ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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LECTURE 5

Geometry

- Geometry of linear systems
- Geometry of nonlinear systems
- Connecting the geometry of linear and nonlinear systems

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

2. *Phase plane analysis and geometry of hyperbolic systems*
   [5 lectures/tutorials]

   Analysis of two-dimensional linear and nonlinear systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling for linear and nonlinear hyperbolic systems. Connecting geometry of nonlinear and linearised hyperbolic systems.

MOTIVATING EXAMPLE (4.2 IN GLENDINNING)

Phase plane example:

$$\dot{x} = x$$

$$\dot{y} = -y + x^2$$

GEOMETRY OF LINEAR SYSTEMS

Let's return to general linear systems in $\mathbb{R}^n$ for a moment to give the following geometric definitions of three key *invariant manifolds/subspaces* for linear systems.

The *flow in the full phase space* is then given by a *linear superposition of motion on these three subspaces*.

These subspaces *also have nonlinear counterparts*, but we will need to consider some aspects more carefully, e.g. linear superposition fails (in the original coordinates anyway).

GEOMETRY OF LINEAR SYSTEMS - STABLE MANIFOLD

Suppose $x \in \mathbb{R}^n$ is a stationary solution to the linear system $\dot{x} = Ax$.

The *stable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^s(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, negative part*.

TERMINOLOGY: LINEAR SUBSPACES AND SPANS

A *linear subspace* of $\mathbb{R}^n$ is a subset E of $\mathbb{R}^n$ which contains the *zero vector* $0 \in \mathbb{R}^n$ and which is *closed* under *vector addition* $u + v$, where $u, v \in \mathbb{R}^n$, and *scalar multiplication* cu, where $c \in \mathbb{R}, u \in \mathbb{R}^n$.

The *span* of a set of vectors is the set generated by *all linear combinations* of those vectors i.e. $\text{span}\{u, v, ...\} = \{x \in \mathbb{R}^n \mid x = au + bv + ..., a, b \in \mathbb{R}, u, v... \in \mathbb{R}^n\}$

GEOMETRY OF LINEAR SYSTEMS - UNSTABLE MANIFOLD

Similarly:

The *unstable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^u(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, positive part*.

GEOMETRY OF LINEAR SYSTEMS

Finally:

The *centre manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^c(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *zero real part*.

TERMINOLOGY: WHAT'S A MANIFOLD?

For our purposes it suffices to think of a *manifold* embedded in $\mathbb{R}^n$ as a subset of $\mathbb{R}^n$, each point of which satisfies $m \leq n$ *constraints*.

This means that, given regularity conditions, a manifold is an *$(n - m)$-dimensional object embedded in n dimensional space*.

Example: a circle is a 1-dimensional manifold which is embedded in $\mathbb{R}^2$ and satisfies one constraint. So, the unit circle can be thought of as e.g. $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$.

TERMINOLOGY: WHAT'S A MANIFOLD?

A key property is that locally an $(n - m)$-dimensional manifold embedded in n-dimensional space *'looks like' a small 'open ball' of dimension $\mathbb{R}^{(n-m)}$*.

E.g. a curve embedded in $\mathbb{R}^2$ can be considered as 'pieced together' from small segments of $\mathbb{R}$, a sphere in $\mathbb{R}^3$ can be considered as 'pieced together' from small 'patches' of $\mathbb{R}^2$

The more general definition throws away the 'background' space and works with the 'intrinsic' $(n - m)$-dimensional object itself.

RECALL: HARTMAN-GROBMAN

We have previously considered how the *existence and stability of hyperbolic fixed points* are preserved during linearisation.

We now want consider the *differences in local dynamics between a nonlinear system and its linearisation in more detail*.

We'll look at how to do this using the *stable manifold theorem* and then using series expansions to approximate local stable/unstable manifolds.

GEOMETRY: STABLE AND UNSTABLE MANIFOLDS

Above we defined the *stable and unstable manifolds for linear systems*. (For non-hyperbolic there is also a centre manifold)

Now we want to give the definitions for *nonlinear hyperbolic fixed points*.

GEOMETRY: STABLE MANIFOLD (LOCAL)

Given some neighbourhood U of a stationary point x, the *local stable manifold* on U for a nonlinear system $W^s_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to \infty, \phi(y,t) \in U \text{ for all } t \geq ($$

What does this mean?

GEOMETRY: UNSTABLE MANIFOLD (LOCAL)

Similarly, given some neighbourhood U of a stationary point x, the *local unstable manifold* on U for a nonlinear system $W^u_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to -\infty, \phi(y,t) \in U \text{ for all } t \leq$$

What does this mean?

GLOBAL MANIFOLDS

Note that if we want *global* versions then we can *'glue' together* all the flows starting at points in the local stable/unstable manifolds. That is,

$$W^s(0) = \bigcup_{t \geq 0} \phi(W^s_{loc}(0), t)$$

$$W^u(0) = \bigcup_{t \leq 0} \phi(W^u_{loc}(0), t)$$

STABLE MANIFOLD THEOREM

What's the *connection between these linear and nonlinear stable/unstable manifolds?* We have the following theorem (for local manifolds).

Suppose the origin is a *hyperbolic fixed point* for $\dot{x} = f(x)$ in $\mathbb{R}^n$ and that $E^s(0)$ and $E^u(0)$ are the stable and unstable manifolds of the linearised system $\dot{x} = Df(0)x$.

Then…

STABLE MANIFOLD THEOREM

…there exist *local stable and unstable manifolds* $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the *same dimension* as $E^s(0)$ and $E^u(0)$, respectively, and which are (respectively) *tangent* to E^s and E^u at the origin.

These manifolds are equally smooth/unsmooth as the original function f.

