P Variational Principles in Acoustics

with A. Cummings

Introduction

In this chapter, some applications of *variational principles* in acoustics are discussed and several examples are given, mainly in duct acoustics. This subject area is not necessarily restrictive, since the reader will be able to see how, by extension, the ideas may be applied to other types of problems. In *Sect. B.11* of this book, *Hamilton's Principle* is described. In its application to particles, the time average of the *Lagrangian* of a system, $L = E_{kin} - E_{pot}$, is minimised and so its first variation is equated to zero, *viz.* $\delta \langle L \rangle = 0$ if there is no external work input. In spatially distributed systems, it is the space-time average of the *Lagrange density* that is minimised. Hamilton's Principle gives rise to *Lagrange's equations*, otherwise known as the *Lagrange-Euler* Eqs. or the *Euler* Eqs. (see, for example, chapter 3 of the book by Morse and Feshbach (1953)). One may apply Hamilton's Principle to sound waves in the absence of dissipation. Here, the Lagrange density is given by:

$$\Lambda = (\rho_0/2)\,|\mathbf{u}|^2 - (\kappa/2)p^2,$$

where ρ_0 is fluid density, $\mathbf{u}$ is acoustic particle velocity, p is sound pressure and κ is isentropic fluid compressibility, $1/\rho_0 c_0^2$, where c_0 is the sound speed. To proceed on the basis of the classical Lagrange equations, one must express both p and $\mathbf{u}$ in terms of the velocity potential ϕ , where $p = \rho_0 \partial\phi/\partial t$, $\mathbf{u} = -\nabla\phi$ (see chapter 6 of the book by Morse and Ingård (1968)). Substitution of Λ into the appropriate Lagrange equation then yields the acoustic wave equation:

$$\nabla^2\phi - (1/c_0^2)\partial^2\phi/\partial t^2 = 0 \qquad \text{or, in terms of sound pressure,}$$

$$\nabla^2 p - (1/c_0^2)\partial^2 p/\partial t^2 = 0\,.$$

Hamilton's Principle is not in fact restricted to conservative systems, and may also include a potential related to dissipative forces (see, for example, the book by Achenbach (1973)).

Hamilton's Principle and Lagrange's Eqs. will yield the governing differential equation(s) of a physical system, but not their solution. Variational methods can be employed to find approximate solutions, by the use of *trial functions*. A trial function is one that contains a number of arbitrary coefficients, which can be altered to change the shape of the function. Values of these coefficients can be found by substituting the trial function into a *functional* (such as the space-time average of the Lagrange density), and then finding a stationary value of this with respect to each of these coefficients in turn.

One can employ variational methods to solve eigenvalue problems too. Such methods will be illustrated in the various examples given in this chapter. Rather than using Lagrange's Eqs. as the starting point of the analysis, an alternative technique is to find a variational principle such that the Euler Eqs. are the governing differential Eqs. of the problem and as many as possible of the prevailing physical boundary conditions. This approach is adopted here in the case of simple harmonic time dependence. The comments of Morse and Feshbach (1953) (see p. 1107), on variational methods *vis-à-vis* perturbation techniques are worth repeating, "the variational method... permits the exploitation of any information bearing on the problem such as might be available from purely intuitional considerations". There is considerable versatility in the application of variational techniques in engineering acoustics particularly and, indeed, intuition can provide valuable information in the solution of problems, as will be seen here.

It should be noted that the trial functions employed in each of the various examples described here could, in principle, be extended to a full finite element discretization. This would give greater numerical accuracy, but at the expense of considerably increased computational effort. In the present context, it is preferable to maintain a level of relative simplicity in the analysis.

P.1 Eigenfrequencies of a Rigid-Walled Cavity and Modal Cut-on Frequencies of a Uniform Flat-Oval Duct with Zero Mean Fluid Flow

The problem of finding the eigenfrequencies of a rigid-walled cavity will first be considered, together with the related problem of modal cut-on frequencies in a rigid-walled "flat-oval" duct, a problem of some importance in the acoustics of air-conditioning ducts. In Morse and Feshbach (1953) (see p. 1112) it is shown that, for the scalar Helmholtz equation, $\nabla^2\psi + k^2\psi = 0$ (ψ being a scalar field variable in a volume Ω satisfying homogeneous Dirichlet or Neumann boundary conditions on the bounding surface, and k being the wavenumber), a variational principle exists such that

$$\delta k^2 = \delta\Big[-\iiint_\Omega \psi\nabla^2\psi d\Omega \Big/ \iiint_\Omega \psi^2 d\Omega\Big] = 0. \tag{1}$$

By the use of Green's theorem, it is shown, Morse and Feshbach (1953), that

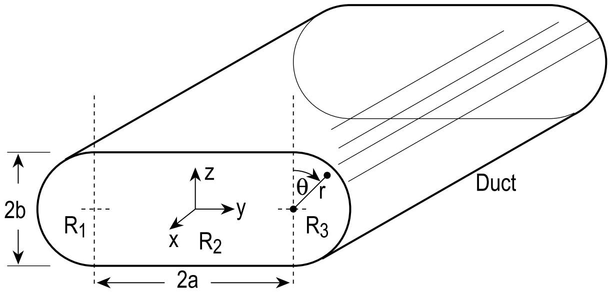
$$k^2 = \iiint_\Omega (\nabla\psi)^2 d\Omega \Big/ \iiint_\Omega \psi^2 d\Omega. \tag{2}$$

This expression may be applied to the acoustic eigenmodes of a rigid-walled cavity, in which case p replaces ψ and k_0^2 ($k_0 = \omega/c_0$ being the acoustic wavenumber, where ω is the radian frequency) replaces k^2. If the cavity is of irregular shape, then a trial function for the sound pressure amplitude distribution, $\tilde{p}(\mathbf{x}) = p(\mathbf{x}) + \varepsilon\eta(\mathbf{x})$ ($\mathbf{x}$ being a position vector, ε being a small parameter and η being an arbitrary function), may be used to replace the (presumably unknown) exact form of sound pressure amplitude $p(\mathbf{x})$ in this expression. One then has:

$$\tilde{k}_0^2 = k_0^2 + O(\varepsilon^2) = \iiint_\Omega \nabla\tilde{p} \cdot \nabla\tilde{p}\, d\Omega \Big/ \iiint_\Omega \tilde{p}^2 d\Omega \tag{3}$$

giving the approximate value of the acoustic wavenumber of a particular eigenmode. If a suitable function for $\tilde{p}$ can be found for a particular mode, this expression may be used to find the approximate eigenfrequency.

It is also possible to apply this formula to a uniform rigid-walled duct of flat-oval cross-section, see Cummings and Chang "Noise Breakout from Flat-Oval Ducts" (1986).



This has two opposite flat sides (width 2a) and two opposite semi-circular sides (diameter 2b) as depicted.

Only the (1,0), (0,1) and (1,1) modes will be discussed here, since higher modes than these present problems in the choice of trial functions. If the mode functions are represented as $p_i = X_i(x)P_i(y,z)\exp(j\omega t)$, then the cut-on frequency for the i-th mode can be found from the acoustic wavenumber $\tilde{k}_i$ corresponding to the trial function $\tilde{P}_i$ for the cross-sectional sound pressure pattern, given by:

$$\tilde{k}_i^2 = \iint_R \nabla_t\tilde{P}_i \cdot \nabla_t\tilde{P}_i\, dR \Big/ \iint_R \tilde{P}_i^2 dR\ , \tag{4}$$

where $R = R_1 + R_2 + R_3$ and the subscript "t" on ∇ signifies a gradient in two dimensions, on the duct cross-section. Suitably continuous trial functions for these modal pressure patterns, satisfying the rigid-wall boundary condition, were found in Cummings and Chang, "Sound Propagation in a Flat-Oval Waveguide" (1986) from intuition and experience (it was assumed that the nodal lines were straight, and this assumption was verified by experiment). These trial functions are as follows:

$$\tilde{P}_{10} = \begin{cases} -A & \text{on } R_1 \\ A\sin(\pi y/2a) & \text{on } R_2 \\ A & \text{on } R_3 \end{cases}\ , \tag{5a}$$

$$\tilde{P}_{01} = \begin{cases} A\left(2r/b - r^2/b^2\right)\cos\theta & \text{on } R_{1,3} \\ A\left(2z/b - \operatorname{sgn}(z)\cdot z^2/b^2\right) & \text{on } R_2 \end{cases} \tag{5b}$$

$$\tilde{P}_{11} = \begin{cases} -A\left(2r/b - r^2/b^2\right)\cos\theta & \text{on } R_1 \\ A\left(2z/b - \mathrm{sgn}(z)\cdot z^2/b^2\right)\sin(\pi y/2a) & \text{on } R_2. \\ A\left(2r/b - r^2/b^2\right)\cos\theta & \text{on } R_3 \end{cases}, \tag{5c}$$

