B General Linear Fluid Acoustics

The medium does not support shear stresses, except viscous shear. The medium parameters are constant in time; stationary flow does not exist, or its velocity is low enough, to be neglected in its influence on the sound field; see Ch. N, *"Flow Acoustics"*, for sound fields in flows.

B.1 Fundamental Differential Equations

No viscous and/or caloric losses:

Conservation of mass: $$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \vec{v} = \rho_0\, q \cdot \delta(r - r_q). \tag{1}$$

Conservation of impulse: $$\rho_0 \frac{\partial v}{\partial t} = -\operatorname{grad} p. \tag{2}$$

Equation of state: $$p = c_0^2 \cdot \rho. \tag{3}$$

Relation between pressure and particle velocity: $$\vec{v} = \frac{j}{k_0 Z_0} \operatorname{grad} p. \tag{4}$$

Homogeneous wave equation for a harmonic wave: $$\Delta p + k_0^2 \cdot p = 0. \tag{5}$$

Helmholtz's wave equation for harmonic wave with monopole source at r_q: $$\left(\Delta + k_0^2\right) p = -j\, k_0 Z_0 \cdot q \cdot \delta(r - r_q). \tag{6}$$

Adiabatic sound velocity: $$c_0^2 = \frac{\kappa P_0}{\rho_0}. \tag{7}$$

p = sound pressure;
$\vec{v}$ = particle velocity;
ρ = density;
Δ = Laplace operator;
P_0 = atmospheric pressure;
q = volume flow density of monopole source;
r = space co-ordinate;
r_q = source position;
δ = Dirac delta function;
for other symbols, see "Conventions"

Boundary conditions: (on both sides of boundary)

- Matching of sound pressures,
- Matching of normal particle velocities;

or: (for waves on both sides)
- Matching of phase velocities parallel to boundary and
- Matching of normal field admittances on both sides of boundary.

Medium with viscous and caloric losses:
▶ *See also:* Mechel (1995)

Field quantities, pressure p, density ρ and (absolute) temperature T, are composed of stationary parts (with subscript $_0$) and oscillating parts (with subscript $_1$). Velocities v are oscillating particle velocities. The sound field is composed of three coupled waves: the *density wave* (index ρ), the *viscous shear wave* (index ν) and the *heat wave* (index α).

Impulse equation:
$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\,\text{grad}\,p_1 - \nu\,\Delta\vec{v} - \frac{1}{3}\,\nu\,\text{grad div}\,\vec{v} = 0. \quad (8)$$

Heat balance:
$$\frac{\partial T_1}{\partial t} + (\kappa - 1)\,T_0\,\text{div}\,\vec{v} - \alpha\,\Delta T_1 = 0. \quad (9)$$

Conservation of mass:
$$\frac{\partial \rho_1}{\partial t} + \rho_0 \text{div}\,\vec{v} = 0.$$

Equation of state:
$$\frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} = 0. \quad (10)$$

Heat conduction inside a bordering medium (index i):
$$\frac{\partial T_{i1}}{\partial t} - \alpha_i\,\Delta T_{i1} = 0. \quad (11)$$

p_0	=	atmospheric pressure;
ρ_0	=	stationary density;
T_0	=	absolute temperature;
c_0	=	adiabatic speed of sound;
κ	=	adiabatic exponent;
ν	=	kinematic viscosity;
α	=	temperature conductivity
	=	$\Lambda/(\rho_0 c_p)$;
Λ	=	heat conductivity;
c_p	=	specific heat at constant pressure

Field composition with potentials (according to Rayleigh)
$$\vec{v} = -\text{grad}\,\Phi + \text{rot}\,\vec{\Psi} \quad (12)$$

Φ is a scalar potential;
$\vec{\Psi}$ is a vector potential with
$$\text{rot}\left(\text{grad}\,\Phi\right) \equiv 0 \quad ; \quad \text{div}\,\vec{\Psi} \equiv 0. \quad (13)$$

With vector identity
$$\Delta = \text{grad div} - \text{rot rot} \quad (14)$$

one gets:
$$-\text{grad}[j\omega\,\Phi - \frac{p_1}{\rho_0} - \frac{4}{3}\nu\,\Delta\Phi] + \text{rot}[j\omega\,\vec{\Psi} - \nu\,\Delta\vec{\Psi}] \equiv 0. \quad (15)$$

Both terms vanish individually (Rayleigh's postulate):
$$j\omega\,\Phi - \frac{p_1}{\rho_0} - \frac{4}{3}\nu\,\Delta\Phi = 0 \;;\; j\omega\,\vec{\Psi} - \nu\,\Delta\vec{\Psi} = 0. \quad (16)$$

Equivalent to two wave equations:
$$(\Delta + k_\nu^2)\,\vec{\Psi} = 0 \quad ; \quad (\Delta + k_\rho^2)\,(\Delta + k_\alpha^2)\,\Phi = 0. \quad (17)$$

Characteristic (plane) wave numbers:

- for viscous wave: $$k_\nu^2 = -j\frac{\omega}{\nu}; \tag{18}$$
- for density wave k_ρ and thermal wave k_α:

$$\left.\begin{matrix} k_\rho^2 \\ k_\alpha^2 \end{matrix}\right\} = j\omega \frac{-\left[\frac{c_0^2}{\omega} + j\left(\alpha + \frac{4}{3}\nu\right)\right] \pm \sqrt{\left[\frac{c_0^2}{\omega} + j\left(\alpha + \frac{4}{3}\nu\right)\right]^2 - 4j\alpha\left(\frac{c_0^2}{\kappa\omega} + \frac{4}{3}j\nu\right)}}{2\alpha\left(\frac{c_0^2}{\kappa\omega} + \frac{4}{3}j\nu\right)}. \tag{19}$$

Approximations to wave numbers: $$\left.\begin{matrix} k_\rho^2 \\ k_\alpha^2 \end{matrix}\right\} \approx j\frac{\kappa\omega}{2\alpha}\left(-1 \pm \sqrt{1 - 4j\frac{\alpha\omega}{\kappa c_0^2}}\right), \tag{20}$$

or with lower degree of precision: $k_\rho^2 \approx (\omega/c_0)^2 \quad ; \quad k_\alpha^2 \approx -j\kappa\omega/\alpha = \kappa\nu/\alpha \cdot k_\nu^2.$

Decomposition of scalar potential for density wave Φ_ρ and thermal wave Φ_α:

$$\Phi = \Phi_\rho + \Phi_\alpha$$

with wave equations: $$(\Delta + k_\rho^2)\,\Phi_\rho = 0 \quad ; \quad (\Delta + k_\alpha^2)\,\Phi_\alpha = 0. \tag{21}$$

Relative variation of density: $$\frac{\rho_1}{\rho_0} = \frac{j}{\omega}\left[k_\rho^2\Phi_\rho + k_\alpha^2\Phi_\alpha\right]. \tag{22}$$

Relative variation of pressure: $$\frac{p_1}{p_0} = \Pi_\rho\Phi_\rho + \Pi_\alpha\Phi_\alpha.$$

with sound pressure coefficients: $$\Pi_{\rho,\alpha} = \frac{\kappa}{c_0^2}\left(j\omega + \frac{4}{3}\nu\, k_{\rho,\alpha}^2\right) = \frac{j\,k_{\rho,\alpha}^2}{\omega}\,\frac{\kappa\omega - j\alpha k_{\rho,\alpha}^2}{\omega - j\alpha k_{\rho,\alpha}^2}. \tag{23}$$

Relative variation of temperature: $$\frac{T_1}{T_0} = \Theta_\rho\Phi_\rho + \Theta_\alpha\Phi_\alpha$$

with temperature coefficients: $$\Theta_{\rho,\alpha} = \frac{4}{3}\frac{\kappa\nu}{c_0^2}k_{\rho,\alpha}^2 + j\left(\omega\frac{\kappa}{c_0^2} - \frac{k_{\rho,\alpha}^2}{\omega}\right) = \frac{j(\kappa-1)k_{\rho,\alpha}^2}{\omega - j\alpha k_{\rho,\alpha}^2}. \tag{24}$$

Approximations to wave numbers and coefficients:

with wave number definitions: $$k_0^2 = \left(\frac{\omega}{c_0}\right)^2 \quad ; \quad k_\nu^2 = -j\frac{\omega}{\nu}\;;\; k_{\alpha0}^2 = -j\frac{\kappa\omega}{\alpha}; \tag{25}$$

$$\left.\begin{matrix} k_\rho^2 \\ k_\alpha^2 \end{matrix}\right\} = \frac{\left[\frac{1}{k_0^2} + \frac{4}{3k_\nu^2} + \frac{\kappa}{k_{\alpha0}^2}\right] \mp \sqrt{\left[\frac{1}{k_0^2} + \frac{4}{3k_\nu^2} + \frac{\kappa}{k_{\alpha0}^2}\right]^2 - \frac{4\kappa}{k_{\alpha0}^2}\left(\frac{1}{\kappa k_0^2} + \frac{4}{3k_\nu^2}\right)}}{\frac{2\kappa}{k_{\alpha0}^2}\left(\frac{1}{\kappa k_0^2} + \frac{4}{3k_\nu^2}\right)}. \tag{26}$$

$$\approx \tfrac{1}{2}k_{\alpha0}^2 \cdot \left(1 \mp \sqrt{1 - 4\frac{k_0^2}{k_{\alpha0}^2}}\right) \quad \approx \begin{cases} k_0^2 \\ k_{\alpha0}^2 \cdot (1 - k_0^2/k_{\alpha0}^2) \end{cases} \approx \begin{cases} k_0^2 \\ k_{\alpha0}^2 \end{cases}$$

Approximations:

$$k_\alpha^2 \approx k_{\alpha 0}^2 \, \frac{1+\kappa\left(1+\dfrac{4\,\mathrm{Pr}}{3}-\dfrac{1}{\kappa}\right)\dfrac{k_0^2}{k_{\alpha 0}^2}}{1+\dfrac{4\kappa^2\,\mathrm{Pr}}{3}\,\dfrac{k_0^2}{k_{\alpha 0}^2}} = k_{\alpha 0}^2 \, \frac{1+1,2165\,\dfrac{k_0^2}{k_{\alpha 0}^2}}{1+1,8259\,\dfrac{k_0^2}{k_{\alpha 0}^2}}, \tag{27}$$

$$\frac{\Theta_\rho}{\Theta_\alpha} \approx -(\kappa-1)\,\frac{k_0^2}{k_{\alpha 0}^2}, \tag{28}$$

$$\begin{aligned} \Pi_\rho &\approx \kappa\, j\, \frac{k_\rho^2}{\omega} \approx \kappa\, j\, \frac{k_0^2}{\omega}, \\ \Pi_\alpha &\approx \frac{jk_{\alpha 0}^2}{\omega}\, \frac{\kappa\left(1-\dfrac{4\kappa\,\mathrm{Pr}}{3}\right)\dfrac{k_0^2}{k_{\alpha 0}^2}}{1+\kappa\,\dfrac{k_0^2}{k_{\alpha 0}^2}} \approx \frac{jk_0^2}{\omega}\,\kappa\left(1-\frac{4\kappa\,\mathrm{Pr}}{3}\right), \end{aligned} \tag{29}$$

$$\frac{\Pi_\alpha}{\Pi_\rho} \approx 1-\frac{4\kappa\,\mathrm{Pr}}{3} = -0.3033. \tag{30}$$

$\mathrm{Pr} = \nu/\alpha$ Prandtl number

Boundary conditions with v_t= tangential velocity, v_n= normal velocity, T_i= temperature behind the boundary; Λ= heat conductivity of the medium with the sound wave, Λ_i = heat conductivity of the medium behind the boundary:

$$\begin{aligned} v_t &= 0 \quad ; \quad v_n = v_{n,i}\,, \\ T_1 &= T_{1,i} \quad ; \quad \Lambda\frac{\partial}{\partial n}T_1 = \Lambda_i\frac{\partial}{\partial n}T_{1,i}. \end{aligned} \tag{31}$$

Isothermal boundary condition: $\quad T_1 = 0.$ (32)

Adiabatic boundary condition: $\quad \dfrac{\partial}{\partial n}T_1 = 0.$ (33)

B.2 Material Constants of Air

▸ *See also:* Mechel (1995); VDI-Wärmeatlas (1984)

For definitions of symbols see *Sect. B.1* of this chapter and Table 1. Regressions (range see Fig. 1) using measured data are given in the form $f(T) = \sum_{i=-4}^{4} a_i \cdot T^{i/2}$ for the material constants of dry air as functions of (absolute) temperature T (in Kelvin degrees K).

The atmospheric pressure is assumed to be P_0=1 [bar]=10^5 [Pa]. The range of application of the regressions is 100 K $\leq T \leq$ 1500 K.

Table 1 Material constants of air at standard conditions (20° C; 1 bar)

Quantity	Symbol	Value	Dimension	Remark
Molekular weight	M	28.96	kg/kmol	Dry air
Gas constant	R	287.10	J/kgK	Ideal gas
Density	ρ_0	1.1886	kg/m^3	
Sound velocity	c_0	343.30	m/s	$c_0^2 = \kappa P_0/\rho_0$
Dynamical viscosity	η	$17.99 \cdot 10^{-6}$	$N \cdot s/m^2$	
Kinematic viscosity	ν	$15.13 \cdot 10^{-6}$	m^2/s	$\nu = \eta/\rho_0$
Adiabatic exponent	κ	1.401	–	$\kappa = c_p/c_v$
Specific heat	c_p	$1.007 \cdot 10^{3}$	J/kgK	P const.
Temperature expansion	ß	$3.421 \cdot 10^{-3}$	1/K	
Heat cconductivity	Λ	0.02603	W/mK	
Temperature conductivity	$\propto$	$21.74 \cdot 10^{-6}$	m^2/s	$\propto = \Lambda/(\rho_0 \cdot c_p)$
Prandtl number	Pr	0.6977	–	$Pr = \nu/\propto$

Interrelations are:

Prandtl number: $$\mathrm{Pr} = \nu/\alpha. \tag{1}$$

Specific heat at constant volume: $$c_v = c_p - \frac{ß^2 T}{K\rho_0}. \tag{2}$$

Isothermal compressibility: $$K = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T. \tag{3}$$

V	=	volume;
P	=	static pressure;
β	=	coefficient of thermal volume expansion

Temperature dependence of Prandtl number:

$$\begin{aligned} \mathrm{Pr} = {} & 0.66000 + 6.5853 \cdot 10^{-6} \cdot (T - 700) + 3.97457 \cdot 10^{-7} \cdot (T - 700)^2 \\ & -1.43416 \cdot 10^{-12} \cdot (T - 700)^4 + 3.05114 \cdot 10^{-18} \cdot (T - 700)^6. \end{aligned} \tag{4}$$

Sound velocity:

$$\begin{aligned} c_0 &= \sqrt{\frac{\kappa P_0}{\rho_0}} = \sqrt{\frac{\tilde{p}}{\tilde{\rho}}} = \frac{1}{\sqrt{K_0 \rho_0}} \approx \sqrt{\frac{\langle v^2 \rangle}{3}} = \sqrt{\frac{\kappa R T}{M}} \\ &= 108.28\sqrt{\frac{T}{M}} \approx 333_{m/s} + 0.6 \cdot \Theta_C \approx 20,05\sqrt{T_K}. \end{aligned} \tag{5}$$

Sound velocity of a mixture of two gas components (x is the concentration of the component with primes):

$$c_x^2 = \frac{R\,T}{x \cdot M + (1-x) \cdot M'} \; \frac{x \cdot c_p + (1-x) \cdot c'_p}{x \cdot c_p/\kappa + (1-x) \cdot c'_p/\kappa'}. \tag{6}$$

κ	=	adiabatic exponent;
P_0	=	atmospheric pressure;
ρ_0	=	atmospheric density;
$\tilde{p}$	=	sound pressure;
$\tilde{\rho}$	=	oscillating density;
K_0	=	compressibility;
$< v^2 >$	=	average square of molecular velocities;
R	=	universal gas constant;
M	=	molecular weight,
Θ	=	temperature in Celsius;
c_p	=	specific heat at constant pressure

Example for measured data (points) and regression (curve):

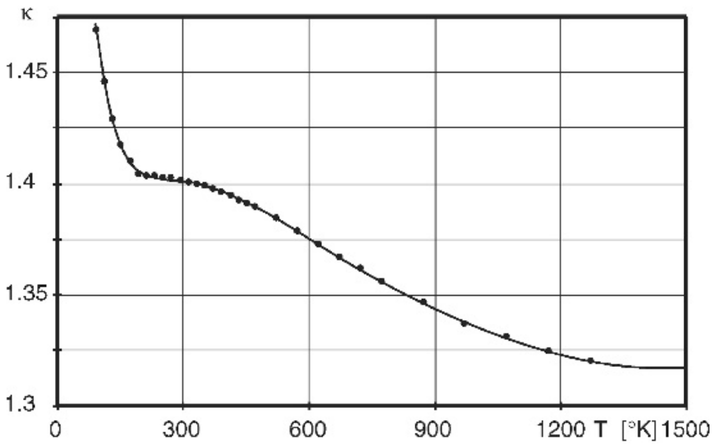


Figure 1 Adiabatic exponent κ as function of absolute temperature T. Points: measured; curve: regression

Table 2 Regression coefficients for material data as functions of (absolute) Temperature T

Quantity	a_0	$a_{\pm 1}$	$a_{\pm 2}$	$a_{\pm 3}$	$a_{\pm 4}$
ρ_0 kg/m^3	-29.2987	1.38519 363.205	-0.0384181 $-2.08219 \cdot 10^3$	$5.78952 \cdot 10^{-4}$ $6.48716 \cdot 10^3$	$-3.65858 \cdot 10^{-6}$ $3.25451 \cdot 10^3$
η Ns/m^2	$-3.30199 \cdot 10^{-4}$	$1.39487 \cdot 10^{-5}$ $4.35462 \cdot 10^{-3}$	$-2.29854 \cdot 10^{-7}$ -0.0294172	$1.43167 \cdot 10^{-9}$ 0.0740619	$4.55963 \cdot 10^{-12}$ 0.03768996
ν m^2/s	$1.04734 \cdot 10^{-4}$	$-1.00547 \cdot 10^{-5}$ $-2.16340 \cdot 10^{-4}$	$4.03090 \cdot 10^{-7}$ $-3.69703 \cdot 10^{-3}$	$-3.87707 \cdot 10^{-9}$ 0.0183863	$6.20832 \cdot 10^{-11}$ $9.00314 \cdot 10^{-3}$
κ –	25.9651	-1.08207 *) -313.593	0.0273543 $2.04477 \cdot 10^3$	$-3.79526 \cdot 10^{-4}$ $-4.89956 \cdot 10^3$	$2.26518 \cdot 10^{-6}$ $-2.50299 \cdot 10^3$
c_p J/kgK	$1.66918 \cdot 10^4$ *)	-983.174 $-1.29648 \cdot 10^5$	32.7843 $4.55412 \cdot 10^5$	-0.540032 $-1.55411 \cdot 10^5$	$3.48332 \cdot 10^{-3}$ $-1.17469 \cdot 10^5$
Λ W/mK	7.13849	-0.400186 -72.5736	0.0129070 386.078	$-2.17156 \cdot 10^{-4}$ -778.310	$1.4793566 \cdot 10^{-6}$ -403.616
$\propto$ m^2/s	0.0128841	$-7.09636 \cdot 10^{-4}$ -0.133306	$2.21478 \cdot 10^{-5}$ 0.723019	$-3.54709 \cdot 10^{-7}$ -1.49085	$2.33895 \cdot 10^{-9}$ -0.771509
ß 1/K	0.0762123	$4.36358 \cdot 10^{-3}$ -0.695016	$1.37872 \cdot 10^{-4}$ 3.55119	$-2.28121 \cdot 10^{-6}$ 0.516673	$1.54530 \cdot 10^{-8}$ -0.098786

B.3 General Relation for Field Admittance and Intensity

► *See also:* Mechel, Vol. I, Ch. 3 (1989)

The vector component G_n in a direction n of the field admittance G is defined by

$$G_n = \frac{v_n}{p} = \frac{j}{k_0 Z_0} \frac{\partial p/\partial n}{p}. \tag{1}$$

If the sound pressure is described by magnitude and phase

$$p(r) = |p(r)| \cdot e^{j\,\varphi(r)}, \tag{2}$$

the field admittance is given by

$$G_n(r) = \frac{1}{k_0 Z_0}\left[-\frac{\partial}{\partial n}\varphi(r) + j \cdot \frac{\partial}{\partial n} \ln\,\big(|p(r)|\big)\right]. \tag{3}$$

Near an absorbing wall the reactance of the wall admittance determines the slope of sound pressure level by the term $\ln(|p(r)|)$ ("admittance rule").

