There are two important theorems for hyperbolic stationary points, the stable manifold theorem and Hartman's theorem. The first shows that the local structure of hyperbolic stationary points of nonlinear flows, in terms of the existence and transversality of local stable and unstable manifolds, is the same as the linearized flow, and the second asserts that there is a continuous invertible map in some neighbourhood of the stationary point which takes the nonlinear flow to the linear flow preserving the sense of time.

Let $U$ be some neighbourhood of a stationary point, $x$. Then, by analogy with the definition of the invariant manifolds for linear systems we can define the local stable manifold of $x$, $W^s_{loc}(x)$, and the local unstable manifold of $x$, $W^u_{loc}(x)$, by

$$W^s_{loc}(x) = \{y \in U | \varphi(y,t) \to x \textit{ as } t \to \infty,\ \varphi(y,t) \in U \textit{ for all } t \geq 0\}$$

and

$$W^u_{loc}(x) = \{y \in U | \varphi(y,t) \to x \textit{ as } t \to -\infty,\ \varphi(y,t) \in U \textit{ for all } t \leq 0\}.$$

The stable manifold theorem states that these manifolds exist and are of the same dimension as the stable and unstable manifolds of the linearized equation $\dot{y} = Df(x)y$ if $x$ is hyperbolic, and that they are tangential to the linearized manifolds at $x$.

(4.7) THEOREM (STABLE MANIFOLD THEOREM)

*Suppose that the origin is a hyperbolic stationary point for $\dot{x} = f(x)$ and $E^s$ and $E^u$ are the stable and unstable manifolds of the linear system $\dot{x} = Df(0)x$. Then there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the same dimension as $E^s$ and $E^u$ respectively. These manifolds are (respectively) tangential to $E^s$ and $E^u$ at the origin and as smooth as the original function $f$.*

Note that in Chapter 3 the centre manifold of a stationary point was also defined, but that for a hyperbolic stationary point the centre manifold is empty. We shall return to the problems of finding nonlinear centre manifolds in Chapters 7 and 8, where the basic ideas of bifurcation theory are introduced. The content of this theorem is illustrated in Fig. 4.1. The proof is, unfortunately, long and technical and we leave this to the end of this chapter, since we will be much more concerned with the use of this theorem.



One further point can be made without difficulty: suppose that $x_0$ is a hyperbolic stationary point, then there are three possibilities. Either $W^s_{loc}(x_0) = \emptyset$, or $W^u_{loc}(x_0) = \emptyset$, or both manifolds are non-empty. These three possibilities are given names: $x_0$ is called a source, sink or saddle respectively. From the definition of the linear stable and unstable manifolds and the stable manifold theorem it should be obvious that these definitions can be made in terms of the eigenvalues of the Jacobain matrix at $x_0$ in the following way.

(4.8) DEFINITION

*Suppose that $x_0$ is a hyperbolic stationary point of $\dot{x} = f(x)$ and let $Df(x_0)$ denote the Jacobian matrix of $f$ evaluated at $x_0$. Then $x_0$ is a sink if all the eigenvalues of $Df(x_0)$ have strictly negative real parts and a source if all the eigenvalues of $Df(x_0)$ have strictly positive real parts. Otherwise $x_0$ is a saddle.*

We shall see in Sections 4.3 and 4.5 that for small perturbations of the defining equations, a source remains a source, a sink remains a sink and a saddle remains a saddle. Furthermore, as one would expect, if $x_0$ is a sink then it is asymptotically stable.

If we choose a coordinate system for which the linear part of the differential equation at the origin is in normal form we can always arrange

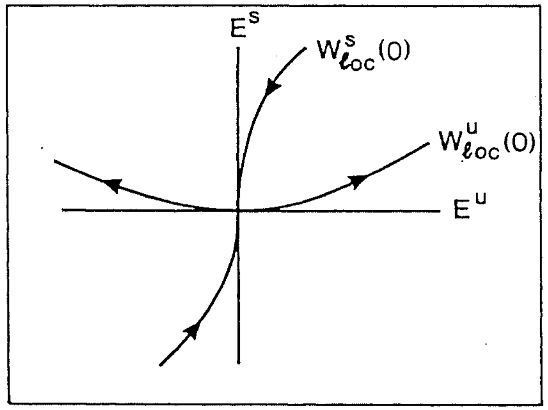


Fig. 4.1 Stable and unstable manifolds.

