1. Units and Measurements

Can you recall?

1. What is a unit?
2. Which units have you used in the laboratory for measuring
   (i) length (ii) mass (iii) time (iv) temperature?
3. Which system of units have you used?

1.1 Introduction:

Physics is a quantitative science, where we measure various physical quantities during experiments. In our day to day life, we need to measure a number of quantities, e.g., size of objects, volume of liquids, amount of matter, weight of vegetables or fruits, body temperature, length of cloth, etc. A measurement always involves a comparison with a standard measuring unit which is internationally accepted. For example, for measuring the mass of a given fruit we need standard mass units of 1 kg, 500 g, etc. These standards are called units. The measured quantity is expressed in terms of a number followed by a corresponding unit, e.g., the length of a wire is written as 5 m where m (metre) is the unit and 5 is the value of the length in that unit. Different quantities are measured in different units, e.g. length in metre (m), time in seconds (s), mass in kilogram (kg), etc. The standard measure of any quantity is called the unit of that quantity.

1.2 System of Units:

In our earlier standards we have come across various systems of units namely

(i) CGS: Centimetre Gram Second system

(ii) MKS: Metre Kilogram Second system

(iii) FPS: Foot Pound Second system.

(iv) SI: System International

The first three systems namely CGS, MKS and FPS were used extensively till recently. In 1971, the 14th International general conference on weights and measures recommended the use of 'International system' of units. This international system of units is called the SI units. As the SI units use decimal system, conversion within the system is very simple and convenient.

1.2.1 Fundamental Quantities and Units:

The physical quantities which do not depend on any other physical quantities for their measurements are known as fundamental quantities. There are seven fundamental quantities: length, mass, time, temperature, electric current, luminous intensity and amount of substance.

Fundamental units: The units used to measure fundamental quantities are called fundamental units. The fundamental quantities, their units and symbols are shown in the Table 1.1.

Table 1.1: Fundamental Quantities with their SI Units and Symbols

Fundamental quantity	SI units	Symbol
1) Length	metre	m
2) Mass	kilogram	kg
3) Time	second	s
4) Temperature	kelvin	K
5) Electric current	ampere	A
6) Luminous Intensity	candela	cd
7) Amount of substance	mole	mol

1.2.2 Derived Quantities and Units:

In physics, we come across a large number of quantities like speed, momentum, resistance, conductivity, etc. which depend on some or all of the seven fundamental quantities and can be expressed in terms of these quantities. These are called derived quantities and their units, which can be expressed in terms of the fundamental units, are called derived units.

For example,

SI unit of velocity

$$= \frac{\text{Unit of displacement}}{\text{Unit of time}} = \frac{\text{m}}{\text{s}} = \text{m s}^{-1}$$

Unit of momentum = (Unit of mass)×(Unit of velocity)

$= kg\ m/s = kg\ m\ s^{-1}$

The above two units are derived units.

Supplementary units : Besides, the seven fundamental or basic units, there are two more units called supplementary units: (i) Plane angle $d\theta$ and (ii) Solid angle $d\Omega$

(i) Plane angle ($d\theta$) : This is the ratio of the length of an arc of a circle to the radius of the circle as shown in Fig. 1.1 (a). Thus $d\theta = ds/r$ is the angle subtended by the arc at the centre of the circle. It is measured in radian (rad). An angle θ in radian is denoted as θ^c.

(ii) Solid angle ($d\Omega$) : This is the 3-dimensional analogue of $d\theta$ and is defined as the area of a portion of surface of a sphere to the square of radius of the sphere. Thus $d\Omega = dA/r^2$ is the solid angle subtended by area ds at O as shown in Fig. 1.1 (b). It is measured in steradians (sr). A sphere of radius r has surface area $4\pi r^2$. Thus, the solid angle subtended by the entire sphere at its centre is $\Omega = 4\pi r^2/r^2 = 4\pi$ sr.

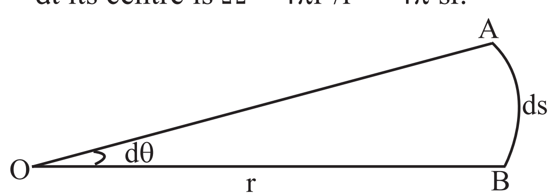


Fig 1.1 (a): Plane angle $d\theta$.

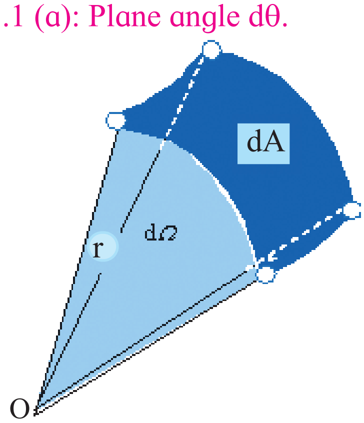


Fig 1.1 (b): Solid angle $d\Omega$.

Example 1.1: What is the solid angle subtended by the moon at any point of the Earth, given the diameter of the moon is 3474 km and its distance from the Earth 3.84×10^8 m.

Solution: Solid angle subtended by the moon at the Earth

$$= \frac{\text{Area of the disc of the moon}}{(\text{moon - earth distance})^2}$$

$$= \frac{\pi\times(1.737\times10^3)^2}{(3.84\times10^5)^2}$$

$$= 6.425\times10^{-5}\ sr$$

Do you know ?

The relation between radian and degree is

π radians $= \pi^c = 180°$

$$\therefore\ 1 \text{ radian} = \frac{180}{\pi} = \frac{180}{3.1415} = 57.297°$$

$$\text{Similarly } 1° = \frac{\pi}{180} = \frac{3.1415}{180} = 1.745\times10^{-2} \text{ rad}$$

$$1° = 60',\ 1' = 2.91\times10^{-4} \text{ rad}$$

$$\text{and } 1' = 60'',\ 1'' = 4.847\times10^{-6} \text{ rad}$$

1.2.3 Conventions for the use of SI Units:

(1) Unit of every physical quantity should be represented by its symbol.

(2) Full name of a unit always starts with smaller letter even if the name is after a person, e.g., 1 newton, 1 joule, etc. But symbol for unit named after a person should be in capital letter, e.g., N after scientist Newton, J after scientist Joule, etc.

(3) Symbols for units do not take plural form for example, force of 20 N and not 20 newtons or not 20 Ns.

(4) Symbols for units do not contain any full stops at the end of recommended letter, e.g., 25 kg and not 25 kg..

(5) The units of physical quantities in numerator and denominator should be written as one ratio for example the SI unit of acceleration is m/s^2 or $m\ s^{-2}$ but not m/s/s.

(6) Use of combination of units and symbols for units is avoided when physical quantity is expressed by combination of two. e.g., The unit J/kg K is correct while joule/kg K is not correct.

(7) A prefix symbol is used before the symbol of the unit.

Thus prefix symbol and units symbol constitute a new symbol for the unit which can be raised to a positive or negative power of 10.

1ms = 1 millisecond = 10^{-3}s
1µs = 1 microsecond = 10^{-6}s
1ns = 1 nanosecond = 10^{-9}s
Use of double prefixes is avoided when single prefix is available
10^{-6}s =1µs and not 1mms.
10^{-9}s = 1ns and not 1mµs

(8) Space or hyphen must be introduced while indicating multiplication of two units e.g., m/s should be written as m s^{-1} or m-s^{-1} and not as ms^{-1} (because ms will be read as millisecond).

1.3 Measurement of Length:

One fundamental quantity which we have discussed earlier is length. To measure the length or distance the SI unit used is metre (m). In 1960, a standard for the metre based on the wavelength of orange-red light emitted by atoms of krypton was adopted. By 1983 a more precise measurement was developed. It says that a metre is the length of the path travelled by light in vacuum during a time interval of 1/299792458 second. This was possible as by that time the speed of light in vacuum could be measured precisely as c = 299792458 m/s.

Some typical distances/lengths are given in Table 1.2.

Table 1.2: Some Useful Distances

Measurement	Length in metre
Distance to Andromeda Galaxy (from Earth)	2×10^{22} m
Distance to nearest star (after Sun) Proxima Centuari (from Earth)	4×10^{16} m
Distance to Pluto (from Earth)	6×10^{12} m
Average Radius of Earth	6×10^{6} m
Height of Mount Everest	9×10^{3} m
Thickness of this paper	1×10^{-4} m
Length of a typical virus	1×10^{-8} m
Radius of hydrogen atom	5×10^{-11} m
Radius of proton	1×10^{-15} m

1.3.1 Measurements of Large Distance: Parallax method

Large distance, such as the distance of a planet or a star from the Earth, cannot be measured directly with a metre scale, so a parallax method is used for it.

Let us do a simple experiment to understand what is parallax.

Hold your hand in front of you and look at it with your left eye closed and then with your right eye closed. You will find that your hand appears to move against the background. This effect is called parallax. Parallax is defined as the apparent change in position of an object due to a change in the position of the observer. By measuring the parallax angle (θ) and knowing the distance between the eyes E_1 and E_2 as shown in Fig. 1.2, we can determine the distance of the object from us, i.e., OP as E_1E_2/θ.

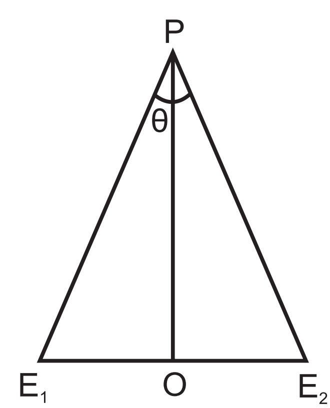


Fig.1.2: Parallax method for determining distance.

As the distances of planets from the Earth are very large, we can not use two eyes method as discussed here. In order to make simultaneous observations of an astronomical object, we select two distant points on the Earth.

Consider two positions A and B on the surface of Earth, separated by a straight line at distance b as shown in Fig. 1.3. Two observers at these two points observe a distant planet S simultaneously. We measure the angle ∠ASB between the two directions along which the planet is viewed at these two points. This angle, represented by symbol θ, is the parallax angle.

As the planet is far away, i.e., the distance of the planet from the Earth is very large in comparison to b, $b/D << 1$ and, therefore, θ is very small.

We can thus consider AB as the arc of length b of the circle and D its radius.

AB = b and AS = BS = D and $\theta \cong AB/D$, where θ is in radian

$$\therefore D = b/\theta \qquad \text{--- (1.1)}$$

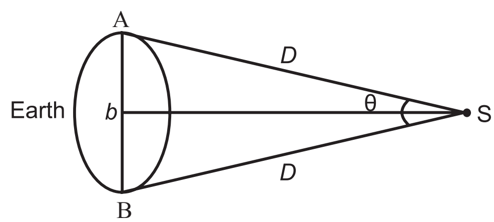


Fig.1.3: Measurement of distances of planets

1.3.2 Measurement of Distance to Stars:

Sun is the star which is closest to the Earth. The next closest star is at a distance of 4.29 light years. The parallax measured from two most distance points on the Earth for stars will be too small to be measured and for this purpose we measure the parallax between two farthest points (i.e. 2 AU apart, see box below) along the orbit of the Earth around the Sun (see figure in example 1.2 below).

1.3.3 Measurement of the Size of a Planet or a Star:

If d is the diameter of a planet, the angle subtended by it at any single point on the Earth is called angular diameter of the planet. Let α be the angle between the two directions when two diametrically opposite points of the planet are viewed through a telescope as shown in Fig. 1.4. As the distance D of the planet is large (assuming it has been already measured), we can calculate the diameter of the planet as

$$\alpha = \frac{d}{D}$$

$$\therefore d = \alpha D \qquad \text{--- (1.2)}$$

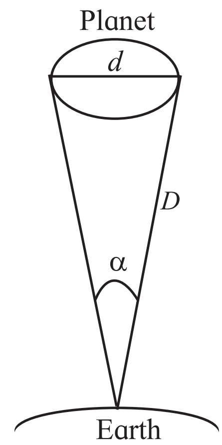


Fig. 1.4: Measurement of size of a planet

1.3.4 Measurement of Very Small Distances:

When we intend to measure the size of the atoms and molecules, the conventional apparatus like Vernier calliper or screw guage will not be useful. Therefore, we use electron microscope or tunnelling electron microscope to measure the size of atoms.

Do you know ?

For measuring large distances, astronomers use the following units.

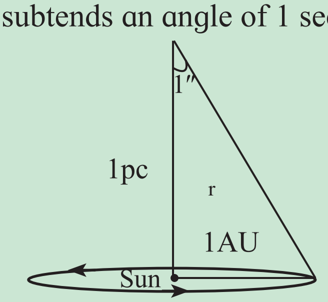
1 astronomical unit (AU) = 1.496×10^{11}m

1 light year = 9.467×10^{15}m

1 parsec (pc) = 3.08×10^{16}m ≅ 3.26 light years

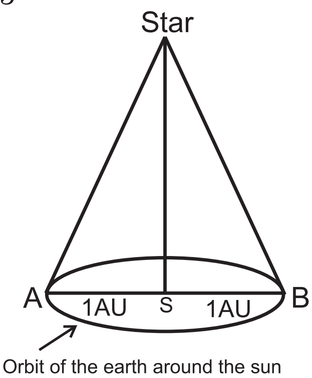
A light year is the distance travelled by light in one year. The astronomical unit (AU) is the mean distance between the centre of the Earth and the centre of the Sun.

A parsec (pc) is the distance from where 1AU subtends an angle of 1 second of arc.



$$r = \frac{1AU}{(1")^c} = \frac{1.496\times10^{11}}{4.847\times10^{-6}} = 3.086\times10^{16}\text{m}$$

Example 1.2: A star is 5.5 light years away from the Earth. How much parallax in arcsec will it subtend when viewed from two opposite points along the orbit of the Earth?



Solution: Two opposite points A and B along the orbit of the Earth are 2 AU apart. The angle subtended by AB at the position of the star is = AB/distance of the star from the Earth

$$= \frac{2AU}{5.5\,ly} = \frac{2\times1.496\times10^{11}\,m}{5.5\times9.46\times10^{15}\,m} = 5.75\times10^{-6}\,rad$$

$$= 5.75\times10^{-6}\times57.297\times60\times60\ arcsec$$

$$= 1.186\ arcsec$$

Do you know ?

Small distances are measured in units of (i) fermi (1F = 10^{-15} m) in SI system. Thus, 1F is one femtometre (fm) or (ii) Angstrom ($1\,\mathring{A} = 10^{-10}$ m).

For measuring sizes using a microscope we need to select the wavelength of light to be used in the microscope such that it is smaller than the size of the object to be measured. Thus visible light (wavelength from 4000 $\mathring{A}$ to 7000 $\mathring{A}$) can measure sizes down to about 4000 $\mathring{A}$. If we want to measure even smaller sizes we need to use even smaller wavelength and so the use of electron microscope is necessary. As you will study in the XIIth standard, each material particle corresponds to a wave. The approximate wavelength of the electrons in an electron microscope is about 0.6 $\mathring{A}$ so that one can measure atomic sizes ≈ 1 $\mathring{A}$ using this microscope.

Example 1.3: The moon is at a distance of 3.84×10^8 m from the Earth. If viewed from two diametrically opposite points on the Earth, the angle subtended at the moon is 1° 54'. What is the diameter of the Earth?

Solution: Angle subtended

$\theta = 1°\ 54' = 114' = 114\times2.91\times10^{-4}$ rad

$= 3.317\times10^{-2}$ rad

Diameter of the Earth = θ × distance to the moon from the Earth

$= 3.317\times10^{-2}\times3.84\times10^{8}$ m

$= 1.274\times10^{7}$m

1.4 Measurement of Mass:

Since 1889, a kilogram was the mass of a shiny piece of platinum-iridium alloy kept in a special glass case at the International Bureau of weights and measures. This definition of mass has been modified on 20th May 2019, the reason being that the carefully kept platinum-iridium piece is seen to pick up micro particles of dirt and is also affected by the atmosphere causing its mass to change. The new measure of kilogram is defined in terms of magnitude of electric current. We know that electric current can be used to make an electromagnet. An electromagnet attracts magnetic materials and is thus used in research and in industrial applications such as cranes to lift heavy pieces of iron/steel. Thus the kilogram mass can be described in terms of the amount of current which has to be passed through an electromagnet so that it can pull down one side of an extremely sensitive balance to balance the other side which holds one standard kg mass.

While dealing with mass of atoms and molecules, kg is an inconvenient unit. Therefore, their mass is measured in atomic mass unit. It will be easy to compare mass of any atom in terms of mass of some standard atom which has been decided internationally to be C^{12} atom. The (1/12)th mass of an unexcited atom of C^{12} is called atomic mass unit (amu).

1 amu = 1.6605402×10^{-27} kg with an uncertainty of 10 in the last two decimal places.

1.5 Measurement of Time:

The SI unit of time is the second (s). For many years, duration of one mean Solar day was considered as reference. A mean Solar day is the average time interval from one noon to the next noon. Average duration of a day is taken as 24 hours. One hour is of 60 minutes and each minute is of 60 seconds. Thus a mean Solar day = 24 hours = $24 \times 60 \times 60$ = 86400 s. Accordingly a second was defined as 1/86400 of a mean Solar day.

It was later observed that the length of a Solar day varies gradually due to the gradual slowing down of the Earth's rotation. Hence, to get more standard and nonvarying (constant) unit for measurement of time, a cesium atomic clock is used. It is based on periodic vibrations produced in cesium atom. In cesium atomic clock, a second is taken as the time needed for 9,192,631,770 vibrations of the radiation (wave) emitted during a transition between two hyperfine states of Cs^{133} atom.

Do you know ?

Why is only carbon used and not any other element for defining atomic mass unit? Carbon 12 (C^{12}) is the most abundant isotope of carbon and the most stable one. Around 98% of the available carbon is C^{12} isotope.

Earlier, oxygen and hydrogen were used as the standard atoms. But various isotopes of oxygen and hydrogen are present in higher proportion and therefore it is more accurate to use C^{12}.

1.6 Dimensions and Dimensional Analysis:

As mentioned earlier, a derived physical quantity can be expressed in terms of some combination of seven basic or fundamental quantities. For convenience, the basic quantities are represented by symbols as 'L' for length, 'M' for mass, 'T' for time, 'K' For temperature, 'I' for current, 'C' for luminous intensity and 'mol' for amount of mass.

The dimensions of a physical quantity are the powers to which the concerned fundamental units must be raised in order to obtain the unit of the given physical quantity.

When we represent any derived quantity with appropriate powers of symbols of the fundamental quantities, then such an expression is called dimensional formula. This dimensional formula is expressed by square bracket and no comma is written in between any of the symbols.

Illustration:

(i) Dimensional formula of velocity

$$\text{Velocity} = \frac{\text{displacment}}{\text{time}}$$

$$\text{Dimensions of velocity} = \frac{[L]}{[T]} = [L^1M^0T^{-1}]$$

ii) Dimensional formula of velocity gradient

$$\text{velocity gradient} = \frac{\text{velocity}}{\text{distance}}$$

Dimensions of velocity gradient

$$= \frac{[LT^{-1}]}{[L]} = [L^0M^0T^{-1}]$$

iii) Dimensional formula for charge.

charge = current $\times$ time

Dimensions of charge = [I] [T] = $[L^0M^0T^1I^1]$

Table 1.3: Some Common Physical Quantities their, SI Units and Dimensions

S. No	Physical quantity	Formula	SI unit	Dimensional formula
1	Density	$\rho = M/V$	kilogram per cubic metre (kg/m^3)	$[L^{-3}M^1T^0]$
2	Acceleration	$a = v/t$	metre per second square (m/s^2)	$[L^1M^0T^{-2}]$
3	Momentum	$P = mv$	kilogram metre per second (kg m/s)	$[L^1M^1T^{-1}]$
4	Force	$F = ma$	kilogram metre per second square (kg m/s^2) or newton (N)	$[L^1M^1T^{-2}]$
5	Impulse	$J = F.\,t$	newton second (Ns)	$[L^1M^1T^{-1}]$
6	Work	$W = F.s$	joule (J)	$[L^2M^1T^{-2}]$
7	Kinetic Energy	$KE = 1/2\ mv^2$	joule (J)	$[L^2M^1T^{-2}]$
8	Pressure	$P = F/A$	kilogram per metre second square (kg/ms^2)	$[L^{-1}M^1T^{-2}]$

Table 1.3 gives the dimensions of various physical quantities commonly used in mechanics.

1.6.1 Uses of Dimensional Analysis:

(i) To check the correctness of physical equations: In any equation relating different physical quantities, if the dimensions of all the terms on both the sides are the same then that equation is said to be dimensionally correct. This is called the principle of homogeneity of dimensions. Consider the first equation of motion.

$v = u + at$

Dimension of L.H.S $= [v] = [LT^{-1}]$

$[u] = [LT^{-1}]$

$[at] = [LT^{-2}]\,[T] = [LT^{-1}]$

Dimension of R.H.S $= [LT^{-1}] + [LT^{-1}]$

$[L.H.S] = [R.H.S]$

As the dimensions of L.H.S and R.H.S are the same, the given equation is dimensionally correct.

(ii) To establish the relationship between related physical quantities: The period T of oscillation of a simple pendulum depends on length l and acceleration due to gravity g. Let us derive the relation between T, l, g :

Suppose $T \propto l^a$

and $T \propto g^b$

$\therefore T \propto l^a g^b$

$T = k\, l^a g^b$,

where k is constant of proportionality and it is a dimensionless quantity and a and b are rational numbers. Equating dimensions on both sides,

$[L^0M^0T^1] = k\,[L^1]^a\,[LT^{-2}]^b$

$= k\,[L^{a+b}\,T^{-2b}]$

$[L^0T^1] = k\,[L^{a+b}T^{-2b}]$

Comparing the dimensions of the corresponding quantities on both the sides we get

$a + b = 0$

$\therefore a = -b$

and

$-2b = 1$

$\therefore b = -1/2$

$\therefore a = -b = -(-1/2)$

$\therefore a = 1/2$

$\therefore T = k\, l^{1/2}\, g^{-1/2}$

$\therefore T = k\sqrt{l/g}$

The value of k is determined experimentally and is found to be 2π

$\therefore T = 2\pi\sqrt{l/g}$

(iii) To find the conversion factor between the units of the same physical quantity in two different systems of units: Let us use dimensional analysis to determine the conversion factor between joule (SI unit of work) and erg (CGS unit of work).