The time-averaged intensity of a harmonic wave is

$$I_n = \frac{1}{2}\,p \cdot v_n^* = \frac{1}{2}\,G_n^* \cdot |p|^2. \tag{4}$$

With the admittance relation follows:

$$I_n = -\frac{|p(r)|^2}{2\,k_0 Z_0} \cdot \left[\frac{\partial\varphi(r)}{\partial n} + j \cdot \frac{\partial}{\partial n} \ln\,\big(|p(r)|\big)\right]; \tag{5}$$

the real part of I_n is the effective intensity, and the imaginary part the reactive intensity.

*) See Preface to the 2nd edition.

In vector notation:	$\vec{G}(r) = \frac{1}{k_0 Z_0}\left[-\text{grad}\,\varphi(r) + j\cdot\text{grad}\,\left(\ln\,|p(r)|\right)\right],$ $\vec{I} = -\frac{|p(r)|^2}{2\,k_0 Z_0}\cdot\left[\text{grad}\,\varphi(r) + j\cdot\text{grad}\,\left(\ln\,|p(r)|\right)\right].$ (6)*)
System of two coupled differential equations for magnitude and phase of sound pressure (with the relations from above):	$\Delta|p(r)| + k_0^2\left(1 - Z_0^2\cdot\text{Re}\{G(r)\}\right)\cdot|p(r)| = 0,$ $\Delta\varphi(r) - 2\,k_0^2 Z_0^2\cdot\text{Re}\{G(r)\}\cdot\text{Im}\{G(r)\} = 0.$
If a sound field has no sources or sinks, then	$\text{div Re}\{I\} = 0.$ (7)
The effective intensity $I_{eff} = \text{Re}\{I\}$ has the rotation	$\text{rot}\,I_{eff} = \frac{-1}{k_0 Z_0}|p(r)|\cdot\text{grad}\,\varphi(r)\times\ \text{grad}\,\left(|p(r)|\right)$

(with × for the cross product of vectors). It follows that rot I_{eff}=0 if phase $\varphi(r)$ and magnitude $|p(r)|$ have parallel gradients (as in a plane wave).

B.4 Integral Relations

▶ *See also:* Pierce (1981) and others

Consider two different sound fields p_1, p_2 in a volume V with a bounding surface S (with outwards directed surface element $d\vec{s}$). Green's integral is then

$$\iiint_V \left(p_1\cdot\Delta p_2 - p_2\cdot\Delta p_1\right)\,d\mathbf{r} = \oiint_S \left(p_1\cdot\vec{\nabla}p_2 - p_2\cdot\vec{\nabla}p_1\right)\cdot d\vec{s}. \tag{1}$$

The fields may differ either by different source strengths and/or locations, and/or by different boundary conditions on S, and/or are different forms (modes) for the same sources and boundaries. The surface S is either soft (p(S)=0) or hard ($\partial p/\partial n$=0) on parts S_0 or locally reacting on parts S_a with surface admittance G, or parts S_∞ are at infinity, where the fields obey Sommerfeld's condition.

With the fundamental relations of *Sect. B.1* it follows that

$$\oiint_S p_1\cdot\vec{v}_2\cdot d\vec{s} - \oiint_S p_2\cdot\vec{v}_1\cdot d\vec{s} = \iiint_V p_1\cdot q_2\cdot\delta(r - r_2)\,dr - \iiint_V p_2\cdot q_1\cdot\delta(r - r_1)\,dr \tag{2}$$

if field p_1 has a source with volume flow q_1 at r_1 and field p_2 has a source with volume flow q_2 at r_2. Integration over the Dirac delta functions gives

$$\oiint_S p_1\cdot\vec{v}_2\cdot d\vec{s} - \oiint_S p_2\cdot\vec{v}_1\cdot d\vec{s} = p_1(r_2)\cdot q_2 - p_2(r_1)\cdot q_1. \tag{3}$$

The *reciprocity principle* follows if both p_1 and p_2 everywhere satisfy the same boundary conditions:

$$p_1(r_2)\cdot q_2 = p_2(r_1)\cdot q_1. \tag{4}$$

*) See Preface to the 2nd edition.

If both fields are source free, then $$\oint\!\!\oint_S p_1 \cdot \vec{v}_2 \cdot d\vec{s} - \oint\!\!\oint_S p_2 \cdot \vec{v}_1 \cdot d\vec{s} = 0. \quad (5)$$

If, additionally, they satisfy the same boundary condition on a part, e.g. S_a, of the surface S, then $$\iint_{S_a+S_\infty} p_1 \cdot \vec{v}_2 \cdot d\vec{s} - \iint_{S_a+S_\infty} p_2 \cdot \vec{v}_1 \cdot d\vec{s} = 0. \quad (6)$$

If S_a is hard for p_1 and/or soft for p_2, then $$\iint_{S_a+S_\infty} p_1 \cdot \vec{v}_2 \cdot d\vec{s} - \iint_{S_\infty} p_2 \cdot \vec{v}_1 \cdot d\vec{s} = 0, \quad (7)$$

and if they obey the same far field conditions, then $$\iint_{S_a} p_1 \cdot \vec{v}_2 \cdot d\vec{s} = 0. \quad (8)$$

If one or both fields have sources, the relevant source terms appear on the right-hand sides.

B.5 Green's Functions and Formalism

► *See also:* Skudrzyk (1971)

In a loss-free medium it is convenient to formulate the wave equation for the sound pressure field p. The particle velocity is then $$v = \frac{j}{k_0 Z_0} \operatorname{grad} p. \quad (1)$$

Let r be a general co-ordinate.
The homogeneous wave equation is $$\Delta p(r) + k_0^2 \cdot p(r) = 0. \quad (2)$$

The inhomogeneous wave equation with a source of volume flow q(r) is $$\Delta p(r) + k_0^2 \cdot p(r) = -j\, k_0 Z_0\, q(r). \quad (3)$$

Here q(r) is the rate of volume generation per unit volume and unit time.

Green's formalism uses a potential function g for the field (instead of the sound pressure function)

i.e. $$\vec{v} = -\operatorname{grad} g \quad ; \quad p = \rho_0 \frac{\partial g}{\partial t}. \quad (4)$$

The Green's function $g(r|r_q, \omega)$ is the solution of the *inhomogeneous* wave equation for a time harmonic excitation by a point source in r_q of unit strength, which satisfies specified boundary conditions, with the Dirac delta function: $$\Delta\, g(r|r_q, \omega) + k_0^2\, g(r|r_q, \omega) = -\,\delta(r - r_q) \quad (5)$$

$$\delta(r - r_q) = \delta(x - x_q) \cdot \delta(y - y_q) \cdot \delta(z - z_q).$$

Any solution h(r) of the homogeneous wave equation, satisfying the boundary conditions, can be added to give a solution $G(r|r_q)$: $$G(r|r_q) = g(r|r_q) + h(r). \quad (6)$$

The Green's function of a point source in free space is

$$g(r|r_q, \omega) = \frac{e^{-j\, k_0 R}}{4\pi R} \quad ; \quad R = \sqrt{(x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2}. \quad (7)$$

The volume flow of the source is given by $\lim_{R\to 0}\left(-4\pi R^2 \frac{\partial g}{\partial R}\right)$. (8)

From this it follows in three dimensions: $\lim_{r\to r_q}\left(g(r|r_q,\omega)\right) = \frac{1}{4\pi R}$; (9a)

in two dimensions: $\lim_{r\to r_q}\left(g(r|r_q,\omega)\right) = \frac{1}{2\pi}\ln|r-r_q|$; (9b)

in one dimension: $\left(\frac{\partial g}{\partial x}\right)_{x_q+\varepsilon} - \left(\frac{\partial g}{\partial x}\right)_{x_q-\varepsilon} = -1,$ (9c)

i.e., the one-dimensional Green's function has a discontinuity in slope at $x=x_q$.

Green's functions are reciprocal: $g(r|r_q,\omega) = g(r_q|r,\omega)$. (10)

The sound pressure field $p(r,\omega)$ in a finite space with given boundary conditions and a volume source distribution $f(r,\omega)$ has to be a solution of the wave equation

$$\Delta p(r,\omega) + k_0^2 p(r,\omega) = -f(r,\omega). \tag{11}$$

The solution can be expressed with Green's functions of the infinite space as

$$\begin{aligned} p(r,\omega) = &\iiint f(r_q,\omega)\cdot g(r|r_q,\omega)\,dV_q \\ &+ \iint\left[g(r|r_q,\omega)\frac{\partial}{\partial n_q}p(r_q,\omega) - p(r_q,\omega)\frac{\partial}{\partial n_q}g(r|r_q,\omega)\right]dS_q, \end{aligned} \tag{12}$$

where the subscript q indicates the variable for differentiation and integration. This is the *Helmholtz–Huygens equation*. The surface integral simplifies if either $g(r|r_q,\omega)$ or its normal derivative vanishes.

Green's functions may be defined also for non-harmonic sources but with time functions with unit spectral density, i.e. for time the function $\delta(t-t_0)$:

$$g(r|r_q, t-t_0) := g(r,t|r_q,t_0) := \int_0^\infty g(r|r_q,\omega)\cdot e^{j\omega(t-t_0)}\frac{d\omega}{2\pi}. \tag{13}$$

Then: $$p(r,t) = \int_{-\infty}^{t} f(r_q,t_0)\cdot g(r|r_q,t-t_0)\,dt_0, \tag{14}$$

where $f(r_q,t_0)$ is the time function of the source having the spectrum $f(r,\omega)$.

The Green's function $g(r|r_q,\omega) = \frac{e^{-jk_0r+j\omega t}}{4\pi R}$ (15a)

belongs to the time function $\frac{1}{4\pi R}\delta\left(t-r/c_0\right)$. (15b)

Green's functions in closed spaces can be expanded in *modes* Ψ_n; these are solutions of the homogeneous wave equation

$$\Delta\Psi_n + k_n^2\Psi_n = 0 \tag{16}$$

satisfying the boundary conditions (and Sommerfeld's far field condition if the space is infinite in one dimension). The wave number k_n (instead of k_0) recalls that in a finite size space harmonic solutions exist only if the frequency is a resonant frequency. The modes are orthogonal, and may be made orthonormal, i.e.

$$\int \Psi_n \cdot \Psi_m \, dV = \delta_{nm} = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases} \tag{17}$$

assuming the boundary conditions have one of the following forms:

$$\Psi_n = 0 \quad \text{or} \quad \partial\Psi_n/\partial n = 0 \quad \text{or} \quad \Psi_n = -\alpha \cdot \partial\Psi_n/\partial n. \tag{18}$$

Then Green's function can be expanded:

$$g(r|r_q, \omega) = \sum_n A_n \Psi_n = \sum_n \frac{\Psi_n(r) \cdot \Psi_n(r_q)}{k_n^2 - k_0^2} \quad ; \quad A_n = \frac{\Psi_n(r_q)}{k_n^2 - k_0^2}. \tag{19}$$

The residues at the poles $\quad k_0 = \pm k_n \quad$ are $\quad \pm\frac{1}{2\,k_n}\Psi_n(r) \cdot \Psi_n(r_q).$ (20)

If the space is infinite, complex modes are convenient. The orthogonality integral then should be [instead of (17)]:

$$\int \Psi_n \cdot \Psi_m^* \, dV = \delta_{nm} = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m, \end{cases} \tag{21}$$

(the asterisk indicates the complex conjugate).

The Green's function then is

$$g(r|r_q, \omega) = \sum_n \frac{\Psi_n(r) \cdot \Psi_n^*(r_q)}{k_n^2 - k_0^2}. \tag{22}$$

If one sets the condition that the physical solution should be real, the relations follow:

$$\begin{aligned} &\Psi_n(r) = \Psi_{-n}^*(r), \\ &\Psi_n(r) + \Psi_{-n}(r) = \Psi_n(r) + \Psi_n^*(r) = 2\,\mathrm{Re}\{\Psi_n(r)\}. \end{aligned} \tag{23}$$

The eigenvalues k_n need not be a discrete set of values, but may be continuous. Then the Green's function is

$$g(r|r_q, \omega) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\Psi(r) \cdot \Psi^*(r_q)}{k_n^2 - k_0^2} \left(\frac{\partial k_n}{\partial n}\right)^{-1} dk_n. \tag{24}$$

The form of Green's functions for continuous eigenvalues is similar to *integral transforms*:

$$\begin{aligned} &S_F(x) = \int F(z) \cdot \Psi^*(x, z) \cdot w(z)\, dz, \\ &F(z) = \int S_F(x) \cdot \Psi^*(x, z) \cdot w(x)\, dx. \end{aligned} \tag{25}$$

The weight function w(z) is often introduced by the co-ordinate system; generally it represents the density of eigenvalues in z space. The following orthogonality and normalizing relations are used:

$$\begin{aligned} &w(k) \int \Psi(k,z)\cdot\Psi^*(x,z)\cdot w(z)\,dz = \delta(k-x),\\ &w(k) \int \Psi(k,z)\cdot\Psi^*(x,\zeta)\cdot w(k)\,dk = \delta(z-\zeta). \end{aligned} \tag{26}$$

In particular, the Dirac delta function is represented by

$$\delta(r-r_q) = \int_{-\infty}^{+\infty} \Psi(k,r)\cdot\Psi^*(x,r_q)\cdot w(k)\cdot w(r_q)\,dk; \tag{27}$$

thus

$$S_F(x) = \frac{-w(r_q)\cdot\Psi^*(x,r_q)}{k^2-x^2}, \tag{28}$$

and the Green's function becomes

$$w(r_q)\cdot g(r,r_q|\omega) = F(r) = \int_{-\infty}^{+\infty} \frac{\Psi(k,r)\cdot\Psi^*(x,r_q)\cdot w(k)\cdot w(r_q)}{k^2-x^2}\,dx. \tag{29}$$

Some examples of Green's functions are given below.

A set of plane waves:

Substitute above $x \to \vec{\kappa}$ indicating a wave number vector; denote with $\vec{r}$ the co-ordinate vector of a point. A set of plane waves is represented by

$$\Psi(\vec{\kappa},\vec{r}) = A(\vec{\kappa})\cdot e^{-j\,\vec{\kappa}\cdot\vec{r}} \tag{30}$$

(with the scalar product $\vec{\kappa}\cdot\vec{r}$ in the exponent). The density $w(\kappa)$ is unity. The amplitudes are $A(\vec{\kappa}) = 1/(2\pi)$ (for normalisation). In a two-dimensional space (x,y) with $\vec{\kappa}\cdot\vec{r} = \kappa_x x + \kappa_y y$ the Green's function becomes

$$g(r,r_q|\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{-j\,\vec{\kappa}\cdot(\vec{r}-\vec{r}_q)}}{\kappa^2-k_0^2}\,\frac{d\kappa_x\,d\kappa_y}{4\pi^2} \quad ; \quad \kappa^2 = \kappa_x^2+\kappa_y^2. \tag{31}$$

It can be shown that the integral goes over to the Hankel function of the second kind:

$$g(r,r_q|\omega) = \frac{-j}{4}\,H_0^{(2)}(k_0R) \quad ; \quad R = |r-r_q|\,. \tag{32}$$

Cylindrical waves: wave

A set of eigenfunctions of the Bessel differential equation is

$$\begin{aligned} &\psi_n(r) = J_m(\alpha_n r/a) \quad ; \quad J_m(\alpha_n) = 0,\\ &k_n = (\alpha_n/a) \quad ; \quad n = 0,\ 1,\ 2,\ldots, \end{aligned} \tag{33}$$

where α_n are zeros of the Bessel function $J_m(z)$ of order m. The orthogonality relation is

$$\int_0^a J_m(\alpha_n r/a) \cdot J_m(\alpha_\ell r/a)\, r\, dr = \begin{cases} 0 \quad ; \quad \ell \neq n, \\ -\frac{a^2}{2} J_{m+1}(\alpha_n) \cdot J_{m-1}(\alpha_n) \quad ; \quad \ell = n. \end{cases} \tag{34}$$

If $a \to \infty$ the eigenvalues become continuous. One sets

$$\Psi(k, z) = A \cdot J_m(kz) \quad ; \quad A = 1, \tag{35}$$

where A=1 follows from the normalisation.

Two-dimensional infinite space in polar co-ordinates:

Two-dimensional eigenfunctions of the wave equation satisfying the normalisation conditions (26) in polar co-ordinates (r,φ) are

$$\sqrt{w(\kappa)\, w(r)}\Psi(\kappa, z) = \sqrt{\frac{\kappa r}{2\pi}} J_m(\kappa r) \cdot e^{-j\, m\varphi}\,. \tag{36}$$

The Green's function becomes

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{1}{8\pi^2} \sum_{m=0}^{\infty} \delta_m \cos(m(\varphi - \varphi_q)) \int_{-\infty}^{+\infty} \frac{J_m(\kappa r) \cdot J_m(\kappa r_q)}{\kappa^2 - k_0^2}\, \kappa\, d\kappa. \tag{37}$$

One gets after evaluation of the integral

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{-j}{4} \sum_{m=0}^{\infty} \delta_m \cos(m(\varphi - \varphi_q)) \begin{cases} J_m(k_0 r) \cdot H_m^{(2)}(k_0 r_q) \quad ; \quad r \leq r_q, \\ J_m(k_0 r_q) \cdot H_m^{(2)}(k_0 r) \quad ; \quad r \geq r_q. \end{cases} \tag{38}$$

Three-dimensional infinite space:

The Green's function is

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{e^{-j\, k_0 R}}{4\pi R} \quad ; \quad R = |\vec{r} - \vec{r}_q|. \tag{39}$$

Green's function in spherical harmonics:

In the spherical co-odinates r, φ, ϑ the Green's function is

$$\begin{aligned} g(\vec{r}, \vec{r}_q|\omega) &= \frac{e^{-j\, k_0 R}}{4\pi R} = -\frac{j\, k_0}{4\pi}\, h_0^{(2)}(k_0 R) \\ &= -\frac{j\, k_0}{4\pi} \sum_{n=0}^{\infty} (2n+1) \cdot \sum_{m=0}^{n} \delta_m \frac{(n-m)\,!}{(n+m)\,!} \cos(m(\varphi - \varphi_q)) \\ &\quad \cdot P_n^m(\cos\vartheta_q) \cdot P_n^m(\cos\vartheta) \\ &\quad \cdot \begin{cases} j_n(k_0 r) \cdot h_n^{(2)}(k_0 r_q) \quad ; \quad r < r_q, \\ j_n(k_0 r_q) \cdot h_n^{(2)}(k_0 r) \quad ; \quad r > r_q, \end{cases} \end{aligned} \tag{40}$$

where $j_m(z)$, $h_m^{(2)}(z)$ are spherical Bessel and Hankel functions and $P_n^m(x)$ are associated Legendre functions.

If the source distance r_q goes to infinity, one gets for the plane wave incident from the spherical directions φ_q, ϑ_q (in the spherical angles φ, ϑ)

$$e^{-j\vec{k}_0\cdot\vec{r}} = \sum_{n=0}^{\infty} (-j)^n (2n+1)\cdot j_n(k_0 r) \times \sum_{m=0}^{n} \delta_m \frac{(n-m)!}{(n+m)!} \cos(m(\varphi-\varphi_q)) \cdot P_n^m(\cos\vartheta_q) \cdot P_n^m(\cos\vartheta). \tag{41}$$

Green's function in cylindrical co-ordinates:

In the cylindrical co-odinates r, ϑ, z the Green's function is

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{-1}{8\pi^3} \int_0^{2\pi} d\alpha \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{+j\kappa_r r \cos(\alpha-\vartheta) - j\kappa_r r_q \cos(\alpha-\vartheta_q) - j\kappa_z(z-z_q)}}{k_0^2-\kappa_r^2-\kappa_z^2} \kappa_r \, d\kappa_r \, d\kappa_z. \tag{42a}$$

Performing the integration over κ_z with $\kappa_z = \pm\sqrt{k_0^2-\kappa_r^2} = \pm\sigma$

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{j}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\infty} \frac{\kappa_r}{\sigma} e^{-j\kappa_r r \cos(\alpha-\vartheta) + j\kappa_r r_q \cos(\alpha-\vartheta_q) - j\kappa_z(z-z_0)} \cdot e^{-j\sigma|z-z_q|} \, d\kappa_r. \tag{42b}$$

With the exponentials expressed by Bessel functions, one gets

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{-j}{4\pi} \sum_{m\geq 0} \delta_m \cos(m(\vartheta-\vartheta_q)) \int_0^{\infty} \frac{\kappa_r}{\sigma} J_m(\kappa_r r) \cdot J_m(\kappa_r r_q) \cdot e^{-j\sigma|z-z_q|} \, d\kappa_r \tag{43}$$

with

$$\sigma = \begin{cases} \sqrt{k_0^2-\kappa_r^2} & \text{if} \quad 0<\kappa_r<k_0, \\ -j\sqrt{\kappa_r^2-k_0^2} & \text{if} \quad 0<k_0<\kappa_r. \end{cases} \tag{44}$$

For r_q=0 (43) reduces to the term with m=0.