CALCULATING THE MANIFOLDS - THE 'MANIFOLD EQUATION'

The basic idea is to substitute the defining equation $y = U(x)$ or $x = V(y)$ into the governing equations and use the *chain rule* applied *along the manifold*:

$$\frac{dy}{dt}(x, U(x)) = \frac{dy}{dx}(x)\frac{dx}{dt}(x, U(x))$$

from which to find $U(x)$.

Let's call this the *'manifold equation'*. We usually solve it locally by assuming a *power series solution* - justified by the SMT!

POWER SERIES EXPANSIONS IN TWO-DIMENSIONAL SYSTEMS

Even on small neighbourhoods of our fixed points, *our manifolds are no-longer straight lines* (or hyperplanes etc in higher dims) as in the linear case - they are *curves* (or surfaces in higher-dimensions).

We can, however, use the information from the previous theorem to (try to) *compute local expressions for these curves*.

POWER SERIES EXPANSIONS FOR ONE-DIMENSIONAL MANIFOLDS

Assume a stable/unstable manifold of interest can be described by a curve $y = U(x)$ (or $x = V(y)$).

We can try to approximate this by a *local series expansion* of the form

$$y = U(x) = \sum_{n=0}^{\infty} a_n x^n$$

POWER SERIES: STEPS

- *Assume* the manifold can be described by $y = U(x)$ (or the other way around).
- *Substitute* $y = U(x)$ into our x and y equations to give $\dot{x} = f_1(x, U(x))$ and $\dot{y} = f_2(x, U(x))$.
- Use $y = U(x)$ *again*, along with the *chain rule* for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dU}{dx}\dot{x}$.

POWER SERIES: STEPS

- Use the above relationships along with an assumed *power series* expansion such as $U(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain *two expressions* in x for $\dot{y}$
- *Equate* powers of x to determine the unknown coefficients.
- Make sure to use the fact that the stable/unstable manifold *passes through* the fixed point and *is tangent* to the linearised stable/unstable manifold to determine the *first two terms* of the series.

EngSci 711 L05

- Geometry of Hyperbolic Systems } arbitrary dimensions in general
  - Linear
  - Nonlinear } connecting

Goals

- Given an ODE system,
- Find stable/unstable linear subspaces E^s, E^u
- Find stable/unstable nonlinear subspaces
  ie stable/unstable manifolds, W^s, W^u
- Sketch local trajectories

Examples

(+ Assignment 1 / Tutorial 2)

- Exam 2016

Question 5 (20 marks)

Consider the system

$$\begin{aligned} \dot{x} &= y(2x - y) \\ \dot{y} &= x^2 - y \end{aligned}$$

(a) Find the two fixed points of this system. Show your working. You do not need to classify these.

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin (0,0).

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0,0)$ that is correct up to and including cubic order.

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.

- Exam 2017

Question 4 (16 marks)

Consider the system

$$\begin{aligned} \dot{x} &= 2xy + x^3 \\ \dot{y} &= -y - x^2 \end{aligned}$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system. (1 mark)

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0, 0)$. (2 marks)

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories. (3 marks)

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0, 0)$ that is correct up to and including cubic order. (8 marks)

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable. (2 marks)

Note: these focus on centre (non-hyperbolic) manifold
→ today we do same for stable & unstable manifolds. } hyperbolic
we return to centre manifold case later → same basic ideas tho'.

Example (4.2 in Glendinning)

Consider analysing

$$\boxed{\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y + x^2 \end{aligned}}$$

Key steps

- Find fixed points
- Linearise near fixed points
- Classify fixed points

} so far

- Sketch a 'local picture' of flow near fixed points
- Build up a more 'global' picture
  - extend/join flows
  - look for other 'global' objects (like periodic orbits etc)

} need to understand the local 'geometry'
- linearised
- nonlinear
↓ link

Local flow/trajectories?

Linear systems: superposition of flows on eigenspaces
linear subspaces

Nonlinear systems: similar idea
→ decompose into flows on 'nonlinear subspaces'
└ 'manifolds'
→ these are 'curved' versions of linear subspaces!

Key idea: decompose into motions on/in invariant sets/spaces:

$$\boxed{\begin{array}{c} x \in M \implies \varphi(x,t) \in M \text{ for all } t \\ \text{'once in/on, stay in/on'} \end{array}}$$

Recall linear separation: real & distinct example.

Given a linear system

$$\boxed{\dot{x} = Ax}, \quad x \in \mathbb{R}^n$$

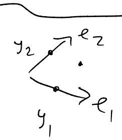
& suppose the eigenvalues are real & distinct

⇒ It can be shown (see supplement & later lectures) that changing coordinates to y using

$$\boxed{x = Py} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ e_1 & e_2 & \cdots & e_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

$$= \underbrace{y_1 \begin{bmatrix} \vdots \\ e_1 \\ \vdots \end{bmatrix} + y_2 \begin{bmatrix} \vdots \\ e_2 \\ \vdots \end{bmatrix} + \dots y_n \begin{bmatrix} \vdots \\ e_n \\ \vdots \end{bmatrix}}$$

ie



y is coord rel. to eigenbasis

leads to a diagonal system

$$\boxed{\dot{y} = \Lambda y}, \text{ where } \Lambda = P^{-1}AP$$

$$\& \ \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

→

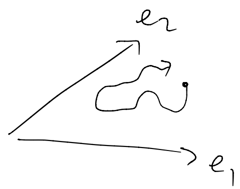
(similar results apply more generally).

$$\Rightarrow \begin{bmatrix}\dot y_1\\ \dot y_2\\ \vdots\\ \dot y_n\end{bmatrix} = \begin{bmatrix}\lambda_1 &&&\\ &\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{bmatrix}\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned}\dot y_1 &= \lambda_1 y_1\\ \dot y_2 &= \lambda_2 y_2\\ &\vdots\\ \dot y_n &= \lambda_n y_n\end{aligned}\right\}$$ motion is |separated| into independent motions

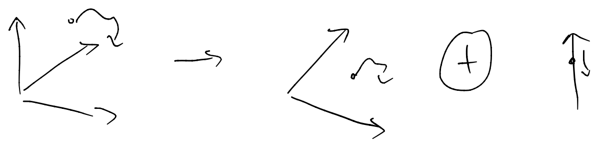
eg can solve each separately. 'Decoupled'!