for the differential equation to be of the form

$$\dot{x} = Ax + g_1(x,y) \quad \dot{y} = -By + g_2(x,y) \tag{4.21}$$

where $x \in \mathbf{R}^{n_u}$, $y \in \mathbf{R}^{n_s}$ (where $n_u$ is the dimension of the local unstable manifold and $n_s$ is the dimension of the local stable manifold, $n_u + n_s = n$) and both the square matrices $A$ and $B$ have eigenvalues with positive real parts. The functions $g_i(x,y)$, $i = 1,2$, contain the nonlinear parts of the equation, so they vanish, together with their first derivatives at the origin, $(x,y) = (0,0)$. Hence

$$E^s(0,0) = \{(x,y)|x=0\} \text{ and } E^u(0,0) = \{(x,y)|y=0\}.$$ here

Since the stable and unstable manifolds are smooth and are tangential to these manifolds at the origin they can be described as the graphs of functions, so the stable manifold is given by

$$x_i = S_i(y), \quad i = 1, \dots, n_u \tag{4.22}$$

where

$$\frac{\partial S_i}{\partial y_j}(0) = 0, \ 1 \le i \le n_u, 1 \le j \le n_s \tag{4.23}$$

since the manifold is tangential to $E^s$ at 0. Similarly we can write the unstable manifold (again locally) as

$$y_j = U_j(x), \quad \frac{\partial U_j}{\partial x_i}(0) = 0, \ 1 \le i \le n_u, 1 \le j \le n_s. \tag{4.24}$$

This observation allows us to approximate the stable and unstable manifolds by expanding the functions $S_i$ and $U_j$ as power series. Consider $U_j$ (the argument is the same for $S_i$). We begin by expanding $U_j$ as a power series in $x$, so

$$U_j(x) = \sum_{r\ge 2} \sum_{m \in M_r} u_{mj} x^m \tag{4.25}$$

where the notation is as in the previous section. If $B$ is diagonal then with eigenvalues $(\lambda_i)$, $i = 1, 2, \dots, n_s$ then

$$\dot{y}_j = -\lambda_j y_j + g_{2j}(x,y) \tag{4.26}$$

and on the unstable manifold $y = U(x)$ so

$$\dot{y}_j = -\lambda_j U_j(x) + g_{2j}(x, U(x)). \tag{4.27}$$

On the other hand

$$\dot{y}_j = \frac{d}{dt} U_j(x) = \sum_{k=1}^{n_u} \dot{x}_k \frac{\partial}{\partial x_k} U_j(x). \tag{4.28}$$



Comparing the right hand sides of (4.27) and (4.28) we find that

$$-\lambda_j U_j(x) + g_{2j}(x, U(x)) = \sum_{k=1}^{n_u} \dot{x}_k \frac{\partial}{\partial x_k} U_j(x) \tag{4.29}$$

and we can now substitute the series expansion for the functions $U_j$ into these equations and equate coefficients of powers of $x$ in order to get a set of simultaneous equations for the coefficients $u_{mi}$ which can be solved to arbitrary order. An example may make this clearer.

**Example 4.2**

Consider the equations

$$\dot{x} = x, \quad \dot{y} = -y + x^2.$$

This has a unique stationary point at $(x,y) = (0,0)$ and the equation is already in normal form near the stationary point. The linearized equation is

$$\dot{x} = x, \quad \dot{y} = -y,$$

giving a saddle at the origin with invariant linear subspaces

$$E^s(0,0) = \{(x,y)|x=0\} \text{ and } E^u(0,0) = \{(x,y)|y=0\}.$$

By the stable manifold theorem we know that the nonlinear system has a local unstable manifold of the form

$$y = U(x), \quad \frac{\partial U}{\partial x}(0) = 0$$ In this case

and so we try a series expansion for $U$,

$$U(x) = \sum_{k \ge 2} u_i x^i.$$

Now,

$$\dot{y} = -y + x^2 = -\sum_{k\ge 2} u_i x^i + x^2$$

on the unstable manifold and also

$$\dot{y} = \dot{x}\frac{\partial U}{\partial x}(x) = \sum_{k \ge 2} k u_k x^k.$$

Equating terms of order $x^2$, $x^3$ and so on gives

$$-u_2 + 1 = 2u_2, \text{ and } -u_k = ku_k, \ k \ge 3.$$

Hence $u_2 = \frac{1}{3}$, $u_k = 0$ for $k \geq 3$ and so

$\frac{1}{3}x^2$

$$W^u_{loc}(0,0) = \{(x,y)|y = \tfrac{1}{3}\cancel{x^3}\}.$$

A similar exercise shows that $W^s_{loc}(0,0) = E^s(0,0)$.