Let 1 J = x erg

Dimensional formula for work is $[L^2M^1T^{-2}]$ Substituting in the above equation, we can write

$[L_1^{\,2}M_1^{\,1}T_1^{\,-2}] = x\,[L_2^{\,2}M_2^{\,1}T_2^{\,-2}]$

$$x = \frac{[L_1^{\,2}M_1^{\,1}T_1^{\,-2}]}{[L_2^{\,2}M_2^{\,1}T_2^{\,-1}]}$$

$$\text{or,}\quad x = \left(\frac{L_1}{L_2}\right)^2\left(\frac{M_1}{M_2}\right)^1\left(\frac{T_1}{T_2}\right)^{-2}$$

where suffix 1 indicates SI units and 2 indicates CGS units.

In SI units, L, M, T are expressed in m, kg and s and in CGS system L, M, T are represented in cm, g and s respectively.

$$\therefore\ x = \left(\frac{\text{kg}}{\text{g}}\right)^1\left(\frac{\text{m}}{\text{cm}}\right)^2\left(\frac{\text{s}}{\text{s}}\right)^{-2}$$

$$\text{or}\ \ x = \left(10^3\,\frac{\text{g}}{\text{g}}\right)^1\left((100)\frac{\text{cm}}{\text{cm}}\right)^2(1)^{-2}$$

$\therefore\ x = (10^3)\,(10^4) = 10^7$

$\therefore$ 1 joule $= 10^7$ erg

Example 1.4: A calorie is a unit of heat and it equals 4.2 J, where 1 J = kg m^2 s^{-2}. A distant civilisation employs a system of units in which the units of mass, length and time are α kg, β m and γ s. Also J' is their unit of energy. What will be the magnitude of calorie in their units?

Solution: Let us write the new units as A, B and C for mass, length and time respectively. We are given

$1\text{ A} = \alpha \text{ kg}$

$1\text{ B} = \beta \text{ m}$

$1\text{ C} = \gamma \text{ s}$

$1\text{ cal} = 4.2\text{ J} = 4.2\text{ kg m}^2\text{ s}^{-2}$

$$= 4.2\left(\frac{A}{\alpha}\right)\left(\frac{B}{\beta}\right)^2\left(\frac{C}{\gamma}\right)^{-2}$$

$$= \frac{4.2\gamma^2}{\alpha\beta^2}\text{ AB}^2\text{ C}^{-2}$$

$$= \frac{4.2\gamma^2}{\alpha\beta^2}\text{ J}'$$

Thus in the new units, 1 calorie is $= \dfrac{4.2\gamma^2}{\alpha\beta^2}\text{ J}'$

1.6.2 Limitations of Dimensional Analysis:

1) The value of dimensionless constant can be obtained with the help of experiments only.
2) Dimensional analysis can not be used to derive relations involving trigonometric, exponential, and logarithmic functions as these quantities are dimensionless.
3) This method is not useful if constant of proportionality is not a dimensionless quantity.

   Illustration : Gravitational force between two point masses is directly proportional to product of the two masses and inversely proportional to square of the distance between the two

   $$\therefore F \propto \frac{m_1 m_2}{r^2}$$

   $$\text{Let } F = G\frac{m_1 m_2}{r^2}$$

   The constant of proportionality 'G' is NOT dimensionless. Thus earlier method will not work.
4) If the correct equation contains some more terms of the same dimension, it is not possible to know about their presence using dimensional equation. For example, with standard symbols, the equation $S = \frac{1}{2}at^2$ is dimensionally correct. However, the complete equation is $S = ut + \frac{1}{2}at^2$

1.7 Accuracy, Precision and Uncertainty in Measurement:

Physics is a science based on observations and experiments. Observations of various physical quantities are made during an experiment. For example, during the atmospheric study we measure atmospheric pressure, wind velocity, humidity, etc. All the measurements may be accurate, meaning that the measured values are the same as the true values. Accuracy is how close a measurement is to the actual value of that quantity. These measurements may be precise, meaning that multiple measurements give nearly identical values (i.e., reproducible results). In actual measurements, an observation may be both accurate and precise or neither accurate nor precise. The goal of the observer should be to get accurate as well as precise measurements.

Possible uncertainties in an observation may arise due to following reasons:

1) Quality of instrument used.

2) Skill of the person doing the experiment.

3) The method used for measurement.

4) External or internal factors affecting the result of the experiment.

Can you tell?

If ten students are asked to measure the length of a piece of cloth up to a mm, using a metre scale, do you think their answers will be identical? Give reasons.

1.8 Errors in Measurements:

Faulty measurements of physical quantity can lead to errors. The errors are broadly divided into the following two categories :

a) Systematic errors : Systematic errors are errors that are not determined by chance but are introduced by an inaccuracy (involving either the observation or measurement process) inherent to the system. Sources of systematic error may be due to imperfect calibration of the instrument, and sometimes imperfect method of observation.

Each of these errors tends to be in one direction, either positive or negative. The sources of systematic errors are as follows:

(i) Instrumental error: This type of error arises due to defective calibration of an instrument, for example an incorrect zeroing of an instrument will lead to such kind of error ('zero' of a thermometer not graduated at proper place, the pointer of weighting balance in the laboratory already indicating some value instead of showing zero when no load is kept on it, an ammeter showing a current of 0.5 amp even when not connected in circuit, etc).

(ii) Error due to imperfection in experimental technique: This is an error due to defective setting of an instrument. For example the measured volume of a liquid in a graduated tube will be inaccurate if the tube is not held vertical.

(iii) Personal error: Such errors are introduced due to fault of the observer. Bias of the observer, carelessness in taking observations etc. could result in such errors. For example, while measuring the length of an object with a ruler, it is necessary to look at the ruler directly from above. If the observer looks at it from an angle, the measured length will be wrong due to parallax.

Systematic errors can be minimized by using correct instrument, following proper experimental procedure and removing personal error.

b) Random errors: These are the errors which are introduced even after following all the procedures to minimize systematic errors. These type of errors may be positive or negative. These errors can not be eliminated completely but we can minimize them by repeated observations and then taking their mean (average). Random errors occur due to variation in conditions in which experiment is performed. For example, the temperature may change during the course of an experiment, pressure of any gas used in the experiment may change, or the voltage of the power supply may change randomly, etc.

1.8.1 Estimation of error:

Suppose the readings recorded repeatedly for a physical quantity during a measurement are

$a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n$.

Arithmetic mean a_{mean} is given by

$$a_{mean} = \frac{a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n}{n}$$

$$a_{mean} = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad \text{--- (1.3)}$$

This is the most probable value of the quantity. The magnitude of the difference between mean value and each individual value is called **absolute error** in the observation.

Thus for 'a_1', the absolute error Δa_1 is given by

$$\Delta a_1 = |a_{mean} - a_1|,$$

for a_2,

$$\Delta a_2 = |a_{mean} - a_2|$$

and so for a_n it will be

$$\Delta a_n = |a_{mean} - a_n|$$

The arithmetic mean of all the absolute errors is called **mean absolute error** in the measurement of the physical quantity.

$$\Delta a_{mean} = \frac{\Delta a_1 + \Delta a_2 + \ldots\ldots + \Delta a_n}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \Delta a_i \qquad \text{--- (1.4)}$$

The measured value of the physical quantity a can then be represented by

$a = a_{mean} \pm \Delta a_{mean}$ which tells us that the actual value of 'a' could be between $a_{mean} - \Delta a_{mean}$ and $a_{mean} + \Delta a_{mean}$. The ratio of mean absolute error to its arithmetic mean value is called **relative error**.

$$\text{Relative error} = \frac{\Delta a_{mean}}{a_{mean}} \qquad \text{--- (1.5)}$$

When relative error is represented as percentage it is called **percentage error**.

$$\text{Percentage error} = \frac{\Delta a_{mean}}{a_{mean}} \times 100 \qquad \text{--- (1.6)}$$

Activity :

Perform an experiment using a Vernier callipers of least count 0.01cm to measure the external diameter of a hollow cylinder. Take 3 readings at different position on the cylinder and find (i) the mean diameter (ii) the absolute mean error and (iii) the percentage error in the measurement of diameter.

Example 1.5: The radius of a sphere measured repeatedly yields values 5.63m, 5.54 m, 5.44 m, 5.40 m and 5.35 m. Determine the most probable value of radius and the mean absolute, relative and percentage errors.

Solution: Most probable value of radius is its arithmetic mean

$$= \frac{5.63+5.54+5.44+5.40+5.35}{5}\text{ m}$$

$$= 5.472 \text{ m}$$

Mean absolute error

$$= \frac{1}{5}\left\{\begin{array}{l} |5.63-5.472|+|5.54-5.472| \\ +|5.44-5.472|+|5.40-5.472| \\ +|5.35-5.472| \end{array}\right\}\text{m}$$

$$= \frac{0.452}{5} = 0.0904 \text{ m}$$

$$\text{Relative error } = \frac{0.0904}{5.472} = 0.017$$

% error = 1.7%

1.8.2 Combination of errors:

When we do an experiment and measure various physical quantities associated with the experiment, we must know how the errors from individual measurement combine to give errors in the final result. For example, in the measurement of the resistance of a conductor using Ohms law, there will be an error in the measurement of potential difference and that of current. It is important to study how these errors combine to give the error in the measurement of resistance.

a) Errors in sum and in difference:

Suppose two physical quantities A and B have measured values A ± ΔA and B ± ΔB, respectively, where ΔA and ΔB are their mean absolute errors. We wish to find the absolute error ΔZ in their sum.

$$Z=A+B$$

$$Z \pm \Delta Z = (A \pm \Delta A)+(B \pm \Delta B)$$

$$= (A+B) \pm \Delta A \pm \Delta B$$

$$\pm \Delta Z = \pm \Delta A \pm \Delta B,$$

For difference, i.e., if Z = A-B,

$$Z \pm \Delta Z = (A \pm \Delta A)-(B \pm \Delta B)$$

$$= (A-B) \pm \Delta A \mp \Delta B$$

$$\pm \Delta Z = \pm \Delta A \pm \Delta B,$$

There are four possible values for ΔZ, namely (+ ΔA - ΔB), (+ΔA+ΔB), (-ΔA-ΔB), (-ΔA+ΔB). Hence maximum value of absolute error is ΔZ = ΔA+ΔB in both the cases.

When two quantities are added or subtracted, the maximum absolute error in the final result is the *sum* of the absolute errors in the individual quantities.

b) Errors in product and in division:

Suppose Z=AB and measured values of A and B are (A ± ΔA) and (B ± ΔB) Then

$$Z \pm \Delta Z = (A \pm \Delta A)(B \pm \Delta B)$$

$$= AB \pm A\Delta B \pm B\Delta A \pm \Delta A\Delta B$$

Dividing L.H.S by Z and R.H.S. by AB we get

$$\left(1 \pm \frac{\Delta z}{z}\right) = \left(1 \pm \frac{\Delta B}{B} \pm \frac{\Delta A}{A} \pm \left(\frac{\Delta A}{A}\right)\left(\frac{\Delta B}{B}\right)\right)$$

Since ΔA/A and ΔB/B are very small we shall neglect their product. Hence maximum *relative* error in Z is

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B} \qquad \text{--- (1.7)}$$

This formula also applies to the division of two quantities.

Thus, when two quantities are multiplied or divided, the maximum relative error in the result is the sum of *relative* errors in each quantity.

c) Errors due to the power (index) of measured quantity:

Suppose

$$Z = A^3 = A.A.A$$

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta A}{A} + \frac{\Delta A}{A}$$

from the multiplication rule above.

Hence the maximum relative error in Z $=A^3$ is three times the relative error in A. So if $Z = A^n$

$$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A} \qquad \text{--- (1.8)}$$

In general, if $Z = \frac{A^p B^q}{C^r}$

$$\frac{\Delta Z}{Z} = p\frac{\Delta A}{A} + q\frac{\Delta B}{B} + r\frac{\Delta C}{C} \qquad \text{--- (1.9)}$$

The quantity in the formula which has large power is responsible for maximum error.

Example 1.6: In an experiment to determine the volume of an object, mass and density are recorded as $m = (5 \pm 0.15)$ kg and $\rho = (5 \pm 0.2)$ kg m^{-3} respectively. Calculate percentage error in the measurement of volume.

Solution : We know,

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\therefore \text{Volume} = \frac{\text{Mass}}{\text{Density}} = \frac{M}{\rho}$$

Maximum percentage error in volume

$$= \left(\frac{\Delta m}{m} + \frac{\Delta \rho}{\rho}\right) \times 100$$

$$= \left(\frac{0.15}{5} + \frac{0.2}{5}\right) \times 100$$

$$= (0.03 + 0.04) \times 100$$

$$= (0.07) \times 100 = 7\%$$

Example 1.7: The acceleration due to gravity is determined by using a simple pendulum of length $l = (100 \pm 0.1)$ cm. If its time period is $T = (2 \pm 0.01)$ s, find the maximum percentage error in the measurement of g.

Solution: The time period of oscillation of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Squaring both sides

$$T^2 = 4\pi^2 l / g$$

$$\therefore g = 4\pi^2 \frac{l}{T^2}$$

Now $\Delta l = 0.1$, $l = 100$ cm, $\Delta T = 0.01$ s, $T = 2$ s

$$\text{Maximum Percentage error} = \frac{\Delta g \times 100}{g}$$

$$= \left(\frac{\Delta l}{l} + \frac{2\Delta T}{T}\right) \times 100$$

$$= \left(\frac{0.1}{100} + \frac{2 \times 0.01}{2}\right) \times 100$$

$$= (0.001 + 0.01) \times 100 = 1.1$$

Maximum percentage error in measurement of g is 1.1%

1.9 Significant Figures:

In the previous sections, we have studied various types of errors, their origins and the ways to minimize them. Our accuracy is limited to the least count of the instrument used during the measurement. Least count is the smallest measurement that can be made using the given instrument. For example with the usual metre scale, one can measure 0.1 cm as the least value. Hence its least count is 0.1cm.

Suppose we measure the length of a metal rod using a metre scale of least count 0.1cm. The measurement is done three times and the readings are 15.4, 15.4, and 15.5 cm. The most probable length which is the arithmetic mean as per our earlier discussion is 15.43. Out of this we are certain about the digits 1 and 5 but are not certain about the last 2 digits because of the least count limitation.

The number of digits in a measurement about which we are certain, plus one additional digit, the first one about which we are not certain is known as significant figures or significant digits.

Thus in above example, we have 3 significant digits 1, 5 and 4.

The larger the number of significant figures obtained in a measurement, the greater is the accuracy of the measurement. If one uses the instrument of smaller least count, the number of significant digits increases.

Rules for determining significant figures

1) All the nonzero digits are significant, for example if the volume of an object is 178.43 cm^3, there are five significant digits which are 1,7,8,4 and 3.
2) All the zeros between two nonzero digits are significant, eg., m = 165.02 g has 5 significant digits.
3) If the number is less than 1, the zero/zeroes on the right of the decimal point and to the left of the first nonzero digit are not significant e.g. in <u>0.00</u>1405, the underlined zeros are not significant. Thus the above number has four significant digits.
4) The zeros on the right hand side of the last nonzero number are significant (but for this, the number must be written with a decimal point), e.g. 1.500 or 0.01500 have both 4 significant figures each.

On the contrary, if a measurement yields length L given as

L = 125 m = 12500 cm = 125000 mm, it has only three significant digits.

To avoid the ambiguities in determining the number of significant figures, it is necessary to report every measurement in scientific notation (i.e., in powers of 10) i.e., by using the concept of order of magnitude.

The magnitude of any physical quantity can be expressed as $A\times10^n$ where 'A' is a number such that $0.5 \leq A<5$ and 'n' is an integer called the **order of magnitude**.

(i) radius of Earth = 6400 km

= 0.64×10^7m

The order of magnitude is 7 and the number of significant figures are 2.

(ii) Magnitude of the charge on electron e = 1.6×10^{-19} C

Here the order of magnitude is -19 and the number of significant digits are 2.

Internet my friend

1. videolectures.net/mit801f99_lewin_lec01/
2. hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

Definitions of SI Units

Till May 20, 2019 the *kilogram* did not have a definition; it was mass of the prototype cylinder kept under controlled conditions of temperature and pressure at the SI museum at Paris. A rigorous and meticulous experimentation has shown that the mass of the *standard* prototype for the *kilogram* has changed in the course of time. This shows the acute necessity for standardisation of units. The new definitions aim to improve the SI without changing the size of any units, thus ensuring continuity with existing measurements. In November 2018, the 26th General Conference on Weights and Measures (CGPM) unanimously approved these changes, which the International Committee for Weights and Measures (CIPM) had proposed earlier that year. These definitions came in force from 20 May 2019.

(i) As per new SI units, each of the seven fundamental units (metre, kilogram, etc.) uses ***one*** of the following 7 constants which are proposed to be having ***exact values*** as given below:

The Planck constant,
$h = 6.62607015 \times 10^{-34}$ joule-second (J s ***or*** kg m^2 s^{-1}).

The elementary charge,
$e = 1.602176634 \times 10^{-19}$ coulomb (C ***or*** A s).

The Boltzmann constant,
$k = 1.380649 \times 10^{-23}$ joule per kelvin (J K^{-1} ***or*** kg m^2 s^{-2} K^{-1}).

The Avogadro constant (number),
$N_A = 6.02214076 \times 10^{23}$ reciprocal mole (mol^{-1}).

The speed of light in vacuum,
c = 299792458 metre per second (m s^{-1}).

The ground state hyperfine structure transition frequency of Caesium-133 atom,
$\Delta\nu_{Cs}$ = 9192631770 hertz (Hz ***or*** s^{-1}).

The luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd} = 683 lumen per watt (lm·W^{-1}) = 683 cd sr s^3 kg^{-1} m^{-2}, where sr is steradian; the SI unit of solid angle.

(ii) Definitions of the units *second* and *mole* are based only upon their respective constants, for example (a) the *second* uses ground state hyperfine structure transition frequency of Caesium-133 atom to be exactly 9192631770 hertz. Thus, the *second* is defined as 9192631770 periods of that transition, (b) the *mole* uses Avogadro's number to be $N_A = 6.02214076 \times 10^{23}$. Thus, one *mole* is that amount of substance which contains exactly $6.02214076 \times 10^{23}$ molecules.

(iii) Definitions of all the other fundamental units use one constant each and at least one other fundamental unit, for example, the *metre* makes use of speed of light in vacuum as a constant and *second* as fundamental unit. Thus, *metre* is defined as the distance traveled by light in vacuum in exactly 1/299792458 *second*.

(iv) The figures show the dependency of various units upon their respective constants and other units (wherever used). The arrows arriving at that unit refer to the constant and the fundamental unit (or units, wherever used) for defining that unit. The arrows going away from a unit indicate other units which use this unit for their definition.

For example, as described above, in Fig (a), i) the arrows directed to *metre* are from *second* and *c*. The arrows going away from the *metre* indicate that the *metre* is used in defining the *kilogram* the *candela* and the *kelvin,* (ii) the newly defined unit *kilogram* uses Planck constant, the *metre* and the *second*, while the *kilogram* itself is used in defining the *kelvin* and the *candela.* This definition relates the *kilogram* to the equivalent mass of the energy of a photon given its frequency, via the Planck constant.

Fig (a) refers to new definitions while the Fig (b) refers to the corresponding definitions before 20 May 2019. Interested students may compare them to know which definitions are modified and how.

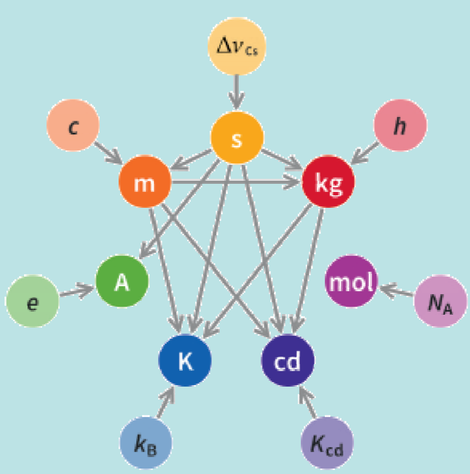


Fig (a) New SI

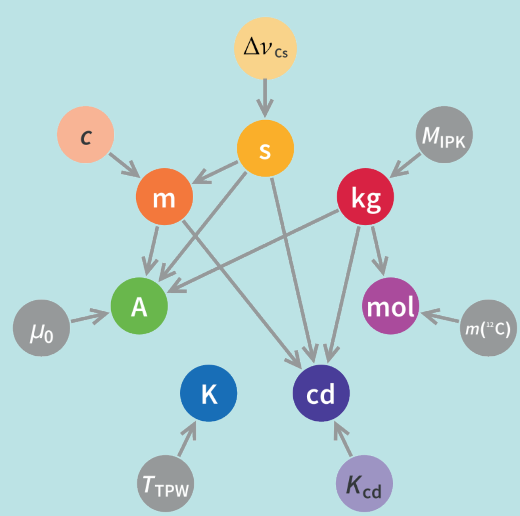


Fig (b) Old SI

Exercises

1. Choose the correct option.

i) $[L^1M^1T^{-2}]$ is the dimensional formula for
(A) Velocity (B) Acceleration
(C) Force (D) Work

ii) The error in the measurement of the sides of a rectangle is 1%. The error in the measurement of its area is
(A) 1% (B) (½)%
(C) 2% (D) None of the above.

iii) Light year is a unit of
(A) Time (B) Mass
(C) Distance (D) Luminosity

iv) Dimensions of kinetic energy are the same as that of
(A) Force (B) Acceleration
(C) Work (D) Pressure

v) Which of the following is not a fundamental unit?
(A) cm (B) kg
(C) centigrade (D) volt

2. Answer the following questions.

i) Star A is farther than star B. Which star will have a large parallax angle?

ii) What are the dimensions of the quantity $l\sqrt{l/g}$, l being the length and g the acceleration due to gravity?

iii) Define absolute error, mean absolute error, relative error and percentage error.

iv) Describe what is meant by significant figures and order of magnitude.

v) If the measured values of two quantities are A ± ΔA and B ± ΔB, ΔA and ΔB being the mean absolute errors. What is the maximum possible error in A ± B? Show that if $Z = \frac{A}{B}$

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

vi) Derive the formula for kinetic energy of a particle having mass *m* and velocity v using dimensional analysis

3. Solve numarical examples.

i) The masses of two bodies are measured to be 15.7 ± 0.2 kg and 27.3 ± 0.3 kg. What is the total mass of the two and the error in it?

[Ans : 43 kg, ± 0.5 kg]

ii) The distance travelled by an object in time (100 ± 1) s is (5.2 ± 0.1) m. What is the speed and it's relative error?