Actual motion is 'sum' of motions along each eigendirection



} motion in e_1, e_2 plane

This motivates decomposing into groups

eg 

→

In particular into |+ve/−ve groups| (real, distinct ex.)

or |stable/unstable| groups ($Re\,\lambda > 0, < 0$) (general case)

– real, distinct example: suppose w.log. that

$\lambda_1, \dots \lambda_m < 0$ } stable,
$\lambda_{m+1}, \dots \lambda_n > 0$ } unstable, then:

$$\begin{bmatrix}\begin{bmatrix}\dot y_1\\ \dot y_2\\ \vdots\\ \dot y_m\end{bmatrix}\\ \begin{bmatrix}\dot y_{m+1}\\ \vdots\\ \dot y_n\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\lambda_1 &&\\ &\boxed{\text{stable}}&\\&&\lambda_m\end{bmatrix} & 0\\ 0 & \begin{bmatrix}\lambda_{m+1}&&\\&\boxed{\text{unstable}}&\\&&\lambda_n\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}y_1\\ \vdots\\ y_m\end{bmatrix}\\ \begin{bmatrix}y_{m+1}\\ \vdots\\ y_n\end{bmatrix}\end{bmatrix}$$

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ie two separated/independent parts:

'stable motion' & 'unstable motion'

in 'subspace' 'spanned by $Re(\lambda_i) < 0$ eigenvectors'

in 'subspace' 'spanned by $Re(\lambda_i) > 0$ eigenvectors'

Idea:

– applies in general (stable/unstable)
  ↳ but need generalised eigenvectors

– can extend to nonlinear
  ↳ need to be careful
    ↳ subspace?
    ↳ decoupled?

'subspace'?

Linear subspace of $\mathbb{R}^n$: some 'mathy intuition'

a subset E of $\mathbb{R}^n$

where
- $0 \in E$ } goes through zero
  - ↳ note: 0 is a vector eg $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$
- if $u, v \in E$ then $u + v \in E$
- if $u \in E$ & $a \in \mathbb{R}$ then $au \in E$

} linear combinations 'stay in' space

ie : lines & (hyper) planes

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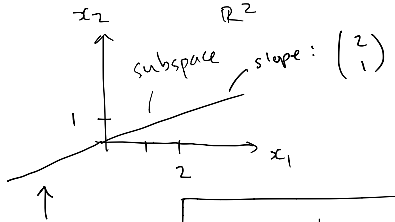
eg



can define as : $\{ x \in \mathbb{R}^2 \mid x = c \cdot u,\ c \in \mathbb{R},\ u = \binom{1}{2} \}$ ← span

or $\{ x \in \mathbb{R}^2 \mid x_2 = \frac{1}{2} x_1 \}$ ← constraint

so

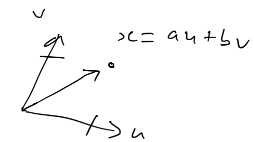
either : • $\boxed{\text{span}}$ of u (see next page) — explicit

or • a $\boxed{\text{subset}}$ of $\mathbb{R}^2$ satisfying one linear $\boxed{\text{constraint}}$ on components x_1, x_2 — implicit

} both ideas can be generalised to nonlinear setting
- second easier for us
- but first useful in linear case

Span? of a set of vectors $\{u, v, \dots\}$

→ just the set of all linear combinations of the given vectors



eg

$$\boxed{\text{span}\{u, v, w\} = \{ x \mid x = au + bv + cw;\ a, b, c \in \mathbb{R} \}}$$

→ 'explicit' construction

Q : what is the span of $\{ (1,0)^T, (0,1)^T \}$?

A : all of $\mathbb{R}^2$

Challenge : prove !

Linear subspaces – via linear $\boxed{\text{constraints}}$

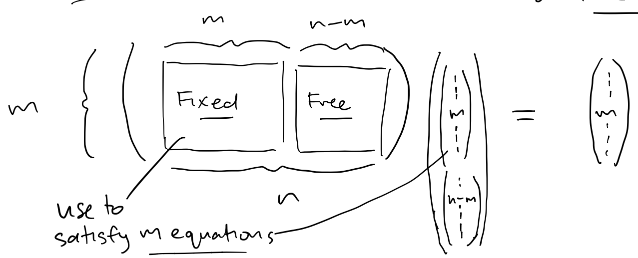
- can also think of defining a linear subspace via linear constraints on vectors in a larger space

$$\text{eg} \quad \boxed{\begin{array}{l}\{x \mid x_1 = \tfrac{1}{2}x_2\} \text{ or } \{x \mid x_1 - \tfrac{1}{2}x_2 = 0\} \\ \text{ie } \{x \mid \text{constraint}\}\end{array}}$$

- 'implicit' definition : 'what's left over' ('null space')
- eg $\boxed{x \in \mathbb{R}^n}$, impose $\boxed{\text{m linear constraints}}$ (independent)

  → get $\boxed{n-m}$ 'degrees of freedom' remaining or 'free' } $\boxed{\text{left over}}$

Underdetermined linear system : implicitly define a 'free subspace'



→ if $m = n$ then just define a unique point

→ if $m < n$ get 'infinitely' many sol^{n}s, lying on a (hyper) plane of dim $n-m$

→ nontrivial null space (implicit : null complement to range space)

So.... back to ODEs.

Linear subspaces (assume FP is at 0.)