Point source above hard or soft plane:

The Green's function for a hard plane is

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{e^{-jk_0 r}}{4\pi r} + \frac{e^{-jk_0 r'}}{4\pi r'}, \tag{45}$$

where r is the distance from the source to the field point and r′ is the distance from the image source (in a mirror-reflected position relative to the plane) to the field point.

If the plane is soft, then

$$g(\vec{r}, \vec{r}_q|\omega) = \frac{e^{-jk_0 r}}{4\pi r} - \frac{e^{-jk_0 r'}}{4\pi r'}. \tag{46}$$

Point source above a locally reacting plane:

The plane is at x=0; the source at x_q, y_q, z_q; the image source at $-x_q, y_q, z_q$. Let $\vec{n}\cdot\vec{r} = n_x x + n_y y + n_z z$ be the scalar product of the wave direction vector $\vec{n}$ with the co-ordinate vector $\vec{r}$. A plane wave, reflected at the plane, can be represented as

$$p_r = R\cdot e^{-j\,k_0(n_x x+n_y y+n_z z)} \quad ; \quad R = \frac{\zeta\, n_x - 1}{\zeta\, n_x + 1}, \tag{47}$$

where ζ is the normalised surface impedance of the plane.

The Green's function in Cartesian co-ordinates is

$$\begin{aligned} g(\vec{r},\vec{r}_q|\omega) &= \frac{1}{8\pi^3}\int\limits_{-\infty}^{+\infty} \frac{e^{j\,[\kappa_x x+\kappa_y(y-y_q)+\kappa_z(z-z_q)]}}{\kappa^2-k_0^2}\left(e^{-j\,\kappa_x x_q} + R\, e^{+j\,\kappa_x x_q}\right)\, d^3\kappa \\ \kappa^2 &= \kappa_x^2+\kappa_y^2+\kappa_z^2 \quad ; \quad d^3\kappa = d\kappa_x\cdot d\kappa_y\cdot d\kappa_z \end{aligned} \tag{48}$$

and in cylindrical co-ordinates:

$$\begin{aligned} g(\vec{r},\vec{r}_q|\omega) = \frac{-j}{4\pi}\sum_{m\geq 0}\delta_m\,\cos\left(m(\vartheta-\vartheta_q)\right) \\ \times\int\limits_0^{\infty}\frac{\kappa}{\kappa_x}J_m(\kappa' r)\cdot J_m(\kappa' r_q)\,\left(e^{-j\,\kappa_x|x-x_q|} + R\, e^{-j\,\kappa_x|x+x_q|}\right)\, d\kappa' \end{aligned} \tag{49}$$

where $\kappa'^2 = k_0^2 - \kappa_x^2$.

An approximate expression (if field point and/or source are distant to the plane) is

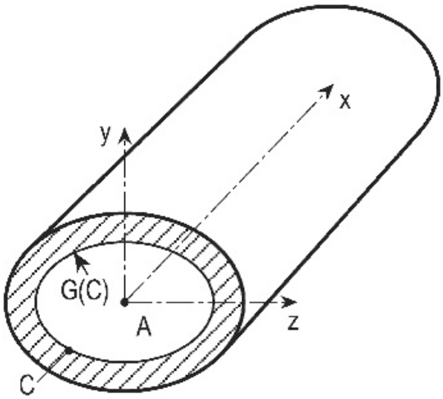
$$g(\vec{r},\vec{r}_q|\omega) = \frac{1}{4\pi}\left[\frac{e^{-j\,k_0|\vec{r}-\vec{r}_q|}}{|\vec{r}-\vec{r}_q|} + R\frac{e^{+j\,k_0|\vec{r}-\vec{r}'_q|}}{|\vec{r}-\vec{r}'_q|}\right], \tag{50}$$

where $\vec{r}$ is the vector of the field point, $\vec{r}_q$ the vector to the original source, and $\vec{r}'_q$ the vector to the mirror source.

B.6 Orthogonality of Modes in a Duct with Locally Reacting Walls

▶ *See also:* Mechel, Vol. III, Ch. 26 (1998)

Consider a duct whose interior contour follows a co-ordinate surface of a separable system of co-ordinates and whose contour surface is either totally or in parts locally reacting with an admittance G (the other parts are either hard or soft). Let the cross-section normal to the axial co-ordinate x be A, and let r be the one- or two-dimensional co-ordinate normal to x.



Let $p_m(x, r) = T_m(r) \cdot R_m(x)$ be a mode in the duct, i.e. a field which satisfies the homogeneous wave equation and the boundary conditions, with the transversal function $T_m(r)$ and the axial function $R_m(x)$.

Such modes are orthogonal over the cross section A, i.e.

$$\int_A T_m(r) \cdot T_n(r) \cdot g(r)\, dr = \delta_{m,n} \cdot N_m, \qquad (1)$$

where g(r) is the weight function induced by some co-ordinate systems; it is independent of the mode order m; $\delta_{m,n}$ is the Kronecker symbol, and N_m is the *norm* of the mode.

The orthogonality of modes, under the conditions mentioned, holds whatever the value of G is, and also if the medium in the duct has losses (i.e. k_0, Z_0 complex). They form a complete set of solutions (see Morse/Feshbach 1953, part I, Sect. 6.3, pp. 738 et sqq.) if the defining boundaries normal to r are either hard or soft or locally reacting, and if in this case the derivative $\partial p/\partial r$ does not appear in the separated wave equation of the co-ordinate r. Modes may be one-, two-, or three-dimensional according to the number of pairs of walls that define the boundary conditions.

B.7 Orthogonality of Modes in a Duct with Bulk Reacting Walls

► *See also:* Mechel, Vol. III, Ch. 27 (1998); Cummings (1989)

Assume a duct like that in › *Sect. B.6*, but whose duct lining is laterally (bulk) reacting, and whose outer wall (behind the lining) is hard. The field in the interior volume of the duct, with cross section A_1, is marked with an index i=(1), the field in the lining with an index i=(2), and its cross section is A_2. Let the characteristic wave number and wave impedance in A_1 be, respectively, k_0 and Z_0, and let the characteristic propagation constant and wave impedance of the lining material in A_2 be, respectively, Γ_a, Z_a. The transversal functions of a mode $T_m^{(i)}(r)$ are different in the two areas; its axial function $R_m(x)$ is the same.

The modes are orthogonal over the cross section A_1+A_2 with the mode norm in the case of a single homogeneous layer of the lining in a cylindrical duct:

$$N_m = \frac{1}{j\,k_0 Z_0} \iint_{A_1} \left(T_m^{(1)}(r)\right)^2 dr + \frac{1}{\Gamma_a Z_a} \iint_{A_2} \left(T_m^{(2)}(r)\right)^2 dr. \qquad (1)$$

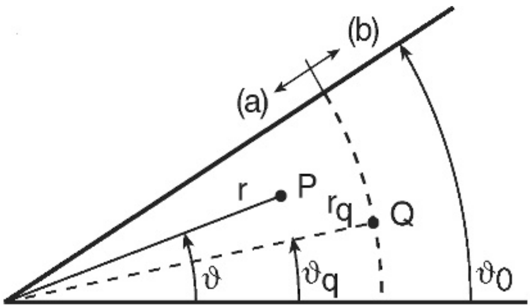
In the case of multiple layers, an integral must be added for each layer.

B.8 Source Conditions

See *Sects. B.1, B.4, B.5*.

A special form of the boundary conditions, the source condition, must be satisfied if the sound field p(r) is excited by a sound source. Commonly used are volume flow sources $q(r_q)$ either as a point source in three-dimensional fields, or as a line source in two-dimensional fields, or, more generally, as a source distribution on a surface S_q, or as a source distribution in a volume V_q. In the case of distributed sources, $q(r_q)$ is the spatial density of emanating volume flow.

The source condition requires that the integral of the outward normal velocity over a small spherical surface around a point source, or over a narrow cylindrical surface around a line source, or on S_q around distributed sources, equal the given source strength q. This form of the source condition is sometimes difficult to evaluate. A form more suitable to evaluation shall be given.



First, consider a *point source or a line source*. This case is illustrated with a line source (for simplicity); a point source is treated similarly. Let the source be located at (r_q, ϑ_q) in a cylindrical co-ordinate system (r, ϑ). In general ϑ stands for a co-ordinate over which orthogonal modes exist (i.e. the modes satisfy the homogeneous wave equation, Sommerfeld's condition, and the boundary conditions at the surfaces normal to ϑ). The line source is located at Q. This defines two zones: zone (a) with $0 \le r \le r_q$ and zone (b) with $r_q \le r < \infty$.

The modes have the form
$$p_m(r,\vartheta) = T(\eta_m\vartheta)\cdot R_m(r). \tag{1}$$

They are orthogonal over $(0,\vartheta_0)$ with norms N_m.

The radial functions $R_m(r)$ are formulated so that they are continuous at $r=r_q$, but discontinuous in their radial derivatives.

The source condition can be written in the form
$$Z_0\left[v_r(r_q+0)-v_r(r_q-0)\right] \overset{!}{=} \frac{k_0 Z_0\, q}{k_0 r_q}\cdot\delta(\vartheta-\vartheta_q). \tag{2}$$

The Dirac delta function may be expanded in modes:
$$\delta(\vartheta-\vartheta_q) = \sum_{m\ge 0} b_m \cdot \cos(\eta_m\vartheta). \tag{3}$$

By application on both sides of
$$\int_0^{\vartheta_0} \ldots \cdot \cos(\eta_m\vartheta)\, d\vartheta\,, \tag{4}$$

follow the b_m from $$\delta(\vartheta-\vartheta_q)=\frac{1}{(2)\,\vartheta_0}\sum_{m\geq 0}\frac{\cos(\eta_m\vartheta_q)}{N_m}\cdot\cos(\eta_m\vartheta). \quad (5)$$

Factor (2) is applied if the source is on a boundary, else $(2)\to 1$.

If the sources $q(\vartheta)$ are *distributed* over the surface at $r=r_q$, this distribution is synthesised with the modes having norms as above.

B.9 Sommerfeld's Condition

► *See also:* Skudrzyk (1971)

If the field extends to infinity, it must approach zero there, unless it is a plane wave in a loss-free medium. A sufficient condition is a medium with losses.

Otherwise: $$\lim_{r\to\infty}\left[r\left(\frac{\partial p}{\partial r}+j\,k_0 p\right)\right]=0. \quad (1)$$

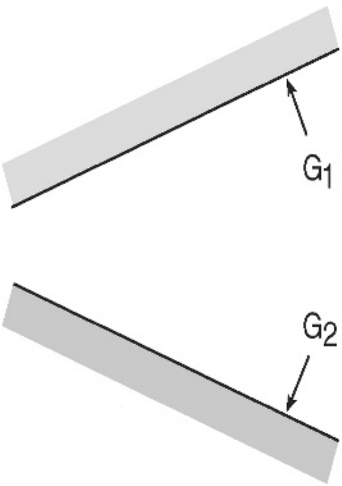
A weaker but simpler condition is, with A an arbitrary constant, $$\lim_{r\to\infty}|r\,p|<A.$$

B.10 Principles of Superposition

► *See also:* Ochmann/Donner (1994); Mechel (2000)

Some principles of superposition may help to reduce more general problems to a repetition of simpler standard tasks.

First principle of superposition (by Mechel): (*unsymmetry superposition*)



Two opposite walls, normal to the same co-ordinate, locally react with different admittances G_1, G_2. The sound fields at the walls have the corresponding indices 1,2.

The boundary conditions at these surfaces are (with normal particle velocity components v_i):

$$v_1 = G_1 \cdot p_1; \qquad v_2 = G_2 \cdot p_2 .$$

Set (with G_1, G_2 selected so that $\mathrm{Re}\{G_a\} \geq 0$): G_s is the symmetrical and G_a the antisymmetrical part of the boundary conditions.

$$G_s = \frac{1}{2}(G_1 + G_2), \quad G_a = \frac{1}{2}(G_1 - G_2). \tag{1}$$

Suppose the sound fields p_s, p_a are known for the two symmetrical linings G_s, G_a, respectively, on each side, i.e. with the boundary conditions at both flanks: p_s is the symmetrical solution belonging to G_s, p_a the antisymmetrical solution belonging to G_a.

$$v_s = G_s \cdot p_s = \frac{1}{2}(G_1 + G_2) \cdot p_s, \quad v_a = G_a \cdot p_a = \frac{1}{2}(G_1 - G_2) \cdot p_a. \tag{2}$$

It follows immediately that

$$v_{s1,2} + v_{a1,2} = G_1 \cdot \left(p_{s1,2} + p_{a1,2}\right), \quad v_{s1,2} - v_{a1,2} = G_2 \cdot \left(p_{s1,2} - p_{a1,2}\right). \tag{3}$$

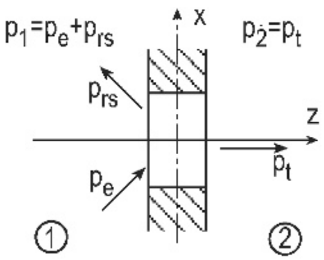
Comparing this with the boundary conditions of the original task, one sees the correspondence:

$$p_1 = p_{s1,2} + p_{a1,2}, \quad p_2 = p_{s1,2} - p_{a1,2}.$$

The desired solution is evidently $p = p_s + p_a$, because both lines formally merge at the walls.

Second principle of superposition (by Ochmann): (*symmetry superposition*)

Suppose the object has a plane of symmetry. The medium is steady across the plane of symmetry, and no sound transmissive foil or sheet is in that plane. Let a co-ordinate z be normal to the plane of symmetry, directed from the side of incidence to the side of transmission, with z=0 in the plane of symmetry. Co-ordinate transversals to z are represented by x.



An index 1 marks the half-space with the incident wave p_e and a reflected and/or backscattered wave p_{rs}; an index 2 marks the half-space with the transmitted wave p_t. The fields in the two half-spaces are $p_1(x,z) = p_e(x,z) + p_{rs}(x,z)$, $p_2(x,z) = p_t(x,z)$.

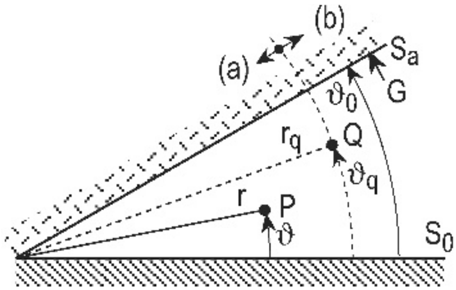
Replace the original task by two subtasks; in the first one the sound transmissive parts of the plane of symmetry are assumed to be hard, in the second one they are assumed to be soft. Both conditions are marked by upper indices (h), (s), respectively. Solve the problems of reflection and/or backscattering for the two subtasks.

The sound field components of the original task then are

$$\begin{aligned} p_{rs}(x,z) &= \frac{1}{2}\left(p_{rs}^{(h)}(x,z) + p_{rs}^{(s)}(x,z)\right); \quad z \le 0, \\ p_t(x,z) &= \frac{1}{2}\left(p_{rs}^{(h)}(x,-z) - p_{rs}^{(s)}(x,-z)\right); \quad z \ge 0. \end{aligned} \tag{4}$$

Third principle of superposition (by Mechel): (*hard-soft superposition*)

The task: Find the sound field p_a with (part of) the boundaries absorbent with local reaction, described by a wall admittance G. Suppose the solutions are known for the same source and geometry, but all walls are ideally reflecting, i.e. either hard or soft or mixed with both types. The third principle of superposition composes p_a for the absorbent boundary with such solutions.



The example assumes a line source at Q in a wedge-shaped space with one hard flank at $\vartheta = 0$, and one locally absorbing flank at $\vartheta = \vartheta_0$. The standard situation with a soft flank at $\vartheta = 0$ is treated similarly; other situations are treated after application of the first and second principles of superposition.

It is assumed that the field p_h is known, for which the flank at $\vartheta = \vartheta_0$ is hard, and that p_s is known, with a soft flank at $\vartheta = \vartheta_0$. Both fields satisfy the source condition at Q individually (see Sect. B.8).

The desired field p_a then is $\quad p_a(r,\vartheta) = \dfrac{1}{1+G\cdot X(r)}\left[p_h(r,\vartheta) + G\cdot X(r)\cdot p_s(r,\vartheta)\right]$

with the "cross impedance" $\quad X(r) = \dfrac{p_h(r,\vartheta_0)}{v_{sn}(r,\vartheta_0)} = -j\,k_0 Z_0\,\dfrac{p_h(r,\vartheta_0)}{\mathrm{grad}_n p_s(r,\vartheta_0)}. \qquad (5)$

The index n indicates the vector component normal to the absorbing wall and directed into it.

X(r) is an impedance, formed with the sound pressure if the flank at $\vartheta = \vartheta_0$ is hard divided by the normal particle velocity if the flank is soft. The third principle of superposition returns an exact solution if X is constant with respect to the co-ordinate on the absorbing wall (r in the example); otherwise an approximation to p_a is obtained.

B.11 Hamilton's Principle

► *See also:* Cremer/Heckl (1995); Morse/Feshbach (1953)

Let E_{kin} be the kinetic (effective) energy of a vibrating system, associated with oscillating masses, and E_{pot} its (effective) potential energy, associated with displacements against stresses; further let W be the (effective) work done by external forces on the system.

Lagrange function: $L = E_{kin} - E_{pot}$

Hamilton's principle: If the system starts to oscillate from reasonable initial conditions, the form of oscillation which it assumes is such that the time average of its Lagrange function is an extreme if the form of the oscillation is varied (δ stands for such variations):

$$\delta \int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2} \delta W\,dt = 0. \tag{1}$$

If the work W of external forces is constant over time intervals, and time average values of L and W are used, then $\delta\langle L\rangle + \delta\langle W\rangle = 0$. If the system is adiabatic, i.e. $W = 0$, Hamilton's principle requires $\delta\langle L\rangle = 0$. The form of the system's oscillation is governed by amplitudes either of system elements or of field components, such as modes. The variation is applied to these amplitudes a_m. On the other hand, many systems have to obey boundary conditions, which are constraints in terms of variational methods. These boundary conditions are formulated as equations $g_k(a_m) = 0$, and they are introduced into Hamilton's principle using the *Lagrange multipliers* λ_k (see Morse/Feshbach, 1953, part I, Sect. 3.1), leading to the form of Hamilton's principle suited for application to mechanical systems:

$$\frac{1}{T}\int_0^T \left(E_{kin} - E_{pot}\right)\,dt + \sum_k \left(\lambda_k^* \cdot g_k + \lambda_k \cdot g_k^*\right) = \min. \tag{2}$$

The λ_k are treated in the application of the principle like the amplitudes a_m, i.e. they are parts of the variation.

This expression is formulated as a function $f(a_m, \lambda_k)$. The energies will be sums with products $a_m \cdot a_n^*$ as factors. The minimum is found where the following equations hold:

$$\frac{\partial f}{\partial a_n^*} = 0 \quad ; \quad \frac{\partial f}{\partial \lambda_k^*} = 0. \tag{3}$$

This gives a set of linear equations for the a_m, λ_k.

