13. Fresnel's Equations for Reflection and Transmission

Incident, transmitted, and reflected beams

Boundary conditions: tangential fields are continuous

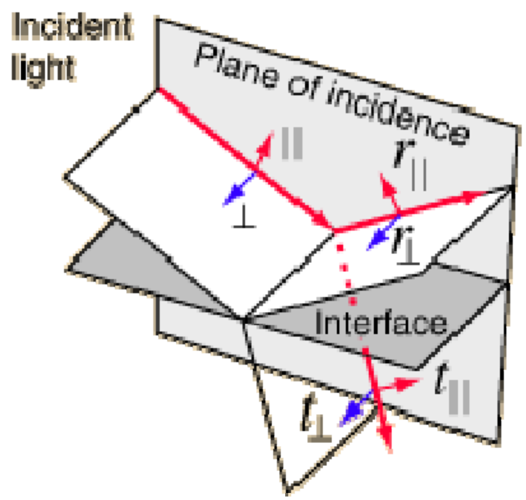
Reflection and transmission coefficients

The "Fresnel Equations"

Brewster's Angle

Total internal reflection

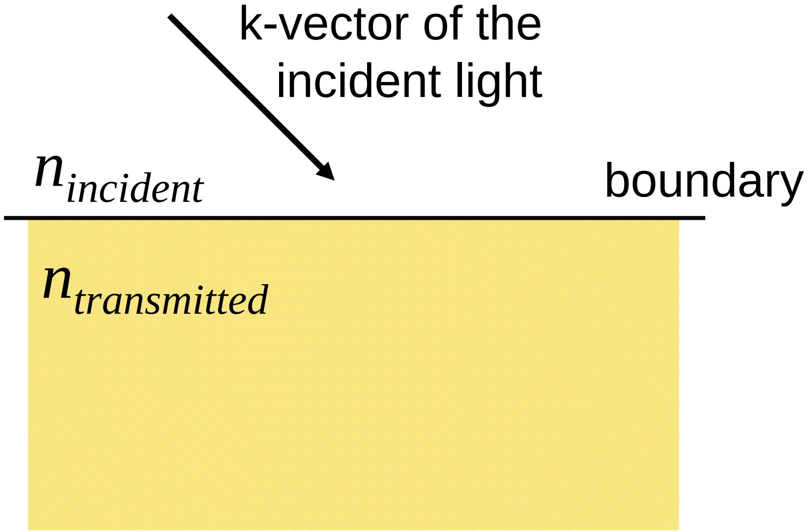
Power reflectance and transmittance



Augustin Fresnel
1788-1827

Posing the problem

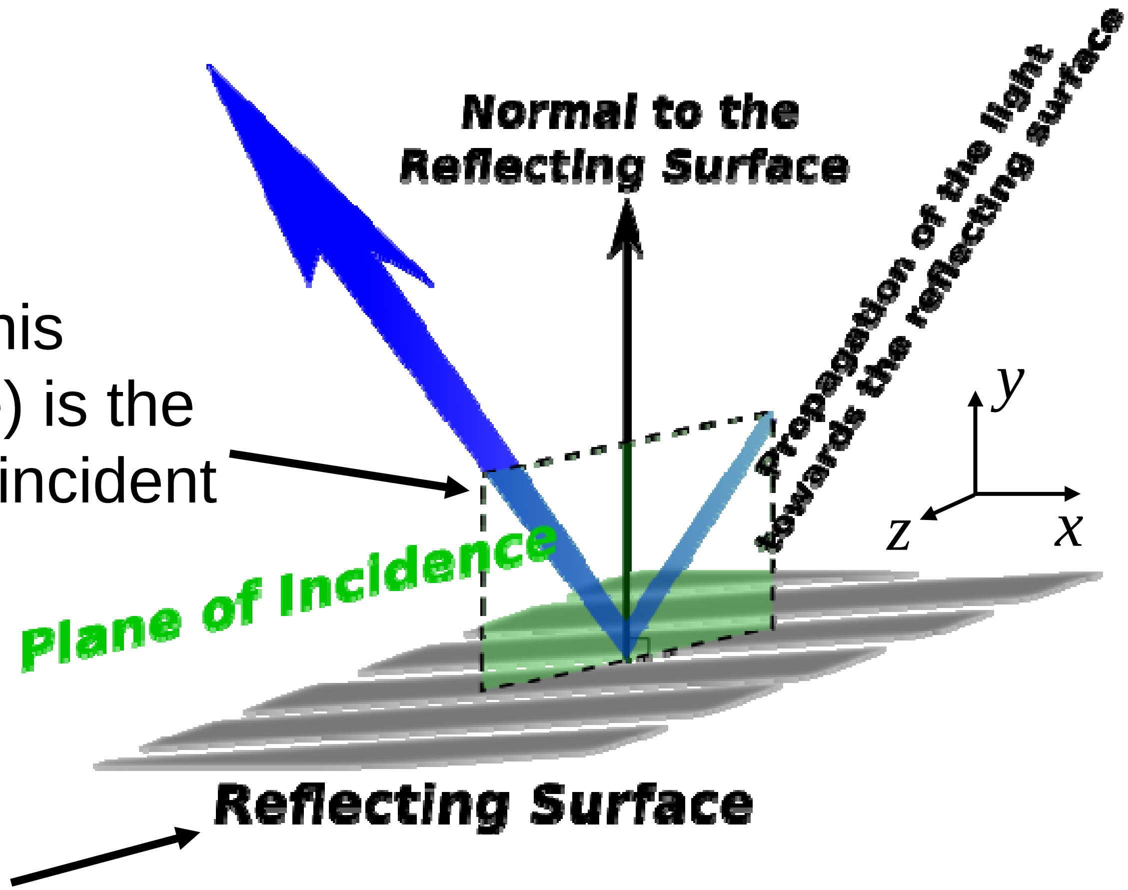
What happens when light, propagating in a uniform medium, encounters a smooth interface which is the boundary of another medium with a different refractive index?



First we need to define some terminology.

Definitions: Plane of Incidence and plane of the interface

Plane of incidence (in this illustration, the yz plane) is the plane that contains the incident and reflected k-vectors.



