Q Elasto-Acoustics

with W. Maysenhölder

Some fundamental relations, and relations concerning sound transmission through plates may be found also in the Ch. *"I. Sound Transmission"*.

Q.1 Fundamental Equations of Motion

► *See also:* Achenbach (1975); Maysenhölder (1994)

Used notation (including some quantities of later Sections):

$x_i (i = 1, 2, 3)$ or x, y, z	Cartesian co-ordinates of position;
u_i	displacement;
$v_i = \partial u_i / \partial t$	velocity;
c	phase velocity;
C	group velocity;
λ_w	wavelength (various subscripts);
k	wavenumber;
k_i	wavevector;
ε_{ij}	strain;
σ_{ij}	stress;
p	sound pressure;
ρ	mass density;
λ, μ	Lamé's constants ($\mu \equiv$ shear modulus);
K	compression (bulk) modulus;
E	Young's modulus;
σ	Poisson's ratio for isotropic media (no subscript);
ν_{ij}	Poisson's ratios for anisotropic media;
C_{ijkl}	elasticity tensor;
e_{kin}	kinetic energy density;
e_{pot}	potential energy density;
L	Lagrangian density;
I_i	intensity;
δ_{ij}	Kronecker's delta (identity matrix)

Sometimes vectors will be written with arrows, e. g. $\vec{x}$ for x_i.

The summation rule is applied:

$$C_{ijkl}\,\varepsilon_{kl} \equiv \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\,\varepsilon_{kl} \qquad \varepsilon_{kk} \equiv \sum_{k=1}^{3} \varepsilon_{kk} \qquad \text{(summation over repeated subscripts).}$$

Unlike electrodynamics the theory of elasticity is genuinely non-linear, since the exact relation between strain field ε_{ij} and displacement field u_i is non-linear. This chapter, however, is confined to the linearised theory, i.e. with

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

and the generalised version of Hooke's law

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}\,. \tag{2}$$

The decomposition of the strain tensor,

$$\varepsilon_{ij} = \frac{\Delta V}{3\,V}\,\delta_{ij} + \tilde{\varepsilon}_{ij}\,, \tag{3}$$

is invariant with respect to co-ordinate transformations and therefore physically essential. The first term involving the trace ($\varepsilon_{kk} = \Delta V/V$) represents a change of volume without change of shape, whereas the strain deviator $\tilde{\varepsilon}_{ij}$ with zero trace describes pure shear deformations (change of shape without change of volume). The corresponding decomposition for the stress tensor is $\sigma_{ij} = -p\,\delta_{ij} + \tilde{\sigma}_{ij}$ with the pressure p ($\sigma_{kk} = -3p$) and the stress deviator $\tilde{\sigma}_{ij}$ with zero trace for the shear stresses. In isotropic media Hooke's law decomposes into the part for (isotropic) compression, $p = -K \cdot \Delta V \big/ V$ with compression (bulk) modulus K, and the shear part $\tilde{\sigma}_{ij} = 2\,\mu\,\tilde{\varepsilon}_{ij}$ with shear modulus μ. Both parts are combined in the convenient form $\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\,\mu\,\varepsilon_{ij}$ with the Lamé constant $\lambda = K - \frac{2}{3}\mu$.

If dissipation effects are ignored, the Lagrangian density is given by $L = e_{kin} - e_{pot}$ with

$$e_{kin} = \frac{1}{2}\,\rho\,v_i\,v_i\,, \quad e_{pot} = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2}\,\varepsilon_{ij}\,C_{ijkl}\,\varepsilon_{kl} = \frac{1}{2}\,\frac{\partial u_i}{\partial x_j}\,C_{ijkl}\,\frac{\partial u_k}{\partial x_l}\,, \tag{4}$$

where the last equal sign is justified because of the symmetries of the elastic tensor,

$$C_{ijkl} = C_{jikl}\,, \quad C_{ijkl} = C_{klij}\,.$$

The equation of motion (for time-independent material properties) may be obtained from Hamilton's principle,

$$\delta \int L\,dt\,dx_1\,dx_2\,dx_3 = 0 \tag{5}$$

leading to the Lagrange-Euler equations:

$$\frac{d}{d\,t}\left(\frac{\partial L}{\partial v_i}\right) + \frac{d}{d\,x_j}\left(\partial L \Big/ \partial\left(\frac{\partial u_i}{\partial x_j}\right)\right) - \frac{\partial L}{\partial u_i} = 0^{*)} \tag{6}$$

which for the considered case result in the partial differential equations:

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = C_{ijkl}\,\frac{\partial^2 u_k}{\partial x_j\,\partial x_l} + \frac{\partial C_{ijkl}}{\partial x_j}\,\frac{\partial u_k}{\partial x_l}\,. \tag{7}$$

*) See Preface to the 2nd edition.

The last term vanishes for homogeneous media with position-independent material properties.

Alternative derivation:
The right-hand side of (7) equals the divergence $\partial \sigma_{ij} / \partial x_j$ of the stress tensor, which is zero for local elastostatic equilibrium in the absence of external forces. According to D'Alembert's principle, the combination of this balance of forces with the inertia term (left-hand side of (7)) again yields the above equation of motion. In the case of external body forces, like gravity, a volume density of forces [N/m^3] must be added.

In the special case of locally isotropic media one obtains with

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \tag{8}$$

the simplified equation of motion:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial \lambda}{\partial x_i} \frac{\partial u_j}{\partial x_j} + \frac{\partial \mu}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mu}{\partial x_j} \frac{\partial u_j}{\partial x_i}, \tag{9}$$

where the last three terms vanish in homogeneous media.

Q.2 Anisotropy and Isotropy

► *See also:* Helbig, pp. 68–92 (1994); Jones, pp. 56–70 (1999); Lai et al., pp. 293–314 (1993)

For an explicit description of anisotropic elasticity the fourth-rank tensor C_{ijkl} is often transformed to the symmetric 6x6-matrix c_{IJ} with subscript relations

$$ij \rightarrow I: \quad 11 \rightarrow 1, \quad 22 \rightarrow 2, \quad 33 \rightarrow 3, \quad 23 \rightarrow 4, \quad 31 \rightarrow 5, \quad 12 \rightarrow 6$$

(contracted or Voigt's notation). The most general (triclinic) anisotropy is described by a fully occupied matrix (21 independent elastic constants), which is shown here with four-subscript entries in order to illustrate the above subscript relations:

$$\{c_{IJ}\} = \underset{\text{triclinic}}{\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{1322} & C_{1222} \\ & & C_{3333} & C_{2333} & C_{1333} & C_{1233} \\ & & & C_{2323} & C_{1323} & C_{1223} \\ & \text{sym} & & & C_{1313} & C_{1213} \\ & & & & & C_{1212} \end{pmatrix}}, \tag{1}$$

"sym" means symmetric completion. Since c_{IJ} is not a tensor, transformation of Hooke's law (see *Sect. Q1*) is not trivial:

$$\sigma_I = c_{IJ} \varepsilon_J \quad \text{with} \quad \begin{aligned} \{\sigma_I\} &= \left(\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{13} \quad \sigma_{12} \right) \\ \{\varepsilon_J\} &= \left(\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad 2\varepsilon_{23} \quad 2\varepsilon_{13} \quad 2\varepsilon_{12} \right) \end{aligned}. \tag{2}$$

Similarly, with the compliance tensor S_{ijkl}, which is the inverse of C_{ijkl}, the contracted form is:

$$\varepsilon_J = s_{IJ} \sigma_I; \qquad s_{IJ} = c_{IJ}^{-1}. \tag{3}$$

The most general anisotropy admissible in thin-plate theory, which implies the middle plane of the plate to be a plane of symmetry, needs 13 independent elastic constants (monoclinic), whereas orthotropic anisotropy, characterised by three mutually perpendicular planes of symmetry, requires nine:

$$\underset{\substack{\text{monoclinic}\\ \text{plane of symmetry:}\ x_3\ =\ \text{const}}}{\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ & c_{22} & c_{23} & 0 & 0 & c_{26} \\ & & c_{33} & 0 & 0 & c_{36} \\ & & & c_{44} & c_{45} & 0 \\ & \text{sym} & & & c_{55} & 0 \\ & & & & & c_{66} \end{pmatrix}} \quad \underset{\substack{\text{orthotropic}\\ (\equiv \text{orthorhombic})}}{\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & \text{sym} & & & c_{55} & 0 \\ & & & & & c_{66} \end{pmatrix}} . \tag{4}$$

In “engineering notation” orthotropic anisotropy is expressed by three Young’s moduli E_i, six Poisson numbers ν_{ij}, and three shear moduli G_{ij}. Their physical meaning may be deduced from the compliance representation:

$$\{ s_{IJ} \} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix} , \tag{5}$$

which also provides for the three relations between the six Poisson numbers due to $s_{IJ} = s_{JI}$. In terms of the c_{IJ}, Maysenhölder (1996):

$$E_1 = \frac{N}{c_{22}\,c_{33} - c_{23}^2} , \quad E_2 = \frac{N}{c_{11}\,c_{33} - c_{13}^2} , \quad E_3 = \frac{N}{c_{11}\,c_{22} - c_{12}^2} , \tag{6}$$

$$N = c_{11}\,c_{22}\,c_{33} - c_{11}\,c_{23}^2 - c_{22}\,c_{13}^2 - c_{33}\,c_{12}^2 + 2\,c_{12}\,c_{13}\,c_{23}, \tag{7}$$

$$\nu_{21} = \frac{c_{12}\,c_{33} - c_{13}\,c_{23}}{c_{11}\,c_{33} - c_{13}^2}, \quad \nu_{31} = \frac{c_{13}\,c_{22} - c_{12}\,c_{23}}{c_{11}\,c_{22} - c_{12}^2}, \quad \nu_{32} = \frac{c_{23}\,c_{11} - c_{12}\,c_{13}}{c_{11}\,c_{22} - c_{12}^2},$$

$$\nu_{12} = \frac{c_{12}\,c_{33} - c_{13}\,c_{23}}{c_{22}\,c_{33} - c_{23}^2}, \quad \nu_{13} = \frac{c_{13}\,c_{22} - c_{12}\,c_{23}}{c_{22}\,c_{33} - c_{23}^2}, \quad \nu_{23} = \frac{c_{23}\,c_{11} - c_{12}\,c_{13}}{c_{11}\,c_{33} - c_{13}^2}. \tag{8}$$

Backward transformation:

$$c_{11} = \frac{1 - \nu_{23}\,\nu_{32}}{\Delta}\,E_1, \quad c_{22} = \frac{1 - \nu_{13}\,\nu_{31}}{\Delta}\,E_2, \quad c_{33} = \frac{1 - \nu_{12}\,\nu_{21}}{\Delta}\,E_3,$$

$$c_{12} = \frac{\nu_{21} + \nu_{31}\,\nu_{23}}{\Delta}\,E_1, \quad c_{23} = \frac{\nu_{32} + \nu_{12}\,\nu_{31}}{\Delta}\,E_2, \quad c_{13} = \frac{\nu_{13} + \nu_{23}\,\nu_{12}}{\Delta}\,E_3, \tag{9}$$

$$c_{44} = G_{23}, \quad c_{55} = G_{31}, \quad c_{66} = G_{12},$$

$$\Delta = 1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{31}\,\nu_{13} - 2\,\nu_{21}\,\nu_{32}\,\nu_{13}.$$

Elastic stability requires all diagonal elements of both c_{IJ} and s_{IJ} as well as N and Δ to be positive. In addition, $\nu_{IJ}^2 < E_I/E_J$ for $I \neq J$. For further constraints on Poisson's ratios see Jones, p. 69 (1999).

Further frequent cases with even lower anisotropies (i.e. higher symmetries) are transversely isotropic (equivalent to hexagonal) and cubic with five and three independent elastic constants, respectively:

$$\underset{\substack{\text{with } c_{66} = \frac{1}{2}\,(c_{11} - c_{12}) \\ \text{transverse isotropy (hexagonal)}}}{\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & \text{sym} & & & c_{44} & 0 \\ & & & & & c_{66} \end{pmatrix}} \quad \underset{\text{cubic}}{\begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{11} & c_{12} & 0 & 0 & 0 \\ & & c_{11} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & \text{sym} & & & c_{44} & 0 \\ & & & & & c_{44} \end{pmatrix}}. \tag{10}$$

The engineering notation is also used for transversely isotropic materials ($E_1 = E_2, G_{31} = G_{23}, \nu_{32} = \nu_{31}, G_{12} = E_1/[2(1 + \nu_{21})]$; further $\nu_{12} = \nu_{21}, \nu_{13}/E_1 = \nu_{31}/E_3, \nu_{23}/E_1 = \nu_{32}/E_3$; $\nu_{13} = \nu_{23}$; therefore E_1, ν_{21}, E_3, ν_{31} and G_{23} provide a complete description. For cubic materials E_1, ν_{21} and G_{23} suffice (however, $G_{23} \neq E_1/[2(1 + \nu_{21})]$, if not isotropic!).

The elastic properties of cubic materials are conveniently expressed by one modulus of compression K (bulk modulus) for changes of volume with constant shape and two shear moduli μ and μ' for changes of shape with constant volume:

$$K = \frac{1}{3}\,(c_{11} + 2\,c_{12})\,, \qquad \mu' = \frac{1}{2}\,(c_{11} - c_{12})\,, \qquad \mu = c_{44}. \tag{11}$$

This anisotropy of shear deformation may be characterised by the dimensionless measure:

$$a = \frac{\mu' - \mu}{\mu' + \mu} \quad \text{with} \quad -1 < a < 1. \tag{12}$$

With $a = 0$ one proceeds from cubic anisotropy to isotropy, which is determined by two independent parameters. Several pairs are in common use, e.g. compression (bulk) modulus K and shear modulus μ, Lamé's constants λ and μ, finally Young's modulus

E and Poisson's ratio σ. The dependencies are summarised in the table below. See also Table 4 in Sect. I.8.

Elastic stability requires $K > 0$ and $\mu > 0$, leading to:

$$\lambda > -\frac{2}{3}\mu\,,\quad E > 0\,,\quad -1 < \sigma < \frac{1}{2}\,,\quad c_{11} > 0,\quad -\frac{1}{2}c_{11} < c_{12} < c_{11}. \tag{13}$$

Table 1 Interrelations between isotropy parameters. For a more extensive table including eight additional pair combinations see Thurston, p. 74 (1964)

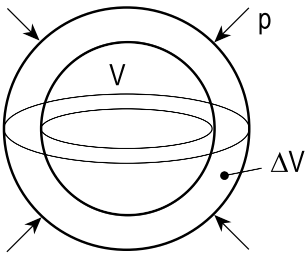
From: To:	K,μ	λ,μ	E,σ	$c_{11},c_{12}\,(c_{44})$
$K =$	K	$\lambda + \frac{2}{3}\mu$	$\frac{E}{3(1-2\sigma)}$	$\frac{1}{3}(c_{11} + 2c_{12})$
$\mu =$	μ	μ	$\frac{E}{2(1+\sigma)}$	$\frac{1}{2}(c_{11} - c_{12}) = c_{44}$
$\lambda = c_{12} =$	$K - \frac{2}{3}\mu$	λ	$\frac{\sigma E}{(1+\sigma)(1-2\sigma)}$	c_{12}
$E =$	$\frac{9K\mu}{3K+\mu}$	$\frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$	E	$\frac{(c_{11}+2c_{12})(c_{11}-c_{12})}{c_{11}+c_{12}}$
$\sigma =$	$\frac{3K-2\mu}{6K+2\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	σ	$\frac{c_{12}}{c_{11}+c_{12}}$
$c_{11} =$	$K + \frac{4}{3}\mu$	$\lambda + 2\mu$	$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$	c_{11}

Moduli associated with elementary deformations and corresponding waves:

Compression modulus (bulk modulus) K:

$$p = -K\frac{\Delta V}{V} \tag{14}$$

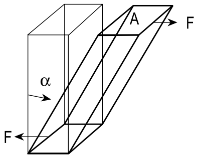
K^{-1} = compressibility; no associated wave.



Shear modulus μ: $\frac{F}{A} = \mu\,\alpha$ (α: engineering shear strain) (15)

Velocity of transversal waves (shear waves and torsional waves):

$$c_T = \sqrt{\mu/\rho}. \tag{16}$$

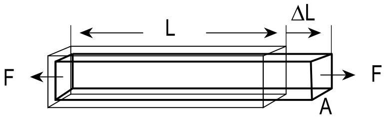


Young's modulus E *and Poisson's ratio (lateral contraction)* σ:

$$\frac{F}{A} = E\,\frac{\Delta L}{L} \quad \text{(x-direction along L)} \tag{17}$$

$$\sigma = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} \tag{18}$$

Velocity of (quasi-) longitudinal waves in bars: $c_{QL} = \sqrt{E/\rho}.$ (19)

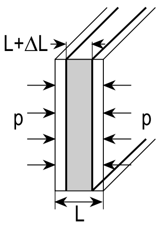


Modulus for longitudinal waves D:

$$-p = \sigma_{xx} = D\,\varepsilon_{xx} = c_{11}\,\varepsilon_{xx} \tag{20}$$

Velocity of (pure) longitudinal waves (with compression/dilatation and shear):

$$c_L = \sqrt{D/\rho}. \tag{21}$$



Q.3 Interface Conditions, Reflection and Refraction of Plane Waves

▸ *See also:* Auld, Vol. II, pp. 1–62 (1990)

At the interface (boundary) between two elastic media I and II the requirements for tight contact are continuity of displacements, $u_i^{I} = u_i^{II}$, and balance of forces

$$\left(\sigma_{ij}^{I} - \sigma_{ij}^{II}\right) n_j = 0 \tag{1}$$

with unit normal vector n_i on the interface. If the medium II is a fluid with sound pressure p^{II} at the interface, then $\sigma_{ij}^{I}\, n_j = -p^{II}\, n_i$, since $\sigma_{ij}^{II} = -p^{II}\delta_{ij}$. With the usual slip assumption only the displacement component normal to the interface, $n_i\ u_i$, must be continuous.

A plane elastic wave incident on a plane interface between two homogeneous elastic solids with different material properties (tight contact) generates up to three reflected waves and up to three refracted (transmitted) waves, since there are up to three different wave velocities in an anisotropic medium. The propagation directions of these scattered waves are given by Snell's law, the relations for their amplitudes are called Fresnel equations. The boundary conditions require that the wavevector component tangent to the boundary is the same for all waves. This immediately leads to Snell's law:

$$\frac{\sin \Theta_i}{c_i} = \frac{\sin \Theta_0}{c_0} = \frac{1}{c_{trace}} \tag{2}$$

with phase velocities c and angles Θ $(0 \leq \Theta < \pi)$ between wavevector $\vec{k}$ and normal $\vec{n}$. Subscript i : reflected and refracted waves; subscript 0: incident wave. For $\sin \Theta_i > 1$, the i-th wave becomes evanescent (exponentially decaying perpendicular to the interface) and propagates with trace velocity c_{trace} along the interface (e.g. for total reflection).

Since in anisotropic media the phase velocities are no longer independent from the propagation direction (see *Sect. Q.9.1*), the evaluation of Snell's law is more involved. A helpful geometrical technique makes use of slowness surfaces, which provide the magnitude of $\vec{k}/\omega$ as a function of the direction of $\vec{k}$ [Helbig, pp. 32–38 (1994); Auld, Vol. I, pp. 393–413 (1990)].

In anisotropic media, where the direction of energy propagation (see *Sect. Q.5*) may be different from the wavevector direction, the incident wave should be characterised by the former rather than by the latter. The energy directions need not lie in the plane defined by the wavevectors! [Rokhlin/Bolldand/Adler (1986); Lanceleur/Ribeiro/De Belleval (1993)].

For Fresnel equations see e.g. [Achenbach, pp. 168–187 (1975) and Auld, Vol. II, pp. 21–43 (1990)].

Q.4 Material Damping

▸ *See also:* Gaul (1999)

The conventional viscoelastic generalisation of the relation $\tilde{\sigma}_{ij}(t) = 2\,\mu\,\tilde{\varepsilon}_{ij}(t)$ between stress and strain deviators for pure shear deformation in an isotropic solid (μ: shear modulus; see *Sect. Q.1*) is given by:

$$\sum_{k=0}^{M} p_k \frac{d^k}{d t^k}\,\tilde{\sigma}_{ij}(t) = \sum_{k=0}^{N} q_k \frac{d^k}{d t^k}\,\tilde{\varepsilon}_{ij}(t) \tag{1}$$

with integer k and real coefficients p_k and q_k. Classical viscoelastic models are the Zener model:

$$\tilde{\sigma}_{ij}(t) + p_1 \frac{d\,\tilde{\sigma}_{ij}(t)}{d\,t} = q_0\,\tilde{\varepsilon}_{ij}(t) + q_1 \frac{d\,\tilde{\varepsilon}_{ij}(t)}{d\,t} \tag{2}$$

and – as its descendants – the Kelvin-Voigt model ($p_1 = 0$) and the Maxwell model ($q_0 = 0$). In the following, however, the further generalisation to time derivatives of fractional order will be considered:

$$\sum_{k=0}^{M} p_k \frac{d^{\alpha_k}}{d\,t^{\alpha_k}} \tilde{\sigma}_{ij}(t) = \sum_{k=0}^{N} q_k \frac{d^{\beta_k}}{d\,t^{\beta_k}} \tilde{\varepsilon}_{ij}(t), \tag{3}$$

which leads in many cases to improved curve-fitting of measured data with less parameters ($\alpha_0 = \beta_0 = 0;\ \ 0 < \alpha_k, \beta_k < 1$). An extension to fractional orders beyond one is possible. An alternative formulation in terms of relaxation functions reads:

$$\tilde{\sigma}_{ij}(t) = 2 \int_{-\infty}^{t} G(t-\tau) \frac{d\,\tilde{\varepsilon}_{ij}(\tau)}{d\,\tau}\, d\tau, \quad \tilde{\varepsilon}_{ij}(t) = 2 \int_{-\infty}^{t} J(t-\tau) \frac{d\,\tilde{\sigma}_{ij}(\tau)}{d\,\tau}\, d\tau \tag{4}$$

with $\displaystyle \frac{d}{d\,t} \int_{0}^{t} 2\,G(\tau)\,J(t-\tau)\, d\tau = 1.$

The relaxation modulus G(t) and the creep compliance J(t) describe the fading memory of the (linear viscoelastic) material with respect to the loading history. Any elastic modulus M may be generalised in the same manner.

