(2) Iteration Method:

→ In iteration method the basic idea is to expand the recurrence and express it as a summation of terms dependent only on 'n' and the initial conditions.

→ **Example 1:** Consider the recurrence $T(n) = 3T(\frac{n}{4}) + n$

Solution: We iterate it as follows:

$$T(n) = n + 3T(n/4) \quad \text{--- (i)}$$

$$T(n/4) = n/4 + 3T(n/16) \quad \text{--- (ii)}$$

$$T(n/16) = n/16 + 3T(n/64) \quad \text{--- (iii)}$$

put eq^n (ii) in eq^n (i), we get

$$T(n) = n + 3[n/4 + 3T(n/16)] \quad \text{--- (iv)}$$

put eq^n (iii) in eq^n (iv), we get

$$T(n) = n + 3[n/4 + 3[n/16 + 3T(n/64)]]$$

$$\Rightarrow T(n) = n + 3(n/4) + 9(n/16) + 27T(n/64)$$

$$\Rightarrow T(n) \leq n + \frac{3n}{4} + \frac{9n}{16} + \cdots + 3^i T(n/4^i)$$

The series terminates when $\frac{n}{4^i} = 1 \Rightarrow n = 4^i$

or $i = \log_4 n$

$$T(n) \leq n + \frac{3n}{4} + \frac{9n}{16} + \frac{27n}{64} + \cdots + 3^{\log_4 n} T(1)$$

$$\Rightarrow T(n) \leq n + \frac{3n}{4} + \frac{9n}{16} + \frac{27n}{64} + \cdots + 3^{\log_4 n} \Theta(1)$$

$$\Rightarrow T(n) \leq n\left(1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots\right) + \Theta(n^{\log_4 3})$$

as $3^{\log_4 n} = n^{\log_4 3}$

$$\Rightarrow T(n) \leq n \cdot \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i + \Theta(n^{\log_4 3})$$

$$\Rightarrow T(n) \leq n \cdot \frac{1}{1-\frac{3}{4}} + O(n) \quad \text{as } \log_4 3 < 1$$

i.e. $\Theta(n^{\log_4 3}) = O(n)$

$$\Rightarrow T(n) \leq 4n + O(n)$$

$$\Rightarrow T(n) = O(n)$$

→ **Example 2:** $T(n) = T(n/2) + 1$. Show the recurrence is bounded by $O(\log n)$

Solution:

$$T(n) = T(n/2) + 1 \quad \text{--- (1)}$$

$$T(n/2) = T(n/4) + 1 \quad \text{--- (2)}$$

$$T(n/4) = T(n/8) + 1 \quad \text{--- (3)}$$

put eq^n (2) in eq^n (1) and we get

$T(n) = T(n/4) + 1 + 1$ — ④

put eqⁿ ③ in eqⁿ ④, we get

$T(n) = T(n/8) + 1 + 1 + 1$

$\Rightarrow T(n) = T(n/2^3) + 1 + 1 + 1$

$\Rightarrow T(n) = T(n/2^k) + 1 + 1 \cdots k$ times

$\Rightarrow T(n) - T(n/2^k) = 1 + 1 + \cdots + k$ times $= k$

$\Rightarrow T(n) - T(n/2^k) = k$

Assume $n = 2^k$

Taking log both the sides, we get

$\log n = k \log 2^k = k \log 2$

$\Rightarrow \log_2 n = k \quad (\because \log 2 = 1)$

$\Rightarrow T(n) - T(2^k/2^k) = k$

$\Rightarrow T(n) - T(1) = k$

$\Rightarrow T(n) = T(1) + k$

put $k = \log_2 n$

$\Rightarrow T(n) = 1 + \log_2 n \quad (\because T(1) = 1)$

$\Rightarrow 1 + \log_2 n \leq 2 \log_2 n$

So, $T(n) = O(\log_2 n)$

→ **Example 3:** Consider the recurrence $T(n) = T(n-1) + 1$ and $T(1) = \Theta(1)$. Solve i

Solution! $T(n) = T(n-1) + 1$

$T(n-1) = T(n-2) + 1 + 1$

$T(n-2) = T(n-3) + 1 + 1 + 1 = T(n-3) + 3$

$T(n-3) = T(n-4) + 1 + 1 + 1 + 1 = T(n-4) + 4$

$T(n-4) = T(n-5) + 5$

$\cancel{T(n-(k-1))}\ T(n) = T(n-k) + k \Rightarrow T(n-(k-1)) = T(n-k) + k$

where $k = n-1$ $\qquad \Rightarrow T(n-n+1+1) = T(n-n+1) + n - 1$

$\cancel{T(n-k) = T(n-(k+1)) + (k+1)}$ $\Rightarrow T(2) = T(1) + n - 1$

$\cancel{= T(n-(n-k+1)) + n-1+k}$

$T(n-k) = T(1) = \Theta(1)$

$T(n) = \Theta(1) + k$

$= \Theta(1) + (n-1) = 1 + n - 1 = n$

$T(n) = \Theta(n)$

$T(n) = T(n-n+1) + n - 1$ $\qquad T(n-(k-1)) = T(n-k) + k$

$T(n) = T(1) + n - 1$ $\qquad T(n-k+1) = T(n-k) + k$

$\Rightarrow \cancel{T(1) = T(1) = \Theta(1)}$ $\qquad \Rightarrow T(n-(n-1)+1) = T(n-k) + k$

$\Rightarrow T(n) = \Theta(1) + n - 1$ $\qquad \Rightarrow T(n-n+1+1) = T(n-k) + k$

$\boxed{T(n) = \Theta(n)}$