- stable $E^s(0)$: span of eigenvectors/generalised eigenvectors associated with $\boxed{Re(\lambda) < 0}$
- unstable $E^u(0)$: span .... $\boxed{Re(\lambda) > 0}$
- centre $E^c(0)$: span .... $\boxed{Re(\lambda) = 0}$

  ↳ if $Re(\lambda)$ & $Im(\lambda) = 0$ ie $\boxed{\lambda = 0}$, also called 'slow'

Side note : in general we need to allow complex eigenvectors &/or generalised eigenvectors

→ I'll avoid as much as possible but see linear algebra handout for fun!

Example cont'd.

$$\left.\begin{array}{l}\dot{x} = x \\ \dot{y} = -y + x^2\end{array}\right\} \quad Df(x,y) = \begin{pmatrix} 1 & 0 \\ 2x & -1 \end{pmatrix}$$

$$\Rightarrow Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{'}A\text{'}$$

Cont'd.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$tr = \lambda_1 + \lambda_2 = 0$

$det = \lambda_1 \lambda_2 = -1$

$\Rightarrow \lambda_1 = -\lambda_2$
$= 1$

$\Rightarrow$ diagonal $\Rightarrow$ $\lambda_1 = 1$, $\lambda_2 = -1$

(see Linear Algebra supplement)

$\lambda = 1$ } unstable dir

$\lambda = -1$ } stable dir.

Find E^u. → find eigvector

$$[A - \lambda I]u = 0$$

if $\lambda = 1$, $A - \lambda I$

$$= \begin{pmatrix} 1-1 & 0 \\ 0 & -1-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

ie $$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

choose arb. value

$\Rightarrow$ $u_2 = 0$, $u_1 =$ free } $e^u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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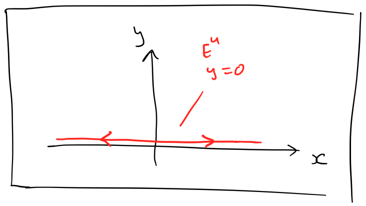
ie $E^u = \{ (x,y) \mid y = 0 \}$ implicit

$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ explicit.

ie

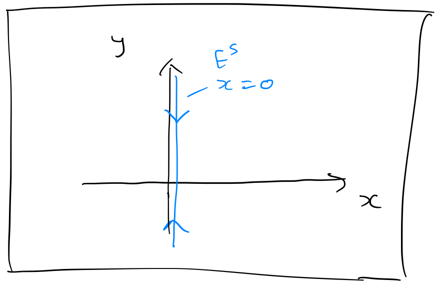


Find E^s (stable, $\lambda = -1$)

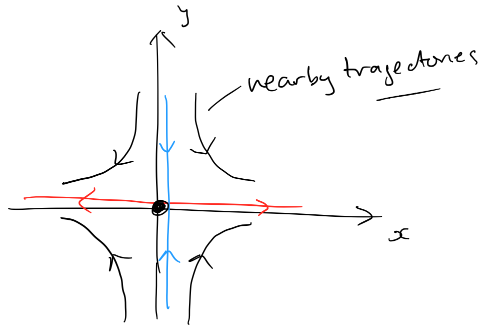
Similarly → exercise!

should get $E^s = \{ (x,y) \mid x = 0 \}$

or $E^s = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

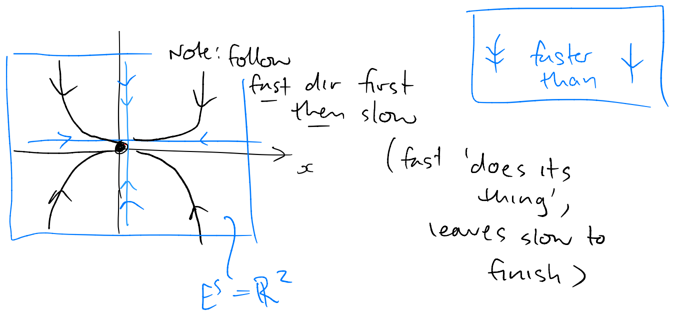


Put together: Local picture

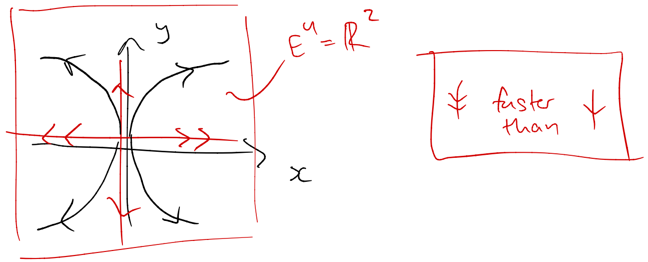


Notes:

– if $\boxed{\text{both}}$ stable/unstable, span = whole plane!



(fast 'does its thing', leaves slow to finish)



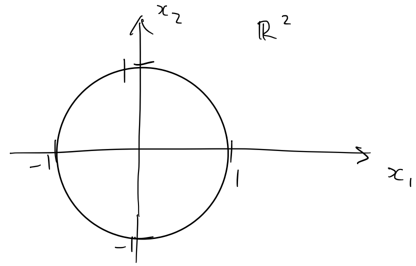
– Rule of thumb: 'fast' direction dominates away from origin, trajectories approach/leave origin tangent to 'slow' direction

Manifolds – 'nonlinear subspaces'

- Basic idea:
  ↳ subset W of $\mathbb{R}^n$ (here), instead of 'E' for linear
  defined by m (possibly) $\boxed{\text{nonlinear constraints}}$ } easier to generalise than 'span'

- Example:
  $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ (nonlinear constraint)

  → defines a circle!