The acoustic wavenumbers at cut-on for these three modes are determined by inserting these trial functions into equation (4) and evaluating the two integrals. The cut-on frequencies ($\tilde{f}_{mn} = \tilde{k}_{mn}c_0/2\pi$) are then given as follows:

$$\tilde{f}_{10} = \frac{c_0}{4a}\sqrt{\frac{1}{1+\pi b/2a}};\quad \tilde{f}_{01} = \frac{c_0}{2\pi}\sqrt{\frac{5\pi/4 + 16a/3b}{11\pi b^2/30 + 32ab/15}};$$
$$\tilde{f}_{11} = \frac{c_0}{2\pi}\sqrt{\frac{5\pi/4 + 4\pi^2 b/15a + 8a/3b}{11\pi b^2/30 + 16ab/15}}. \tag{6a–c}$$

Experimental data were taken on a flat-oval cavity, to find the cut-on frequencies for a duct with a = b = 50 mm, and comparisons were also made with predictions from a finite difference numerical scheme (Cummings and Chang, "Sound Propagation in a Flat-Oval Waveguide" (1986)). The comparisons between experiment and numerical prediction are summarised in the table below (the % figure referring to the numerical prediction accuracy as compared to the measured cut-on frequency).

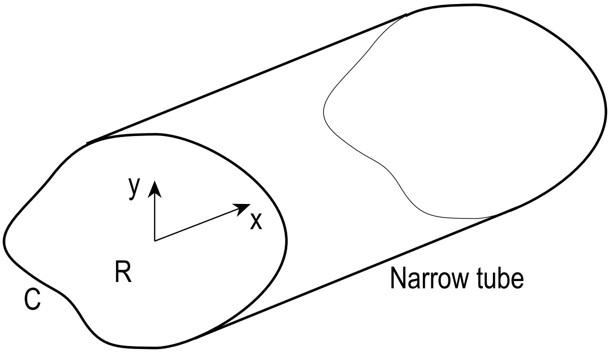
Mode numbers (m,n)	Measured $\tilde{f}_{mn}$ (Hz)	$\tilde{f}_{mn}$ from FD method (Hz)	$\tilde{f}_{mn}$ from equs. (6a–c) (Hz)
(1,0)	959.9	970 (+1 %)	1084 (+13 %)
(0,1)	1829.9	1845 (+0.8 %)	1859 (+1.5 %)
(1,1)	2206	2232 (+1.2 %)	2257 (+2.3 %)

It can be seen that the prediction accuracy of the variational formulae is only modest for the (1,0) mode, but is much better for the (0,1) and (1,1) modes. In the case of the (1,0) mode, the assumed constant sound pressure distributions in the half-cylindrical parts of the duct would not be a particularly accurate representation of the actual pattern. As one would expect, the predicted frequencies are always too high as is the case, for example, in the use of Rayleigh's method in vibration analysis.

P.2 Sound Propagation in a Uniform Narrow Tube of Arbitrary Cross-Section with Zero Mean Fluid Flow

See also *Sects. J.1, J.2, J.3* for sound propagation in flat or circular capillaries.

This second example is not, strictly speaking, an acoustics problem at all (since the thermodynamic processes involved are not isentropic) but relates to wave propagation in narrow tubes, a topic connected with the acoustics of porous media. The procedure here will be to write a functional that has the correct Euler Eqs. and then to proceed to find suitable trial functions (see Cummings (1993)).



The geometry of the problem is as depicted; x is the axial co-ordinate and **y** is a position vector in the transverse plane.

The tube is assumed to have a uniform cross-section and rigid heat-conducting walls, which are at a constant temperature. Subject to the usual boundary-layer approximations, appropriate forms of the linearised Navier-Stokes and thermal energy Eqs. may be written, for simple-harmonic time dependence, as:

$$(\nabla_t^2 - j\omega/\nu)u = (-j\,k_x/\mu)\,p \quad ; \quad (\nabla_t^2 - j\,\omega\rho_0 C_p/K)T = (-j\,\omega/K)\,p. \tag{1}$$

Here, ν, μ, C_p and K are the fluid kinematic viscosity, dynamic viscosity, specific heat at constant pressure and thermal conductivity respectively, k_x is the axial wavenumber of the fluid wave, T is the temperature perturbation and p is the sound pressure. It will be noted that the above two Eqs. are isomorphic and also have identical boundary conditions, namely zero axial velocity and temperature perturbations at the duct wall. Stinson (1991) wrote both these Eqs. in the form:

$$(\nabla_t^2 - j\,\omega/\eta)\psi = -j\,\omega/\eta, \tag{2}$$

where $\psi \equiv (\omega\rho_0/k_x p)u$, $\eta \equiv \nu$ in the velocity equation and $\psi \equiv (\rho_0 C_p/p)T, \eta \equiv \nu/\,\mathrm{Pr}$ in the temperature equation, Pr being the fluid Prandtl number. The axial wavenumber can be found in Stinson (1991) as:

$$k_x = (\omega/c_0)\{[\gamma - (\gamma - 1)F(\nu/\mathrm{Pr})]/F(\nu)\}^{1/2}, \tag{3}$$

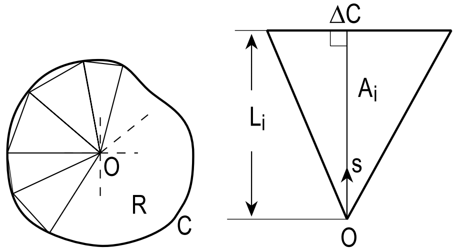
where $F(\eta)$ is defined as $\langle\psi\rangle$, *viz.* the average of ψ over R, the cross-section of the tube, and γ is the ratio of principal specific heats. The problem now is to find an approximate solution to this equation that satisfies the Dirichlet boundary condition $\psi = 0$ on C.

A variational principle for this problem may be obtained as follows, if ψ is expressed as $\psi = \Psi(\mathbf{y})\exp[j\,(\omega t - k_x x)]$. A functional may be defined, see Cummings (1993),

$$\Phi = \frac{1}{2}\iint_R [\nabla_t\Psi\cdot\nabla_t\Psi + (j\,\omega/\eta)\Psi^2 - (j\,2\omega/\eta)\Psi]\,dR. \tag{4}$$

By putting $\delta\Phi = 0$ and using Green's formula, it is easily shown that the Euler Eqs. for Φ are the governing differential equation (2) and the "natural" Neumann boundary condition $\nabla\Psi\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the outward unit normal to the tube surface. This natural boundary condition is not, in fact, the aforementioned physical boundary condition $\Psi = 0$ on C, and this latter condition must be imposed as a "forced" boundary condition with the proviso that $\delta\Psi = 0$ on C. This is done by choosing a trial function $\tilde{\Psi}$ such that $\tilde{\Psi} = 0$ and $\delta\tilde{\Psi} = 0$ on C. For the sake of simplicity, two trial functions will be used here, the first intended to apply in the low frequency limit where the velocity and temperature perturbations are quasi-steady, and the second at high frequencies where both viscous and thermal boundary layers are thin.

Low frequencies:



The tube cross-section is divided, approximately, into triangles as shown above. At point O, both velocity and temperature perturbations are assumed to have their maximum value. At sufficiently low frequencies in a narrow circular tube, both temperature and velocity fields exhibit a parabolic radial distribution of the field variable (as can be seen by solving equation (2)), and so an obvious choice for a low frequency trial function is:

$$\tilde{\Psi} = \Psi_0(1 - \zeta^2), \tag{5}$$

where $\zeta = s/L_i$. It can be noted that $\delta\vec{\tilde{\Psi}} = \partial\vec{\tilde{\Psi}}/\partial\Psi_0 = 1 - \zeta^2$, which is zero everywhere on C, thus satisfying the aforementioned requirement for the forced boundary condition. The location of O is obvious in certain cases (e.g. tubes with circular or square cross-section) but may be less so in other cases. It is suggested in Cummings (1993) that, in general, O be located at the centre of the *largest possible* inscribed circle within R. For a single triangular area element, the contribution to Φ is:

$$\Phi_i = \frac{1}{2}[\Psi_0^2 \Delta C/L_i + (j\,\omega/\eta)\Psi_0^2 A_i/3 - (j\,2\omega/\eta)\Psi_0 A_i/2], \tag{6}$$

and therefore

$$\Phi = \sum_i \Phi_i = \frac{1}{2}\left(\Psi_0^2 \oint_C dC/L + j\,\omega\Psi_0^2 R/3\eta - j\,\omega\Psi_0 R/\eta\right). \tag{7}$$

