(4) The Master Method

→ The master method is used for solving the following type of recurrence

$T(n) = aT(\frac{n}{b}) + f(n)$ with $a \geq 1$ and $b > 1$

→ In this, problem is divided into 'a' subproblems, each of size n/b where a and b are positive constants.

→ The cost of dividing the problem and combining the results of the subproblem is described by the function $f(n)$.

Here, 'n' is the problem size.

The Master Theorem

→ Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence.

$T(n) = aT(n/b) + f(n)$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

Then $T(n)$ can be bounded asymptotically as follows.

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and If $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$\therefore a.f(n/b) \leq c \cdot f(n)$ is called the "regularity" condition.

→ **Example 1:** $T(n) = 4T(n/2) + n$

$a = 4, \; b = 2, \; f(n) = n$

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$\Rightarrow f(n) = O(n^{\log_2 4 - \epsilon})$

$\Rightarrow f(n) = O(n^{2 - \epsilon})$

$\Rightarrow f(n) = O(n^{2-1})$

$\Rightarrow f(n) = O(n)$ (proved)

Hence, we can apply case 1 and the solution is

$T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 4}) = \Theta(n^2)$

→ **Example 2:** $T(n) = 4T(n/2) + n^2$.

$a = 4, \; b = 2, \; f(n) = n^2$

Apply case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$\Rightarrow f(n) = O(n^{\log_2 4 - \epsilon})$

$\Rightarrow f(n) = O(n^{2 - \epsilon})$

$\Rightarrow f(n) = O(n^{2-1}) = O(n)$ (Not satisfied)

Now apply case 2: $f(n) = \theta(n^{\log_b a})$

$\Rightarrow f(n) = \theta(n^{\log_2 4})$

$\Rightarrow f(n) = \theta(n^2)$ (Satisfied)

The solution is:

$T(n) = \theta(n^{\log_b a} \log n)$

$\Rightarrow T(n) = \theta(n^{\log_2 4} \cdot \log n)$

$\Rightarrow T(n) = \theta(n^2 \log n)$

→ Example 3: $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2,\ f(n) = n^3$

Apply case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$\Rightarrow f(n) = O(n^{\log_2 4 - \epsilon})$

$\Rightarrow f(n) = O(n^{2-\epsilon})$

$\Rightarrow f(n) = O(n^{2-1}) = O(n)$ (Not satisfied)

Now apply case 2:

$f(n) = \theta(n^{\log_b a})$

$\Rightarrow f(n) = \theta(n^{\log_2 4})$

$\Rightarrow f(n) = \theta(n^2)$ (Not satisfied.)

Now apply case 3:

$f(n) = \Omega(n^{\log_b a + \epsilon})$

$\Rightarrow f(n) = \Omega(n^{\log_2 4 + \epsilon})$

$\Rightarrow f(n) = \Omega(n^{2+\epsilon})$

$\Rightarrow f(n) = \Omega(n^{2+1})$

$\Rightarrow f(n) = \Omega(n^3)$ ~~(Satisfied)~~ (Satisfied)

Condition 1 satisfied. Now check the 2nd condition.

$a \cdot f(n/b) \leq c \cdot f(n)$

$\Rightarrow 4 \cdot (n/2)^3 \leq c \cdot n^3$

$\Rightarrow \cancel{4} \cdot \frac{n^3}{\cancel{8}\,2} \leq c \cdot n^3$

$\Rightarrow n^3/2 \leq c \cdot n^3$ (Satisfied)

Condition 2 satisfied. Both the conditions are satisfied.

The solution is:

$T(n) = \theta(n^3)$

→ **Example 4:** Solve the recurrence using master method

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

$a = 9,\ b = 3,\ f(n) = n$

Apply case 1: $f(n) = O(n^{\log_b a - \epsilon})$

$\Rightarrow f(n) = O(n^{\log_3 9 - \epsilon})$

$\Rightarrow f(n) = O(n^{2-\epsilon})$

$\Rightarrow f(n) = O(n^{2-1}) = O(n)$ (satisfied)

The solution is :

$$T(n) = \Theta(n^{\log_b a})$$

$$\Rightarrow T(n) = \Theta(n^{\log_3 9})$$

$$\Rightarrow T(n) = \Theta(n^2)$$

→ **Example 5:** Solve $T(n) = T\left(\frac{2n}{3}\right) + 1$ by master method.

Solution: $a = 1,\ b = 3/2,\ f(n) = 1$

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$

$\Rightarrow f(n) = n^{\log_b a} = 1$ (satisfied)

Hence, case 2 applies.

The solution is : $T(n) = \Theta(n^{\log_b a} \lg n)$

$\Rightarrow T(n) = \Theta(1 . \lg n)$

$\Rightarrow T(n) = \Theta(\lg n)$

→ **Example 6:** Solve the recurrence $T(n) = 3T(n/4) + n\lg n$

Solution: $a = 3,\ b = 4,\ f(n) = n\lg n$

$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$

$f(n) = \Omega(n^{\log_4 3 + \epsilon})$ where $\epsilon = 0.2$

case 3 applies, Now for regularity condition

$a \cdot f(n/b) \leq c \cdot f(n)$

$\Rightarrow 3 \cdot f(n/4) \leq c \cdot f(n)$

$\Rightarrow 3 \cdot (n/4) \log(n/4) \leq c \cdot n \log n$

$\Rightarrow 3n/4 \log(n/4) \leq \frac{3n}{4} \log n$ ($\because c = 3/4$)

The solution is :

$$T(n) = \Theta(n \log n)$$

→ **Example 7:** Solve $T(n) = 16T(n/4) + n^3$

Solution: $a = 16$, $b = 4$, $f(n) = n^3$

$$n^{\log_b a} = n^{\log_4 16} = n^2$$

$$f(n) = n^3 = n^{\log_4 16 + \epsilon} = n^{2+1} = n^3 \quad (\because \epsilon = 1)$$

Now check regularity condition

$$16 f\left(\frac{n^3}{4^3}\right) \leq c \cdot f(n^3)$$

$$\Rightarrow 16 \cdot \frac{n^3}{64} \leq c \cdot n^3$$

$$\Rightarrow n^3/4 \leq cn^3$$ for $c = 1/4$ regularity cond^n holds.

Hence case 3 is satisfied. So, the solution is:

$$T(n) = \theta(f(n))$$

$$\Rightarrow T(n) = \theta(n^3)$$