); Költzsch (1974, 1998)

The power law of aerodynamic sound sources is presented based on developments with the help of dimensional analysis, generalised for multipoles of arbitrary order and of variable number of space dimensions, for compact and non-compact aerodynamic multipoles.

Order of multipoles N:

Monopole source $\rightarrow N = 0$;
Dipole source $\rightarrow N = 1$;
Quadrupole source $\rightarrow N = 2$.

Power law of compact aerodynamic multipoles:

$$(\rho - \rho_0)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right)^{\frac{n-1}{2}} M_t^{N+\frac{n+1}{2}} \quad (1)$$

with:

$\rho - \rho_0$	acoustic density fluctuation in far field
ρ_0	density of ambient fluid
$\bar{\rho}_q$	mean density inside flow
L	scale of coherent regions in flow, characteristic length scale
$M_t = v'/c_0$	Mach number, ratio of characteristic turbulence velocity v' to speed of sound in uniform environment, measure of compactness
N	order of multipoles
n	number of space dimensions in which wave field spreads

Measure of compactness $\rightarrow \quad M_t > kL\,;\ \dfrac{L}{\lambda} < \dfrac{1}{2\pi} M_t$

in general: M_t is much less than 2π, that is $\dfrac{L}{\lambda} \ll 1$;
acoustic wavelength is much greater than the length scale characteristic of the turbulent source flow;

it follows: such sources are acoustically compact.

The following overview presents (in the three-dimensional case):

- power laws for acoustic density fluctuations,
- sound power P,
- acoustic-aerodynamic efficiency η (ratio of sound power to flow power):

Assumptions: $M_t \sim M = \dfrac{U}{c_0}$ with: U a characteristic flow velocity;

$\eta_{ac} = \dfrac{P_{ac}}{P_{mech}}$ acoustic-aerodynamic efficiency;

with: P_{ac} sound power;

P_{mech} flow mechanical power: $P_{mech} \sim \bar{\rho}_q L^2 U^3$.

General: $$(\rho - \rho_0)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right) M^{2+N} \sim \bar{\rho}_q \left(\frac{L}{r}\right) U^2 M^N \quad (2)$$

$$P \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^{4+2N} \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^3 M^{1+2N} \quad (3)$$

$$\eta_{ac} \sim \frac{\bar{\rho}_q}{\rho_0} M^{1+2N} \quad (4)$$

Monopole: $$\left(\rho - \rho_0\right)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right) M^2 \sim \bar{\rho}_q \left(\frac{L}{r}\right) U^2 \tag{5}$$

$$P \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^4 \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^3 M \tag{6}$$

$$\eta_M \sim \frac{\bar{\rho}_q}{\rho_0} M \tag{7}$$

Dipole: $$\left(\rho - \rho_0\right)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right) M^3 \sim \bar{\rho}_q \left(\frac{L}{r}\right) U^2 M \tag{8}$$

$$P \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^6 \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^3 M^3 \tag{9}$$

$$\eta_{ac} \sim \frac{\bar{\rho}_q}{\rho_0} M^3 \tag{10}$$

Quadrupole: $$\left(\rho - \rho_0\right)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right) M^4 \sim \bar{\rho}_q \left(\frac{L}{r}\right) U^2 M^2 \tag{11}$$

$$P \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^8 \sim \frac{\bar{\rho}_q}{\rho_0} L^2 U^3 M^5 \tag{12}$$

$$\eta_{ac} \sim \frac{\bar{\rho}_q}{\rho_0} M^5 \tag{13}$$

Power law of non-compact aerodynamic multipoles:

Measure of compactness $\rightarrow$ $M_t < kL$; $\frac{L}{\lambda} > \frac{1}{2\pi} M_t$; acoustic wavelength λ is smaller than $2\pi\, L/M_t$; $\rightarrow$ such sources are acoustically non-compact.

Power law: $$\left(\rho - \rho_0\right)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right)^{\frac{n-1}{2}} M_t \tag{14}$$

Power law of moving aerodynamic sources:

$$\left(\rho - \rho_0\right)(\vec{x}) \sim \bar{\rho}_q \left(\frac{L}{r}\right) \frac{M^{2+N}}{\left|1 - M_{qr}\right|^{1+N}} \tag{15}$$

with: $M_{qr} = \frac{\left(x_i - y_i\right)}{\left|x_i - y_i\right|} M_{qi}$ projection of Mach number vector M_{qi} in direction of sound radiation $\vec{r} = \vec{x} - \vec{y}$

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