Taking $\delta\Phi = 0$ is equivalent to putting $\partial\Phi/\partial\Psi_0 = 0$, and this gives:

$$\Psi_0 = (j\,\omega R/2\eta)/\left(\oint_C dC/L + j\,\omega R/3\eta\right), \tag{8}$$

and $$F(\eta) = \Psi_0/2 = (j\,\omega R/4\eta)/\left(\oint_C dC/L + j\,\omega R/3\eta\right). \tag{9a}$$

The integral over C can readily be evaluated in most cases of interest and an expression for k_x found from eq. (3). For a regular polygon

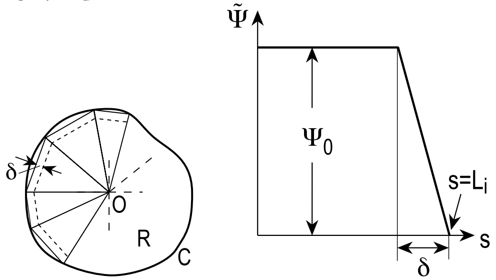
$$F(\eta) = (j\,\omega\, r_h^2/4\eta)\big/(2 + j\,\omega\, r_h^2/3\eta) \tag{9b}$$

with r_h = hydraulic radius (= 2× area/perimeter).

It can be assumed that the low frequency approximation is valid at frequencies where both viscous and thermal boundary layer thicknesses are greater than the largest value of L (L_{max}, say). By using expressions for these boundary layer thicknesses (see Morse and Ingård (1968), p. 286), respectively $\delta_v = \sqrt{2\nu/\omega}$, $\delta_t = \sqrt{2\nu/\omega\,\mathrm{Pr}}$, one may then express the upper limiting frequency for the low frequency model as:

$$f_1 = \min(\nu/\pi L_{max}^2, \nu/\pi L_{max}^2\,\mathrm{Pr}). \tag{10}$$

High frequencies:



At high frequencies, one may assume that the field variables have an approximately constant value in the central part of the tube, and that a boundary layer exists between this region and the wall. The most basic approximation for the boundary layer profile is linear, as shown above. Here, δ is the boundary layer thickness (viscous or thermal, as the case may be). Now the contribution to Φ from a single element is:

$$\begin{aligned}\Phi_i = \frac{1}{2}[&\Psi_0^2 A_i(2\varepsilon_i - \varepsilon_i^2)/\delta^2 + (j\,\omega/\eta)\Psi_0^2 A_i(1 - 4\varepsilon_i/3 + \varepsilon_i^2/2)\\ &- (j\,2\omega/\eta)\Psi_0 A_i(1 - \varepsilon_i + \varepsilon_i^2/3)],\end{aligned} \tag{11}$$

where $\varepsilon_i = \delta/L_i$, and so:

$$\Phi = \frac{1}{2}[\Psi_0^2 I_1/\delta^2 + (j\,\omega/\eta)\Psi_0^2 I_2 - (j\,2\omega/\eta)\Psi_0 I_3], \tag{12}$$

where

$$\begin{aligned}&I_1 = \oint_C (\delta - \delta^2/2L)\,dC; \quad I_2 = \oint_C (L/2 - 2\delta/3 + \delta^2/4L)\,dC;\\ &I_3 = \oint_C (L/2 - \delta/2 + \delta^2/6L)\,dC.\end{aligned} \tag{13}$$

Putting $\delta\Phi = 0$ now gives:

$$\Psi_0 = (j\,\omega/\eta)I_3/[I_1/\delta^2 + (j\,\omega/\eta)I_2]; \quad F(\eta) = (j\,\omega/\eta R)I_3^2[I_1/\delta^2 + (j\,\omega/\eta)I_2]. \tag{14}$$

In the case of a regular polygon, one may define $\varepsilon = \delta/L = \delta/r_h$ and utilise the fact that $\delta = \sqrt{2\eta/\omega}$ to write: (15)

$$F(\eta) = 2j\,(1-\varepsilon+\varepsilon^2/3)^2/[(2\varepsilon-\varepsilon^2)+2j\,(1-4\varepsilon/3+\varepsilon^2/2)]. \tag{16}$$

This expression should be valid at frequencies above a limit:

$$f_2 = \max(\nu/\pi L^2_{min}, \nu/\pi L^2_{min}\,Pr). \tag{17}$$

A circular section tube: (see also *Sect. J.3*)

A circle is the limiting case of a regular polygon with an infinite number of sides. Accordingly, at low frequencies, the axial wavenumber here is immediately found by putting $r_h = a$, the tube radius, in (9) and utilising (3). At high frequencies, k_x is found by putting $\varepsilon = \delta/a$ in (16) (with δ expressed in terms of η from (15)) and using (3). The predictions for both the real and imaginary parts of k_x are in close agreement with the exact solution, given in Stinson (1991) or *Sect. J.3*. The region $f_1 < f < f_2$, in which neither the low frequency nor high frequency approximation is valid, is fairly narrow in most cases of practical interest, since $f_2/f_1 = Pr$. It is easy to connect the two curves graphically.

A parallel slit : (see also *Sect. J.1*)

Neither of the trial functions above is appropriate in the case of the slit, and equivalent trial functions in one dimension only – with a parabolic profile and a linear boundary layer profile respectively – may be employed in this case. A process analogous to that above yields:

$$F(\eta) = (j\,\omega a^2/12\eta)/(1+j\,\omega a^2/10\eta) \tag{18}$$

at low frequencies (where a is the *width* of the slit) and

$$F(\eta) = j\,(1-\varepsilon)^2/[\varepsilon+j\,(1-4\varepsilon/3)] \tag{19}$$

(with $\varepsilon = \delta/a$) at high frequencies. The low frequency formula is in excellent agreement with the exact solution (Stinson (1991) or *Sect. J.1*) and the high frequency formula yields good accuracy in this comparison, though this degenerates slightly as f_2 is approached from above.

Other geometries:

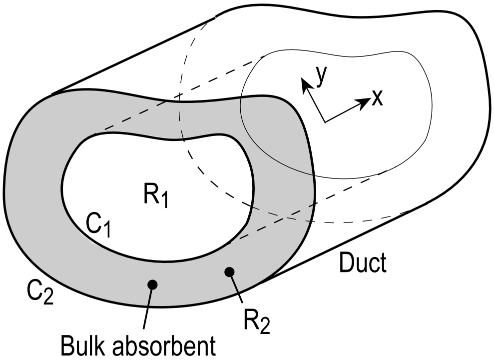
In Cummings (1993), tubes with equilateral triangular, square, rectangular, hexagonal and semi-circular sections are examined and in all cases, the method described above yields predictions that are in at least reasonable agreement with other reported exact or numerical solutions. The detailed formulae for each case can readily by obtained from the above expressions.

P.3 Sound Propagation in a Uniform, Rigid-Walled, Duct of Arbitrary Cross-Section with a Bulk-Reacting Lining and no Mean Fluid Flow: Low Frequency Approximation

See also Ch. J for circular and rectangular ducts.

Consider the above problem, in which a rigid-walled duct of uniform but arbitrary cross-sectional geometry has an internal lining of "bulk-reacting" sound absorbent (initially considered to be isotropic), describable as an equivalent fluid, and characterised by a complex characteristic impedance and acoustic wavenumber. The duct contains no mean fluid flow. It is required to find an approximate expression for the axial wavenumber of the fundamental "coupled mode" in this duct, such that the axial wavenumbers for the sound fields in R_1 and R_2 (the central passage and lining respectively) are identical. Astley (1990) has (in a more general formulation) derived a variational principle with a functional

$$\Phi = \frac{1}{2}\iint_{R_1}(\nabla_t P\cdot\nabla_t P-\kappa^2P^2)dR_1+(\rho_0/2\rho_a)\iint_{R_2}(\nabla_t P\cdot\nabla_t P-\kappa_a^2P^2)dR_2, \tag{1}$$



where the sound pressure in the fundamental mode in R_1 and R_2 is expressed in the form $p = P(y)\exp\big(j(\omega t - k_x x)\big)$, and $\kappa^2 = k_0^2 - k_x^2, \kappa_a^2 = k_a^2 - k_x^2$, k_a being the (complex) effective acoustic wavenumber in the absorbent. The Euler Eqs. of this functional are the Helmholtz Eqs. in R_1 and R_2 (*viz.*, $(\nabla_t^2+\kappa^2)P = 0$ and $(\nabla_t^2+\kappa_a^2)P = 0$), together with the conditions of continuity of normal particle displacement on C_1 and zero particle displacement on C_2. Note that continuity of sound pressure on C_1 is not a natural boundary condition, and must therefore be satisfied by the trial function; the imposition of a forced boundary condition is not necessary here. The simplest trial function for the sound pressure is just a constant, representing a plane wave, *viz.* $\tilde{P} = P_0$ (this satisfies the requirement of continuity of sound pressure on C_1). This function would normally be a reasonable approximation to reality at low frequencies. Inserting this expression into (1) and taking $\delta\Phi = 0$ (which simply involves taking $\partial\Phi/\partial P_0 = 0$) then yields an explicit dispersion relationship:

$$k_x = k_a\sqrt{\frac{1+\phi\rho_a k_0^2/\rho_0 k_a^2}{1+\phi\rho_a/\rho_0}}, \tag{2}$$

where ρ_a is the (complex) effective density of the absorbent and $\phi = R_1/R_2$. Clearly, this formula gives the correct limiting behaviour as $\phi \to 0$ and $\phi \to \infty$ (namely $k_x \to k_a$ and $k_x \to k_0$ respectively). Cummings (1991) has reported a version of this formula that is valid for an anisotropic absorbent, with acoustic wavenumber k_{ax} in the axial direction and k_{ay} in any transverse direction (independent of direction in the transverse plane), and effective density ρ_{ay} in any transverse direction:

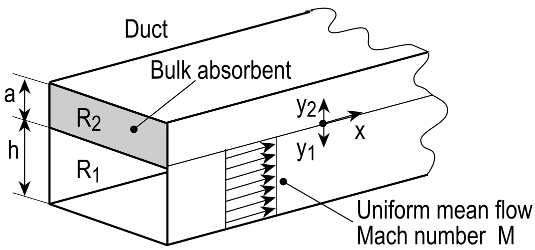
$$k_x = k_{ax}\sqrt{\frac{1 + \phi\rho_{ay}k_0^2/\rho_0 k_{ay}^2}{1 + \phi\rho_{ay}k_{ax}^2/\rho_0 k_{ay}^2}}, \tag{3}$$

which, again, can be seen to give the correct limiting behaviour as $\phi \to 0$ and $\phi \to \infty$.

P.4 Sound Propagation in a Uniform, Rigid-Walled, Rectangular Flow Duct Containing an Anisotropic Bulk-Reacting Wall Lining or Baffles

The duct is depicted below. In its simplest form, the lining geometry involves one layer of bulk absorbent (treated as an equivalent fluid), placed against one wall of the duct. The bulk acoustic properties of the absorbent are assumed to be different in the x and y directions. A uniform mean gas flow (Mach number M) is assumed to be present in the remaining part of the duct cross-section. To treat a baffle silencer, one would assume the baffle width to be 2a and the "airway" width to be 2h . The following analysis would then be representative of the fundamental coupled acoustic mode in this arrangement. For the sake of simplicity, the sound field will be assumed to be two-dimensional. The extension to three dimensions is trivial. Cummings (1992) has reported a variational statement of this problem. If it is assumed that the sound fields in R_1 and R_2 can be expressed in the form $p = P(y_{1,2})\exp\left(j(\omega t - k_x x)\right)$, a variational functional may be defined:

$$\begin{aligned}\Phi = {} & \frac{1}{2(1-MK)^2}\iint_{R_1}\{\nabla_t P\cdot\nabla_t P - k_0^2[(1-MK)^2 - K^2]P^2\}dR_1 \\ & + \frac{\rho_0}{2\rho_{ay}}\iint_{R_2}[\nabla_t P\cdot\nabla_t P - k_0^2(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2 P^2]dR_2,\end{aligned} \tag{1}$$



where $K = k_x/k_0$, $\gamma_{ax} = k_{ax}/k_0$ and $\gamma_{ay} = k_{ay}/k_0$. The Euler Eqs. of this functional are: the convected wave equation in R_1, the Helmholtz equation in the lining, the rigid-wall boundary condition on the duct walls and continuity of normal particle displacement on the interface between the lining and the airway. Suitable trial functions for the sound pressure may be written separately for regions R_1 and R_2:

$$\tilde{P}_1 = A + \sum_{m=1}^{M} B_m \sin[(2m-1)\pi y_1/2h] \quad ; \quad \tilde{P}_2 = A + \sum_{n=1}^{N} C_n \sin[(2n-1)\pi y_2/2a]. \quad (2a,b)$$

It is desirable to use separate functions here for $\tilde{P}_1$ and $\tilde{P}_2$ since the normal gradient of the sound pressure is discontinuous at $y_1 = y_2 = 0$, and this discontinuity cannot be represented by a finite number of terms. Only odd integers are included in the arguments of the sine functions because it is necessary for the trial functions to satisfy the rigid-wall boundary condition at $y_1 = h,\ y_2 = a$. Note that the boundary condition of continuity of sound pressure at $y_1 = 0,\ y_2 = 0$ (not one of the natural boundary conditions) is satisfied by these trial functions. By truncating these summations at appropriate values of M and N, solutions of the desired accuracy may be achieved. An example with $M = 3,\ N = 3$ (i.e. a seven degree-of-freedom trial function) will be discussed in detail here. Equations (2a,b) are inserted into (1), with $\nabla_t\tilde{P} \cdot \nabla_t\tilde{P} \equiv (d\tilde{P}/dy_{1,2})^2$, and the appropriate integrations are carried out. Next, putting $\delta\Phi = 0$ involves taking $\partial\Phi/\partial A = 0,\ \partial\Phi/\partial B_i = 0,\ \partial\Phi/\partial C_i = 0$ (with $i = 1, \ldots 3$). This gives rise to a homogeneous system of simultaneous linear Eqs. in $A,\ B_i,\ C_i$:

$$\begin{aligned}
&A\{[k_0h/(1-MK)^2][(1-MK)^2-K^2] + k_0a(\rho_0/\rho_{ay})[(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2]\} \\
&+ B_1\{[k_0h/(1-MK)^2](2/\pi)[(1-MK)^2-K^2]\} \\
&+ B_2\{[k_0h/(1-MK)^2](2/3\pi)[(1-MK)^2-K^2]\} \\
&+ B_3\{[k_0h/(1-MK)^2](2/5\pi)[(1-MK)^2-K^2]\} \\
&+ C_1\{k_0a(\rho_0/\rho_{ay})(2/\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + C_2\{k_0a(\rho_0/\rho_{ay})(2/3\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} \\
&+ C_3\{k_0a(\rho_0/\rho_{ay})(2/5\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} = 0,
\end{aligned} \quad (3a)$$

$$\begin{aligned}
&A\{-(4/\pi)[(1-MK)^2-K^2]\} + B_1\{(\pi/2k_0h)^2 - [(1-MK)^2-K^2]\} + B_2\{0\} + B_3\{0\} \\
&+ C_1\{0\} + C_2\{0\} + C_3\{0\} = 0,
\end{aligned} \quad (3b)$$

$$\begin{aligned}
&A\{-(4/3\pi)[(1-MK)^2-K^2]\} + B_1\{0\} + B_2\{9(\pi/2k_0h)^2 - [(1-MK)^2-K^2]\} \\
&+ B_3\{0\} + C_1\{0\} + C_2\{0\} + C_3\{0\} = 0,
\end{aligned} \quad (3c)$$

$$\begin{aligned}
&A\{-(4/5\pi)[(1-MK)^2-K^2]\} + B_1\{0\} + B_2\{0\} + B_3\{25(\pi/2k_0h)^2 \\
&-[(1-MK)^2-K^2]\} + C_1\{0\} + C_2\{0\} + C_3\{0\} = 0,
\end{aligned} \quad (3d)$$

$$\begin{aligned}
&A\{-(4/\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + B_1\{0\} + B_2\{0\} + B_3\{0\} + C_1\{(\pi/2k_0a)^2 \\
&-(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + C_2\{0\} + C_3\{0\} = 0,
\end{aligned} \quad (3e)$$

$$\begin{aligned}
&A\{-(4/3\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + B_1\{0\} + B_2\{0\} + B_3\{0\} + C_1\{0\} + C_2\{9(\pi/2k_0a)^2 \\
&-(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + C_3\{0\} = 0,
\end{aligned} \quad (3f)$$

$$A\{-(4/5\pi)(1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} + B_1\{0\} + B_2\{0\} + B_3\{0\} + C_1\{0\} + C_2\{0\} + C_3\{25(\pi/2k_0a)^2 - (1-K^2/\gamma_{ax}^2)\gamma_{ay}^2\} = 0. \tag{3g}$$

Equations (3a–g) may be written in the form:

$$[A_{ij}](\ A \quad B_1 \quad B_2 \quad B_3 \quad C_1 \quad C_2 \quad C_3\ )^T = \{0\}, \tag{4a}$$

and a dispersion relation follows as:

