Defuzzification Techniques

What is defuzzification?

- Defuzzification means the fuzzy to crisp conversion.

- **Example 1:**

  Suppose, T_{HIGH} denotes a fuzzy set representing **temperature is High.**

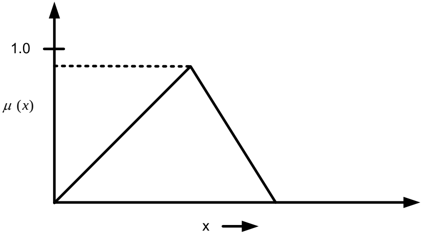
  T_{HIGH} is given as follows.

  T_{HIGH} = (15,0.1), (20, 0.4), (25,0.45), (30,0.55), (35,0.65), (40,0.7), (45,0.85),(50,0.9)

- What is the crisp value that implies for the high temperature?

Example 2: Fuzzy to crisp

As an another example, let us consider a fuzzy set whose membership finction is shown in the following figure.



What is the crisp value of the fuzzy set in this case?

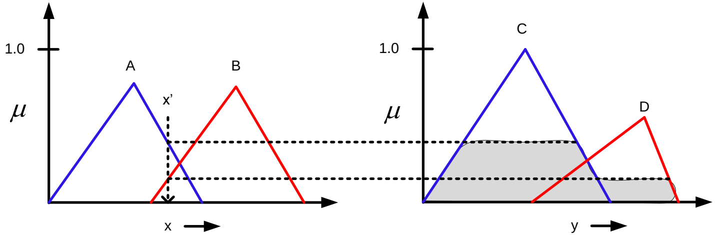
Example 3: Fuzzy to crisp

Now, consider the following two rules in the fuzzy rule base.

R1: If x **is** A then y **is** C

R2: If x **is** B then y **is** D

A pictorial representation of the above rule base is shown in the following figures.



What is the crisp value that can be inferred from the above rules given an input say x'?

Why defuzzification?

The fuzzy results generated can not be used in an application, where decision has to be taken only on crisp values.

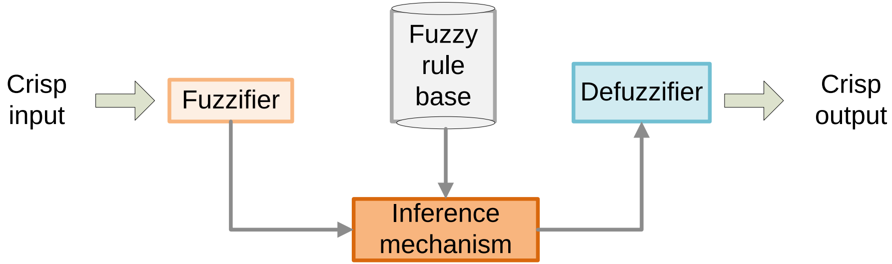
Example:

$$\text{If } T_{HIGH} \text{ then rotate } R_{FIRST}.$$

Here, may be input T_{HIGH} is fuzzy, but action **rotate** should be based on the crisp value of R_{FIRST}.

Generic structure of a Fuzzy system

Following figures shows a general fraework of a fuzzy system.



Defuzzification Techniques

Defuzzification methods

A number of defuzzification methods are known. Such as

1. **Lambda-cut method**
2. **Weighted average method**
3. **Maxima methods**
4. **Centroid methods**

Lambda-cut method

Lambda-cut method

Lmabda-cut method is applicable to derive crisp value of a fuzzy set or relation. Thus

- Lambda-cut method for fuzzy set
- Lambda-cut method for fuzzy relation

In many literature, Lambda-cut method is also alternatively termed as *Alph-cut method*.

Lamda-cut method for fuzzy set

1. In this method a fuzzy set A is transformed into a crisp set A_λ for a given value of λ $(0 \leq \lambda \leq 1)$

2. In other-words, $A_\lambda = \{x | \mu_A(x) \geq \lambda\}$

3. That is, the value of Lambda-cut set A_λ is x, when the membership value corresponding to x is greater than or equal to the specified λ.