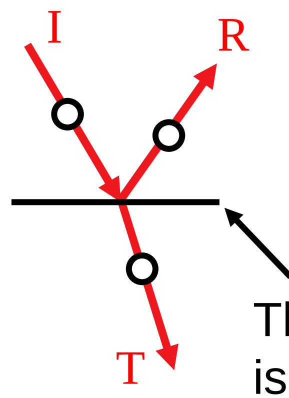
Plane of the interface (y=0, the xz plane) is the plane that defines the interface between the two materials

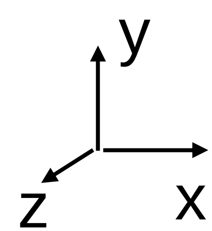
Definitions: “S” and “P” polarizations

A key question: which way is the E-field pointing?
There are two distinct possibilities.

1. “S” polarization is the perpendicular polarization, and it **sticks up** out of the plane of incidence

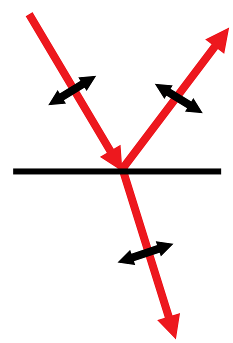
Here, the plane of incidence (z=0) is the plane of the diagram.





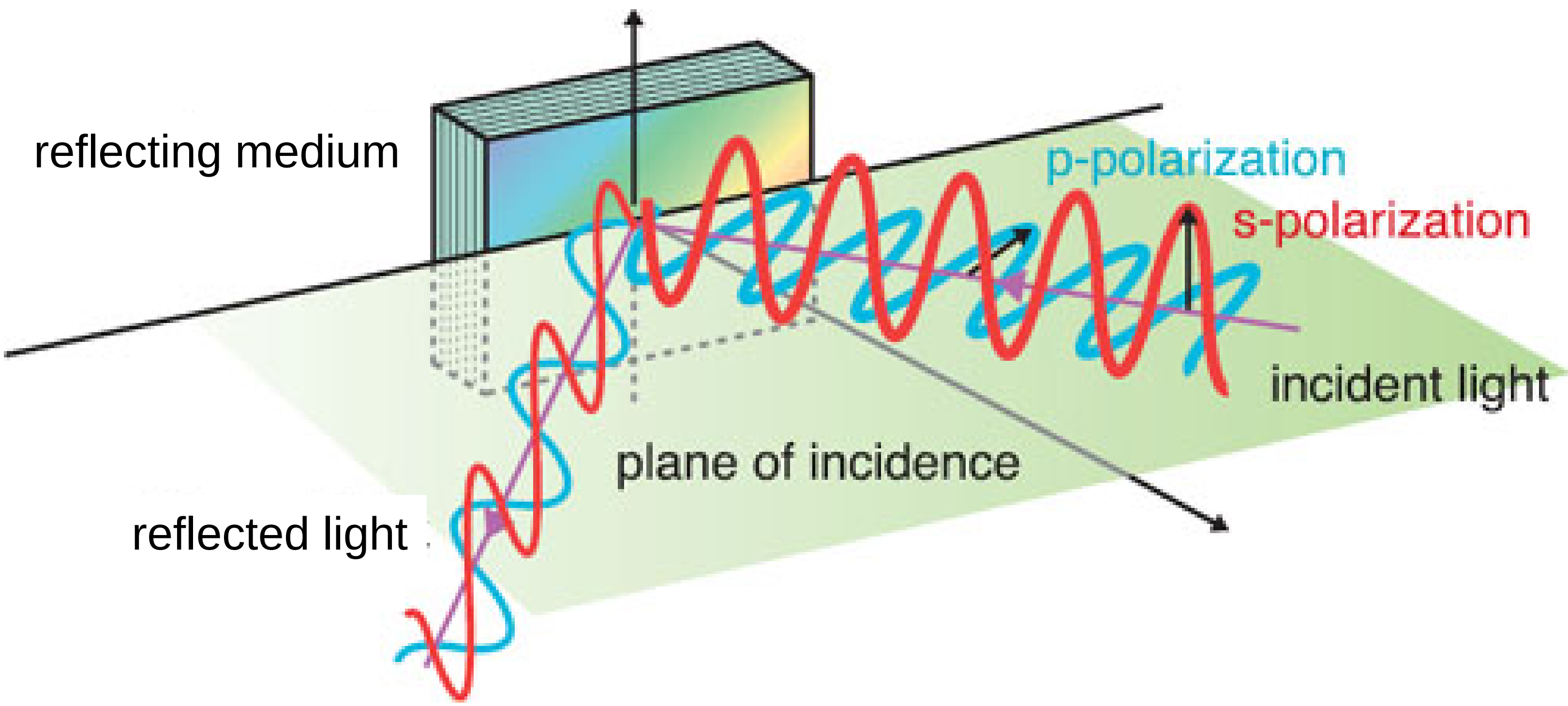
The plane of the interface (y=0) is perpendicular to this page.

2. “P” polarization is the parallel polarization, and it lies **parallel** to the plane of incidence.



Definitions: "S" and "P" polarizations

Note that this is a different use of the word "polarization" from the way we've used it earlier in this class.

