(1) SUBSTITUTION METHOD :

→ In substitution method, we guess a bound or answer of the problem and then use mathematical induction to prove that our guess is correct.

→ It involves guessing the form of the solution and then using mathematical induction to find the constants and show that the solution works. This method can be applied only when it is easy to guess the form of the answer.

→ The substitution method can be used to establish either upper or lower bounds on a recurrence.

→ **Example 1:** Consider the recurrence $T(n) = 2T(n/2) + n$
we have to show that it is asymptotically bound by $O(n\log n)$

Solution: The recurrence is $T(n) = 2T(n/2) + n$ —— ①

Let guess the solution of recurrence is

$$T(n) = O(n\log n)$$

Here, we have to prove that $T(n) \leq c\,n\log n$ for an appropriate choice of the constant $c > 0$.

$$T(n) \leq c n \log n$$

$$T(n/2) \leq c(n/2)\log(n/2) \quad —— ② \qquad \text{(By Induction)}$$

put eqn ② on eqn ①

$$T(n) \leq 2\left(c\cdot\left(\tfrac{n}{2}\right)\log(n/2)\right) + n$$

$$\leq cn\log(n/2) + n$$

$$= cn\log n - cn\log 2 + n$$

$$= cn\log n - cn + n \quad (\because \log 2 = 1)$$

$$T(n) \leq cn\log n, \quad c \geq 1$$

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By mathematical induction, we have to show that our solution holds for the boundary conditions:

$T(n) \leq cn\log n$

Put $n=1$, $T(1) \leq c \cdot 1 \cdot \log 1 = 0$ $(\because \log 1 = 0)$

Thus for any value of c, this will not hold.

By asymptotic definition, we need to prove

$T(n) \leq cn\log n$ for $n \geq n_0$

$T(2) \leq c \cdot 2 \cdot \log 2$

$T(3) \leq c \cdot 3 \cdot \log 3$

Here, n can't be 1.

But n can be 2 or 3.

Thus we can replace $T(1)$ by $T(2)$ and $T(3)$ as the base case.

The relation holds for $c \geq 2$. Thus $T(n) \leq cn\log n$ holds

Hence our guess $T(n) = O(n\log n)$ is true.

→ Example 2: Consider the recurrence $T(n) = T(\lfloor \frac{n}{2} \rfloor) + 1$

we have to show that it is asymptotically bound by $O(\log n)$

Solution: $T(n) = O(\log n)$

$T(n) \leq c \cdot \log n$

$T(n/2) \leq c \cdot \log n/2$

Now, the recurrence relation becomes

$T(n) \leq c \cdot \log(\lfloor \frac{n}{2} \rfloor) + 1$

$\leq c \cdot \log(n/2) + 1 = c\log n - c\log_2 2 + 1 = c\log n - c + 1$

$\leq c\log n$ for $c \geq 1$

$T(n) = O(\log n)$

→ Example 3: Show that solution of $T(n) = 2T(n/2) + n$ is $O(n^2)$

Solution: $T(n) = O(n^2)$

$T(n) \leq c \cdot n^2$

$T(n/2) \leq c(n/2)^2$ — ①

~~But then~~ put eqn ① in recurrence relation

$T(n) \leq 2 \cdot c(n/2)^2 + n$

$\Rightarrow T(n) \leq \frac{cn^2}{2} + n$

$T(n) \leq n^2$

$T(n) = O(n^2)$

→ Example 4: Consider the recurrence $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 16) + n$

we have to show that it is asymptotically bound by $O(n\log n)$

Solution:
$$T(n) = O(n\log n)$$
$$T(n) \leq c \cdot n \log n$$
$$T(n/2) \leq c \cdot (\tfrac{n}{2}) \log (\tfrac{n}{2}) \quad \text{— ①}$$

put eqn ① in recurrence relation

$$T(n) \leq 2 \cdot \cancel{\left[c\left(\lfloor \tfrac{n}{2} \rfloor + 16\right) \log \left(\lfloor \tfrac{n}{2} \rfloor + 16 \right) \right] + n} \; 2\left\{ c \cdot \tfrac{n}{2} \log (\tfrac{n}{2}) + 16 \right\} + n$$

$$= cn \log (\tfrac{n}{2}) + 32 + n = cn\log n - cn \log 2 + 32 + n$$

$$\cancel{T(n) \leq (cn + 32) \log \tfrac{n}{2}}$$

$$= cn \log n - cn + 32 + n = cn \log n - (c-1)n + 32$$
$$\leq cn \log n \quad (\text{for } c \geq 1)$$

$$\therefore T(n) = O(n \log n)$$

→ Example 5: Consider the recurrence

$$T(n) = \begin{cases} 1 & n = 1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + n & n > 1 \end{cases}$$

find the asymptotic bound on T.

Solution: we guess the solution is $O(n \log n)$
$$T(n) \leq cn \log n$$
$$T(n/2) \leq c(n/2) \cdot \log (n/2) \quad \text{— ①}$$

put eqn ① in recurrence relation
$$T(n) \leq 2c\lfloor \tfrac{n}{2} \rfloor \log \lfloor \tfrac{n}{2} \rfloor + n$$

$$\leq cn\log n - cn \log 2 + n$$
$$\leq cn \log n - cn + n = cn \log n - n(c-1)$$
$$T(n) \leq cn \log n \quad \forall\, c \geq 1$$
$$T(n) = O(n \log n)$$

Hence our given $T(n) = O(n \log n)$ is true.