4. This Lambda-cut set A_λ is also called alpha-cut set.

Lambda-cut for a fuzzy set : Example

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

Then $A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$

and

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$

Lambda-cut sets : Example

Two fuzzy sets P and Q are defined on x as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

(a) $P_{0.2}$, $Q_{0.3}$

(b) $(P \cup Q)_{0.6}$

(c) $(P \cup \overline{P})_{0.8}$

(d) $(P \cap Q)_{0.4}$

Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.
Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of $\lambda = 0, 0.2, 0.9, 0.5$

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Some properties of λ-cut sets

If A and B are two fuzzy sets, defined with the same universe of discourse, then

1. $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$
2. $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
3. $(\overline{A})_\lambda \neq \overline{A}_\lambda$ except for value of $\lambda = 0.5$
4. For any $\lambda \leq \alpha$, where α varies between 0 and 1, it is true that $A_\alpha \subseteq A_\lambda$, where the value of A_0 will be the universe of discourse.

Some properties of λ-cut relations

If R and S are two fuzzy relations, defined with the same fuzzy sets over the same universe of discourses, then

5. $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$
6. $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
7. $(\overline{R})_\lambda \neq \overline{R}_\lambda$
8. For $\lambda \leq \alpha$, where α between 0 and 1 , then $R_\alpha \subseteq R_\lambda$

Summary: Lambda-cut methods

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into crisp set (or relation).

Output of a Fuzzy System

Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with n-rules.

$$
\begin{array}{l}
R_1\text{: If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\
R_2\text{: If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
R_n\text{: If } x \text{ is } A_n \text{ then } y \text{ is } B_n
\end{array}
$$

In this case, the output y for a given input $x = x_1$ is possibly B = $B_1 \cup B_2 \cup .....B_n$

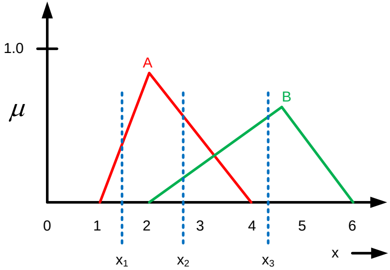
Output fuzzy set : Illustration

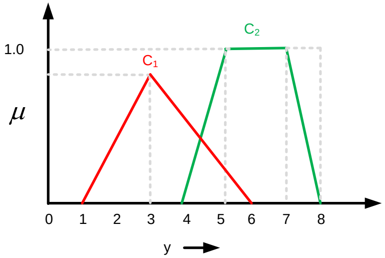
Suppose, two rules R_1 and R_2 are given as follows:

1. R_1: If x is A_1 then y is C_1
2. R_2: If x is A_2 then y is C_2

Here, the output fuzzy set $C = C_1 \cup C_2$.

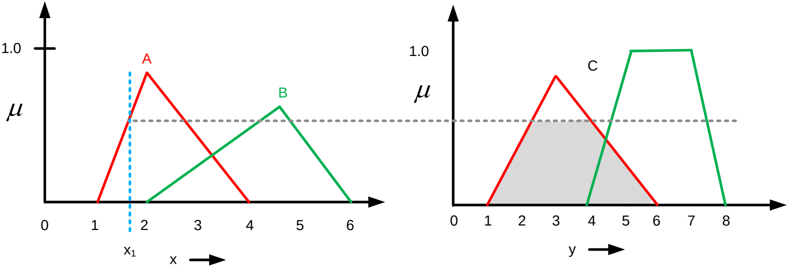
For instance, let us consider the following:





Output fuzzy set : Illustration

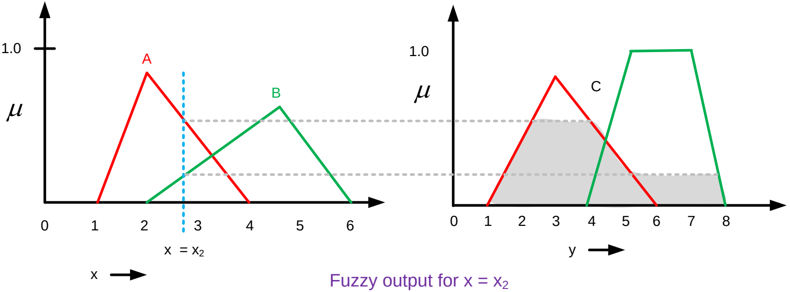
The fuzzy output for $x = x_1$ is shown below.



Fuzzy output for $x = x_1$

Output fuzzy set : Illustration

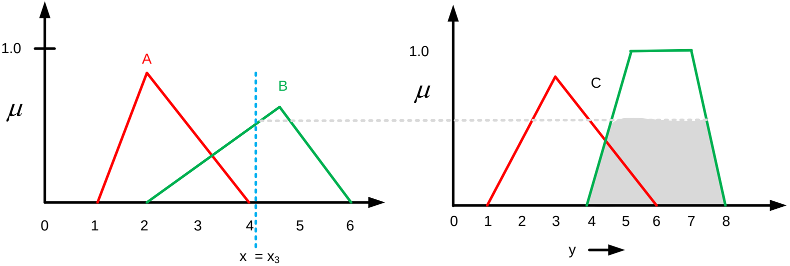
The fuzzy output for $x = x_2$ is shown below.



Fuzzy output for x = x_2

Output fuzzy set : Illustration

The fuzzy output for $x = x_3$ is shown below.



Fuzzy output for x = x_3

Defuzzification Methods

Following defuzzification methods are known to calculate crisp output in the situations as discussed in the last few slides

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Defuzzification Technique
Maxima Methods

Maxima methods

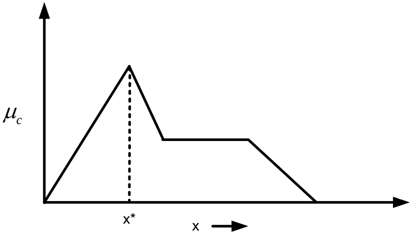
Following defuzzification methods are known to calculate crisp output.

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Maxima method : Height method

This method is based on Max-membership principle, and defined as follows.

$$\mu_C(x^*) \geq \mu_C(x) \text{ for all } x \in X$$

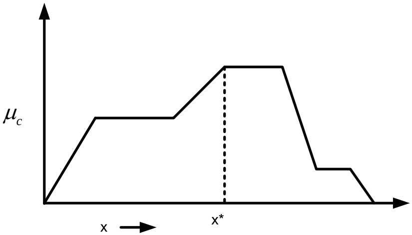


Note:

1. Here, x^* is the height of the output fuzzy set C.

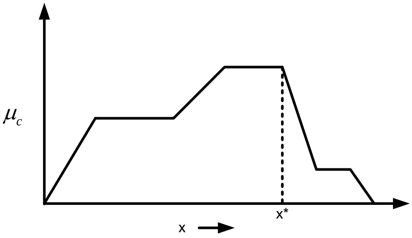
2. This method is applicable when height is unique.

FoM: First of Maxima : $x^* = min\{x | C(x) = max_w C\{w\}\}$



Maxima method : LoM

LoM : Last of Maxima : $x^* = max\{x|C(x) = max_w C\{w\}\}$



Maxima method : MoM

$$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where, $M = \{x_i | \mu(x_i) = h(C)\}$ where $h(C)$ is the height of the fuzzy set C

MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

Young = $\{$(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3) $\}$

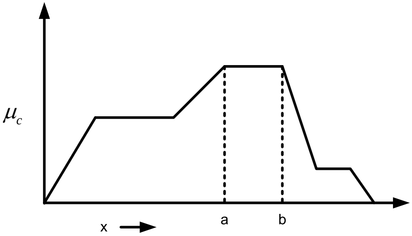
Then the crisp value of **Young** using MoM method is

$x^* = \frac{20+25}{2} = 22.5$

Thus, a person of 22.5 years old is treated as young!

MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



$$x^* = \frac{a+b}{2}$$

Note:

- Thus, MoM is also synonymous to middle of maxima.
- MoM is also general method of Height.

Defuzzification Technique
Centroid Methods

Cenroid methods

Following defuzzification methods are known to calculate crisp output.