In distributed systems and/or wave fields, the integration is not only over time but also over space. If the sound field is described by a velocity potential function $\psi(r)$, the *Lagrange density* Λ is

$$\begin{aligned} L &= \iiint_V \Lambda(r)\,dr, \\ \Lambda(r) &= E_{kin}(r) - E_{pot}(r) = \frac{\rho_0}{2}\left[|\mathrm{grad}\,\psi|^2 - \frac{1}{c_0^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right] = \frac{\rho_0}{2}\left[|\mathrm{grad}\,\psi|^2 + k_0^2\,\psi^2\right]. \end{aligned} \tag{4}$$

B.12 Adjoint Wave Equation

L and Λ here must not be confused with these symbols in *Sect. B.11*.

The wave equation is a second-order linear differential equation, with p,q possibly functions of r:

$$L\,(f(r)) = f''(r) + p \cdot f'(r) + q \cdot f(r) = 0. \qquad (1)$$

The *adjoint wave* equation is

$$\Lambda\left(g(r)\right) = g''(r) - p \cdot g'(r) + (q - p') \cdot g(r) = 0. \qquad (2)$$

Both satisfy the identity

$$g \cdot L(f) - f \cdot \Lambda(g) = \frac{d\,P(g,f)}{dr}. \qquad (3)$$

P(g,f) is the *bilinear concomitant*. If Λ(g)=0 can be solved, then solutions of L(f)=0 are

$$\begin{aligned} f_1(r) &= g(r) \cdot e^{-\int p\,dr}, \\ f_2(r) &= f_1(r) \cdot \int \frac{e^{-\int p(s)\,ds}}{g^2}\,dr. \end{aligned} \qquad (4)$$

The general solution is

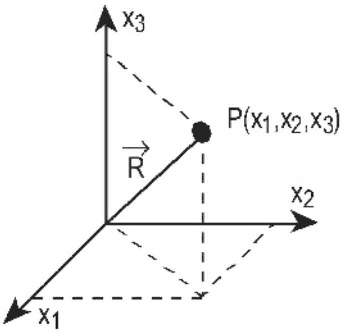
$$f(r) = a \cdot f_1 \cdot (r) + b \cdot f_2(r).$$

In the special case q(r)=dp(r)/dr,

$$\begin{aligned} g(r) &= \int e^{-\int p(s)\,ds}\,dr, \\ f_1(r) &= \frac{g(r)}{g'(r)} \quad ; \quad f_2(r) = \frac{1}{g'(r)}. \end{aligned} \qquad (5)$$

B.13 Vector and Tensor Formulation of Fundamentals

Co-ordinate systems:
Let (x_1, x_2, x_3) be a rectilinear, orthogonal coordinate system. The vector components of a point P are given by $\vec{R} = \overrightarrow{OP} = [x_1, x_2, x_3]$.



Let (u_1, u_2, u_3) be a curvilinear, orthogonal coordinate system. The co-ordinate surfaces are given by

$$u_1(x_1, x_2, x_3) = \text{const},$$

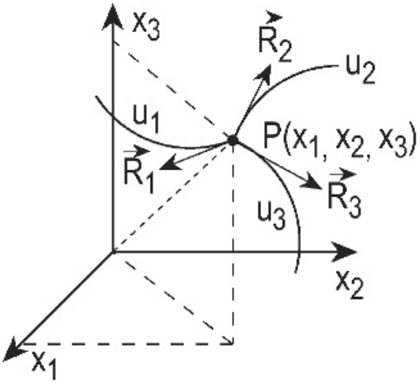
$$u_2(x_1, x_2, x_3) = \text{const},$$

$$u_3(x_1, x_2, x_3) = \text{const}.$$

The intersection of two co-ordinate surfaces is a co-ordinate line.

Tangent vectors at co-ordinate lines:

$$\vec{R}_1 = \frac{\partial \vec{R}}{\partial u_1} = \left[\frac{\partial x_1}{\partial u_1}, \frac{\partial x_2}{\partial u_1}, \frac{\partial x_3}{\partial u_1} \right],$$
$$\vec{R}_2 = \frac{\partial \vec{R}}{\partial u_2} = \left[\frac{\partial x_1}{\partial u_2}, \frac{\partial x_2}{\partial u_2}, \frac{\partial x_3}{\partial u_2} \right], \tag{1}$$
$$\vec{R}_3 = \frac{\partial \vec{R}}{\partial u_3} = \left[\frac{\partial x_1}{\partial u_3}, \frac{\partial x_2}{\partial u_3}, \frac{\partial x_3}{\partial u_3} \right].$$



Normal vectors on co-ordinate surfaces:

$$\vec{N}^1 = \mathrm{grad}\, u_1 = \left[\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial x_3} \right],$$
$$\vec{N}^2 = \mathrm{grad}\, u_2 = \left[\frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3} \right], \tag{2}$$
$$\vec{N}^3 = \mathrm{grad}\, u_3 = \left[\frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3} \right].$$

If the $\vec{R}_i$ form the basis vectors of a system of co-ordinates, then the $\vec{N}^i$ are the basis of the "reciprocal" system, with

$\vec{R}_i \bullet \vec{N}^k = \delta_{i,k} \quad ; \quad \delta_{i,k} = \begin{cases} 1\ ;\ i = k \\ 0\ ;\ i \neq k \end{cases}$ with the "dot product" or "scalar product".

Unitary tensors:

$$g_{ik} = \vec{R}_i \bullet \vec{R}_k = g_{ik} \qquad \text{covariant co-ordinates},$$
$$g_i^k = \vec{R}_i \bullet \vec{N}^k \qquad \text{mixed co-ordinates}, \tag{3}$$
$$g^{ik} = \vec{N}^i \bullet \vec{N}^k = g^{ki} \qquad \text{contravariant co-ordinates}$$

with $g^{ij} \bullet g_{jk} = g^{i1} \cdot g_{1k} + g^{i2} \cdot g_{2k} + g^{i3} \cdot g_{3k} = \delta_{i,k}$. The determinant of g_{ik} is the square of the scalar triple product $g = \det(g_{ik}) = \vec{R}_1 \vec{R}_2 \vec{R}_3$.

Vector components of a vector $\vec{a}$:

covariant components: $a_i = \vec{a} \bullet \vec{R}_i;$

contravariant components: $a^i = \vec{a} \bullet \vec{R}^i;$ (4)

vector representation in a covariant basis: $\vec{a} = a^1 \cdot \vec{R}_1 + a^2 \cdot \vec{R}_2 + a^3 \cdot \vec{R}_3 = \vec{a}^i \bullet \vec{R}_i;$

vector representation in a contravariant basis: $\vec{a} = a_1 \cdot \vec{R}^1 + a_2 \cdot \vec{R}^2 + a_3 \cdot \vec{R}^3 = \vec{a}_i \bullet \vec{R}^i,$

It follows that

$$a^i = g^{i1} a_1 + g^{i2} a_2 + g^{i3} a_3 = g^{ij} a_j ,$$
$$a_i = g_{i1} a^1 + g_{i2} a^2 + g_{i3} a^3 = g_{ij} a^j , \tag{5}$$

where the last notations use the "summation rule" (summation over multiple indices).

Thus: $\vec{N}^i = g^{ij} \vec{R}_j \quad ; \quad \vec{R}_i = g_{ij} \vec{N}^j.$ (6)

Transformation between systems of co-ordinates $U(u^1, u^2, u^3) \to V(v^1, v^2, v^3)$:

With definitions:

$$\Delta = \frac{\partial\left(v^1, v^2, v^3\right)}{\partial\left(u^1, u^2, u^3\right)} = \begin{vmatrix} \frac{\partial v^1}{\partial u^1} & \frac{\partial v^1}{\partial u^2} & \frac{\partial v^1}{\partial u^3} \\ \frac{\partial v^2}{\partial u^1} & \frac{\partial v^2}{\partial u^2} & \frac{\partial v^2}{\partial u^3} \\ \frac{\partial v^3}{\partial u^1} & \frac{\partial v^3}{\partial u^2} & \frac{\partial v^3}{\partial u^3} \end{vmatrix} = \det\left(A_k^i\right) \neq 0 \tag{7}$$

and
$$A_k^i = \frac{\partial v^i}{\partial u^k} \quad ; \quad B_i^k = \frac{\partial u^k}{\partial v^i} \tag{8}$$

$$\Delta^{-1} = \det\left(B_i^k\right) ;$$

follows that
$$\sum_{j=1}^{3} A_j^i \cdot B_k^j = \sum_{j=1}^{3} B_j^i \cdot A_k^j = \delta_{i,k} , \tag{9}$$

$$\vec{R}_i = \sum_k B_i^k \vec{R}_i = \sum_k A_i^k \vec{R}_k \quad ; \quad \vec{N}^i = \sum_k A_k^i \vec{N}^k = \sum_k B_k^i \vec{N}^k ,$$
$$A_i^k = \vec{R}_i \bullet \vec{N}^k \quad ; \quad B_i^k = \vec{R}_k \bullet \vec{N}^i \tag{10}$$

and:

$$a^i = \sum_k A_k^i a^k = \sum_k B_k^i a^k \quad ; \quad a_i = \sum_k B_i^k a_k = \sum_k A_i^k a_k. \tag{11}$$

Vector algebra:

Consider the vectors

$$\vec{a} = \sum_i a^i \cdot \vec{R}_i = \sum_i a_i \cdot \vec{R}^i;$$
$$\vec{b} = \sum_i b^i \cdot \vec{R}_i = \sum_i b_i \cdot \vec{R}^i; \tag{12}$$
$$\vec{c} = \sum_i c^i \cdot \vec{R}_i = \sum_i c_i \cdot \vec{R}^i.$$

Scalar product:

$$\vec{a} \bullet \vec{b} = \sum_{i} a^i b_i = \sum_{i} a_i b^i = \sum_{i,k} g_{ik} a^i b^k = \sum_{i,k} g^{ik} a_i b_k = \frac{-1}{g} \begin{vmatrix} g_{11} & g_{12} & g_{13} & b_1 \\ g_{21} & g_{22} & g_{23} & b_2 \\ g_{31} & g_{32} & g_{33} & b_3 \\ a_1 & a_2 & a_3 & 0 \end{vmatrix} . \tag{13}$$

Length of a vector:

$$a = \ |\vec{a}| \ = \sqrt{\vec{a} \bullet \vec{a}} = \sum_{i,k} \sqrt{g_{ik} a^i a^k} = \sum_{i,k} \sqrt{a^i a_k} = \sum_{i,k} \sqrt{g^{ik} a_i a_k} . \tag{14}$$

Cosine of the *angle between two vectors*:

$$\cos(\vec{a}, \vec{b}) = \frac{\vec{a} \bullet \vec{b}}{ab} = \frac{\sum\limits_{i,k} g_{ik} a^i b^k}{\sum\limits_{i,k} \sqrt{g_{ik} a^i a^k} \sum\limits_{i,k} \sqrt{g_{ik} b^i b^k}} . \tag{15}$$

Vector (cross) product:

$$\vec{a} \times \vec{b} = \sqrt{g} \begin{vmatrix} \vec{R}^1 & \vec{R}^2 & \vec{R}^3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \frac{1}{\sqrt{g}} \begin{vmatrix} \vec{R}_1 & \vec{R}_2 & \vec{R}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . \tag{16}$$

Vector triple product:

$$\vec{a}\vec{b}\vec{c} = \sqrt{g} \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{vmatrix} = \frac{1}{\sqrt{g}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{17}$$

Derivatives of basis vectors:

$$\text{Notation:} \quad \vec{R}_{ik} = \vec{R}_{ki} = \frac{\partial^2 \vec{R}}{\partial u_i \, \partial u_k} \quad ; \quad \vec{R}_{mn} = \vec{R}_{nm} = \frac{\partial^2 \vec{R}}{\partial v_m \, \partial v_n} . \tag{18}$$

Transformation:

$$\vec{R}_{mn} = \sum_{i,k} B^i_m B^k_n \, \vec{R}_{ik} + \sum_{s} \frac{\partial B^s_n}{\partial u_m} \vec{R}_s . \tag{19}$$

Christoffel symbols of second kind: $\left\{ {j \atop ik} \right\}$:

$$\vec{R}_{ik} = \sum_{j} \left\{ {j \atop ik} \right\} \vec{R}_j \quad \text{or} \quad \left\{ {j \atop ik} \right\} = \vec{R}_{ik} \bullet \vec{R}^j . \tag{20}$$

Transformation:

$$\left\{ {r \atop mn} \right\} = \sum_{i,j,k} B^i_m B^k_n B^j_r \left\{ {j \atop ik} \right\} + \sum_{s} \frac{\partial B^s_n}{\partial u_m} \vec{R}_s . \tag{21}$$

Christoffel symbols of first kind: $\{ikj\}$:

$$\vec{R}_{ik} = \sum_j \{ikj\}\,\vec{R}^j \quad \text{or} \quad \{ikj\} = \vec{R}_{ik} \bullet \vec{R}_k . \tag{22}$$

Transformation:

$$\{mnr\} = \sum_{i,j,k} B^i_m B^k_n B^j_r \,\{ikj\} + \sum_{s,t} g_{st} \frac{\partial B^s_n}{\partial u_m} B^t_r . \tag{23}$$

Relations with unitary tensors of the co-ordinate systems:

$$\left\{ \begin{matrix} j \\ ik \end{matrix} \right\} = \sum_s g^{sj}\,\{iks\} \quad ; \quad \{ikj\} = \sum_s g_{sj} \left\{ \begin{matrix} s \\ ik \end{matrix} \right\}, \tag{24}$$

$$\left\{ \begin{matrix} j \\ ik \end{matrix} \right\} = \frac{1}{2} \sum_s g^{sj} \left(\frac{\partial g_{is}}{\partial u_i} + \frac{\partial g_{is}}{\partial u_k} - \frac{\partial g_{ik}}{\partial u_s} \right). \tag{25}$$

Derivative of a vector along a curve:

Let a curve be defined by the equations $u^i = u^i(\tau) \;\; ; \;\; i = 1, 2, 3$,
with the parameter τ varying along the curve.

Let further be a vector $\vec{a} = \sum_i a^i \vec{R}_i = \sum_i a^i \vec{R}^i$

with functions $a^i = a^i\,(u_1(\tau), u_2(\tau), u_2(\tau)) \quad ; \quad a_i = a_i\,(u_1(\tau), u_2(\tau), u_2(\tau))$.

The complete derivative of the vector components is

$$\begin{aligned} \frac{D\,a^i}{d\tau} &= \sum_{j,k} \left(\frac{\partial a^i}{\partial u_k} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} a^j \right) \frac{du_k}{d\tau} = \sum_k \frac{du_k}{d\tau} \nabla_k a^i , \\ \frac{D\,a_i}{d\tau} &= \sum_{j,k} \left(\frac{\partial a_i}{\partial u_k} - \left\{ \begin{matrix} j \\ ik \end{matrix} \right\} a_j \right) \frac{du_k}{d\tau} = \sum_k \frac{du_k}{d\tau} \nabla_k a_i \end{aligned} \tag{26}$$

with the notation $\nabla_k a^i = \dfrac{\partial a^i}{\partial u_k} + \sum_j \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} a^j \quad ; \quad \nabla_k a_i = \dfrac{\partial a_i}{\partial u_k} - \sum_j \left\{ \begin{matrix} j \\ ik \end{matrix} \right\} a_j$. (27)

Derivative of a tensor:

$$\begin{aligned} \nabla_j a^{ik} &= \frac{\partial a^{ik}}{\partial u_j} + \sum_s \left\{ \begin{matrix} i \\ js \end{matrix} \right\} a^{sk} + \left\{ \begin{matrix} k \\ js \end{matrix} \right\} a^{is} , \\ \nabla_j a^k_i &= \frac{\partial a^k_i}{\partial u_j} + \sum_s \left\{ \begin{matrix} k \\ js \end{matrix} \right\} a^s_i - \left\{ \begin{matrix} s \\ ji \end{matrix} \right\} a^k_s , \\ \nabla_j a_{ik} &= \frac{\partial a_{ik}}{\partial u_j} - \sum_s \left\{ \begin{matrix} s \\ ji \end{matrix} \right\} a_{sk} - \left\{ \begin{matrix} s \\ jk \end{matrix} \right\} a_{is} . \end{aligned} \tag{28}$$

It holds that

$$\begin{aligned} \nabla_j \left(a^i b_k\right) &= \nabla_j \left(a^i\right) b_k + a^i \nabla_j \left(b_k\right) , \\ \nabla_j \left(\delta_{ik}\right) &= \nabla_j g_{ik} = \nabla_j g^k_i = \nabla_j g^{ik} = 0 . \end{aligned} \tag{29}$$

Orthonormal basis vectors:

Orthonormal basis vectors are called $\vec{e}_i$; $i=1,2,3$.

The basis vector components are $\vec{R}_i = H_i\,\vec{e}_i$; $\vec{R}^i = h_i\,\vec{e}_i$,
with

$$
\begin{aligned}
H_i &= |\vec{R}_i| = \left(\sum_k \left(\frac{\partial x_k}{\partial u_i}\right)^2\right)^{1/2} ; \\
h_i &= |\vec{R}^i| = \left(\sum_k \left(\frac{\partial u_k}{\partial x_i}\right)^2\right)^{1/2} = 1/H_i \quad ; \quad \vec{e}_i \bullet \vec{e}_j = \delta_{i,j};
\end{aligned}
\tag{30}
$$

$$
g_{ij} = H_i H_j\,\delta_{i,j} \quad ; \quad g^{ij} = \tfrac{1}{H_i H_j}\delta_{i,j};
$$

$$
\{g_{ij}\} = \begin{pmatrix} H_1^2 & 0 & 0 \\ 0 & H_2^2 & 0 \\ 0 & 0 & H_3^2 \end{pmatrix} \quad ; \quad g = \det(g_{ij}) = H_1 H_2 H_3 \,. \tag{31}
$$

Vector components: $\vec{a} = \sum_i a_i^*\,\vec{e}_i \quad ; \quad a_i^* = \vec{a} \bullet \vec{e}_i = \dfrac{a_i}{H_i} = H_i\,a^i.$ (32)

Scalar product: $\vec{a} \bullet \vec{b} = \sum_i \dfrac{a_i b_i}{H_i^2} = \sum_i a_i^* b_i^*.$ (33)

Vector (cross) product:

$$
\vec{a} \times \vec{b} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1\vec{e}_1 & H_2\vec{e}_2 & H_3\vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1^* & a_2^* & a_3^* \\ b_1^* & b_2^* & b_3^* \end{vmatrix} . \tag{34}
$$

Vector triple product:

$$
\vec{a}\vec{b}\vec{c} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1^* & a_2^* & a_2^* \\ b_1^* & b_2^* & b_3^* \\ c_1^* & c_2^* & c_3^* \end{vmatrix} . \tag{35}
$$

Differential operators:

The *gradient* of a scalar function is a vector:

$$
\text{grad}\,\varphi = \vec{\nabla}\varphi = \sum_i \frac{1}{H_i}\frac{\partial \varphi}{\partial u_i}\,\vec{e}_i = \sum_i (\text{grad}_i\,\varphi)\,\vec{e}_i . \tag{36}
$$

Nabla operator (a vector): $\vec{\nabla} = \left[\dfrac{1}{H_1}\dfrac{\partial}{\partial u_1}, \dfrac{1}{H_2}\dfrac{\partial}{\partial u_2}, \dfrac{1}{H_3}\dfrac{\partial}{\partial u_3}\right]$.

