3.4.2 Boundary Conditions of Sturm–Liouville Problems

Sturm–Liouville Operators as Hermitian Operators. Let L be the Sturm–Liouville operator in (3.14), and $f(x)$ and $g(x)$ be two functions having continuous second derivatives on the interval $a \leq x \leq b$, then

$$\langle Lf \,|g\rangle = \int_a^b \left\{ -\frac{1}{w}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}\right)+q\right] f\right\}^* gw \; \mathrm{d}x.$$

Since $p, \; q, \; w$ are real, the integral can be written as

$$\langle Lf \,|g\rangle = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}f^*\right) g \; \mathrm{d}x - \int_a^b qf^*g \; \mathrm{d}x.$$

With integration by parts,

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}f^*}{\mathrm{d}x}\right) g \; \mathrm{d}x = p\frac{\mathrm{d}f^*}{\mathrm{d}x}g\bigg|_a^b - \int_a^b p\frac{\mathrm{d}f^*}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x,$$

and

$$\int_a^b p\frac{\mathrm{d}f^*}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x = \int_a^b \frac{\mathrm{d}f^*}{\mathrm{d}x}p\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x = f^*p\frac{\mathrm{d}g}{\mathrm{d}x}\bigg|_a^b - \int_a^b f^*\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}g}{\mathrm{d}x}\right)\mathrm{d}x.$$

It follows that

$$\langle Lf \,|g\rangle = -\, p\frac{\mathrm{d}f^*}{\mathrm{d}x}g\bigg|_a^b + f^*p\frac{\mathrm{d}g}{\mathrm{d}x}\bigg|_a^b - \int_a^b f^*\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}g}{\mathrm{d}x}\right)\mathrm{d}x - \int_a^b qf^*g \; \mathrm{d}x,$$

or

$$\begin{aligned}\langle Lf \,|g\rangle &= \left[p\left(f^*\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}f^*}{\mathrm{d}x}g\right)\right]_a^b + \int_a^b f^*\left\{-\frac{1}{w}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}\right)+q\right] g\right\} w \; \mathrm{d}x \\ &= \left[p\left(f^*\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}f^*}{\mathrm{d}x}g\right)\right]_a^b + \langle f \,|Lg\rangle \,.\end{aligned}$$

It is clear that if

$$\left[p\left(f^*\frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}f^*}{\mathrm{d}x}g\right)\right]_a^b = 0, \tag{3.15}$$

then

$$\langle Lf \,|g\rangle = \langle f \,|Lg\rangle \,.$$

In other words, if the function space consists of functions that satisfy (3.15), then the Sturm–Liouville operator L is Hermitian in that space.

Sturm–Liouville Problems. It is customary to refer to the Sturm–Liouville equation and the boundary conditions together as the Sturm–Liouville problem. Since the operator is Hermitian, the eigenfunctions of the Sturm–Liouville