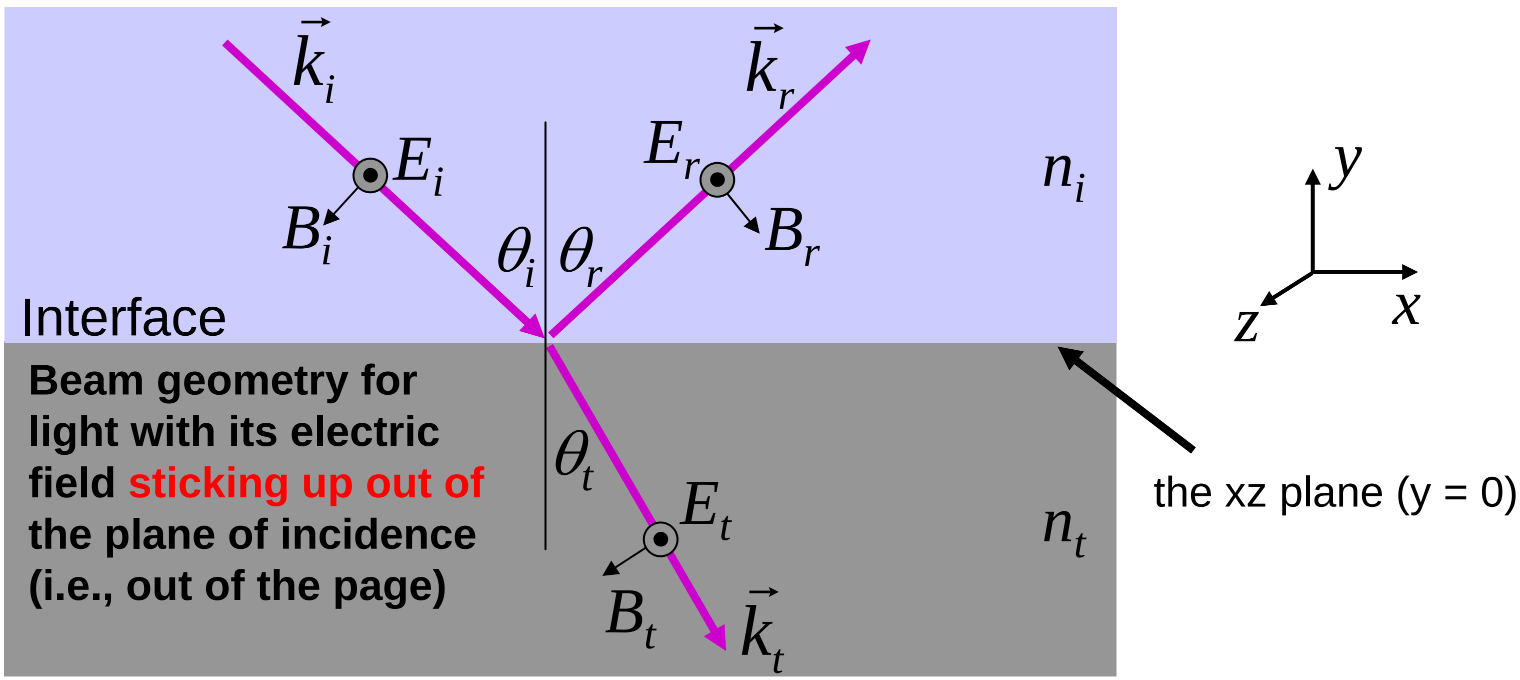


The amount of reflected (and transmitted) light is different for the two different incident polarizations.

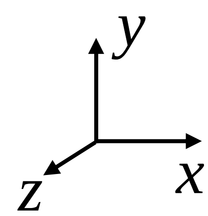
Fresnel Equations—Perpendicular E field

Augustin Fresnel was the first to do this calculation (1820's).

We treat the case of s-polarization first:



Boundary Condition for the Electric Field at an Interface: s polarization

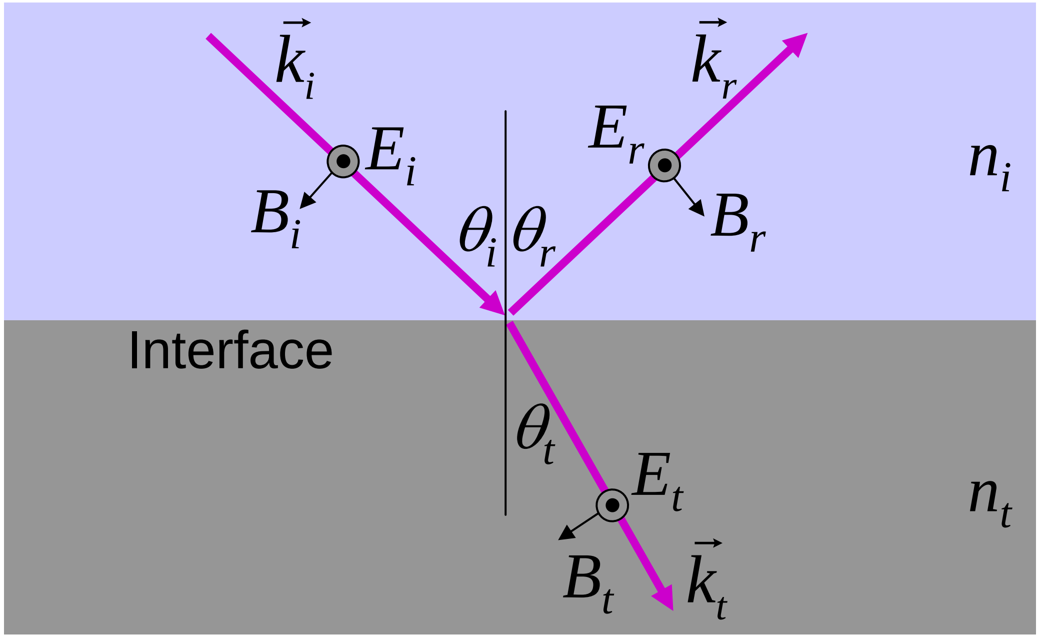


The Tangential Electric Field is Continuous

In other words,

The component of the E-field that lies in the xz plane is continuous as you move across the plane of the interface.

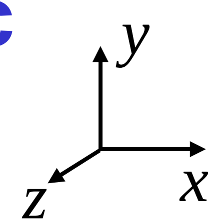
Here, all E-fields are in the z-direction, which is in the plane of the interface.



So: $E_i(y = 0) + E_r(y = 0) = E_t(y = 0)$

(We're not explicitly writing the x, z, and t dependence, but it is still there.)

Boundary Condition for the Magnetic Field at an Interface: s polarization

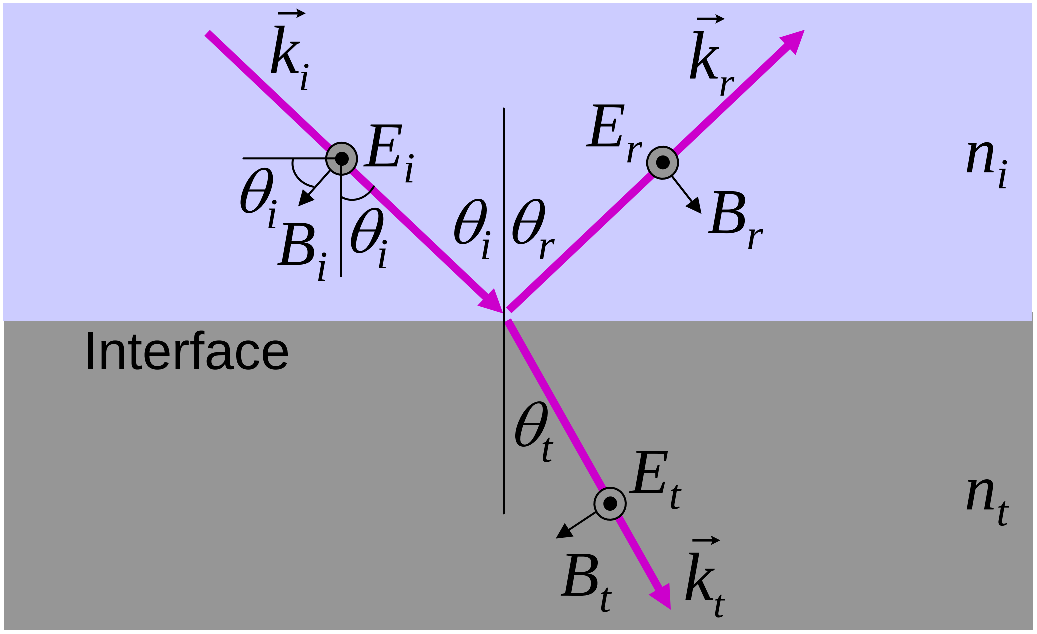


The Tangential Magnetic Field is Continuous*

In other words,

The total B-field in the plane of the interface is continuous.