The *divergence* of a vector is a scalar:

$$\operatorname{div}\vec{a}=\vec{\nabla}\bullet\vec{a}=\frac{1}{H_1H_2H_3}\sum_i\frac{\partial}{\partial u_i}\left(H_1H_2H_3\cdot a^i\right)=\frac{1}{H_1H_2H_3}\sum_i\frac{\partial}{\partial u_i}\left(\frac{H_1H_2H_3}{H_i}\cdot a_i^*\right). \quad (37)$$

The *rotation* of a vector is a vector:

$$\begin{aligned}\operatorname{rot}\vec{a}=\vec{\nabla}\times\vec{a}&=\frac{1}{H_1H_2H_3}\begin{pmatrix} H_1\vec{e}_1 & H_2\vec{e}_2 & H_3\vec{e}_3\\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3\\ a_1 & a_2 & a_3\end{pmatrix}\\ &=\frac{1}{H_1H_2H_3}\begin{pmatrix} H_1\vec{e}_1 & H_2\vec{e}_2 & H_3\vec{e}_3\\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3\\ H_1a_1^* & H_2a_2^* & H_3a_3^*\end{pmatrix}\end{aligned} \quad (38)$$

The *Laplacian* of a scalar function:

$$\Delta\varphi=(\vec{\nabla}\bullet\vec{\nabla})\varphi=\frac{1}{H_1H_2H_3}\sum_i\frac{\partial}{\partial u_i}\left(\frac{H_1H_2H_3}{H_i^2}\,\frac{\partial\varphi}{\partial u_i}\right) \quad (39)$$

The Laplacian of a vector is a vector: $\Delta\vec{a}=\operatorname{grad}(\operatorname{div}\vec{a})-\operatorname{rot}(\operatorname{rot}\vec{a})$. (40)

Identities:

$$\begin{aligned}
&\operatorname{grad}(U_1U_2)=U_1\cdot\operatorname{grad}U_2+U_2\cdot\operatorname{grad}U_1,\\
&\operatorname{grad}\left(\vec{V}_1\bullet\vec{V}_2\right)=\left(\vec{V}_1\bullet\operatorname{grad}\right)\vec{V}_2+\left(\vec{V}_2\bullet\operatorname{grad}\right)\vec{V}_1+\vec{V}_1\times\operatorname{rot}\vec{V}_2+\vec{V}_2\times\operatorname{rot}\vec{V}_1,\\
&\operatorname{div}\left(U\cdot\vec{V}\right)=U\cdot\operatorname{div}\vec{V}+\vec{V}\cdot\operatorname{grad}U,\\
&\operatorname{div}\left(\vec{V}_1\times\vec{V}_2\right)=\vec{V}_2\bullet\operatorname{rot}\vec{V}_1-\vec{V}_1\bullet\operatorname{rot}\vec{V}_2,\\
&\operatorname{rot}\left(U\cdot\vec{V}\right)=U\cdot\operatorname{rot}\vec{V}+\operatorname{grad}U\times\vec{V},\\
&\operatorname{rot}\left(\vec{V}_1\times\vec{V}_2\right)=\left(\vec{V}_2\bullet\operatorname{grad}\right)\vec{V}_1-\left(\vec{V}_1\bullet\operatorname{grad}\right)\vec{V}_2+\vec{V}_1\operatorname{div}\vec{V}_2-\vec{V}_2\operatorname{div}\vec{V}_1,\\
&\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{V}\right)=\operatorname{div}\ \operatorname{rot}\vec{V}=0,\\
&\vec{\nabla}\times\left(\vec{\nabla}U\right)=\operatorname{rot}\ \operatorname{grad}U=0,\\
&\vec{\nabla}\bullet\left(\vec{\nabla}U\right)=\operatorname{div}\ \operatorname{grad}U=\Delta U.
\end{aligned} \quad (41)$$

Some co-ordinate systems (see *Sects. B.10, B.13* for more systems):

A vector: $\vec{V}$; a scalar function: U.

Cartesian co-ordinates: [x,y,z]

Line, surface, and volume elements:

$$
\begin{aligned}
(ds)^2 &= dx^2 + dy^2 + dz^2, \\
dF_x &= dy \cdot dz \; ; \; dF_y = dz \cdot dx \; ; \; dF_z = dx \cdot dy, \\
dV &= dx \cdot dy \cdot dz, \\
a_i &= a^i = a_i^*.
\end{aligned}
\tag{42}
$$

Differential operators:

$$\mathrm{grad}\, U = \frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z, \tag{43}$$

$$\mathrm{div}\, \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},$$

$$\mathrm{rot}\, \vec{V} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}, \tag{44}$$

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}. \tag{45}$$

Circular cylindrical co-ordinates*:* $[r, \varphi, z]$

A vector: $\vec{V} = [V_r, V_\varphi, V_z]$; a scalar function U.

Transformation:

$$
\begin{aligned}
x &= r \cdot \cos\varphi & r &= \sqrt{x^2 + y^2} \\
y &= r \cdot \sin\varphi & \varphi &= \arctan(y/x) \\
z &= z & z &= z
\end{aligned}
$$

Line, surface, and volume elements:

$$
\begin{aligned}
(ds)^2 &= dr^2 + r^2 d\varphi^2 + dz^2 \\
dF_r &= r\, d\varphi \cdot dz \; ; \\
dF_\varphi &= dr \cdot dz \; ; \; dF_z = r\, dr\, d\varphi \\
dV &= r\, dr \cdot d\varphi \cdot dz \\
H_r &= 1 \quad ; \quad H_\varphi = r \quad ; \quad H_z = 1
\end{aligned}
\tag{46}
$$

Differential operators:

$$
\begin{aligned}
\mathrm{grad}\, U &= \frac{\partial U}{\partial x}\vec{e}_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\vec{e}_\varphi + \frac{\partial U}{\partial z}\vec{e}_z \\
\mathrm{div}\, \vec{V} &= \frac{1}{r}\left(\frac{\partial (rV_x)}{\partial r} + \frac{\partial V_\varphi}{\partial \varphi}\right) + \frac{\partial V_z}{\partial z} \\
\mathrm{rot}\, \vec{V} &= \left(\frac{1}{r}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\vec{e}_\varphi + \frac{1}{r}\left(\frac{\partial (rV_\varphi)}{\partial r} - \frac{\partial V_r}{\partial \varphi}\right)\vec{e}_z \\
\Delta U &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}
\end{aligned}
\tag{47}
$$

Spherical co-ordinates: $[r, \varphi, \vartheta]$

A vector: $\vec{V} = [V_r, V_\varphi, V_\vartheta]$; a scalar function U.

Transformation:

$$\begin{aligned} x &= r \cdot \sin\vartheta \cdot \cos\varphi; \qquad & r &= \sqrt{x^2+y^2+z^2}, \\ y &= r \cdot \sin\vartheta \cdot \sin\varphi; \qquad & \vartheta &= \arctan\left(\sqrt{x^2+y^2}/z\right), \\ z &= r \cdot \cos\vartheta; \qquad & \varphi &= \arctan(y/x). \end{aligned} \tag{48}$$

Line, surface, and volume elements:

$$\begin{aligned} (ds)^2 &= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2, \\ dF_r &= r^2 \sin\vartheta \, d\varphi \cdot d\vartheta; \quad dF_\varphi = r\,dr \cdot d\vartheta; \quad dF_\vartheta = r\,dr\,d\varphi, \\ dV &= r^2 \sin\vartheta \cdot dr \cdot d\varphi \cdot d\vartheta, \\ H_r &= 1 \quad; \quad H_\varphi = r\sin\vartheta \quad; \quad H_\vartheta = r. \end{aligned} \tag{49}$$

Differential operators:

$$\begin{aligned} \operatorname{grad} U &= \frac{\partial U}{\partial r}\vec{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial U}{\partial\varphi}\vec{e}_\varphi + \frac{1}{r}\frac{\partial U}{\partial\vartheta}\vec{e}_\vartheta, \\ \operatorname{div}\vec{V} &= \frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, V_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial\varphi}, \\ \operatorname{rot}\vec{V} &= \frac{1}{r\sin\vartheta}\left(\frac{\partial\left(\sin\vartheta\, V_\varphi\right)}{\partial\vartheta} - \frac{\partial V_\vartheta}{\partial\varphi}\right)\vec{e}_r \\ &\quad + \left(\frac{1}{r\sin\vartheta}\frac{\partial V_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(rV_\varphi)}{\partial r}\right)\vec{e}_\vartheta + \frac{1}{r}\left(\frac{\partial(rV_\vartheta)}{\partial r} - \frac{\partial V_r}{\partial\vartheta}\right)\vec{e}_\varphi, \\ \Delta U &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 U}{\partial\varphi^2}. \end{aligned} \tag{50}$$

Differential relations of acoustics:

Field variables (overbar: total quantity; index 0: stationary value)

density:	$\bar{\rho}$	=	$\rho_0 + \rho$
pressure:	$\bar{p}$	=	$p_0 + p$
temperature:	$\bar{T}$	=	$T_0 + T$
entropy:	$\bar{S}$	=	$S_0 + S$
velocity:	$\bar{v}$	=	$v_0 + v$

η = dynamic viscosity;
μ = volume viscosity;
Λ = heat conductivity

Total time derivative:

$$\frac{D\ldots}{Dt} = \frac{\partial\ldots}{\partial t} + (\bar{v} \bullet \operatorname{grad})\ldots. \tag{51}$$

Equation of continuity:

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\cdot\operatorname{div}\bar{v} = 0. \tag{52}$$

linearised and v_0=0: $$\frac{\partial \rho}{\partial t} + \rho_0 \,\mathrm{div}\, v = 0. \tag{53}$$

Navier-Stokes equation: $$\begin{aligned}\bar{\rho}\,\frac{D\bar{v}}{Dt} &= -\mathrm{grad}\,(\bar{p} + \bar{\rho}\Phi - (\mu + \tfrac{4}{3}\eta)\,\mathrm{div}\,\bar{v}) \\ &\quad -\eta\,\mathrm{rot}\,(\mathrm{rot}\,\bar{v}),\end{aligned} \tag{54}$$

with Φ= potential of an external force per unit mass (e.g. gravity);

linearised and v_0=0; Φ=0: $$\begin{aligned}\rho_0\frac{\partial v}{\partial t} &= -\mathrm{grad}\,p + (\mu + \tfrac{4}{3}\eta)\,\mathrm{grad}\,(\mathrm{div}\,v) \\ &\quad -\eta\,\mathrm{rot}\,(\mathrm{rot}\,v)\,.\end{aligned} \tag{55}$$

Apply $$\left[\mathrm{grad}\,(\mathrm{div}\,v)\right]_x = \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \Delta v_x + [\mathrm{rot}\,(\mathrm{rot}\,v)]_x \tag{56}$$

and compose: $$v = v_\ell + v_t \quad \text{with} \quad \begin{cases}\mathrm{rot}\,v_\ell = 0 \\ \mathrm{div}\,v_t = 0\end{cases} \tag{57}$$

ℓ	=	molecular mean free path length;
κ	=	adiabatic exponent;
C_v	=	specific heat at constant volume;
C_p	=	specific heat at constant pressure;
η	=	dynamic viscosity

This leads to the following two differential equations:

$$\begin{aligned}\rho_0\frac{\partial v_\ell}{\partial t} &= -\mathrm{grad}\,p + (\mu + \tfrac{4}{3}\eta)\cdot\Delta v_\ell\,, \\ \rho_0\frac{\partial v_t}{\partial t} &= -\eta\cdot\mathrm{rot}\,(\mathrm{rot}\,v_t)\,.\end{aligned} \tag{58}$$

Energy equations:

A) *Heat conduction*: $\vec{J}_h = -\Lambda\cdot\mathrm{grad}\,T$ with Λ= heat conductibility.

From molecular gas dynamics: $$\Lambda = 1.6\,\frac{\ell\rho_0 c_0 C_v}{\sqrt{\kappa}} \approx \frac{5}{3}\eta C_v. \tag{59}$$

Energy balance with heat conduction: $$\frac{\partial T}{\partial t} = \frac{\Lambda}{\rho_0 C_p}\,\mathrm{div}\,\mathrm{grad}\,T. \tag{60}$$

B) *Viscous energy loss*

per unit volume: $$D = \sum_{i,k}\frac{\partial v_i}{\partial x_k}D_{ik}. \tag{61}$$

Shear stresses by viscosity $$\begin{aligned}D_{ii} &= -\left(\mu - \frac{2}{3}\eta\right)\mathrm{div}\,v - 2\eta\frac{\partial v_i}{\partial x_i} \\ D_{ik} &= -\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\end{aligned} \tag{62}$$

C) *Balance of internal energy* E per unit mass:

$$\frac{dE}{dt} = \frac{1}{\rho_0}\frac{dU}{dt} = \frac{1}{\rho_0}\left[\mathrm{div}\ (\Lambda\,\mathrm{grad}\,T) + D - P\cdot \mathrm{div}\,u\right] \tag{63}$$

with:	E	=	internal energy per unit mass,
	U	=	internal energy per unit volume,
	D	=	viscous energy loss,
	P	=	static pressure

D) *Balance of entropy* S per unit mass:

$$\rho\frac{dS}{dt} = \frac{D}{T} + \frac{1}{T}\mathrm{div}\ (\Lambda\,\mathrm{grad}\,T) = \frac{D}{T} + \frac{\Lambda}{T^2}|\mathrm{grad}\,T|^2 + \mathrm{div}\left(\frac{\Lambda}{T}\mathrm{grad}\,T\right). \tag{64}$$

E) *Balance of heat* Q per unit volume:

$$\frac{dQ}{dt} = \rho_0 T\frac{dS}{dt} = D + \mathrm{div}\ (\Lambda\,\mathrm{grad}\,T) = D + \frac{\Lambda}{T}|\mathrm{grad}\,T|^2 + T\cdot\mathrm{div}\left(\frac{\Lambda}{T}\mathrm{grad}\,T\right). \tag{65}$$

Equation of state:
For an ideal gas: $\bar{p} = \bar{\rho}\cdot R_0\cdot\bar{T},$ (66)
with R_0= gas constant.

Equation of state for the mass density variation $\tilde{\rho}$ in a sound wave
(sound field quantities with a $^{\sim}$, stationary quantities with a $^{-}$, atmospheric values with $_0$):

$$\tilde{\rho} = \left(\frac{\partial\bar{\rho}}{\partial\bar{p}}\right)_T\cdot\tilde{p} + \left(\frac{\partial\bar{\rho}}{\partial\bar{T}}\right)_p\cdot\tilde{T} = \kappa\,\rho_0 K_S\left(\tilde{p} - \alpha\tilde{T}\right) \xrightarrow[\text{ideal gas}]{} \frac{\rho_0}{P_0}\tilde{p} - \frac{\rho_0}{T_0}\tilde{T} \approx \frac{\rho_0}{P_0}\tilde{p}. \tag{67}$$

Equation of state for the entropy variation $\tilde{S}$ in a sound wave:

$$\tilde{S} = \left(\frac{\partial\bar{S}}{\partial\bar{p}}\right)_T\cdot\tilde{p} + \left(\frac{\partial\bar{S}}{\partial\bar{T}}\right)_p\cdot\tilde{T} = \frac{C_p}{T_0}\left(\tilde{T} - \frac{\kappa-1}{\alpha\,\kappa}\tilde{p}\right) \xrightarrow[\text{ideal gas}]{} C_p\left(\frac{\tilde{T}}{T_0} - \frac{\kappa-1}{\kappa}\frac{\tilde{p}}{P_0}\right). \tag{68}$$

Thermodynamic relations:

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \qquad \text{isothermal compressibility;} \tag{69}$$

$$K_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S \qquad \text{isotrope compressibility;} \tag{70}$$

$$ß = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \qquad \text{thermal expansion coefficient;} \tag{71}$$

$$\alpha = \left(\frac{\partial p}{\partial T}\right)_V = \frac{ß}{K_T} \qquad \text{thermal pressure coefficient;} \tag{72}$$

$$\kappa = \frac{C_p}{C_v} \qquad \text{adiabatic exponent;} \tag{73}$$

$$K_T = \kappa K_s; \tag{74}$$

$$\kappa - 1 = \frac{T\,ß^2}{K_s \rho C_p}; \tag{75}$$

$$c_0^2 = \frac{1}{\rho_0 K_s} = \frac{1}{\kappa \rho_0 K_T} \quad \text{adiabatic sound velocity } c_0; \tag{76}$$

$$K_s = \frac{1}{\kappa\,\rho_0}\left(\frac{\partial \rho}{\partial P}\right)_T = \frac{1}{\rho_0 c_0^2} \xrightarrow[\text{ideal gas}]{} \frac{1}{\kappa P_0}; \tag{77}$$

$$ß = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_P \xrightarrow[\text{ideal gas}]{} -\frac{1}{T_0}; \tag{78}$$

$$\alpha = \left(\frac{\partial p}{\partial T}\right)_V = \frac{ß}{K_t} = \frac{ß\,\rho_0 c_0^2}{\kappa} \xrightarrow[\text{ideal gas}]{} \frac{P_0}{T_0}; \tag{79}$$

$$\ell_h = \frac{\Lambda}{\rho_0 c_0\, C_p} \approx 1.6\,\frac{\ell}{\sqrt{\kappa}} \tag{80}$$

characteristic mean free molecular path length for heat conduction effects; ℓ mean free path

$$\ell_v = \frac{\eta}{\rho_0 c_0} \approx \frac{\ell}{\sqrt{\kappa}} \tag{81}$$

characteristic mean free molecular path length for shear viscosity effects;

$$\ell_v' = \frac{\mu + {}^4\!/\!_3\eta}{\rho_0 c_0} = \left(\frac{4}{3} + \frac{\mu}{\eta}\right)\ell_v \tag{82}$$

characteristic mean free molecular path length for shear and bulk viscosity;

Linearised fundamental equations for a density wave (time factor $e^{+j\omega t}$):

$$\text{with} \quad \begin{aligned} k_\rho^2 &\approx \left(\frac{\omega}{c_0}\right)^2 \left[1 + \frac{\omega}{c_0}\ell_v' - j(\kappa-1)\frac{\omega}{c_0}\,\ell_h\right]; \\ \Delta\tilde{p} &= -k_\rho^2 \tilde{p}\,; \quad \frac{\partial \tilde{p}}{\partial t} = j\,\omega \cdot \tilde{p}\,; \quad c_0^2 = \frac{1}{\rho_0 K_s}; \end{aligned} \tag{83}$$

temperature variation: $$\tilde{T} = \frac{\kappa - 1}{\alpha\,\kappa}\left(1 + j\,\frac{\omega}{c_0}\,\ell_h\right) \cdot \tilde{p}; \tag{84}$$

density variation: $$\tilde{\rho} = \frac{1}{c_0^2}\left(1 - j(\kappa-1)\frac{\omega}{c_0}\,\ell_h\right) \cdot \tilde{p}; \tag{85}$$

entropy variation: $$\tilde{S} = j\,\frac{C_p}{T_0}\,\frac{\kappa-1}{\alpha\,\kappa}\,\frac{\omega}{c_0}\,\ell_h \cdot \tilde{p}; \tag{86}$$

longitudinal particle velocity: $$v_\ell = \left(\frac{j}{\omega\rho_0} - \frac{\ell_v'}{\rho_0 c_0}\right) \cdot \operatorname{grad}\tilde{p}. \tag{87}$$

Linearised fundamental equations for a temperature wave:

$$\text{with} \qquad k_T^2 \approx \frac{-j\,\omega}{c_0 \ell_h}\;; \quad \Delta\tilde{T} = -k_T^2 \tilde{T}\;; \quad \frac{\partial \tilde{T}}{\partial t} = j\,\omega \cdot \tilde{T}; \tag{88}$$

$$\text{pressure variation:} \qquad \tilde{p} = \frac{j\,\kappa\alpha\omega}{c_0}\left(\ell'_v - \ell_h\right) \cdot \tilde{T}; \tag{89}$$

$$\text{density variation:} \qquad \tilde{\rho} = \frac{-\alpha\kappa}{c_0^2}\left(1 + \frac{j\,\kappa\omega}{c_0}\left(\ell_h - \ell'_v\right)\right) \cdot \tilde{T}; \tag{90}$$

$$\text{entropy variation:} \qquad \tilde{S} = \frac{C_p}{T_0}\left(1 + j\,(\kappa - 1)\frac{\omega}{c_0}\left(\ell_h - \ell'_v\right)\right) \cdot \tilde{T}. \tag{91}$$

B.14 Boundary Condition at a Moving Boundary

▸ *See also:* Kleinstein/Gunzburger (1976)

A boundary separates two media with density and sound velocity ρ_1, c_1 and ρ_2, c_2, respectively (other quantities are distinguished with the same indices). The boundary moves with a velocity U_0 normal to its surface. A wave is incident from the side with ρ_1, c_1. The co-ordinate normal to the surface is x, directed $1 \rightarrow 2$.