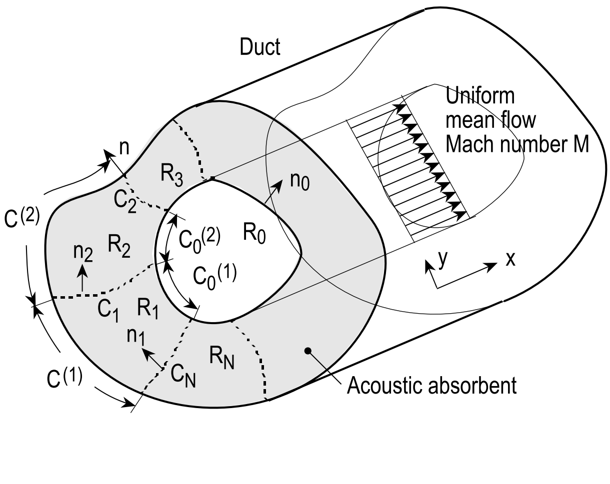
$$\left|[A_{ij}]\right| = 0. \tag{4b}$$

The elements of the (7×7) square matrix in (4a, b) are the coefficients of $A,\ B_1, \ldots C_3$ in Eqs. (3a–g). The solution of (4b) for k_x is readily accomplished by standard numerical techniques such as Newton's method or Muller's method, given a suitable starting value for k_x. This can be done for the fundamental mode or higher modes of propagation, though the most accurate results will be obtained for the fundamental mode. The seven degree-of-freedom trial function yields accurate results for the fundamental mode, Cummings (1992), even up to frequencies of several kHz. Better accuracy could be obtained by taking more terms in the summations in (2a, b). Of course, there is an exact dispersion relationship for this problem, involving circular functions (see, e.g., Cummings (1976)), and therefore the Rayleigh-Ritz method described here is to be regarded as an alternative method rather than one to be used out of necessity in this particular case. The mode shape (e.g., from $B_1/A,\ B_2/A, \ldots$ and Eqs. (2a, b)) may readily be found from Eqs. (3a–g), once k_x has been determined.

P.5 Sound Propagation in a Uniform, Rigid-Walled, Flow Duct of Arbitrary Cross-Section, with an Inhomogeneous, Anisotropic Bulk Lining

This is a more difficult problem than that described in Section P.4, and is one which does not, in general, give rise to an exact dispersion relationship.

General formulation:



The duct geometry is shown in the diagram. The duct is of uniform but arbitrary cross-section, as is the lining, which is assumed to behave as an equivalent fluid. The bulk acoustic properties of the lining are assumed different in the x direction and in any transverse direction, independent of direction, in the transverse $\mathbf{y}$ plane.

The lining is divided into sections $1, 2, \ldots, N$. Within each of these sections, the properties are assumed uniform, but may vary between sections. The outer duct wall is rigid and the central passage carries a uniform mean gas flow.

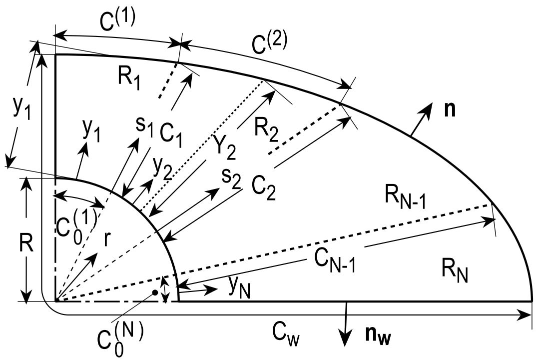
As in *Sections P.3 and P.4*, coupled mode solutions will be sought for the sound field, which will be taken to have the form $p = P(\mathbf{y}) \exp\left(j(\omega t - k_x x)\right)$, where the acoustic pressure amplitude is defined piecewise in the various parts of the cross-section. It will prove convenient to find a variational functional that will have, as its Euler equations, not only the governing wave Eqs. in R_0 and R_i together with the rigid-wall boundary condition on $C^{(i)}$ and continuity of normal particle displacement on $C_0^{(I)}$, but also continuity of normal particle displacement and sound pressure on C_i, since the last two boundary conditions are not readily satisfied by the trial function in what follows. Such a functional is (see Cummings (1995)):

$$\begin{aligned}\Phi = {} & \frac{1}{2(1-MK)^2} \iint_{R_0} \{\nabla_t P \cdot \nabla_t P - k_0^2[(1-MK)^2 - K^2]P^2\}\, dR_0 \\ & + \sum_{i=1}^{N} \frac{\rho_0}{2\rho_{ayi}} \iint_{R_i} [\nabla_t P_i \cdot \nabla_t P_i - k_0^2(1 - K^2/\gamma_{axi}^2)\gamma_{ayi}^2 P_i^2]\, dR_i \\ & + \rho_0\omega^2 \sum_{i=1}^{N} \int_{C_i} \xi_i (P_{i+1} - P_i)\, dC_i,\end{aligned} \tag{1}$$

where (again) $K = k_x/k_0$ and the summations are over the N sub-regions R_i of absorbent, separated by the boundaries C_i. The last summation ensures that the aforementioned two boundary conditions on C_i are natural boundary conditions.

Application to a duct with two cross-sectional lines of symmetry:

One important application of this formulation is to dissipative vehicle exhaust silencers of oval cross-section. These almost invariably have a central circular-section gas flow passage and a cross-section with two lines of symmetry at right angles. The lining material can be not only anisotropic, but also inhomogeneous, in its bulk properties.



For the fundamental acoustic mode, it is clear that only one-quarter of the duct cross-section need be considered, and the appropriate geometry is shown in the figure.

The portion C_W of the boundary is effectively rigid, from considerations of symmetry. At fairly low frequencies, it should be reasonable to assume a purely radial variation is acoustic pressure in both R_0 and in each segment of lining R_i (though without any oscillatory behaviour), together with a circumferential variation between the lining segments. Accordingly, the lining is segmented radially, as shown. The functional in (1) still applies, except that the second summation is taken to $N-1$ rather than N, and a lower limit of 2 is thus imposed on N. The Euler Eqs. now include the rigid-wall boundary condition on C_W. A composite trial function for this geometry is appropriate, having the following form:

$$\begin{aligned}
&\tilde{P}_0 = a + b\cos(\pi r/2R) && \text{in} \quad R_0, \\
&\tilde{P}_i = a + c_i \sin(\pi y_i/2Y_i) && \text{in} \quad R_i \quad (i = 1, \ldots, N), \\
&\tilde{\xi}_i = d_i \sin(\pi s_i/2C_i) && \text{on} \quad C_i \quad (i = 1, \ldots, N-1).
\end{aligned} \tag{2a–c}$$

Here, a, b, c_i and d_i are (complex) arbitrary constants, r, y_i and s_i are co-ordinates (as shown) and R, Y_i and C_i are the lengths shown in the diagram. Continuity of pressure on $C_0^{(i)}$ (not a natural boundary condition) is satisfied by (2a,b). It is convenient to divide the quarter of the absorbent into N *equal segments*.

The next step is to insert the trial functions (2a–c) into (1) and carry out the integrations. The area integrals over the sub-regions of absorbent as depicted above are not easily found, and so it will be assumed that these radial segments are equivalent to segments of circular annuli, having an inner radius R and an outer radius $R+Y_i$, with the radial sound pressure distribution as specified in (2b). This greatly simplifies these area integrals. The integrals over C_i are readily found exactly for the geometry above. The variational functional may now be written:

$$\Phi = a^2 f_1 + b^2 f_2 + abf_3 + \sum_{i=1}^{N} c_i^2 g_i + a\sum_{i=1}^{N} c_i h_i + \sum_{i=1}^{N-1} d_i c_{i+1} u_i - \sum_{i=1}^{N-1} d_i c_i v_i, \tag{3}$$

where

$$\begin{aligned}
f_1 = &-(\pi/8)[k_0R/(1-MK)]^2[(1-MK)^2 - K^2] \\
&-(\pi/4N)\sum_{i=1}^{N}(\rho_0/\rho_{ayi})(1-K^2/\gamma_{axi}^2)\gamma_{ayi}^2[k_0^2RY_i + (k_0Y_i)^2/2],
\end{aligned} \tag{4a}$$

$$\begin{aligned}
f_2 = &[\pi/4(1-MK)^2]\{(\pi^2/4)(1/4+1/\pi^2) - (k_0R)^2[(1-MK)^2 \\
&-K^2](1/4-1/\pi^2)\},
\end{aligned} \tag{4b}$$