→ a linear subspace is just a special kind of manifold:
- generated by linear constraints

→ manifolds can 'curve' & 'close', but locally look linear

Connection between linear & nonlinear

- Hartman-Grobman
  - → existence & stability of FP for hyperbolic:

Linear FP	⟷	nonlinear FP

- stable manifold theorem } today
  - → existence & tangency of manifolds for hyperbolic:

Linear geometry (stable/unstable subspaces)	⟷	nonlinear geometry (stable/unstable manifolds)

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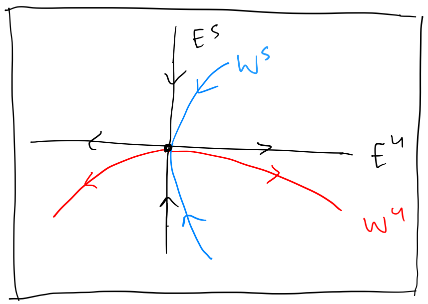
ie $E^s \to W^s$, $E^u \to W^u$ } for hyperbolic use above } today

$E^c \to ?$ } need 'centre manifold theorem' } later



Key: linear & nonlinear manifolds

- same $\boxed{\text{dimension}}$
- $\boxed{\text{tangent}}$ at FP.
- each is $\boxed{\text{invariant}}$ for respective (linear/nonlinear) system

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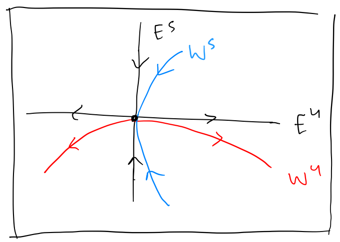
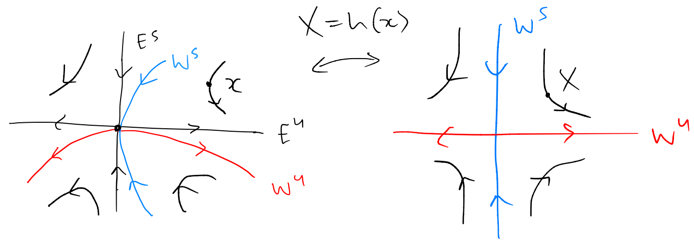
Side Note: Linear vs nonlinear

Consider:



HG & SMT: Local nonlinear change of coord:



→ Nonlinear manifolds of original system are linear manifolds of a nonlinear transformation of the original system!

[see eg Roberts ch. 10].

(still need nonlinear change of coord)

→ we will mainly consider W^s, W^u etc as nonlinear manifolds of the original system

Defining nonlinear stable/unstable manifolds

- same 'issue' as stability in nonlinear case
  - → can't define in terms of eigenvalues/eigenvectors } linear ideas
  - → want 'truly nonlinear'/general definitions
  - → BUT: as before, want to then relate the general case back to linear case } Stable manifold theorem

- Intuition:
  - stable as an $\boxed{\text{invariant}}$ set near origin/FP such that! $\boxed{\text{points} \rightarrow \text{FP}}$ as $\boxed{t \rightarrow \infty}$

  - unstable: $\boxed{\text{invariant}}$ set near origin/FP such that: $\boxed{\text{points} \rightarrow \text{FP}}$ as $\boxed{t \rightarrow -\infty}$

    └ trick: unstable as $t \rightarrow \infty$ is stable as $t \rightarrow -\infty$!

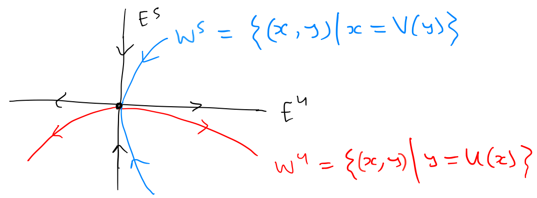
Local vs global

- These are local definitions
- But, as invariant manifolds, can trace out/piece together/extend to global objects
- see videos for Lorenz (UoA math dep.)

'Manifold Equation'

Let's calculate our manifolds & the flows on these! (Semi-quantitative)

- Idea: exists either $y = U(x)$ or $x = V(y)$ } local (or multivariable surface if higher dim) expressions for manifolds:



- Substitute in & use chain rule

1. subs: $\dot{x} = f_1(x,y) = f_1(x, U(x))$
   $\dot{y} = f_2(x,y) = f_2(x, U(x))$

2. chain: (along $y = U(x)$) $\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} \Leftrightarrow \boxed{\dot{y} = \dot{x} \cdot \frac{dy}{dx}}$

Gives the 'Manifold Equation'

$$\boxed{\dot{y}(x) = \dot{x}(x) \cdot \frac{dy}{dx}(x)}$$

Solution is $\boxed{y = U(x)}$

Power Series Sol^n

We only really want 'local' sol^n to a given order

$\longrightarrow$ use a power series expansion

Example cont'd: unstable W^u for

$$\dot{x} = x$$
$$\dot{y} = -y + x^2$$

$$y = U(x) = a_0 + a_1 x + a_2 x^2 + \dots \quad (1)$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + \dots \quad (2).$$

$$\dot{x}(x) = f_1(x, y(x)) = x \quad (3)$$

$$\dot{y}(x) = f_2(x, y(x)) = -(a_0 + a_1 x + \dots) + x^2$$

$$= -a_0 - a_1 x + (1 - a_2)x^2 + \dots \quad (4)$$

Gives:

$$\dot{y} = \frac{dy}{dx} \cdot \dot{x} = \left[a_1 + 2a_2 x + \dots \right] x \quad \text{(manifold eq}^n\text{)}$$

$$= -a_0 - a_1 x + (1 - a_2)x^2$$

Cont'd.

So

$$a_1 x + 2a_2 x^2 + \dots = -a_0 - a_1 x + (1 - a_2)x^2$$

---

solving:

1. use
   a) pass through origin
   b) tangent to E^u at origin } distinguishes E^u & E^s
   c) equate coefficients for rest.