[Ans : 0.052 m s^{-1}, ± 0.0292 m s^{-1}]

iii) An electron with charge e enters a uniform. magnetic field $\vec{B}$ with a velocity $\vec{v}$. The velocity is perpendicular to the magnetic field. The force on the charge e is given by

$|\vec{F}| = Bev$ Obtain the dimensions of $\vec{B}$.

[Ans: $[L^0M^1T^{-2}I^{-1}]$]

iv) A large ball 2 m in radius is made up of a rope of square cross section with edge length 4 mm. Neglecting the air gaps in the ball, what is the total length of the rope to the nearest order of magnitude?

[Ans : $\approx 10^6$ m = 10^3km]

v) Nuclear radius R has *a* dependence on the mass number (A) as $R = 1.3\times10^{-16}A^{1/3}$ m. For a nucleus of mass number A=125, obtain the order of magnitude of R expressed in metre.

[Ans : -15]

vi) In a workshop a worker measures the length of a steel plate with a Vernier callipers having a least count 0.01 cm. Four such measurements of the length yielded the following values: 3.11 cm, 3.13 cm, 3.14 cm, 3.14 cm. Find the mean length, the mean absolute error and the percentage error in the measured value of the length.

[Ans: 3.13 cm, 0.01 cm, 0.32%]

vii) Find the percentage error in kinetic energy of a body having mass 60.0 ± 0.3 g moving with a velocity 25.0 ± 0.1 cm/s.

[Ans: 1.3%]

viii) In Ohm's experiments , the values of the unknown resistances were found to be 6.12 Ω , 6.09 Ω, 6.22 Ω, 6.15 Ω. Calculate the mean absolute error, relative error and percentage error in these measurements.

[Ans: 0.04 Ω ,0.0065 Ω , 0.65%]

ix) An object is falling freely under the gravitational force. Its velocity after travelling a distance h is v. If v depends upon gravitational acceleration g and distance, prove with dimensional analysis that $v = k\sqrt{gh}$ where k is a constant.

x) $v = at + \frac{b}{t+c} + v_0$ is a dimensionally valid equation. Obtain the dimensional formula for a, b and c where v is velocity, t is time and v_0 is initial velocity.

[Ans: a- $[L^1M^\circ T^{-2}]$, b- $[L^1M^\circ T^\circ]$, c- $[L^\circ M^\circ T^1]$]

xi) The length, breadth and thickness of a rectangular sheet of metal are 4.234 m, 1.005 m, and 2.01 cm respectively. Give the area and volume of the sheet to correct significant figures.

[Ans: 4.255 m^2, 8.552 m^3]

xii) If the length of a cylinder is l = (4.00±0.001) cm, radius r = (0.0250 ±0.001) cm and mass m = (6.25±0.01) gm. Calculate the percentage error in the determination of density.

[Ans: 8.185%]

xiii) When the planet Jupiter is at a distance of 824.7 million kilometer from the Earth, its angular diameter is measured to be 35.72" of arc. Calculate the diameter of the Jupiter.

[Ans: 1.428×10^5 km]

xiv) If the formula for a physical quantity is $X = \frac{a^4 b^3}{c^{1/3} d^{1/2}}$ and if the percentage error in the measurements of a, b, c and d are 2%, 3%, 3% and 4% respectively. Calculate percentage error in X.

[Ans: 20%]

xv) Write down the number of significant figures in the following: 0.003 m^2, 0.1250 gm cm^{-2}, 6.4 x 10^6 m, 1.6 x 10^{-19} C, 9.1 x 10^{-31} kg.

[Ans: 1, 4, 2, 2, 2]

xvi) The diameter of a sphere is 2.14 cm. Calculate the volume of the sphere to the correct number of significant figures.

[Ans: 5.13 cm^3]

***

2. Mathematical Methods

Can you recall?

1. What is the difference between a scalar and a vector?
2. Which of the following are scalars or vectors?
   (i) displacement (ii) distance travelled (iii) velocity
   (iv) speed (v) force (vi) work done (vii) energy

2.1 Introduction:

You will need certain mathematical tools to understand the topics covered in this book. Vector analysis and elementary calculus are two among these. You will learn calculus in details, in mathematics, in the XIIth standard. In this Chapter, you are going to learn about vector analysis and will have a preliminary introduction to calculus which should be sufficient for you to understand the physics that you will learn in this book.

2.2 Vector Analysis:

In the previous Chapter, you have studied different aspects of physical quantities like their division into fundamental and derived quantities and their units and dimensions. You also need to understand that all physical quantities may not be fully described by their magnitudes and units alone. For example if you are given the time for which a man has walked with a certain speed, you can find the distance travelled by the man, but you cannot find out where exactly the man has reached unless you know the direction in which the man has walked.

Therefore, you can say that some physical quantities, which are called scalars, can be described with magnitude alone, whereas some other physical quantities, which are called vectors, need to be described with magnitude as well as direction. In the above example the distance travelled by the man is a scalar quantity while the final position of the man relative to his initial position, i.e., his displacement can be described by magnitude and direction and is a vector quantity. In this Chapter you will study different aspects of scalar and vector quantities.

2.2.1 Scalars:

Physical quantities which can be completely described by their magnitude are called scalars, i.e. they are specified by a number and a unit. For example when we say that a given metal rod has a length 2 m, it indicates that the rod is two times longer than a certain standard unit *metre*. Thus the number 2 is the magnitude and metre is the unit; together they give us a complete idea about the length of the rod. Thus length is a scalar quantity. Similarly mass, time, temperature, density, etc., are examples of scalars. Scalars can be added or subtracted by rules of simple algebra.

2.2.2 Vectors:

Physical quantities which need magnitude as well as direction for their complete description are called vectors. Examples of vectors are displacement, velocity, force etc.

A vector can be represented by a directed line segment or by an arrow. The length of the line segment drawn to scale gives the magnitude of the vector, e.g., displacement of a body from P to Q can be represented as P $\longrightarrow$ Q, where the starting point P is called the tail and the end point Q (arrow head) is called the head of the vector. Symbolically we write it as $\overrightarrow{PQ}$. Symbolically vectors are also represented by a single capital letter with an arrow above it, e.g., $\vec{X}$, $\vec{A}$, etc. Magnitude of a vector $\vec{X}$ is written as $|\vec{X}|$.

Let us see a few examples of different types of vectors.

(a) Zero vector (Null vector): A vector having zero magnitude with a particular direction (arbitrary) is called zero vector. Symbolically it is represented as $\vec{0}$.

(1) Velocity vector of a stationary particle is a zero vector.

(2) The acceleration vector of an object

moving with uniform velocity is a zero vector.

(b) **Resultant vector:** The resultant of two or more vectors is that single vector, which produces the same effect, as produced by all the vectors together.

(c) **Negative vector (opposite vector):** A negative vector of a given vector is a vector of the same magnitude but opposite in direction to that of the given vector.

In Fig. 2.1, $\vec{B}$ is a negative vector to $\vec{A}$.

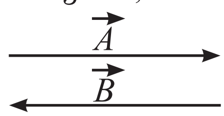


Fig. 2.1: Negative vector.

(d) **Equal vector:** Two vectors A and B representing same physical quantity are said to be equal if and only if they have the same magnitude and direction. Two equal vectors are shown in Fig. 2.2.

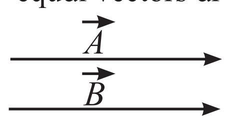


Fig. 2.2: Equal vectors.

(e) **Position vector:** A vector which gives the position of a particle at a point with respect to the origin of a chosen co-ordinate system is called the position vector of the particle.

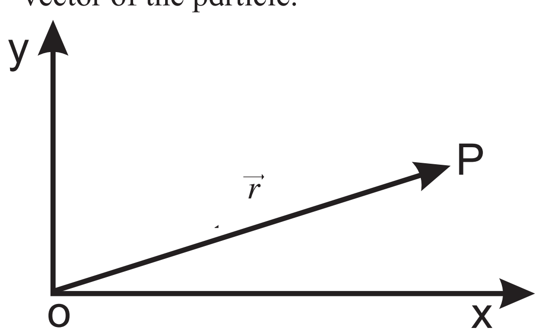


Fig 2.3: Position vector.

In Fig 2.3, $\vec{r} = \overrightarrow{OP}$ is the position vector of the particle present at P.

(f) **Unit vector:** A vector having unit magnitude in a given direction is called a unit vector in that direction. If $\vec{M}$ is a non-zero vector i.e. its magnitude $M = |\vec{M}|$ is not zero, the unit vector along $\vec{M}$ is written as $\hat{u}_M$ and is given by

$$\vec{M} = \hat{u}_M M \qquad \text{--- (2.1)}$$

$$or,\ \hat{u}_M = \frac{\vec{M}}{M} \qquad \text{--- (2.2)}$$

Hence $\hat{u}_M$ has magnitude unity and has the same direction as that of $\vec{M}$. We use $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively, as unit vectors along the x, y and z directions of a rectangular (three dimensional) coordinate system.

$$\hat{u}_x = \hat{i}, \hat{u}_y = \hat{j} \text{ and } \hat{u}_z = \hat{k}$$

$$\therefore \hat{i} = \frac{\vec{x}}{x}, \hat{j} = \frac{\vec{y}}{y} \text{ and } \hat{k} = \frac{\vec{z}}{z} \qquad \text{--- (2.3)}$$

Here $\vec{x}$, $\vec{y}$ and $\vec{z}$ are vectors along x, y and z axes, respectively.

2.3 Vector Operations:

2.3.1 Multiplication of a Vector by a Scalar:

Multiplying a vector $\vec{P}$ by a scalar quantity, say s, yields another vector. Let us write

$$\vec{Q} = s\vec{P} \qquad \text{--- (2.4)}$$

$\vec{Q}$ will be a vector whose direction is the same as that of $\vec{P}$ and magnitude is s times the magnitude of $\vec{P}$.

2.3.2 Addition and Subtraction of Vectors:

The addition or subtraction of two or more vectors of the same type, i.e., describing the same physical quantity, gives rise to a single vector, such that the effect of this single vector is the same as the net effect of the vectors which have been added or subtracted .

It is important to understand that only vectors of the same type (describing same physical quantity) can be added or subtracted e.g. force $\vec{F_1}$ and force $\vec{F_2}$ can be added to give the resultant force $\vec{F} = \vec{F_1} + \vec{F_2}$. But a force vector can not be added to a velocity vector.

It is easy to find addition of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ having the same or opposite direction but different magnitudes. If individual vectors are parallel (i.e., in the same direction), the magnitude of their resultant is the addition of individual magnitudes, i.e., $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

and direction of the resultant is the same as that of the individual vectors as shown in Fig 2.4 (a). However, if the individual vectors are anti-parallel (i.e., in the opposite direction), the magnitude of their resultant is the difference of the individual magnitudes, and the direction is that of the larger vector i.e., $|\overline{AC}| = ||\overline{AB}| - |\overline{BC}||$ as shown in Fig. 2.4 (b).

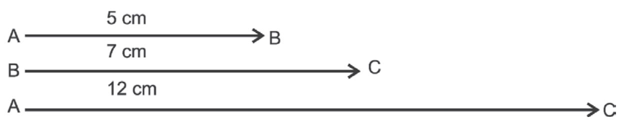


Fig. 2.4 (a): Resultant of parallel displacements.

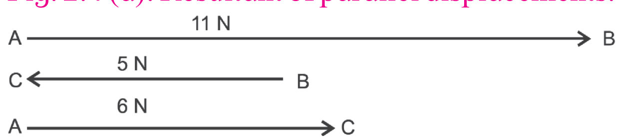


Fig 2.4 (b): Resultant of anti-parallel forces.

2.3.3 Triangle Law for Vector Addition:

When vectors of a given type do not act in the same or opposite directions, the resultant can be determined by using the triangle law of vector addition which is stated as follows:

If two vectors describing the same physical quantity are represented in magnitude and direction by the two sides of a triangle taken in order, then their resultant is represented in magnitude and direction by the third side of the triangle drawn in the opposite sense (from the starting point of first vector to the end point of the second vector).

Let $\vec{A}$ and $\vec{B}$ be two vectors in the plane of paper as shown in Fig. 2.5 (a). The sum of these two vectors can be obtained by using the triangle law described above as shown in Fig. 2.5 (b). The resultant vector is indicated by $\vec{C}$.

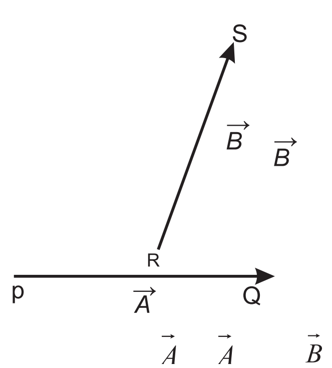


Fig. 2.5 (a): Two vectors $\vec{A}$ and $\vec{B}$ in a plane,

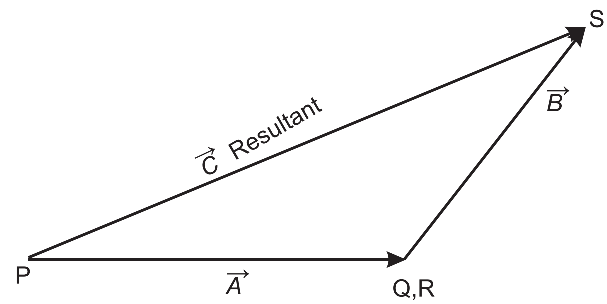


Fig. 2.5 (b): Resultant vector $\vec{C} = \vec{A} + \vec{B}$.

We can use the triangle law for showing that

(a) Vector addition is commutative.

For any two vectors $\vec{P}$ and $\vec{Q}$,

$\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$ --- (2.5)

Figure 2.6 (a) shows addition of the two vector $\vec{P}$ and $\vec{Q}$ in two different ways. Triangle OAB shows $\vec{P} + \vec{Q} = \vec{R} = \overline{OB}$, while triangle OCB shows $\vec{Q} + \vec{P} = \vec{R} = \overline{OB}$.

$\therefore \vec{P} + \vec{Q} = \vec{Q} + \vec{P}$

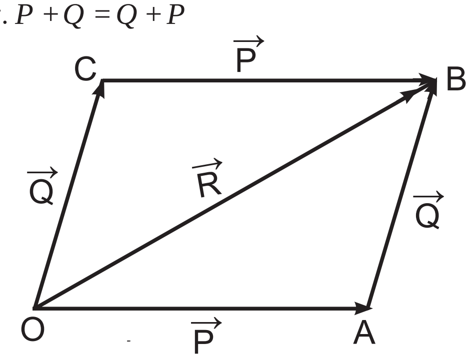


Fig. 2.6 (a): Commutative law.

(b) Vector addition is associative

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors then

$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

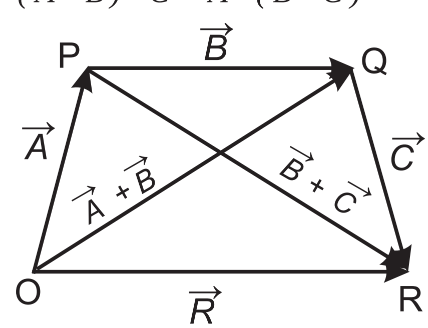


Fig. 2.6 (b): Associative law.

Figure 2.6 (b) shows addition of 3 vectors

$\vec{A}$, $\vec{B}$ and $\vec{C}$ in two different ways to give resultant $\vec{R}$.

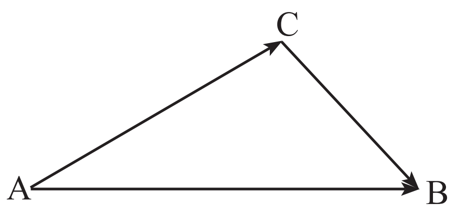
$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} \quad \text{--- from triangle OQR}$$

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) \quad \text{--- from triangle OPR}$$

i.e., $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$ --- (2.6)

Thus the Associative law is proved.

Example 2.1: Express vector $\overrightarrow{AC}$ in terms of vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$ shown in the following figure.



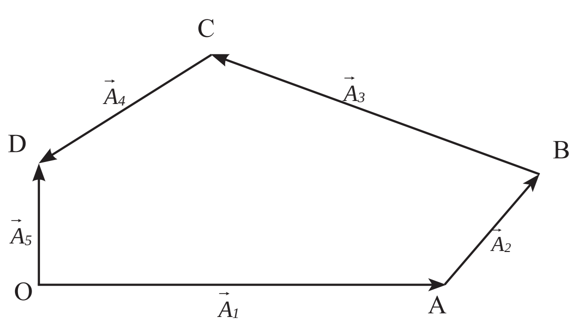
Solution: Using the triangle law of addition of vectors we can write

$$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$$

$$\therefore \overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{CB}$$

Example 2.2: From the following figure, determine the resultant of four forces

$\vec{A}_1, \vec{A}_2, \vec{A}_3$ and $\vec{A}_4$



Solution: Join $\overrightarrow{OB}$ to complete Δ OAB as shown in (a)

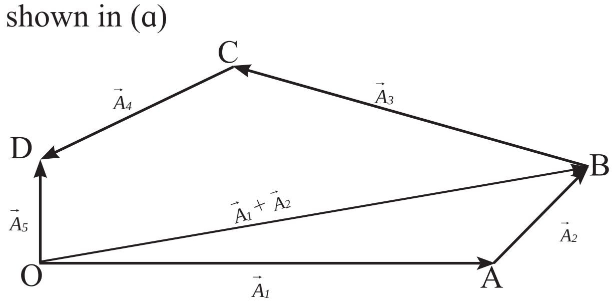


Fig. (a)

Now, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{A}_1 + \vec{A}_2$

Join $\overrightarrow{OC}$ to complete triangle OBC as shown in (b).

Now, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3$

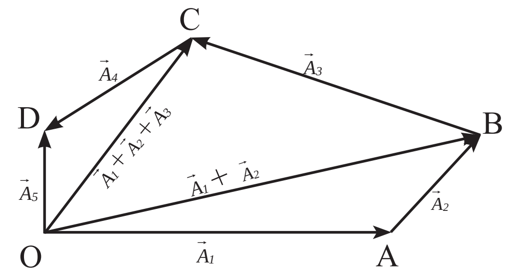


Fig. (b)

From triangle OCD,

$$\overrightarrow{OD} = \vec{A}_5 = \overrightarrow{OC} + \overrightarrow{CD} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4$$

Thus $\overrightarrow{OD}$ is the resultant of the four vectors, $\vec{A}_1, \vec{A}_2, \vec{A}_3$ and $\vec{A}_4$, represented by

$\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$, respectively.

2.3.4 Law of parallelogram of vectors:

Another geometrical method of adding two vectors is called parallelogram law of vector addition which is stated as follows:

If two vectors of the same type, originating from the same point (tails at the same point) are represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant vector is given in magnitude and direction by the diagonal of the parallelogram starting from the same point as shown in Fig. 2.7.

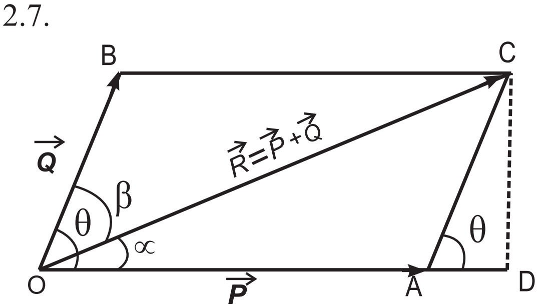


Fig 2.7: Parallelogram law of vector addition.

In Fig. 2.7, vector $\overrightarrow{OA} = \vec{P}$ and vector $\overrightarrow{OB} = \vec{Q}$, represent two vectors originating from point O, inclined to each other at an angle θ. If we complete the parallelogram, then according to this law, the diagonal $\overrightarrow{OC} = \vec{R}$ represents the resultant vector.

To find the magnitude of $\vec{R}$, drop a perpendicular from C to reach OA (extended) at D. In right angled triangle ODC, by application by Pythagoras theorem,

$$OC^2 = OD^2+DC^2$$

$$= (OA+AD)^2 + DC^2$$

$$OC^2 = OA^2+2OA{\cdot}AD+AD^2+DC^2$$

In the right angled triangle ADC, by application of Pythagoras theorem

$$AD^2+DC^2=AC^2$$

$$\therefore OC^2=OA^2+2OA.\ AD+ AC^2 \quad \text{--- (2.7)}$$

Also,

$$\overrightarrow{OA} = \vec{P},\ \overrightarrow{AC} = \overrightarrow{OB} = \vec{Q} \text{ and } \overrightarrow{OC} = \vec{R}$$

In Δ ADC, $\cos\theta = AD/AC$

$\therefore AD=AC\cos\theta = Q\cos\theta$

Substituting in Eq. (2.7)

$$R^2= P^2+Q^2+2\,P\,Q\cos\theta.$$

$$\therefore R = \sqrt{P^2+Q^2+2\,P\,Q\cos\theta} \quad \text{--- (2.8)}$$

Equation (2.8) gives us the magnitude of resultant vector $\vec{R}$.

To find the direction of the resultant vector $\vec{R}$, we will have to find the angle (α) made by $\vec{R}$ with $\vec{P}$.

$$\text{In } \Delta ODC,\ \tan\alpha = \frac{DC}{OD}$$

$$= \frac{DC}{OA+AD} \quad \text{--- (2.9)}$$

From the figure, $\sin\theta = \dfrac{DC}{AC}$

$$\therefore DC = AC\sin\theta = Q\sin\theta$$

Also,

$AD = AC\cos\theta = Q\cos\theta$

and $OA = \vec{P}$,

Substituting in Eq. (2.9), we get

$$\tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta}$$

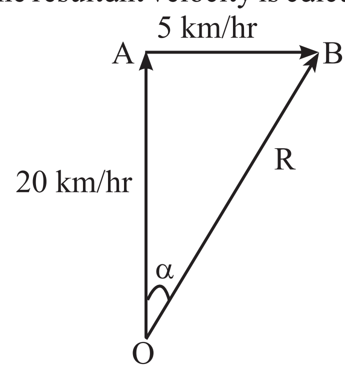
$$\therefore \alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P+Q\cos\theta}\right) \quad \text{--- (2.10)}$$

Equation (2.10) gives us the direction of resultant vector $\vec{R}$.