For time-harmonic fields, $\tilde{\varepsilon}_{ij}(t) = \mathrm{Re}\left\{ \tilde{\varepsilon}_{ij}(\omega)\ \exp\left[j\omega\, t \right] \right\},$

one obtains:

$$\tilde{\sigma}_{ij}(\omega) = 2\,\mu(\omega)\ \tilde{\varepsilon}_{ij}(\omega) \quad \text{with} \quad 2\,\mu(\omega) = \sum_{k=0}^{N} q_k\,(j\,\omega)^{\beta_k} \Big/ \sum_{k=0}^{M} p_k\,(j\,\omega)^{\alpha_k}. \tag{5}$$

The decomposition $\mu(\omega) = \mu'(\omega) + j\,\mu''(\omega) = \mu'(\omega)[\,1 + j\,\eta(\omega)\,]$ of the complex shear modulus introduces the storage modulus μ'(ω) (real part), the loss modulus $\mu''(\omega)$ (imaginary part) and the loss factor $\eta(\omega)$, which may be interpreted as the ratio of the dissipated energy D(ω) per cycle to the 2π-fold of the stored energy U(ω):

$$\eta\,(\omega) = \frac{D(\omega)}{2\,\pi\,U(\omega)} = \frac{\mu''(\omega)}{\mu'(\omega)} = \frac{1}{Q(\omega)} = \frac{\Lambda}{\pi} = \tan\ \delta(\omega) \tag{6}$$

with quality factor Q(ω), logarithmic decrement Λ(ω) and loss tangent tan δ(ω). The relationship between the complex modulus $\mu(\omega)$ and the relaxation modulus G(t) amounts to:

$$\mu'(\omega) = \mu_0 + \omega \int_{0}^{\infty} \hat{G}(\tau)\ \sin(\omega\tau)\ d\tau, \quad \mu''(\omega) = \omega \int_{0}^{\infty} \hat{G}(\tau) \cos(\omega\tau)\ d\tau \tag{7}$$

with real static or equilibrium modulus $\mu_0 = \mu(\omega = 0) = G(t = \infty)$ and $\hat{G}(t) = G(t) - \mu_0$, hence $\hat{G}(\infty) = 0$.

Because of causality and linearity the real and imaginary parts of any modulus $M = M' + jM''$ are connected via the Kramers-Kronig relations:

$$M'(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} \frac{\Omega M''(\Omega)}{\Omega^2 - \omega^2}\, d\Omega, \quad M''(\omega) = -\frac{2}{\pi} P \int_{0}^{\infty} \frac{\Omega\, M'(\Omega)}{\Omega^2 - \omega^2}\, d\Omega, \tag{8}$$

where P denotes Cauchy's principal value [Beltzer, p. 20 (1988)]. Regarding practical applications of these relations see, e.g. [Mobley et al. (2000)]. The often utilised (non-viscoelastic) model with M' and M" independent of frequency ('constant hysteresis damping model', 'hysteretic model' or 'structural damping model') violates causality.

The elastic-viscoelastic correspondence principle states that the solution of a viscoelastic problem may be obtained from the solution of the corresponding elastic problem. In the case of time-harmonic problems this means a straightforward substitution of the elastic moduli by the corresponding frequency-dependent and complex viscoelastic moduli. Otherwise, Fourier transformations have to be performed before and after this substitution. (An alternative realisation uses Laplace transforms and impact response functions.)

Example: Five-parameter fractional-derivative model (generalised Zener model)

The complex shear modulus for a highly damped polymer,

$$\mu(\omega) = \mu_0 \frac{1 + b\,(j\,\omega)^{\beta}}{1 + a\,(j\,\omega)^{\alpha}} \tag{9}$$

with $\mu_0 = 87\,\mathrm{kPa}, a = 0.039, \alpha = 0.39, b = 0.38, \beta = 0.64$, yields a good fit of experimental data of $\mu'(\omega)$ and $\eta(\omega)$ from 1 Hz to almost 10 kHz.

Special case: Four-parameter fractional-derivative model [Pritz (1996)]

Thermodynamic constraints like non-negative rate of dissipated energy and the requirement of a finite viscoelastic wave speed impose the conditions

$$\mu > 0\,, \qquad b > a > 0\,, \qquad 0 < \alpha = \beta < 2 \tag{10}$$

on the generalised Zener model, thus leading to the simplified version with four parameters. For some modulus M it may be written as:

$$M(\omega) = \frac{M_0 + (j\,\omega\tau_r)^{\alpha} M_\infty}{1 + (j\,\omega\tau_r)^{\alpha}} \quad \text{or} \quad \frac{M(\omega) - M_\infty}{M_0 - M_\infty} = \frac{1}{1 + (j\,\omega\tau_r)^{\alpha}} \tag{11}$$

(Cole-Cole equation) with the static modulus M_0, the high-frequency limit M_∞ and a relaxation time τ_r. With normalised frequency $\nu = \omega\tau_r$, $\Psi = \alpha\pi/2$ and ratio $\gamma = M_\infty/M_0$:

$$\frac{M'}{M_0} = \frac{1 + (\gamma + 1)\,\nu^{\alpha} \cos\Psi + \gamma\,\nu^{2\alpha}}{1 + 2\,\nu^{\alpha} \cos\Psi + \nu^{2\alpha}}, \qquad \frac{M''}{M_0} = \frac{(\gamma - 1)\,\nu^{\alpha} \sin\Psi}{1 + 2\nu^{\alpha} \cos\Psi + \nu^{2\alpha}}, \tag{12}$$

$$\eta = \frac{(\gamma - 1)\,\nu^{\alpha} \sin\Psi}{1 + (\gamma + 1)\,\nu^{\alpha} \cos\Psi + \gamma\,\nu^{2\alpha}}. \tag{13}$$

M' is monotonically increasing with frequency, while M" and η have maxima:

$$M''_{\max} = \frac{(\Delta M/2)\,\sin\Psi}{1 + \cos\Psi} \qquad \text{at} \qquad \omega = 1/\tau_r, \tag{14}$$

$$\eta_{\max} = \frac{(\gamma - 1)\,\sin\Psi}{2\sqrt{\gamma} + (\gamma + 1)\,\cos\Psi} \qquad \text{at} \qquad \omega = \frac{1}{\tau_r\,\sqrt[2\alpha]{\gamma}} \tag{15}$$

($\Delta M = M_\infty - M_0$). The half-value bandwidths are for M" and for η:

$$\left(\frac{\omega_2}{\omega_1}\right)_{M''} = \sqrt[\alpha]{\frac{A + \sqrt{A^2 - 1}}{A - \sqrt{A^2 - 1}}} \qquad \text{with} \qquad A = 2 + \cos\Psi, \tag{16}$$

$$\left(\frac{\omega_2}{\omega_1}\right)_{\eta} = \sqrt[\alpha]{\frac{B + \sqrt{B^2 - 4\gamma}}{B - \sqrt{B^2 - 4\gamma}}} \qquad \text{with} \qquad B = 4\sqrt{\gamma} + (\gamma + 1)\cos\Psi. \tag{17}$$

The model is causal and applicable for materials with one symmetric loss peak.

Example: Young's modulus E of dense PVC foam (446 kg/m³)

$E_0 = 1.82$ MPa, $E_\infty = 1.14$ GPa, $\alpha = 0.335$ and $\tau_r = 21.3$ ns have been determined from measurements between about 100 Hz and 10 kHz. Hence the loss factor maximum is $\eta_{max} = 0.53$ at 500 Hz.

Q.5 Energy

Q.5.1 General Relations

▸ *See also:* Auld, Vol. I., Acoustic fields and waves in solids. pp. 144–145 (1990)

With the density of power $P_{ext} = F_i v_i$ supplied by an external density of forces F_i and the density of dissipated power P_{diss} the energy balance is expressed by:

$$\frac{d}{dt}\left(e_{kin} + e_{pot}\right) + \frac{\partial S_i}{\partial x_i} + P_{diss} = P_{ext} \tag{1}$$

with the energy flux density $S_i = -\sigma_{ij} v_j$, also called acoustic Poynting vector or Kirchhoff vector.

Q.5.2 Surface Intensity

At the free surface $z = 0$ of an elastic solid the stress components σ_{xz}, σ_{yz} and σ_{zz} vanish, hence, with the in-surface velocity components v_x and v_y, the Kirchhoff vector becomes:

$$\vec{S} = -\begin{pmatrix} \sigma_{xx} v_x + \sigma_{xy} v_y \\ \sigma_{xy} v_x + \sigma_{yy} v_y \\ 0 \end{pmatrix}. \tag{2}$$

This quantity or its time-average, the surface intensity, can be experimentally determined from the five measurable values of ε_{xx}, ε_{xy}, ε_{yy}, v_x and v_y, if the elastic properties at the surface are known. In the isotropic case, [Pavic (1987)]

$$S_x = -2\mu\left\{\left(\frac{\varepsilon_{xx} + \sigma\,\varepsilon_{yy}}{1 - \sigma}\right) v_x + \varepsilon_{xy} v_y\right\}, \tag{3}$$

$$S_y = -2\mu\left\{\varepsilon_{xy} v_x + \left(\frac{\sigma\,\varepsilon_{xx} + \varepsilon_{yy}}{1 - \sigma}\right) v_y\right\}, \tag{4}$$

whereas in the anisotropic case the three needed stress components (including three non-measurable strain components) have to be evaluated from the system of six linear equations:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & \text{sym} & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\,\varepsilon_{yz} \\ 2\,\varepsilon_{zx} \\ 2\,\varepsilon_{xy} \end{pmatrix} . \tag{5}$$

Q.5.3 Time-Harmonic Wavefields

► *See also:* Maysenhölder, pp 10–28 (1994); Auld, Vol. I, Acoustic fields and waves in solids (1990)

For harmonic time dependence the field quantities are advantageously written in complex notation, e.g. $\sigma_{ij}(x_i, t) = \sigma_{ij}(x_i)\exp(j\omega t)$, with the local amplitude $\sigma_{ij}(x_i)$ being complex as well. The physical quantity is understood to be the real (or imaginary) part of the complex representation. However, for energy quantities which are products of two field quantities, one has to be careful. Here the complex representation conveniently yields the time average, e.g., of the Kirchhoff vector,

$$\begin{aligned} \langle S_i\,(x_i,\,t)\rangle_t &= \left\langle \mathrm{Re}\left\{-\sigma_{ij}\,(x_i,\,t)\right\}\ \mathrm{Re}\left\{v_j\,(x_i,\,t)\right\}\right\rangle_t \\ &= \frac{1}{2}\,\mathrm{Re}\left\{-\sigma_{ij}\,(x_i)\,v_j^*\,(x_i)\right\} = I_i\,(x_i)\,, \end{aligned} \tag{6}$$

which is called intensity. Sometimes, a complex intensity is defined (spatial dependencies will be dropped from now on):

$$I_i^C = -\frac{1}{2}\,\sigma_{ij}\,v_j^* = I_i + j\,Q_i \quad \text{with} \quad Q_i = \frac{1}{2}\,\mathrm{Im}\left\{-\sigma_{ij}\,v_j^*\right\}. \tag{7}$$

The sign of the reactive intensity Q_i depends on the convention of the time factor (here: $\exp(j\omega t)$), whereas the sign of the (active or effective) intensity I_i does not. With the complex external power density $P_{ext}^C = F_i\,v_i^*$ the divergence relations

$$\begin{aligned} \frac{\partial I_i}{\partial x_i} &= -\langle P_{diss}\rangle_t + \mathrm{Re}\left\{\left\langle P_{ext}^C\right\rangle_t\right\} \quad \text{and} \\ \frac{\partial Q_i}{\partial x_i} &= -2\,\omega\,\langle L\rangle_t + \mathrm{Im}\left\{\left\langle P_{ext}^C\right\rangle_t\right\} \\ (L &= e_{kin} - e_{pot}) \end{aligned} \tag{8}$$

hold for general time-harmonic fields in solids (P_{diss} is real). (Unlike with airborne sound fields, in elasto-acoustic fields Q_i is not in general proportional to the gradient of the time-averaged potential energy density, and the curl of Q_i is not generally zero.)

Q.5.4 Rayleigh's Principle

► *See also:* Pierce (1985); Maysenhölder (1993)

Rayleigh's principle states equipartition of potential and kinetic energy for time-harmonic sound fields without dissipation under certain circumstances, e.g.,

- after space and time averaging in freely vibrating finite solids.

For solids of infinite extent in at least one dimension the rule is called *Rayleigh's principle for propagating waves* and holds in homogeneous media, e.g.,

- for a plane wave in an unbounded isotropic or anisotropic solid (no averaging required);
- for a plate wave in a thick isotropic or anisotropic plate after averaging over time and thickness;
- for a beam wave in a thick isotropic or anisotropic straight beam after averaging over time and cross-section;
- for a bending wave in a thin isotropic plate or beam [Cremer/Heckl, p. 103 (1996)] (no averaging required).

and in periodically inhomogeneous media:

- for a Bloch wave after averaging over time and unit cell.

Q.5.5 Energy Velocity and Group Velocity

► *See also:* Lighthill (1965), Maysenhölder (1993)

The velocity of energy transport, defined by $\dfrac{I_i}{\langle e_{kin} + e_{pot} \rangle_t}$,

and the group velocity $C_i = \dfrac{\partial \omega}{\partial k_i}$

of a plane wave with wavevector k_i are equal in homogeneous anisotropic solids.

Therefore, $I_i = \langle e_{kin} + e_{pot} \rangle_t \, C_i.$

The same is true for Bloch waves in periodically inhomogeneous media after spatial averaging over a unit cell and for the fundamental waves in homogeneous plates and straight beams after averaging over the thickness or the cross-section, respectively.

Q.6 Random Media

► *See also:* Sornette (1989)

The random inhomogeneities in a three-dimensional elastic medium may be characterised by an elastic mean free path $\ell_e \approx (nq_s)^{-1}$ with the density n of (non-absorptive) scatterers with scattering cross section q_s. The elasto-acoustic energy propagation over a distance d obeys different laws in different regimes:

I. Wavelike propagation: Weak disorder; $d < \ell_e$:

$$I_i = \langle e_{kin} + e_{pot} \rangle_t \, C_i^{eff}. \tag{1}$$

Propagation takes place like in homogeneous media with an effective velocity C_i^{eff}. Homogenisation techniques (effective medium theories) are applicable. Even without dissipation there is an attenuation of the wave due to scattering into other directions.

II. Diffusion: Moderate disorder; $d > \ell_e$ (multiple scattering); wavelength $\lambda_w < \ell_e$:

$$I_i = -D(\omega)\,\frac{\partial\left\langle e_{kin} + e_{pot}\right\rangle_t}{\partial x_i} \tag{2}$$

with frequency-dependent diffusion coefficient

$$D(\omega) \approx D_0\left(1-\gamma\left(\frac{\lambda_w}{\ell_e}\right)^2\right) \quad ; D_0 = \ell_e c/3 \tag{3}$$

(c: velocity between scatterers) and γ a numerical factor of order unity. The transmission factor of a plate of thickness h is proportional to D/h. If $(\lambda_w/\ell_e)^2$ cannot be neglected in $D(\omega)$, one enters the regime of weak localisation (contributions of coherent interference effects; transition to III).

III. Anderson localisation: Strong disorder; $d > \ell_e$; $\lambda_w > \ell_e$ *(Ioffe-Regel criterion).*

Energy propagation is slower than diffusive ($D = 0$) due to coherent interference effects. Vibrations decay exponentially with localisation length ξ. The transmission factor of a plate is proportional to $\exp(-h/\xi)$.

Q.7 Periodic Media

► *See also:* Maysenhölder, Ch. 5 (1994)

A periodically inhomogeneous medium may be constructed by infinite repetition of a unit cell defined by linearly independent basis vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Its volume is $V_0 = \vec{a}_1 \cdot \vec{a}_2 \times \vec{a}_3$. A position $\vec{r}$ in a unit cell and an equivalent position in another unit cell are connected by a lattice vector $\vec{g}_n = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$ (n_i: integer). Any function f with the periodicity of the lattice, $f\left(\vec{r} + \vec{g}_n\right) = f(\vec{r})$, may be Fourier expanded:

$$f(\vec{r}) = \sum_M f^M \exp\left[j\,\vec{G}^M \cdot \vec{r}\right] \tag{1}$$

with Fourier coefficients:

$$f^M = \frac{1}{V_0}\int f(\vec{r})\exp\left[-j\,\vec{G}^M \cdot \vec{r}\right] d^3r. \tag{2}$$

The sum is over all reciprocal lattice points, the integral over one unit cell.

$$\vec{G}^M = M_1\vec{A}_1 + M_2\vec{A}_2 + M_3\vec{A}_3 \tag{3}$$

(M_i: integer) is a vector of the reciprocal lattice with the basis vectors:

$$\vec{A}_1 = \frac{2\pi}{V_0}(\vec{a}_2 \times \vec{a}_3) \quad ; \quad \vec{A}_2 = \frac{2\pi}{V_0}(\vec{a}_3 \times \vec{a}_1) \quad ; \quad \vec{A}_3 = \frac{2\pi}{V_0}(\vec{a}_1 \times \vec{a}_2); \\ \vec{a}_i \cdot \vec{A}_j = 2\pi\,\delta_{ij}. \tag{4}$$

Analogous to plane waves in homogeneous media the fundamental wave solutions for periodic media are Bloch waves:

$$\vec{u}(\vec{r}, t) = \vec{p}(\vec{r})\; e^{j\left(\omega t - \vec{k}\cdot\vec{r}\right)} \tag{5}$$

with the periodic function $\vec{p}\,(\vec{r}+\vec{g}_n)=\vec{p}\,(\vec{r})$. With the Fourier coefficients of the reciprocal density, $\tilde{\rho}=\rho^{-1}$, the Fourier transform of the equation of motion (see Sect. Q.1) for Bloch waves attains the usual form of an eigenvalue problem:

$$\omega^2 p_i^N=\sum_{L,M}\tilde{\rho}^{\,N-M}C_{ijkl}^{M-L}\left(k_l+G_l^L\right)\left(k_j+G_j^M\right)p_k^L. \tag{6}$$

Capital Latin superscripts again run over all points of the reciprocal lattice. Local isotropy in terms of the Lamé constants λ and μ leads to [Sigalas/Economou (1992)]:

$$\begin{aligned}\omega^2 p_i^N=\sum_{L,M}\tilde{\rho}^{\,N-M}\Big\{\Big[&\lambda^{M-L}\left(k_j+G_j^L\right)\left(k_i+G_i^M\right)\\ &+\mu^{M-L}\left(k_i+G_i^L\right)\left(k_j+G_j^M\right)\Big]p_j^L\\ &+\mu^{M-L}\left(k_j+G_j^L\right)\left(k_j+G_j^M\right)p_i^L\Big\}.\end{aligned} \tag{7}$$

Analytical expressions for energy density and intensity of Bloch waves and their averages over a unit cell including low-frequency limits are given in [Maysenhölder, Ch. 5 (1994)].

Special case: Locally isotropic medium with one-dimensional periodicity along the x-direction (density ρ, stiffness $\zeta=\lambda+2\mu$, spatial period h). Consider longitudinal waves in x-direction (one-dimensional problem).

a) *Example with analytical solution*:

Unit cell: two homogeneous layers ($n=1,2$) of thicknesses h_n with ρ_n and ζ_n. For given frequency ω the Bloch-wavenumber k follows from [Beltzer, pp. 216–219 (1988)]:

$$\cos(kh)=\cos(\omega t_1)\cos(\omega t_2)-\Delta\sin(\omega t_1)\sin(\omega t_2) \tag{8}$$

with travel times $t_n=h_n/c_n$, phase velocities $c_n=\sqrt{\zeta_n/\rho_n}$ and

$$\Delta=\frac{\rho_1\zeta_1+\rho_2\zeta_2}{2\sqrt{\rho_1\rho_2\zeta_1\zeta_2}}. \tag{9}$$

$|\cos(kh)|\le 1$: pass bands; $|\cos(kh)|>1$: stop bands.

b) *Low-frequency limit (homogenisation)*:

Real formulation for $\zeta(-x)=\zeta(x)$ with spatial averages $\langle\zeta(x)\rangle=\zeta_0$ and $\langle\rho(x)\rangle=\rho_0$. For $k\ll 2\pi/h$ the displacement field of a Bloch wave may be approximated by a 'phase-modulated' wave:

$$\begin{aligned}u(x,t)&\propto\cos(kx-\omega t)-k\,q(x)\sin(kx-\omega t)\\ &\approx\cos\left[k\left(x+q(x)\right)-\omega t\right]\end{aligned} \tag{10}$$

with anti-symmetric periodic 'modulation function'

$$q(x)=-x+\zeta_{\mathrm{eff}}\int_0^x\frac{dx'}{\zeta(x')},\quad q(0)=q\left(\frac{h}{2}\right)=0. \tag{11}$$

The effective properties are:

$$\rho_{\text{eff}} = \rho_0, \quad \zeta_{\text{eff}} = \left\langle \frac{1}{\zeta(x)} \right\rangle^{-1}, \quad c_{\text{eff}} = \sqrt{\frac{\zeta_{\text{eff}}}{\rho_0}}. \tag{12}$$

Explicitly for the example a):

$$\zeta_{\text{eff}} = \frac{h}{\dfrac{h_1}{\zeta_1} + \dfrac{h_2}{\zeta_2}}, \tag{13}$$

$$q(x) = \begin{cases} \left(\dfrac{\zeta_{\text{eff}}}{\zeta_1} - 1\right) x & 0 \le x \le \dfrac{h_1}{2} \\ & \text{for} \\ \left(\dfrac{\zeta_{\text{eff}}}{\zeta_2} - 1\right) x + \dfrac{h_1 \zeta_{\text{eff}}}{2} \left(\dfrac{1}{\zeta_1} - \dfrac{1}{\zeta_2}\right) & \dfrac{h_1}{2} < x \le \dfrac{h}{2} \end{cases}. \tag{14}$$

For continuous variation of the stiffness, e.g. $\zeta(x) = \zeta_0 + \zeta_s \cos(2\pi x/h) > 0$, one obtains:

$$\zeta_{\text{eff}} = \sqrt{\zeta_0^2 - \zeta_s^2}, \quad q(x) = -x + \frac{h}{\pi} \arctan\left[\sqrt{\frac{1 - \zeta_s/\zeta_0}{1 + \zeta_s/\zeta_0}} \tan\frac{\pi x}{h}\right]. \tag{15}$$

Q.8 Homogenisation

▸ *See also:* Beltzer, pp. 187–202 (1988)

Homogenisation substitutes an inhomogeneous medium by an ‘equivalent’ homogeneous medium with effective material properties in the limit of low frequencies. The definition of effective elastic constants is by spatial averages:

$$\langle \sigma_{ij} \rangle = C_{ijkl}^{\text{eff}} \langle \varepsilon_{kl} \rangle \quad ; \quad \text{in general} \quad C_{ijkl}^{\text{eff}} \neq \langle C_{ijkl} \rangle . \tag{1}$$

In general, the effective density may become complex [Viktorova/Tyutekin (1998)] or even a tensor [Helbig, pp. 298, 322–323 (1994)], however, in all cases listed below $\rho_{\text{eff}} = \langle \rho \rangle$.

Q.8.1 Bounds on Effective Moduli

For an inhomogeneous elastic medium consisting of N phases with volume fractions ϕ_n and elastic constants $C_{ijkl}^{(n)}$ the *Voigt and Reuss averages,*

$$C_{ijkl}^{\text{Voigt}} = \sum_{n=1}^{N} \phi_n C_{ijkl}^{(n)}, \qquad C_{ijkl}^{\text{Reuss}} = \left[\sum_{n=1}^{N} \frac{\phi_n}{C_{ijkl}^{(n)}}\right]^{-1}, \quad C_{ijkl}^{\text{Reuss}} \le C_{ijkl}^{\text{eff}} \le C_{ijkl}^{\text{Voigt}}, \tag{2}$$

are rigorous bounds for the effective elastic constants, which hold for arbitrary structures. These bounds are useful for small ϕ_n and small differences between the $C_{ijkl}^{(n)}$.