Here, all B-fields are in the xy-plane, so we take the x-components:



$$-B_i(y=0)\cos\theta_i + B_r(y=0)\cos\theta_r = -B_t(y=0)\cos\theta_t$$

*It's really the tangential B/μ, but we're using $\mu_i = \mu_t = \mu_0$

Reflection and Transmission for Perpendicularly Polarized Light

Ignoring the rapidly varying parts of the light wave and keeping only the complex amplitudes:

$$E_{0i} + E_{0r} = E_{0t}$$

$$-B_{0i}\cos(\theta_i) + B_{0r}\cos(\theta_r) = -B_{0t}\cos(\theta_t)$$

But $B = E/(c_0/n) = nE/c_0$ and $\theta_i = \theta_r$.

Substituting into the second equation:

$$n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t E_{0t}\cos(\theta_t)$$

Substituting for E_{0t} using $E_{0i} + E_{0r} = E_{0t}$:

$$n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t)$$

Reflection & Transmission Coefficients for Perpendicularly Polarized Light

Rearranging $n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t)$ yields:

$$E_{0r}\left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right] = E_{0i}\left[n_i \cos(\theta_i) - n_t \cos(\theta_t)\right]$$

Solving for E_{0r} / E_{0i} yields the reflection coefficient :

$$\boxed{r_\perp = E_{0r} / E_{0i} = \left[n_i \cos(\theta_i) - n_t \cos(\theta_t)\right] / \left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right]}$$

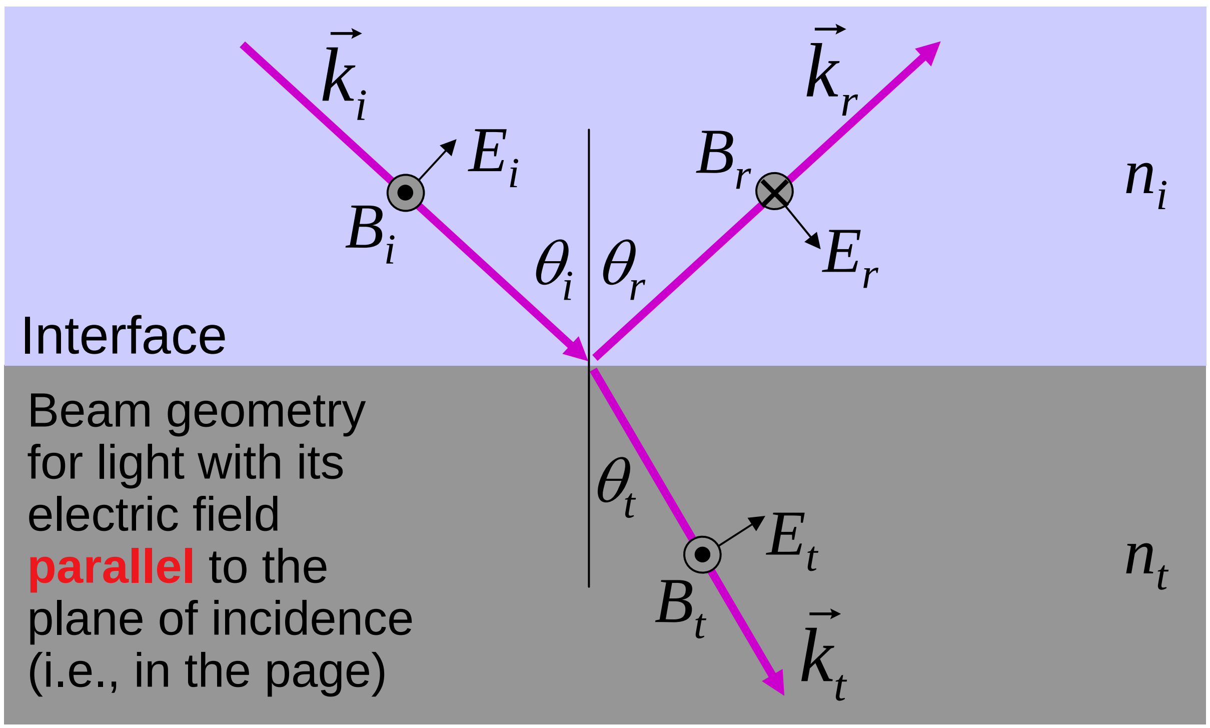
Analogously, the transmission coefficient, E_{0t} / E_{0i}, is

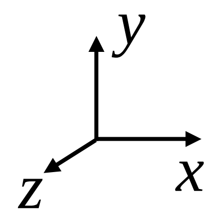
$$\boxed{t_\perp = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / \left[n_i \cos(\theta_i) + n_t \cos(\theta_t)\right]}$$

These equations are called the Fresnel Equations for perpendicularly polarized (s-polarized) light.

Fresnel Equations—Parallel electric field

Now, the case of P polarization:





Note that Hecht uses a different notation for the reflected field, which is confusing!

Ours is better!

This leads to a difference in the signs of some equations...

Note that the reflected magnetic field must point into the screen to achieve $\vec{E} \times \vec{B} \propto \vec{k}$ for the reflected wave. The x with a circle around it means "into the screen."