---

a & b) (From SM Theorem)

$$y = a_0 + a_1 x + \dots$$

$$y(0) = 0 \Rightarrow \boxed{a_0 = 0}$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + \dots$$

$$\frac{dy}{dx}(0) = 0 \Rightarrow \boxed{a_1 = 0}$$

tip: do this straight away!

---

c).

$$2a_2 x^2 + \dots = (1 - a_2)x^2$$

$$\Rightarrow 2a_2 = 1 - a_2$$

$$\Rightarrow \boxed{a_2 = \frac{1}{3}}$$

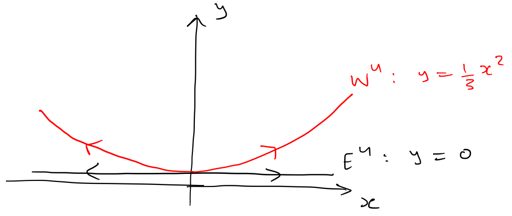
So we have:

$$y = \frac{1}{3}x^2 \quad \text{ie,}$$

Unstable manifold:

$$\boxed{W^u = \{ (x,y) \mid y = \tfrac{1}{3}x^2 \}}$$

So:



Note:

$$\dot{x}(u(x)) = x \quad \begin{matrix} <0 \text{ if } x<0 \\ >0 \text{ if } x>0 \end{matrix}$$

$$\begin{aligned} \dot{y}(u(x)) &= -y(x) + x^2 \\ &= -\frac{1}{3}x^2 + x^2 \\ &= \frac{2}{3}x^2 > 0 \end{aligned}$$

flow on unstable

Exercise: use $x = V(y)$ & find stable manifold!

(I'll go over in tutorial....)

Glendinning: stable manifold theorem and example 4.2

There are two important theorems for hyperbolic stationary points, the stable manifold theorem and Hartman's theorem. The first shows that the local structure of hyperbolic stationary points of nonlinear flows, in terms of the existence and transversality of local stable and unstable manifolds, is the same as the linearized flow, and the second asserts that there is a continuous invertible map in some neighbourhood of the stationary point which takes the nonlinear flow to the linear flow preserving the sense of time.

Let U be some neighbourhood of a stationary point, x. Then, by analogy with the definition of the invariant manifolds for linear systems we can define the local stable manifold of x, $W^s_{loc}(x)$, and the local unstable manifold of x, $W^u_{loc}(x)$, by

$$W^s_{loc}(x) = \{y \in U | \varphi(y,t) \to x \text{ as } t \to \infty,\ \varphi(y,t) \in U \text{ for all } t \geq 0\}$$

and

$$W^u_{loc}(x) = \{y \in U | \varphi(y,t) \to x \text{ as } t \to -\infty,\ \varphi(y,t) \in U \text{ for all } t \leq 0\}.$$

The stable manifold theorem states that these manifolds exist and are of the same dimension as the stable and unstable manifolds of the linearized equation $\dot{y} = Df(x)y$ if x is hyperbolic, and that they are tangential to the linearized manifolds at x.

(4.7) THEOREM (STABLE MANIFOLD THEOREM)

Suppose that the origin is a hyperbolic stationary point for $\dot{x} = f(x)$ and E^s and E^u are the stable and unstable manifolds of the linear system $\dot{x} = Df(0)x$. Then there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the same dimension as E^s and E^u respectively. These manifolds are (respectively) tangential to E^s and E^u at the origin and as smooth as the original function f.

Note that in Chapter 3 the centre manifold of a stationary point was also defined, but that for a hyperbolic stationary point the centre manifold is empty. We shall return to the problems of finding nonlinear centre manifolds in Chapters 7 and 8, where the basic ideas of bifurcation theory are introduced. The content of this theorem is illustrated in Fig. 4.1. The proof is, unfortunately, long and technical and we leave this to the end of this chapter, since we will be much more concerned with the use of this theorem.

One further point can be made without difficulty: suppose that x_0 is a hyperbolic stationary point, then there are three possibilities. Either $W^s_{loc}(x_0) = \emptyset$, or $W^u_{loc}(x_0) = \emptyset$, or both manifolds are non-empty. These three possibilities are given names: x_0 is called a source, sink or saddle respectively. From the definition of the linear stable and unstable manifolds and the stable manifold theorem it should be obvious that these definitions can be made in terms of the eigenvalues of the Jacobain matrix at x_0 in the following way.

(4.8) DEFINITION

Suppose that x_0 is a hyperbolic stationary point of $\dot{x} = f(x)$ and let $Df(x_0)$ denote the Jacobian matrix of f evaluated at x_0. Then x_0 is a sink if all the eigenvalues of $Df(x_0)$ have strictly negative real parts and a source if all the eigenvalues of $Df(x_0)$ have strictly positive real parts. Otherwise x_0 is a saddle.

We shall see in Sections 4.3 and 4.5 that for small perturbations of the defining equations, a source remains a source, a sink remains a sink and a saddle remains a saddle. Furthermore, as one would expect, if x_0 is a sink then it is asymptotically stable.

If we choose a coordinate system for which the linear part of the differential equation at the origin is in normal form we can always arrange

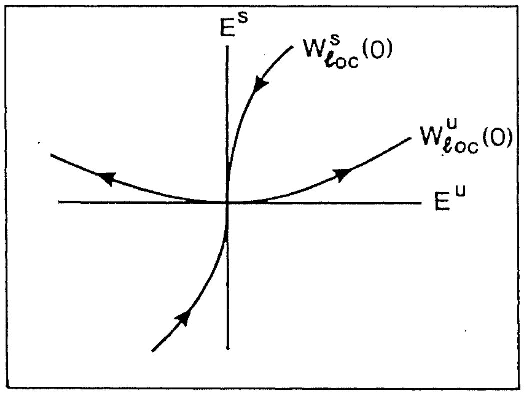


Fig. 4.1 Stable and unstable manifolds.