Narrower bounds are available for special cases, e.g. the *Hashin-Shtrikman bounds* for $N = 2$ statistically distributed isotropic phases (K_n, μ_n: compression and shear moduli; $\phi_2 \equiv \phi$):

$$\frac{\left(K_1 + \tilde{K}\right)\phi}{K_1 + \tilde{K} + (K_2 - K_1)\,(1-\phi)} \le \frac{K_{eff} - K_1}{K_2 - K_1} \le \frac{\left(K_1 + \tilde{\tilde{K}}\right)\phi}{K_1 + \tilde{\tilde{K}} + (K_2 - K_1)\,(1-\phi)}, \tag{3}$$

$$\frac{\left(\mu_1 + \tilde{\mu}\right)\phi}{\mu_1 + \tilde{\mu} + \left(\mu_2 - \mu_1\right)\,(1-\phi)} \le \frac{\mu_{eff} - \mu_1}{\mu_2 - \mu_1} \le \frac{\left(\mu_1 + \tilde{\tilde{\mu}}\right)\phi}{\mu_1 + \tilde{\tilde{\mu}} + \left(\mu_2 - \mu_1\right)\,(1-\phi)}, \tag{4}$$

where, if $(\mu_2 - \mu_1)(K_2 - K_1) \ge 0$, then

$$\tilde{K} = \frac{4\,\mu_1}{3},\quad \tilde{\tilde{K}} = \frac{4\,\mu_2}{3},\quad \tilde{\mu} = \frac{3}{2}\left(\frac{1}{\mu_1} + \frac{10}{9\,K_1 + 8\,\mu_1}\right)^{-1},\ \tilde{\tilde{\mu}} = \frac{3}{2}\left(\frac{1}{\mu_2} + \frac{10}{9\,K_2 + 8\,\mu_2}\right)^{-1}, \tag{5}$$

while, if $(\mu_2 - \mu_1)(K_2 - K_1) \le 0$, then

$$\tilde{K} = \frac{4\,\mu_2}{3},\quad \tilde{\tilde{K}} = \frac{4\,\mu_1}{3},\quad \tilde{\mu} = \frac{3}{2}\left(\frac{1}{\mu_1} + \frac{10}{9\,K_2 + 8\,\mu_1}\right)^{-1},\ \tilde{\tilde{\mu}} = \frac{3}{2}\left(\frac{1}{\mu_2} + \frac{10}{9\,K_1 + 8\,\mu_2}\right)^{-1}. \tag{6}$$

For $\mu_1 = \mu_2$ the bounds for K_{eff} coincide and yield the exact K_{eff}.

Q.8.2 Effective Moduli for Particular Structures

a) Voigt and Reuss averages for polycrystals with statistical orientation of the grains (all of the same anisotropic material) are approximations for compression and shear moduli of an effective isotropic material:

$$K_{Voigt} = \frac{1}{9} C_{iikk}\,, \qquad \mu_{Voigt} = \frac{1}{10}\left(C_{ikik} + \frac{1}{3} C_{iikk}\right). \tag{7}$$

Reuss averages are obtained accordingly with the compliance tensor (inverse of C_{ijkl}).

For a material with cubic grains:

$$K_{Voigt} = \frac{1}{3}\left(c_{11} + 2\,c_{12}\right) = K\,, \qquad \mu_{Voigt} = \frac{1}{5}\left(c_{11} - c_{12} + 3\,c_{44}\right) = \frac{1}{5}\left(2\,\mu' + 3\,\mu\right). \tag{8}$$

In this case $K_{Reuss} = K_{Voigt} = K_{eff}$.

b) Spherical inclusions (K_s, μ_s, radius r) randomly dispersed in a homogeneous matrix with K_m and μ_m; wavelengths much larger than r. Neglecting multiple scattering leads to the approximation for small volume fractions ϕ of the spheres:

$$K_{eff} = K_m + \frac{\left(3\,K_m + 4\,\mu_m\right)\,\left(K_s - K_m\right)\phi}{\left(3\,K_s + 4\,\mu_m\right) - 3\,\left(K_s - K_m\right)\phi}, \tag{9}$$

$$\mu_{eff} = \mu_m + \frac{\mu_m\,\left(15\,K_m + 20\,\mu_m\right)\,\left(\mu_s - \mu_m\right)\,\phi}{6\,\mu_s\,\left(K_m + 2\,\mu_m\right) + \mu_m\,\left(9\,K_m + 8\,\mu_m\right) - 6\,\left(K_m + 2\,\mu_m\right)\,\left(\mu_s - \mu_m\right)\,\phi}. \tag{10}$$

c) Composite sphere assembly: A sphere composed of two isotropic materials (K_s, μ_s up to inner radius a; K_m, μ_m from a to outer radius b; $\phi = a/b$) has an effective compression modulus K_{eff}, which is equal to the above expression for spherical inclusions. An assembly of such composite spheres with different sizes, but common ratio ϕ, possesses the same K_{eff} in the long-wavelength limit (exact results).

d) Periodically spaced fibres parallel to the x-axis in isotropic matrix; ϕ = volume fraction of the fibres. Both the fibre material and the effective medium are transversely isotropic. Approximation for the effective moduli in 'engineering notation' according to Skelton/James, p. 151 (1997):

$$E_{xx} = \phi\, E^f_{xx} + (1-\phi)\, E_m, \qquad E_{yy} = E_{zz} = \frac{E_m}{1-(1-E_m/E^f_{yy})\sqrt{\phi}}, \tag{11}$$

$$G_{xy} = G_{xz} = \frac{G_m}{1-(1-G_m/G^f_{xy})\sqrt{\phi}}, \qquad G_{yz} = \frac{G_m}{1-(1-G_m/G^f_{yz})\sqrt{\phi}}, \tag{12}$$

$$\nu_{xy} = \nu_{xz} = \phi\, \nu^f_{xy} + (1-\phi)\, \nu_m, \qquad \left(\nu_{yz} = -1 + \frac{E_{yy}}{2\, G_{yz}}\right). \tag{13}$$

(E_m, ν_m, $G_m = E_m/[2(1+\nu_m)]$: Young's modulus, Poisson's ratio and shear modulus of isotropic matrix; superscript f denotes fibre properties.) An alternative for isotropic fibre materials (with E_f, ν_f and $G_f = E_f/[2(1+\nu_f)]$ are the Halpin-Tsai equations [Jones, pp. 151–158 (1999)]:

$$E_{xx} = \phi\, E_f + (1-\phi)\, E_m, \qquad \nu_{xy} = \nu_{xz} = \phi\, \nu_f + (1-\phi)\, \nu_m, \tag{14}$$

$$\frac{M}{M_m} = \frac{1+\xi\,\eta\,\phi}{1-\eta\,\phi} \quad \text{with} \qquad \eta = \frac{(M_f/M_m)-1}{(M_f/M_m)+\xi}, \tag{15}$$

where M is one of the quantities E_{yy}, G_{xy} or ν_{yz} of the composite and M_m or M_f are the corresponding quantities E, G or ν for the matrix and the fibre materials. The three ξ's are adjustable parameters depending on fibre and packing geometry and can range from 0 to ∞. Example: Circular fibres in a square array: Use $\xi = 2$ for E_{yy} and $\xi = 1$ for G_{xy}.

e) Symmetric stack of layers with unidirectional fibre reinforcement (for properties of transversely isotropic fibres see above). Fractional layer thickness h_n, total thickness h; the fibre directions relative to the global x-axis may be different for different layers, but lie all in the x-y-plane of the layers. By symmetric is meant that the top half of the stack is a mirror image of the bottom half. After transformation to global co-ordinates the elastic constants (now 'monoclinic-like') of the n-th layer are denoted by c_{nIJ}, the constants of the monoclinic effective medium by c_{IJ} (no summation convention):

$$c_{33} = \left[\sum_{n=1}^{N} \frac{h_n}{c_{n33}}\right]^{-1}, \qquad c_{13} = \frac{1}{2}\sum_{n=1}^{N} h_n\, c_{n13}\left(1+\frac{c_{33}}{c_{n33}}\right), \tag{16}$$

$$c_{23} = \frac{1}{2}\sum_{n=1}^{N} h_n\, c_{n23}\left(1+\frac{c_{33}}{c_{n33}}\right), \qquad c_{36} = \frac{1}{2}\sum_{n=1}^{N} h_n\, c_{n36}\left(1+\frac{c_{33}}{c_{n33}}\right), \tag{17}$$

$$c_{IJ} = \sum_{n=1}^{N} h_n\, c_{nIJ} \quad \{I, J = 1, 2, 6\}, \qquad c_{IJ} = \sum_{n=1}^{N} \frac{h_n\, c_{nIJ}}{\Delta\, \Delta_n} \quad \{I, J = 4, 5\}, \tag{18}$$

$$\Delta = \left[\sum_{n=1}^{N} \frac{h_n\, c_{n44}}{\Delta_n}\right]\left[\sum_{n=1}^{N} \frac{h_n\, c_{n55}}{\Delta_n}\right] - \left[\sum_{n=1}^{N} \frac{h_n\, c_{n45}}{\Delta_n}\right]^2, \quad \Delta_n = c_{n44}\, c_{n55} - (c_{n45})^2. \tag{19}$$

The formulae are valid for wavelengths greater than the total thickness of the stack [Skelton/James, pp. 159–160 (1997)].

f) Periodic stack of transversely isotropic layers. Notation as in the previous case (h = period). Note that in both *e)* and *f)* the constituent layers are transversely isotropic, however, in *e)* the axes of symmetry lie – possibly in different directions – in the x-y-plane, whereas now the axis of symmetry is the z-axis for all layers:

$$c_{33} = \left[\sum_{n=1}^{N} \frac{h_n}{c_{n33}}\right]^{-1}, \quad c_{13} = c_{33} \sum_{n=1}^{N} h_n \frac{c_{n13}}{c_{n33}}, \tag{20}$$

$$\begin{aligned} c_{11} &= \frac{(c_{13})^2}{c_{33}} + \sum_{n=1}^{N} h_n \left(c_{n11} - \frac{(c_{n13})^2}{c_{n33}}\right), \\ c_{55} &= \left[\sum_{n=1}^{N} \frac{h_n}{c_{n55}}\right]^{-1}, \quad c_{66} = \sum_{n=1}^{N} h_n\, c_{n66}. \end{aligned} \tag{21}$$

The formulae are valid for wavelengths greater than the total thickness of the period [Helbig, p. 313 (1994)]. If all constituent layers are isotropic, then c_{55} cannot be greater than c_{66} (because of the Cauchy-Schwartz-Kolmogorov inequality). For the inverse problem, the determination of the c_{nIJ} from the c_{IJ}, with N = 2 or 3 see [Helbig, pp. 324–336 (1994)].

For further homogenisation results see *Sect. I.13* (sandwich panels) and *Sect. Q.7* (periodic media).

Q.9 Plane Waves in Unbounded Homogeneous Media

▸ *See also:* Maysenhölder, pp. 50–51, 91–100 (1994); Beltzer, pp. 95–109 (1988)

Q.9.1 Anisotropic Media

The equation of motion for homogeneous anisotropic media (see *Sect. Q.1*) can be satisfied by the plane-wave ansatz $u_i = A_i \exp[(j(\omega t - k_m x_m)]$ with real amplitude A_i defining the polarisation, wavenumber k and wavevector direction e_i, leading to Christoffel's equation:

$$\left(\Gamma_{il} - \delta_{il}\rho\, c^2\right) A_l = 0 \quad \text{with} \quad \Gamma_{il} = C_{ijkl}\, e_j e_k = \Gamma_{li}{}^{*)} \tag{1}$$

*) See Preface to the 2nd edition.

(Γ_{il}: Christoffel's tensor). For a given direction e_i this is a cubic eigenvalue problem for eigenvalues ρc^2 and eigenvectors A_i ($c = \omega/k$: magnitude of phase velocity). Since without dissipative effects the symmetric Γ_{il} is also real, the polarisations are mutually orthogonal or can be chosen as such in case of degenerate eigenvalues. 'Pure modes' are – according to one definition – waves with purely longitudinal or purely transversal polarisation. Another definition requires the group velocity or intensity to be parallel to the wavevector. A wave which is not a pure mode may be termed quasi-longitudinal or quasi-transversal, if its polarisation is close to one of the pure polarisation.

The group velocity C_i is equal to the velocity of energy transport, which is in general not parallel to e_i:

$$C_j = \frac{\partial c}{\partial e_j} = \frac{A_i\, C_{ijkl}\, A_l\, e_k}{\rho\, A_m^2\, c}; \qquad C_i\, e_i = c. \tag{2}$$

Time average of energy density: $$\left\langle e_{kin} + e_{pot} \right\rangle_t = 2\, \langle e_{kin} \rangle_t = \frac{1}{2}\, \rho\, \omega^2\, A_i^2\,; \tag{3}$$

Intensity: $$I_j = \left\langle e_{kin} + e_{pot} \right\rangle_t\, C_j = \frac{\omega^2}{2\,c}\, A_i\, C_{ijkl}\, A_l\, e_k\,. \tag{4}$$

The slowness vector ℓ_m, which is useful for reflection and refraction problems (see *Sect. Q.3*) is defined by:

$$\ell_m = \frac{e_m}{c} = \frac{k_m}{\omega}: \qquad \exp[(j(\omega t - k_m x_m)] = \exp[(j\omega(t - \ell_m x_m)].$$

Two-dimensional example with cubic (i.e. quadratic) anisotropy:

Wave vector direction: $$e_i = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}; \tag{5}$$

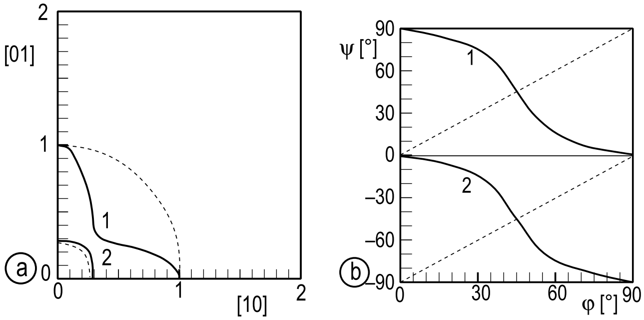
Polarisation angle: $$\psi = \arctan \frac{A_2}{A_1}\,; \tag{6}$$

Eigenvalues:

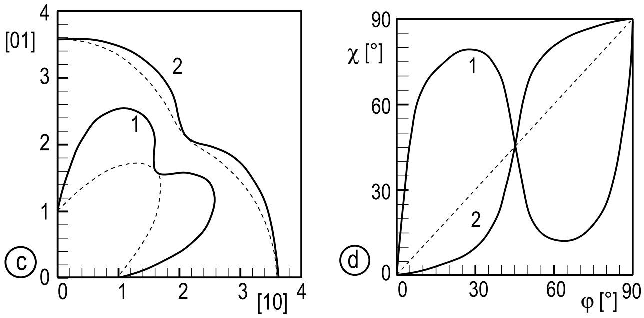
$$\rho\, c^2 = \frac{1}{2}\left\{ c_{11} + c_{44} \pm \sqrt{\frac{1}{2}\,(c_{11} - c_{44})^2\,\left(1 + \cos 4\varphi\right) + \frac{1}{2}\,(c_{12} + c_{44})^2\,\left(1 - \cos 4\varphi\right)} \right\}; \tag{7}$$

Polarisations: $$\frac{A_2}{A_1} = \frac{\rho\, c^2 - c_{11} \cos^2\varphi - c_{44} \sin^2\varphi}{(c_{12} + c_{44}) \cos\varphi \sin\varphi}\,. \tag{8}$$

The figures below are drawn for compression modulus $K = 1$, shear modulus $\mu' = 9$ and shear modulus $\mu = 1$ (fictitious material, arbitrary units; anisotropy $a = 0.8$). The two eigensolutions (modes) are numbered according to increasing phase velocity.



a) Slowness diagram (polar diagram of the slowness); [10]: x_1-direction; [01]: x_2-direction; dotted circles: isotropic case. b) Polarisation ψ. Dotted lines: longitudinal polarisation $\psi = \varphi$, and transversal polarisation $\psi = \varphi - 90^\circ$



c) Group velocity (polar diagram). Dotted lines: phase velocity. d) Intensity direction χ. Dotted line: propagation direction φ

Q.9.2 Isotropic Media

In isotropic materials Christoffel's equation leads to two types of solutions: longitudinal and transversal plane waves with wave speeds

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_T = \sqrt{\frac{\mu}{\rho}}. \tag{9}$$

The displacement field of longitudinal waves is curl free (changes of shape and volume without rotations), that of transversal waves is divergence free (change of shape without volume change). The strain and stress amplitudes for waves propagating in the x-direction with velocity amplitude v_0 are for longitudinal polarisation:

$$-\frac{v_0}{c_L}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad -\rho\, v_0\, c_L \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\sigma}{1-\sigma} & 0 \\ 0 & 0 & \dfrac{\sigma}{1-\sigma} \end{pmatrix} \tag{10}$$

and for transversal polarisation along the y-direction:

$$-\frac{v_0}{2\,c_T}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad -\frac{1}{2}\rho\, v_0\, c_T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

The formulas for energetic quantities are of the same form for both wave types:

$$e_{kin} = e_{pot} = \frac{1}{2}\rho\,|v_0|^2\cos^2(kx-\omega t)\,,\qquad S_x = \rho\,|v_0|^2\,c\,\cos^2(kx-\omega t)\,, \tag{12}$$

$$\langle e_{kin}\rangle_t = \langle e_{pot}\rangle_t = \frac{1}{4}\rho|v_0|^2 \qquad I_x = \frac{1}{2}\rho\,|v_0|^2\,c = 2\,\langle e_{kin}\rangle_t\,c. \tag{13}$$

(Add subscript $_L$ or $_T$ to the wave number k and the wave speed c as required.).

All y- and z-components and the reactive intensity vanish.

Q.10 Waves in Bounded Media

Q.10.1 Plate Waves

▸ *See also:* Maysenhölder (1990) Plate waves may be classified into two groups: shear waves and Lamb waves. Each group is further subdivided into a symmetric family (mirror symmetry of the displacement vector with respect to the plane $z = 0$) and an anti-symmetric family (sign change of the displacement vector after the mirror operation). The *shear waves*, sometimes called SH ('Shear-Horizontal') waves, possess non-vanishing displacements in the y direction only:

$$u_y = A\cos\left(\frac{n\pi z}{h}\right),\quad n = 0, 2, 4, \ldots\quad \text{(symmetric family);} \tag{1}$$

$$u_y = A\sin\left(\frac{n\pi z}{h}\right),\quad n = 1, 3, 5, \ldots\quad \text{(anti-symmetric family).} \tag{2}$$

(A: Amplitude; phase factor $\exp[(j(\omega t - kx)]$ omitted.) The expressions for phase velocities c and group velocities C are valid for both families ($c_T = \sqrt{\mu/\rho}$):

$$c^2 = c_T^2\left[1+\left(\frac{n\pi}{kh}\right)^2\right] = \frac{c_T^2}{1-\left(\frac{n\,c_T}{2fh}\right)^2},\qquad C\cdot c = c_T^2. \tag{3}$$

Conversely, $u_y \equiv 0$ for *Lamb waves*. With the abbreviations

$$\alpha_1 = \sqrt{1-\left(\frac{c}{c_L}\right)^2},\quad \alpha_2 = \sqrt{1-\left(\frac{c}{c_T}\right)^2},\quad \alpha_x = \frac{2\,\alpha_1\,\alpha_2}{1+\alpha_2^2}, \tag{4}$$

$$\alpha_z = \frac{2}{1+\alpha_2^2},\quad R_s = \frac{\sinh(\alpha_1 kh/2)}{\sinh(\alpha_2 kh/2)},\quad R_a = \frac{\cosh(\alpha_1 kh/2)}{\cosh(\alpha_2 kh/2)}, \tag{5}$$

the x- and z-components of the displacement fields are for the symmetric family:

$$u_x = j\,A\,[\cosh(\alpha_1 kz) - \alpha_x R_s \cosh(\alpha_2 kz)]\,,$$

$$u_z = \alpha_1 A\,[\sinh(\alpha_1 kz) - \alpha_z R_s \sinh(\alpha_2 kz)]\,, \tag{6}$$

and for the anti-symmetric family:

$$u_x = j\,A\,[\sinh(\alpha_1 kz) - \alpha_x R_a \sinh(\alpha_2 kz)]\,,$$

$$u_z = \alpha_1 A \ [\cosh(\alpha_1 kz) - \alpha_z R_a \ \cosh(\alpha_2 kz)] \, . \tag{7}$$

Since measurements are usually confined to the surfaces of the plate, the displacement ratios u_x/u_z at $z = h/2$ are of particular interest [Maysenhölder (1987)]:

$$\left.\frac{u_x}{u_z}\right|_{z=h/2} = \frac{\left(1+\alpha_2^2\right)\coth(\alpha_1 kh/2) - 2\,\alpha_1\,\alpha_2\ \coth(\alpha_2 kh/2)}{-j\,\alpha_1\ \left(\alpha_2^2 - 1\right)} \quad \text{(symmetric family)}, \tag{8}$$

$$\left.\frac{u_x}{u_z}\right|_{z=h/2} = \frac{\left(1+\alpha_2^2\right)\tanh(\alpha_1 kh/2) - 2\,\alpha_1\,\alpha_2\ \tanh(\alpha_2 kh/2)}{-j\,\alpha_1\ \left(\alpha_2^2 - 1\right)} \text{(anti-symmetric family)}. \tag{9}$$

Usually, the phase velocities have to be determined numerically from the transcendental Rayleigh-Lamb frequency equations:

$$\frac{4\,\alpha_1\,\alpha_2}{\left(1+\alpha_2^2\right)^2} = \left[\frac{\tanh(\pi\,\alpha_2 f\,h/c)}{\tanh(\pi\,\alpha_1 f\,h/c)}\right]^{\pm 1} \tag{10}$$

(f: frequency) with the plus sign for the symmetric family and the minus sign for the anti-symmetric family. For dispersion diagrams see e.g. [Auld, pp. 76–87 Vol. II, "Acoustic fields and waves in solids" (1990); Cremer/Heckl, p. 143 (1996)]. The orthogonality relation for two Rayleigh-Lamb modes (1) and (2) with common frequency, but different wavenumbers,

$$\int_{-h/2}^{h/2} \left(u_x^{(1)}\,\sigma_{xx}^{(2)} - u_z^{(2)}\,\sigma_{zx}^{(1)}\right)\,dz \ = 0, \tag{11}$$

holds even for the corresponding modes of a layered plate with z-dependent Lamé constants [Murphy/Li/Chin-Bing (1994)].