Reflection & Transmission Coefficients for Parallel Polarized Light

For parallel polarized light, $B_{0i} - B_{0r} = B_{0t}$

and $E_{0i}\cos(\theta_i) + E_{0r}\cos(\theta_r) = E_{0t}\cos(\theta_t)$

Solving for E_{0r} / E_{0i} yields the reflection coefficient, $r_{||}$:

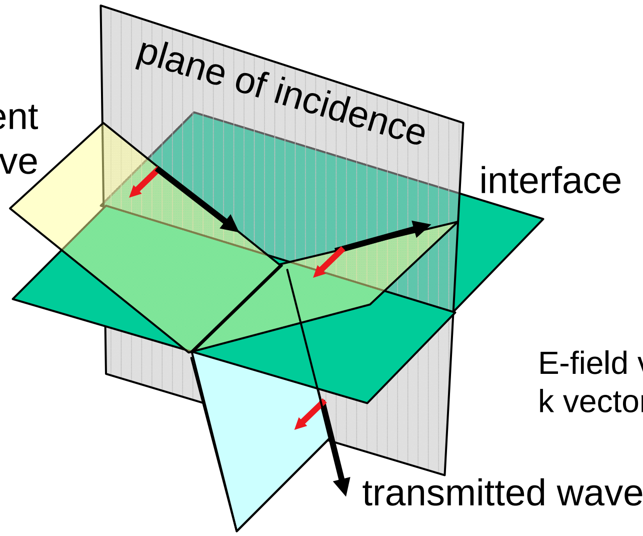
$$\boxed{r_{||} = E_{0r} / E_{0i} = \left[n_i \cos(\theta_t) - n_t \cos(\theta_i)\right] / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]}$$

Analogously, the transmission coefficient, $t_{||} = E_{0t} / E_{0i}$, is

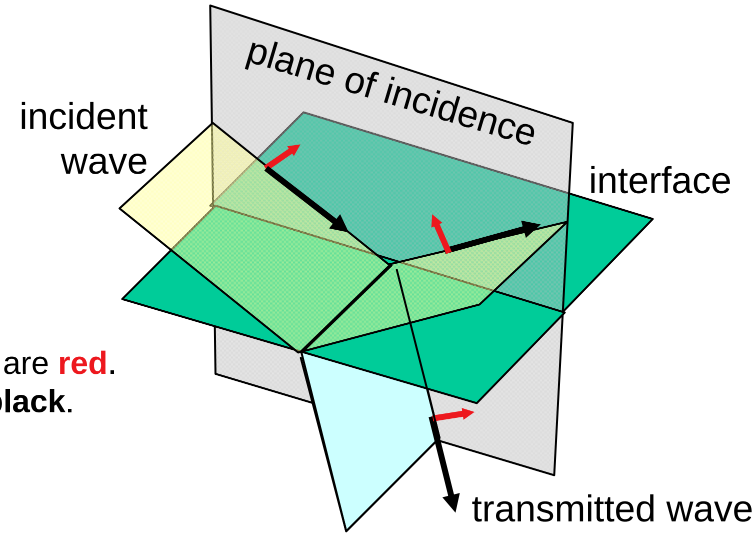
$$\boxed{t_{||} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / \left[n_i \cos(\theta_t) + n_t \cos(\theta_i)\right]}$$

These equations are called the Fresnel Equations for parallel polarized (p-polarized) light.

To summarize…



E-field vectors are **red**.
k vectors are **black**.



s-polarized light:

$$r_{\perp} = \frac{n_i \cos(\theta_i) - n_t \cos(\theta_t)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)}$$

$$t_{\perp} = \frac{2 n_i \cos(\theta_i)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)}$$

p-polarized light:

$$r_{\parallel} = \frac{n_i \cos(\theta_t) - n_t \cos(\theta_i)}{n_i \cos(\theta_t) + n_t \cos(\theta_i)}$$

$$t_{\parallel} = \frac{2 n_i \cos(\theta_i)}{n_i \cos(\theta_t) + n_t \cos(\theta_i)}$$

And, for <u>both</u> polarizations: $n_i \sin(\theta_i) = n_t \sin(\theta_t)$

Reflection Coefficients for an Air-to-Glass Interface

The two polarizations are indistinguishable at $\theta = 0°$

Total reflection at $\theta = 90°$ for both polarizations.

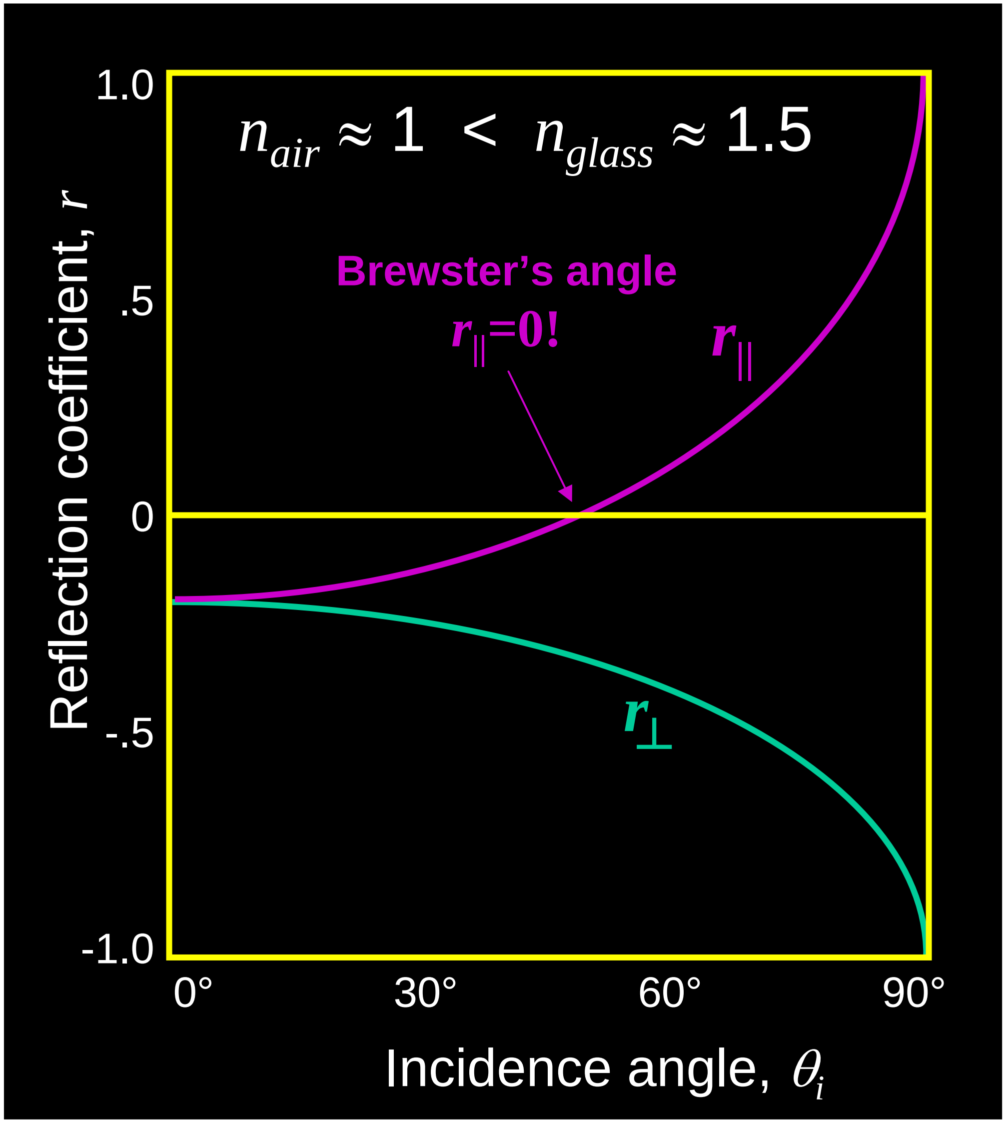
Zero reflection for parallel polarization at 56.3°