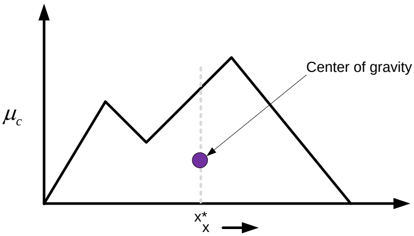
- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

Centroid method : CoG

1. The basic principle in CoG method is to find the point x^* where a vertical line would slice the aggregate into two equal masses.
2. Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$

3. Graphically,



Centroid method : CoG

Note:

1. x^* is the x-coordinate of center of gravity.
2. $\int \mu_C(x)dx$ denotes the area of the region bounded by the curve μ_C.
3. If μ_C is defined with a discrete membership function, then CoG can be stated as :

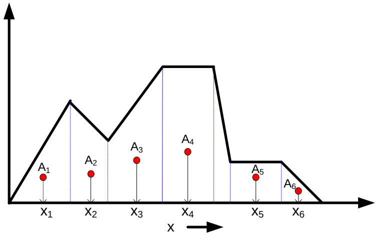
$$x^* = \frac{\sum_{i=1}^{n} x_i.\mu_C(x_i)}{\sum_{i=1}^{n} \mu_C(x_i)} \;;$$

4. Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

CoG : A geometrical method of calculation

Steps:

1. Divide the entire region into a number of small regular regions (e.g. triangles, trapizoid etc.)

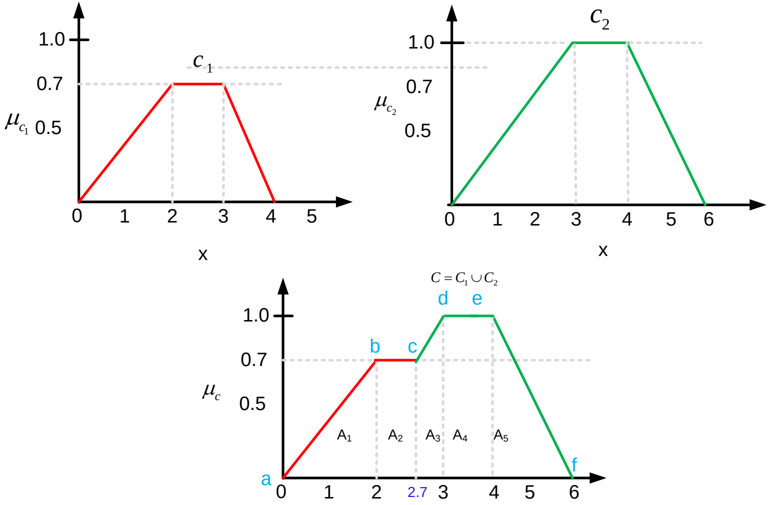


2. Let A_i and x_i denotes the area and c.g. of the i-th portion.
3. Then x^* according to CoG is

$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

where n is the number of smaller geometrical components.

CoG: An example of integral method of calculation



CoG: An example of integral method of calculation

$$\mu_c(x) = \begin{cases} 0.35x & 0 \leq x < 2 \\ 0.7 & 2 \leq x < 2.7 \\ x - 2 & 2.7 \leq x < 3 \\ 1 & 3 \leq x < 4 \\ (-0.5x + 3) & 4 \leq x \leq 6 \end{cases}$$

For $A_1 : y - 0 = \frac{0.7}{2}(x - 0)$, or $y = 0.35x$

For $A_2 : y = 0.7$

For $A_3 : y - 0 = \frac{1-0}{3-2}(x - 2)$, or $y = x - 2$

For, $A_4 : y = 1$

For, $A_5 : y - 1 = \frac{0-1}{6-4}(x - 4)$, or $y = -0.5x + 3$

CoG: An example of integral method of calculation

Thus, $x^* = \frac{\int x.\mu_c(x)dx}{\int \mu_c(x)dx} = \frac{N}{D}$

$N = \int_0^2 0.35x^2dx + \int_2^{2.7} 0.7x^2dx + \int_{2.7}^3 (x^2 - 2x)dx + \int_3^4 xdx + \int_4^6 (-0.5x^2 + 3x)dx$

= 10.98

$D = \int_0^2 0.35xdx + \int_2^{2.7} 0.7xdx + \int_{2.7}^3 (x - 2)dx + \int_3^4 dx + \int_4^6 (-0.5x + 3)dx$