If c is known, the corresponding group velocity C may be obtained analytically [Maysenhölder (1992)] (with the same meaning of ± as above):

$$C = \frac{c}{1 - \dfrac{f}{c}\dfrac{dc}{df}} \quad \text{with} \quad \frac{f}{c}\,\frac{dc}{df} = \frac{\pm Y}{X_\pm \pm Z}, \tag{12}$$

$$X_+ = T_1^2\,X, \quad X_- = T_2^2\,X, \quad T_1 = \tanh(\pi\,\alpha_1 f\,h/c), \quad T_2 = \tanh(\pi\,\alpha_2 f\,h/c),$$

$$X = \frac{4\,c^3}{\pi\,c_T^2 f\,h\,N^2}\left[\frac{4\,\alpha_1^2\,\alpha_2^2}{N} - \alpha_1^2 - \left(\frac{\alpha_2\,c_T}{c_L}\right)^2\right], \quad N = 1 + \alpha_2^2 = 2 - \left(\frac{c}{c_T}\right)^2,$$

$$Y = \alpha_1\,T_1\,\alpha_2^2\,K_2 - \alpha_2\,T_2\alpha_1^2\,K_1, \qquad Z = \alpha_1\,T_1\,K_2 - \alpha_2\,T_2\,K_1,$$

$$K_1 = \cosh^{-2}(\pi\,\alpha_1 f\,h/c), \qquad K_2 = \cosh^{-2}(\pi\,\alpha_2 f\,h/c).$$

The group velocity C together with Rayleigh's principle for propagating waves (see ➲ *Sect. Q.5.4*) may be used for the calculation of the average intensity of a plate wave:

$\langle I_x \rangle = 2\,\langle w_{kin} \rangle\, C$, with the time average w_{kin} of the kinetic energy density and the average over the plate thickness denoted by $\langle \ldots \rangle$. For both families of shear waves:

$$\langle w_{kin} \rangle = \frac{\pi^2}{2}\; |A|^2\; \rho\, f^2. \tag{13}$$

With $\quad S_m = \dfrac{\sinh(\alpha_m\, k\, h)}{\alpha_m\, k\, h} \qquad (m = 1,\; 2) \quad$ for symmetric Lamb waves : (14)

$$\begin{aligned}\langle w_{kin} \rangle = \frac{\pi^2}{2}\; |A|^2\; \rho\, f^2\; \Big\{ & (S_1 + 1) + |\alpha_x|^2\; |R_s|^2\; (S_2 + 1) \\ & + |\alpha_1|^2\; \big[|S_1 - 1| + |\alpha_z|^2\, |R_s|^2\, |S_2 - 1| \big] - 4\,\alpha_1^2\,\alpha_z\, S_1 \Big\}\,,\end{aligned} \tag{15}$$

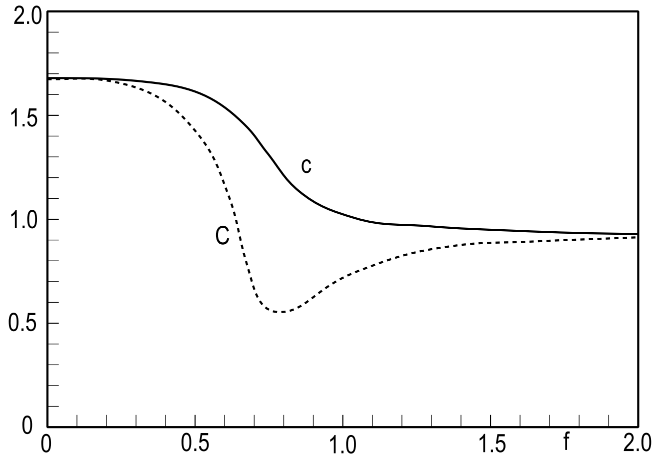
and for anti-symmetric Lamb waves:

$$\begin{aligned}\langle w_{kin} \rangle = \frac{\pi^2}{2}\; |A|^2\; \rho\, f^2\; \Big\{ & |S_1 - 1| + |\alpha_x|^2\, |R_a|^2\, |S_2 - 1| \\ & + |\alpha_1|^2\; \big[(S_1 + 1) + |\alpha_z|^2\; |R_a|^2\; (S_2 + 1) \big] - 4\; |\alpha_1|^2\; \alpha_z\, S_1 \Big\}\,.\end{aligned} \tag{16}$$

(Note the little difference in the last term. It is essential for imaginary α_1, i.e. for $c > c_L$.) The y- and z-components of the intensity are everywhere zero for all wave families.

Special case: *Quasi-longitudinal mode*:

This is the fundamental symmetric Lamb wave, which exhibits predominantly longitudinal character at low frequencies.



Phase velocity c and group velocity C of the quasi-longitudinal mode for Poisson's ratio $\sigma = 0.3$. Units: c_T for velocities, c_T/h for frequency

At $f = c_T/\sqrt{2}h$ phase and group velocity are independent of Poisson's ratio σ and analytically known (Lamé wave): $c = c_T\sqrt{2}$, $C = c_T/\sqrt{2}$. This implies $kh = \pi$ (wavelength $= 2h$) and

$$u_x = -A\ \cos\left(\frac{\pi z}{h}\right), \qquad u_z = j\, A \sin\left(\frac{\pi z}{h}\right), \tag{17}$$

$$\langle w_{kin}\rangle = \frac{\pi^2}{2}\left|\frac{A}{h}\right|^2 \mu, \qquad \langle I_x\rangle = \frac{\pi^2}{\sqrt{2}}\left|\frac{A}{h}\right|^2 \mu\, c_T. \tag{18}$$

The strain field of this wave is pure shear and $u_x \equiv 0$ at the plate surfaces!

Low-frequency approximation:

$$c_{QL} = C_{QL} = c_T\sqrt{\frac{2}{1-\sigma}}, \qquad \frac{u_z}{u_x} = \frac{-j\,\sigma}{1-\sigma} k\, z, \qquad \langle w_{kin}\rangle = \pi^2\ |A|^2\ \rho\, f^2. \tag{19}$$

For Poisson's ratio $\sigma = 0.3$ the error of $\langle I_x\rangle = 2\ \langle w_{kin}\rangle\ C$ in this approximation is smaller than 20% (10%; 5%) for $fh/c_T < 0.4$ (0.3; 0.1).

Low-frequency expansions up to the third non-vanishing order [Maysenhölder (1987)]:

$$\begin{aligned} c^2 &= c_{QL}^2\left[1 - \frac{\sigma^2}{12\ (1-\sigma)^2}\ (h\,k)^2 - \frac{\sigma^2\ \left(6\ - 10\,\sigma - 7\,\sigma^2\right)}{720\ (1-\sigma)^4}\ (h\,k)^4\right] \\ &= c_{QL}^2\left[1 - \frac{\pi^2\,\sigma^2}{3\ (1-\sigma)^2}\left(\frac{h\,f}{c_{QL}}\right)^2 - \frac{\pi^4\,\sigma^2\ \left(6\ - 10\,\sigma - 2\,\sigma^2\right)}{45\ (1-\sigma)^4}\left(\frac{h\,f}{c_{QL}}\right)^4\right]. \end{aligned} \tag{20}$$

High-frequency approximation ($\approx$ a Rayleigh wave on each plate surface):

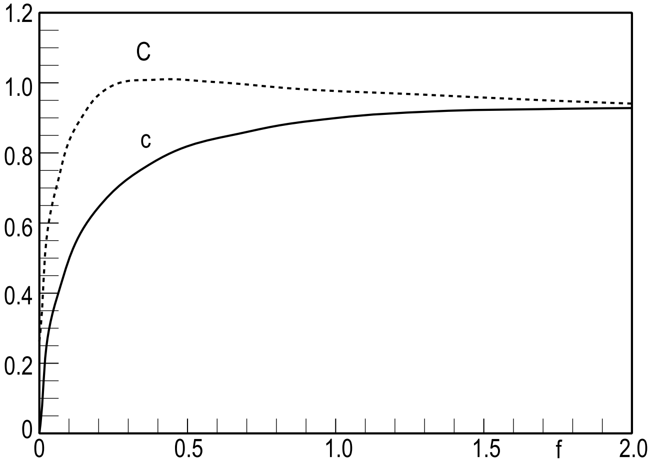
$$c = C = c_R, \qquad \frac{u_x}{u_z} = \frac{j\left[\exp\left(\alpha_1 kz\right) - \alpha_x\ \exp\left(\alpha_2 kz\right)\right]}{\alpha_1\left[\exp\left(\alpha_1 kz\right) - \alpha_z\ \exp\left(\alpha_2 kz\right)\right]}, \tag{21}$$

$$\langle w_{kin}\rangle = \frac{\pi}{8}\ |A|^2\ \rho\ \frac{f\, c_R}{h}\ e^{\alpha_1 kh}\ \left\{\frac{1}{\alpha_1} + \alpha_1\,(1 - 4\,\alpha_z) + (\alpha_1\alpha_z)^2\left(\frac{1}{\alpha_2} + \alpha_2\right)\right\}. \tag{22}$$

For Poisson's ratio $\sigma = 0.3$ the error of $\langle I_x\rangle = 2\ \langle w_{kin}\rangle\ C$ in this approximation is smaller than 20% (10%; 5%) for $fh/c_T > 1.5$ (1.9; 3.0).

Special case: *Bending mode*:

This is the fundamental anti-symmetric Lamb wave, which coincides with the bending wave in thin plates at low frequencies.



Phase velocity c and group velocity C of the bending mode for Poisson's ratio $\sigma = 0.3$. Units: c_T for velocities, c_T/h for frequency

Low-frequency approximation (lowest order; corresponding to thin plate theory):

$$c_B = \frac{\sqrt{2\,\pi\, f\, h\, c_T}}{\sqrt[4]{6\,(1-\sigma)}}, \quad C_B = 2\, c_B, \quad k_B = \sqrt[4]{6\,(1-\sigma)}\sqrt{\frac{2\,\pi\, f}{h\, c_T}}, \tag{23}$$

$$\lambda_B = \frac{\sqrt{2\,\pi\, h\, c_T}}{\sqrt[4]{6\,(1-\sigma)}\,\sqrt{f}}, \quad \frac{u_x}{u_z} = -j\,kz, \quad \langle w_{kin}\rangle = \pi^2\ |A|^2\ \rho\, f^2. \tag{24}$$

For Poisson's ratio $\sigma = 0.3$ the error of $\langle I_x\rangle = 2\ \langle w_{kin}\rangle\ C$ in this approximation is smaller than 20% (10%; 5%) for $fh/c_T < 0.13$ (0.065; 0.007). (For formulas with bending stiffness B see *Sect. Q.10.3*)

Low-frequency expansions up to the third non-vanishing order [Maysenhölder (1987)]:

$$\begin{aligned} c^2 &= \frac{c_{QL}^2}{12}\ (h\,k)^2\ \left[1 - \frac{17-7\,\sigma}{60\ (1-\sigma)}\ (h\,k)^2 + \frac{489 - 418\,\sigma + 62\,\sigma^2}{5040\ (1-\sigma)^2}\ (h\,k)^4\right] \\ &= \frac{\pi\, c_{QL}}{\sqrt{3}}\ (h\,f)\ \left[1 - \frac{\pi\sqrt{3}\ (17-7\,\sigma)}{30\ (1-\sigma)}\ \left(\frac{h\,f}{c_{QL}}\right) \right. \\ &\quad \left. + \frac{\pi^2\ \left(3711\ - 3362\,\sigma + 211\,\sigma^2\right)}{4200\ (1-\sigma)^2}\ \left(\frac{h\,f}{c_{QL}}\right)^2\right]. \end{aligned} \tag{25}$$

The high-frequency approximation (≈ a Rayleigh wave on each plate surface) is the same as with the quasi-longitudinal mode. For Poisson's ratio $\sigma = 0.3$ the error of $\langle I_x\rangle = 2\ \langle w_{kin}\rangle\ C$ in this approximation is smaller than 20% (10%; 5%) for $fh/c_T > 0.67$ (0.77; 3.0).

For additional formulas and diagrams including z-dependence of displacements, energy densities and intensities see [Maysenhölder (1990)].

Q.10.2 Rayleigh Waves

The wave speed c_R of Rayleigh waves on a force-free surface of an isotropic half-space is obtained from the positive solution of the equation for γ^2:

$$(2-\gamma^2)^2 - 4\sqrt{(1-\zeta^2\gamma^2)(1-\gamma^2)} = 0, \quad \gamma = \frac{c_R}{c_T}, \quad \zeta^2 = \left(\frac{c_T}{c_L}\right)^2 = \frac{1-2\sigma}{2-2\sigma}. \qquad (26)$$

There is exact one such solution within the bounds $0 < c_R < c_T$. Eliminating the square root leads to the more familiar form $\gamma^6 - 8\gamma^4 + 8(3-2\zeta^2)\gamma^2 - 16(1-\zeta^2) = 0$, which, however, has additional extraneous solutions [Achenbach, pp. 189–191 (1975)].

The non-vanishing displacement components of Rayleigh waves propagating along the x-direction on a half-space $z \le 0$ may be obtained by superposition of the two fundamental symmetric and anti-symmetric Lamb modes (for the α's with $c = c_R$ see *Sect. Q.10.1*):

$$\begin{aligned} u_x &= j\tilde{A}\left[\exp(\alpha_1 kz) - \alpha_x \exp(\alpha_2 kz)\right], \\ u_z &= \alpha_1 \tilde{A}\left[\exp(\alpha_1 kz) - \alpha_z \exp(\alpha_2 kz)\right]. \end{aligned} \qquad (27)$$

This describes elliptical trajectories at arbitrary depth. The sense of rotation changes at a depth of about 0.2 wavelengths, where $u_x = 0$.

Displacement ratio at the surface: $$\left.\frac{u_x}{u_z}\right|_{z=0} = \frac{j(1-\alpha_x)}{\alpha_1(1-\alpha_z)}; \qquad (28)$$

Time average of kinetic energy per unit width in y-direction:

$$W_{kin} = \int_{-\infty}^{0} w_{kin}\, dz = \frac{\pi}{4}\rho c_R f\, |\tilde{A}|^2 \left\{\frac{1}{\alpha_1} + \alpha_1(1-4\alpha_z) + (\alpha_1\alpha_z)^2\left(\frac{1}{\alpha_2} + \alpha_2\right)\right\}; \qquad (29)$$

Time average of energy flow per unit width in y-direction:

$$\int_{-\infty}^{0} I\, dz = 2\, W_{kin}\, c_R\,. \qquad (30)$$

There are various approximations to the velocity of Rayleigh waves for the range $0 < \sigma < \frac{1}{2}$, some of which are discussed in [Mozhaev (1991)]. The Bergmann-Viktorov equation:

$$\gamma = \frac{c_R}{c_T} = \frac{0.87 + 1.12\,\sigma}{1+\sigma} \qquad (31)$$

is accurate to within 0.5 %. For surface waves on anisotropic half-spaces see also [Ting/Barnett (1997)].

Q.10.3 Waves in Thin Plates

Wave equations for thin plates with thickness h (thickness direction = z-direction) and mass density ρ; excitation terms like force densities, moment densities or pressure differences are omitted. Solutions: Phase velocities $c = \omega/k$ of waves propagating along the x-direction (u, w: displacements in x- and z-direction) with phase factor $\exp[j(\omega t - kx)]$.

Nota bene: The symbols c_{QL} and B have different meanings for plates and beams!

The intensity I has dimension Wm^{-2}; the mean energy flow per unit width in a plate is $I \cdot h$.

Quasi-longitudinal waves (in-plane waves), [Cremer/Heckl, p. 86 (1996)]:

For a homogeneous isotropic plate:

$$-\frac{\partial^2 u}{\partial t^2} + c_{QL}^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad c_{QL} = \sqrt{\frac{E}{\rho(1-\sigma^2)}}\,; \tag{32}$$

$$\text{Transversal contraction: } \frac{\hat{w}}{\hat{u}} = \left(\frac{\sigma}{1-\sigma}\right)\frac{\pi h}{\lambda}. \quad I_{QL} = 2\pi^2\,|u_0|^2\,\rho f^2\,c_{QL} \tag{33}$$

($\hat{w}$, $\hat{u}$: maximum transversal and longitudinal displacements; u_0: amplitude of u). For range of validity see *Sect. Q.10.1*.

Bending waves:

Inhomogeneous, locally monoclinic plate ($x \equiv x_1, y \equiv x_2$), [Maysenhölder (1998)]:

$$\rho h \frac{\partial^2 w}{\partial t^2} + \sum_{\alpha,\beta,\gamma,\delta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(B_{\alpha\beta\gamma\delta}\frac{\partial^2 w}{\partial x_\gamma \partial x_\delta}\right) = 0 \tag{34}$$

with generalised bending stiffnesses:

$$B_{\alpha\beta\gamma\delta} = \frac{h^3}{12}\left(C_{\alpha\beta\gamma\delta} - \frac{C_{\alpha\beta 33}\,C_{\gamma\delta 33}}{C_{3333}}\right); \tag{35}$$

C_{ijkl}: elasticity tensor. ρ, h and $B_{\alpha\beta\gamma\delta}$ may depend on x_1 and x_2.

Inhomogeneous, locally isotropic plate, [Pierce (1993)]:

$$\begin{aligned} \rho h \frac{\partial^2 w}{\partial t^2} &+ \frac{\partial^2}{\partial x^2}\left(\frac{B\,\partial^2 w}{\partial x^2}\right) + 2\frac{\partial^2}{\partial x \partial y}\left(\frac{B\,\partial^2 w}{\partial x \partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{B\,\partial^2 w}{\partial y^2}\right) \\ &+ \frac{\partial^2}{\partial x^2}\left(\frac{\sigma B\,\partial^2 w}{\partial y^2}\right) - 2\frac{\partial^2}{\partial x \partial y}\left(\frac{\sigma B\,\partial^2 w}{\partial x \partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\sigma B\,\partial^2 w}{\partial x^2}\right) = 0 \end{aligned} \tag{36}$$

with the usual bending stiffness:

$$B = \frac{E h^3}{12(1-\sigma^2)} = \frac{\mu h^3}{6(1-\sigma)} = \frac{\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)}. \tag{37}$$

Homogeneous orthotropic plate, [Heckl (1960)]:

$$\rho h \frac{\partial^2 w}{\partial t^2} + \left(B_x \frac{\partial^4}{\partial x^4} + 2 B_{xy} \frac{\partial^4}{\partial x^2 \partial y^2} + B_y \frac{\partial^4}{\partial y^2} \right) w = 0 . \tag{38}$$

The bending stiffnesses are, [Haberkern, personal communication], in terms of the Voigt constants:

$$B_x = \frac{h^3}{12} \left[c_{11} - \frac{c_{13}^2}{c_{33}} \right] , \qquad B_{xy} = \frac{h^3}{12} \left[c_{12} + 2 c_{66} - \frac{c_{13}\, c_{23}}{c_{33}} \right] ,$$
$$B_y = \frac{h^3}{12} \left[c_{22} - \frac{c_{23}^2}{c_{33}} \right] \tag{39}$$

and with engineering constants (see *Sect. Q.2*):

$$B_x = \frac{h^3}{12} \left[\frac{E_x}{1 - \nu_{xy}\, \nu_{yx}} \right] , \qquad B_{xy} = \frac{h^3}{12} \left[2 G_{xy} + \frac{E_x\, \nu_{yx}}{1 - \nu_{xy}\, \nu_{yx}} \right] ,$$
$$B_y = \frac{h^3}{12} \left[\frac{E_y}{1 - \nu_{xy}\, \nu_{yx}} \right] . \tag{40}$$

A plane wave propagating at an angle Φ with the x-direction experiences a bending stiffness:

$$B(\Phi) = B_x\, (\cos \Phi)^4 + 2 B_{xy}\, (\cos \Phi \sin \Phi)^2 + B_y\, (\sin \Phi)^4 . \tag{41}$$

$(B(45°) = \frac{1}{4} \left(B_x + 2 B_{xy} + B_y \right)$; phase velocity as for an isotropic plate with $B = B(\Phi)$).
The extremal values of $B(\Phi)$ from 0° to 90° are $B(0°) = B_x$, $B(90°) = B_y$ and

$$B(\Phi_e) = \frac{B_x B_y - B_{xy}^2}{B_x - 2 B_{xy} + B_y} \quad \text{with} \quad \Phi_e = \frac{1}{2} \arccos \frac{-B_x + B_y}{B_x - 2 B_{xy} + B_y} \ \text{(for real } \Phi_e). \tag{42}$$

In order that $B(\Phi) > 0$ for all Φ: $\quad B_x > 0, B_y > 0\,, B_{xy} > -\sqrt{B_x B_y}$. (43)

Transition to isotropic case with $\quad B_x = B_{xy} = B_y = B = \dfrac{E\, h^3}{12\,(1 - \sigma^2)}$. (44)

Homogeneous isotropic plate – classical theory:

$$\rho h \frac{\partial^2 w}{\partial t^2} + B \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w = 0, \qquad c_B = \sqrt[4]{\frac{B}{\rho h}} \sqrt{\omega} \tag{45}$$

$$B = \frac{E\, h^3}{12\,(1 - \sigma^2)} = \frac{\mu\, h^3}{6\,(1 - \sigma)} = \frac{\mu\,(\lambda + \mu)\, h^3}{3\,(\lambda + 2\mu)} :$$

$$\text{with} \quad c_B = \sqrt[4]{\frac{E}{12 \rho\,(1 - \sigma^2)}} \sqrt{\omega h} = \frac{\sqrt{\omega\, h\, c_T}}{\sqrt[4]{6\,(1 - \sigma)}} ,$$
$$\lambda_B = \sqrt[4]{\frac{E}{3 \rho\,(1 - \sigma^2)}} \sqrt{\frac{\pi h}{f}} = \frac{\sqrt{2 \pi\, c_T\, h / f}}{\sqrt[4]{6\,(1 - \sigma)}} . \tag{46}$$

The accuracy of c_B is better than 10 %, if the bending wavelength $\lambda_B > 6h$ (Cremer-Heckl limit [Cremer/Heckl, p. 162 (1996)]). The non-propagating solutions with imaginary speed $\pm jc_B$ and imaginary wavenumber $\pm jk_B$ are called nearfields.

Amplitude ratio: $$\frac{\hat{u}}{\hat{w}} = \frac{\pi h}{\lambda_B};\tag{47}$$

Bending wave intensity: $$I_B = 2\pi^2\,|w_0|^2\,\rho f^2\,C_B;\tag{48}$$

Group velocity: $$C_B = 2c_B\tag{49}$$

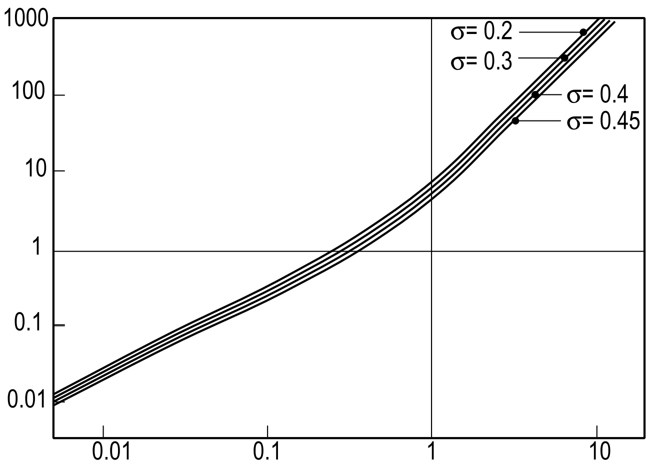
($\hat{u}$, $\hat{w}$: maximum longitudinal and transversal displacements; w_0: amplitude of w).

Measurement of the intensity of bending waves along the x-direction (without nearfields) by two accelerometers according to :

$$I_B = -\frac{\sqrt{B\rho h}}{\omega h}\,\mathrm{Re}\left\{a\int\frac{\partial a^*}{\partial x}\,dt\right\},\qquad a = \frac{\partial^2 w}{\partial t^2};\tag{50}$$

$\partial a^*/\partial x$ being approximated by a finite difference. The error of I_B relative to the exact result $\langle I_x\rangle$ (see *Sect. Q.10.1*):

$$\delta = \frac{I_B - \langle I_x\rangle}{\langle I_x\rangle}$$



δ is shown in the diagram for various Poisson ratios (frequency in units of c_T/h) [Maysenhölder (1990)]

Critical frequency f_{cr}, where $c_B = c_0$ (phase velocity in an ambient fluid):

$$f_{cr} = \frac{c_0^2}{2\pi}\sqrt{\frac{\rho h}{B}} = \frac{c_0^2}{\pi h}\sqrt{\frac{3\rho(1-\sigma^2)}{E}}\tag{51}$$

Coincidence frequency $$f_c = \frac{f_{cr}}{\sin^2\Theta}, \tag{52}$$

where the trace velocity $c_0/\sin\Theta$ of a plane wave incident on the plate with polar angle Θ equals the wave speed c_B of free bending waves.

Wavenumber relations: $$k_B = \frac{2\pi}{\lambda_B} = \frac{\omega}{c_B} = \sqrt[4]{6\,(1-\sigma)}\sqrt{\frac{\omega}{h\,c_T}} = k_0\sqrt{\frac{f_{cr}}{f}}. \tag{53}$$

Homogeneous isotropic plate – Timoshenko-Mindlin model, [Beltzer, p. 160 (1988)]: (including rotatory inertia and transverse shear effects)

$$\rho\,h\frac{\partial^2 w}{\partial t^2} + \left(B\,\nabla^2 - \frac{\rho\,h^3}{12}\,\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{\rho}{\kappa^2\,\mu}\,\frac{\partial^2}{\partial t^2}\right)\,w = 0, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{54}$$

with μ: shear modulus; κ: factor near unity. For a wave propagating in x-direction:

$$\frac{\partial^2 w}{\partial t^2} + \frac{B}{\rho\,h}\,\frac{\partial^4 w}{\partial x^4} - \left(\frac{B}{\kappa^2\,\mu\,h} + \frac{h^2}{12}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2} + \frac{\rho\,h^2}{12\,\kappa^2\,\mu}\,\frac{\partial^4 w}{\partial t^4} = 0. \tag{55}$$

The compact form of the dispersion relation:

$$\left(1 - \frac{c^2}{c_\kappa^2}\right)\left(\frac{c_{QL}^2}{c^2} - 1\right) = \frac{12}{(kh)^2} \tag{56}$$

with $c_\kappa^2 = \kappa^2\,c_T^2 = \kappa^2\mu/\rho$ may be transformed to quadratic equations for $\tilde{c}^2 = c^2/c_{QL}^2$:

$$\frac{\tilde{c}^4}{\tilde{c}_\kappa^2} - \left(\frac{1}{\tilde{c}_\kappa^2} + 1 + \frac{12}{(kh)^2}\right)\,\tilde{c}^2 + 1 = 0, \qquad \left(\frac{1}{\tilde{c}_\kappa^2} - \frac{12\,c_{QL}^2}{(\omega h)^2}\right)\tilde{c}^4 - \left(\frac{1}{\tilde{c}_\kappa^2} + 1\right)\,\tilde{c}^2 + 1 = 0 \tag{57}$$

with $\tilde{c}_\kappa = c_\kappa/c_{QL}$. The smaller root c^2 of the dispersion relation is the desired solution. The choice $c_\kappa = c_R$ (Rayleigh velocity) assures the correct value in the limit of high frequencies ($\kappa = 0.925$ for steel).

Q.10.4 Waves in Thin Beams

Wave equations for thin, straight, isotropic, homogeneous beams with cross-sectional area A. Young's modulus E, Poisson's ratio σ and mass density ρ; excitation terms like forces or moments are omitted. Solutions: Phase velocities $c = \omega/k$ of waves propagating along the x-direction (= beam axis) with wavelengths λ_{QL} or λ_B and phase factor $\exp[j(\omega t - kx)]$.

Nota bene: The symbols c_{QL} and B have different meanings for plates and beams!

The intensity I has dimension Wm^{-2}; the mean total energy flow in a beam is $I \cdot A$.

Quasi-longitudinal waves, [Cremer/Heckl, p. 82–85 (1996)]:

$$-\frac{\partial^2 u}{\partial t^2} + c_{QL}^2\,\frac{\partial^2 u}{\partial x^2} = 0, \qquad c_{QL} = \sqrt{\frac{E}{\rho}}, \qquad I_{QL} = 2\,\pi^2\,|u_0|^2\,\rho\,f^2 c_{QL} \tag{58}$$

u: longitudinal displacement; u_0: amplitude of u. Valid for arbitrary cross-section, if greatest thickness $d \ll \lambda_{QL}$. (For $d >> \lambda_{QL}, c = c_L$.)

Lateral contraction for quadratic or circular cross-section:

$$\frac{\hat{w}}{\hat{u}} = \sigma \frac{\pi d}{\lambda_{QL}} \tag{59}$$

($\hat{w}$, $\hat{u}$: maximum lateral and longitudinal displacements).

Torsional waves, [Cremer/Heckl, pp. 90–94 (1996)]:

$$-\frac{\partial^2 \Phi}{\partial t^2} + c_\Phi^2 \frac{\partial^2 \Phi}{\partial x^2} = 0, \quad c_\Phi = \sqrt{\frac{T}{\Theta}}, \quad I_\Phi = 2\pi^2 \, |\Phi_0|^2 \, \Theta f^2 c_\Phi \, ; \tag{60}$$

Φ: angle of rotation of a cross-section; Φ_0: amplitude of Φ; T: torsional stiffness (rigidity) defined by $M_x = T \cdot \partial \Phi / \partial x$ for arbitrary cross-section (M_x: torsional moment), Θ: moment of inertia per unit length about the x-axis. For a hollow circular cylinder with inner radius r_i and outer radius r_o:

$$T = \frac{\pi}{2} \mu \left(r_o^4 - r_i^4\right), \quad \Theta = \frac{\pi}{2} \rho \left(r_o^4 - r_i^4\right), \quad c_\Phi = \sqrt{\frac{\mu}{\rho}} = c_T \, ; \tag{61}$$

(μ: shear modulus). This is the exact solution for arbitrary frequency (lowest torsional mode) including the solid cylinder ($r_i = 0$), Gazis (1959).

Cross-sections without rotational symmetry do not remain plane ('warping'). Therefore the above torsional wave solution is only approximate; it always yields $c_\Phi < c_T$.

Results for rectangular cross-sections with dimensions $a \geq b$:

$$\Theta = \rho \frac{a b^3 + a^3 b}{12} = \frac{\rho A^2}{12} \left(\frac{a}{b} + \frac{b}{a}\right), \quad T = s \mu a b^3 = s \mu A^2 \frac{b}{a} \tag{62}$$

a/b	1	1.5	2	3	6	10	∞
s	0.141	0.196	0.229	0.263	0.298	0.312	1/3
c_Φ	0.920 c_T	$2 \frac{b}{a} c_T \sqrt{\frac{3s}{1 + (a/b)^{-2}}}$					$2 \frac{b}{a} c_T$

Hence $c_\Phi \approx 2 \frac{b}{a} c_T$ within 7 % for $a/b \geq 6$.

Bending waves – Bernoulli-Euler model, [Cremer/Heckl, pp. 95–99 (1996)] :

$$\frac{\partial^2 w}{\partial t^2} + \frac{B}{\rho A} \frac{\partial^4 w}{\partial x^4} = 0, \quad c_B = \sqrt[4]{\frac{B}{\rho A}} \sqrt{\omega}, \quad I_B = 2\pi^2 \, |w_0|^2 \, \rho f^2 C_B \tag{63}$$

($C_B = 2 c_B$).

w: displacement along z-direction (displacement vector is in x-z-plane);
w_0: amplitude of w;
B = EJ: bending stiffness;
J: second moment of cross-sectional area A about the neutral axis:

$$J = \int_A z^2 \, dy \, dz \quad \text{with} \quad \int_A z \, dy \, dz = 0$$

(Select z = 0 accordingly for the neutral axis.).

Hollow circular cylinder:

$$J = \frac{\pi}{4}\left(r_o^4 - r_i^4\right) = \frac{A}{4}\left(r_o^2 + r_i^2\right), \quad c_B = \sqrt[4]{\frac{E\,(r_i^2 + r_o^2)}{4\,\rho}}\,\sqrt{\omega}. \tag{64}$$

Rectangular cross-section:

$$J = \frac{h^3\, b}{12}, \qquad c_B = \sqrt[4]{\frac{E\,h^2}{12\,\rho}}\,\sqrt{\omega} \tag{65}$$

(h: thickness in the z-direction, b: thickness in the y-direction). For a solid cylinder ($r_i = 0$) the Bernoulli-Euler model agrees with the exact solution only for $r_o/\lambda_B < 0.1$ [Beltzer, pp. 155–156 (1988)].

Bending waves – Timoshenko model, [Junger/Feit, pp. 201–205 (1986)]:
(including rotatory inertia and transverse shear effects)

$$\frac{\partial^2 w}{\partial t^2} + \frac{B}{\rho A}\frac{\partial^4 w}{\partial x^4} - \left(\frac{B}{\kappa^2 \mu A} + \frac{J}{A}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho J}{\kappa^2 \mu A}\frac{\partial^4 w}{\partial t^4} = 0, \tag{66}$$

κ: factor near unity depending on the shape of the cross-section.

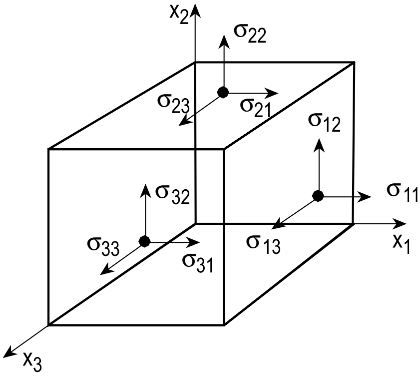
The smaller root c^2 of the dispersion relation:

$$\begin{aligned} &\left(\frac{1}{c_\kappa^2} - \frac{A}{J\,\omega^2}\right) c^4 - \left(\frac{c_{QL}^2}{c_\kappa^2} + 1\right) c^2 + c_{QL}^2 = 0 \quad \text{or} \\ &\frac{c^4}{c_\kappa^2} - \left(\frac{c_{QL}^2}{c_\kappa^2} + 1 - \frac{A}{J\,k^2}\right) c^2 + c_{QL}^2 = 0 \end{aligned} \tag{67}$$

with $c_\kappa = \kappa\, c_T$, is the desired solution. The Rayleigh velocity c_R is a convenient choice for c_κ, which is the high frequency limit of c ($\kappa = 0.925$ for steel). For rectangular cross-section ($A/J = 12/h^2$) the dispersion relation is the same as for Timoshenko-Mindlin plates, however, with the different definition of c_{QL} for plates (see *Sect. Q.10.3*).

Q.11 Moduli of Isotropic Materials and Related Quantities

Notice:
Some notations in this and the following Sections (written by F.P. Mechel) are different from corresponding notations in the previous Sections of this chapter (by W. Maysenhölder).



Co-ordinates of a point: $\mathbf{r} = \{x, y, z\} = \{x_i\}$
displacement: $\mathbf{u} = \{u_i\} = \{\delta x_i\}$
strain: $\boldsymbol{\varepsilon} = \{\varepsilon_{ik}\}$
force: $\mathbf{F} = \{F_i)$
stress: $\boldsymbol{\sigma} = \{\sigma_{ik}\}$
pressure: $p = -\sigma_{ii}{}^{*)}$

$$\sum_i \varepsilon_{ii} = \operatorname{div} \mathbf{u} \quad ; \quad \varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{1}$$

Lamé constants λ, μ :

$$\sigma_{ik} = \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) = 2\mu \cdot \varepsilon_{ik} \quad ; \quad i \neq k \,, \tag{2}$$

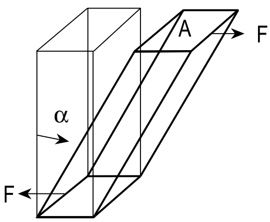
$$\sigma_{ii} = \lambda \cdot \operatorname{div} \mathbf{u} + 2\,\mu \cdot \varepsilon_{ii} \,. \tag{3}$$

With losses: $\lambda \to \lambda(1 + j\eta) \quad ; \quad \mu \to \mu(1 + j\eta)$

Shear modulus S :

$$\sigma = F/A \quad ; \quad S = \sigma/\alpha \,, \tag{4}$$

$$S = \mu = \frac{E}{2\,(1+\sigma)} \,. \tag{5}$$

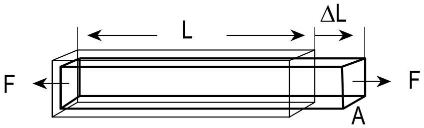


Free shear wave velocity (ρ = material density): $c_S = \sqrt{S/\rho}\,.$ (6)

*) See Preface to the 2nd edition.

Young's modulus E :

$$-\sigma = F/A = E \cdot \Delta L/L = E \cdot s_{xx} \tag{7}$$



Lateral contraction:

$$\sigma = -\varepsilon_{xx}/\varepsilon_{yy} \ , \tag{8}$$

$$E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu} = 2\mu(1+\sigma) = 2S\,(1+\sigma) \ . \tag{9}$$

Free bar longitudinal wave velocity: $c_E = \sqrt{E/\rho}\,.$ (10)

Poisson's lateral contraction σ (Poisson ratio):

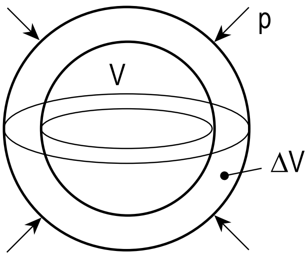
$$\begin{aligned} &\sigma = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda+\mu)} = \frac{E}{2S} - 1 \quad ; \quad -1 < \sigma < 0.5 \ , \\ &\frac{\mu}{\lambda} = \frac{1-2\sigma}{2\sigma} \,. \end{aligned} \tag{11}$$

Compression modulus K :

$$K = -\frac{p}{dV/V} \,, \tag{12}$$

$$K = \lambda + \frac{2}{3}\mu = S\left[\frac{2}{3} + \frac{2\sigma}{1-2\sigma}\right] = \frac{E}{3\,(1-2\sigma)} = \frac{1}{C} \,, \tag{13}$$

C = compressibility.

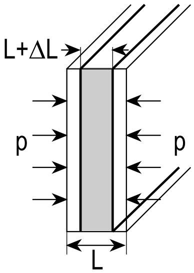


Dilatation modulus D: (1-dimensional deformation)

$$p = -D \cdot \varepsilon_{xx} \ , \tag{14}$$

$$D = \lambda + 2\mu = 2S\,\frac{1-\sigma}{1-2\sigma} = E\,\frac{1-\sigma}{(1+\sigma)\,(1-2\sigma)} = \frac{1}{3}\,\frac{1-\sigma}{1+\sigma}\,K \ . \tag{15}$$

Free dilatational wave velocity: $$c_D = \sqrt{D/\rho}\,. \tag{16}$$



Bar bending modulus B_{st}:

$$B_{St} = E\,\frac{b \cdot h^3}{12}\,. \tag{17}$$

Free bar bending wave velocity: $$c_{B_{St}} = \sqrt[4]{\omega^2 B_{St}/m'} \tag{18}$$

(m' = mass per bar length).

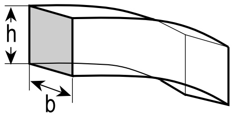
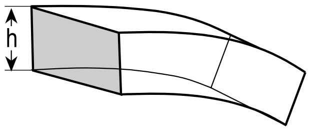


Plate bending modulus B :

$$B = E\,\frac{h^3}{12\,(1-\sigma^2)} = S\,\frac{h^3}{6\,(1-\sigma)} = \frac{h^3}{3}\,\frac{\mu\,(\lambda+\mu)}{\lambda+2\mu}\,. \tag{19}$$

Free plate bending wave velocity: $$c_B = \sqrt[4]{\omega^2 B/m''}\,. \tag{20}$$



Relations for free plate bending velocity c_B :

$$c_B = \sqrt[4]{\omega^2 B/m''} = 1.347\sqrt{f\,h}\,\sqrt[4]{\frac{E}{\rho\,(1-\sigma^2)}} = 1.391\sqrt{f\,c_E\,h} = \frac{\sqrt{2\pi\,f\,c_E\,h}}{\sqrt[4]{12\,(1-\sigma^2)}}\,. \tag{21}$$

Relations for free plate bending wave number $k_B = \omega/c_B$ (for $h \leq \lambda_B/6$):

$$k_B = \frac{\omega}{c_B} = \frac{2\pi}{\lambda_B} = \sqrt{\omega}\,\sqrt[4]{m''/B} = \sqrt{\frac{\omega}{c_E h}}\,\sqrt[4]{12\,(1-\sigma^2)} = \frac{1}{h}\sqrt{k_E h}\,\sqrt[4]{12\,(1-\sigma^2)}$$
$$= k_0\sqrt{f_{cr}/f} \xrightarrow[\sigma=0.35]{} 4.515\sqrt{\frac{f}{c_E h}} \tag{22}$$

(f_{cr} = critical frequency)

Relations at coincidence: The free plate bending wave speed c_B agrees with the trace speed $c_0/\sin\vartheta$ of a plane wave incident on the plate with a polar angle ϑ.

Coincidence frequency: $$f_c = f_{cr}/\sin^2\vartheta = \frac{c_0^2}{2\pi h\,\sin^2\vartheta}\sqrt{\frac{12\,\rho\,(1-\sigma^2)}{E}} \tag{23}$$

Critical frequency (at $\vartheta = \pi/2$; $c_0 = c_B$):

$$f_{cr} = \frac{c_0^2}{2\pi}\sqrt{\frac{m''}{B}} = \frac{c_0^2}{2\pi h}\sqrt{\frac{12\,\rho\,(1-\sigma^2)}{E}} = \frac{c_0^2}{2\pi h\, c_E}\sqrt{12\,(1-\sigma^2)} \quad ; \quad \lambda_{cr} = c_0/f_{cr} \tag{24}$$

$$k_B = k_0\sqrt{f_{cr}/f} \tag{25}$$

Effective bending moduli for sandwich panels :
(sheets and boards with subscripts 1,2 ; adhesive layers without subscript)

Table 1 Sandwich panels

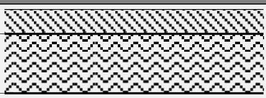
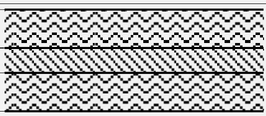
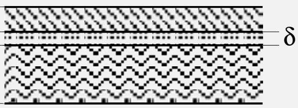
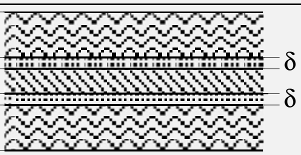
No.	Sandwich	Connection
1	h_2 h_1 	fix connection
2	$h_1/2$ h_2 $h_1/2$ 	fix connection
3	h_2 h_1 δ 	connection with shear $\delta \geq \frac{3.5\cdot 10^{-3}}{h_1}\frac{G}{E_1} - 1.3\cdot 10^{-12}$ [m] $E_2 h_2 \geq 2\cdot 10^7$ [Pa m]
4	$h_1/2$ h_2 $h_1/2$ δ δ 	connection with shear $\delta \geq \frac{0.25\cdot 10^{-3}}{h_1}\frac{G}{E_1}$ [m] h_i in [m]; G, E_i in [Pa]

Table 2 Effective bending moduli B for sandwiches from Table 1

No.	Effective bending modulus	Remark
1	$B = B_2 \dfrac{1 + 2\dfrac{E_1}{E_2}\left[2\left(\dfrac{h_1}{h_2}\right) + 3\left(\dfrac{h_1}{h_2}\right)^2 + 2\left(\dfrac{h_1}{h_2}\right)^3\right] + \left(\dfrac{h_1}{h_2}\right)^2\left(\dfrac{E_1}{E_2}\right)^4}{1 + \dfrac{h_1}{h_2}\dfrac{E_1}{E_2}}$	
2	$B \approx B_1\left(1 + \dfrac{h_2}{h_1}\right)^3 + B_2$	
3	$B \approx B_1 + B_2 + 3G\delta\dfrac{h_1^2}{4} + \dfrac{E_1 h_1 E_2 h_2 (h_1/2 + h_2/2)^2 \cdot g}{E_1 h_1 + g \cdot (E_1 h_1 + E_2 h_2)}$ $- 3G\delta\dfrac{h_1 + h_2}{4}\dfrac{E_1 h_1^2 + 2gE_2 h_1 h_2}{E_1 h_1 + g \cdot (E_1 h_1 + E_2 h_2)}$	$\delta \ll h_1$ $g = \dfrac{G}{\delta E_1 h_1 \omega}\sqrt{B/m}$
4	$B = B_2 + 2E_1\dfrac{(h_1/2)^3}{12(1 - \sigma_1^2)}$	

Q.12 Modes of Rectangular Plates

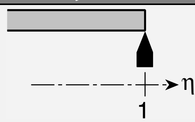
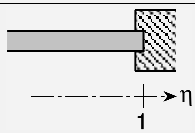
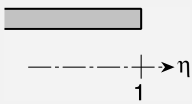
► *See also:* Mechel, Vol. II, Ch. 27 (1995); Gorman (1982)

A rectangular plate in the x,y-plane has the dimensions a in x-direction and b in y-direction; its shape factor is $ß = b/a$. The non-dimensional co-ordinates are $\xi = x/a$, $\eta = y/b$. Two oppo-site borders are supposed to be simply supported, the other borders may have different supports. Distinguish between the wave number k_B of the "free bending wave" (in an infinite plate) and k_b the bending wave number in the finite, supported plate.

Used abbreviations (with integer m):

$$\mu_m = m\,\pi \quad ; \quad \gamma_m = \sqrt{|(m\pi)^2 - (k_b a)^2|} \quad ; \quad \delta_m = \sqrt{(m\pi)^2 + (k_b a)^2}\,. \tag{1}$$

Table 1 Classical boundary conditions for plates

Fixation	Condition	Symbol
Simply supported S	$u(\xi, 1) = \dfrac{\partial^2 u(\xi, 1)}{\partial \eta^2} = 0$	
Clamped C	$u(\xi, 1) = \dfrac{\partial u(\xi, 1)}{\partial \eta} = 0$	
Free F	$\dfrac{\partial^2 u}{\partial \eta^2} + \sigma ß^2 \dfrac{\partial^2 u}{\partial \xi^2} = 0$ $\dfrac{\partial^3 u}{\partial \eta^3} + (2 - \sigma)ß^2 \dfrac{\partial^3 u}{\partial \eta \partial \xi^2} = 0$	

Bending wave equation for the plate displacement $u(\xi,\eta)$:

$$\left[\frac{\partial^4}{\partial\eta^4}+2ß\frac{\partial^4}{\partial\eta^2\partial\xi^2}+ß^4\frac{\partial^4}{\partial\xi^4}-ß^4(k_b a)^4\right]u(\xi,\eta)=0 \tag{2}$$

Supposed the plate is simply supported at $\xi = 0$ and $\xi = 1$, the plate displacement field can be formulated as:

$$u(\xi,\eta)=\sum_{m=1}^{\infty}Y_m(\eta)\cdot\sin(m\pi\xi) \tag{3}$$

with the wave equation after insertion:

$$\left[\frac{d^4}{d\eta^4}-2ß^2\mu_m^2\frac{d^2}{d\eta^2}+ß^4(\mu_m^4-(k_b a)^4)\right]Y_m(\eta)=0. \tag{4}$$

General solutions (with yet undetermined A_m, D_m):

$$\mu_m^2>(k_b a)^2:\quad \text{i.e.}\quad m>a/(\lambda_b/2)\,,$$

$$\begin{aligned}Y_m(\eta)&=A_m\cdot\cosh\left(ß\eta\sqrt{\mu_m^2+(k_b a)^2}\right)+B_m\cdot\sinh\left(ß\eta\sqrt{\mu_m^2+(k_b a)^2}\right)\,,\\&\quad+C_m\cdot\cosh\left(ß\eta\sqrt{\mu_m^2-(k_b a)^2}\right)+D_m\cdot\sinh\left(ß\eta\sqrt{\mu_m^2-(k_b a)^2}\right)\,,\end{aligned} \tag{5}$$

$$\mu_m^2<(k_b a)^2:\quad \text{i.e.}\quad m<a/(\lambda_b/2)\,,$$

$$\begin{aligned}Y_m(\eta)&=A_m\cdot\cosh\left(ß\eta\sqrt{\mu_m^2+(k_b a)^2}\right)+B_m\cdot\sinh\left(ß\eta\sqrt{\mu_m^2+(k_b a)^2}\right)\,,\\&\quad+C_m\cdot\cos\left(ß\eta\sqrt{(k_b a)^2-\mu_m^2}\right)+D_m\cdot\sin\left(ß\eta\sqrt{(k_b a)^2-\mu_m^2}\right)\,.\end{aligned} \tag{6}$$

Eigenvalues must be found for $k_b a$; they follow from the boundary conditions. These are:

$$\text{simply supported (S):}\qquad Y_m(\eta)=\frac{d^2Y_m(\eta)}{d\eta^2}=0, \tag{7}$$

$$\text{clamped (C):}\qquad Y_m(\eta)=\frac{dY_m(\eta)}{d\eta}=0, \tag{8}$$

$$\text{free (F):}\qquad \frac{\partial^2u}{\partial\eta^2}+\sigma ß^2\frac{\partial^2u}{\partial\xi^2}=0\quad;\quad\frac{\partial^3u}{\partial\eta^3}+(2-\sigma)ß^2\frac{\partial^3u}{\partial\eta\,\partial\xi^2}=0. \tag{9}$$

The boundary conditions give a system of homogeneous equations for the amplitudes; for a non-trivial solution the determinant must vanish; this is the eigenwert equation for k_b.