"Brewster's angle"

The value of this angle depends on the value of the ratio n_i/n_t:

$$\theta_{Brewster} = \tan^{-1}(n_t/n_i)$$

Sir David Brewster
1781 - 1868



Reflection Coefficients for a Glass-to-Air Interface

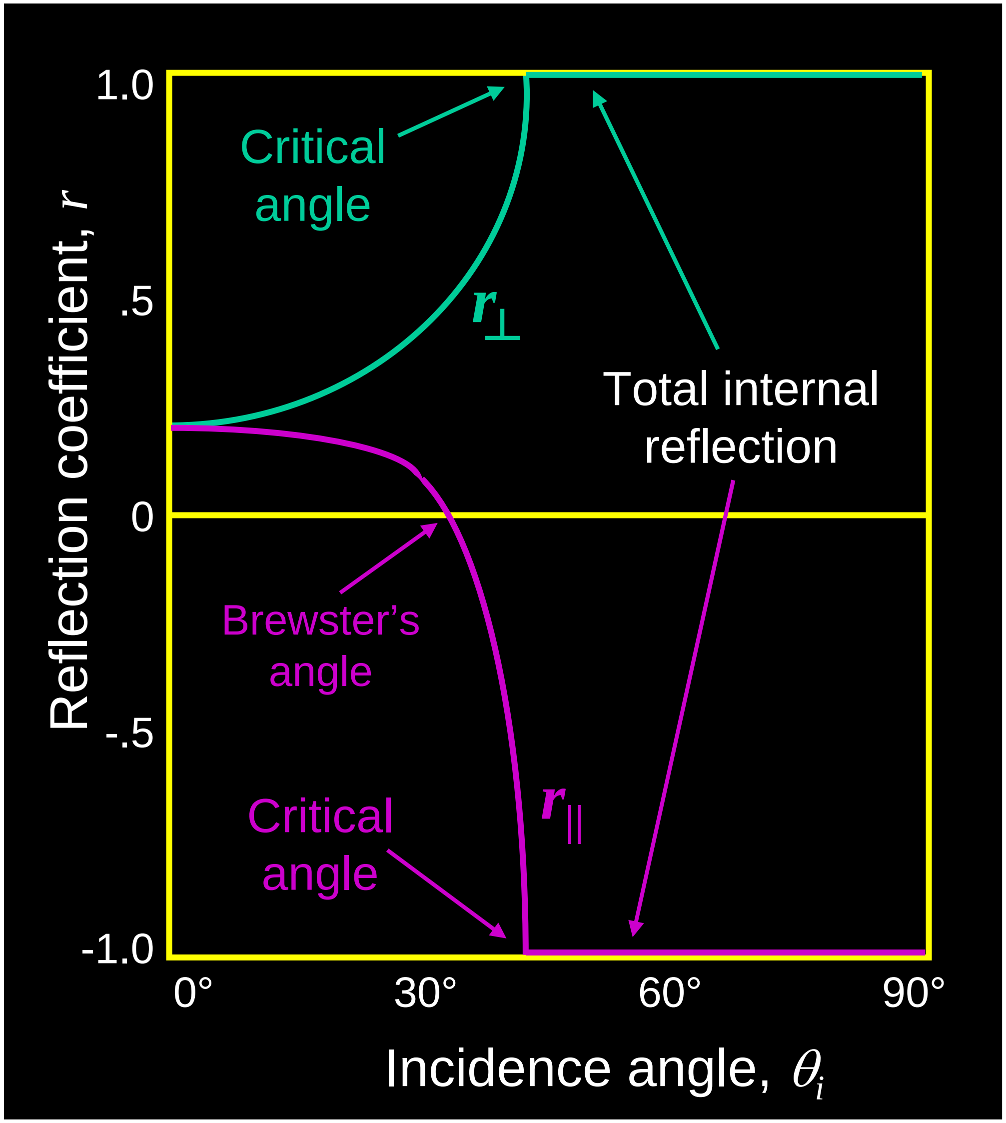
$$n_{glass} > n_{air}$$

Total internal reflection above the **"critical angle"**

$$\theta_{crit} \equiv \sin^{-1}(n_t / n_i)$$

$\approx$ 41.8° for glass-to-air

(The sine in Snell's Law can't be greater than one!)

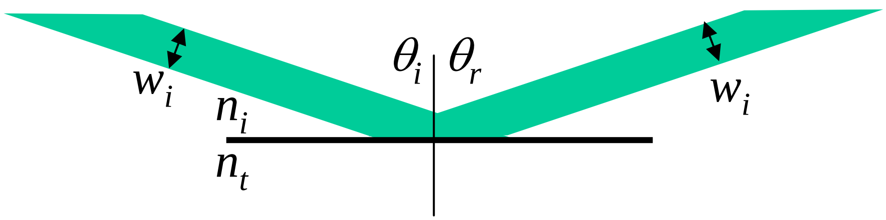


Reflectance (R)

$$I = \left(n\frac{\varepsilon_0 c_0}{2}\right)\left|E_0\right|^2$$

$R \equiv$ Reflected Power / Incident Power $= \dfrac{I_r A_r}{I_i A_i}$

A = Area



Because the angle of incidence = the angle of reflection, the beam's area doesn't change on reflection.