One-dimensional wave equation (in fixed co-ordinates) for density ρ_i on both sides i=1,2:

$$\left(\partial/\partial t + U_0\,\partial/\partial x\right)^2 \rho = c_i^2\,\partial^2\rho/\partial x^2 \tag{1}$$

with general solutions

$$\begin{aligned} \rho_1 &= F_1\left(\omega_1 t - \frac{\omega_1 x}{c_1(1+M_1)}\right) + G_1\left(\bar{\omega}_1 t + \frac{\bar{\omega}_1 x}{c_1(1-M_1)}\right) \\ \rho_1 &= F_2\left(\omega_2 t - \frac{\omega_2 x}{c_2(1+M_2)}\right) \end{aligned} \tag{2}$$

M_1	=	U_0/c_1;
M_2	=	U_0/c_2;
F_1	=	incident wave;
G_1	=	reflected wave;
F_2	=	transmitted wave

$$\text{Boundary conditions:} \qquad \begin{aligned} \delta v &= 0 \rightarrow \left(\partial/\partial t + U_0\,\partial/\partial x\right)\,\delta v = 0 \\ \delta p &= 0 \rightarrow \left(\partial/\partial t + U_0\,\partial/\partial x\right)\,\delta p = 0 \end{aligned} \tag{3}$$

This leads to the Doppler shifted frequencies:

$$\frac{\omega_1}{1+M_1} = \frac{\bar{\omega}_1}{1-M_1} = \frac{\omega_2}{1+M_2}. \tag{4}$$

Wave numbers:

$$k_1 = \omega_1 / (c_1 + U_0),$$

$$\bar{k}_1 = -\bar{\omega}_1 / (c_1 - U_0), \tag{5}$$

$$k_2 = \omega_2 / (c_2 + U_0).$$

Rule of conservation of wave numbers:

$$\partial k / \partial t + \partial \omega / \partial x = 0 \quad \text{or} \quad -U_0 \cdot \delta k + \delta\omega = 0 \quad \text{or} \quad U_0 = \frac{\delta\omega}{\delta k}. \tag{6}$$

Applications:

A) The boundary is a shock front in the undisturbed medium: i.e. $\omega_2 = k_2 = 0$; it follows that $U_0 = \omega_1 / k_1 =$ phase velocity in the medium $i = 1$.

B) A shock front with a jump in the state of the medium:

Shock front equation of gas dynamics: $$U_0^2 = \delta\left(p + \rho v^2\right) / \delta\rho, \tag{7}$$

in the limit of small amplitudes, i.e. $$\rho_2 \approx \rho_1;\ p_2 \approx p_1;\ v_1 \approx v_{2,} \tag{8}$$

it follows that $$U_0 = \frac{\delta\omega}{\delta k} \to \frac{d\omega}{dk} = \text{group velocity.} \tag{9}$$

C) Stationary shock front, i.e. $U_0 = 0$,

follows with $\omega_1 = \omega_2$; $$k_2 = k_1 \frac{c_1}{c_2} \frac{1 + M_1}{1 + M_2}. \tag{10}$$

D) Shock front with velocity U_0 and $U_2 =$ flow velocity behind shock:

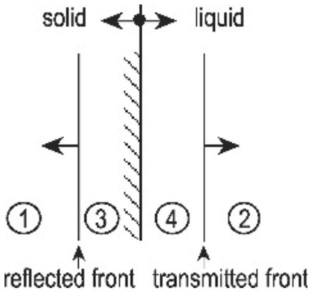
$$\omega_1 = k_1 c_1 \quad ; \quad \omega_2 = (c_2 + U_2)\, k_2\,,$$

$$\frac{\omega_2}{\omega_1} = \left(1 + \frac{U_2}{c_2}\right) \frac{1 + M_1}{1 + M_2} \quad ; \quad \frac{k_2}{k_1} = \frac{c_1}{c_2} \frac{1 + M_1}{1 + M_2}\,. \tag{11}$$

B.15 Boundary Conditions in Liquids and Solids

► *See also:* Gottlieb (1975)

Let a plane pressure front be parallel to a plane boundary. Let the density and sound velocities in a solid on both sides be ρ_i, c_i, respectively, and tensions and velocities σ_i, u_i; $i = 1,2$; $i = 1$ input side. In a liquid let $p_i = -\sigma_i$ and v_i be the sound pressure and particle velocity, respectively.



A) In a homogeneous solid with

$$\rho_1, c_1: \quad \sigma_1 - \sigma_2 = \rho_1 c_1 (u_2 - u_1). \tag{1}$$

B) In a homogeneous liquid with

$$\rho_1, c_1: \quad p_1 - p_2 = \rho_1 c_1 (v_1 - v_2). \tag{2}$$

C) At a solid-liquid interface

$$\begin{aligned} v_4 = u_3 &= \frac{-\sigma_1 - p_2 + \rho_1 c_1 \cdot u_1 + \rho_2 c_2 \cdot v_2}{\rho_1 c_1 + \rho_2 c_2}, \\ -p_4 = \sigma_3 &= \frac{\rho_2 c_2 \cdot \sigma_1 - \rho_1 c_1 \cdot p_2 - \rho_1 c_1 \cdot \rho_2 c_2 \cdot (u_1 - v_2)}{\rho_1 c_1 + \rho_2 c_2}. \end{aligned} \tag{3}$$

B.16 Corner Conditions

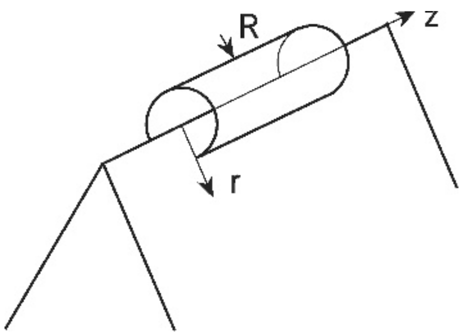
▶ *See also:* Felsen/Marcuvitz (1973)

Two-dimensional corner:

Consider a field $f(r) \cdot g(\varphi, z)$. The condition in the corner at $r = 0$ is

$$\int_0^R |f(r)|^2 \cdot r \cdot dr = \text{finite}, \tag{1}$$

from which follows $|f(r)|^2 \leq \frac{1}{r^{2(1-\alpha)}}$, for $r \to 0$, α small, positive.

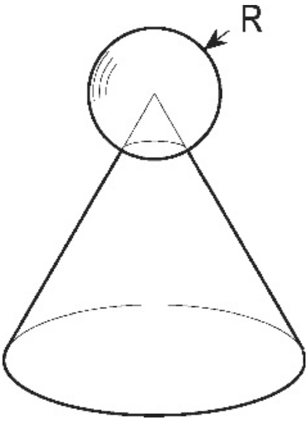


Three-dimensional corner:

Consider a field f(r). The condition in the corner at r = 0 is

$$\int_0^R |f(r)|^2 \cdot r^2 \cdot dr = \text{finite}, \tag{2}$$

from which follows $|f(r)|^2 \le \frac{1}{r^{3(1-2\alpha/3)}}$, for $r \to 0$, α small, positive.



B.17 Surface Wave at Locally Reacting Plane

► *See also:* Mechel, Vol. I, Ch. 11 (1989)

Surface waves are well known in elastic bodies (e.g. as Rayleigh waves). Here surface waves in a fluid are considered, but not those which, as a consequence of a surface wave in an elastic boundary, are produced in the fluid. Synonyms are "guided wave", because surface waves may follow curved boundaries, "creeping wave" in the scattering at cylinders and spheres as they are slow waves propagation around the scattering objects.

Consider a plane boundary in the x, z plane, with air in the half-space $y \ge 0$. Let the surface be characterised either by a surface impedance Z or by surface admittance $G = 1/Z$.

A wave of the form $$p(x, y) = P_0 \cdot e^{-\Gamma_x x} \cdot e^{-\Gamma_y y} \cdot e^{j\omega t} \tag{1}$$

satisfies the wave equation if $$\Gamma_x^2 + \Gamma_y^2 = -k_0^2, \tag{2}$$

the radiation condition if $$\Gamma_x' := \mathrm{Re}\{\Gamma_x\} \ge 0 \ ; \ \Gamma_y' := \mathrm{Re}\{\Gamma_y\} \ge 0 \tag{3}$$

and the boundary condition if $$Z_0 G \overset{!}{=} Z_0 G_y := -\left.\frac{Z_0 v_y}{p}\right|_{y=0} = \frac{j\Gamma_y}{k_0} = \frac{-\Gamma_y'' + j\Gamma_y'}{k_0}. \tag{4}$$

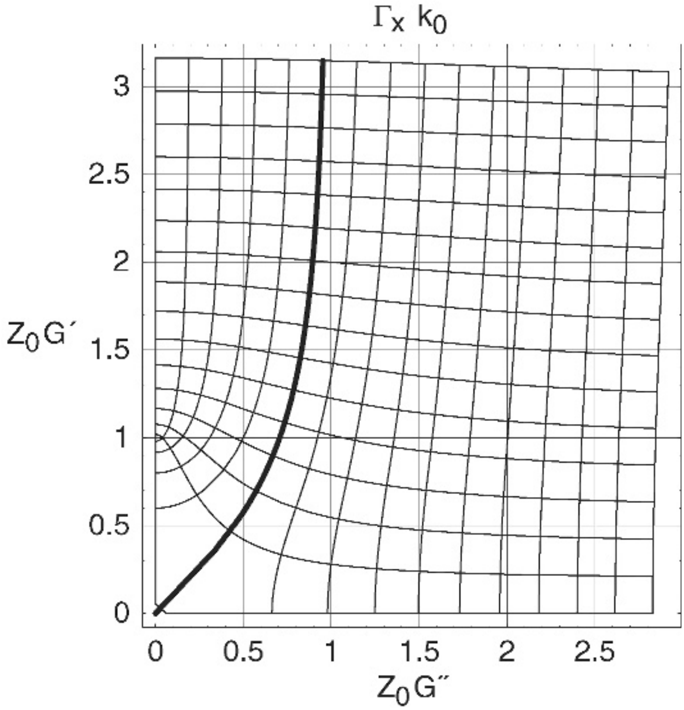
If one compares the last relation with the general admittance relation of Sect. B.3, in which the sound field is written as $p(x,y) = |p(x,y)| \cdot e^{j\varphi(x,y)}$,

which reads
$$Z_0G = \frac{1}{k_0}\left[-\frac{\partial}{\partial n}\varphi(x,y) + j\cdot\frac{\partial}{\partial n}\ln|p(x,y)|\right], \quad (5)$$

one gets
$$\Gamma_y'' = \frac{\partial}{\partial y}\varphi(x,y) \quad ; \quad \Gamma_y' = \frac{\partial}{\partial y}\ln|p(x,y)|. \quad (6)$$

These are just the definitions of Γ_y'', Γ_y' as phase and level measures. Thus a surface wave is a wave type which satisfies the fundamental equations and the boundary condition "by definition".

The graph shows curves $\mathrm{Re}\{\Gamma_x/k_0\}$ = const (nearly horizontal lines) and curves $\mathrm{Im}\{\Gamma_x/k_0\}$ = const (nearly vertical lines) in the complex plane of $Z_0G = Z_0G' + j\cdot Z_0G''$. The parameter steps $\Delta\mathrm{Re}\{\Gamma_x/k_0\}$, $\Delta\mathrm{Im}\{\Gamma_x/k_0\}$ are 0.2 over the values 0, ..., 3. The curve $\mathrm{Im}\{\Gamma_x/k_0\} = 1$ is thick. Values $\mathrm{Im}\{\Gamma_x/k_0\} < 1$ are on the left of the curve for $\mathrm{Im}\{\Gamma_x/k_0\} = 1$. The waves there are "fast"; the waves on the right of that curve are "slow". Because $\mathrm{Re}\{\Gamma_x/k_0\} > 0$, the waves are attenuated along the surface.



B.18 Surface Wave Along a Locally Reacting Cylinder

► *See also:* Mechel, Vol. I, Ch. 11 (1989)

The topic here is a surface wave *along* a cylinder, not *around* a cylinder. The cylinder has a diameter 2a and is locally reacting at its surface with the normalised radial impedance

$W = Z/Z_0 = 1/(Z_0 G)$ (G = admittance). The wave is supposed to have an axial symmetry. It is formulated as

$$p(r,z) = P_0 \cdot K_0(\Gamma_r r) \cdot e^{-\Gamma_z z} \quad ; \quad \Gamma_r^2 + \Gamma_z^2 = -k_0^2 ,$$
$$Z_0 v_r(r,z) = \frac{j}{k_0} \text{grad}_r p = \frac{-j\,\Gamma_r}{k_0} P_0 \cdot K_1(\Gamma_r r) \cdot e^{-\Gamma_z z} \tag{1}$$

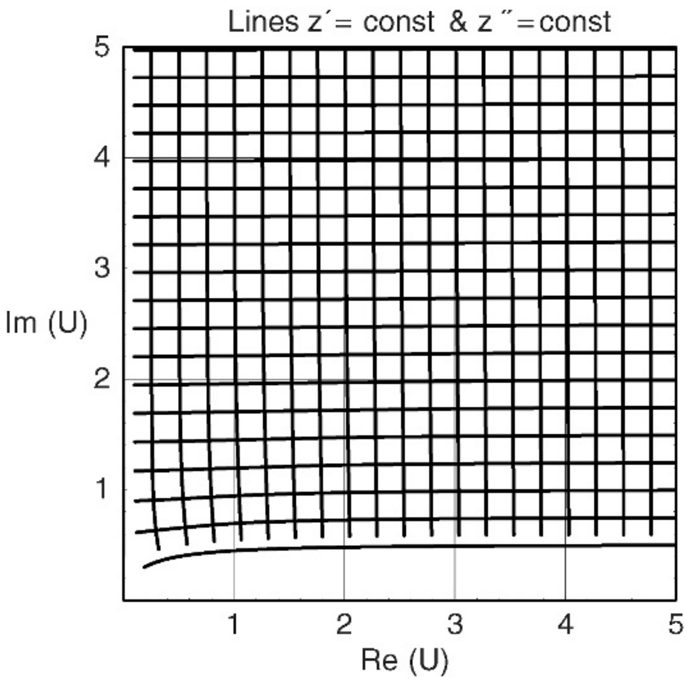
with the modified Bessel function $K_0(z)$ of the second kind of zero order. The boundary condition at $r = a$ leads to the characteristic equation for $\Gamma_r a$:

$$\Gamma_r a \cdot \frac{K_1(\Gamma_r a)}{K_0(\Gamma_r a)} = -j\,k_0 a \cdot Z_0 G = -j \cdot U. \tag{2}$$

Start values for the numerical solution are $\Gamma_r a / k_0 a \approx -j\,Z_0 G$.

With its solution the axial propagation costant Γ_z is evaluated from

$$\frac{\Gamma_z a}{k_0 a} = j\sqrt{1 + \left(\Gamma_r a / k_0 a\right)^2}. \tag{3}$$



Lines for constant real or imaginary parts of $z = \Gamma_r a = z' + j \cdot z''$ in the plane of $U = k_0 a \cdot Z_0 G$; for $z', z'' = 0$ to 5; $\Delta z = 0.2$ (The parameter values at the lines are approximately equal to the co-ordinate values of U)

B.19 Periodic Structures, Admittance Grid

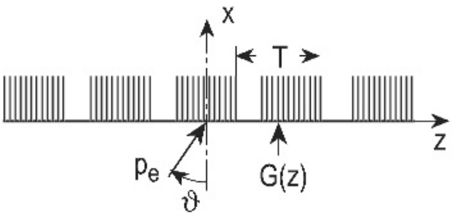
► *See also:* Mechel, Vol. I, Ch. 12 (1989)

An object with a periodic surface is a special case of an object with an inhomogeneous surface (other inhomogeneous surfaces which are amenable to analysis are those in which either the scale of the inhomogeneities and their distances is small compared to λ_0, then the average admittance is relevant, or the inhomogeneities are at large distances from each other, then scatter matrices can be set up). The method to be applied with periodic structures will be displayed in this and the next sections with some typical examples.

In principle, the quantities that describe the periodic surface, such as its surface admittance or the sound field at the surface, are synthesised with a Fourier series. The Fourier terms are waves which have different names in the literature: "spatial harmonics" (used here), "Hartree harmonics" (often used in microwave technology), or "Bloch waves" (used in solid state physics). The most important quality of these waves is their orthogonality over a period, which makes them suited for the synthesis of field quantities.

Plane surface with periodic admittance function G(z) *and incident plane wave*:

Consider a plane with a periodic surface admittance G(z) and a plane wave p_e incident with a polar angle ϑ (the wave vector in the x,z plane).



The plane wave p_e is

$$p_e(x,z) = P_e \cdot e^{-j(k_x x + k_z z)}$$
$$k_x = k_0 \cos\vartheta \quad ; \quad k_z = k_0 \sin\vartheta. \tag{1}$$

The field in the half-space $x \leq 0$ is written $p(x,z) = p_e(x,z) + p_s(x,z)$ with the scattered wave p_s formulated as

$$p_s(x,z) = \sum_{n=-\infty}^{+\infty} A_n \cdot e^{\gamma_n x} \cdot e^{-j\beta_n z} \quad ; \quad \gamma_n^2 = \beta_n^2 - k_0^2 \quad ; \quad \mathrm{Re}\{\gamma_n\} \geq 0. \tag{2}$$

The relation for γ_n ensures the (term-wise) satisfaction of the wave equation, and the condition for γ_n the satisfaction of Sommerfeld's far field condition. The scattered field p_s can be written as a product $p_s(x,z) = e^{-j\beta_0 z} \cdot S(x,z)$ of a propagation factor $e^{-j\beta_0 z}$ and a factor S(x,z) which must be periodic in z: $S(x,z) = S(x, z+T)$.

This gives for the wave numbers in the z direction

$$\beta_n = \beta_0 + n\frac{2\pi}{T} \quad ; \quad n = 0, \pm 1, \pm 2, \ldots . \tag{3}$$

The spatial harmonic with the order n=0 evidently must agree in its z pattern with the trace of the incident wave at the surface: $ß_0 = k_z = k_0 \sin\vartheta$.

Thus: $$ß_n = k_0 \left(\sin\vartheta + n\,\lambda_0/T\right) \quad ; \quad \gamma_n^2 = k_0^2 \left[\left(\sin\vartheta + n\,\lambda_0/T\right)^2 - 1\right], \tag{4}$$

and the sound field in $x \leq 0$:

$$\begin{aligned} p(x,z) = \Bigg[& P_e \cdot e^{-j\,k_0 x\cdot\cos\vartheta} + A_0 \cdot e^{+j\,k_0 x\cdot\cos\vartheta} + \\ & + \sum_{n\neq 0} A_n \cdot e^{k_0 x\sqrt{(\sin\vartheta + n\,\lambda_0/T)^2 - 1}} \cdot e^{-j\,(2n\pi/T)\,z} \Bigg] \cdot e^{-j\,k_0 z\,\sin\vartheta}. \end{aligned} \tag{5}$$

The second term in the brackets is a homogeneously reflected plane wave; the terms in the sum are higher scattered waves. The exponent of the exponential factor with x under the sum must be zero or imaginary if the spatial harmonic should extend to infinity, i.e. the harmonic is "radiating". The condition for radiating harmonics (order n_s) is

$$-\frac{T}{\lambda_0}\,(1 + \sin\vartheta) \leq n_s \leq \frac{T}{\lambda_0}\,(1 - \sin\vartheta)\,. \tag{6}$$

At (and near) the lower limit the harmonic propagates in the opposite z direction of the incident wave; at (and near) the upper limit the harmonic propagates in the same z direction as the incident wave (if the limits are reached exactly, the harmonic propagates as a plane wave parallel to the surface). The lower limit is attained (or surpassed) the first time for $n_s < 0$ with $1/2 \leq T/\lambda_0 \leq 1$; the upper limit for $n_s > 0$ with $1 \leq T/\lambda_0 < \infty$. A radiating harmonic does not exist for $T/\lambda_0 < 1/2$. The non-radiating harmonics shape the near field at the surface.

