Introduction

Number theory is a branch of mathematics that studies integers. There are many questions involving integers that are difficult to solve even though they may seem simple at first glance. We will discuss a few important concepts in the number theory that every programmer should know.

Prime numbers and factors

Factors: A number x is said to be a **factor** or **divisor** of another number y if x divides y. If x divides y then we also write it as x | y.
For Example: The factors of number 36 are 1,2,3,4,6,9,12,18 and 36.

Prime number: A number *n>1* is said to be **prime** if its factors are only 1 and n.
For Example: 2,3,5,7,11 are all prime numbers.

Composite number: A number *n>1* is said to be **composite** if it has at least one more factor other than 1 and itself. In other words, a number *n>1* is composite if it is not prime.
For Example: 12 is a composite number as the factors of 12 other than 1 and 12 are 2,3,4 and 6.

Prime factorization: For every number *n>1*, there exists a **unique** prime factorization of the number i.e n can be expressed as the product of distinct primes that are the factors of the number raised to some positive exponent.
In other words, $\mathbf{n = p_1^{x1}p_2^{x2}p_3^{x3}p_4^{x4}...p_n^{xn}}$, where p1,p2,....,pn are distinct primes that are factors of the number with their respective exponents(positive integers) being x1,x2,....,xn.

For Example: The prime factorization of the number 120: $2^3.3^1.5^1$

- The number of factors of the number n is given as the **product** of the (1+exponents) in the prime factorization of the number as for each prime p_i and its exponent x_i, we have (x_i+1) choices to include the prime p_i in the factorization of the number.
  For Example: The number of factors of the number 120 are 4.2.2 = 16 i.e 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.

- For the prime factorization of the number n given as: $p_1^{e1}.p_2^{e2}.p_3^{e3}...p_k^{ek}$, where pi are distinct prime numbers, the sum of the factors, is given as the **product** of the terms $(p_i^{ei+1} - 1)/(p_i - 1)$ for 1<=i<=k:
  $((p_1^{e1+1} - 1)/(p_1 - 1)).((p_2^{e2+1} - 1)/(p_2 - 1)).((p_3^{e3+1} - 1)/(p_3 - 1))...((p_k^{ek+1} - 1)/(p_k - 1))$.

  Proof:

  If there is a single prime factor of a number, let's denote it by p with exponent e, then the sum of the divisors of the number will be:
  Sum of Divisors(S):

$= p^0 + p^1 + p^2 + ... + p^e$ (A geometric progression)
$= (p^{e+1} - 1)/(p - 1)$ (Sum of terms of GP)

Now, let there be two distinct prime factors of the number given as p_1^{e1} and p_2^{e2}, then the sum of each combination of these factors can be given as:
Sum of Divisors(S):
$= (p_1^0 + p_1^1 + p_1^2 + ... + p_1^{e1}).(p_2^0 + p_2^1 + p_2^2 + ... + p_2^{e2})$
$= ((p_1^{e1+1} - 1)/(p_1 - 1)).((p_2^{e2+1} - 1)/(p_2 - 1))$

Hence, similar we can extend the above formula for the prime factorization of the number n given as: $p_1^{e1}.p_2^{e2}.p_3^{e3}...p_k^{ek}$ where p_i are distinct prime numbers, the sum of the factors is given as the **product** of the terms $(p_i^{ei+1} - 1)/(p_i - 1)$ for 1<=i<=k:
$((p_1^{e1+1} - 1)/(p_1 - 1)).((p_2^{e2+1} - 1)/(p_2 - 1)).((p_3^{e3+1} - 1)/(p_3 - 1))...((p_k^{ek+1} - 1)/(p_k - 1))$.

For Example: The sum of factors of the number 6 =>
Prime factorization: $2^1.3^1$
$= ((2^2 - 1)/(2 - 1)).((3^2 - 1)/(3 - 1))$
= 3.4
= 12
=> 1+2+3+6 = 12

Conjectures involving prime numbers:

Conjectures are basically mathematical statements that have not been yet proven rigorously, conjectures are based upon observations, and when rigorously proved they become theorems. Some of the conjectures involving primes are:

- **Twin Prime conjecture:** Twin prime conjecture also known as the polignac's conjecture states that there are infinitely many twin primes i.e primes that differ by 2. For Example: {3,5},{5,7},{11,13} are twin primes.

- **Goldbach conjecture:** Goldbach conjecture states that every even integer greater than 2 can be expressed as the sum of two primes a and b. For Example: 4 = 2+2, 6 = 3+3, 12 = 7+5.

- **Legendre conjecture:** Legendre conjecture states that there always exists a prime number between n and $(n+1)^2$, where n is some positive integer. For Example: The primes between 3^2 and 4^2 are 11,13,17,19.