Also, n is the same for both incident and reflected beams.

So: $\boxed{R = r^2}$ since $\dfrac{\left|E_{0r}\right|^2}{\left|E_{0i}\right|^2} = r^2$

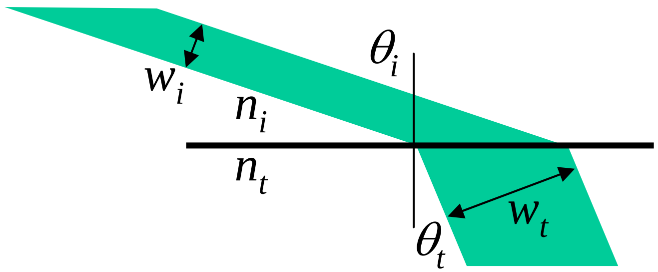
Transmittance (T)

$$I = \left(n \frac{\varepsilon_0 c_0}{2} \right) |E_0|^2$$

$T \equiv$ Transmitted Power / Incident Power $= \dfrac{I_t A_t}{I_i A_i}$

A = Area

If the beam has width w_i:



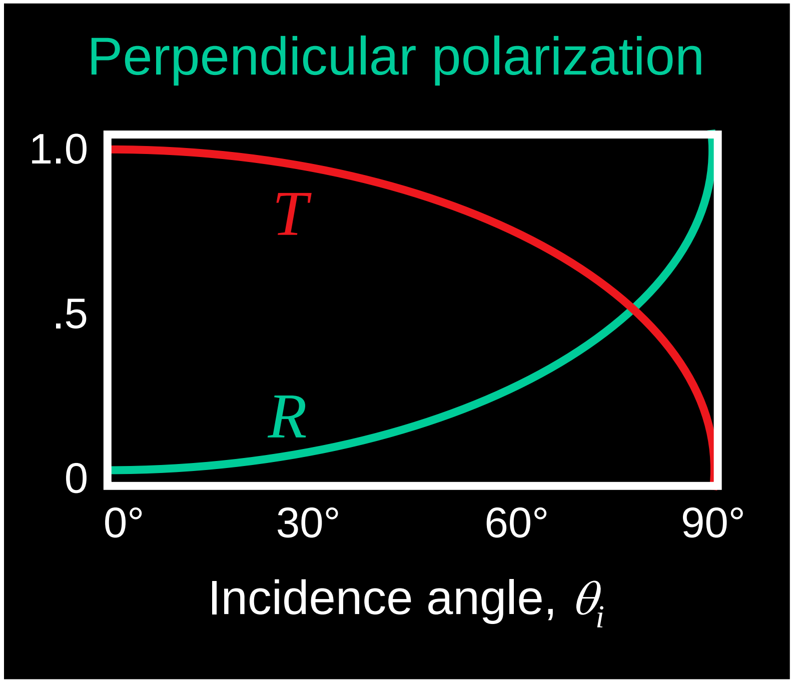
$$\frac{A_t}{A_i} = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)}$$

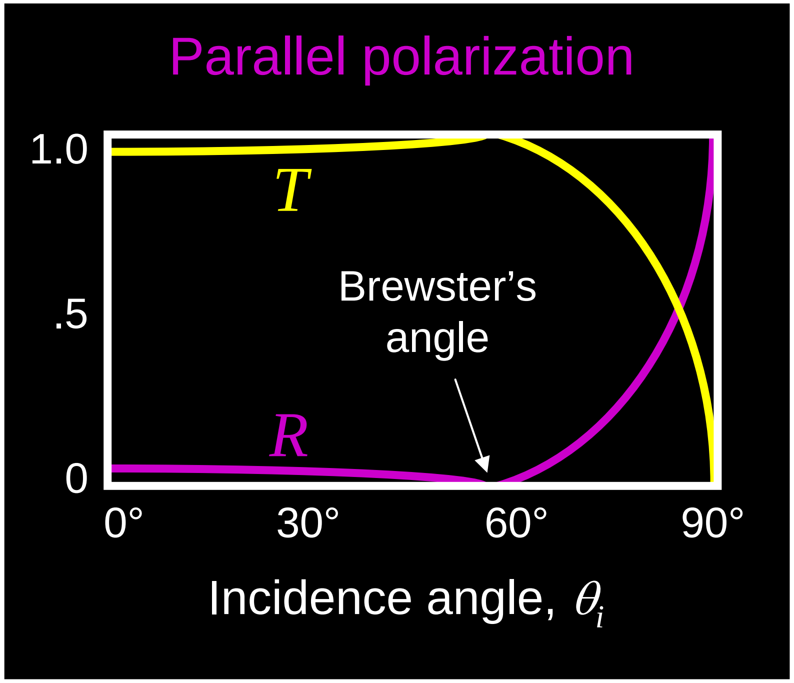
The beam expands (or contracts) in one dimension on refraction.

$$T = \frac{I_t A_t}{I_i A_i} = \frac{\left(n_t \frac{\varepsilon_0 c_0}{2} \right) |E_{0t}|^2}{\left(n_i \frac{\varepsilon_0 c_0}{2} \right) |E_{0i}|^2} \left[\frac{w_t}{w_i} \right] = \frac{n_t |E_{0t}|^2 w_t}{n_i |E_{0i}|^2 w_i} = \frac{n_t w_t}{n_i w_i} t^2 \quad \text{since} \quad \frac{|E_{0t}|^2}{|E_{0i}|^2} = t^2$$

$$\Rightarrow \boxed{T = \left[\frac{\left(n_t \cos(\theta_t) \right)}{\left(n_i \cos(\theta_i) \right)} \right] t^2}$$