The amplitudes A_n are determined from the boundary condition $Z_0 v_x(0,z) \overset{!}{=} G(z)\cdot p(0,z)$ at the surface. One expands:

$$G(z) = \sum_{n=-\infty}^{+\infty} g_n \cdot e^{-j\,(2n\pi/T)\,z} \quad ; \quad g_n = \frac{1}{T}\int_{-T/2}^{+T/2} G(z)\cdot e^{+j\,(2n\pi/T)\,z}\,dz, \tag{7}$$

or, alternatively,

$$\begin{aligned} G(z) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(2n\pi\,z/T\right) + j\cdot b_n \cdot \sin\left(2n\pi\,z/T\right)\right]; \\ a_n &= \frac{2}{T}\int_{-T/2}^{+T/2} G(z)\cdot\cos\left(2n\pi\,z/T\right)\,dz; \\ b_n &= \frac{2}{T}\int_{-T/2}^{+T/2} G(z)\cdot\sin\left(2n\pi\,z/T\right)\,dz; \\ g_n &= \frac{1}{2}\left[a_n + j\cdot b_n\right] \quad ; \quad n = \pm 1,\ \pm 2, \ldots \quad ; \quad g_0 = \frac{a_0}{2}. \end{aligned} \tag{8}$$

The boundary condition gives for $m = 0$

$$A_0\left(g_0 + \cos\vartheta\right) + \sum_{n\neq 0} A_n \cdot g_{-n} = P_e \cdot \left(\cos\vartheta - g_0\right) \tag{9}$$

and for m≠0

$$A_0\, g_m + \sum_{n\neq 0} A_n \cdot \left[g_{m-n} - j\,\delta_{m,n}\sqrt{\left(\sin\vartheta + m\,\lambda_0/T\right)^2 - 1}\right] = -P_e \cdot g_m, \tag{10}$$

with the Kronecker symbol $\delta_{m,n}$. This is a linear, inhomogeneous system of equations for the amplitudes A_n.

The special case G(z) = const leads to $A_{n\neq 0} = 0$ and the known reflection factor

$$\frac{A_0}{P_e} = r_0 = \frac{\cos\vartheta - g_0}{\cos\vartheta - g_0}. \tag{11}$$

The absorbed effective power (on a period length) is

$$\Pi' = \frac{T}{2Z_0}\left[\left(|P_e|^2 - |A_0|^2\right)\cos\vartheta - \sum_{n_s\neq 0} |A_{n_s}|^2 \sqrt{1-\left(\sin\vartheta + n_s\,\lambda_0/T\right)^2}\,\right]. \tag{12}$$

Referring this to the incident effective power $\quad \Pi'_e = \dfrac{T}{2Z_0}|P_e|^2 \cdot \cos\vartheta \qquad (13)$

gives the absorption coefficient:

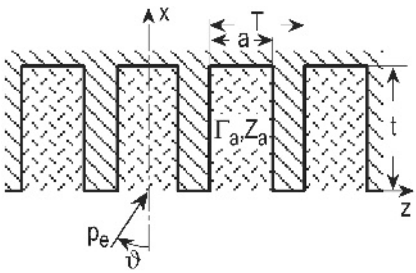
$$\alpha(\vartheta) = \frac{\Pi'}{\Pi'_e} = 1 - \left|\frac{A_0}{P_e}\right|^2 - \frac{1}{\cos\vartheta}\sum_{n_s\neq 0}\left|\frac{A_{n_s}}{P_e}\right|^2 \sqrt{1-\left(\sin\vartheta + n_s\,\lambda_0/T\right)^2}. \tag{14}$$

The last term is a correction for the absorption of a homogeneous surface (represented by the first two terms) due to the structured surface; only radiating spatial harmonics enter into this correction. This is plausible because only the radiating harmonics transport energy into the far field.

Grooved wall with narrow, absorber-filled grooves:

Consider, as a simple example, a plane wall with rectangular grooves, width a, distance T, and depth t, the grooves being filled with a porous material with characteristic values Γ_a, Z_a.

A plane wave p_e is incident under a polar angle ϑ (the wave vector in the x,z plane).



The grooves are narrow ($a < \lambda_0/4$) so that only a plane wave can be assumed to exist in the grooves. Then the grooves can be characterised by an admittance G_s in the groove orifice, and the admittance of the arrangement is $G(z) = 0$ in front of the ribs between the grooves:

$$G_s = \frac{1}{Z_{an}} \tanh(k_0 t \cdot \Gamma_{an}) \quad ; \quad \Gamma_{an} = \Gamma_a / k_0 \quad ; \quad Z_{an} = Z_a / Z_0. \tag{15}$$

The Fourier coefficients of $G(z)$ are

$$g_0 = \frac{a}{T} G_s \quad ; \quad g_m = g_{-m} = \frac{a}{T} G_s \frac{\sin(m\pi a/T)}{m\pi a/T} \quad ; \quad m = 1, 2, 3, \ldots . \tag{16}$$

The system of equations for the amplitudes A_n of the spatial harmonics becomes (with $P_e = 1$)

$m = 0$:

$$A_0 \cdot \left(s(0) + \frac{T}{a} Z_s \cos(\vartheta) \right) + \sum_{n \geq 1} (A_n + A_{-n}) \cdot s(n) = \frac{T}{a} Z_s \cos(\vartheta) - s(0)$$

$m = \pm 1, \pm 2, \ldots$:

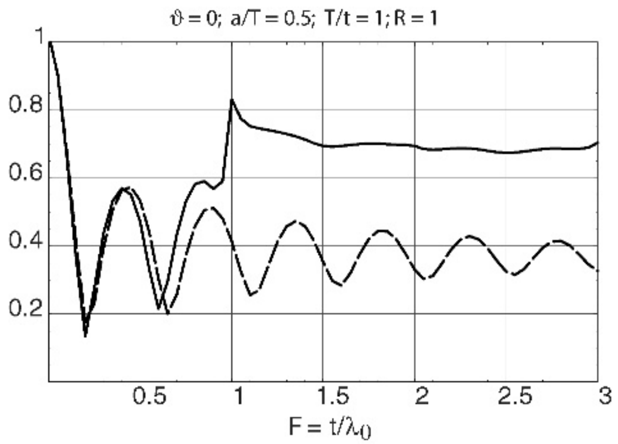
$$A_0 \cdot s(m) + \sum_{n=\pm 1, \pm 2, \ldots} A_n \left[s(m-n) + \delta_{m,n} \frac{T}{a} Z_s \begin{cases} \sqrt{1 - (\sin\vartheta + m_s \lambda_0 / T)^2} \\ -j\sqrt{(\sin\vartheta + m_s \lambda_0 / T)^2 - 1} \end{cases} \right] = -s(m) \tag{17}$$

with $Z_s = 1/G_s$ and the abbreviations:

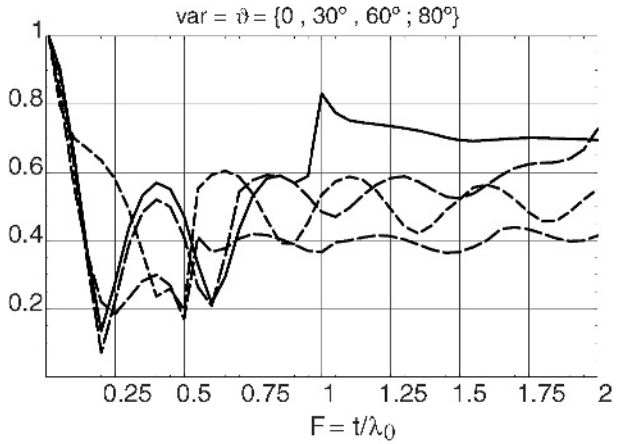
$$s(m) = \frac{\sin(m\pi a/T)}{m\pi a/T} \quad ; \quad s(0) = 1 \quad ; \quad s(-m) = s(m). \tag{18}$$

The upper form after the brace holds if $m \leq m_s$, with m_s the limit of orders of radiating harmonics; otherwise the lower form holds.

For $a/T \to 1$ it follows that $A_{n \neq 0} = 0$ and $A_0 = (Z_s \cos\vartheta - 1)/(Z_s \cos\vartheta + 1)$. This is the analytic justification for making a homogeneous (bulk reacting) absorber layer locally reacting by thin partition walls with small distances.



Magnitude of the reflection factor $|r|$ of wall with grooves for normal incidence. Full line: periodic surface; dashed: homogeneous



Magnitude of the reflection factor |r| of wall with grooves for oblique incidence, as a periodic surface for a list of ϑ values (dashes become shorter for increasing list place). $a/T = 0.5; T/t = 1; R = 1$

The examples shown below use the parameters $F = f \cdot t/c_0 = t/\lambda_0; R = \Xi \cdot t/Z_0$ with the flow resistivity Ξ of the porous material (glass fibres) in the grooves; a/T; T/t. The first graph shows the magnitude of the reflection factor |r| for a homogeneous surface (dashed) and a periodic surface (full).

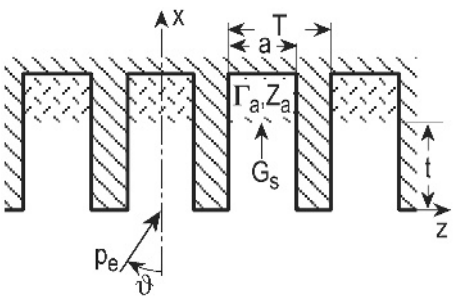
B.20 Plane Wall with Wide Grooves

► *See also:* Mechel, Vol. I, Ch. 12 (1989)

In contrast to the previous *Sect. B.19* the grooves are no longer narrow; higher modes may exist in them.

The ground of the grooves is supposed to be terminated with an admittance G_s (e.g. produced by a porous layer there).

Where possible, the relations are taken from the previous *Sect. B.19*.



The grooves are numbered $\nu = 0, \pm 1, \pm 2, \ldots$ and a co-ordinate $z_\nu = z - \nu \cdot T$ is used in the νth groove with $-a/2 \le z_\nu \le +a/2$. The field in the groove is formulated as

$$p_k(x, z_\nu) = e^{-j\,ß_0 \cdot \nu T} \sum_{m \ge 0} \left[B_m \cdot e^{-j\,\kappa_m x} + C_m \cdot e^{+j\,\kappa_m x}\right] \cdot \cos\left(m\pi\left(\frac{z_\nu}{a} - \frac{1}{2}\right)\right) \tag{1}$$

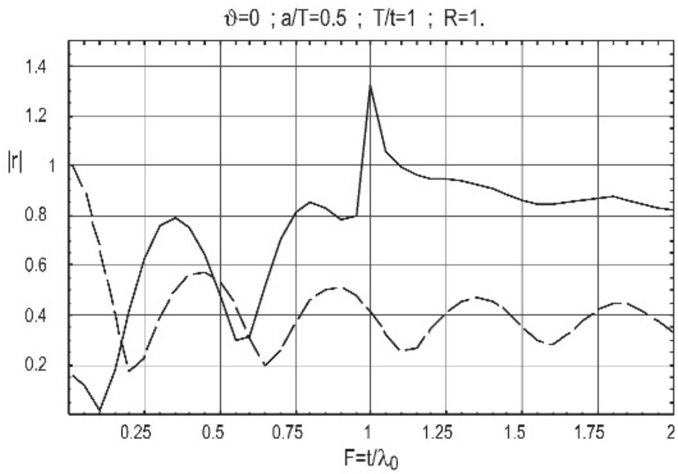
with

$$\kappa_m = \begin{cases} \sqrt{k_0^2 - (m\pi/a)^2} \ge 0 & ; \quad k_0 \ge m\pi/a\,, \\ -j\sqrt{(m\pi/a)^2 - k_0^2} \ge 0 & ; \quad k_0 < m\pi/a\,. \end{cases} \tag{2}$$

The amplitudes C_m of the groove modes reflected at the ground are

$$C_m = -B_m \cdot \frac{G_s - \kappa_m/k_0}{G_s + \kappa_m/k_0} \cdot e^{-2j\,\kappa_m t} := B_m \cdot R_m. \tag{3}$$

The R_m are modal reflection factors "measured" in the groove orifice.



Magnitude of the reflection factor $|r|$ of a wall with wide grooves. The grooves are completely filled with glass fibre material; t = groove depth; $R = \Xi \cdot t/Z_0$. Full: with spatial harmonics; dashed: homogeneous

The boundary conditions in the plane x=0 lead to the inhomogeneous linear system of equations ($m = 0, \pm 1, \pm 2, \ldots$) for the amplitudes A_n of the spatial harmonics, which exist in the half-space $x < 0$ (with $\delta_{n.m}$= Kronecker symbol; $\delta_0 = 1$; $\delta_{m>0} = 2$):

$$\begin{aligned} &\sum_{n=-\infty}^{+\infty} A_n \left[\frac{a}{2T} \sum_{\mu=0}^{\infty} \frac{\delta_\mu}{2} \frac{\kappa_\mu}{k_0} \frac{1 - R_\mu}{1 + R_\mu} (-1)^\mu s_{-\mu,n} \cdot s_{\mu,m} - \delta_{n,m} \frac{j\,\gamma_m}{k_0}\right] \\ &\quad = P_e \cdot \left[\delta_{0,m} \cdot \cos\vartheta - \frac{a}{2T} \sum_{\mu=0}^{\infty} \frac{\delta_\mu}{2} \frac{\kappa_\mu}{k_0} \frac{1 - R_\mu}{1 + R_\mu} (-1)^\mu s_{-\mu,0} \cdot s_{\mu,m}\right] \end{aligned} \tag{4}$$

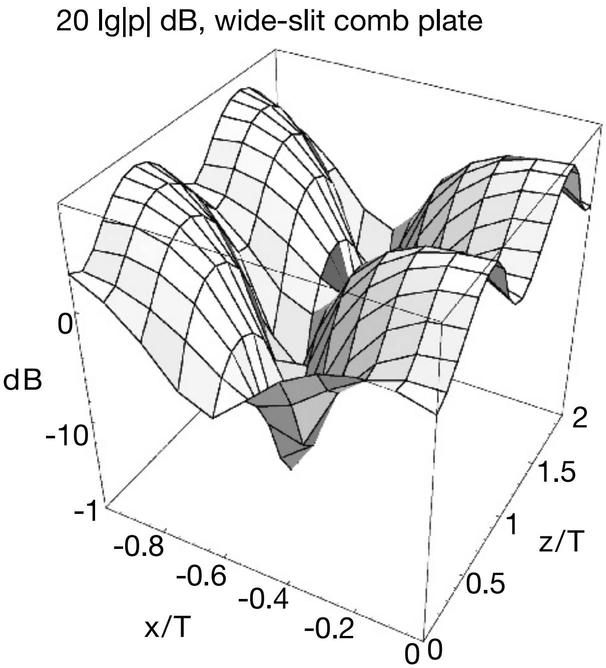
with the abbreviation

$$s_{m,n} = e^{j\,m\pi/2}\left[\frac{\sin\,(m\pi/2 - ß_n a/2)}{m\pi/2 - ß_n a/2} + (-1)^m \frac{\sin\,(m\pi/2 + ß_n a/2)}{m\pi/2 + ß_n a/2}\right]. \tag{5}$$

The amplitudes B_m follow with the A_n from

$$B_m = \frac{(-1)^m \delta_m}{2\,(1 + R_m)}\left[P_e \cdot s_{-m,0} + \sum_{n=0,\pm 1,\pm 2\ldots} A_n \cdot s_{-m,n}\right]. \tag{6}$$

The reflection factor r of the arrangement follows with the A_n as in the previous *Sect. B.19*.



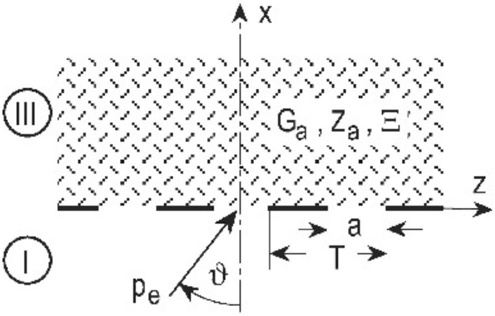
Sound pressure level $20 \cdot \lg|p|$ in front of a wall (at $x/T = 0$) with wide grooves, completely filled with glass fibre material. $F = T/\lambda_0 = 0.75$; $\vartheta = 45°$; $a/T = 0.5$; $T/t = 0.25$; $R = \Xi \cdot t/Z_0 = 1$. One spatial harmonic is radiating, therefore the periodicity extends to far distances

B.21 Thin Grid on Half-Infinite Porous Layer

► *See also:* Mechel, Vol. I, Ch. 12 (1989)

A thin grid with slits of width a at mutual distance T covers a half space of porous absorber material with flow resistivity Ξ and characteristic values Γ_a, Z_a (or, in a normalised form, $\Gamma_{an} = \Gamma_a/k_0$, $Z_{an} = Z_a/Z_0$).

A plane wave p_e is incident at a polar angle ϑ (wave vector in the x,z plane).



Field formulation in the zone I ($x \le 0$):

$$p_I(x,z) = p_e(x,z) + p_s(x,z) = P_e \cdot e^{-j(k_x x + k_z z)} + \sum_{n=-\infty}^{+\infty} A_n \cdot e^{\gamma_n x} \cdot e^{-j\beta_n z}; \tag{1}$$

$$\begin{aligned} &k_x = k_0 \cos\vartheta \quad ; \quad k_z = k_0 \sin\vartheta \quad ; \quad k_x^2 + k_z^2 = k_0^2, \\ &\beta_0 = k_z = k_0 \sin\vartheta \quad ; \quad \beta_n = \beta_0 + 2n\pi/T = k_0(\sin\vartheta + n\lambda_0/T), \\ &\gamma_n^2 = \beta_n^2 - k_0^2 \quad ; \quad \gamma_0 = j k_0 \cos\vartheta \quad ; \gamma_n = k_0\sqrt{(\sin\vartheta + n\lambda_0/T)^2 - 1}. \end{aligned} \tag{2}$$

Radiating spatial harmonics with order n_s in the limits:

$$-\frac{T}{\lambda_0}(1 + \sin\vartheta) \le n_s \le \frac{T}{\lambda_0}(1 - \sin\vartheta). \tag{3}$$

Field formulation in the zone III ($x \ge 0$):

$$\begin{aligned} p_{III}(x,z) &= \sum_{n=-\infty}^{+\infty} D_n \cdot e^{-\varepsilon_n x} \cdot e^{-j\beta_n z} = e^{-j\beta_0 z} \sum_{n=-\infty}^{+\infty} D_n \cdot e^{-\varepsilon_n x} \cdot e^{-j(2n\pi/T)z}, \\ v_{IIIx}(x,z) &= \frac{k_0}{\Gamma_a Z_a} e^{-j\beta_0 z} \sum_{n=-\infty}^{+\infty} D_n \frac{\varepsilon_n}{k_0} \cdot e^{-\varepsilon_n x} \cdot e^{-j(2n\pi/T)z}, \end{aligned} \tag{4}$$

$$\varepsilon_n^2 = \beta_n^2 + \Gamma_a^2 \quad ; \quad \frac{\varepsilon_n}{k_0} = \sqrt{(\sin\vartheta + n\lambda_0/T)^2 + \Gamma_{an}^2}. \tag{5}$$

The boundary conditions on the front and back side of the grid, together with the orthogonality of the spatial harmonics, lead to the following linear inhomogeneous system of equations ($m = 0, \pm 1, \pm 2, \ldots$):

$$\text{m=0:} \quad A_0\left(\frac{T}{a}\Gamma_{an}Z_{an}\cos\vartheta + \frac{\varepsilon_0}{k_0}\right) + \sum_{n\neq 0} A_n \frac{\varepsilon_n}{k_0} s(n) = P_e\left(\frac{T}{a}\Gamma_{an}Z_{an}\cos\vartheta - \frac{\varepsilon_0}{k_0}\right); \tag{6}$$

$$\text{m}\neq\text{0:} \quad A_0 \cdot \frac{\varepsilon_0}{k_0} s(m) + \sum_{n\neq 0} A_n\left[\frac{\varepsilon_n}{k_0} s(m-n) - \delta_{m,n} \cdot \frac{T}{a}\Gamma_{an}Z_{an}\left(j\frac{\gamma_m}{k_0}\right)\right] = -P_e \cdot \frac{\varepsilon_0}{k_0} s(m); \tag{7}$$

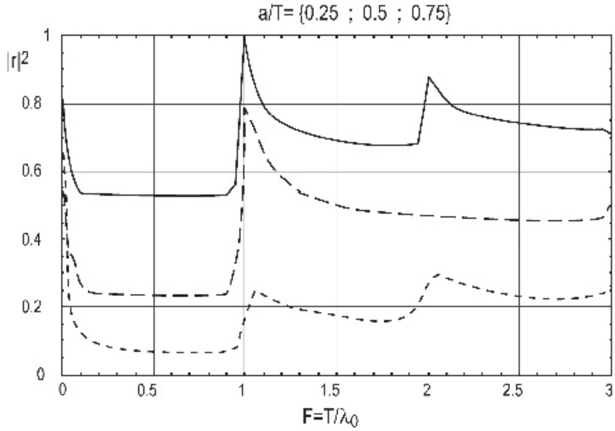
with the Kronecker symbol $\delta_{m,n}$ and the abbreviation

$$s(n) := \frac{\sin(n\pi a/T)}{n\pi a/T}. \tag{8}$$

The amplitudes D_n follow from $D_0 = P_e + A_0$; $D_{n\neq 0} = A_n$.
Special case $a/T \to 0$: (i.e. hard plane) it follows that $A_0 = P_e$; $A_{n\neq 0} = 0$;
special case $a/T \to 1$: (i.e. open absorber) it follows that $A_{n\neq 0} = 0$ and

$$\frac{A_0}{P_e} = \left(\Gamma_{an} Z_{an} \frac{\cos\vartheta}{\sqrt{\sin^2\vartheta + \Gamma_{an}^2}} - 1\right) \Big/ \left(\Gamma_{an} Z_{an} \frac{\cos\vartheta}{\sqrt{\sin^2\vartheta + \Gamma_{an}^2}} + 1\right), \tag{9}$$

which is just the reflection factor at a semi-infinite absorber layer.