If β is the angle between $\vec{R}$ and $\vec{Q}$, it can be similarly derived that $\beta = \tan^{-1}\left(\dfrac{P\sin\theta}{Q+P\cos\theta}\right)$

Example 2.3: Water is flowing in a stream with velocity 5 km/hr in an easterly direction relative to the shore. Speed of a boat is relative to still water is 20 km/hr. If the boat enters the stream heading North, with what velocity will the boat actually travel?

Solution: The resultant velocity $\vec{R}$ of the boat can be obtained by adding the two velocities using Δ OAB shown in the figure. Magnitude of the resultant velocity is calculated as follows:



$$R = \sqrt{20^2+5^2}$$

$$= \sqrt{425} = 20.61\ \text{km / hr}$$

The direction of the resultant velocity is

$$= \tan^{-1}\left(\frac{5}{20}\right) = \tan^{-1}(0.25)$$

$$=14^0 04'$$

$\therefore$ The velocity of the boat is 20.61 km/hr in a direction $14^0 04'$ east of north.

2.4 Resolution of vectors:

A vector can be written as a sum of two or more vectors along certain fixed directions. Thus a vector $\vec{V}$ can be written as

$$\vec{V} = V_1\hat{\alpha} + V_2\hat{\beta} + V_3\hat{\gamma} \quad \text{--- (2.11)}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are unit vectors along chosen directions. V_1, V_2 and V_3 are known as components of $\vec{V}$ along the three directions $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$.

The process of splitting a given vector into its components is called resolution of the vector. The components can be found along directions at any required angles, but if these components are found along the directions which are mutually perpendicular, they are called rectangular components.

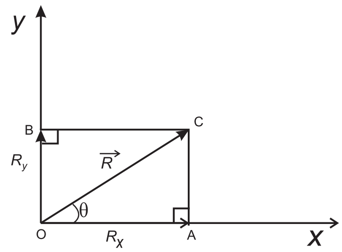


Fig. 2.8 : Resolution of a vector.

Let us see how to find rectangular components in two dimensions.

Consider a vector $\vec{R} = \overrightarrow{OC}$, originating from the origin of a rectangular co-ordinate system as shown in Fig. 2.8.

Drop perpendiculars from C that meet the x-axis at A and y-axis of at B.

$\overrightarrow{OA} = \vec{R}_x$ and $\overrightarrow{OB} = \vec{R}_y$; $\vec{R}_x$ and $\vec{R}_y$ being the components of $\overrightarrow{OC}$ along the x and y axes, respectively.

Then by the law of parallelogram of vectors,

$$\vec{R} = \vec{R}_x + \vec{R}_y$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the x and y axes respectively, and R_x and R_y are the magnitudes of the two components of $\vec{R}$.

Let θ be the angle made by $\vec{R}$ with the x-axis, then

$$\cos\theta = \frac{R_x}{R}$$

$$\therefore R_x = R\cos\theta \qquad \text{--- (2.12)}$$

$$\sin\theta = \frac{R_y}{R}$$

$$\therefore R_y = R\sin\theta \qquad \text{--- (2.13)}$$

Squaring and adding Eqs. (2.12) and (2.13), we get

$$R^2\cos^2\theta + R^2\sin^2\theta = R_x^{\,2} + R_y^{\,2}$$

$$\therefore R^2 = R_x^{\,2} + R_y^{\,2}$$

$$\text{or, } R = \sqrt{R_x^{\,2} + R_y^{\,2}} \qquad \text{--- (2.14)}$$

Equation (2.14) gives the magnitude of $\vec{R}$. To find the direction of $\vec{R}$, from Fig. 2.8,

$$\tan\theta = \frac{R_y}{R_x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) \qquad \text{--- (2.15)}$$

Similarly, if $\vec{R}_x$, $\vec{R}_y$ and $\vec{R}_z$ are the rectangular components of $\vec{R}$ along the x, y and z axes of the rectangular Cartesian co-ordinate system in three dimensions, then

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

$$\text{or, } |\vec{R}| = \sqrt{R_x^{\,2} + R_y^{\,2} + R_z^{\,2}} \qquad \text{---- (2.16)}$$

If two vectors are equal, it means that their corresponding components are also equal and vice versa.

If $\vec{A} = \vec{B}$

i.e., if $A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, then

$$A_x = B_x,\ A_y = B_y \text{ and } A_z = B_z$$

Example 2.4: Find a unit vector in the direction of the vector $3\hat{i} + 4\hat{j}$

Solution:

Let $\vec{V} = 3\hat{i} + 4\hat{j}$

Magnitude of $\vec{V} = |\vec{V}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\vec{V} = \hat{\alpha}\,|\vec{V}|$, where $\hat{\alpha}$ is a unit vector along $\vec{V}$.

$$\hat{\alpha} = \frac{\vec{V}}{|\vec{V}|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

Example 2.5: Given $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$, what are the magnitudes of the two vectors? Are these two vectors equal?

Solution:

$$|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

The magnitudes of $\vec{a}$ and $\vec{b}$ are equal. However, their corresponding components are not equal i.e., $a_x \neq b_x$ and $a_y \neq b_y$. Hence, the two vectors are not equal.

2.5 Multiplication of Vectors:

We saw that we can add or subtract vectors of the same type to get resultant vectors of the same type. However, when we multiply vectors of the same or different types, we get a new physical quantity which may either be a scalar (scalar product) or a vector (vector product). Also note that the multiplication of a scalar with a scalar is always a scalar and the multiplication of scalar with a vector is always a vector. Let us now study the characteristics of a scalar product and vector product of two vectors.

2.5.1 Scalar Product (Dot Product):

The scalar product or dot product of two nonzero vectors $\vec{P}$ and $\vec{Q}$ is defined as the product of magnitudes of the two vectors and the cosine of the angle θ between the two vectors. The scalar product of $\vec{P}$ and $\vec{Q}$ is written as,

$$\vec{P}\cdot\vec{Q} = PQ\cos\theta, \qquad \text{--- (2.17)}$$

where θ is the angle between $\vec{P}$ and $\vec{Q}$.

Characteristics of scalar product

(1) The scalar product of two vectors is equivalent to the product of magnitude of one vector with the magnitude of the component of the other vector in the direction of the first.

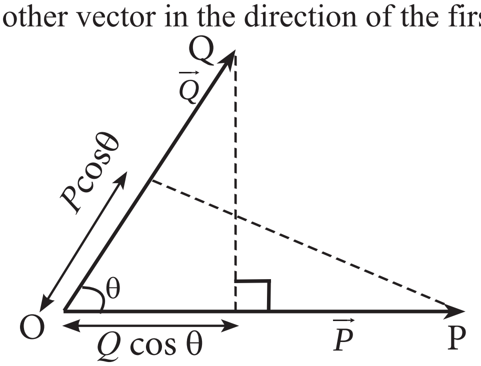


Fig. 2.9: Projection of vectors.

From Fig. 2.9,

$\vec{P}\cdot\vec{Q} = PQ\cos\theta$

$= P\,(Q\cos\theta)$

$= P$ (component of $\vec{Q}$ in the direction of $\vec{P}$)

Similarly $\vec{P}\cdot\vec{Q} = Q\,(P\cos\theta)$

$= Q$ (component of $\vec{P}$ in the direction of $\vec{Q}$)

(2) Scalar product obeys the commutative law of vector multiplication.

$$\vec{P}\cdot\vec{Q} = P\,Q\cos\theta = Q\,P\cos\theta = \vec{Q}\cdot\vec{P}$$

(3) Scalar product obeys the distributive law of multiplication

$$\vec{P}\cdot(\vec{Q}+\vec{R}) = \vec{P}\cdot\vec{Q}+\vec{P}\cdot\vec{R}$$

(4) Special cases of scalar product $\vec{P}\cdot\vec{Q} = P\,Q\cos\theta$

(i) If θ = 0, i.e., the two vectors $\vec{P}$ and $\vec{Q}$ are parallel to each other, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos\theta = P\,Q$$

Thus, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Do you know ?

Scalar and vector products are very useful in physics. They make mathematical formulae and their derivation very elegant.

Figure below shows a toy car pulled through a displacement $\vec{s}$. The force $\vec{F}$ responsible for this is not in the direction of $\vec{s}$ but is at an angle θ to it. Component of displacement along the direction of force $\vec{F}$ is $S\cos\theta$. According to the definition, the work done by a force is the product of the force and the displacement in the direction of force. $\therefore W = FS\cos\theta$. According to the definition of scalar product,

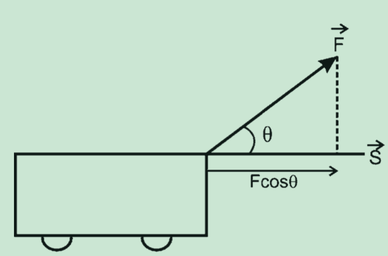
$$\vec{F}\cdot\vec{s} = F\,S\cos\theta$$

$$\therefore W = \vec{F}\cdot\vec{s}$$

Also $W = F\,(S\cos\theta) = (F\cos\theta)\,S$

Hence dot or scalar product is the product of magnitude of one of the vectors and component of the other vector in the direction of the first.

Power is the rate of doing work on a body by an external force $\vec{F}$ assumed to be constant in time. If $\vec{v}$ is the velocity of the body under the action of the force then power P is given by the scalar product of $\vec{F}$ and $\vec{v}$ i.e., $P = \vec{F}\cdot\vec{v}$.



(ii) If θ = 180°, i.e., the two vectors $\vec{P}$ and $\vec{Q}$ are anti-parallel, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos 180° = -P\,Q$$

(iii) If θ = 90°, i.e., the two vectors are perpendicular to each other, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos 90° = 0$$

Thus, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(5) If $\vec{P}=\vec{Q}$ then $\vec{P}.\vec{Q} = P^2 = Q^2$

(6) Scalar product of vectors expressed in terms of rectangular components :

Let $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

and $\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$

Then $\vec{P}\cdot\vec{Q} = P_x\,Q_x + P_y\,Q_y + P_z\,Q_z$

Proof :

$$\vec{P}\cdot\vec{Q} = (P_x\hat{i} + P_y\hat{j} + P_z\hat{k})\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$= P_x\hat{i}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$+ P_y\hat{j}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$+ P_z\hat{k}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$= (\hat{i}\cdot\hat{i})P_xQ_x + (\hat{i}\cdot\hat{j})P_xQ_y + (\hat{i}\cdot\hat{k})P_xQ_z$$

$$+ (\hat{j}\cdot\hat{i})P_yQ_x + (\hat{j}\cdot\hat{j})P_yQ_y + (\hat{j}\cdot\hat{k})P_yQ_z$$

$$+ (\hat{k}\cdot\hat{i})P_zQ_x + (\hat{k}\cdot\hat{j})P_zQ_y + (\hat{k}\cdot\hat{k})P_zQ_z$$

Since, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$

$$\therefore \vec{P}\cdot\vec{Q} = P_xQ_x + 0 + 0$$

$$+ 0 + P_yQ_y + 0$$

$$+ 0 + 0 + P_zQ_z$$

$$\therefore \vec{P}\cdot\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

(7) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, where $\vec{a} \neq 0$, it is not necessary that $\vec{b} = \vec{c}$. Using the distributive law, we can write $\vec{a}\cdot\left(\vec{b}-\vec{c}\right) = 0$. It implies that either $\vec{b}-\vec{c}$ = 0 or $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$. It does not necessarily imply that $\vec{b}-\vec{c} = 0$

Example 2.6: Find the scalar product of the two vectors

$$\vec{v}_1 = \hat{i} + 2\hat{j} + 3\hat{k} \text{ and } \vec{v}_2 = 3\hat{i} + 4\hat{j} - 5\hat{k}$$

Solution:

$$\vec{v}_1\cdot\vec{v}_2 = (\hat{i} + 2\hat{j} + 3\hat{k})\cdot(3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$= 1\times 3 + 2\times 4 + 3\times(-5)$$

$$= -4$$

as $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,$

and $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$

2.5.2 Vector Product (cross product):

The vector product or cross product of two vectors ($\vec{P}$ and $\vec{Q}$) is a vector whose magnitude is equal to the product of magnitudes of the two vectors and sine of the smaller angle (θ) between the two vectors. The direction of the product vector is given by $\hat{u}_r$ which is a unit vector perpendicular to the plane containing the two vectors and is given by the right hand screw rule. This is shown in Fig. 2.10 (a) and (b)

a) $\vec{R} = \vec{P}\times\vec{Q} = PQ\sin\theta\,\hat{u}_r$ --- (2.18)

b) $\vec{S} = \vec{Q}\times\vec{P} = PQ\sin\theta\,\hat{u}_s$ --- (2.19)

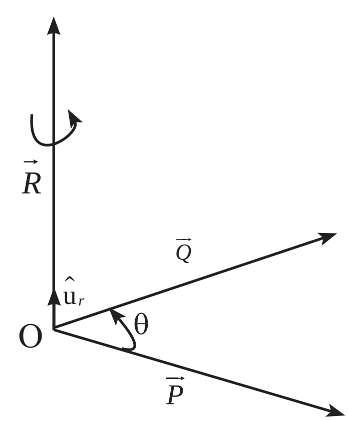


Fig. 2.10 (a): Vector product $\vec{R} = \vec{P}\times\vec{Q}$.

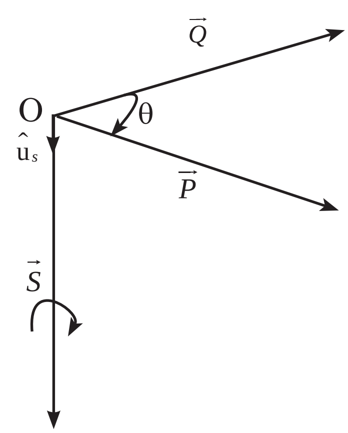


Fig. 2.10 (b): Vector product $\vec{S} = \vec{Q}\times\vec{P}$.

According to the right hand screw rule, if the screw is rotated in a direction from $\vec{P}$ to $\vec{Q}$ through the smaller angle, then the direction in which the tip of the screw advances is the direction of $\vec{R}$, perpendicular to the plane containing $\vec{P}$ and $\vec{Q}$. One example of vector or cross product is the force $\vec{F}$ experienced by a charge q moving with velocity $\vec{v}$ through a uniform magnetic field of magnetic induction $\vec{B}$. It is an empirical law (experimentally determined) given by $\vec{F} = q\vec{v} \times \vec{B}$.

Do you know ?

1. As linear displacement $\vec{x}$ is the distance travelled by a body along the line of travel, angular displacement $\vec{\theta}$ is the angle swept by a body about a given axis. The rate of change of angular displacement is the angular velocity denoted by $\vec{\omega}$. If a body is rotating about as axis, it possesses an angular velocity $\vec{\omega}$. If at a point at a distance $\vec{r}$ from the axis of rotation the body has linear velocity $\vec{v}$, then $\vec{v} = \vec{\omega} \times \vec{r}$.
2. An external force is needed to move a body from one point to other. Similarly to rotate a body about an axis passing through it, torque is required. Torque is a vector with its direction along the axis of rotation and magnitude describing the turning effect of force $\vec{F}$ acting on the body to rotate it about the given axis. Torque $\vec{\tau}$ is given as $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{r}$ being the perpendicular distance of a point on the body where the force is applied from the axis of rotation.

Characteristics of Vector Product:

(1) Vector product does not obey commutative law of multiplication.

$$\vec{P} \times \vec{Q} \neq \vec{Q} \times \vec{P} \qquad \text{--- (2.20)}$$

However, $|\vec{P} \times \vec{Q}| = |\vec{Q} \times \vec{P}|$ i.e., the magnitudes are the same but the directions are opposite to each other.

(2) The vector product obeys the distributive law of multiplication.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad \text{--- (2.21)}$$

(3) Special cases of cross product

$$|\vec{P} \times \vec{Q}| = P\,Q \sin\theta \qquad \text{--- (2.22)}$$

(i) If $\theta = 0$. i.e., if the two nonzero vectors are parallel to each other, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \cdot 0 = 0$

(ii) If θ = 180°, i.e., if the two nonzero vectors are anti-parallel, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \sin 180° = P\,Q \sin\pi = 0$

(iii) If θ = 90°, i.e., if the two nonzero vectors are perpendicular to each other, the magnitude of their vector product is equal to the product of magnitudes of the two vectors.

$$|\vec{P} \times \vec{Q}| = P\,Q \sin 90° = P\,Q$$

Thus $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$

(4) If $\vec{P} = \vec{Q}$ then $|\vec{P} \times \vec{Q}| = |\vec{P} \times \vec{P}| = |\vec{Q} \times \vec{Q}| = 0$.

Thus $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(5) Let $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

and $\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$

$$\begin{aligned}
\vec{P} \times \vec{Q} &= \left(P_x\hat{i} + P_y\hat{j} + P_z\hat{k}\right) \times \left(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}\right) \\
&= P_xQ_x\left(\hat{i} \times \hat{i}\right) + P_xQ_y\left(\hat{i} \times \hat{j}\right) + P_xQ_z\left(\hat{i} \times \hat{k}\right) \\
&\quad + P_yQ_x\left(\hat{j} \times \hat{i}\right) + P_yQ_y\left(\hat{j} \times \hat{j}\right) + P_yQ_z\left(\hat{j} \times \hat{k}\right) \\
&\quad + P_zQ_x\left(\hat{k} \times \hat{i}\right) + P_zQ_y\left(\hat{k} \times \hat{j}\right) + P_zQ_z\left(\hat{k} \times \hat{k}\right)
\end{aligned}$$

Now $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$, and

$$\hat{i} \times \hat{k} = -\hat{j},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}.$$

$$\begin{aligned}
\therefore \vec{P} \times \vec{Q} &= 0 + P_xQ_y\hat{k} - P_xQ_z\hat{j} \\
&\quad - P_yQ_x\hat{k} + 0 + P_yQ_z\hat{i} \\
&\quad + P_zQ_x\hat{j} - P_zQ_y\hat{i} + 0 \\
&= (P_yQ_z - P_zQ_y)\hat{i} \\
&\quad + (P_zQ_x - P_xQ_z)\hat{j} \\
&\quad + (P_xQ_y - P_yQ_x)\hat{k}
\end{aligned}$$

This can be written in a determinant form as

$$\vec{P} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad \text{--- (2.23)}$$

(6) The magnitude of cross product of two vectors is numerically equal to the area of a parallelogram whose adjacent sides represent the two vectors.

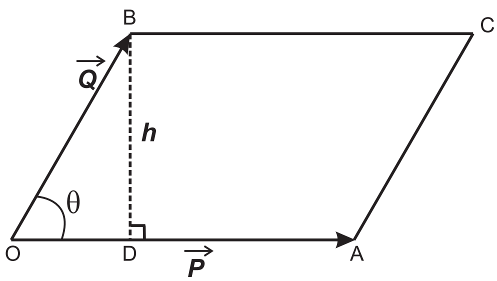


Fig 2.11: Area of parallelogram and vector product.

As shown in fig. 2.11,

$\vec{P} = \overrightarrow{OA}$, $\vec{Q} = \overrightarrow{OB}$, $\vec{P}$ and $\vec{Q}$ are inclined at an angle θ.

Perpendicular BD, of length h drawn on OA, gives the height of the parallelogram with OA as base.

Area of parallelogram

= base × height

$= \text{OA} \times \text{BD}, \text{ as } \sin\theta = \dfrac{\text{BD}}{\text{OB}}$

$= P\,Q\sin\theta$

$= \left|\vec{P} \times \vec{Q}\right|$

= magnitude of the vector product --- (2.24)

Example 2.7: The angular momentum $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is a position vector and $\vec{p}$ is linear momentum of a body.

If $\vec{r} = 4\hat{i} \times 6\hat{j} - 3\hat{k}$ and $\vec{p} = 2\hat{i} + 4\hat{j} - 5\hat{k}$, find $\vec{L}$

Solution:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$\therefore \vec{L} = (-30 + 12)\hat{i} + (-6 + 20)\hat{j} + (16 - 12)\hat{k}$$

$$= -18\hat{i} + 14\hat{j} + 4\hat{k}.$$

Example 2.8: If $\vec{A} = 5\hat{i} + 6\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 3\hat{k}$, determine the angle between $\vec{A}$ and $\vec{B}$.

Solution: $\vec{A} \cdot \vec{B} = A\,B\cos\theta = A_xB_x + A_yB_y + A_zB_z$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{A\,B}$$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

$$\cos\theta = \frac{(5)(2) + (6)(-2) + (4)(3)}{\sqrt{25 + 36 + 16}\sqrt{4 + 4 + 9}}$$

$$= \frac{10}{\sqrt{77}.\sqrt{17}} = 0.2764$$

$$\theta = \cos^{-1} 0.2765 = 73°58'$$

Example 2.9: Given $\vec{P} = 4\hat{i} - \hat{j} + 8\hat{k}$ and $\vec{Q} = 2\hat{i} - m\hat{j} + 4\hat{k}$, find m if $\vec{P}$ and $\vec{Q}$ have the same direction.

Solution: Since $\vec{P}$ and $\vec{Q}$ have the same direction, their corresponding components must be in the same proportion, i.e.,

$$\frac{P_x}{Q_x} = \frac{P_y}{Q_y} = \frac{P_z}{Q_z}$$

$$\frac{4}{2} = \frac{-1}{-m} = \frac{8}{4}$$

$$\therefore m = \frac{1}{2}$$

2.6 Introduction to Calculus:

Calculus is the study of continuous (not discrete) changes in mathematical quantities. This branch of mathematics was first developed by G.W Leibnitz and Sir Issac Newton in the 17^{th} century and is extensively used in several branches of science. You will study calculus in mathematics in XII^{th} standard. Here we will learn the basics of the two branches of calculus namely differential and integral calculus. These are necessary to understand the topics covered in this book.

2.6.1 Differential Calculus:

Let us consider a function $y = f(x)$. Here x is called an independent variable and $f(x)$ gives the value of y for different values of x and is the dependent variable. For example x could be the position of a particle moving along x-axis and $y = f(x)$ could be its velocity at that position x. We can thus draw a graph of y against x as shown in Fig. 2.12 (a). Let A and B be two points on the curve giving values of y at $x = x_0$ and $x = x_0 + \Delta x$, where Δx is a small increment in x. The slope of the straight line joining A and B is given by $\tan\theta = \dfrac{\Delta y}{\Delta x}$.

If we make Δx smaller, the point B will come closer to A and if we keep making Δx smaller and smaller, we will ultimately reach a stage when B will coincide with A. This process is called taking the limit Δx going to zero and is written as $\lim\limits_{\Delta x \to 0}$. In this limit the line AB extended on both sides to P and Q will become the tangent to the curve at A, i.e., at

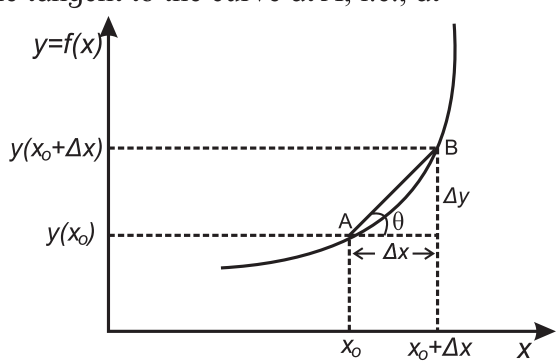


Fig. 2.12 (a): Average rate of change of y with respect to x.

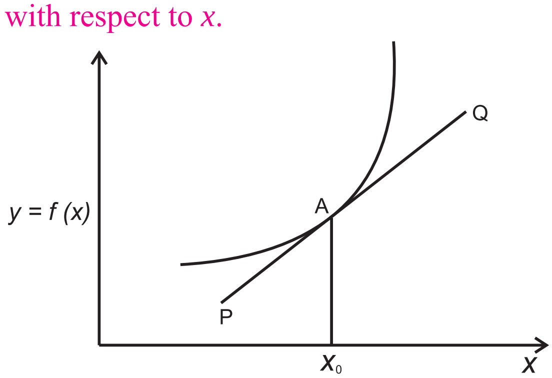


Fig. 2.12 (b): Rate of change of y with respect to x at x_0

$x = x_o$. In this limit both Δx and Δy will go to zero. However, when two quantities tend to zero, their ratio need not go to zero. In fact $\lim\limits_{\Delta x \to 0}\left(\dfrac{\Delta y}{\Delta x}\right)$ becomes the slope of the tangent shown by PQ in Fig. 2.12 (b). This is written as dy/dx at $x = x_o$.

Thus,

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\Delta x \to 0}\frac{(y+\Delta y)-y}{\Delta x}$$

$$\left.\frac{df(x)}{dx}\right|_{x_0} = \lim_{\Delta x \to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

We can drop the subscript zero and write a general formula which will be valid for all values of x as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx} \quad \text{--- (2.25)}$$

In XII[th] standard you will learn about the properties of derivatives and how to find derivatives of different functions. Here we will just list the properties as we will need them in later Chapter s. dy/dx is called the derivative of y with respect to x (which is the rate of change of y with respect to change in x) and the process of finding the derivative is called differentiation. Let $f_1(x)$ and $f_2(x)$ be two different functions of x and let s be a constant. Some of the properties of differentiation are

1. $$\frac{d(sf(x))}{dx} = s\frac{df(x)}{dx} \quad \text{--- (2.26)}$$

2. $$\frac{d}{dx}(f_1(x)+f_2(x)) = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} \quad \text{--- (2.27)}$$

3. $$\frac{d}{dx}(f_1(x)\times f_2(x)) = f_1(x)\frac{df_2(x)}{dx} + f_2(x)\frac{df_1(x)}{dx} \quad \text{--- (2.28)}$$

4. $$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{1}{f_2(x)}\frac{df_1(x)}{dx} - \frac{f_1(x)}{f_2^{\,2}(x)}\frac{df_2(x)}{dx} \quad \text{--- (2.29)}$$

5. If x depends on time another variable t then,

$$\frac{df(x)}{dt} = \frac{df(x)}{dx}\frac{dx}{dt} \quad \text{--- (2.30)}$$

6.

$$\frac{d}{dx}f\left(g[x]\right) = f'\left(g(x)\right)\times g'(x)$$

where $f'\left(g(x)\right) = \dfrac{df}{dg}$

or $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx}$

The derivatives of some simple functions of x are given below.

1. $\frac{d}{dx}(x^n) = n\, x^{n-1}$ --- (2.31)

2. $\frac{d(e^x)}{dx} = e^x$ and $\frac{d(e^{ax})}{dx} = ae^{ax}$ --- (2.32)

3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ --- (2.33)

4. $\frac{d}{dx}(\sin x) = \cos x$ --- (2.34)

5. $\frac{d}{dx}(\cos x) = -\sin x$ --- (2.35)

6. $\frac{d}{dx}(\tan x) = \sec^2 x$ --- (2.36)

7. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ --- (2.37)

8. $\frac{d}{dx}(\sec x) = \tan x \sec x$ --- (2.38)

9. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ --- (2.39)

Example 2.10: Find the derivatives of the functions.

(a) $f(x) = x^8$ (b) $f(x) = x^3 + \sin x$

(c) $f(x) = x^3 \sin x$

Solution :

(a) Using $\frac{dx^n}{dx} = nx^{n-1}$,

$$\frac{d(x^8)}{dx} = 8x^7$$

(b) Using

$$\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} \text{ and}$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d}{dx}(x^3 + \sin x) = \frac{d(x^3)}{dx} + \frac{d(\sin x)}{dx}$$

$$= 3x^2 + \cos x$$

c) Using

$$\frac{d}{dx}(f_1(x)\, f_2(x)) = f_1(x)\frac{df_2(x)}{dx} + \frac{df_1(x)}{dx} f_2(x)$$

and $\frac{d(\sin x)}{dx} = \cos x$

$$\frac{d}{dx}(x^3 \sin x) = x^3 \frac{d(\sin x)}{dx} + \frac{d(x^3)}{dx} \sin x$$

$$= x^3 \cos x + 3x^2 \sin x$$

2.6.2 Integral calculus

Integral calculus is the branch of mathematics dealing with properties of integrals and their applications. Physical interpretation of integral of a function $f(x)$, i.e., $\int f(x)dx$ is the area under the curve $f(x)$ versus x. It is the reverse process of differentiation as we will see below.

We know how to find the area of a rectangle, triangle etc. In Fig. 2.13(a) we have shown y which is a function of x, A and B being two points on it.

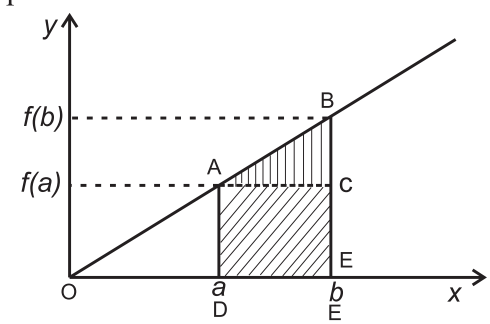


Fig. 2.13 (a): Area under a straight line.

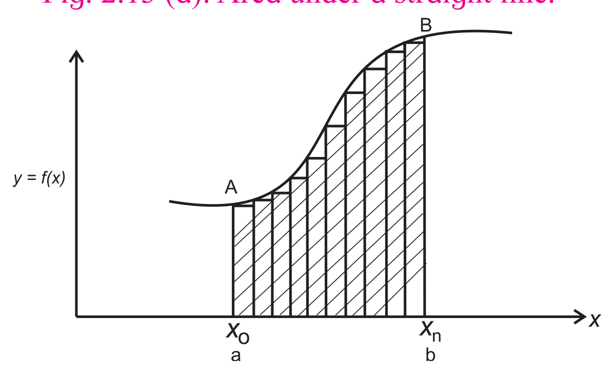


Fig. 2.13 (b): Area under a curve.

The area under the curve (straight line) from $x = a$ to $x = b$ is shown by shaded area. This can be obtained as sum of the area of the rectangle ADEC = $f(a)\ (b-a)$ and the area of the triangle ABC = 1/2 $(b-a)\ (f(b)-f(a))$

Figure 2.13(b) shows another function of x. We do not have a simple formula to calculate the area under this curve. For this calculation, we use a simple trick. We divide the area into a large number of vertical strips as shown in the figure. We assume thickness (width) of each strip to be so small that it can be assumed to be a rectangle as shown in the figure and add the areas of these rectangles. Thus the area under the curve is given by

Area under the curve

$$= \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i)$$

where n is the number of strips and ΔA_i is the area of the i[th] strip.

As the strips are not really rectangles, the area calculated above is not exactly equal to the area under the curve. However as we increase n, the sum of areas of rectangles gets closer to the actual area under the curve and becomes equal to it in the limit $n \to \infty$. Thus we can write,

Area under the curve

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \qquad \text{--- (2.40)}$$

Integration helps us in getting exact area if the change is really continuous, i.e., n is really infinite. It is represented as $\int_{x=a}^{x=b} f(x)dx$ and is called the definite integral of $f(x)$ from $x = a$ to $x = b$.

Thus, $$\int_{x=a}^{x=b} f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \qquad \text{--- (2.41)}$$

The process of obtaining the integral is called integration. We can also write

$$F(x) = \int f(x)dx \qquad \text{--- (2.42)}$$

$F(x)$ is called the indefinite (without any limits on x) integral of $f(x)$. Differentiation is the reverse process to that of integration. Therefore,

$$f(x) = \frac{d}{dx}(F(x)) \qquad \text{--- (2.43)}$$

$$\therefore F(x)\Big|_a^b = F(b) - F(a) = \int_a^b f(x)dx \qquad \text{--- (2.44)}$$

Properties of integration

1. $$\int \left(f_1(x) + f_2(x) \right) dx = \int f_1(x)dx + \int f_2(x)dx \qquad \text{--- (2.45)}$$

2. $$\int K\, f(x)dx = K \int f(x)dx \;\; \text{for } K = \text{constant} \qquad \text{--- (2.46)}$$

Indefinite integrals of some basic functions are given below. Their definite integrals can be obtained by using the Eq. (2.44)

1. $$\int x^n dx = \frac{x^{n+1}}{n+1} \qquad \text{--- (2.47)}$$

2. $$\int \frac{1}{x} dx = \ln x \qquad \text{--- (2.48)}$$

3. $$\int \sin x\, dx = -\cos x \qquad \text{--- (2.49)}$$

4. $$\int \cos x\, dx = \sin x \qquad \text{--- (2.50)}$$

5. $$\int e^x\, dx = e^x \qquad \text{--- (2.51)}$$

Example 2.11: Evaluate the following integrals:

(a) $\int x^8 dx$

(b) $\int_2^5 x^2 dx$

(c) $\int (x + \sin x)\, dx$

Solution: (a) Using formula

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad \int x^8 dx = \frac{x^9}{9}$$

(b) Using Eq. (2.44),

$$\int_2^5 x^2 dx = \frac{x^3}{3}\Big|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125-8}{3} = \frac{117}{3}$$

(c) Using Eq. (2.45),

$$\int \left(f_1(x) + f_2(x) \right) dx = \int f_1(x)dx + \int f_2(x)dx$$

and $\int \sin x\, dx = \cos x$, we get $\int (x + \sin x)\, dx$

$$\int x\, dx + \int \sin x\, dx = \frac{x^2}{2} - \cos x$$

Internet my friend

1. hyperphysics.phy-astr.gsu.edu/hbase/vect.html#veccon
2. hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

Exercises

1. Choose the correct option.

i) The resultant of two forces 10 N and 15 N acting along $+x$ and $-x$-axes respectively, is
(A) 25 N along $+x$-axis
(B) 25 N along $-x$-axis
(C) 5 N along $+x$-axis
(D) 5 N along $-x$-axis

ii) For two vectors to be equal, they should have the
(A) same magnitude
(B) same direction
(C) same magnitude and direction
(D) same magnitude but opposite direction

iii) The magnitude of scalar product of two unit vectors perpendicular to each other is
(A) zero (B) 1
(C) -1 (D) 2

iv) The magnitude of vector product of two unit vectors making an angle of 60° with each other is
(A) 1 (B) 2
(C) 3/2 (D) $\sqrt{3}/2$

v) If $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors, then which of the following is not correct?

(A) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

(B) $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$

(C) $\vec{A}\times\vec{B}=\vec{B}\times\vec{A}$

(D) $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{B}\times\vec{C}$

2. Answer the following questions.

i) Show that $\vec{a}=\dfrac{\hat{i}-\hat{j}}{\sqrt{2}}$ is a unit vector.

ii) If $\vec{v}_1=3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{v}_2=\hat{i}-\hat{j}-\hat{k}$, determine the magnitude of $\vec{v}_1+\vec{v}_2$.

[Ans: 5]

iii) For $\vec{v}_1=2\hat{i}-3\hat{j}$ and $\vec{v}_2=-6\hat{i}+5\hat{j}$, determine the magnitude and direction of $\vec{v}_1+\vec{v}_2$.

$\left[\text{Ans: } 2\sqrt{5},\ \theta=\tan^{-1}\left(-\dfrac{1}{2}\right) \text{with } x\text{-axis}\right]$

iv) Find a vector which is parallel to $\vec{v}=\hat{i}-2\hat{j}$ and has a magnitude 10.

$\left[\text{Ans: } \dfrac{10}{\sqrt{5}}\hat{i}-\dfrac{20}{\sqrt{5}}\hat{j}\right]$

v) Show that vectors $\vec{a}=2\hat{i}+5\hat{j}-6\hat{k}$ and $\vec{b}=\hat{i}+\dfrac{5}{2}\hat{j}-3\hat{k}$ are parallel.

3. Solve the following problems.

i) Determine $\vec{a}\times\vec{b}$, given $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=3\hat{i}+5\hat{j}$.

$\left[\text{Ans: } \hat{k}\right]$

ii) Show that vectors $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b}=3\hat{i}-6\hat{j}+2\hat{k}$ and $\vec{c}=6\hat{i}+2\hat{j}-3\hat{k}$ are mutually perpendicular.

iii) Determine the vector product of $\vec{v}_1=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{v}_2=\hat{i}+2\hat{j}-3\hat{k}$,

$\left[\text{Ans: } -7\hat{i}+5\hat{j}+\hat{k}\right]$

iv) Given $\vec{v}_1=5\hat{i}+2\hat{j}$ and $\vec{v}_2=a\hat{i}-6\hat{j}$ are perpendicular to each other, determine the value of a.

$\left[\text{Ans: } \dfrac{12}{5}\right]$

v) Obtain derivatives of the following functions:
(i) $x\sin x$ (ii) $x^4+\cos x$
(iii) $x/\sin x$

$\left[\text{Ans: (i) } \sin x+x\cos x,\ \text{(ii) } 4x^3-\sin x,\ \text{(iii) } \dfrac{1}{\sin x}-\dfrac{x\cos x}{\sin^2 x}\right]$

vi) Using the rule for differentiation for quotient of two functions, prove that

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\sec^2 x$$

vii) Evaluate the following integral:
(i) $\int_0^{\pi/2}\sin x\,dx$ (ii) $\int_1^5 x\,dx$

$\left[\text{Ans: (i) } 1, \text{(ii) } 12\right]$

***

3. Motion in a Plane

Can you recall?

1. What is meant by motion?
2. What is rectilinear motion?
3. What is the difference between displacement and distance travelled?
4. What is the difference between uniform and nonuniform motion?

3.1 Introduction:

We see objects moving all around us. Motion is a change in the position of an object with time. We have come across the motion of a toy car when pushed along some particular direction, the motion of a cricket ball hit by a batsman for a sixer and the motion of an aeroplane from one place to another. The motion of objects can be divided in three categories: (1) motion along a straight line, i.e., rectilinear motion, (2) motion in two dimensions, i.e., motion in a plane and, (3) motion in three dimensions, i.e., motion in space. The above cited examples correspond to three types of motions, respectively. You have studied rectilinear motion in earlier standards. In rectilinear motion the force acting on the object and the velocity of the object both are along one and the same line. The distances are measured along the line only and we can indicate distances along the +ve and –ve axes as being positive and negative, respectively. The study of the motion of an object in a plane or in space becomes much easier and the corresponding equations become more elegant if we use vector quantities. In this Chapter we will first recall basic facts about rectilinear motion. We will use vector notation for this study as it will be useful later when we will study the motion in two dimensions. We will then study the motion in two dimensions which will be restricted to projectile motion only. Circular motion, i.e., the motion of an object around a circular path will be introduced here and will be studied in detail in the next standard.

3.2 Rectilinear Motion:

Consider an object moving along a straight line. Let us assume this line to be along the *x*-axis. Let $\vec{x}_1$ and $\vec{x}_2$ be the position vectors of the body at times t_1 and t_2 during its motion. The following quantities can be defined for the motion.

1. **Displacement:** The displacement of the object between t_1 and t_2 is the difference between the position vectors of the object at the two instances. Thus, the displacement is given by

$$\vec{s} = \Delta\vec{x} = \vec{x}_2 - \vec{x}_1 \qquad \text{--- (3.1)}$$

Its direction is along the line of motion of the object. Its dimensions are that of length. For example, if an object has travelled through 1 m from time t_1 to t_2 along the +ve *x*-direction, the magnitude of its displacement is 1 m and its direction is along the +ve *x*-axis. On the other hand, if the object travelled along the +ve *y* direction through the same distance in the same time, the magnitude of its displacement is the same as before, i.e., 1 m but the direction of the displacement is along the +ve *y*-axis.

2. **Path length**: This is the actual distance travelled by the object during its motion. It is a scalar quantity and its dimensions are also that of length. If an object travels along the *x*-axis from x = 2 m to x = 5 m then the distance travelled is 3 m. In this case the displacement is also 3 m and its direction is along the +ve *x*-axis. However, if the object now comes back to x = 4, then the distance through which the object has moved increases to 3 + 1 = 4 m. Its initial position was x = 2 m and the final position is now x = 4 m and thus, its displacement is Δx = 4 – 2 = 2 m, i.e., the magnitude of the displacement is 2 m and its direction is along the +ve *x*-axis. If the object now moves to x =1, then the distance travelled, i.e., the path length increases to 4 + 3 =

7 m while the magnitude of displacement becomes 2 – 1 = 1 m and its direction is along the negative *x*-axis.

3. **Average velocity**: This is defined as the displacement of the object during the time interval over which average velocity is being calculated, divided by that time interval. As displacement is a vector quantity, the velocity is also a vector quantity. Its dimensions are $[L^1 M^0 T^{-1}]$.

   If the position vectors of the object are $\vec{x}_1$ and $\vec{x}_2$ at times t_1 and t_2 respectively, then the average velocity is given by

   $$\vec{v}_{av} = \frac{\vec{x}_2 - \vec{x}_1}{(t_2 - t_1)} \qquad \text{--- (3.2)}$$

   For example, if the positions of an object are $x = +2$ m and $x = +4$ m at times $t = 0$ and $t = 1$ minute respectively, the magnitude of its average velocity during that time is $v_{av} = (4 - 2)/(1 - 0) = 2$ m per minute and its direction will be along the +ve *x*-axis, and we write $\vec{v}_{av} = 2\hat{i}$ m/min where $\hat{i}$ is a unit vector along *x*-axis.

4. **Average speed**: This is defined as the total path length travelled during the time interval over which average speed is being calculated, divided by that time interval.

   Average speed = v_{av} = path length/time interval. It is a scalar quantity and has the same dimensions as that of velocity, i.e., $[L^1 M^0 T^{-1}]$.

   If the rectilinear motion of the object is only in one direction along a line, then the magnitude of its displacement will be equal to the distance travelled and so the magnitude of average velocity will be equal to the average speed. However if the object reverses its direction (the motion remaining along the same line) then the magnitude of displacement will be smaller than the path length and the average speed will be larger than the magnitude of average velocity.

5. **Instantaneous velocity**: Instantaneous velocity of an object is its velocity at a given instant of time. It is defined as the limiting value of the average velocity of the object over a small time interval (Δt) around t when the value of the time interval (Δt) goes to zero.

   $$\vec{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{x}}{\Delta t}\right) = \frac{d\vec{x}}{dt}, \qquad \text{--- (3.3)}$$

   $\frac{d\vec{x}}{dt}$ being the derivative of $\vec{x}$ with respect to t (see Chapter 2).

6. **Instantaneous speed:** Instantaneous speed is the speed of an object at a given instant of time t. It is the limiting value of the average speed of the object taken over a small time interval (Δt) around t when the time interval goes to zero. In such a limit, the path length will be equal to the magnitude of the displacement and so the instantaneous speed will always be equal to the magnitude of the instantaneous velocity of the object.

Always Remember:

For uniform rectilinear motion, i.e., for an object moving with constant velocity along a straight line

1. The average and instantaneous velocities are equal.
2. The average and instantaneous speeds are the same and are equal to the magnitude of the velocity.

For nonuniform rectilinear motion

1. The average and instantaneous velocities are different.
2. The average and instantaneous speeds are different.
3. The average speed will be different from the magnitude of average velocity.

Example 3.1: A person walks from point P to point Q along a straight road in 10 minutes, then turns back and returns to point R which is midway between P and Q after further 4 minutes. If PQ is 1 km, find the average speed and velocity of the person in going from P to R.

Solution: The path length travelled by the person is 1.5 km while the displacement is the distance between R and P which is 0.5 km. The time taken for the motion is 14 min.

The average speed = 1.5 / 14 = 0.107 km/min = 6.42 km/hr.

The magnitude of the average velocity = 0.5/14 = 0.0357 km/min = 2.142 km/hr.

Graphical Study of Motion

We can study the motion of an object by using graphs showing its position as a function of time. Figure 3.1 shows the graphs of position as a function of time for five different types of motion of an object. Figure 3.1(a) shows an object at rest, for which the *x-t* graph is a horizontal straight line. Since the position is not changing, displacement of the object zero. Velocity is displacement (which is zero) divided by time interval or the derivative of displacement with respect to time. It can be obtained from the slope of the line plotted in the figure which is zero.

Figure 3.1(b) shows *x-t* graph for an object moving with constant velocity along the +ve *x*-axis. Since velocity is constant, displacement is proportional to elapsed time. The slope of the straight line is +ve, showing that the velocity is along the +ve *x*-axis. As the motion is uniform, the average velocity is same as the instantaneous velocity at all times. Also, the speed is equal to the magnitude of the velocity.

Figure 3.1(c) shows the *x-t* graph for a body moving with uniform velocity but along the -ve *x*-axis, the slope of the line being -ve. Figure 3.1(d) shows the *x-t* graph of an object having oscillatory motion with constant speed. The direction of velocity changes from +ve to -ve and vice versa over fixed intervals of time.

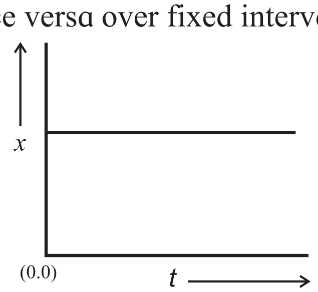


Fig 3.1 (a): Object at rest.

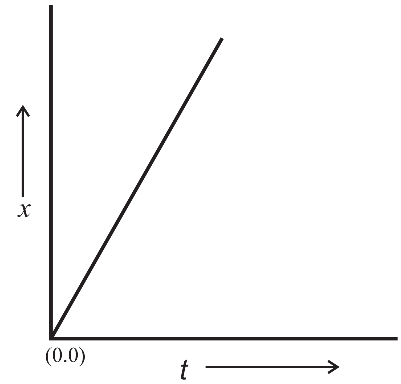


Fig 3.1 (b): Object with uniform velocity along +ve *x*-axis.

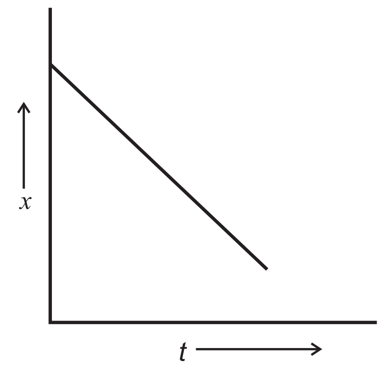


Fig 3.1 (c): Object with uniform velocity along -ve *x*-axis.

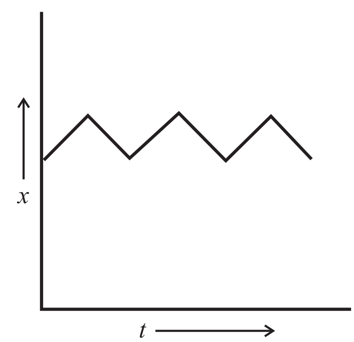


Fig 3.1 (d): Object performing oscillatory motion.

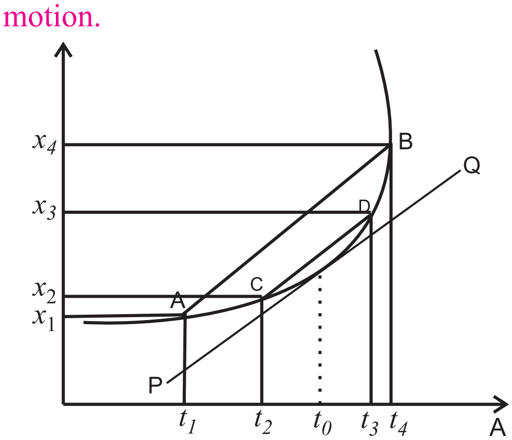


Fig.3.1 (e): Object in nonuniform motion.

Figure 3.1(e) shows the motion of an

object with nonuniform velocity. Its velocity changes with time and, therefore, the average and instantaneous velocities are different. Figure shows the average velocity over time interval from t_1 to t_4 around time t_0, which can be seen from Eq. (3.2) to be the slope of line AB. For a smaller time interval from t_2 to t_3, the average velocity is the slope of the line CD. If we keep reducing the time interval around t_0, we will ultimately come to a limit, when the time interval will go to zero and lines AB, CD... will go over to the tangent to the curve at t_0. The instantaneous velocity at t_0 will thus be equal to the slope of the tangent PQ at t_0 (see Eq. (3.3)).

7. Acceleration: Acceleration is defined as the rate of change of velocity with time. It is a vector quantity and its dimensions are [$L^1 M^0 T^{-2}$]. The average acceleration of an object having velocities $\vec{v}_1$ and $\vec{v}_2$ at times t_1 and t_2 is given by

$$\vec{a} = \frac{(\vec{v}_2 - \vec{v}_1)}{(t_2 - t_1)} \quad \text{--- (3.4)}$$

Instantaneous acceleration is the limiting value of the average acceleration when the time interval goes to zero. It is given by

$$\vec{a} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} \quad \text{--- (3.5)}$$

The instantaneous acceleration at a given time is the slope of the tangent to the velocity versus time curve at that time. Figure 3.2 shows the velocity versus time (v - t) graphs for four different cases. Figure 3.2(a) represents the motion of an object with zero acceleration, i.e., constant velocity. The shaded area under the velocity-time graph over some time interval t_1 to t_2, shown in Figs. 3.2(a) is equal to $v_0 (t_2 - t_1)$ which is the magnitude of the displacement of the object from t_1 to t_2. Figure 3.2(b) is the velocity-time graph for an object moving with constant +ve acceleration (magnitude of velocity uniformly increasing with time). Figure 3.2(c) shows similar motion but the object has -ve acceleration, i.e., the acceleration is opposite to the direction of velocity which, therefore, decreases uniformly with time. The area under both the curves between two instants of time is the displacement of the object during that time interval (as shown below). Figure 3.2(d) shows the motion of an object having nonuniform acceleration. The average acceleration between t_1 and t_2 around t_0 and the instantaneous accelerations at t_0 for the object are shown by straight lines AB and CD respectively.

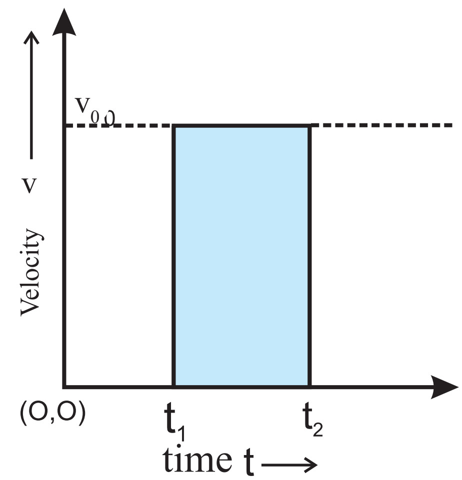


Fig 3.2 (a): Object moving with constant velocity.

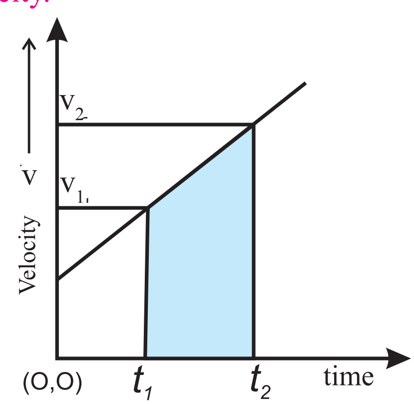


Fig 3.2 (b): Object moving with velocity (v) along +ve x-axis with uniform acceleration along the same direction.

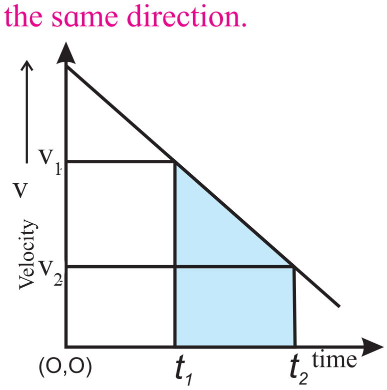


Fig 3.2 (c): Object moving with velocity (v) with negative uniform acceleration.

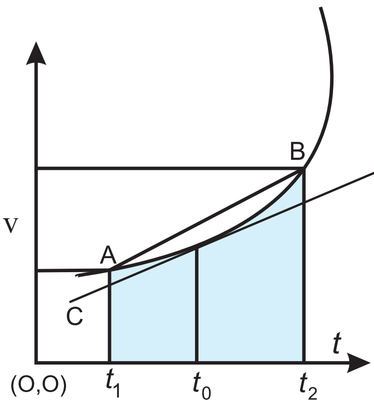


Fig. 3.2 (d): Object moving with nonuniform acceleration.

The area under the velocity-time curves in Figs. 3.2(a) to (d) can be written using the definition of integral given in Chapter 2 as

$$\text{Area} = \int_{t_1}^{t_2} \text{v}\,dt = \int_{t_1}^{t_2} \frac{dx}{dt}dt = \int_{t_1}^{t_2} dx = x(t_2) - x(t_1) \quad \text{--- (3.6)}$$

= displacement of the object from t_1 to t_2.

Always Remember:

For uniform acceleration, for a rectilinear motion:

1. Velocity-time graph is linear.
2. The area under the velocity-time graph between two instants of time t_1 and t_2 gives the displacement of the object during that time interval.
3. The slope of the velocity-time graph is the acceleration of the object

For nonuniform acceleration in a rectilinear motion:

1. Velocity-time graph is nonlinear.
2. The area under the velocity-time graph between two instants of time t_1 and t_2 gives the displacement of the object during that time interval.
3. The instantaneous acceleration of the object at a given time is equal to the slope of the tangent to the curve at that point.

While using the concept of area under the curve, the origin of the velocity axis (for v-t graph) must be zero.

Equations of Motion for Uniform Acceleration:

We can graphically derive Newton's equations of motion for an object moving with uniform acceleration. Consider an object having position $x = 0$ at $t = 0$. Let the velocity at $t = 0$ be u and at time t be v. The graphical representation of motion is shown in Fig. 3.3. The acceleration is given by the slope of the line AB. Thus,

$$\text{Acceleration, } a = \frac{\text{v}-u}{t-0} = \frac{\text{v}-u}{t}$$

$$\therefore \text{v} = u + at \quad \text{--- (3.7)}$$

This is the first equation of motion.

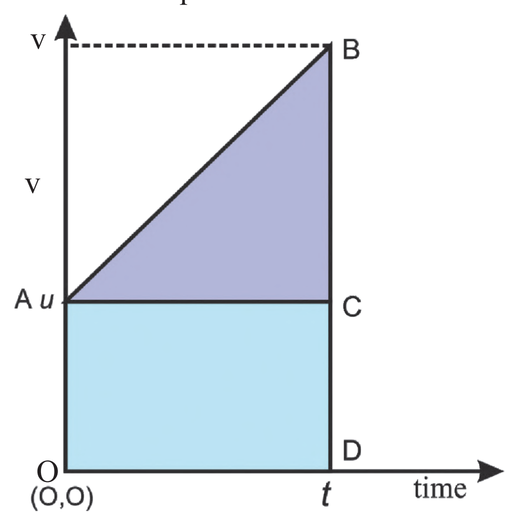


Fig.3.3: Derivation of equation of motion for motion with uniform acceleration.

As we know, the area under the curve in velocity-time graph is the displacement of the object. Thus displacement s = area of the quadrilateral OABD. = area of triangle ABC + area of rectangle OACD.

$$= \frac{1}{2}(\text{v}-u)t + ut$$

Using Eq. (3.7), $s = ut + \frac{1}{2}at^2$ --- (3.8)

This is the second equation of motion.

As the acceleration is constant, the velocity is increasing linearly with time and we can use average velocity v_{av}, to calculate the displacement using Eq. (3.7) as

$$s = \text{v}_{av}t = \left(\frac{\text{v}+u}{2}\right)t = \frac{(\text{v}+u)(\text{v}-u)}{2a}$$

$$\therefore s = \left(\text{v}^2 - u^2\right)/(2a)$$

$$\therefore \text{v}^2 - u^2 = 2a.s \quad \text{--- (3.9)}$$

This is the third equation of motion. Vector notation was not included here as the motion was rectilinear.

The most common example of uniform rectilinear motion with uniform acceleration of an object in day to day life is a freely falling body. When a body starts with zero velocity at a certain height from the ground and falls under the influence of the gravity of the Earth , it is said to be in free fall. The only other force that acts on it is that of the air resistance or friction. For displacements of a few metres, this force is too small and can be neglected. The acceleration of the body is the acceleration due to gravity which is along the vertical direction and can be assumed to be constant over distances which are small compared to the radius of the Earth . Thus the velocity and acceleration are both along the vertical direction and the motion is a uniform rectilinear motion with uniform acceleration.

Do you know ?

The distance travelled by an object starting from rest and having a uniform acceleration in successive seconds are in the ratio 1:3:5:7... Consider a freely falling object. Let us calculate the distances travelled by it in equal intervals of time t_0 (say). This can be done using the second equation of motion $s = u\,t_0 + (1/2)\,g\,t_0^2$. The initial velocity is zero. Therefore, the distance travelled in the first t_0 interval $= (1/2)\,g\,t_0^2$. For simplification let us write $(1/2)\,g = A$. Then the distance travelled in the first t_0 time interval $= d_1 = At_0^2$. In the time interval $2t_0$, the distance travelled $= A(2t_0)^2$. Hence, the distance travelled in the second t_0 interval is $d_2 = A(4t_0^2 - t_0^2) = 3A\,t_0^2 = 3\,d_1$. The distance travelled in time interval $3t_0 = A(3t_0)^2$. Thus, the distance travelled in the 3rd t_0 interval $= d_3 = A(9t_0^2 - 4t_0^2) = 5A\,t_0^2 = 5d_1$. Continuing, one can see that the distances d_1, d_2, d_3 .. are in the ratio 1:3:5:7... This is true for any rectilinear motion, starting from rest, with positive uniform acceleration.

Example 3.2: A stone is thrown vertically upwards from the ground with a velocity 15 m/s. At the same instant a ball is dropped from a point directly above the stone from a height of 30 m. At what height from the ground will the stone and the ball meet and after how much time? (Use $g = 10$ m/s^2 for ease of calculation).

Solution: Let us assume that the stone and the ball meet after time t_0. The distances (not displacements) travelled by the stone and the ball in that time can be obtained from Eq. (3.8) as

$$s_{stone} = 15\,t_0 - \frac{1}{2}\,g\,t_0^2$$

$$s_{ball} = \frac{1}{2}\,g\,t_0^2$$

When they meet, $s_{stone} + s_{ball} = 30$

$$15\,t_0 - \frac{1}{2}\,g\,t_0^2 + \frac{1}{2}\,g\,t_0^2 = 30$$

$$t_0 = 30/15 = 2\text{ s}$$

$$\therefore\ s_{stone} = 15\,(2) - \frac{1}{2}\,(10)\,(2)^2 = 30 - 20 = 10\text{ m}$$

Thus the stone and the ball meet at a height of 10 m.

8. Relative Velocity: You must have often experienced relative motion. The most striking example is when you are going in a train and another train travelling in the same direction along parallel tracks, overtakes you. If you look at that train, it actually seems to be moving much slower than what your train seemed to move and yet it is overtaking you. On the other hand if your train overtakes another train, travelling on a parallel track in the same direction, and you look at that train, you feel that your train has suddenly slowed down. Why does this happen? This is because when you look at the neighbouring train, you are actually experiencing relative motion, i.e., your motion with respect to the other train or the motion of the other train with respect to you. Thus, in the first case as the other train overtakes you what you perceive is the velocity of the train with respect to you, i.e., the difference in the velocities of the two trains which most often is much smaller than the velocity of your train. In the second case, you are moving faster but when you look at that train you only feel your velocity relative to it and, therefore, your velocity appears to be lower than its actual value. We can define relative velocity of object A with respect to object B as the difference between their velocities, i.e.,

$$v_{AB} = v_A - v_B \qquad \text{--- (3.10)}$$

Similarly, the velocity of B with respect to A is given by

$$v_{BA} = v_B - v_A \qquad \text{--- (3.11)}$$

We assume that at time $t = 0$, A and B were at the same point $x = 0$. As they are travelling with different velocities, the distance between them will go on increasing with time in direct proportion to the difference in their velocities, i.e., the relative velocity between them.

Example 3.3: An aeroplane A, is travelling in a straight line with a velocity of 300 km/hr with respect to Earth. Another aeroplane B, is travelling in the opposite direction with a velocity of 350 km/hr with respect to Earth. What is the relative velocity of A with respect to B? What should be the velocity of a third aeroplane C moving parallel to A, relative to the Earth if it has a relative velocity of 100 km/hr with respect to A?

Solution: Let v_A, v_B and v_C be the velocities of the three planes relative to the Earth. Relative velocity of A with respect to B = $v_{AB} = v_A - v_B$ = 300 – (-350) = 650 km/hr

Relative velocity of C with respect to A = v_{CA} = $v_C - v_A$ = 100 km/hr.

Thus, $v_C = v_{CA} + v_A$ = 400 km/hr

3.3 Motion in Two Dimensions-Motion in a Plane:

So far we were considering rectilinear motion of an object. The direction of motion of the object was always along one straight line. Now we will consider the motion of an object in two dimensions, i.e., along a plane. Here, the direction of the force acting on an object will not be in the same line as its initial velocity. Thus, the velocity and acceleration will have different directions. For this reason we have to use vector equations. The definitions of various terms given in section 3.2 will remain valid except that the magnitude of the average velocity and the value of average speed will be different as the magnitude of the displacement need not be equal to the path length. For example, if a particle travels along a circle and comes back to its original position, its displacement will be zero but the path length will be equal to the circumference of the circle.

3.3.1 Average and Instantaneous Velocities:

For studying the motion of an object in two dimensions, for simplicity, we will take the plane to be the *x-y* plane. To describe the position of an object in this plane we will have to specify, both its *x* and *y* coordinates. The definitions of displacement, average and instantaneous velocities, average and instantaneous speeds and acceleration will be the same as those for rectilinear motion except that each of these quantities will now have components along the *x* and *y* directions. Let us assume the object to be at point P at time t_1 as shown in Fig. 3.4 (a).

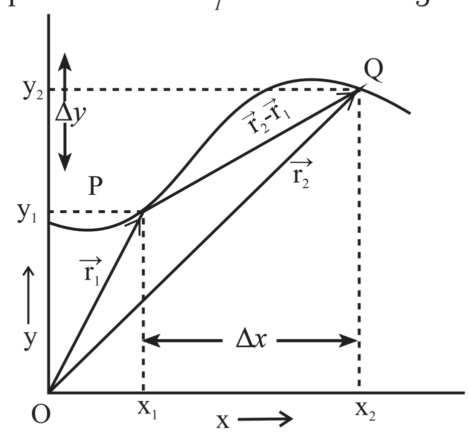


Fig. 3.4 (a) Motion in two dimensions

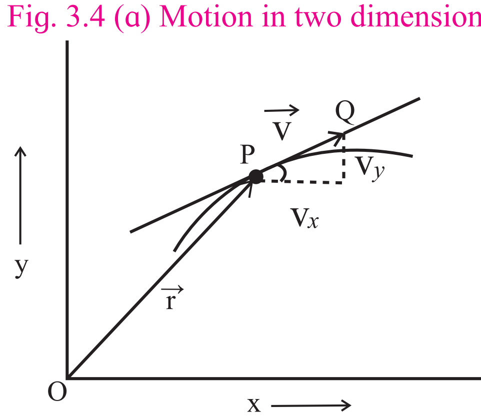


Fig. 3.4 (b) Instantaneous velocity

The position of the object will be described by its position vector $\vec{r}_1$. This can be written in terms of its components along the *x* and *y* axes as

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} \quad \text{--- (3.12)}$$

At time t_2, let the position of the object be Q and its position vector be $\vec{r}_2$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} \quad \text{--- (3.13)}$$

The displacement of the particle from t_1 to t_2 shown by PQ, i.e., in time $t = t_2 - t_1$ is given by

$$\overrightarrow{\Delta r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} \quad \text{--- (3.14)}$$

We can write the average velocity of the object as

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \left(\frac{x_2 - x_1}{t_2 - t_1}\right)\hat{i} + \left(\frac{y_2 - y_1}{t_{21} - t}\right)\hat{j}$$

$$\vec{v}_{av} = (v_{av})_x\hat{i} + (v_{av})_y\hat{j} \quad \text{--- (3.15)}$$

where, $(v_{av})_x = (x_2 - x_1)/(t_2 - t_1)$ and

$$(v_{av})_y = (y_2 - y_1)/(t_2 - t_1) \quad \text{--- (3.16)}$$

Average velocity is a vector whose direction is along $\Delta\vec{r}$ (see Eq. (3.2)), i.e., along the direction of displacement. In terms of its components, the magnitude (v) and direction (the angle θ that the velocity vector makes with the x-axis) can be written as (see Chapter 2)

$$v_{av} = \sqrt{(v_{av})_x^2 + (v_{av})_y^2} \quad \text{and}$$

$$\tan\theta = (v_{av})_y/(v_{av})_x \quad \text{--- (3.17)}$$

Figure 3.4(b) shows the trajectory of an object moving in two dimensions. The instantaneous velocity of the object at point P along the trajectory is along the tangent to the curve at P. This is shown by the vector PQ. Its x and y components v_x and v_y are also shown in the figure.

The instantaneous velocity of the object can be written in terms of derivative as (see Eq. 3.3)

$$\vec{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{r}}{\Delta t}\right) = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} \quad \text{-- (3.18)}$$

The magnitude and direction of the instantaneous velocity are given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \quad \text{--- (3.19)}$$

$$\tan\theta = (dy/dt)/(dx/dt) = dy \quad dx \quad \text{--- (3.20)}$$

which is the slope of the tangent to the curve at the point at which we are calculating the instantaneous velocity.

3.3.2 Average and Instantaneous Acceleration:

Again, the definitions are the same as those for rectilinear motion. Thus, the average acceleration ($\vec{a}_{av}$) of a particle between times t_1 and t_2 can be written as

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \left(\frac{v_{2x} - v_{1x}}{t_2 - t_1}\right)\hat{i} + \left(\frac{v_{2y} - v_{1y}}{t_2 - t_1}\right)\hat{j} \quad \text{- (3.21)}$$

where $\vec{v}_2$ and $\vec{v}_1$ are the velocities of the particle at times t_2 and t_1 respectively.

$$\vec{a}_{av} = (a_{av})_x\hat{i} + (a_{av})_y\hat{j} \quad \text{--- (3.22)},$$

$(a_{av})_x$ and $(a_{av})_y$ being the x and y components of the average acceleration.

The magnitude and direction of the acceleration are given by

$$a_{av} = \sqrt{(a_{av})_x^2 + (a_{av})_y^2} \quad \text{--- (3.23)}$$

and

$$\tan\theta = (a_{av})_y/(a_{av})_x \quad \text{--- (3.24)}$$

The instantaneous acceleration is given by (see Eq. (3.5))

$$\vec{a} = \lim_{t \to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} = \left(\frac{dv_x}{dt}\right)\hat{i} + \left(\frac{dv_y}{dt}\right)\hat{j} \quad \text{-(3.25)}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)\hat{i} + \frac{d}{dt}\left(\frac{dy}{dt}\right)\hat{j} = \left(\frac{d^2x}{dt^2}\right)\hat{i} + \left(\frac{d^2y}{dt^2}\right)\hat{j} \quad \text{--- (3.26)}$$

Thus, the x and y components of the instantaneous acceleration are respectively given by

$$a_x = d^2x/dt^2 \text{ and } a_y = d^2y/dt^2 \quad \text{--- (3.27)}$$

The magnitude and direction of the instantaneous acceleration are given by

$$a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} \quad \text{--- (3.28)},$$

and

$$\tan\theta = (dv_y/dt)/(dv_x/dt) = dv_y/dv_x \quad \text{--- (3.29)}$$

which is the slope of the tangent to the curve in velocity graph, i.e., a plot of v_y versus v_x.

Example 3.4: The position vectors of three particles are given by $\vec{x}_1 = (5\hat{i} + 5\hat{j})$ m, $\vec{x}_2 = (5t\hat{i} + 5t\hat{j})$ m and $\vec{x}_3 = (5t\hat{i} + 10t^2\hat{j})$m as a function of time t. Determine the velocity and acceleration for each, in SI units.

Solution: $\vec{v}_1 = d\vec{x}_1/dt = 0$ as $\vec{x}_1$ does not depend on time t.

Thus, the particle is at rest.

$\vec{v}_2 = d\vec{x}_2/dt = 5\hat{i} + 5\hat{j}$ m/s. $\vec{v}_2$ does not change with time. $\therefore \vec{a}_2 = 0$

$v_2 = \sqrt{5^2 + 5^2} = 5\sqrt{2}\ m/s$, tan $\theta = 5/5 = 1$ or $\theta =$ 45°. Thus, the direction of v_2 makes an angle of 45° to the horizontal.

$\vec{v}_3 = d\vec{x}_3/dt = 5\hat{i} + 20t\hat{j}$.

$\therefore v_3 = \sqrt{5^2 + (20t)^2}\ m/s$. Its direction is along $\theta = \tan^{-1}\left(\frac{20t}{5}\right)$ with the horizontal.

$$\vec{a}_3 = \frac{d\vec{v}_3}{dt} = 20\hat{j}\ m/s^2$$

Thus, the particle 3 is getting accelerated along the y-axis at 20 m/s².

3.3.3 Equations of Motion for an Object travellinging a Plane with Uniform Acceleration:

We have derived equations of motion for an object in rectilinear motion in section 3.2. We will now derive similar equations for a particle moving with uniform acceleration in two dimensions. Let the initial velocity of the object be $\vec{u}$ at $t = 0$ and its velocity at time t be $\vec{v}$. As the acceleration is constant, the average acceleration and the instantaneous acceleration will be equal. By using the definition of acceleration (Eq. (3.21)), we get

$$\vec{a} = (\vec{v} - \vec{u})/(t - 0)$$

$$\text{or } \vec{v} = \vec{u} + \vec{a}\,t \qquad \text{--- (3.30)}$$

which is the same as Eq. (3.7) but is in vector form.

Let the displacement from time $t = 0$ to t be $\vec{s}$. This can be calculated from the average velocity of the object during this time. For constant acceleration, $\vec{v}_{av} = \frac{\vec{u} + \vec{v}}{2}$

$$\therefore \vec{s} = (\vec{v}_{av})t = \left(\frac{\vec{u} + \vec{v}}{2}\right)t = \left(\frac{\vec{u} + \vec{u} + \vec{a}\,t}{2}\right)t$$

$$\therefore \vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2 \qquad \text{--- (3.31),}$$

which is the vector form of Eq. (3.8).

Eq. (3.30) and (3.31) can be resolved into their x and y components so as to get corresponding scalar equations as follows.

$$v_x = u_x + a_x t \qquad \text{--- (3.32)}$$

and $$v_y = u_y + a_y t \qquad \text{--- (3.33)}$$

$$s_x = u_x t + \frac{1}{2}a_x t^2 \qquad \text{--- (3.34)}$$

and $$s_y = u_y t + \frac{1}{2}a_y t^2 \qquad \text{--- (3.35)}$$

We can see that Eqs. (3.32) and (3.34) involve only the x components of displacement, velocity and acceleration while Eqs. (3.33) and (3.35) involve only the y components of these quantities. Thus the two sets of equations are independent of each other and can be solved independently. We can thus see that the motion along the x direction of an object is completely controlled by the x components of velocity and acceleration while that along the y direction is completely controlled by the y components of these quantities. This makes it easy to study the motion in two dimensions which gets converted to two independent rectilinear motions along two perpendicular directions.

Always Remember:

Motion in two dimensions can be resolved into two independent motions in mutually perpendicular directions.

Example 3.5: The initial velocity of an object is $\vec{u} = 5\hat{i} + 10\hat{j}$ m/s. Its constant acceleration is $\vec{a} = 2\hat{i} + 3\hat{j}$ m/s². Determine the velocity and the displacement after 5 s.

Solution:

$$\vec{v} = \vec{u} + \vec{a}\,t$$

$$= (5\hat{i} + 10\hat{j}) + (2\hat{i} + 3\hat{j})(5) = 15\hat{i} + 25\hat{j}$$

$$\therefore v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{15^2 + 25^2} = \sqrt{225 + 625} = \sqrt{850}$$

$$= 29.15\ m/s$$

Direction of $\vec{v}$ with x-axis is $\tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}$

$\left(\frac{25}{15}\right) = \tan^{-1}(1.667) = 59^{o}$

$\vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\ t^2$

$= \left(5\hat{i} + 10\hat{j}\right)(5) + \frac{1}{2}\left(2\hat{i} + 3\hat{j}\right)5^2$

$= 50\hat{i} + (87.5)\,\hat{j}$

$\therefore s = \sqrt{s_x^2 + s_y^2} = \sqrt{50^2 + 87.5^2}$

$= \sqrt{2500 + 7656.25}$

$= \sqrt{10156.25} = 100.78\ \text{m}$

at $\tan^{-1}\frac{87.5}{50} = 60°15'$ with x-axis.

3.3.4 Relative Velocity:

Relative velocity between two objects moving in a plane can be defined in a way similar to that for objects moving along a straight line. The relative velocity of object A having velocity $\vec{v}_A$, with respect to the object B having velocity $\vec{v}_B$, is given by

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B \quad \text{--- (3.36)}$$

Similarly, the relative velocity of object B with respect to object A , is given by

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A \quad \text{--- (3.37)}$$

We can see that the magnitudes of the two relative velocities (v_{AB} and v_{BA}) are equal and their directions are opposite.

Consider a number of objects A, B, C, D ---- Y, Z, moving with respect to the other. Using the symbol v_{AB} for representing the velocity of A relative to B etc, the velocity of A relative to Z can be written as

$$\vec{v}_{AZ} = \vec{v}_{AB} + \vec{v}_{BC} + \vec{v}_{CD} + \ldots + \vec{v}_{XY} + \vec{v}_{YZ}$$

Note the order of subscripts (A→B→C→D---→Z).

Example 3.6: An aeroplane is travelling northward with a speed of 300 km/hr with respect to the Earth, when the wind is blowing from east to west at a speed of 100 km/hr. What is the velocity of the aeroplane with respect to the wind?

Solution: Let the velocity of the aeroplane with respect to Earth be $\vec{v}_{AE}$, velocity of wind with respect to Earth be $\vec{v}_{WE}$. The velocity of aeroplane with respect to wind, $\vec{v}_{AW}$ can be determined by the following expression:

$\vec{v}_{AW} = \vec{v}_{AE} + \vec{v}_{EW} = \vec{v}_{AE} - \vec{v}_{WE} = 100\hat{i} + 300\hat{j}$, considering north along +y axis.

Magnitude of $\vec{v}_{AW} = \sqrt{(10000 + 90000)}$

$= 100\sqrt{10}$ km/hr, and its direction,

$\theta = \tan^{-1}\left(\frac{300}{100}\right) = 71.6°$ is towards north of east.

3.3.5 Projectile Motion:

Any object in flight after being thrown with some velocity is called a projectile and its motion is called projectile motion. We often see projectile motion in our day-to-day life. Children throw stones towards trees for getting tamarind pods or mangoes. A bowler bowls a ball towards a batsman in cricket, a basket ball player throws a ball towards the basket, all these are illustrations of projectile motion. In this motion, we have objects (projectiles) with given initial velocity, moving under the influence of the Earth's gravitational field. The projectile has two components of velocity, one in the horizontal, i.e., along x-direction and the other in the vertical, i.e., along the y direction. The acceleration due to gravity acts only along the vertically downward direction. The horizontal component of velocity, therefore, remains unchanged as no force is acting in the horizontal direction, while the vertical component changes in accordance with laws of motion with $\vec{a}_x$ being 0 and $\vec{a}_y\ (= -\vec{g}\,)$ being the downward acceleration due to gravity (upward is positive). Unless stated otherwise, retarding forces like air resistance, etc., are neglected for the projectile motion.

Let us assume that the initial velocity of the projectile is $\vec{u}$ and its direction makes an angle θ with the horizontal as shown in Fig. 3.5. The projectile is thrown from the ground. We take the x-axis along the ground and y-axis in the vertical direction. The horizontal and vertical components of initial velocity are u

cosθ and u sinθ respectively. The horizontal component remains unchanged in absence of any force acting in that direction, while the vertical component changes according to (Eq. 3.33) with $a_y = -g$ and $u_y = u \sin\theta$.

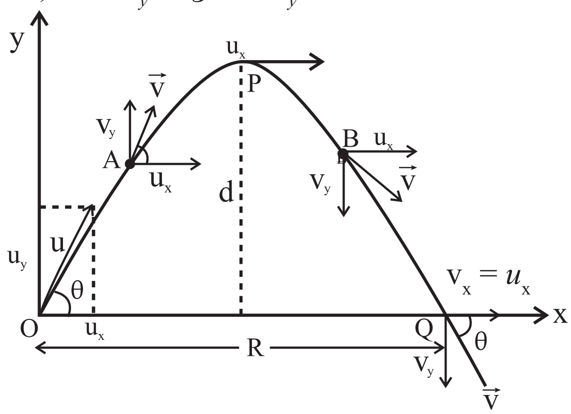


Fig.3.5: Trajectory of a projectile.

Thus, the components of velocity at time t are given by

$$v_x = u_x = u \cos\theta \quad \text{--- (3.38)}$$

$$v_y = u_y - gt = u \sin\theta - gt \quad \text{--- (3.39)}$$

As $0 < \theta < 90°$, the vertical component initially is in the upward direction. Similarly, the displacements of the projectile in the horizontal and vertical directions at time t, according to Eqs. (3.34) and (3.35) are given by

$$s_x = u\cos\theta . t \quad \text{---- (3.40)}$$

$$s_y = u\sin\theta . t - \frac{1}{2} gt^2 \quad \text{--- (3.41)}$$

The direction of motion of the projectile at any time t makes an angle α with the horizontal which is given by

$$\tan \alpha = v_y(t)/v_x(t) \quad \text{--- (3.42)}$$

The vertical velocity keeps on decreasing as the projectile goes up and becomes zero at certain time. At that time the height of the projectile is maximum. The velocity then starts increasing in the downward direction as the particle is now falling under the Earth 's gravitational field with a constant horizontal component of velocity. After a while the projectile reaches the ground. The trajectory of the object is shown in Fig. 3.5. The projectile is assumed to start from the origin of the coordinate system, O. The point of maximum height is indicated by P and the point where it falls down to the ground is indicated by Q. The horizontal and vertical components of velocity are shown at these points as well as at two intermediate points A and B, on the trajectory of the projectile. Note that the horizontal component of velocity remains the same, i.e., u_x, while the vertical component decreases and becomes zero at P. After that it changes its direction, its magnitude increases and becomes equal to u_y again at Q. The horizontal distance covered by the projectile before it falls to the ground is OQ. We can derive the equation of the trajectory of the projectile as follows.

Let the time taken by the projectile to reach the maximum height be t_0. The trajectory of the object being symmetrical, it can be shown by using equations of motion, that the object will take the same time in going up in air and coming down to the ground. At the highest point P, $t = t_0$ and $v_y = 0$. Using Eq. (3.39),

we get, $0 = u \sin\theta - gt_0$

$$t_0 = (u \sin\theta)/g \quad \text{--- (3.43)}$$

∴ Total time in air = $T = 2t_0$ is the ***time of flight***.

The total horizontal distance travelled by the particle in this time T can be obtained by using Eq. (3.40) as

$$R = u_x .\, T = u \cos\theta . 2t_0 = u \cos\theta . (2u \sin\theta)/g$$

$$= 2\, u_x\, u_y/g = u^2(2 \sin\theta \cos\theta)/g$$

$$= u^2 \sin 2\theta/g \quad \text{--- (3.44)}$$

This maximum horizontal distance travelled by the projectile is called the ***horizontal range*** R of the projectile and depends on the magnitude and direction of initial velocity of the projectile as well as the value of acceleration due to gravity at that place.

For maximum horizontal range,

$\sin 2\theta = 1 \;\therefore 2\theta = 90^0 \; or \; \theta = 45^0$

Hence, $R = R_{max} = \dfrac{u^2}{g}$ for $\theta = 45^0$

The ***maximum height*** H reached by the projectile, having certain value of θ, is the distance travelled along the vertical (y) direction in time t_0. This can be calculated by using Eq. (3.41) as

$$H = u \sin\theta . t_0 - \frac{1}{2} g t_0^2$$

$$= u \sin\theta \left(\frac{u \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{u \sin\theta}{g}\right)^2$$

$$= \frac{u^2 \sin^2\theta}{2g} = \frac{u_y^2}{2g} \qquad \text{--- (3.45)}$$

Do you know ?

All the above expressions of T, R, R_{max} and H are valid if the entire motion is governed only by gravitational acceleration g, i.e., retarding forces like air resistance are absent. However, in reality, it is never so. As a result, time of ascent t_a and time of decent t_d are not equal but $t_a > t_d$. Also, in order to achieve maximum horizontal range for given initial velocity, the angle of projection should be greater than 45^0 and the range is much less than $\frac{u^2}{g}$.

Example 3.7: A stone is thrown with an initial velocity components of 20 m/s along the vertical, and 15 m/s along the horizontal direction. Determine the position and velocity of the stone after 3 s. Determine the maximum height that it will reach and the total distance travelled along the horizontal on reaching the ground. (Assume $g = 10$ m/s^2)

Solution: The initial velocity of the stone in x-direction $= u\cos\theta = 15$ m/s and in y-direction $= u\sin\theta = 20$ m/s.

After 3 s, $v_x = u\cos\theta = 15$ m/s and $v_y = u\sin\theta - gt = 20 - 10(3) = -10$ m/s $= 10$ m/s downwards.

$$\therefore v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 10^2}$$
$$= \sqrt{225+100} = \sqrt{325}$$
$$= 18.03\ \text{m/s}$$

$\tan\alpha = v_y / v_x = 10/15 = 2/3$

$\therefore \alpha = \tan^{-1}(2/3) = 33°\ 41'$ with the horizontal.

$s_x = (u\cos\theta)\, t = 15\times3 = 45$ m,

$s_y = (u\sin\theta)\, t - \frac{1}{2}gt^2 = 20\times3 - 5(3)^2 = 15$ m.

Thus the stone will be at a distance 45 m along horizontal and 15 m along vertical direction from the initial position after time 3 s. The velocity is 18.03 m/s making an angle 33° 41' with the horizontal.

The maximum vertical distance travelled is given by $H = (u\sin\theta)^2/(2g) = 20^2/(2\times10) = 20$ m

Maximum horizontal distance travelled

$R = 2.u_x.u_y/g = 2(15)(20)/10 = 60$ m

Equation of motion for a projectile

We can derive the equation of motion of the projectile which is the relation between the displacements of the projectile along the vertical and horizontal directions. This can be obtained by eliminating t between the equations giving these displacements, i.e., Eqs. (3.40) and (3.41).

As the projectile starts from $\vec{x} = 0$, we can write $s_x = x$ and $s_y = y$.

$$\therefore s_x = (u\cos\theta)t \quad \therefore t = \frac{s_x}{u\cos\theta} = \frac{x}{u\cos\theta}$$

$$\therefore y = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$= (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\therefore y = (\tan\theta)x - \frac{1}{2}\left(\frac{g}{u^2\cos^2\theta}\right)x^2 \qquad \text{--- (3.46)}$$

This is the equation of the trajectory of the projectile. Here, u and θ are constants for the given projectile motion. The above equation is of the form

$$y = Ax + Bx^2 \qquad \text{--- (3.47)}$$

which is the equation of a parabola. Thus, the path, i.e., the trajectory of a projectile is a parabola.

3.4 Uniform Circular Motion:

An object moving with *constant speed* along a circular path is said to be in *uniform* circular motion (UCM). Such a motion is only possible if its velocity is *always tangential* to its circular path, *without change in its magnitude*.

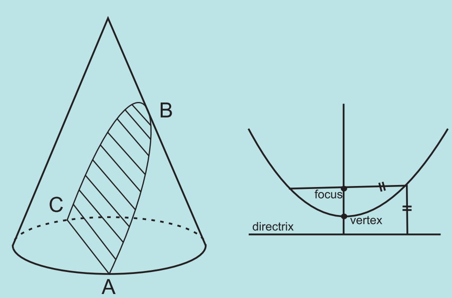
To change the direction of velocity, acceleration is a must. However, if the acceleration or its component is in line with the velocity (along or opposite to the velocity), it will *always* change the speed (magnitude of velocity) in which case it will not continue its uniform circular motion. In order to achieve both these requirements, the acceleration must be (i) perpendicular to the tangential velocity, (ii) of constant magnitude and (iii) always directed towards the centre of the circular trajectory. Such an acceleration is called *centripetal* (centre seeking) acceleration and the force causing this acceleration is *centripetal* force.

Thus, in order to realize a circular motion, there are two requirements; (i) *tangential velocity* and (ii) *centripetal force* of *suitable constant magnitude*.

An example is the motion of the moon going around the Earth in an early circular orbit as a result of the constant gravitational attraction of fixed magnitude felt by it towards the Earth.

Do you know ?

A parabola is a symmetrical open curve obtained by the intersection of a cone with a plane which is parallel to its side. Mathematically, the parabola is described with the help of a point called the focus and a straight line called the directrix shown in the accompanying figure. The parabola is the locus of all points which are equidistant from the focus and the directrix. The chord of the parabola which is parallel to the directrix and passes through the focus is called latus rectum of the parabola as shown in the accompanying figure.



3.4.1 Period, Radius Vector and Angular Speed:

Consider an object of mass *m*, moving with a uniform speed v, along a circle of radius *r*. Let *T* be the time period of revolution of the object, i.e., the time taken by the object to complete one revolution or to travel a distance of $2\pi r$.

Thus, $T = 2\pi r/\mathrm{v}$

$$\therefore \text{Speed } \mathrm{v} = \frac{\text{Distance}}{\text{Time}} = \frac{2\pi r}{T} \quad \text{--- (3.48).}$$

During circular motion of a point object, the position vector of the object from centre of the circle is the radius vector $\vec{r}$. Its magnitude is radius *r* and it is directed away from the centre to the particle, i.e., away from the centre of the circle. As the particle performs UCM, this radius vector describes equal angles in equal intervals of time. At this stage we can define a new quantity called angular speed ω which gives the angle described by the radius vector, per unit time. It is analogous to speed which is distance travelled per unit time.

During one complete revolution, the angle described is 2π and the time taken is period *T*. Hence, the angular speed

$$\omega = \frac{\text{Angle}}{\text{time}} = \frac{2\pi}{T} = \frac{(2\pi)}{\left(\frac{2\pi r}{\mathrm{v}}\right)} = \frac{\mathrm{v}}{r} \quad \text{--- (3.49)}$$

The unit of ω is radian/sec.

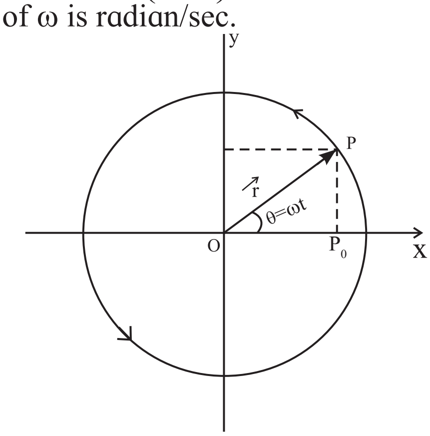


Fig.3.6: Uniform circular motion.

3.4.2 Expression for Centripetal Acceleration:

Figure 3.6 shows a particle P performing a UCM in anticlockwise sense along a circle of radius *r* with angular speed ω and period *T*. Let us choose the coordinates such that this motion is in the *xy*- plane having centre at the origin O. Initially (for simplicity), let the particle be at P_0 on the positive *x*-axis. At a given instant *t*, the radius vector of P makes an angle θ with the *x*-axis.

$$\therefore \theta = \omega t \text{ and so } \frac{d\theta}{dt} = \omega$$

x and *y* components of the radius vector $\vec{r}$ will then be $r\cos\theta$ and $r\sin\theta$ respectively.

$$\therefore \vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$= (r\cos[\omega t])\hat{i} + (r\sin[\omega t])\hat{j} \quad \text{--- (3.50)}$$

Time derivative of position vector $\vec{r}$ gives instantaneous velocity $\vec{v}$ and time derivative of velocity $\vec{v}$ gives instantaneous acceleration $\vec{a}$. Magnitudes of r and ω are constants.

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = r\left(-\omega\sin[\omega t]\hat{i} + \omega\cos[\omega t]\hat{j}\right)$$

$$= r\omega\left(-\sin[\omega t]\hat{i} + \cos[\omega t]\hat{j}\right) \quad \text{--- (3.51)}$$

$$\therefore \vec{a} = \frac{d\vec{v}}{dt} = r\omega\left(-\omega\cos[\omega t]\hat{i} - \omega\sin[\omega t]\hat{j}\right)$$

$$= -\omega^2\left(r\cos[\omega t]\hat{i} + r\sin[\omega t]\hat{j}\right) = -\omega^2\vec{r} \quad \text{--- (3.52)}$$

Here minus sign shows that the acceleration is opposite to that of $\vec{r}$, i.e., towards the centre. This is the centripetal acceleration.

The magnitude of acceleration,

$$a = \omega^2 r = \frac{v^2}{r} = \omega v \quad \text{--- (3.53)}$$

The force providing this acceleration should also be along the same direction, hence centripetal.

$$\therefore \vec{F} = m\vec{a} = -m\omega^2\vec{r} \quad \text{--- (3.54)}$$

Magnitude of $F = m\omega^2 r = \frac{mv^2}{r} = m\omega v$ - (3.55)

Conical pendulum

In a simple pendulum a mass m is suspended by a string of length l and moves along an arc of a vertical circle. If the mass instead revolves in a horizontal circle and the string which makes a constant angle with the vertical describes a cone whose vertex is the fixed point O, then mass-string system is called a conical pendulum as shown in Fig. 3.7. In the absence of friction, the system will continue indefinitely once started.

As shown in the figure, the forces acting on the bob of mass, m, of the conical pendulum are: (i) Gravitational force, mg, acting vertically downwards, (ii) Force due to tension $\vec{T}$ acting along the string directed towards the support. These are the only two forces acting on the bob.

For the bob to undergo horizontal circular motion, (on a circle of radius r) the resultant force must be centripetal, (directed towards the centre of the circle). In other words vertical gravitational force must be balanced.

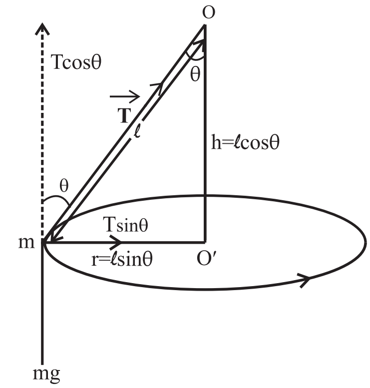


Fig 3.7: Conical pendulum

Thus, we resolve tension $\vec{T}$ into two mutually perpendicular components. Let θ be the angle made by the string with the vertical at any position. The component $T\cos\theta$ is acting vertically upwards. The inclination should be such that $T\cos\theta$ = mg, so that there is no net vertical force.

The resultant force on the bob is then $T\sin\theta$ which is radial or centripetal or directed towards centre O'.

$$\therefore \quad T\sin\theta = mv^2/r$$

$$\tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{(mv^2/r)}{mg} = \frac{v^2}{rg}$$

Since we know $v = \frac{2\pi r}{T}$

$$\therefore \tan\theta = \frac{4\pi^2 r^2}{T^2 rg}$$

$$T = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

$$T = 2\pi\sqrt{\frac{l\sin\theta}{g\tan\theta}} \quad (\because r = l\sin\theta)$$

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}} \quad (\because h = l\cos\theta) \quad \text{--- (3.56)}$$

where l is length of the pendulum and h is the vertical distance of the horizontal circle from the fixed point O.

Example 3.8: An object of mass 50 g moves uniformly along a circular orbit with an angular speed of 5 rad/s. If the linear speed of the particle is 25 m/s, what is the radius of the circle? Calculate the centripetal force acting on the particle.

Do you know ?

1. The centripetal force is not one of the external forces acting on the object. As can be seen from above, the actual forces acting on the bob are T and mg, the resultant of these is the centripetal force. Conversely, if the resultant force is centripetal, motion must be circular.
2. In planetary motion, the gravitational force between Sun and the planets provides the necessary centripetal force for the circular motion.

Solution: The linear speed and angular speed are related by v = ω*r*

$\therefore\ r = v/\omega = 25/5\ \text{m} = 5\ \text{m}.$

Centripetal force acting on the object = $\frac{mv^2}{r} = \frac{0.05 \times 25^2}{5} = 6.25$ N.

Example 3.9: An object is travelling in a horizontal circle with uniform speed. At $t = 0$, the velocity is given by $\vec{u} = 20\hat{i} + 35\hat{j}$ km/s. After one minute the velocity becomes $\vec{v} = -20\hat{i} - 35\hat{j}$. What is the magnitude of the acceleration?

Solution: Magnitude of initial and final velocities =

$$= u = \sqrt{(20)^2 + (35)^2}\ \text{m/s}$$

$$= \sqrt{1625}\ \text{m/s}$$

$$= 40.3\ \text{m/s}$$

As the velocity reverses in 1 min, the time period of revolution is 2 min.

$$T = \frac{2\pi r}{u},\ \text{giving}\ r = \frac{uT}{2\pi}$$

$$a = \frac{u^2}{r} = \frac{u^2 2\pi}{uT} = \frac{2\pi u}{T} = \frac{\cancel{2} \times 3.14 \times 40.3}{\cancel{2} \times 60}$$

$$= 2.11\ \text{m s}^{-2}$$

Internet my friend

1. hyperphysics.phy-astr.gsu.edu/hbase/mot.html#motcon
2. www.college-physics.com/book/mechanics

Exercises

1. Choose the correct option.

i) An object thrown from a moving bus is on example of
   (A) Uniform circular motion
   (B) Rectilinear motion
   (C) Projectile motion
   (D) Motion in one dimension

ii) For a particle having a uniform circular motion, which of the following is constant
   (A) Speed (B) Acceleration
   (C) Velocity (D) Displacement

iii) The bob of a conical pendulum under goes
   (A) Rectilinear motion in horizontal plane
   (B) Uniform motion in a horizontal circle
   (C) Uniform motion in a vertical circle
   (D) Rectilinear motion in vertical circle

iv) For uniform acceleration in rectilinear motion which of the following is not correct?
   (A) Velocity-time graph is linear
   (B) Acceleration is the slope of velocity time graph
   (C) The area under the velocity-time graph equals displacement
   (D) Velocity-time graph is nonlinear

v) If three particles A, B and C are having velocities $\vec{v}_A$, $\vec{v}_B$ and $\vec{v}_C$ which of the following formula gives the relative velocity of A with respect to B
   (A) $\vec{v}_A + \vec{v}_B$ (B) $\vec{v}_A - \vec{v}_C + \vec{v}_B$

(C) $\vec{v}_A - \vec{v}_B$ (D) $\vec{v}_C - \vec{v}_A$

2. Answer the following questions.

i) Separate the following in groups of scalar and vectors: velocity, speed, displacement, work done, force, power, energy, acceleration, electric charge, angular velocity.

ii) Define average velocity and instantaneous velocity. When are they same?

iii) Define free fall.

iv) If the motion of an object is described by $x = f(t)$ write formulae for instantaneous velocity and acceleration.

v) Derive equations of motion for a particle moving in a plane and show that the motion can be resolved in two independent motions in mutually perpendicular directions.

vi) Derive equations of motion graphically for a particle having uniform acceleration, moving along a straight line.

vii) Derive the formula for the range and maximum height achieved by a projectile thrown from the origin with initial velocity $\vec{u}$ at an angel θ to the horizontal.

viii) Show that the path of a projectile is a parabola.

ix) What is a conical pendulum? Show that its time period is given by $2\pi\sqrt{\dfrac{l\cos\theta}{g}}$, where l is the length of the string, θ is the angle that the string makes with the vertical and g is the acceleration due to gravity.

x) Define angular velocity. Show that the centripetal force on a particle undergoing uniform circular motion is $-m\omega^2\vec{r}$.

3. Solve the following problems.

i) An aeroplane has a run of 500 m to take off from the runway. It starts from rest and moves with constant acceleration to cover the runway in 30 sec. What is the velocity of the aeroplane at the take off ?

[Ans: 120 km/hr]

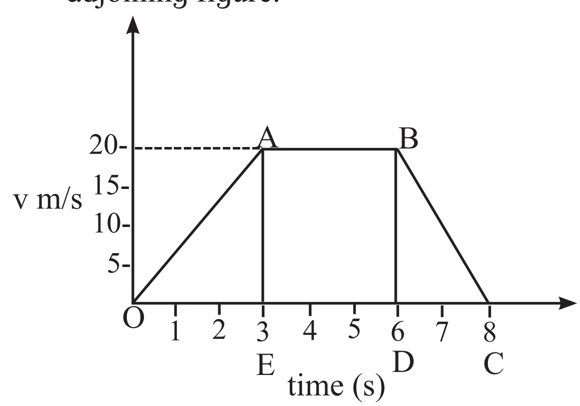
ii). A car moving along a straight road with a speed of 120 km/hr, is brought to rest by applying brakes. The car covers a distance of 100 m before it stops. Calculate (i) the average retardation of the car (ii) time taken by the car to come to rest.

[Ans: 50/9 m/sec^2, 6 sec]

iii) A car travels at a speed of 50 km/hr for 30 minutes, at 30 km/hr for next 15 minutes and then 70 km/hr for next 45 minutes. What is the average speed of the car?

[Ans: 56.66 km/hr]

iv) A velocity-time graph is shown in the adjoining figure.



Determine:

(i) initial speed of the car (ii) maximum speed attained by the car (iii) part of the graph showing zero acceleration (iv) part of the graph showing constant retardation (v) distance travelled by the car in first 6 sec.

[Ans: (i) 0 (ii) 20 m/sec (iii) AB (iv) BC (v) 90 m]

v) A man throws a ball to maximum horizontal distance of 80 meters. Calculate the maximum height reached.

[Ans: 20 m]

vi) A particle is projected with speed v_0 at angle θ to the horizontal on an inclined surface making an angle ϕ ($\phi < \theta$) to the horizontal. Find the range of the projectile along the inclined surface.

[Ans: $R = \dfrac{2v_0^2\cos\theta\sin(\theta-\phi)}{g\cos^2\phi}$]

vii) A metro train runs from station A to B to C. It takes 4 minutes in travelling from station A to station B. The train halts at station B for 20 s. Then it starts from station B and reaches station C in next 3 minutes. At the start, the train accelerates for 10 sec to reach the constant speed of 72 km/hr. The train moving at the constant speed is brought to rest in 10 sec. at next station. (i) Plot the velocity- time graph for the train travelling from the station A to B to C. (ii) Calculate the distance between the stations A, B and C.

[Ans: AB = 4.6 km, BC =3.4 km]

viii) A train is moving eastward at 10 m/sec. A waiter is walking eastward at 1.2m/sec; and a fly is flying toward the north across the waiter's tray at 2 m/s. What is the velocity of the fly relative to Earth

[Ans: 11.4 m/s, 10° due north of east]

ix) A car moves in a circle at the constant speed of 50 m/s and completes one revolution in 40 s. Determine the magnitude of acceleration of the car.

[Ans: 7.85 m s^{-2}]

x) A particle moves in a circle with constant speed of 15 m/s. The radius of the circle is 2 m. Determine the centripetal acceleration of the particle.

[Ans: 112.5 m s^{-2}]

xi) A projectile is thrown at an angle of 30° to the horizontal. What should be the range of initial velocity (u) so that its range will be between 40m and 50 m? Assume g = 10 m s^{-2}.

[Ans: $21.49 \le u \le 24.03$ m s^{-2}]

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4. Laws of Motion

Can you recall?

1. What are different types of motions?
2. What do you mean by kinematical equations and what are they?
3. Newton's laws of motion apply to most bodies we come across in our daily lives.
4. All bodies are governed by Newton's law of gravitation. Gravitation of the Earth results into weight of objects.
5. Acceleration is directly proportional to force for fixed mass of an object.
6. Bodies possess potential energy and kinetic energy due to their position and motion respectively which may change. Their total energy is conserved in absence of any external force.

4.1. Introduction:

If an object continuously changes its position, it is said to be in motion. Mechanics is a branch of Physics that deals with motion. There are basically two branches of mechanics (i) Statics, where we deal with objects at rest or in equilibrium under the action of balanced forces and (ii) Kinetics, which deals with actual motion.

Kinetics can be further divided into two branches **(i) Kinematics:** In kinematics, we describe various motions without discussing their cause. Various parameters discussed in kinematics are distance, displacement, speed, velocity and acceleration. **(ii) Dynamics:** In dynamics we describe the motion along with its cause, which is force and/or torque. Parameters discussed in dynamics are momentum, force, energy, power, etc. in addition to those in kinematics.

It must be understood that motion is strictly a relative concept, i.e., it should always be described in context to a reference frame. For example, if you are in a running bus, neither you nor your co-passengers sitting in the bus are in motion in your reference, i.e., moving bus. However, from the ground reference, bus, you and all the passengers are in motion.

If not random, motions in real life may be understood separately as linear, circular or rotational, oscillatory, etc., or some combinations of these. While describing any of these, we need to know the corresponding forces responsible for these motions. Trajectory of any motion is decided by acceleration $\vec{a}$ and the initial velocity $\vec{u}$.

a) Linear motion: Initial velocity may be zero or non-zero. If initial velocity is zero (starting from rest), acceleration in any direction will result into a linear motion.

   If initial velocity is not zero, the acceleration must be in line with the initial velocity (along the same or opposite direction to that of the initial velocity) for resultant motion to be linear.

b) Circular motion: If initial velocity is not zero and acceleration is throughout perpendicular to the velocity, the resultant motion will be circular.

c) Parabolic motion: If acceleration is constant and initial velocity is *not* in line with the acceleration, the motion is parabolic, e.g., trajectory of a projectile motion.

d) Other combinations of $\vec{u}$ and $\vec{a}$ will result into different more complicated motion.

4.2. Aristotle's Fallacy:

Aristotle (384BC-322BC) stated that "an external force is required to keep a body in uniform motion". This was probably based on a common experience like a ball rolling on a surface stops after rolling through some distance. Thus, to keep the ball moving with constant velocity, we have to continuously apply a force on it. Similar examples can be found elsewhere, like a paper plane flying through air or a paper boat propelled with some initial velocity.

Correct explanation to Aristotle's fallacy was first given by Galileo (1564-1642), which was later used by Newton (1643-1727) in formulating *laws of motion*. Galileo showed that all the objects stop moving because of some resistive or opposing forces like friction, viscous drag, etc. In these examples such forces are frictional force for rolling ball, viscous drag or viscous force of air for paper plane and viscous force of water for the boat.

Thus, in reality, for an uninterrupted motion of a body an additional external force is required for overcoming these opposing forces.

Can you tell?

1. Was Aristotle correct?
2. If correct, explain his statement with an illustration.
3. If wrong, give the correct modified version of his statement.

4.3. Newton's Laws of Motion:

First law: Every inanimate object continues to be in its state of rest or of uniform *unaccelerated* motion unless and until it is acted upon by an *external, unbalanced* force.

Second law: Rate of change of linear momentum of a rigid body is directly proportional to the applied force and takes place in the direction of the applied force. On selecting suitable units, it takes the form $\vec{F} = \frac{d\vec{p}}{dt}$ (where $\vec{F}$ is the force and $\vec{p} = m\vec{v}$ is the linear momentum.

Third law: To every action (force), there is an equal and opposite reaction (force).

Discussion: From Newton's second law of motion, $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$. For a *given* body, mass *m* is constant.

$$\therefore \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \ldots \text{(for constant mass)}$$

Thus, if $\vec{F} = 0$, $\vec{v}$ is constant. Hence if there is no force, velocity will not change. This is nothing but Newton's first law of motion.

Can you tell?

What is then special about Newton's first law if it is derivable from Newton's second law?

4.3.1. Importance of Newton's First Law of Motion:

(i) It shows an equivalence between '*state of rest*' and '*state of uniform motion along a straight line*' as both need a net unbalanced force to change the state. Both these are referred to as '*state of motion*'. The distinction between state of rest and uniform motion lies in the choice of the '*frame of reference*'.

(ii) It defines *force* as an entity (or a physical quantity) that brings about a change in the '*state of motion*' of a body, i.e., force is something that initiates a motion or controls a motion. Second law gives its quantitative understanding or its mathematical expression.

(iii) It defines *inertia* as a fundamental property of every physical object by which the object *resists* any change in its state of motion. Inertia is measured as the mass of the object. More specifically it is called inertial mass, which is the ratio of net force ($|\vec{F}|$) to the corresponding acceleration ($|\vec{a}|$).

4.3.2. Importance of Newton's Second Law of Motion:

(i) It gives mathematical formulation for quantitative measure of force as rate of change of linear momentum.

Do you know ?

Mathematical expression for force must be remembered as $\vec{F} = \frac{d\vec{p}}{dt}$ and ***not*** as $\vec{F} = m\vec{a}$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{dm}{dt}(\vec{v}) + m\left(\frac{d\vec{v}}{dt}\right)$$

$$= \frac{dm}{dt}(\vec{v}) + m\vec{a}$$

For a *given* body, mass is constant, i.e., $\frac{dm}{dt} = 0$ and only in this case, $\vec{F} = m\vec{a}$

In the case of a rocket, both the terms are needed as both mass and velocity are varying.

(ii) It defines *momentum* $\left(\vec{p} = m\vec{v}\right)$ instead of velocity as the fundamental quantity related to motion. What is changed by a force is the momentum and not necessarily the velocity.

(iii) Aristotle's fallacy is overcome by considering *resultant unbalanced* force.

4.3.3 Importance of Newton's Third Law of Motion:

(i) It defines *action* and *reaction* as a pair of equal and opposite forces acting along the same line.

(ii) Action and reaction forces are always on *different* objects.

Consequences:

Action force exerted by a body *x* on body *y*, conventionally written as $\vec{F}_{yx}$, is the force experienced by *y*.

As a result, body *y* exerts *reaction force* $\vec{F}_{xy}$ on body *x*.

In this case, body *x* experiences the force $\vec{F}_{xy}$ only while the body *y* experiences the force $\vec{F}_{yx}$ only.

Forces $\vec{F}_{xy}$ and $\vec{F}_{yx}$ are equal in magnitude and opposite in their directions, but there is no question of cancellation of these forces as those are experienced by *different* objects.

Forces $\vec{F}_{xy}$ and $\vec{F}_{yx}$ *need not* be contact forces. Repulsive forces between two magnets is a pair of action-reaction forces. In this case the two magnets are not in contact. Gravitational force between Earth and moon or between Earth and Sun are also similar pairs of non-contact action-reaction forces.

Example 4.1: A hose pipe used for gardening is ejecting water horizontally at the rate of 0.5 m/s. Area of the bore of the pipe is 10 cm^2. Calculate the force to be applied by the gardener to hold the pipe *horizontally* stationary.

Solution: If ejecting water horizontally is considered as action force on the water, the water exerts a backward force (called recoil force) on the pipe as the reaction force.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

As v, the velocity of ejected water is constant, $F = \frac{dm}{dt}\vec{v}$, where $\frac{dm}{dt}$ is the rate at which mass of water is ejected by the pipe.

As the force is in the direction of velocity (horizontal), we can use scalars. $\therefore F = \frac{dm}{dt}\mathrm{v}$

$$\frac{dm}{dt} = \frac{d\left(V\rho\right)}{dt} = \frac{d\left(Al\rho\right)}{dt} = A\rho\frac{dl}{dt} = A\rho\mathrm{v}$$

where V = volume of water ejected

A = area of cross section of bore = 10 cm^2

ρ = density of water = 1 g/cc

l = length of the water ejected in time t

$\frac{dl}{dt} = \mathrm{v} =$ velocity of water ejected

$= 0.5$ m/s $= 50$ cm/s

$$F = \frac{dm}{dt}\mathrm{v} = \left(A\rho\mathrm{v}\right)\mathrm{v} = A\rho\mathrm{v}^2 = 10\times1\times50^2$$

$= 25000$ dyne $= 0.25$ N

Equal and opposite force must be applied by the gardener.

4.4. Inertial and Non-Inertial Frames of Reference

Consider yourself standing on a railway platform or a bus stand and you see a train or bus moving. According to you, that train or bus is moving or is in motion. As per the experience of the passengers in the train or bus, they are at rest and you are moving (in backward direction). Hence motion itself is a relative concept. To know or describe a motion you need to describe or define some reference. Such a reference is called a *frame of reference*. In the example discussed above, if you consider the platform as the reference, then the passengers and the train are moving. However, if the train is considered as the reference, you and platform, etc. are moving.

Usually a set of coordinates with a suitable origin is enough to describe a frame of reference. If position coordinates of an object are continuously changing with time in a frame of reference, then *that object* is in motion in *that* frame of reference. Any frame of reference in which Newton's first law of motion is applicable is the simplest understanding of an inertial frame of reference. It means, if there is no net force, there is no acceleration. Thus in an inertial frame, a body will move with constant velocity (which may be zero also) if there is no net force acting upon it. In the absence of a net force, if an object suffers an acceleration, that frame of reference is ***not*** an inertial frame and is called as non-inertial frame of reference.

Measurements in one inertial frame can be converted to measurements in another inertial frame by a simple transformation, i.e., by simply using some velocity vectors (relative velocity between the two frames of reference).

Illustration: Imagine yourself inside a car with all windows opaque so that you can not see anything outside. Also consider that there is a pendulum tied inside the car and not set into oscillations . If the car just starts its motion (with reference to outside or ground), you will experience a jerk, i.e., acceleration inside the car even though there is no force acting upon you. During this time, the string of the pendulum may be steady, but ***not*** vertical. During time of acceleration, the car can be considered to be a non-inertial frame of reference. Later on if the car is moving with constant velocity (with reference to the ground), you will not experience any jerky motion within the car and the car can be considered as an inertial frame of reference. In this case, the pendulum string will be vertical, when not oscillating.

Do you know ?

The situations/phenomena that can be explained using Newton's laws of motion fall under Newtonian mechanics. So far as our daily life situations are considered, Newtonian mechanics is perfectly applicable. However, under several extreme conditions we need to use some other theories.

Limitation of Newton's laws of motion

(i) Newton's laws are applicable only in the inertial frames of reference (discussed later). If the body is in a frame of reference of acceleration (a), we need to use a *pseudo* force $\left(-m\vec{a}\right)$ in addition to all the other forces while writing the force equations.

(ii) Newton's laws are applicable for *point* objects.

(iii) Newton's laws are applicable to rigid bodies. A body is said to be *rigid* if the relative distances between its particles do not change for any deforming force.

(iv) For objects moving with speeds comparable to that of light, Newton's laws of motion do not give results that match with the experimental results and Einstein's special theory of relativity has to be used.

(v) Behaviour and interaction of objects having atomic or molecular sizes cannot be explained using Newton's laws of motion, and quantum mechanics has to be used.

A rocket in intergalactic space (gravity free space between galaxies) with all its engines shut is closest to an ideal inertial frame. However, Earth's acceleration in the reference frame of the Sun is so small that any frame attached to the Earth can be used as an inertial frame for any day-to-day situation or in our laboratories.

4.5 Types of Forces:

4.5.1. Fundamental Forces in Nature:

All the forces in nature are classified into following **four** interactions that are termed as fundamental forces.

(i) Gravitational force: It is the attractive force between two (point) masses separated by a distance. Magnitude of gravitational force between point masses m_1 and m_2 separated by distance r is given by $F = \frac{Gm_1m_2}{r^2}$

where G = 6.67×10^{-11} SI units. Between two point masses (particles) separated by a given distance, this is the weakest force having infinite range. This force is always attractive. Structure of the universe is governed by this force.

Common experience of this force for us is gravitational force exerted by Earth on us, which we call as our weight W.