Reflectance and Transmittance for an Air-to-Glass Interface

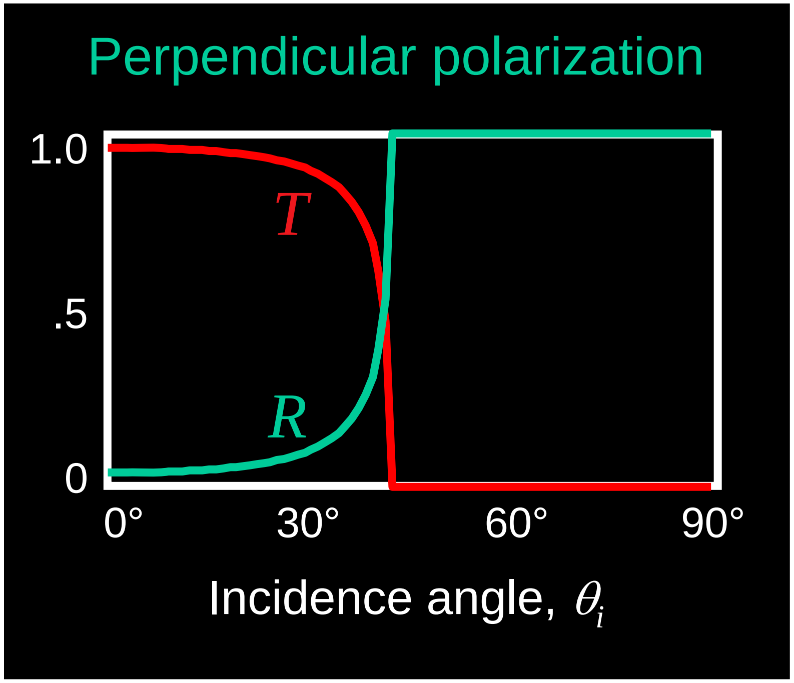


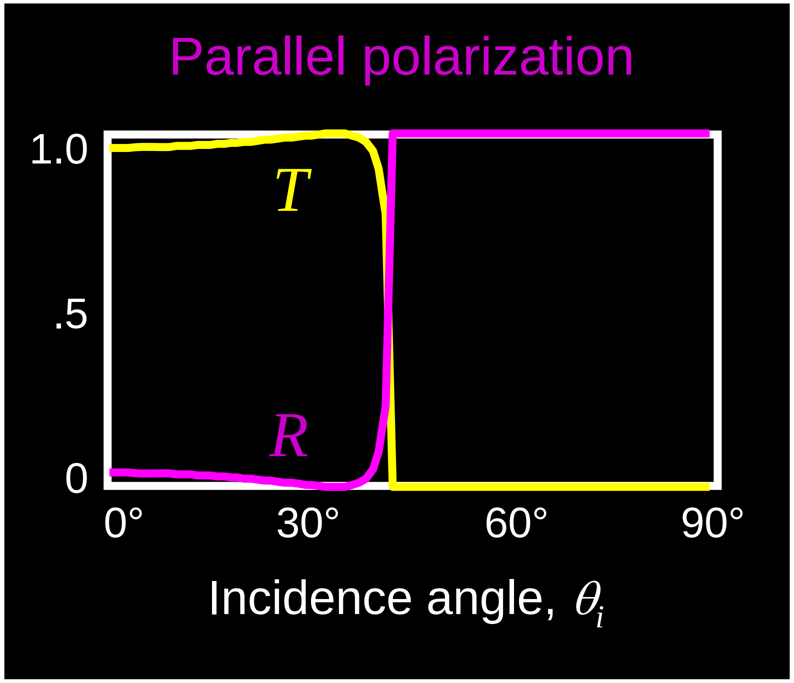


Note that it is NOT true that: $r + t = 1$.

But, it is ALWAYS true that: $R + T = 1$

Reflectance and Transmittance for a Glass-to-Air Interface





Note that the critical angle is the same for both polarizations.

And still, $R + T = 1$

Reflection at normal incidence, $\theta_i = 0$

When $\theta_i = 0$, the Fresnel equations reduce to:

$$R = \left(\frac{n_t - n_i}{n_t + n_i} \right)^2 \qquad T = \frac{4 n_t n_i}{\left(n_t + n_i\right)^2}$$

For an air-glass interface ($n_i = 1$ and $n_t = 1.5$),

R = 4% and T = 96%

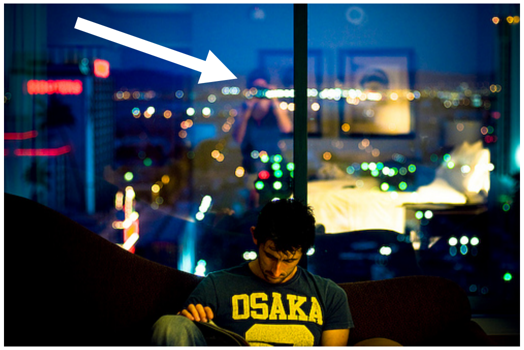
The values are the same, whichever direction the light travels, from air to glass or from glass to air.

This 4% value has big implications for photography.

“lens flare”

Where you've seen Fresnel's Equations in action

Windows look like mirrors at night (when you're in a brightly lit room).



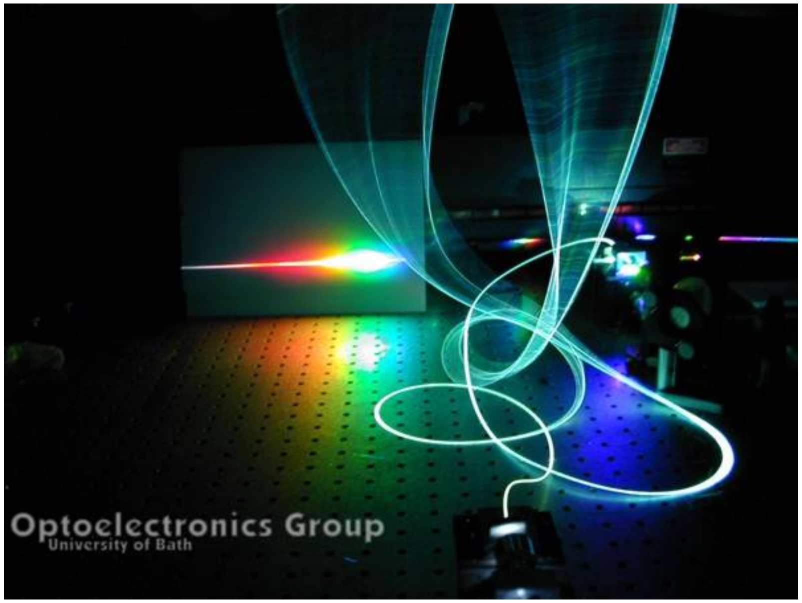
One-way mirrors (used by police to interrogate bad guys) are just partial reflectors (actually, with a very thin aluminum coating).

Disneyland puts ghouls next to you in the haunted house using partial reflectors (also aluminum-coated one-way mirrors).

Smooth surfaces can produce pretty good mirror-like reflections, even though they are not made of metal.

Fresnel's Equations in optics

Optical fibers only work because of total internal reflection.



Many lasers use Brewster's angle components to avoid reflective losses: