ADIABATIC ELIMINATION FOR LASER EQUATIONS VIA CENTER MANIFOLD THEORY

Gian-Luca Oppo*

Department of Physics, Drexel University
Philadelphia, Pa, 19104 U.S.A.

Antonio Politi

Istituto Nazionale di Ottica
Largo E. Fermi 6, 50125 Firenze, Italy

The adiabatic elimination of variables which are irrelevant to the dynamical evolution of laser output, is a widely used procedure in Quantum Optics [1]. This techinique allowed the development of simple models capable of describing the time evolution of laser sources. In spite of their wide use, the adiabatic reduction is often not based on a solid mathematical framework. Naive procedures like setting the time derivatives of the fast variables equal to zero have been proven to be unsatisfactory and, in some instances, incorrect [2]. The aim of this communication is to provide a mathematical theory based on the center manifold [3], for a correct reduction of the number of variables and to show that its application does not require ad hoc treatements. More important, the center manifold technique allows one to derive a two dimensional system of equations capable of describing oscillatory states in antithesis with what the community of Quantum Optics has believed for a long time [4].

The single mode Maxwell-Bloch equations for a system of homogeneously broadened two-level atoms, written for the slowly varying (complex) amplitudes of the field F and polarization R, and for the population inversion Δ are

$$\frac{dF}{dt} = -k\left[(1+i\theta)F - R\right]$$

$$\frac{dR}{dt} = -(1+i\delta)R + F\Delta \qquad (1)$$

$$\frac{d\Delta}{dt} = -\gamma\left[\Delta - A + \mathrm{Re}(FR^*)\right]$$

where the time and the decay rates k, γ have been normalized to the decay rate of the polarization, A is the pump parameter, and θ, δ are the field-cavity and field-atom detunings. We can rewrite Eqs.(1) considering the amplitudes and phases of the complex variables. In doing so, we choose a reference frame which rotates at the instantaneous laser frequency so that the amplitude of the field turns out to be real [5]. The remaining equations are

$$\frac{dE}{dt} = -k\left[E - \mathrm{Re}(P)\right]$$

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$$\frac{dP}{dt} = - \left[1 + i\,(\delta - k\theta + \frac{k\,\mathrm{Im}(P)}{E})\right]P + E\Delta \tag{2}$$

$$\frac{d\Delta}{dt} = - \gamma\left[\Delta - A + E\,\mathrm{Re}(P)\right]$$

where P is the only complex variable left. In order to remove the apparent singularity on the r.h.s. of the equation for P, we introduce the variables

$$U = \frac{P}{E} \quad ; \quad S = 2\ln E \tag{3}$$

so that system (2) becomes

$$\frac{dS}{dt} = - 2k\left[1 - \mathrm{Re}(U)\right]$$

$$\frac{dU}{dt} = - (1 - k + i\,(\delta - k\theta) + kU)U + \Delta \tag{4}$$

$$\frac{d\Delta}{dt} = - \gamma\left[\Delta - A + e^{S}\mathrm{Re}(U)\right]$$

A first step toward the normal form of the laser equations is made by shifting the origin of the coordinate to the steady state of interest. With obvious meaning of symbols, one obtains

$$\frac{ds}{dt} = 2k\,\mathrm{Re}(u)$$

$$\frac{du}{dt} = - (\sigma + ku)u + d \tag{5}$$

$$\frac{dd}{dt} = \gamma\left\{D - d - De^{s}\left[1 + \mathrm{Re}(u)\right]\right\}$$

where we introduced the new parameters

$$D = A - 1 - \left(\frac{\delta - k\theta}{1 + k}\right)^2 \quad ; \quad \sigma = (1 + k) + i\,\frac{(\delta - k\theta)(1 - k)}{1 + k} \tag{6}$$

The normal form of the system (5) is obtained by diagonalizing the linear part of the differential equations. However, for a correct application of the global center manifold, we have to individuate a suitable smallness parameter to be added, as a fictitious variable, to the set of equations (5). The case of a slow relaxation of the population inversion is interesting for two reasons, one theoretical and the other applicative. On one hand, Lugiato et al. [2] have shown that this is an anomalous case where standard techniques of adiabatic elimination cannot be applied. On the other hand, the case specified by the condition $\gamma<<1$ includes a large class of lasers (ruby, CO_2, semiconductor, Nd:YAG, Co:MgF_2, NMR, etc.) and it is obviously important to develop models which better describe their dynamics. By introducing the smallness parameter

$$\mu = \sqrt{\frac{\gamma\sigma\sigma^*}{2Dk\,\mathrm{Re}(\sigma)}} \tag{7}$$

and by defining the new set of variables along the eigenvector directions of the linear matrix obtained from $\gamma=0$

$$w = \frac{d}{\mu D}$$

$$z = - \left(\frac{\sigma}{\mu D}\right)u + w \tag{8}$$

$$\alpha = 2k\,\mathrm{Re}\left(\frac{u}{\sigma}\right) + s$$

Eqs.(5) read as

$$\frac{dz}{dt} = -\,\sigma z + \left(\frac{\mu k D}{\sigma}\right)(w - z)^2 + \frac{dw}{dt}$$

$$\frac{d\alpha}{dt} = 2\mu k D\,\mathrm{Re}\left(\frac{w}{\sigma} - \mu k D\,\frac{(w - z)^2}{\sigma^3}\right) \qquad (9)$$

$$\frac{dw}{dt} = \frac{2\mu k D\,\mathrm{Re}(\sigma)}{\sigma\sigma^*}\left\{1 - \mu w - \exp\left[\alpha + 2\mu k D\,\mathrm{Re}\left(\frac{z - w}{\sigma^2}\right)\right]\left[1 - \mu D\,\mathrm{Re}\left(\frac{z - w}{\sigma}\right)\right]\right\}$$