$$f_3 = -[k_0R/(1-MK)]^2[(1-MK)^2 - K^2](1-2/\pi), \tag{4c}$$

$$\begin{aligned}
g_i = &\{(\pi^2/4)(R/2Y_i + 1/4 - 1/\pi^2) - (1-K^2/\gamma_{axi}^2)\gamma_{ayi}^2[k_0^2RY_i/2 \\
&+(k_0Y_i)^2(1/4+1/\pi^2)]\}(\pi\rho_0/4N\rho_{ayi}),
\end{aligned} \tag{4d}$$

$$h_i = -(1 - K^2/\gamma_{axi}^2)\gamma_{ayi}^2(k_0Y_i/\pi)(2k_0Y_i/\pi + k_0R)(\pi\rho_0/N\rho_{ayi}), \tag{4e}$$

$$u_i = \left\{ \begin{array}{ll} [2\rho_0\omega^2 Y_{i+1}/\pi(Y_{i+1}^2/C_i^2 - 1)]\cos(\pi C_i/2Y_{i+1}), & Y_{i+1} \neq C_i \\ \rho_0\omega^2 C_i/2, & Y_{i+1} = C_i \end{array} \right\}, \tag{4f}$$

$$v_i = \left\{ \begin{array}{ll} [2\rho_0\omega^2 Y_i/\pi(Y_i^2/C_i^2 - 1)]\cos(\pi C_i/2Y_i), & Y_i \neq C_i \\ \rho_0\omega^2 C_i/2, & Y_i = C_i \end{array} \right\}. \tag{4g}$$

Now $\delta\Phi = 0$ is equivalent to taking

$$\partial\Phi/\partial a = 0,\ \partial\Phi/\partial b = 0\,;\ \partial\Phi/\partial c_i = 0\ (i = 1, \ldots, N)\,;\ \partial\Phi/\partial d_i = 0\ (1 = 1, \ldots, N-1).$$

This gives rise to a system of linear equations:

$$\begin{gathered} 2af_1 + bf_3 + \sum_{i=1}^{N} c_i h_i = 0; \quad 2bf_2 + af_3 = 0; \quad 2c_1 g_1 + ah_1 - d_1 v_1 = 0\,; \\ 2c_i g_i + ah_i + d_{i-1}u_{i-1} - d_i v_i = 0 \quad (i = 2, \ldots, N-1); \quad 2c_N g_N + ah_N + d_{N-1}u_{N-1} = 0\,; \\ c_{i+1}u_i - c_i v_i = 0 \quad (i = 1, \ldots, N-1). \end{gathered} \tag{5a–f}$$

There are $2N+1$ of these Eqs. in the $2N+1$ unknowns, a, b,..., and the system of Eqs. may be written:

$$[A_{ij}]\ (a \quad b \quad c_1 \ldots \quad d_1 \ldots)^T = \{0\}, \tag{6a}$$

where A_{ij} are the coefficients in (5a–f) and, as in *Sect. P.5*, a dispersion relation may be written with the determinant:

$$\left|[A_{ij}]\right| = 0. \tag{6b}$$

This may be solved numerically for k_x, as in the example in *Sect. P.5*.

From Eqs. (5a–f), expressions for the ratios of coefficients, b/a, c_i/a and d_i/a, may easily be found, from which the mode shape may be determined via Eqs. (2a–c):

$$b/a = -f_3/2f_2 \quad ; \quad c_1/a = (f_3^2/2f_2 - 2f_1)\Big/\left\{ h_1 + \sum_{i=1}^{N-1}\left[\prod_{j=1}^{i}(v_j/u_j)\right] h_{i+1} \right\} \quad ; \tag{7a–b}$$

$$c_i/a = (v_{i-1}/u_{i-1})c_{i-1}/a\,, \quad (i = 2, \ldots, N) \quad ; \quad d_1/a = (2g_1c_1/a + h_1)/v_1\,; \tag{7c–d}$$

$$\begin{gathered} d_i/a = (2g_ic_i/a + h_i + u_{i-1}d_{i-1}/a)/v_i\,, \quad (i = 2, \ldots, N-2); \\ d_{N-1}/a = -(2g_Nc_N/a + h_N)/u_{N-1}. \end{gathered} \tag{7e–f}$$

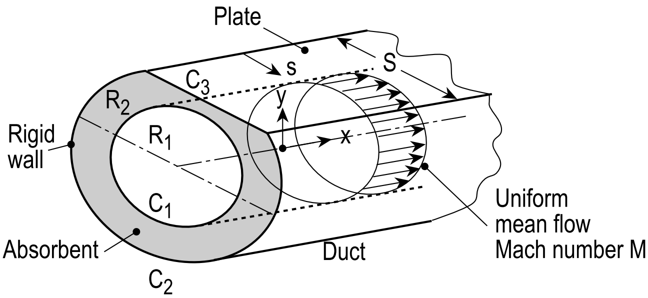
Clearly, some degree of substitution is required in implementing these formulae, e.g. c_1/a in (7d) has first to be found from (7b).

This method has been shown by Cummings (1995), to yield predictions that are in good agreement with measured data. Where mean flow is present, it may be necessary to account for the effects of mean gas flow, within the absorbent itself, on the bulk properties of the absorbent (see Cummings (1995)). This aspect of the problem is, however, beyond the scope of this chapter and in any case does not affect the details of the analysis. The effects of a perforated tube, separating the gas flow passage from the absorbent, may – if desired – also be incorporated in the formulation provided a suitable model for the perforate impedance is available. This feature is, however, omitted from the above formulation for the sake of simplicity.

In the case of a circular duct with a uniform (isotropic or anisotropic) lining of constant thickness and a circular gas flow passage, the above formulation may be used, with $N = 2$. Otherwise it may be more convenient to treat this case separately. The method of *Sect. P.5* may be applied here, with the same functional and a similar trial function. The area integrals have a different form, of course, but the formulation is simple and straightforward. This approach has been shown in Cummings (1992) to yield excellent results, and it has the possible advantage over the exact solution that Bessel functions do not have to be computed.

P.6 Sound Propagation in a Uniform Duct of Arbitrary Cross-Section with one or more Plane Flexible Walls, an Isotropic Bulk Lining and a Uniform Mean Gas Flow

The duct is depicted below. It is shown as having one flexible wall, consisting of a flat elastic plate, though the analysis that follows is equally valid for an arbitrary number of such walls. For simplicity, it will be assumed that there is just one flexible wall; the extension of the treatment to the case of multiple flexible walls is obvious. The other parts of the duct wall are rigid. There is a uniform gas flow in the central passage.



In this problem, there is wave motion not only in the fluid in the duct and the lining, but also in the plate forming the flexible wall. Coupled mode solutions are sought, and accordingly the axial wavenumbers in the central gas flow passage (R_1), the lining (R_2) and the plate (C_3) are all identical.

General formulation:

It will be assumed that the sound pressure can be represented as $p = P(\mathbf{y})\exp\big(j(\omega t - k_x x)\big)$ and the outward plate displacement as $u = U(s)\exp\big(j(\omega t - k_x x)\big)$. Then the three governing differential Eqs. in R_1, R_2 and the plate are, respectively:

$$\nabla_t^2 P + k_0^2[(1 - MK)^2 - K^2]P = 0; \quad \nabla_t^2 P + (k_a^2 - k_x^2)P = 0;$$
$$g[(d^2/ds^2 - k_x^2)^2 U - k_p^4 U] = P_p(s), \qquad (1a\text{–}c)$$

where $P_p(s)$ is the transverse factor in the acoustic pressure difference (inside-outside) across the plate, forcing its motion (it is assumed the axial dependence of this pressure difference is the same as that of p and u), g is the flexural rigidity of the plate and the plate wavenumber is $k_p = (m\omega^2/g)^{1/4}$ (m being the mass/unit area of the plate). Other notation here is as in *Section P.3*.

A variational statement of this problem has been made by Astley (1990), and a functional defined:

$$
\begin{aligned}
\Phi = \rho_0\omega^2 \int_{C_3} &\{(g/2)[(d^2U/ds^2)^2 + 2k_x^2(dU/ds)^2 + (k_x^4 - k_p^4)U^2] - UP_p\}ds \\
&+ \frac{1}{2(1-MK)^2} \iint_{R_1} \{\nabla_t P\cdot \nabla_t P - k_0^2[(1-MK)^2 - K^2]P^2\}dR_1 \\
&+ \frac{\rho_0}{2\rho_a} \iint_{R_2} [\nabla_t P \cdot \nabla_t P - (k_a^2 - k_x^2)P^2]dR_2.
\end{aligned} \tag{2}
$$

The Euler Eqs. of this functional are obtained by taking variations of Φ with respect to P and U: (1a, b) (provided P is continuous on C_1), (1c) (with the constraint of zero displacement at the edges of the flexible wall), the rigid-wall boundary condition on C_2, equality of the normal plate displacement and the normal acoustic particle displacement in the internal sound field on C_3 (provided that the normal gradient of sound pressure is allowed to vary freely at the outer surface of the flexible wall) and continuity of normal particle displacement on C_1 (provided that the normal gradient of sound pressure is allowed to vary independently on C_1 in R_1 and R_2).