Later in this book (Chapter 12) we will see that a great deal of interesting dynamics is controlled by the behaviour of the stable and unstable manifolds of stationary points; for this we need to extend the local manifolds to obtain global stable and unstable manifolds defined by

$$W^u(0) = \bigcup_{t \leq 0} \varphi(W^u_{loc}(0), t) \text{ and } W^s(0) = \bigcup_{t \geq 0} \varphi(W^s_{loc}(0), t).$$

The second result of this section is associated with a weakening of the requirements of Poincaré's Linearization Theorem. In the previous section we looked for a change of variable such that the equation in the new variable is locally just the linear flow. This turned out to be quite a tough condition to meet, but in Example 4.1 we saw that even when the linearization has resonant eigenvalues the flow was remarkably similar to the linear flow (at least for the hyperbolic stationary point considered). This suggests that an alternative strategy might be to look for a map from the nonlinear flow to the linear flow in a neighbourhood of the stationary point, which takes trajectories of the nonlinear flow to trajectories of the linear flow.

(4.9) THEOREM (HARTMAN'S THEOREM)

*If $x = 0$ is a hyperbolic stationary point of $\dot{x} = f(x)$ then there is a continuous invertible map, $h$, defined on some neighbourhood of $x = 0$ which takes orbits of the nonlinear flow to those of the linear flow $exp(tDf(0))$. This map can be chosen so that the parametrization of orbits by time is preserved.*

Note that the map is only continuous (not necessarily differentiable) and so it does not distinguish between, for example, a logarithmic spiral (cf. (3.31)) and the phase portrait obtained when the Jacobian at the stationary point has real eigenvalues. If we want greater smoothness we find ourselves involved once again in problems of resonance.



## 4.3 Persistence of hyperbolic stationary points

Another important feature of hyperbolic stationary points is the fact that they persist under small perturbations of the defining differential equations. Hence if the origin is a hyperbolic stationary point of $\dot{x} = f(x)$ and $v$ is any smooth vector field on $\mathbf{R}^n$ then for sufficiently small $\epsilon$ the equation

$$\dot{x} = f(x) + \epsilon v(x) \tag{4.30}$$

has a hyperbolic stationary point near the origin of the same type as the hyperbolic point of the unperturbed equation. This robustness, together with the results of the previous section, shows that the dynamics in a neighbourhood of a hyperbolic stationary point is not radically altered by small perturbations. This will be of crucial importance when we come to consider bifurcation theory in Chapters 7 and 8. To see this, suppose that $f(0) = 0$ and look for stationary points of the perturbed system. They satisfy

$$f(x) + \epsilon v(x) = 0. \tag{4.31}$$

Expanding this equation about $x = 0$ (or using the implicit function theorem) gives

$$[Df(0) + \epsilon Dv(0)]x + \epsilon v(0) + O(|x|^2) = 0 \tag{4.32}$$

with solutions

$$x = -\epsilon[Df(0) + \epsilon Dv(0)]^{-1}v(0) + O(\epsilon^2) \tag{4.33}$$

provided $[Df(0) + \epsilon Dv(0)]$ is invertible. Now, if $x = 0$ is a hyperbolic stationary point, the eigenvalues of $Df(0)$ are bounded away from zero and hence the eigenvalues of $[Df(0) + \epsilon Dv(0)]$ are bounded away from zero for sufficiently small $\epsilon$. So $\det[Df(0) + \epsilon Dv(0)] \neq 0$ for sufficiently small $\epsilon$ and hence this matrix is invertible. We now want to show that the stationary point of the perturbed equation is also hyperbolic. By continuity in $\epsilon$, there is a neighbourhood of $\epsilon = 0$ for which the real parts of the eigenvalues of $[Df(x) + \epsilon Dv(x)]$ are all non-zero for sufficiently small $x$. In particular, no eigenvalue can cross the imaginary axis and so the number of eigenvalues on the right of the imaginary axis and on the left of the imaginary axis is the same for all $x$ sufficiently small in this neighbourhood of $\epsilon = 0$. Now simply choose $\epsilon$ small enough so that it is in this neighbourhood of $\epsilon = 0$ and the stationary point of the perturbed equation has sufficiently small $|x|$. Then for all values of $\epsilon$ sufficiently small the stationary point of the perturbed equation is hyperbolic.