→ Example 6: Show that solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $\Omega(n \log n)$. Conclude that the solution is $\Theta(n \log n)$

Solution: Let the solution to $T(n)$ is $\Omega(n \log n)$, then we need to prove $T(n) \geq cn\log n$

$$T(n/2) \geq c(n/2) \log (n/2) \quad \text{— ①}$$

put eqn ① in recurrence relation.

$T(n) \geq 2T(\lfloor n/2 \rfloor) + n$

$\Rightarrow T(n) \geq 2\left(c\left(\frac{n}{2}\right)\log(n/2)\right) + n$

$\Rightarrow T(n) \geq cn\log n - cn\log 2 + n$

$\Rightarrow T(n) \geq cn\log n - cn + n$

for $c \geq 1$

$T(n) \geq cn\log n$

$\therefore T(n) = \Omega(n\log n)$

~~for~~ ~~0<~~ Beet for large value of n and c

$T(n) \leq cn\log n$ also holds

so, $T(n) = \Theta(n\log n)$

<u>Changing variable Concept:</u>

→ <u>Example 7:</u> Solve $T(n) = 2T(\sqrt{n}) + 1$ by making a change of variables

<u>Solution:</u> Let $m = \log n$

$n = 2^m$

$n^{1/2} = 2^{m/2}$

$\sqrt{n} = 2^{m/2}$

Thus, recurrence relation becomes

$T(2^m) = 2T(2^{m/2}) + 1$

Changing the recurrence $T(2^m)$ to $S(m)$, we get

$S(m) = 2S(m/2) + 1$ —— ①

Let the solution to the recurrence is $O(\log m)$

$S(m) \leq c \cdot \log m$

$S(m/2) \leq c \log m/2$ —— ②

put eq^n ② in eq^n ①, it becomes

$S(m) \leq 2 \cdot c \cdot \log m/2 + 1$

$\leq 2c\log m - 2c\log 2 + 1$

$\leq 2c\log m - 2c + 1 \quad (\because \log 2 = 1)$

$S(m) \leq 2c\log m$

$S(m) = O(\log m)$

$T(2^m) = O(\log\log n)$

Therefore, $T(n) = O(\log\log n)$

<u>Example 8:</u> Consider the following recurrence

$T(n) = 2T(\sqrt{n}) + \log n$

Solve it by changing variable.

Solution: $T(n) = 2T(\sqrt{n}) + \log n \longrightarrow ①$

suppose $m = \log_2 n \Rightarrow n = 2^m$

$\therefore n^{1/2} = 2^{m/2}$

$\Rightarrow \sqrt{n} = 2^{m/2}$

put the values in eq^n ①, we get

$$T(2^m) = 2T(2^{m/2}) + m$$

changing the recurrence $T(2^m)$ to $S(m)$, we get

$$S(m) = 2S(m/2) + m \text{ --- } ②$$

Let the solution of eq^n ② is $O(m \log m)$

$$S(m) = O(m \log m)$$

$$S(m) \leq c \cdot m \log m$$

$$S(m/2) \leq c \cdot \frac{m}{2} \log \frac{m}{2} \text{ --- } ③$$

put eq^n ③ in eq^n ②; it becomes

$$S(m) \leq 2 \cdot c \frac{m}{2} \log \frac{m}{2} + m$$

$\Rightarrow S(m) \leq cm \log m - cm \log 2 + m$

$\Rightarrow S(m) \leq cm \log m - cm + m \quad (\because \log 2 = 1)$

$\Rightarrow S(m) \leq cm \log m - m(c-1)$

$\Rightarrow S(m) \leq cm \log m$

$\Rightarrow S(m) = O(m \log m)$

$\Rightarrow T(2^m) = O(\log n \log \log n)$

$\Rightarrow T(n) = O(\log n \log \log n)$

→ Show that solution to $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$ is $O(n \log n)$

sol^n: Given that $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$

we guess that solution is $O(n \log n)$. Then we have to prove that

$$T(n) \leq cn \log n$$

~~$\Rightarrow T(n/2) \leq 2c(\frac{n}{2})$~~

$\Rightarrow T(\lfloor n/2 \rfloor) \leq c \lfloor \frac{n}{2} \rfloor \log \lfloor \frac{n}{2} \rfloor$

$\Rightarrow T((n/2)+17) \leq c(\frac{n}{2}) \log(\frac{n}{2}) + 17$

$\Rightarrow T(n) \leq 2\left[c\left(\frac{n}{2}\right) \log\left(\frac{n}{2}\right) + 17\right] + n$

$\leq cn \log(\frac{n}{2}) + 34 + n$

$\leq cn \{\log n - cn \log 2\} + 34 + n$

$\leq cn \log n - cn + 34 + n$

$\leq cn \log n - n(c-1) + 34$

$\leq cn \log n - b$

$T(n) = O(n \log n)$

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