Example 3.3.2. (a) Let the weight function be equal to unity $w(x) = 1$, find the required boundary conditions for the differential operator $L = \mathrm{d}^2/\mathrm{d}x^2$ to be Hermitian over the interval $a \le x \le b$. (b) Show that if the solutions of $Ly = \lambda y$ in the interval $0 \le x \le 2\pi$ satisfy the boundary conditions $y(0) = y(2\pi),\ y'(0) = y'(2\pi)$, (where y' means the derivative of y with respect to x), then the operator L in this interval is Hermitian. (c) Find the complete set of eigenfunctions of L.

Solution 3.3.2. (a) Let $y_i(x)$ and $y_j(x)$ be two functions in this space. Integrating the inner product $\langle y_i | L y_j \rangle$ by parts gives

$$\langle y_i | L y_j \rangle = \int_a^b y_i^* \frac{\mathrm{d}^2 y_j}{\mathrm{d}x^2} \mathrm{d}x = \left[y_i^* \frac{\mathrm{d}y_j}{\mathrm{d}x} \right]_a^b - \int_a^b \frac{\mathrm{d}y_i^*}{\mathrm{d}x} \frac{\mathrm{d}y_j}{\mathrm{d}x} \mathrm{d}x.$$

Integrating the second term on the right-hand side by parts again yields

$$\int_a^b \frac{\mathrm{d}y_i^*}{\mathrm{d}x} \frac{\mathrm{d}y_j}{\mathrm{d}x} \mathrm{d}x = \left[\frac{\mathrm{d}y_i^*}{\mathrm{d}x} y_j \right]_a^b - \int_a^b \frac{\mathrm{d}^2 y_i^*}{\mathrm{d}x^2} y_j \mathrm{d}x.$$

Thus

$$\langle y_i | L y_j \rangle = \left[y_i^* \frac{\mathrm{d}y_j}{\mathrm{d}x} \right]_a^b - \left[\frac{\mathrm{d}y_i^*}{\mathrm{d}x} y_j \right]_a^b + \langle L y_i | y_j \rangle .$$

Therefore L is Hermitian provided

$$\left[y_i^* \frac{\mathrm{d}y_j}{\mathrm{d}x} \right]_a^b - \left[\frac{\mathrm{d}y_i^*}{\mathrm{d}x} y_j \right]_a^b = 0.$$

(b) Because of the boundary conditions $y(0) = y(2\pi),\ y'(0) = y'(2\pi)$,

$$\left[y_i^* \frac{\mathrm{d}y_j}{\mathrm{d}x} \right]_0^{2\pi} = y_i^*(2\pi) y_j'(2\pi) - y_i^*(0) y_j'(0) = 0,$$

$$\left[\frac{\mathrm{d}y_i^*}{\mathrm{d}x} y_j \right]_0^{2\pi} = y_i^{*\prime}(2\pi) y_j(2\pi) - y_i^{*\prime}(0) y_j(0) = 0.$$

Therefore L is Hermitian in this interval, since

$$\langle y_i | L y_j \rangle = \left[y_i^* \frac{\mathrm{d}y_j}{\mathrm{d}x} \right]_a^b - \left[\frac{\mathrm{d}y_i^*}{\mathrm{d}x} y_j \right]_a^b + \langle L y_i | y_j \rangle = \left\langle y_i \middle| L^+ y_j \right\rangle .$$

(c) To find the eigenfunctions of L, we must solve the differential equation

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} = \lambda y(x),$$