= 3.445

Thus, $x^* = \frac{10.98}{3.445} = 3.187$

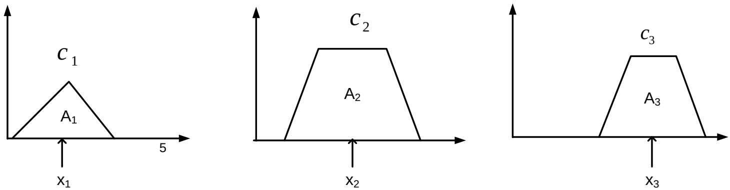
Centroid method : CoS

If the output fuzzy set $C = C_1 \cup C_2 \cup ....C_n$, then the crisp value according to CoS is defined as

$$x^* = \frac{\sum_{i=1}^{n} x_i.A_{c_i}}{\sum_{i=1}^{n} A_{c_i}}$$

Here, A_{c_i} denotes the area of the region bounded by the fuzzy set C_i and x_i is the geometric center of the area A_{c_i}.

Graphically,

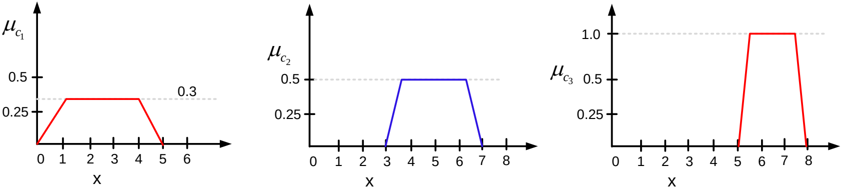


Note:

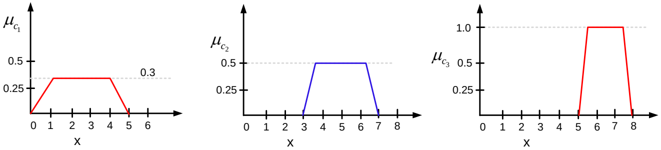
1. In CoG method, the overlapping area is counted once, whereas, in CoS , the overlapping is counted twice or so.
2. In CoS, we use the center of area and hence, its name instead of center of gravity as in CoG.

CoS: Example

Consider the three output fuzzy sets as shown in the following plots:



CoS: Example



In this case, we have

$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3 + 5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4 + 2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1 \times (3 + 1), x_3 = 6.5$$

Thus, $x^* = \frac{\frac{1}{2}\times0.3\times(3+5)\times2.5+\frac{1}{2}\times0.5\times(4+2)\times5+\frac{1}{2}\times1\times(3+1)\times6.5}{\frac{1}{2}\times0.3\times(3+5+\frac{1}{2}\times0.5\times(4+2)+\frac{1}{2}\times1\times(3+1)} = 5.00$

Note:

The crisp value of $C = C_1 \cup C_2 \cup C_3$ using CoG method can be found to be calculated as $x^* = 4.9$

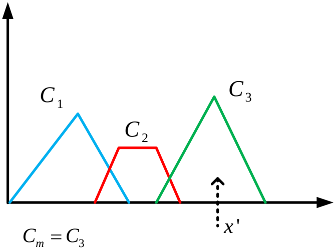
Centroid method: Certer of largest area

If the fuzzy set has two subregions, then the center of gravity of the subregion with the largest area can be used to calculate the defuzzified value.

Mathematically, $x^* = \frac{\int \mu_{c_m}(x).x' dx}{\int \mu_{c_m}(x) dx}$;

Here, C_m is the region with largest area, x' is the center of gravity of C_m.

Graphically,



Weighted Average Method

Cenroid methods

Following defuzzification methods are known to calculate crisp output.