Reflection coefficient $|r|^2$ for a thin grid on an infinite glass fibre layer for different ratios a/T (dashes are shorter with increasing values).
$\vartheta = 0$; $R = \Xi T/Z_0 = 1$

Special case: Ignore all higher spatial harmonics, i.e. $A_{n\neq 0}=0$:

$$\frac{A_0}{P_e} = \left(\frac{T}{a}\Gamma_{an} Z_{an} \frac{\cos\vartheta}{\sqrt{\sin^2\vartheta + \Gamma_{an}^2}} - 1\right) \Big/ \left(\frac{T}{a}\Gamma_{an} Z_{an} \frac{\cos\vartheta}{\sqrt{\sin^2\vartheta + \Gamma_{an}^2}} + 1\right). \tag{10}$$

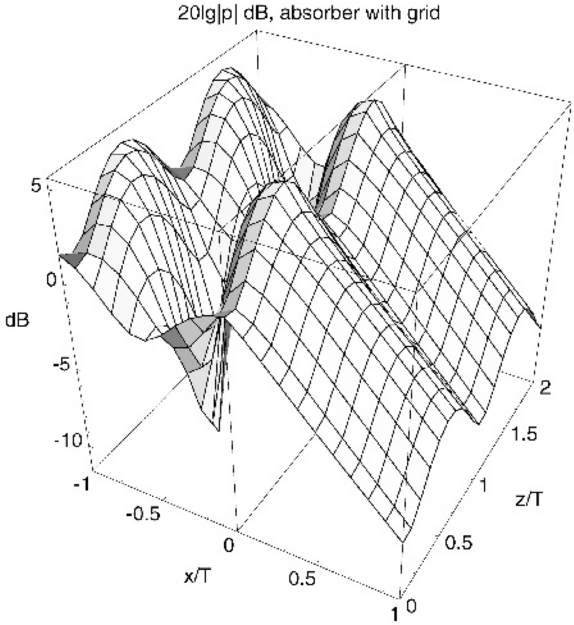
Special case: The material in zone III is air: i.e. $\Gamma_{an} = j$; $Z_{an} = 1$; $\varepsilon_n/k_0 \to \gamma_n/k_0$:

$$\begin{aligned} &\sum_n A_n \cdot j\frac{\gamma_n}{k_0}\left[\ s(m-n) + \delta_{m,n} \cdot \frac{T}{a}\ \right] \\ &\quad = P_e \cos\vartheta \cdot \left[\ s(m) + \delta_{0,m} \cdot \frac{T}{a}\ \right] \quad ; \quad m = 0,\ \pm 1,\ \pm 2, \ldots\ . \end{aligned} \tag{11}$$

The reflection coefficient $|r|^2$ is evaluated by

$$|r|^2 = \left|\frac{A_0}{P_e}\right|^2 + \frac{1}{\cos\vartheta} \sum_{n=n_s\neq 0} \left|\frac{A_n}{P_e}\right|^2 \sqrt{1 - (\sin\vartheta + n\lambda_0/T)^2}. \tag{12}$$

Parameters in the examples shown below are ϑ, a/T, $R = \Xi \cdot T/Z_0$, $F = T/\lambda_0$.



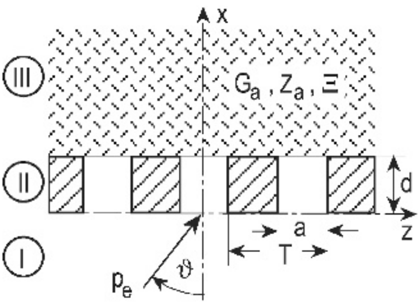
Sound pressure level in front of ($x/T < 0$) and in ($x/T > 0$) the absorber, covered with a thin grid.
$\vartheta = 45°$; $T/\lambda_0 = 0.75$; $a/T = 0.5$; $R = \Xi T/Z_0 = 1$

B.22 Grid of Finite Thickness with Narrow Slits on Half-Infinite Porous Layer

▶ *See also:* Mechel, Vol. I, Ch. 12 (1989)

In contrast to the object in the previous *Sect. B.21*, the grid now has a finite thickness d, and the slits of the grid are assumed to be narrow so that only a plane wave must be assumed in the slits. The slits form the new zone II. The slit and grid period at $z = 0$ can be taken as the representatives for the other slits.

The field formulations in zones I and III remain as in *Sect. B.21*.



The field in the νth slit, with $z = \nu \cdot T + z_\nu$; $\nu = 0, \pm1, \pm2, \ldots$, is formulated as

$$\begin{aligned} p_{II}(x, z_\nu) &= e^{-j\,ß_0 z_\nu}\left[B \cdot e^{-jk_0 x} + C \cdot e^{+jk_0 x}\right] \\ Z_0 v_{xII}(x, z_\nu) &= e^{-j\,ß_0 z_\nu}\left[B \cdot e^{-jk_0 x} - C \cdot e^{+jk_0 x}\right] \end{aligned} \tag{1}$$

The boundary conditions lead to

$$\begin{aligned} \frac{B}{P_e} &= \frac{\sin(ß_0 a/2)}{ß_0 a/2}\,\frac{(1+S_a)\,e^{+j\,k_0 d}}{(S+S_a)\,\cos(k_0 d) + j\,(1+SS_a)\,\sin(k_0 d)}\,, \\ \frac{C}{P_e} &= -\frac{\sin(ß_0 a/2)}{ß_0 a/2}\,\frac{(1-S_a)\,e^{-j\,k_0 d}}{(S+S_a)\,\cos(k_0 d) + j\,(1+SS_a)\,\sin(k_0 d)} \end{aligned} \tag{2}$$

with the abbreviations

$$S := \frac{a}{T}\sum_{n=-\infty}^{+\infty} j\frac{k_0}{\gamma_n}\left(\frac{\sin(ß_n a/2)}{ß_n a/2}\right)^2 \quad ; \quad S_a := \frac{a}{T}\,\frac{\Gamma_a Z_a}{k_0 Z_0}\sum_{n=-\infty}^{+\infty}\frac{k_0}{\varepsilon_n}\left(\frac{\sin(ß_n a/2)}{ß_n a/2}\right)^2 . \tag{3}$$

With B and C it follows that

$$\begin{aligned} A_0 &= P_e - (B-C)\,\frac{a}{T\,\cos\vartheta}\,\frac{\sin(ß_0 a/2)}{ß_0 a/2}\,, \\ A_m &= -j\,(B-C)\,\frac{a}{T}\,\frac{k_0}{\gamma_m}\,\frac{\sin(ß_m a/2)}{ß_m a/2} \quad ; \quad m = \pm1,\ \pm2, \ldots\,, \end{aligned} \tag{4}$$

$$D_m = \frac{a}{T}\,\frac{\Gamma_a Z_a}{k_0 Z_0}\left[B \cdot e^{-jk_0 d} - C \cdot e^{+jk_0 d}\right]\frac{k_0}{\varepsilon_m}\,\frac{\sin(ß_m a/2)}{ß_m a/2} \quad ; \quad m = 0,\ \pm1,\ \pm2, \ldots . \tag{5}$$

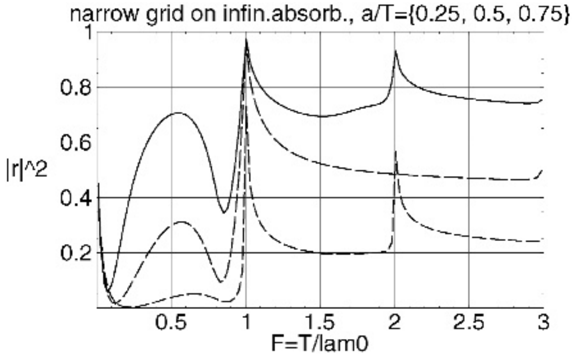
The reflection coefficient $|r|^2$ follows from

$$\begin{aligned} |r|^2 = &\left|\,1 - \frac{a/T}{\cos\vartheta}\,\frac{\sin(ß_0 a/2)}{ß_0 a/2}\,(B-C)\right|^2 \\ &+ \frac{a/T}{\cos\vartheta}\,|B-C|^2\left[\mathrm{Re}\{S\} - \frac{a/T}{\cos\vartheta}\left(\frac{\sin(ß_0 a/2)}{ß_0 a/2}\right)^2\right] . \end{aligned} \tag{6}$$

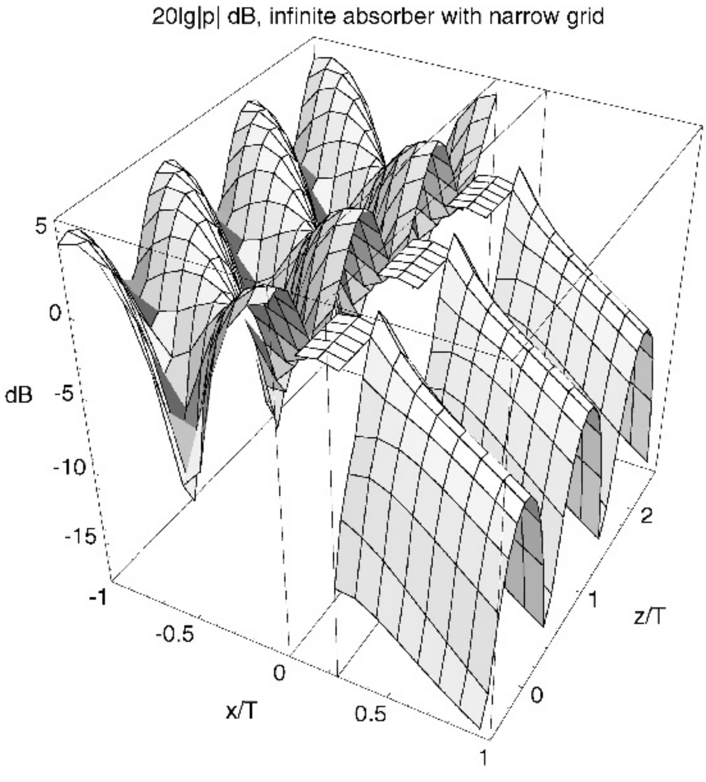
In the special case of air, instead of a porous material, behind the grid

$$\begin{aligned} \frac{B}{P_e} &= 2\frac{\sin(ß_0 a/2)}{ß_0 a/2}\,\frac{(1+S)\,e^{+j\,k_0 d}}{(1+S)^2\,e^{+j\,k_0 d} - (1-S)^2\,e^{-j\,k_0 d}}\,, \\ \frac{C}{P_e} &= -2\frac{\sin(ß_0 a/2)}{ß_0 a/2}\,\frac{(1-S)\,e^{-j\,k_0 d}}{(1+S)^2\,e^{+j\,k_0 d} - (1-S)^2\,e^{-j\,k_0 d}}\,. \end{aligned} \tag{7}$$

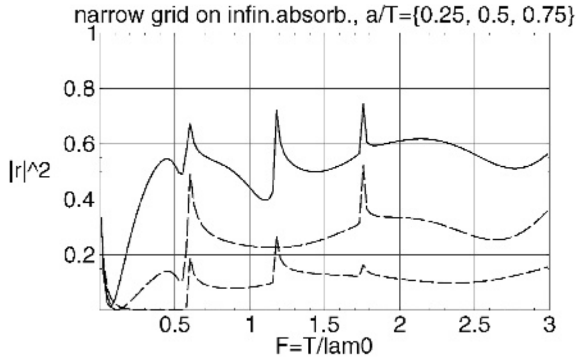
The parameters in the following examples are ϑ, a/T, d/T, $R = \Xi \cdot T/Z_0$, $F = T/\lambda_0$ (equivalences: |r|^2 $\to$ $|r|^2$; lam0 $\to$ λ_0).



Reflection coefficient $|r|^2$ of a porous half-space covered with a grid with finite thickness for some ratios a/T (the dash becomes shorter for larger a/T).
$\vartheta = 0; d/T = 0.25; R = \Xi T/Z_0 = 1$



Sound pressure level in front of, in, and behind the grid. $\vartheta = 45°$, $F = T/\lambda_0 = 0.75$, $a/T = 0.5$, $d/T = 0.25$, $R = 1$.
The ratio $a/T = 0.5$ is too large for the assumption of only a plane wave in the slits. This assumption produces just a least square error matching at the orifices



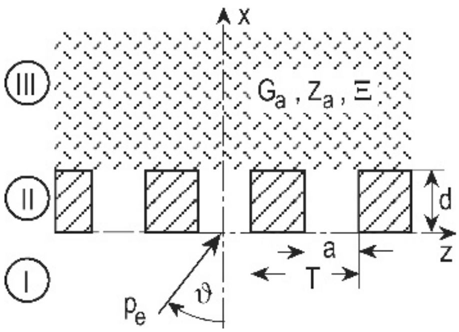
As above, but for $\vartheta = 45°$

B.23 Grid of Finite Thickness with Wide Slits on Half-Infinite Porous Layer

► *See also:* Mechel, Vol. I, Ch. 12 (1989)

The object is the same as in the previous *Sect. B.22*, but the slit channels are no longer assumed to be narrow, i.e. higher modes are assumed in the slits.

The field formulations remain the same as in *Sect. B.21*.



The field in the νth slit, $\nu = 0, \pm 1, \pm 2, \ldots$, with $z_\nu = z - \nu \cdot T$, is formulated as

$$p_{II}(x, z_\nu) = e^{-j\,\beta_0 \nu T} \sum_{m \geq 0} \left[B_m \cdot e^{-j\,\kappa_m x} + C_m \cdot e^{+j\,\kappa_m x} \right] \cdot \cos\left(m\pi (z_\nu / a - 1/2) \right)$$

$$\kappa_m = \begin{cases} \sqrt{k_0^2 - (m\pi/a)^2} & ; \quad m \leq m_g \\ -j\sqrt{(m\pi/a)^2 - k_0^2} & ; \quad m > m_g \end{cases} \quad ; \quad m_g = \mathrm{Int}\,(k_0 a/\pi) \tag{1}$$

For the auxiliary amplitudes X_m, Y_m $(m \geq 0)$

$$X_m := B_m - C_m \quad ; \quad Y_m := B_m \cdot e^{-j\,\kappa_m d} - C_m \cdot e^{+j\,\kappa_m d}$$

$$B_m = \frac{X_m \cdot e^{+j\,\kappa_m d} - Y_m}{e^{+j\,\kappa_m d} - e^{-j\,\kappa_m d}} \quad ; \quad C_m = \frac{X_m \cdot e^{-j\,\kappa_m d} - Y_m}{e^{+j\,\kappa_m d} - e^{-j\,\kappa_m d}} \tag{2}$$

a combined system of equations (m = 0, 1, 2...) is derived from the boundary conditions:

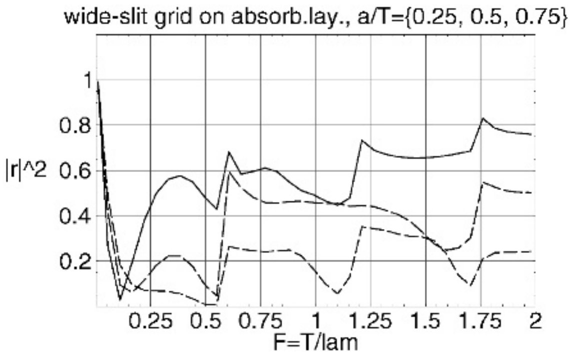
$$\sum_{n\geq 0} X_n \left[\frac{\kappa_n}{k_0} \sum_{\nu=0,\pm 1,\ldots} \left(j\,\frac{k_0}{\gamma_\nu}\right) \cdot s_{-m,\nu} \cdot s_{n,\nu} + (-1)^m \frac{\delta_{m,n}}{\delta_m} \frac{4T}{a} \frac{1+e^{-2j\,\kappa_m d}}{1-e^{-2j\,\kappa_m d}} \right]$$
$$= \frac{4T}{a} \left[P_e \cdot s_{-m,0} + \frac{2(-1)^m}{\delta_m} \frac{e^{-j\,\kappa_m d}}{1-e^{-2j\,\kappa_m d}} \cdot Y_m \right],$$
$$\sum_{n\geq 0} Y_n \left[\frac{\kappa_n}{k_0} \sum_{\nu=0,\pm 1,\ldots} \frac{k_0}{\varepsilon_\nu} \cdot s_{-m,\nu} \cdot s_{n,\nu} + (-1)^m \frac{\delta_{m,n}}{\delta_m} \frac{4T}{a} \frac{k_0 Z_0}{\Gamma_a Z_a} \frac{1+e^{-2j\,\kappa_m d}}{1-e^{-2j\,\kappa_m d}} \right]$$
$$= \frac{8T}{a} \frac{k_0 Z_0}{\Gamma_a Z_a} \frac{(-1)^m}{\delta_m} \frac{e^{-j\,\kappa_m d}}{1-e^{-2j\,\kappa_m d}} \cdot X_m \tag{3}$$

with $\delta_{m,n}$ the Kronecker symbol; $\delta_0 = 1, \delta_{n>0} = 2$; and the abbreviation

$$s_{m,n} := e^{j\,m\pi/2} \left[\frac{\sin\,(m\pi/2 - ß_n a/2)}{m\pi/2 - ß_n a/2} + (-1)^m \frac{\sin\,(m\pi/2 + ß_n a/2)}{m\pi/2 + ß_n a/2} \right], \tag{4}$$

with $s_{m,n} = s_{-m,n}$ and:

$$s_{m,n} = 2 \frac{ß_n a/2}{(ß_n a/2)^2 - (m\pi/2)^2} \cdot \begin{cases} \sin\,(ß_n a/2) & ; \quad m = \text{even} \\ -j\,\cos\,(ß_n a/2) & ; \quad m = \text{odd} \end{cases} . \tag{5}$$



Reflection coefficient of a thick layer of glass fibres, covered with a grid with wide slits for some ratios a/T (the dashes are shorter for higher a/T). $\vartheta = 45°$; $d/T = 0.25$; $R = \Xi \cdot T/Z_0 = 1$

The amplitudes A_n, D_n (n=0, ±1, ±2,...) follow from

$$A_n = j\frac{k_0}{\gamma_n} \left[\delta_{0,\nu} \cdot P_e \cdot \cos\,\vartheta - \frac{a}{2T} \sum_{m\geq 0} (B_m - C_m) \frac{\kappa_m}{k_0} \cdot s_{m,n} \right],$$
$$D_n = \frac{a}{2T} \frac{\Gamma_a Z_a}{k_0 Z_0} \frac{k_0}{\varepsilon_n} e^{+\varepsilon_n d} \sum_{m\geq 0} \left(B_m e^{-j\,\kappa_m d} - C_m e^{+j\,\kappa_m d} \right) \frac{\kappa_m}{k_0} \cdot s_{m,n}. \tag{6}$$

The reflection coefficient $|r|^2$ is evaluated as in the previous Sect. B.22.

The parameters in the shown examples are ϑ, a/T, d/T, $R = \Xi \cdot T/Z_0$, $F = T/\lambda_0$ (equivalences: |r|^2 $\rightarrow$ $|r|^2$; lam $\rightarrow$ λ_0).

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