Low frequency approximation:

As in *Section P.3*, the trial function for the internal acoustic field embodies a uniform transverse sound pressure distribution in R_1 and R_2, i.e. $P(\mathbf{y}) = P \equiv$ const., representative of the fundamental mode. Furthermore, one may write $U(s) = P_u U^*(s, k_x)$, where $U^*(s, k_x)$ is the *solution* of (1c), with a unit pressure on the right-hand side, subject to the prevailing boundary conditions, and P_u is defined by the foregoing equation. With this trial function, the functional may be written (after some manipulation):

$$
\begin{aligned}
\Phi = j\,k_0 \langle\beta(k_x)\rangle\, S(PP_u - \frac{1}{2}P^2) \\
+ \frac{1}{2}k_0^2P^2 \left\{ R_1 \left[\left(\frac{K}{1-MK}\right)^2 - 1 \right] + R_2 \left(\frac{\rho_0}{\rho_a}\right) [K^2 - (k_a/k_0)^2] \right\},
\end{aligned} \tag{3}
$$

where $\langle\beta(k_x)\rangle$ is the space-average (over C_3) of the dimensionless admittance of the flexible wall, $j\,\omega\rho_0 c_0 U^*(s, k_x)$, and S is the width of the flexible duct wall. This expression is now minimised with respect to P and P_u, and P is equated to P_u on the basis that the external radiation load on the flexible duct wall is negligibly small, to give the dispersion equation:

$$
j\,k_0 \langle\beta(k_x)\rangle\, S + k_0^2 \left\{ R_1 \left[\left(\frac{K}{1-MK}\right)^2 - 1 \right] + R_2 \left(\frac{\rho_0}{\rho_a}\right) [K^2 - (k_a/k_0)^2] \right\} = 0. \tag{4}
$$

This can be solved by an appropriate standard root-finding method. Certain special cases are of interest in Astley (1990), as follows.

(i) A duct with rigid walls and mean gas flow:

In this case, $\langle\beta(k_x)\rangle = 0$ and equation (4) becomes:

$$R_1\left[\left(\frac{K}{1-MK}\right)^2 - 1\right] + R_2\left(\frac{\rho_0}{\rho_a}\right)[K^2 - (k_a/k_0)^2] = 0. \tag{5}$$

(ii) A duct with rigid walls and no mean gas flow:

The result here is identical to that in *Section P.3*.

(iii) A central region of gas flow in a rigid-walled duct, surrounded by a stagnant region:

An example of this is an enclosed gas jet in a duct, such as that which is formed downstream of an abrupt area expansion in a flow duct. Here, $k_a = k_0,\ \rho_a = \rho_0$ and $\langle\beta(k_x)\rangle = 0$. The dispersion equation now becomes:

$$R_1\left[\left(\frac{K}{1-MK}\right)^2 - 1\right] + R_2\,(K^2 - 1) = 0. \tag{6}$$

(iv) A duct with a flexible wall, no mean flow and no lining:

In this case, $M = 0$ and $R_2 = 0$, and equation (4) yields:

$$k_x = k_0\sqrt{1 - jS\langle\beta(k_x)\rangle/k_0R_1}, \tag{7}$$

which is identical to the expression obtained by Cummings (1978).

If the flexible wall is clamped along both edges, an exact solution of equation (1c) exists:

$$\begin{aligned}\langle\beta(k_x)\rangle = j\,\omega\rho_0c_0\Bigg\{&\frac{A_1}{\alpha_1 S}\sin(\alpha_1 S) - \frac{A_2}{\alpha_1 S}[\cos(\alpha_1 S) - 1] + \frac{A_3}{\alpha_2 S}\sinh(\alpha_2 S)\\ &+\frac{A_4}{\alpha_2 S}[\cosh(\alpha_2 S) - 1] + \frac{1}{g(k_x^4 - k_p^4)}\Bigg\},\end{aligned} \tag{8}$$

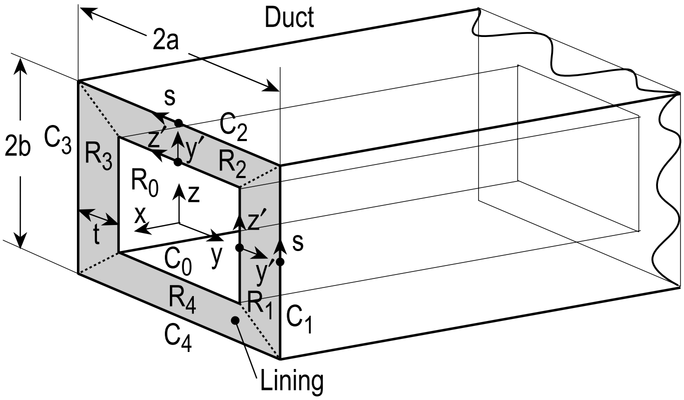
where $\alpha_1 = \sqrt{k_p^2 - k_x^2},\quad \alpha_2 = \sqrt{k_p^2 + k_x^2}$ and

$$\begin{aligned}A_1 = \{&\alpha_1[1 + \cos(\alpha_1 S) - \cosh(\alpha_2 S) - \cos(\alpha_1 S)\cosh(\alpha_2 S)]\\ &+ \alpha_2\sin(\alpha_1 S)\sinh(\alpha_2 S)\}/g(k_x^4 - k_p^4)[2\alpha_1\cos(\alpha_1 S)\cosh(\alpha_2 S) - 2\alpha_1\\ &+ (\alpha_1^2/\alpha_2 - \alpha_2)\sin(\alpha_1 S)\sinh(\alpha_2 S)];\\[1ex] A_2 = \{&\alpha_1\sin(\alpha_1 S)[1 - \cosh(\alpha_2 S)] + \alpha_2\sinh(\alpha_2 S)[1 - \cos(\alpha_1 S)]\}/\\ &g(k_x^4 - k_p^4)[2\alpha_1\cos(\alpha_1 S)\cosh(\alpha_2 S) - 2\alpha_1\\ &+ (\alpha_1^2/\alpha_2 - \alpha_2)\sin(\alpha_1 S)\sinh(\alpha_2 S)];\\[1ex] A_3 = {}&-A_1 - 1/g(k_x^4 - k_p^4);\quad A_4 = -A_2\alpha_1/\alpha_2.\end{aligned} \tag{9}$$

Equation (7) has been shown in Cummings (1978) to give predictions of axial phase speed for the fundamental coupled structural/acoustic mode that are in excellent agreement with measured data, for a duct of square cross-section (and, therefore, effectively having four flexible walls, clamped along their edges, with no rigid walls). Transverse structural resonance effects in the wall are very prominent in the wall admittance expression (8) and, of course, in the axial wavenumber in the duct.

P.7 Sound Propagation in a Rectangular Section Duct with four Flexible Walls, an Anisotropic Bulk Lining and no Mean Gas Flow

The duct geometry and co-ordinate systems are shown in the graph. The open central channel R_0 is surrounded by layers of bulk absorbent, all of thickness t, placed against the four flexible walls. These layers are denoted $R_1, \ldots, R_4$. The perimetral co-ordinates s are local to each of the four walls of the duct, $C_1, \ldots, C_4$, and C_0 is the interface between the central channel and the lining.



A global co-ordinate system x, y, z is centred on the duct axis, and local co-ordinate systems x, y′, z′ are used to define position in the four layers of absorbent. The bulk acoustic properties of the absorbent are different in the y′ and x, z′ directions, and the properties of all layers are identical. Coupled eigenmodes are sought, for the sound pressure in the duct and the outward wall displacement, having the form:

$$p(x, y, z; t) = P(y, z) \cdot \exp\left(j\,(\omega t - k_x x)\right); \quad u(x, s; t) = U(s) \cdot \exp\left(j\,(\omega t - k_x x)\right). \qquad \text{(1a–b)}$$

The acoustic wave Eqs. in R_0 and in the absorbent ($R_1, \ldots, R_4$) yield Helmholtz equations, respectively having the forms:

$$\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} + (k_0^2 - k_x^2)P = 0 \quad ; \quad \frac{\partial^2 P}{\partial y'^2} + \frac{k_{ay}^2}{k_{ax}^2}\frac{\partial^2 P}{\partial z'^2} + \left(k_{ay}^2 - k_x^2 \frac{k_{ay}^2}{k_{ax}^2}\right)P = 0, \qquad \text{(2a–b)}$$

where k_{ax} is the acoustic wavenumber of the absorbent in the x, z′ plane (independent of direction) and k_{ay} is the acoustic wavenumber in the y′ direction. The equation of motion in the duct walls may be written in the form:

$$g\left[\frac{d^4U}{ds^4} - 2k_x^2\frac{d^2U}{ds^2} + (k_x^4 - k_p^4)U\right] = P \quad \text{on } C_1, \ldots, C_4, \tag{2c}$$

where k_p is defined in *Section P.6*.