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods**
  1. Center of gravity method (CoG)
  2. Center of sum method (CoS)
  3. Center of area method (CoA)
- **Weighted average method**

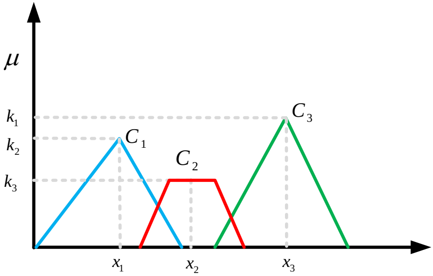
Weighted average method

1. This method is also alternatively called ”Sugeno defuzzification” method.
2. The method can be used only for symmetrical output membership functions.
3. The crisp value accroding to this method is

$$x^* = \frac{\sum_{i=1}^{n} \mu_{C_i}(x_i).(x_i)}{\sum_{i=1}^{n} \mu_{C_i}(x_i)}$$

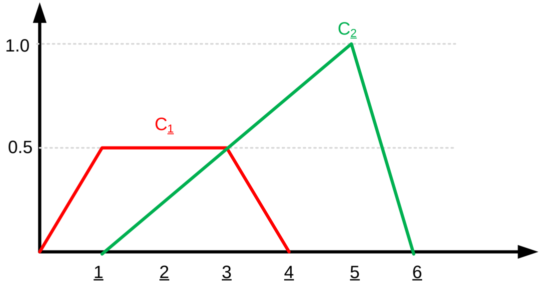
where, $C_1, C_2, ...C_n$ are the output fuzzy sets and (x_i) is the value where middle of the fuzzy set C_i is observed.

Graphically,



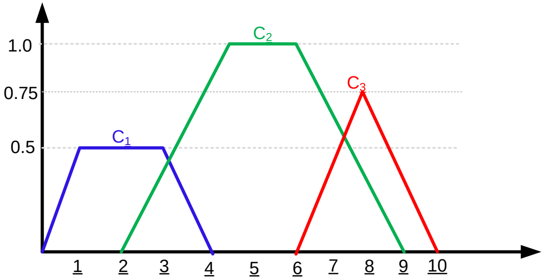
Exercise 1

Find the crisp value of the following using all defuzzified methods.



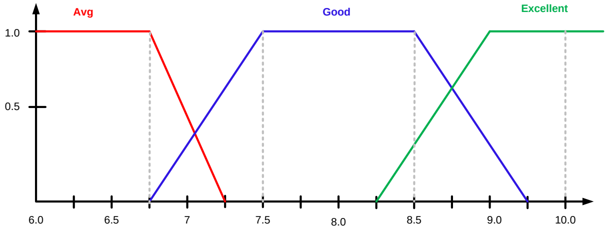
Exercise 1

Find the crisp value of the following using all defuzzified methods.



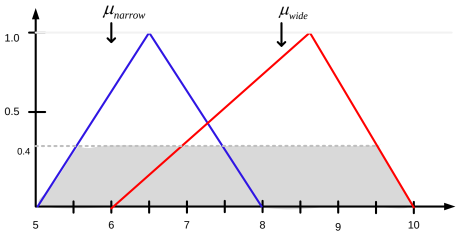
Exercise 3

- The membership function defining a student as Average, Good, and Excellent denoted by respective membership functions are as shown below.



- Find the crisp value of ”Good Student”
- Hint: Use CoG method to the portion ”Good” to calculate it.

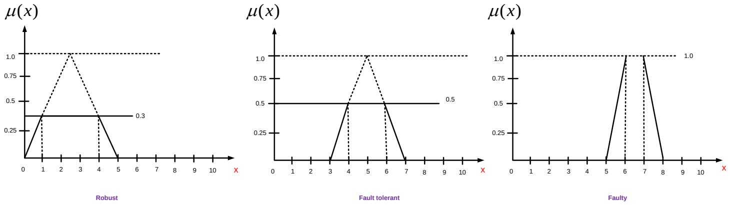
Exercise 4



- The width of a road as narrow and wide is defined by two fuzzy sets, whose membership functions are plotted as shown above.
- If a road with its degree of membership value is 0.4 then what will be its width (in crisp) measure.
- Hint: Use CoG method for the shadded region.

Exercise 5

- The faulty measure of a circuit is defined fuzzily by three fuzzy sets namely Faulty(F), Fault tolerant (FT) and Robust(R) defined by three membership functions with number of faults occur as universe of discourses and is shown below.



- Reliability is measured as $R^* = F \cup FT \cup R$.
  With a certain observation in testing
  $(x, 0.3) \in R, (x, 0.5) \in FT, (x, 0.8) \in F$.
- Calculate the reliability measure in crisp value.
- Calculate with 1) CoS 2) CoG .

Any questions??