Glendinning: stable manifold theorem and example 4.2

for the differential equation to be of the form

$$\dot{x} = Ax + g_1(x,y) \quad \dot{y} = -By + g_2(x,y) \tag{4.21}$$

where $x \in \mathbf{R}^{n_u}$, $y \in \mathbf{R}^{n_s}$ (where n_u is the dimension of the local unstable manifold and n_s is the dimension of the local stable manifold, $n_u + n_s = n$) and both the square matrices A and B have eigenvalues with positive real parts. The functions $g_i(x,y)$, $i = 1,2$, contain the nonlinear parts of the equation, so they vanish, together with their first derivatives at the origin, $(x,y) = (0,0)$. Hence

$$E^s(0,0) = \{(x,y)|x = 0\} \text{ and } E^u(0,0) = \{(x,y)|y = 0\}.$$ here

Since the stable and unstable manifolds are smooth and are tangential to these manifolds at the origin they can be described as the graphs of functions, so the stable manifold is given by

$$x_i = S_i(y), \quad i = 1,\ldots,n_u \tag{4.22}$$

where

$$\frac{\partial S_i}{\partial y_j}(0) = 0, \; 1 \le i \le n_u, 1 \le j \le n_s \tag{4.23}$$

since the manifold is tangential to E^s at 0. Similarly we can write the unstable manifold (again locally) as

$$y_j = U_j(x), \quad \frac{\partial U_j}{\partial x_i}(0) = 0, \; 1 \le i \le n_u, 1 \le j \le n_s. \tag{4.24}$$

This observation allows us to approximate the stable and unstable manifolds by expanding the functions S_i and U_j as power series. Consider U_j (the argument is the same for S_i). We begin by expanding U_j as a power series in x, so

$$U_j(x) = \sum_{r \ge 2} \sum_{m \in M_r} u_{mj} x^m \tag{4.25}$$

where the notation is as in the previous section. If B is diagonal then with eigenvalues (λ_i), $i = 1, 2, \ldots, n_s$ then

$$\dot{y}_j = -\lambda_j y_j + g_{2j}(x,y) \tag{4.26}$$

and on the unstable manifold $y = U(x)$ so

$$\dot{y}_j = -\lambda_j U_j(x) + g_{2j}(x, U(x)). \tag{4.27}$$

On the other hand

$$\dot{y}_j = \frac{d}{dt} U_j(x) = \sum_{k=1}^{n_u} \dot{x}_k \frac{\partial}{\partial x_k} U_j(x). \tag{4.28}$$

Comparing the right hand sides of (4.27) and (4.28) we find that

$$-\lambda_j U_j(x) + g_{2j}(x, U(x)) = \sum_{k=1}^{n_u} \dot{x}_k \frac{\partial}{\partial x_k} U_j(x) \tag{4.29}$$

and we can now substitute the series expansion for the functions U_j into these equations and equate coefficients of powers of x in order to get a set of simultaneous equations for the coefficients u_{mi} which can be solved to arbitrary order. An example may make this clearer.

Example 4.2

Consider the equations

$$\dot{x} = x, \quad \dot{y} = -y + x^2.$$

This has a unique stationary point at $(x,y) = (0,0)$ and the equation is already in normal form near the stationary point. The linearized equation is

$$\dot{x} = x, \quad \dot{y} = -y,$$

giving a saddle at the origin with invariant linear subspaces

$$E^s(0,0) = \{(x,y)|x = 0\} \text{ and } E^u(0,0) = \{(x,y)|y = 0\}.$$

By the stable manifold theorem we know that the nonlinear system has a local unstable manifold of the form

$$y = U(x), \quad \frac{\partial U}{\partial x}(0) = 0$$ in this case

and so we try a series expansion for U,

$$U(x) = \sum_{k \ge 2} u_i x^i.$$

Now,

$$\dot{y} = -y + x^2 = -\sum_{k \ge 2} u_i x^i + x^2$$

on the unstable manifold and also

$$\dot{y} = \dot{x} \frac{\partial U}{\partial x}(x) = \sum_{k \ge 2} k u_k x^k.$$

Equating terms of order x^2, x^3 and so on gives

$$-u_2 + 1 = 2u_2, \text{ and } -u_k = k u_k, \; k \ge 3.$$

Glendinning: stable manifold theorem and example 4.2

Hence $u_2 = \frac{1}{3}$, $u_k = 0$ for $k \geq 3$ and so

$\frac{1}{3}x^2$

$$W^u_{loc}(0,0) = \{(x,y) | y = \tfrac{1}{3}\cancel{x^3}\}.$$

A similar exercise shows that $W^s_{loc}(0,0) = E^s(0,0)$.

Later in this book (Chapter 12) we will see that a great deal of interesting dynamics is controlled by the behaviour of the stable and unstable manifolds of stationary points; for this we need to extend the local manifolds to obtain global stable and unstable manifolds defined by

$$W^u(0) = \bigcup_{t\leq 0} \varphi(W^u_{loc}(0), t) \text{ and } W^s(0) = \bigcup_{t\geq 0} \varphi(W^s_{loc}(0), t).$$

The second result of this section is associated with a weakening of the requirements of Poincaré's Linearization Theorem. In the previous section we looked for a change of variable such that the equation in the new variable is locally just the linear flow. This turned out to be quite a tough condition to meet, but in Example 4.1 we saw that even when the linearization has resonant eigenvalues the flow was remarkably similar to the linear flow (at least for the hyperbolic stationary point considered). This suggests that an alternative strategy might be to look for a map from the nonlinear flow to the linear flow in a neighbourhood of the stationary point, which takes trajectories of the nonlinear flow to trajectories of the linear flow.