A variational statement for this problem is reported by Cummings and Astley (1995), with a functional:

$$\begin{aligned}\Phi = \frac{1}{2}\Bigg(&\iint_{R_0} [(\partial P/\partial y)^2 + (\partial P/\partial z)^2 + (k_x^2 - k_0^2)P^2]\, dR_0 \\ &+ \sum_{i=1}^{4} (\rho_0/\rho_{ay}) \iint_{R_i} [(\partial P/\partial y')^2 + (k_{ay}^2/k_{ax}^2)(\partial P/\partial z')^2 \\ &+ (k_x^2 k_{ay}^2/k_{ax}^2 - k_{ay}^2)P^2]\, dR_i \\ &+ \sum_{i=1}^{4} \rho_0\omega^2 \int_{C_i} \{g[(d^2U/ds^2)^2 + 2k_x^2(dU/ds)^2 + (k_x^4 - k_p^4)U^2] - 2UP\}\, ds\Bigg).\end{aligned} \tag{3}$$

The notation here is essentially that of *Sections P.4 and P.5*. The Euler Eqs. of this functional are the same as those outlined in *Section P.6* (except for the rigid-wall boundary condition), with the additional feature of anisotropy in the absorbent.

A Rayleigh-Ritz approximation for the coupled eigenproblem may be found by the use of trial functions, having (for example) polynomial form, for the sound pressures in the central passage and lining, and for the wall displacement in Cummings and Astley (1995). These trial functions – intended to represent the fundamental coupled mode – are, respectively:

$$\tilde{P} = f_1(y,z)P_1 + f_2(y,z)P_2 + f_3(y,z)P_3 \quad \text{in } R_0, \tag{4a}$$

$$\tilde{P} = g_1(y',z')P_1 + g_2(y',z')P_2 + g_3(y',z')P_3 \quad \text{in } R_1, \ldots, R_4$$

$$(\text{where } f_1 = [1 - (y/a_1)^2][1 - (z/b_1)^2], \quad a_1 = a - t, \quad b_1 = b - t, \tag{4b}$$

$$f_2 = 1 - [1 - (y/a_1)^2][1 - (z/b_1)^2], \quad f_3 = 0, \quad g_1 = 0, \quad g_2 = 1 - (y'/t), \quad g_3 = y'/t),$$

$$\tilde{U} = h_1(s)U_1 + h_2(s)U_2 + h_3(s)\theta \tag{4c}$$

(where $h_1 = 1 - 3(s/b)^2 + 2(s/b)^3$, $h_2 = 0, h_3 = -s[(s/b) - (s/b)^2]$ for $0 \le s \le b$ on the right-hand vertical side of the duct depicted above, with equivalent expressions for the other parts of the walls).

In these equations, the constants P_1, P_2, P_3 have the dimensions of sound pressure and are identified as follows: P_1 is the sound pressure amplitude on the axis of the central passage, P_2 is the (constant) sound pressure amplitude at the liner/air interface and P_3 is the (constant) sound pressure amplitude on the inner surface of the duct wall, according to the assumed form of the trial function for the sound field. In the vibration field, constants U_1, U_2 are the displacement amplitudes at the mid-points of two adjacent sides of the duct wall and constant θ is the amplitude of the angle of rotation of the corner (assumed to remain right-angled) between these sides.

The insertion of Eqs. (4a–c) into (3) yields (after some manipulation) an expression for the functional which may conveniently be written in matrix form, Cummings and Astley (1995)

$$\Phi = \frac{1}{2}\mathbf{U}^{\mathrm{T}}[\mathbf{A} + k_x^2\mathbf{B} + k_x^4\mathbf{C}]\mathbf{U} - \mathbf{P}^{\mathrm{T}}\mathbf{T}\mathbf{U} + \frac{1}{2}\mathbf{P}^{\mathrm{T}}[\mathbf{E} + k_x^2\mathbf{G}]\mathbf{P}, \tag{5}$$

where $\mathbf{P}$ and $\mathbf{U}$ are column vectors containing the acoustic coefficients P_1, P_2, P_3 and the structural coefficients U_1, U_2, θ respectively, and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{T}, \mathbf{E}, \mathbf{G}$ are 3×3 matrices, the elements of which are given by:

$$A_{jk} = \sum_{i=1}^{4} \rho_0\omega^2 g \int_{C_i} [(d^2h_j/ds^2)(d^2h_k/ds^2) - k_p^4 h_j h_k]\, ds, \tag{6a}$$

$$B_{jk} = \sum_{i=1}^{4} \rho_0\omega^2 g \int_{C_i} 2(dh_j/ds)(dh_k/ds)\, ds, \tag{6b}$$

$$C_{jk} = \sum_{i=1}^{4} \rho_0\omega^2 g \int_{C_i} h_j h_k\, ds; \quad T_{jk} = \sum_{i=1}^{4} \rho_0\omega^2 \int_{C_i} g_j h_k\, ds, \tag{6c–d}$$

$$\begin{aligned} E_{jk} = &\iint_{R_0} [(df_j/dy)(df_k/dy) + (df_j/dz)(df_k/dz) - k_0^2 f_j f_k]\, dydz \\ &+ \sum_{i=1}^{4} (\rho_0/\rho_{ay}) \iint_{R_i} [(\partial g_j/\partial y')(\partial g_k/\partial y') \\ &+ (k_{ay}^2/k_{ax}^2)(\partial g_j/\partial z')(\partial g_k/\partial z') - k_{ay}^2 g_j g_k]\, dy'dz', \end{aligned} \tag{6e}$$

$$G_{jk} = \iint_{R_0} f_j f_k dydz + \sum_{i=1}^{4} (\rho_0/\rho_{ay}) \iint_{R_i} (k_{ay}^2/k_{ax}^2) g_j g_k\, dy'dz'. \tag{6f}$$

The integrals in these Eqs. are readily evaluated analytically for the trial functions chosen, but could otherwise be evaluated numerically, for example by Gaussian quadrature.

As before, putting $\delta\Phi = 0$ involves minimising Φ with respect to the acoustic variables P_1, P_2, P_3 and the structural variables U_1, U_2, θ. The former process gives the relationship:

$$[\mathbf{E} + k_x^2\mathbf{G}]\mathbf{P} - \mathbf{T}\mathbf{U} = \mathbf{0}, \tag{7a}$$

and the latter yields:

$$[\mathbf{A} + k_x^2\mathbf{B} + k_x^4\mathbf{C}]\mathbf{U} - \mathbf{T}^T\mathbf{P} = \mathbf{0}. \tag{7b}$$

These two Eqs. constitute a coupled eigenvalue problem in k_x, the coupling occurring *via* the matrix **T**. It is worth noting that, if **T** is removed from Eqs. (7a, b), two uncoupled eigenvalue problems result:

$$[\mathbf{E} + k_x^2\mathbf{G}]\mathbf{P} = \mathbf{0}; \quad [\mathbf{A} + k_x^2\mathbf{B} + k_x^4\mathbf{C}]\mathbf{U} = \mathbf{0}. \tag{8a, b}$$

The first of these relates to acoustic modes in a lined duct with rigid walls, and the second to structural modes in an elastic-walled duct in which the acoustic loading on the walls is neglected.

Solution of the eigenproblem posed in Eqs. (7a, b) is not entirely straightforward, particularly since the coupled mode types are – broadly – divided into "acoustic" type modes, in which the power flow is predominantly in the fluid, and "structural" type modes, in which most of the power flow is in the elastic walls. Astley, who was responsible for the Rayleigh-Ritz formulation reported here, described, in Cummings and Astley (1995), a robust iterative method of solution for this problem, based on the foregoing arguments. Although a detailed description of this is not appropriate here, the interested reader is referred to Cummings and Astley (1995).

The predictive accuracy of the method described here is surprisingly good (see Cummings and Astley (1995), considering the relative crudity of the acoustic and structural trial functions. Better accuracy could be obtained by the use of trial functions containing more degrees of freedom, but one should then also consider a full finite element discretization as an alternative.

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