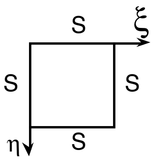
Cases:

(SSSS):

$$Y_m(\eta) = A_{m,n} \cdot \sin(n\pi\eta)\ . \tag{10}$$

Determinant equation:

$$(\gamma_m^2 + \delta_m^2) \cdot \cosh \frac{\delta_m}{2} \cdot \cos \frac{\gamma_m}{2} = 0\ . \tag{11}$$

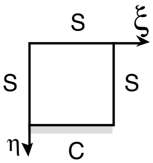


Solutions:

$$(k_b a)^2 = (m\pi)^2 + \frac{(n\pi)^2}{ß^2} \quad ; \quad m,\ n = 1,\ 2, \ldots\ . \tag{12}$$

(SCSS):

$$Y_m(\eta) = -\frac{\sin \gamma_m}{\sinh \delta_m} \cdot \sin(\gamma_m \eta) \cdot \sinh(\delta_m \eta)\ . \tag{13}$$



Determinant equation:

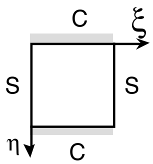
$$\gamma_m \cdot \sinh \delta_m \cdot \cos \gamma_m - ß \cosh \delta_m \cdot \sin \gamma_m = 0\ . \tag{14}$$

The equation must be solved numerically for $k_b a$.

(SCSC):

Symmetrical modes ($m = 1, 3, 5 \ldots$):

$$Y_m(\eta) = -\frac{\cos(\gamma_m/2)}{\cosh(\delta_m/2)} \cdot \cos(\gamma_m \eta) \cdot \cosh(\delta_m \eta)\ . \tag{15}$$



Determinant equation:

$$\gamma_m \cdot \cosh(\delta_m/2) \cdot \sin(\gamma_m/2) + \delta_m \sinh(\delta_m/2) \cdot \cos(\gamma_m/2) = 0\ . \tag{16}$$

Anti-symmetrical modes (m = 2, 4, 6...):

$$Y_m(\eta) = -\frac{\sin(\gamma_m/2)}{\sinh(\delta_m/2)} \cdot \sin(\gamma_m\eta) \cdot \sinh(\delta_m\eta)\ . \tag{17}$$

Determinant equation:

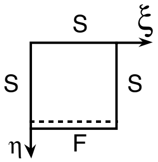
$$\gamma_m \cdot \sinh(\delta_m/2) \cdot \cos(\gamma_m/2) - \delta_m \cosh(\delta_m/2) \cdot \sin(\gamma_m/2) = 0\ . \tag{18}$$

Both equations must be solved numerically for $k_b a$.

(SFSS):

$(k_b a)^2 > (m\pi)^2$:

$$Y_m(\eta) = \sin(\gamma_m\eta) + \frac{(\gamma_m^2 + \sigma\,ß^2 m^2\pi^2)\ \sin\ \gamma_m}{(\delta_m^2 - \sigma\,ß^2 m^2\pi^2)\sinh\ \delta_m} \cdot \sinh(\delta_m\eta)\ . \tag{19}$$



Determinant equation (σ = Poisson ratio):

$$\begin{aligned}&\gamma_m(\gamma_m^2 + (2-\sigma)ß^2m^2\pi^2)(\delta_m^2 - \sigma\,ß^2m^2\pi^2) \cdot \sinh\ \delta_m \cdot \cos\ \gamma_m \\ &- \delta_m(\delta_m^2 - (2-\sigma)ß^2m^2\pi^2)(\gamma_m^2 + \sigma\,ß^2m^2\pi^2)\ \cdot \cosh\ \delta_m \cdot \sin\ \gamma_m = 0\ ,\end{aligned} \tag{20}$$

$(k_b a)^2 < (m\pi)^2$:

$$Y_m(\eta) = \sinh(\gamma_m\eta) - \frac{(\gamma_m^2 - \sigma\,ß^2 m^2\pi^2)\ \sinh\ \gamma_m}{(\delta_m^2 - \sigma\,ß^2 m^2\pi^2)\sinh\ \delta_m} \cdot \sinh(\delta_m\eta)\ . \tag{21}$$

Determinant equation:

$$\begin{aligned}&\gamma_m(\gamma_m^2 - (2-\sigma)ß^2m^2\pi^2)(\delta_m^2 - \sigma\,ß^2m^2\pi^2) \cdot \sinh\ \delta_m \cdot \cosh\ \gamma_m \\ &- \delta_m(\delta_m^2 - (2-\sigma)ß^2m^2\pi^2)(\gamma_m^2 - \sigma\,ß^2m^2\pi^2)\ \cdot \cosh\ \delta_m \cdot \sinh\ \gamma_m = 0\ .\end{aligned} \tag{22}$$

(CCCC):

All sides clamped. Approximate resonance frequencies, after [Mitchel, Hazell (1987)]:

$$\omega_{m,n}^2 = \frac{\pi^4 B}{\rho h}\left[\left(\frac{m+\Delta_m}{a}\right)^2 + \left(\frac{n+\Delta_n}{b}\right)^2\right]^2 \tag{23}$$

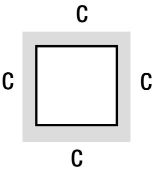
with 'edge effect factors':

$$\Delta_m = \left[2 + \left(na/mb\right)^2\right]^{-1} + 0.17/m \quad ; \quad \Delta_n = \left[2 + \left(ma/nb\right)^2\right]^{-1} + 0.17/n \tag{24}$$

and: h= thickness; a, b = dimensions; ρ = mass density; B = bending stiffness.

In the case of simply supported plates: $\Delta m = \Delta n = 0$.

More combinations of boundary conditions in [Gorman (1982)].



Q.13 Partition Impedance of Plates

▸ *See also:* Mechel (1999)

The partition impedance Z_T is a useful quantity in boundary value problems. It displays its full usefulness if the plate is homogeneous, i.e., has no ribs etc., and is either infinite or at least so large that border effects can be neglected in the given task. Then the sound fields on both sides can be supposed to have the same distribution along the plate.

Suppose a Cartesian co-ordinate system x, y, z with the plate in the plane x, y at the position $z = \zeta$, and the z axis directed from the front side to the back side. The partition impedance for a plate is defined by:

$$Z_T = \frac{p_{front}(x, y, \zeta) - p_{back}(x, y, \zeta)}{v_{plate}(x, y)}. \tag{1}$$

Be $p_{front} = p_e + p_r$ the sum of an incident wave p_e and a reflected wave p_r, and $p_{back} = p_t$ the transmitted wave. All waves $\alpha = e, r, t$ may have the distributions

$$p_\alpha(x, y, z) = P_\alpha \cdot X(x) \cdot Y(y) \cdot Z_\alpha(z), \tag{2}$$

and also the plate velocity has the distribution $v_p(x, y) = V_p \cdot X(x) \cdot Y(y)$. Thus the profile $X(x) \cdot Y(y)$ cancels in Z_T.

It is supposed that the waves p_α satisfy the wave equation, Sommerfeld's far field condition, the source condition (if a source exists) and, possibly, boundary conditions at other boundaries than the plate. There exist three boundary conditions at the plate:

$$\begin{aligned} &\left(p_e + p_r - p_t\right)_{z=\zeta} \overset{!}{=} Z_T \cdot v_p, \\ &\left(v_{e,z} + v_{r,z}\right)_{z=\zeta} \overset{!}{=} v_p \overset{!}{=}! \left(v_{t,z}\right)_{z=\zeta}, \end{aligned} \tag{3}$$

wherein $v_{\alpha,z}$, if the plate is in contact with air, follows from:

$$v_{\alpha,z}(x, y, \zeta) = \frac{j}{k_0 Z_0} \cdot \mathrm{grad}_z \left(p_\alpha(x, y, \zeta)\right), \tag{4}$$

and if the plate is in contact with a porous absorber, from:

$$v_{\alpha,z}(x, y, \zeta) = \frac{-1}{\Gamma_a Z_a} \cdot \mathrm{grad}_z \left(p_\alpha(x, y, \zeta)\right) . \tag{5}$$

The plate has to satisfy the bending wave equation:

$$\left[\Delta_{x,y}\Delta_{x,y} - k_B^4\right] v_p = \frac{j\,\omega}{B} \cdot \delta p, \tag{6}$$

in which Δ is the Laplace operator in the indicated co-ordinates, k_B is the wave number of the free bending wave on the plate, B is the bending stiffness, and $\delta p = p_{front} - p_{back}$ is the driving sound pressure difference. With the relations

$$k_B^4 = \omega^2 \frac{m''}{B} \quad ; \quad \frac{k_0}{k_B} = \sqrt{\frac{f}{fcr}}, \tag{7}$$

in which $\omega = 2\pi f$ is the circular frequency, m'' the surface mass density of the plate, f_{cr} the critical (coincidence) frequency, one immediately gets:

$$\frac{Z_T}{Z_0} = j\,k_0 \frac{m''}{\rho_0} \cdot \left[1 - \left(\frac{f}{f_{cr}}\right)^2 \frac{1}{k_0^4} \cdot \frac{\Delta_{x,y}\Delta_{x,y} v_p}{v_p}\right], \tag{8}$$

or alternatively:

$$\frac{Z_T}{Z_0} = j\,k_0 \frac{m''}{\rho_0} \cdot \left[1 - \left(\frac{f}{f_{cr}}\right)^2 \sin^4\chi\right], \tag{9}$$

where the last two fractions in the brackets are replaced by the sine function of an effective angle of sound incidence χ of the incident wave p_e (defined below).

Energy dissipation in the plate can be taken into account by a loss factor η introducing a complex modulus $B \to B \cdot (1 + j\eta)$. This leads to:

$$\frac{Z_T}{Z_0} = Z_{m''} F \cdot \left[\eta\, F^2 \sin^4\chi + j\,\left(1 - F^2 \sin^4\chi\right)\right] = Z_{m''} F \cdot \left[\eta\,\left(\frac{f}{f_c}\right)^2 + j\left(1 - \left(\frac{f}{f_c}\right)^2\right)\right] \tag{10}$$

$$\text{with} \quad Z_{m''} = \frac{\omega_{cr} m''}{Z_0} \quad ; \quad F = \frac{f}{f_{cr}}, \tag{11}$$

where $Z_{m''}$ is the normalised inertial impedance of the plate at the critical frequency ($\omega_{cr} = 2\pi f_{cr}$), and f_c is the coincidence frequency at the incidence angle χ, with $f_{cr} = f_c \cdot \sin^2\chi$.

It remains to determine:

$$\sin^4\chi = \frac{1}{k_0^4} \cdot \frac{\Delta_{x,y}\Delta_{x,y} v_p}{v_p} = \frac{1}{k_0^4} \cdot \frac{\Delta_{x,y}\Delta_{x,y} v_{z\alpha}(x, y, \zeta)}{v_{z\alpha}(x, y, \zeta)}, \tag{12}$$

where the last form makes use of the boundary condition that the pattern of the waves p_α at the plate agrees with that of v_p. After this determination all wave equations and all boundary conditions are satisfied, therefore the waves p_α make up a solution of the task.

Set $x = x_1, y = x_2, X(x) = X_1(x_1), Y(y) = X_2(x_2)$ and suppose the wave factors $X_i(x_i)$ to have one of the forms:

$$X_i(x_i) = e^{\pm j\, k_{xi} x_i} \quad \text{or} \quad \cos(k_{x_i} x_i) \quad \text{or} \quad \sin(k_{x_i} x_i) \tag{13}$$

or a linear combination thereof. Then

$$\sin^4\chi = \frac{1}{k_0^4} \cdot \frac{\Delta_{x,y}\Delta_{x,y} v_{z\alpha}(x,y,\zeta)}{v_{z\alpha}(x,y,\zeta)} = \frac{\left(k_x^2 + k_y^2\right)^2}{k_0^4}. \tag{14}$$

This corresponds to the possibility, which always exists, to transform the secular equation $k_0^2 = k_x^2 + k_y^2 + k_z^2$ to a form:

$$1 = \left((k_x/k_0)^2 + (k_y/k_0)^2\right) + (k_z/k_0)^2 = \sin^2\chi + \cos^2\chi, \tag{15}$$

where χ evidently is the polar angle of incidence of p_e on the plate. This holds also, if some or all of the k_x, k_y, k_z are complex.

The plates need not be thin in the sense of "thin plate theory"; it is important, that the compressibility of the plate normal to its surface is negligible. For a Timoshenko-Mindlin plate (with shear stress and rotational inertia) the result for Z_T is:

$$Z_T = j\omega m'' \frac{k_B^4 - (k_s^2 - k_L^2)(k_s^2 - k_R^2)}{k_B^4 - k_R^2(k_s^2 - k_L^2)} \quad ; \quad k_R^2 = \frac{12}{\pi^2} k_s^2 \quad ; \quad k_s^2 = k_x^2 + k_y^2 \tag{16}$$

with the characteristic wave numbers k_B, k_L, k_S of the free bending wave with the bending stiffness B, of the longitudinal wave with the plate dilatational stiffness D, and of the shear wave with the shear stiffness S . An equivalent form is:

$$Z_T = j\omega m'' \frac{1 - \dfrac{(k_x^2 + k_y^2)^2}{k_B^4} + \dfrac{h^2}{12}\left[(k_x^2 + k_y^2)\left(1 + \dfrac{c_L^2}{c_T^2}\right) - \dfrac{\omega^2}{c_T^2}\right]}{1 + \dfrac{h^2}{12}\left[(k_x^2 + k_y^2)\dfrac{c_L^2}{c_T^2} - \dfrac{\omega^2}{c_T^2}\right]} \tag{17}$$

with the plate thickness h and the speeds $c_L = \sqrt{E/\rho}$ of the longitudinal wave and $c_T = \sqrt{S/\rho}$ of the shear wave. An approximation has the form:

$$\begin{aligned} Z_T &= j\omega m'' \left[1 - \frac{(k_x^2 + k_y^2)^2}{\hat{k}_B^4}\right], \\ \hat{k}_B^2 &= \frac{4.43}{24}\,\frac{\omega^2 m'' h^2}{B} + \sqrt{\left(\frac{4.43}{24}\,\frac{\omega^2 m'' h^2}{B}\right)^2 - \frac{m''\omega^2}{B}\left(\frac{0.26\,\omega^2 m'' h}{E} - 1\right)}. \end{aligned} \tag{18}$$

Q.14 Partition Impedance of Shells

▸ *See also:* Mechel (1999)

For fundamental considerations about the partition impedance Z_T see the previous *Sect. Q.13*.

Circular cylindrical shell:

In cylindrical co-ordinates r, ϑ, z the sound field near the shell and the vibration velocity of the shell with radius r = a (if necessary after expanding the incident wave in cylindrical waves) be:

$$p(r,\vartheta,z) = R(r)\cdot T(\vartheta)\cdot U(z) \quad ; \quad v_p(a,\vartheta,z) = A\cdot T(\vartheta)\cdot U(z). \tag{1,2}$$

The axial function U(z) may be one of (or a linear combination) of the terms below. The factor R(r) may be one of the cylinder functions and $T(\vartheta)$ a trigonometric function (or a linear combination):

$$Z_m^{(i)}(k_r r) = \begin{cases} J_m(k_r r); & i=1 \\ Y_m(k_r r); & i=2 \\ H_m^{(1)}(k_r r); & i=3 \\ H_m^{(2)}(k_r r); & i=4 \end{cases} \; ; \; T(\vartheta) = \begin{cases} \cos(m\vartheta) \\ \sin(m\vartheta) \end{cases} \; ; \; U(z) = \begin{cases} e^{\pm j k_z z} \\ \cos(k_z z) \\ \sin(k_z z) \end{cases}. \tag{3}$$

The Laplace operators in cylindrical co-ordinates are:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2} \quad ; \quad \Delta_{\vartheta,z} = \frac{1}{a^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}. \tag{4}$$

The wave equation is satisfied by the above field factors, if the secular equation

$$k_0^2 = k_z^2 + k_r^2 \quad ; \quad 1 = (k_z/k_0)^2 + (k_r/k_0)^2 = \sin^2\Theta + \cos^2\Theta \tag{5}$$

holds. The angle Θ is between the wave vector and the radius. The two-dimensional Laplace operator together with the Bessel differential equation for the $Z_m^{(i)}(k_r)$ gives:

$$\Delta_{\vartheta,z} p(a,\vartheta,z) = -\left(\frac{m^2}{a^2} + k_z^2\right)\cdot p(a,\vartheta,z). \tag{6}$$

Therefore:

$$\sin^4\chi = \frac{1}{k_0^4}\frac{\Delta_{\vartheta,z}\Delta_{\vartheta,z} v_p}{v_p} = \frac{1}{k_0^4}\left(\frac{m^2}{a^2} + k_z^2\right)^2 = \left(\frac{m^2}{(k_0 a)^2} + \sin^2\Theta\right)^2. \tag{7}$$

With this quantity the partition impedance Z_T can be evaluated from the previous *Sect. Q.13*, Eqs.(9), (10). Because $T(\vartheta)$ is orthogonal over $0 \le \vartheta \le 2\pi$ for different values of m , the boundary conditions at the shell hold term-wise, if $p(r,\vartheta,z)$ is a sum of multi-pole terms.

Spherical shell :

Suppose spherical co-ordinates r, ϑ, φ and a shell with radius r = a. The field near the shell and the shell vibration velocity have the forms (if necessary after expanding the incident wave in spherical waves):

$$p(r,\vartheta,\varphi) = R(r)\cdot T(\vartheta)\cdot P(\varphi) \quad ; \quad v_p(a,\vartheta,\varphi) = A\cdot T(\vartheta)\cdot P(\varphi) \tag{8}$$

with spherical Bessel functions for R(r) and associated Legendre functions for $T(\vartheta)$ (or linear combinations thereof):

$$R(r) = \begin{cases} j_m(k_0 r) \\ y_m(k_0 r) \\ h_m^{(1)}(k_0 r) \\ h_m^{(2)}(k_0 r) \end{cases} ; \quad T(\vartheta) = \begin{cases} P_m^n(\cos\vartheta) \\ Q_m^n(\cos\vartheta) \end{cases} ; \quad P(\varphi) = \begin{cases} \cos(n\varphi) \\ \sin(n\varphi) \end{cases} . \tag{9}$$

The Laplace operators are:

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{1}{r^2 \tan\vartheta}\frac{\partial}{\partial\vartheta} + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}, \\ \Delta_{\vartheta,\varphi} &= \frac{1}{a^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{a^2\tan\vartheta}\frac{\partial}{\partial\vartheta} + \frac{1}{a^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} \end{aligned} \tag{10}$$

and, with the above separation:

$$\Delta_{\vartheta,\varphi} p(a, \vartheta, \varphi) = -\frac{m(m+1)}{a^2} \cdot p(a, \vartheta, z). \tag{11}$$

Therefore:

$$\sin^4\chi = \frac{1}{k_0^4}\frac{\Delta_{\vartheta,z}\Delta_{\vartheta,z} v_p}{v_p} = \left(\frac{m(m+1)}{(k_0 a)^2}\right)^2 . \tag{12}$$

With this quantity the partition impedance Z_T can be evaluated from the previous *Sect. Q.13*, Eqs.(9), (10). Because $T(\vartheta)$, $P(\varphi)$ are orthogonal over $(0, 2\pi)$ for different values of m or n, respectively, the boundary conditions at the shell hold term-wise, if $p(r, \vartheta, \varphi)$ is a sum of multi-pole terms.

Because of $P_m^n(\cos\vartheta) \equiv 0$; $n > m$, the angle of incidence is $\chi = 0$ for the "breathing sphere" $m = n = 0$, which is plausible.

Q.15 Density of Eigenfrequencies in Plates, Bars, Strings, Membranes

Be n the number of eigenfrequencies in an interval of 1 Hertz.

String of length ℓ: $$n = 2\ell\sqrt{m/T} \tag{1}$$

(m = mass per string length; T = string tension)

Longitudinal wave on a bar of length ℓ: $$n = 2\ell\sqrt{\rho/E} \tag{2}$$

(ρ = material density; E = Young's modulus)

Bending wave on a bar of length ℓ: $$n = \ell\sqrt[4]{m/(\omega^2 B)} \tag{3}$$

(m = mass per bar length: B = bar bending modulus)

Plate, simply supported: $$n = \frac{1}{2} S \sqrt{m/B} \tag{4}$$

(S = plate area; m = surface mass density; B = plate bending stiffness)

Circular membrane: $$n = \pi \frac{S \rho d \cdot f}{T} \tag{5}$$

(S = membrane area; d = membrane thickness; ρ = material density; T = tension per unit length of circumference; f = frequency)

Tube of length ℓ with outer diameter 2a and wall thickness d (simply supported at the ends; ρ = material density; E = Young's modulus):

$$n = \begin{cases} \frac{5\pi}{2} \sqrt[4]{\frac{\rho^3}{E^3}} \sqrt{\omega a^3} \frac{\ell}{d} & ; \quad \omega < \frac{\sqrt{E/\rho}}{a} \\ 2\pi \sqrt{\frac{3\rho}{E}} \frac{\ell a}{d} & ; \quad \omega > \frac{\sqrt{E/\rho}}{a} \end{cases} \tag{6}$$

Q.16 Foot Point Impedances of Forces

▸ *See also:* Fahy (1985); Cremer/Heckl (1996)

Foot point impedances Z of structures for external forces are defined as ratios of

- force of a point source to structure velocity at the point of attack,
- force per length of a line source to average structure velocity at the line of attack,
- force per area of an area source to average structure velocity in the area.

It is advantageous to introduce the *foot point admittance* $G = 1/Z$. Be F the force of a point source or the constant force of a line source, be $v(0,0)$ the structure velocity at the foot point $(0,0)$ of a point source, and $v(0)$ the structure velocity at the line of excitation. The real part of the admittance can always be written as:

$$\mathrm{Re}\{G\} = \mathrm{Re}\{F_0/v(0,0)\} = \frac{1}{\omega\rho\, V_q} \qquad \text{point force,} \tag{1}$$

$$\mathrm{Re}\{G_L\} = \mathrm{Re}\{F/v(0)\} = \frac{1}{\omega\rho\, S_q} \qquad \text{line force.} \tag{2}$$

The quantity V_q with the dimension of a volume is called *source volume*, the quantity S_q with the dimension of an area is called *source area*.

The Table 1 collects values of the foot point admittance G and the source volume V_q for a point force on several objects. The arrows indicate the direction of the force.

h = thickness of plate or membrane;
A = cross section of bar;
ρ = material density;
m' = mass per unit length of bar;
m'' = mass per unit area of plate;
c_L = longitudinal wave velocity;
c_B = bending wave velocity;
c_T = torsional wave velocity;
λ_L = longitudinal wave length;
λ_B = bending wave length;
λ_T = torsional wave length;
ρ_F = fluid density;
c_F = fluid sound speed;
T' = tension of a membrane;
B = plate bending stiffness;
σ = Poisson ratio

Table 1 Foot point admittances

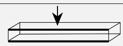
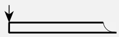
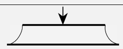
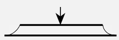
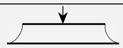
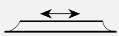
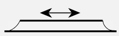
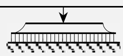
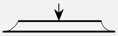
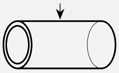
Object		Re{G}	Im{G}	V_q
Bar		$1/(\rho c_L A)$	0	$A\lambda_L/(2\pi)$
Bar, thin		$1/(4m' c_B)$	$-1/(4m' c_B)$	$4A\lambda_B/(2\pi)$
Bar, thin		$1/(m' c_B)$	$-1/(m' c_B)$	$A\lambda_B/(2\pi)$
Bar, Timoshenko		Eq. (3)	Eq. (3)	–
Membrane		$\omega/(4T')$	∞	$h\lambda^2/\pi^2$
Plate, thin		$1\big/(8\sqrt{Bm''})$	0	$2h\lambda_B^2/\pi^2$
Plate, thin		$1\big/(3.5\sqrt{Bm''})$	0	$0.9\,h\lambda_B^2/\pi^2$
Plate with shear stiffness		Eq. (5)	Eq. (5)	–
Plate with tangential force		–	$\approx 2\omega/(Eh)$	$h\lambda_T^2/(3\pi^2)$
Plate, orthotropic		Eq. (7)	0	–
Plate on elastic bed		Eq. (10)	Eq. (10)	–

Table 1 (continued)

Object		Re{G}	Im{G}	V_q
Strip of plate		Eq. (11)	Eq. (11)	–
Tube		eqs. (12,13)	eqs. (12,13)	–
Elastic half space		eqs. (14,15)	eqs. (14,15)	$\approx (\lambda_T/\pi)^3$
Fluid half space		$\frac{\omega^2}{6\pi\rho_F c_F^3}$	∞	$3\lambda_F^3/(4\pi^2)$
Thick plate		eqs. (16, 17)	eqs. (16, 17)	–

Timoshenko bar with central excitation:

$$G = \frac{1}{2\omega m'} \frac{k_T^2 + k_I k_{II}}{k_I + k_{II}} \tag{3}$$

with:

$$k_I^2 = \frac{k_T^2 + k_L^2}{2} + \sqrt{\left(\frac{k_T^2 + k_L^2}{2}\right)^2 + \alpha k_B^2}\,,$$
$$k_{II}^2 = \frac{k_T^2 + k_L^2}{2} - \sqrt{\left(\frac{k_T^2 + k_L^2}{2}\right)^2 + \alpha k_B^2}\,, \tag{4}$$
$$\alpha = 1 - k_T^2 h^2/12\,.$$

Plate with shear stiffness:

$$G = \frac{k_B^4}{8\omega m'' \left[k_I^2 + \frac{1}{2}(k_T^2 + k_L^2)\right]} \left(A_R + j\,A_I\right) \tag{5}$$

(see Eq. (4) for k_I, k_{II}, α), and

with:

$$A_R = \begin{cases} \alpha + k_T^2\left(2k_I^2 - k_L^2 - k_T^2\right)/k_B^4 & ; \quad \alpha > 0 \\ k_T^2\left(2k_I^2 - k_L^2 - k_T^2\right)/k_B^4 & ; \quad \alpha < 0 \end{cases}, \tag{6}$$
$$A_I = -\frac{2}{\pi}\left[\alpha\ \ln\left(\frac{k_B^2}{k_I^2}\sqrt{|\alpha|}\right) + \frac{k_T^2}{k_B^2}\left(k_I^2\ \ln(k_I r) - k_{II}^2\ \ln(k_{II} r)\right)\right]$$

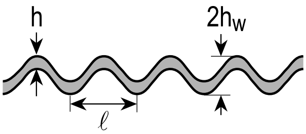
with: r the distance between force point and measuring point.

Foot point admittance of an orthotropic plate with bending stiffnesses B_x, B_z in the x,z-directions, B_σ bending stiffness induced by contraction, B_G bending stiffness induced by the shear modulus G:

$$G = \frac{K(ß)}{4\pi\,(m''^2 B_x B_z)^{1/4}} \approx \frac{1}{8\,(m''^2 B_x B_z)^{1/4}} \quad ; \quad ß = \frac{1}{2}\left(1 - \frac{B_\sigma + 2B_G}{\sqrt{B_x B_z}}\right), \tag{7}$$

where K(ß) is the complete elliptic integral of first kind.

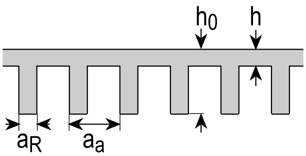
Examples of orthotropic corrugated plates (with E = Young's modulus; S = shear modulus; σ = Poisson ratio):



ℓ much smaller than bending wave length

$$\begin{aligned} &B_x = E \cdot I \quad ; \quad B_z = \frac{\ell}{s}\frac{E\,h^3}{12\,(1-\sigma^2)} \quad ; \quad B_\sigma \approx 0 \quad ; \quad 2S \approx \frac{s}{\ell}\frac{E\,h^3}{12\,(1-\sigma^2)}\,; \\ &s = \ell\left(\frac{\pi\,h_w}{2\ell}\right)^2 \quad ; \quad I = \frac{h_w^2 h}{2}\left(1 - \frac{0.81}{1 + 2.5\,(h_w/2\ell)^2}\right). \end{aligned} \tag{8}$$

Ribbed plate:



a_a much smaller than bending wave length

$$\begin{aligned} &B_x = E \cdot I \quad ; \quad B_z = \frac{E\,h^3}{12}\frac{a_a}{a_a - a_R\left(1 - h^3/h_w^3\right)} \quad ; \quad B_\sigma \approx 0 \quad ; \\ &S \approx \frac{E}{6\,(1+\sigma)}\left(h^3 + h_w^3\, a_R/a_a\right) \quad ; \quad I = \frac{a_a}{3}\left(s_1^2 - s_2^2\right) + \frac{a_R}{3}\left(s_2^2 + s_3^2\right); \\ &s_1 = \frac{1}{2}\frac{a_a h_w^2 + (a_a - a_R)\,h^2}{a_a h_w + (a_a - a_R)h} \quad ; \quad s_2 = s_1 - h \quad ; \quad s_3 = h_w - s_1\,. \end{aligned} \tag{9}$$

Foot point admittance of a thin plate on an elastic bed; the bed with a spring stiffness s (per unit area) is tuned with the surface mass density of the plate m″ to a resonance with $\omega_0^2 = s/m''$:

$$G = \frac{1}{8\sqrt{Bm''}}\begin{cases} \sqrt{1 - \omega_0^2/\omega^2} & ; \quad \omega_0 < \omega \\ \sqrt{\omega_0^2/\omega^2 - 1} & ; \quad \omega_0 > \omega \end{cases}. \tag{10}$$

Foot point admittance of a strip of plate with width l_s and thickness h for a point force at $z = 0, x = x_0$:

$$G = \frac{1}{2\rho h l_s c_B}\left[\frac{1-j}{2}\alpha + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{1-\kappa_n^2}} - \frac{j}{\sqrt{1+\kappa_n^2}}\right)\varphi_n^2(x_0)\right];$$
$$\kappa_n = n\pi/(k_B l_s) \quad ; \quad \alpha = (\varphi_0(x_0))^2; \tag{11}$$

$$\varphi_n(x) = \cos\,(n\pi x/l_s) \quad ; \quad n = 0,\ 1,\ 2, \dots\ .$$

Foot point admittance of a tube with outer radius a and wall thickness h:

$$G \approx (1-j)\left[2\pi a\rho h\sqrt{\omega c_L a/\sqrt{2}}\right]^{-1} \quad ; \quad \nu = \omega a/c_L < 0.77\,h/a; \tag{12}$$

$$\mathrm{Re}\{G\} \approx \begin{cases} \dfrac{0.66}{2.3\,c_L\rho h^2}\sqrt{\omega a/c_L} & ; \quad 0.77\,h/a < \nu < 0.6 \\ \left(2.3\,c_L\rho h^2\right)^{-1} & ; \quad \nu > 2 \end{cases} . \tag{13}$$

Foot point admittance of a force acting in a small circle with radius a on an elastic half space:

$$G \approx \frac{\omega k_T}{S}(1-\sigma)\left(0.19 + j\,0.3/(k_T a)\right) . \tag{14}$$

Foot point admittance of a force acting in a strip of width b on an elastic half space:

$$G \approx \frac{\omega}{S}(1-\sigma)\left\{0.463 + j\,1.5\ \ln\left[\left(1.9 - 15(\sigma-0.25)^2\right)/k_T b\right]\right\} . \tag{15}$$

Foot point admittance of a force acting in a small circle with radius a on a thick plate:

$$G \approx \frac{\omega k_T}{S}\left\{\frac{0.063}{H^2} + \frac{1}{8}\left(\frac{H}{1.6+H}\right)^2 + j\left(0.06 + \frac{0.001}{H^{1.3}}\right)\frac{\lambda_T}{2a}\right\} \quad ; \quad H = k_T h/2 . \tag{16}$$

Foot point admittance of a force acting in a strip of width b on a thick plate:

$$G \approx \frac{\omega}{S}\left\{\frac{1}{8H^{1.5}} + 0.31\left(\frac{H}{1.6+H}\right)^2 + j\left(\frac{-1}{8H^{1.5}} + 0.16\,\ln(\lambda_T/b)\right)\right\} \quad ; \quad H = k_T h/2 . \tag{17}$$

Foot point admittance of a point force acting on an isotropic, thin plate with bending stiffness B and membrane stress T:

$$G = \frac{1}{8\sqrt{1+ß^2}\sqrt{Bm''}}\left[1 + \frac{j}{\pi}\ \ln\left(\frac{\sqrt{1+ß^2}+ß}{\sqrt{1+ß^2}-ß}\right)\right] \quad ; \quad ß = \frac{T}{2\omega\sqrt{Bm''}} . \tag{18}$$

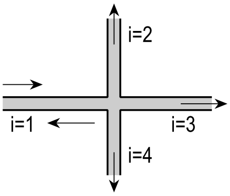
Foot point admittance of a point force acting in the centre of a bar of length ℓ, width w, thickness h, simply supported at both ends:

$$G = \frac{j\,\omega}{4EI\,k_B^3}\left(\tan\,(k_B\ell) - \tanh\,(k_B\ell)\right) \quad ; \quad I = wh^3/12 . \tag{19}$$

Q.17 Transmission Loss at Steps, Joints, Corners

► *See also:* Cremer/Heckl (1996)

Bars or plates $i = 1, 2, \ldots$, are joined to each other at steps (of material and/or cross section), joints, corners. The branch $i = 1$ is the side of excitation, either by longitudinal or bending waves. The other branches are anechoic.



The transmission coefficient $\tau_{1\,i} = \dfrac{\Pi_i}{\Pi_1}$ is the ratio of the effective powers Π_i, with Π_1 the incident power.

Definitions:

$$\kappa = \left(\frac{\rho_2 E_1 K_1^2}{\rho_1 E_2 K_2^2}\right)^{1/4} = \left(\frac{m_2' B_1}{m_1' B_2}\right)^{1/4},$$

$$\psi = \frac{K_2 A_2}{K_1 A_1}\sqrt{\frac{\rho_2 E_2}{\rho_1 E_1}} = \sqrt{\frac{m_2' B_2}{m_1' B_1}}, \qquad (1)$$

$$K_1 = h_1/\sqrt{12} \quad \text{for plates}.$$

h_i	= thickness;
A_i	= cross section (of bars);
ρ_i	= mass density;
m_i'	= mass per length of bars;
E_i	= Young's modulus;
c_i	= group velocity;
K_i	= radius of gyration;
c	= compressional stiffness of interlayer;
m	= blocking mass;
K	= radius of gyration of blocking mass;
S_F	= shear modulus of interlayer;
I_F	= thickness of interlayer;
$s_{i,k}$	= h_i/h_k for plates; $s = s_{12}$;
$s_{i,k}$	= A_i/A_k for bars; $s = s_{12}$

Table 1 Transmission coefficients at steps

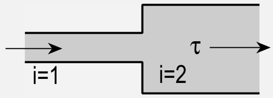
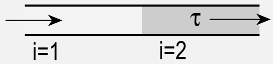
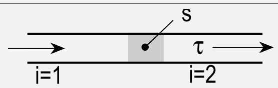
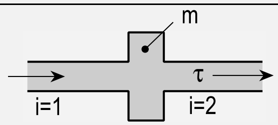
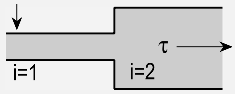
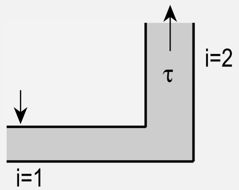
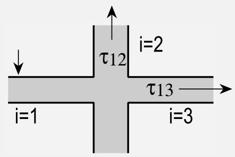
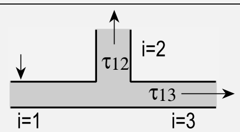
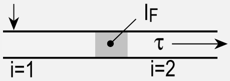
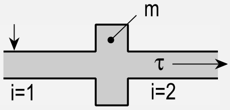
Object	Longitudinal wave	Transmission coefficient
Cross section change		$\tau = 4\left[s_{12}^{1/2} + s_{12}^{-1/2}\right]^{-2}$
Material change		$\tau = 4\left[\left(\frac{E_1\rho_1}{E_2\rho_2}\right)^{1/4} + \left(\frac{E_1\rho_1}{E_2\rho_2}\right)^{-1/4}\right]^{-2}$
Elastic interlayer		$\tau = \left(1 + (f/f_u)^2\right)^{-1}$; $f_u = \frac{c}{\pi A_1\sqrt{E_1\rho_1}}$
Blocking mass		$\tau = \left(1 + (f/f_u)^2\right)^{-1}$; $f_u = \frac{A_1\sqrt{E_1\rho_1}}{\pi m}$
Cross-section change		$\tau = \left[\frac{s^{-5/4} + s^{-3/4} + s^{3/4} + s^{5/4}}{s^{-2}/2 + s^{-1/2} + 1 + s^{1/2} + s^2/2}\right]^2$
Material change		$\tau = \left[\frac{2\sqrt{\kappa\psi}(1+\kappa)(1+\psi)}{\kappa(1+\psi)^2 + 2\psi(1+\kappa^2)}\right]^2$
Corner		$\tau = 2\left[s^{-5/4} + s^{5/4}\right]^{-2}$
Cross		$\tau_{12} = \frac{1}{2}\left[s^{-5/4} + s^{5/4}\right]^{-2}$ $\tau_{13} = \frac{1}{2}\left[1 + 2s^{5/2} + s^5\right]^{-1}$
Branching		$\tau_{12} = \left[\sqrt{2}s^{-5/4} + s^{5/4}/\sqrt{2}\right]^{-2}$ $\tau_{13} = \left[2 + 2s^{5/2} + s^5/2\right]^{-1}$

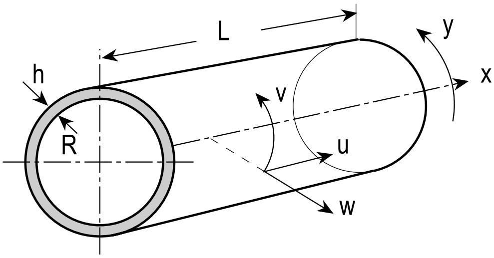
Table 1 continued

Object & Wave	Bending wave	Transmission coefficient
Elastic interlayer		$\tau = \left(1 + (f/f_u)^3\right)^{-1}$ $f_u = \left(\frac{G_F^2}{1.8\pi^2 \rho_1 \sqrt{E_1 \rho_1}\, h_1 l_F^2}\right)^{1/3}$
Blocking masses		$\tau = 1 \quad ; \quad f < 0.5\, f_s$ $\tau = \left[1 + f/f_u\right]^{-1} \quad ; \quad f > 2\, f_s$ $f_s = \frac{K_1}{2\pi K^2}\sqrt{\frac{E_1}{\rho_1}}\ ;\ f_u = \frac{2\rho_1 A_1^2 K_1 \sqrt{E_1 \rho_1}}{\pi\, m^2}$

Q.18 Cylindrical Shell

► *See also:* Dym (1973)

A cylindrical shell is simply supported at its ends. Used co-ordinates are:
$0 \le x \le L$
$0 \le y \le 2\pi R$.



ρ	=	material density;
E	=	Young's modulus;
S	=	shear modulus;
σ	=	Poisson ratio;
u, v, w	=	elongations

Abbreviations:

$$a = \frac{1}{2}(1-\sigma) \quad ; \quad H = \frac{1}{12}(h/R)^2 \ ;$$
$$\Lambda = m\pi R/L \quad ; \quad K^2 = \frac{1-\sigma^2}{E}\rho R^2 \omega^2 \tag{1}$$

Mode shapes: with $m, n = 0, 1, 2, \ldots,$ (2)

$$u(x,y) = A \cdot \cos\frac{m\pi x}{L} \cdot \cos\frac{ny}{R} \quad ; \quad v(x,y) = B \cdot \sin\frac{m\pi x}{L} \cdot \sin\frac{ny}{R};$$
$$w(x,y) = C \cdot \sin\frac{m\pi x}{L} \cdot \cos\frac{ny}{R}\ .$$

Special cases: $n = 0$ axial symmetry;
$n = 1$ no deformation of cross section;
$m = 0$ axial shear motion;
$m = 1$ fundamental bending motion.

General eigenvalue equation (for eigenvalues K):

$$K^6 - (Q_3 + Q_4)K^4 + (Q_1 + Q_2)K^2 - Q_0 = 0 \tag{3}$$

with coefficients:

$$\begin{aligned}
Q_0 = a\ \Big\{&(1-\sigma^2)\Lambda^4 + H\Big[(\Lambda^2+n^2)^4 + \frac{9}{4}(1-\sigma^2)\Lambda^4 \\
&+ 4\Lambda^2 n^2 + n^4 + 6\Lambda^4 n^2 - 8\Lambda^2 n^4 - 2n^6\Big] \\
&+ H^2\left[\frac{1}{4}n^2 - \frac{3}{2}\sigma^2\Lambda^4 n^2 - \frac{1}{2}n^6 + \frac{9}{4}\Lambda^8 + 4\Lambda^6 n^2 + \frac{3}{2}\Lambda^4 n^4 + \frac{1}{4}n^8\right] \\
&+ H^3\left[a(1-a)\Lambda^4 n^4\right]\Big\}\ , \\
Q_1 = a\ \Big\{&(5-4a)\Lambda^2 + n^2 \\
&+ H\left[\frac{9}{4}\Lambda^2 + \left(\frac{1}{a}+\frac{1}{4}\right)n^2 - \left(\frac{2}{a}+4a\right)\Lambda^2 n^2 - \frac{2}{a}n^4 + \frac{1+a}{a}\left(\Lambda^2+n^2\right)^3\right] \\
&+ H^2\left[\frac{9}{4}\Lambda^6 + \left(\frac{11}{4}-a\right)\Lambda^4 n^2 + \left(\frac{3}{4}-a\right)\Lambda^2 n^4 + \frac{1}{4}n^6\right]\Big\}\ , \\
Q_2 = a\ \Big\{&\left(\Lambda^2+n^2\right)^2 + H\left[\frac{9}{4}\Lambda^4 + \left(\frac{3}{2}+a+1/a\right)\Lambda^2 n^2 + \frac{5}{4}n^4\right] + \frac{1}{4}H^2 n^4\Big\}\ , \\
Q_3 = 1 + H&\ \left(\Lambda^2+n^2\right)\ , \\
Q_4 = (1+a)&\left(\Lambda^2+n^2\right) + H\left[\frac{9}{4}\Lambda^2 + (1+a/4)\,n^2\right]\ .
\end{aligned} \tag{4}$$

Eigenfrequencies, from solutions, with Eq. (1) :

$$\omega = \frac{K}{R}\sqrt{\frac{E}{\rho(1-\sigma^2)}}\ . \tag{5}$$

Linear approximation to K (at low frequencies):

$$K^2 \approx \frac{Q_0}{Q_1 + Q_2}\ . \tag{6}$$

Approximation with quadratic correction:

$$K^2 \approx \frac{Q_0}{Q_1 + Q_2} + \frac{Q_0^2(Q_3 + Q_4)}{(Q_1 + Q_2)^3}\ . \tag{7}$$

Amplitude ratios:

$$\begin{aligned}
\frac{A}{C} &= \frac{1}{\Delta}\left\{\left[K^2 - a\left(1+\frac{9}{4}H\right)\Lambda^2 - (1+H)\,n^2\right]\cdot\Lambda\,(aHn^2-\sigma)\right.\\
&\quad\left. - \left(a+\sigma-\frac{3}{4}aH\right)\left[\Lambda n^2 + (1+a)H\Lambda^3 n^2 + H\Lambda n^4\right]\right\}\ ,\\
\frac{B}{C} &= \frac{1}{\Delta}\left\{\left[K^2 - \Lambda^2 - a\left(1+\frac{1}{4}H\right)n^2\right]\cdot\left[n + (1+a)H\Lambda^2 n + Hn^3\right]\right.\\
&\quad\left. - (aHn^2-\sigma)\left(a+\sigma-\frac{3}{4}aH\right)\Lambda^2 n\right\}\ ,\\
\Delta &= K^4 - K^2\left[\left(1+a+\frac{9}{4}aH\right)\Lambda^2 + \left(1+a+\frac{1}{4}aH\right)n^2\right]\\
&\quad + a\left(1+\frac{9}{4}H\right)\Lambda^4 + a\left(1+\frac{5}{4}H+\frac{1}{2}H^2\right)n^4 + \left(2a+H+a^2H+\frac{3}{2}aH\right)\Lambda^2 n^2\ .
\end{aligned} \tag{8}$$

Special case: torsion with axial symmetry $u = w = 0$.

Differential equation: $$\frac{1-\sigma}{2}\left(1+\frac{9}{4}H\right)\frac{\partial^2 v}{\partial x^2} = \frac{1-\sigma}{E}\rho\frac{\partial^2 v}{\partial t^2}\ . \tag{9}$$

Modes: $$v = \begin{cases}\sin\left(\dfrac{m\pi x}{L}\right) & ;\quad \text{simply supported}\\[2ex] \cos\left(\dfrac{m\pi x}{L}\right) & ;\quad \text{free}\end{cases}\ , \tag{10}$$

Eigenvalues: $$\begin{aligned} K^2 &= \frac{1-\sigma}{2}\left(1+\frac{1}{4}H\right)\Lambda^2\ ,\\ \omega^2 &= \frac{E}{2(1+\sigma)\rho}\left(1+\frac{1}{4}H\right)(m\pi/L)^2\ . \end{aligned} \tag{11}$$

Special case: longitudinal vibration with axial symmetry $v = w = 0$.

Differential equation: $$\frac{\partial^2 u}{\partial x^2} = \frac{1-\sigma^2}{E}\rho\frac{\partial^2 u}{\partial t^2}\ . \tag{12}$$

Eigenvalues: $$\begin{aligned} K^2 &= \Lambda^2 = \left(\frac{m\pi R}{L}\right)^2\ ,\\ \omega^2 &= \frac{E}{(1-\sigma^2)\rho}\left(\frac{m\pi}{L}\right)^2\ . \end{aligned} \tag{13}$$

Special case: radial vibration with axial symmetry $u = v = 0$.

Differential equation: $$\frac{w}{R^2} + HR^2\frac{\partial^4 w}{\partial x^4} = \frac{w}{R^2} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} = -\frac{1-\sigma^2}{E}\rho\frac{\partial^2 w}{\partial t^2}\ . \tag{14}$$

Modes: $$w \sim \sin\left(\frac{m\pi x}{L}\right) \quad \text{or} \quad \sim \cos\left(\frac{m\pi x}{L}\right)\ . \tag{15}$$

$$K^2 = 1 + H\Lambda^4 ,$$

Eigenvalues: (16)

$$\omega^2 = \frac{E}{\rho(1-\sigma^2)}\left[\frac{1}{R^2} + \frac{1}{12}h^2\left(\frac{m\pi}{L}\right)^4\right] .$$

Special case: ring-shaped vibration $u = 0$.

Differential equations: (17)

$$(1+H)\frac{\partial^2 v}{\partial y^2} + \frac{1}{R}\frac{\partial w}{\partial y} - 4R\frac{\partial^3 w}{\partial y^3} = \frac{1-\sigma^2}{E}\rho\frac{\partial^2 v}{\partial t^2} ,$$

$$\frac{1}{R}\frac{\partial v}{\partial y} - HR\frac{\partial^3 v}{\partial y^3} + \frac{w}{R^2} + HR^2\frac{\partial^4 w}{\partial y^4} = \frac{1-\sigma^2}{E}\rho\frac{\partial^2 v}{\partial t^2} .$$

Modes: (18)

$$v = V \cdot \sin\left(\frac{ny}{R}\right) \quad ; \quad w = W \cdot \cos\left(\frac{ny}{R}\right) ,$$

$$\frac{V}{W} = \frac{K^2 - (1+H)n^2}{n + Hn^3} .$$

Eigenvalue equation: $K^4 - K^2\left(1+n^2\right)\left(1+Hn^2\right) + Hn^2\left(n^2-1\right)^2 = 0 .$ (19)

Approximation for thin shells : $Hn^2 \ll 1$:

$$K_1^2 = H\frac{n^2(n^2-1)^2}{n^2+1} \quad ; \quad K_2^2 = H(n^2+1);$$

$$\omega_1^2 = \frac{E}{\rho(1-\sigma^2)}\frac{1}{R^2}\frac{n^2(n^2-1)^2}{n^2+1} \quad ; \quad \omega_2^2 = \frac{E}{\rho(1-\sigma^2)}\frac{1}{R^2}(n^2+1) . \tag{20}$$

Special case: ring-shaped vibration without dilatation $nV + W = 0$.

Eigenvalues: (21)

$$K^2 = H\,n^2(n^2-1)^2 ,$$

$$\omega^2 = \frac{E}{\rho(1-\sigma^2)}\frac{1}{R^2}n^2(n^2-1)^2 .$$

Compare with plate in $0 \le x \le L; 0 \le y \le \pi R$; simply supported:

$$K^2 = H\,(n^2 + \Lambda^2)^2 ,$$

$$\omega^2 = \frac{Eh^2}{12\rho(1-\sigma^2)}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{\pi R}\right)^2\right] . \tag{22}$$

Special case: ring-shaped bar, simply supported

with (23)

$A = 2\pi R\,h$ cross section,

$I = \pi R^3 h\,(1+3H)$ moment of inertia.

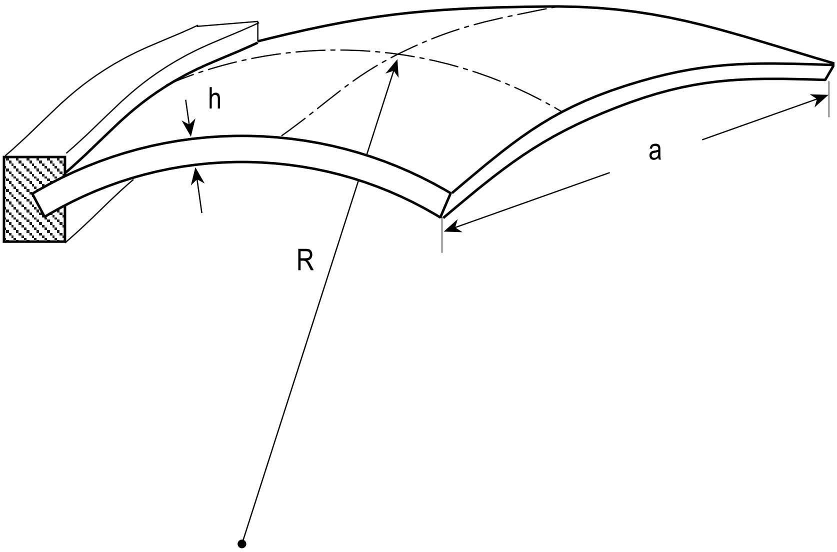
Eigenvalues: $$\omega^2 = \frac{E}{2\rho}\left(R^2 + \frac{1}{4}h^2\right)\left(\frac{m\pi}{L}\right)^2 . \tag{24}$$

Q.19 Similarity Relations for Spherical Shells

► *See also:* Soedel (1973)

The shell be supported anyhow. The bending stiffness is defined as :

$$B = \frac{E\,h^3}{12\,(1-\sigma^2)}. \tag{1}$$



h	=	shell thickness;
s	=	shell dimension;
R	=	radius of curvature;
ρ	=	material density;
E	=	Young's modulus;
B	=	bending stiffness;
σ	=	Poisson ratio

Free bending wave number $k_{B\cap}$ in the shell in relation to the free bending wave number $k_{B||}$ in a plate:

$$k_{B\cap} = \left[\frac{1}{B}\left(\rho h\omega^2 - Eh/R^2\right)\right]^{1/4} = k_{B||}\sqrt{1 - \frac{E}{\rho\omega^2 R^2}}\;. \tag{2}$$

Plates and shells with geometrically similar contours and equal supports have same mode solutions w_n/a if they agree in:

- $\dfrac{a^3}{B}\left(\rho h\omega_n^3 - Eh/R^2\right)$,
- σ.

Relation between the eigenfrequencies ω_{n1}, ω_{n2} of two spherical shells, $i = 1, 2$, which have:
- similar contours,
- equal supports

$$\omega_{n2} = \sqrt{\left(\frac{a_1}{a_2}\right)^3 \frac{E_2}{E_1} \left(\frac{h_2}{h_1}\right)^2 \frac{1-\sigma_1^2}{1-\sigma_2^2} \left(\frac{\rho_1}{\rho_2}\omega_{n1}^2 - \frac{E_1}{\rho_2 R_1^2}\right) + \frac{E_2}{\rho_2 R_2^2}}. \tag{3}$$

Relation between the eigenfrequencies ω_{n2} of a spherical shell and ω_{n1} of a plane plate having
- similar contours,
- equal supports

$$\omega_{n2} = \sqrt{\left(\frac{a_1}{a_2}\right)^3 \frac{E_2}{E_1} \left(\frac{h_2}{h_1}\right)^2 \frac{1-\sigma_1^2}{1-\sigma_2^2} \frac{\rho_1}{\rho_2}\omega_{n1}^2 + \frac{E_2}{\rho_2 R_2^2}}. \tag{4}$$

If material, contour, dimensions and support are equal:

$$\omega_{n2} = \sqrt{\omega_{n1}^2 + \frac{E}{\rho R^2}}. \tag{5}$$

Q.20 Sound Radiation From Plates

► *See also:* Cremer/Heckl (1996); Heckl (1964); Maidanik (1962)

The radiation efficiency Σ, presented below, is the real part of the normalised radiation impedance. See › *Ch. F. "Radiation of Sound"* for radiation impedances.

A plate, infinite if not otherwise stated, be excited by a point or line force, or by a sound field. If the plate has a finite area A it is supposed to be mounted in an infinite hard baffle wall.

The effective sound power Π radiated to one side will be given.

ρ_0	=	density of surrounding medium;
c_0	=	sound velocity in medium;
k_0	=	ω / c_0;
Z_0	=	$\rho_0 c_0$;
h	=	plate thickness;
m''	=	surface mass density of plate;
F_{eff}	=	root mean square of force;
A	=	plate area;
ρ_p	=	plate material density;
E	=	Young's modulus;
σ	=	Poisson ratio;
η	=	plate bending loss factor;
k_B	=	free plate bending wave number;
c_L	=	longitudinal wave speed in plate

Plate excited by a point force:

$$\Pi = \frac{\rho_0}{2\pi c_0 m''^2} F_{eff}^2 \left[1 - \frac{\rho_0 c_0}{\omega m''} \operatorname{arctg} \frac{\omega m''}{\rho_0 c_0} \right] \xrightarrow[k_0 m''/\rho_0 \gg 1]{} \frac{\rho_0}{2\pi c_0} \frac{F_{eff}^2}{m''^2} \xrightarrow[k_0 m''/\rho_0 \ll 1]{} \frac{k_0^2}{6\pi \rho_0 c_0} F_{eff}^2 \,. \tag{1}$$

Radius a of an equivalent piston radiator (i.e. piston radiator with radiation efficiency $\Sigma = 1$ and same effective velocity):

$$a = \sqrt{8/\pi^3} \cdot \lambda_c = 0.286 \cdot \lambda_c \tag{2}$$

with $\lambda_c = c_0 / f_c$ and f_c = coincidence frequency.

Plate excited by a line source (Π and F_{eff} per unit length):

$$\Pi = \frac{\rho_0}{2\,\omega\, m''^2} F_{eff}^2 \left[1 - \frac{\rho_0 c_0}{\omega m''} \left(1 + (\rho_0 c_0 / \omega m'')^2 \right)^{-1/2} \right] \tag{3}$$

Space- and time-averaged squared velocity for excitation of a finite plate by a point source:

$$\langle v^2 \rangle_{s,t} = \frac{k_B^2}{8\,\eta A \omega^2 m''^2} F_{eff}^2 \,. \tag{4}$$

Radiated power of a finite plate, excited by a point force, follows from the definition of radiation efficiency Σ:

$$\Pi = A \cdot \rho_0 c_0 \cdot \Sigma \cdot \langle v^2 \rangle_{s,t} \,, \tag{5}$$

$$\Pi = \frac{\rho_0 c_0 \cdot k_B^2 \cdot \Sigma}{8\,\eta \omega^2 m''^2} F_{eff}^2 \,. \tag{6}$$

Velocity of a plate when excited by a diffuse sound field (p = effective sound pressure):

$$v^2 = \frac{\pi c_0^2 \cdot k_B^2 \cdot \Sigma}{2\,\eta m''^2}\,p \qquad \text{above coincidence frequency,} \tag{7}$$

$$v^2 = \frac{1}{\omega^2 m''^2}\left(2 + \frac{\pi}{2}\,\frac{k_B^2 \cdot \Sigma}{k_0^2 \eta}\right) p \qquad \text{below coincidence frequency.} \tag{8}$$

Ratio of radiated sound power by a point-excited plate to sound field excitation:

- excitation by a point force $\Pi = \alpha \cdot F_{eff}^2$,
- excitation by a diffuse sound field $v^2 = ß \cdot p^2$

follows: $$\frac{\alpha}{ß} = \frac{\rho_0 c_0 k_0^2}{4\pi} = \frac{\rho_0 \omega^2}{4\pi c_0}. \tag{9}$$

Similar relations for a line force (and 2-dimensional sound field):

$$\Pi_L = \alpha_L \cdot F_{Leff}^2 \quad ; \quad v_L^2 = ß_L \cdot p_L^2 \quad ; \quad \frac{\alpha_L}{ß_L} = \frac{\rho_0 c_0 k_0}{4} = \frac{\rho_0 \omega}{4}. \tag{10}$$

Sound power Π_m fed into a plate by a diffuse sound field:

$$\Pi_m = \frac{\pi A k_B^2 \cdot \Sigma}{2\,k_0^2 \omega m''}\,p^2 . \tag{11}$$

Radiated sound power of a finite plate, with area A, periphery U, driven by a point force (approximation):

$$\Pi = \frac{\rho_0}{2\pi c_0 m''^2}\,F_{eff}^2 \left(1 + \frac{U}{2 A k_B \eta}\right) \quad ; \quad f \ll f_c . \tag{12}$$

More general, if radiated power is small compared with internally lost power, i.e. $2\rho_0 c_0 \cdot \Sigma \ll \omega m'' \cdot \eta$:

$$\Pi = \frac{\rho_0}{2\pi c_0 m''^2}\,F_{eff}^2 \left(1 + \frac{2\pi c_0^2 \cdot \Sigma}{2.3\, c_L h\, \omega\, \eta}\right). \tag{13}$$

Approximations for radiation efficiency of point-excited, weakly damped, finite plates:

$$\Sigma \approx \begin{cases} U\lambda_c/(\pi^2 A)\cdot\sqrt{f/f_c} & ; \quad f \ll f_c \\ 0.45\sqrt{U/\lambda_c} & ; \quad f = f_c \\ 1 & ; \quad f \gg f_c \end{cases} \quad ; \quad \lambda_c = c_0/f_c . \tag{14}$$

More precise approximation for $f < f_c$ (Maidanik) :

$$\Sigma = \frac{4}{\pi^2}\,\frac{\lambda_0 \lambda_c}{A}\,\frac{1 - 2\alpha^2}{\alpha\sqrt{1-\alpha^2}} + \frac{U\lambda_c}{4\pi^2 A}\,\frac{(1-\alpha^2)\,\ln\dfrac{1+\alpha}{1-\alpha} + 2\alpha}{(1-\alpha^2)^{3/2}} \quad ; \quad \alpha = \sqrt{f/f_c} \tag{15}$$

with λ_0 = wave length of air-borne sound at frequency f.

Sound pressure far field of a plate excited by a point force $F = \sqrt{2} \cdot F_{eff}$ (R = radius from point of excitation; Θ = polar angle):

Loss-free plate:

$$p(R,\Theta)=\frac{jk_0}{2\pi}F\frac{e^{-j\,k_0R}}{R}\frac{\cos\Theta}{1+\dfrac{jk_0m''}{\rho_0}\cos\Theta\cdot\left(1-(f/f_c)^2\sin^4\Theta\right)}\quad;\quad f<0.7\cdot f_c\,,\tag{16}$$

$$p(R,\Theta)=\frac{jk_0}{2\pi}F\frac{e^{-j\,k_0R}}{R}\frac{\left(1+\varphi(\Theta)\right)\cdot\cos\Theta}{1+\varphi(\Theta)+\dfrac{jk_0m''}{\rho_0}\left\{1+\left[1-\dfrac{1-\sigma}{24}\dfrac{(\pi c_L\sin\Theta)^2}{c_0^2}\right]\cdot\varphi(\Theta)\right\}};$$

$f>0.7\cdot f_c$

with f_c = coincidence frequency;

$$\varphi(\Theta)=\frac{2(k_0h)^2}{\pi^2(1-\sigma)}\left(\sin^2\Theta-(c_0/c_L)^2\right).\tag{17}$$

In the direction $\Theta=0$ normal to the plate:

$$p(R,0)=\frac{jk_0}{2\pi}F\frac{e^{-j\,k_0R}}{R}\frac{\cos\Theta}{1+\dfrac{jk_0m''}{\rho_0}}.\tag{18}$$

In the direction of the angle of coincidence $\Theta_c=\sin^{-1}\sqrt{f_c/f}$:

$$p(R,0)=\frac{jk_0}{2\pi}F\frac{e^{-j\,k_0R}}{R}\sqrt{1-(f_c/f)^2}\,.\tag{19}$$

Plate with bending loss factor $\eta\ll1$:

Losses have negligible influence for $f<f_c$.

Define complex coincidence frequency: $\omega_c=2\pi f_c=\sqrt{12}\dfrac{c_0^2}{c_Lh}\left(1+j\,\eta/2\right)$, (20)

$$p(R,\Theta_c)=\frac{jk_0}{2\pi}F\frac{e^{-j\,k_0R}}{R}\frac{\sqrt{1-(f_c/f)^2}}{1+\eta\dfrac{k_0m''}{\rho_0}\sqrt{1-f_c/f}}.\tag{21}$$

References

Achenbach, J.D.: Wave propagation in elastic solids. North-Holland, Amsterdam (1975)

Auld, B.A.: Acoustic fields and waves in solids. Vol. I and II, Krieger Publishing Company, Malabar, Florida (1990)

Beltzer, A.I.: Acoustics of solids. Springer-Verlag, Berlin (1988)

Cremer, L., Heckl, M.: Körperschall. Springer, Berlin (1996)

Dym, C.L.: Some new results for the vibration of circular cylinders. J.Sound and Vibr. **29**, 189–205 (1973)

Fahy, F.: Sound and Structural Vibration. Academic Press, London (1985)

Gaul, L.: The influence of damping on waves and vibrations. Mechanical Systems and Signal Processing **13**, 1–30 (1999)

Gazis, D.C.: Three-dimensional investigation of the propagation of waves in hollow circular cylinders. I. Analytical formulation. J. Acoust. Soc. Am. **31**, 568–578 (1959) ‘

Gorman, D.J.: Free Vibration Analysis of Rectangular Plates Elesevier/North Holland Inc., N.Y. (1982)

Haberkern, R.: Personal communication

Heckl, M.: Untersuchungen an orthotropen Platten. Acustica **10**, 109–115 (1960)

Heckl, M.: Einige Anwendungen des Reziprozitätsprinzips in der Akustik. Frequenz **18**, 299–304 (1964)

Helbig, K.: Foundations of anisotropy for exploration seismics. Pergamon/Elsevier, Oxford (1994)

Jones, R.M.: Mechanics of composite materials. Taylor & Francis, Philadelphia (1999)

Junger, M.C., Feit, D.: Sound, structures and their interactions. MIT Press, Cambridge MA (1986)

Lai, W.M., Rubin, D., Krempl, E.: Introduction to continuum mechanics. Pergamon Press, Oxford (1993)

Lanceleur, P., Ribeiro, H., De Belleval, J.-F.J.: The use of inhomogeneous waves in the reflection-transmission problem at a plane interface between two anisotropic media. Acoust. Soc. Am. **93**, 1882–1892 (1993)

Lighthill, M.J.: J. Inst. Maths. Appls. **1**, 1–28 (1965)

Maidanik: Response of ribbed panels to reverberant acoustic fields. J.Acoust.Soc.Amer. **34**, 809–826 (1962)

Maysenhölder, W.: Some didactical and some practical remarks on free plate waves. J. Sound Vib. **118**, 531–538 (1987)

Maysenhölder, W.: Rigorous computation of plate-wave intensity. Acustica **72**, 166–179 (1990)

Maysenhölder, W.: Analytical determination of the group velocity of an arbitrary Lamb wave from its phase velocity. Acustica **77**, 208 (1992)

Maysenhölder, W.: Proof of two theorems related to the energy of acoustic Bloch waves in periodically inhomogeneous media. Acustica **78**, 246–249 (1993)

Maysenhölder, W.: Körperschallenergie. Hirzel, Stuttgart (1994)

Maysenhölder, W.: Low-frequency sound transmission through periodically inhomogeneous plates with arbitrary local anisotropy and arbitrary global symmetry. Acustica acta acustica **82**, 628–635 (1996)

Maysenhölder, W.: Sound transmission through periodically inhomogeneous anisotropic plates: Generalizations of Cremer's thin plate theory. Acustica **84**, 668–680 (1998)

Mechel, F.P.: Schallabsorber, Vol. II, Ch. 27: Plate and Membrane Absorbers. Hirzel, Stuttgart (1995)

Mechel, F.P.: About the Partition Impedance of Plates, Shells, and Membranes. Acta Acustica, submitted (1999)

Mitchell, A.K., Hazell, C.R.: A simple frequency formula for clamped rectangular plates. J. Sound Vibr. **118**, 271–281 (1987)

Mobley, J., et al.: Kramers-Kronig relations applied to finite bandwidth data from suspensions of encapsulated microbubbles. J. Acoust. Soc. Am. **108**, 2091–2106 (2000)

Mozhaev, V.G.: Approximate analytical expressions for the velocity of Rayleigh waves in isotropic media and on the basal plane in high-symmetry crystals. Sov. Phys. Acoust. **37**, 186–189 (1991)

Murphy, J.E., Li, G., Chin-Bing, S.A.: Orthogonality relation for Rayleigh-Lamb modes of vibration of an arbitrarily layered elastic plate with and without fluid loading. J. Acoust. Soc. Am. **96**, 2313–2317 (1994)

Pavic, G.: Structural surface intensity: An alternative approach in vibration analysis and diagnosis, J. Sound Vib. **115**, 405–422 (1987)

Pierce, A.D.: Variational formulations in acoustic radiation and scattering. In: Physical Acoustics, Vol. XXII (Underwater Scattering and Radiation), A.D. Pierce, R.N. Thurston (eds.), Academic Press, Boston, 195–371 (1993)

Pierce, A.D.: The natural reference wavenumber for parabolic approximations in ocean acoustics. Comp. & Maths. with Appls. **11**, 831–841 (1985)

Pritz, T.: Analysis of four-parameter fractional derivative model of real solid materials. J. Sound Vib. **195**, 103–115 (1996)

Rokhlin, S.I., Bolland, T.K., Adler, L.: Reflection and refraction of elastic waves on a plane interface between two generally anisotropic media. J. Acoust. Soc. Am. **79**, 906–918 (1986)

Sigalas, M.M., Economou, E.N.: Elastic and acoustic wave band structure. J. Sound Vib. **158**, 377–382 (1992)

Skelton, E.A., James, J.H.: Theoretical acoustics of underwater structures. Imperial College Press, London (1997)

Soedel: A natural frequency analogy between spherically curved panels and flat plates. J. Sound and Vibr. **29**, 457–461 (1973)

Sornette, D.: "Acoustic waves in random media. I. Weak disorder regime. II. Coherent effects and strong disorder regime. III. Experimental situations. Acustica **67**, 199–215 (1989), 251–265; **68**, 15–25 (1989)

Thurston, R.N.: Wave propagation in fluids and normal solids. In: Physical Acoustics, Vol. I (Methods and devices, Part A), W. P. Mason (ed.), Academic Press, New York (1964), 1–110

Ting, T.C.T., Barnett, D.M.: Classifications of surface waves in anisotropic elastic materials. Wave Motion **26**, 207–218 (1997)

Viktorova, R.N., Tyutekin, V.V.: Physical foundations for synthesis of sound absorbers using complex-density composites. Acoust. Phys. **44**, 275–280 (1998)