$$\frac{d\mu}{dt} = 0$$

These equations match the correct form for the application of the center manifold theorem [3]

$$\frac{dY}{dt} = AY + \mu\, f(X,Y,\mu)$$

$$\frac{dX}{dt} = BX + \mu\, g(X,Y,\mu) \qquad (10)$$

$$\frac{d\mu}{dt} = 0$$

where $X \in \mathbf{C}^m$, $Y \in \mathbf{C}^n$, and A (B) is a constant matrix whose eigenvalues have negative (zero) real parts, and $g(0,0,0) = f(0,0,0) = g'(0,0,0) = f'(0,0,0) = 0$ (the prime indicates the total derivative with respect to the variables inside the parenthesis). The Y variables are then adiabatically eliminated through the equation of the center manifold which, under the assumption of Eqs. (10), is proven to exist [3] and is given by

$$Y = H(\mu,X) = \mu A(X) + \mu^2 B(X) + \mu^3 C(X) + \ldots.. \qquad (11)$$

The different functions A, B, C, ... of the slow variables X are obtained by using

$$\frac{dY}{dt} = \frac{dX}{dt}\left(\frac{\partial H(\mu,X)}{\partial X}\right) \qquad (12)$$

The normal form (9) that we recast the laser equations in, shows its usefulness when considering the first-order corrections. Indeed, the slow dynamics takes place at a distance of order μ from the surface z=0. Then, by rescaling the time to

$$\tau = t\left[\frac{\sigma\sigma^*}{2\mu D k\,(1 + k)}\right] \qquad (13)$$

the second and third equations of the system (9) become

$$\frac{d\alpha}{d\tau} = w\left[1 - \frac{\mu k D w}{(\sigma\sigma^*)^2}\left(3\sigma\sigma^* - 2(1 + k)^2\right)\right]$$

$$\frac{dw}{d\tau} = 1 - \mu w - e^{\alpha}\left(1 + \frac{\mu D w\,(1 - k)}{\sigma\sigma^*}\right) \qquad (14)$$

These equations describe the dynamics of the class of lasers characterized by $\gamma \ll 1$ in a far more precise way than the rate equations. For example it is easy to show that the our system correctly predicts the existence of a threshold of oscillations for large enough pump values. By rewriting Eqs. (14) as a second order differential equation for α

$$\frac{d^2\alpha}{d\tau^2} = 1 - e^{\alpha} - \mu \frac{d\alpha}{d\tau}\Big[1 + 2Dk \frac{Re(\sigma^3)}{(\sigma\sigma^*)^2 (1+k)} (1 - e^{\alpha}) +$$

$$+ D\, e^{\alpha} \Big[\frac{1+k}{\sigma\sigma^*} - 2k \frac{Re(\sigma^2)}{(\sigma\sigma^*)^2}\Big]\Big] \qquad (15)$$

the motion is described by slowly damped (enhanced) oscillations in a Toda potential [6]. The second laser threshold is then found by setting the coefficient of $d\alpha/d\tau$ equal to zero

$$D_{th} = \frac{(\sigma\sigma^*)^2}{2k\, Re(\sigma^2) - \sigma\sigma^* (1+k)} \qquad (16)$$

As the sign of D_{th} corresponding to the equilibrium intensity must be positive, we obtain two conditions for the existence of steady oscillations: the well known $k>1$, but also

$$\eta^2 = (\delta - k\theta)^2 < \frac{(1+k)^4}{(k-1)(3k+1)} \qquad (17)$$

which is a NEW condition for the detuning parameter. In the Figure we show the stability diagram obtained by condition (17) separating the oscillatory region from the stable one. Large values of the detuning obviously stabilize the laser action even for large pumpings.

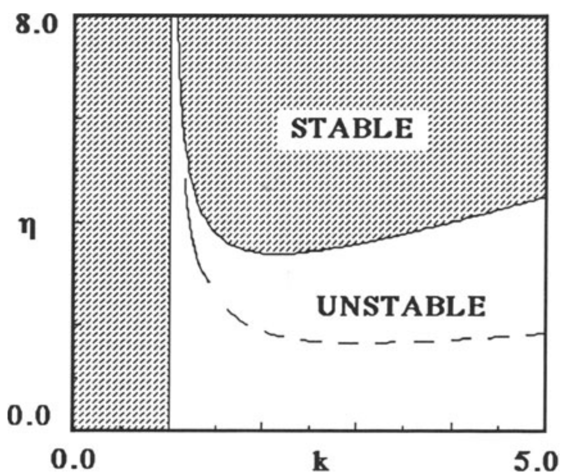

Figure: Stability diagram in the parameter space (k,η). Above the dashed line the Hopf bifurcation is supercritical, and below is subcritical

Without entering into the details of the calculations which are published elsewhere [7], it is important to stress that a third order expansion of the center manifold leads to analytic investigation of the nature of the bifurcation. One of the main results of such an expansion is shown in the Figure where the dashed line separates the subcritical (below) from the supercritical (above) behavior of the Hopf bifurcation. A comparison of the numerical integration of the generalized two dimensional model with the initial five dimensional one (4) has shown an extraordinary agreement of the oscillatiory behavior down to values of the intensity equal to 10^{-43}.

In conclusion we have shown that the adiabatic elimination procedure via center manifold theory is a general one. The difficulty introduced by the detunings and by the rotation due to the phase has been overcome by properly rewriting the equations in a normal form. The actual theory can be straightforwardly extended to include other dynamical effects in lasers such as modulation of losses and pump, signal injection, coupling to rotational bands [8] and to saturable absorbers.

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* Present Address: Department of Physics and Applied Physics, University of Strathclyde, John Anderson Building, G4 0NG Glasgow, Scotland, U.K.

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