* (4.9) THEOREM (HARTMAN'S THEOREM)

If $x = 0$ is a hyperbolic stationary point of $\dot{x} = f(x)$ then there is a continuous invertible map, h, defined on some neighbourhood of $x = 0$ which takes orbits of the nonlinear flow to those of the linear flow $exp(tDf(0))$. This map can be chosen so that the parametrization of orbits by time is preserved.

Note that the map is only continuous (not necessarily differentiable) and so it does not distinguish between, for example, a logarithmic spiral (cf. (3.31)) and the phase portrait obtained when the Jacobian at the stationary point has real eigenvalues. If we want greater smoothness we find ourselves involved once again in problems of resonance.

4.3 Persistence of hyperbolic stationary points

Another important feature of hyperbolic stationary points is the fact that they persist under small perturbations of the defining differential equations. Hence if the origin is a hyperbolic stationary point of $\dot{x} = f(x)$ and v is any smooth vector field on $\mathbf{R}^n$ then for sufficiently small ϵ the equation

$$\dot{x} = f(x) + \epsilon v(x) \tag{4.30}$$

has a hyperbolic stationary point near the origin of the same type as the hyperbolic point of the unperturbed equation. This robustness, together with the results of the previous section, shows that the dynamics in a neighbourhood of a hyperbolic stationary point is not radically altered by small perturbations. This will be of crucial importance when we come to consider bifurcation theory in Chapters 7 and 8. To see this, suppose that $f(0) = 0$ and look for stationary points of the perturbed system. They satisfy

$$f(x) + \epsilon v(x) = 0. \tag{4.31}$$

Expanding this equation about $x = 0$ (or using the implicit function theorem) gives

$$[Df(0) + \epsilon Dv(0)]x + \epsilon v(0) + O(|x|^2) = 0 \tag{4.32}$$

with solutions

$$x = -\epsilon[Df(0) + \epsilon Dv(0)]^{-1}v(0) + O(\epsilon^2) \tag{4.33}$$

provided $[Df(0) + \epsilon Dv(0)]$ is invertible. Now, if $x = 0$ is a hyperbolic stationary point, the eigenvalues of $Df(0)$ are bounded away from zero and hence the eigenvalues of $[Df(0) + \epsilon Dv(0)]$ are bounded away from zero for sufficiently small ϵ. So $\det[Df(0) + \epsilon Dv(0)] \neq 0$ for sufficiently small ϵ and hence this matrix is invertible. We now want to show that the stationary point of the perturbed equation is also hyperbolic. By continuity in ϵ, there is a neighbourhood of $\epsilon = 0$ for which the real parts of the eigenvalues of $[Df(x) + \epsilon Dv(x)]$ are all non-zero for sufficiently small x. In particular, no eigenvalue can cross the imaginary axis and so the number of eigenvalues on the right of the imaginary axis and on the left of the imaginary axis is the same for all x sufficiently small in this neighbourhood of $\epsilon = 0$. Now simply choose ϵ small enough so that it is in this neighbourhood of $\epsilon = 0$ and the stationary point of the perturbed equation has sufficiently small $|x|$. Then for all values of ϵ sufficiently small the stationary point of the perturbed equation is hyperbolic.

Engsci 711

Linear Algebra supplement

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Note: you don't need to know this material for the exam (I don't think!), other than basic stuff used in lectures.

Other the other hand, some *might* be useful for assignments and for e.g. understanding the full range of possible behaviour in linear systems.

Eigenvalues, eigenvectors and all that

Here we recall some basic facts from linear algebra. Some will be stated in full generality, i.e. for $\mathbb{R}^n$, some will be stated for $\mathbb{R}^2$ or $\mathbb{R}^3$. This is because I'm lazy. For more detail see e.g. Glendinning or Kuznetsov (or more generally, any book on linear algebra - I recommend the ones by Strang).

Fact

An $n \times n$ matrix always has n eigenvalues, some of these may be repeated however. The number of times an eigenvalue is repeated is called the multiplicity of that eigenvalue (e.g. the number of times it is a repeated root of the characteristic equation).

Fact

The trace of $n \times n$ matrix is equal to the sum of the n eigenvalues. The determinant is equal to the product of the n eigenvalues.

Fact

If an $n \times n$ matrix is upper or lower triangular, then the eigenvalues are just the diagonal elements. Similarly for a purely diagonal matrix.

Fact

If the eigenvalues of a matrix A are all distinct then the associated eigenvectors are linearly independent (but not necessarily orthogonal). Moreover these are unique up to a scale factor. These eigenvectors can be used to diagonalise A (see below).

Fact

Square matrices with distinct eigenvalues are always diagonalisable. Some matrices with non-distinct eigenvalues are diagonalisable, however - a simple example is the identity matrix.

Fact

Diagonalisability and invertibility are distinct concepts: e.g. you can diagonalise some non-invertible matrices. You can also invert some non-diagonalisable matrices.

Fact

Square matrices are invertible if and only if all eigenvalues are non-zero. Furthermore, all eigenvalues are non-zero if and only if the matrix has a non-zero determinant, if and only if it has linearly independent columns, and if and only if it has linearly independent rows.

Fact

Given a repeated eigenvalue you can sometimes find multiple corresponding standard eigenvectors in the usual way. You just get multiple independent solutions to $(A - \lambda I)x = 0$. This is not guaranteed in general, however (see next result for a condition).

Implication for two-dimensional differential equations If the eigenvalue of a two-dimensional (planar) system is repeated and it *happens to have distinct standard eigenvectors*, then *every* vector in the plane is an eigenvector.

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5). This is called a 'star node' (see figure below). These are on the margin between being a spiral and being a standard node. So they are *sort of* structurally unstable cases in the sense of being boundary cases. On the other hand the actual *existence and stability* of the fixed point is *unchanged* (even though it may change from a spiral to a node, for example, both will have the *same stability*) so they do *still count as hyperbolic fixed points* (another way to see this is to note that a stable spiral and a stable node are topologically equivalent, i.e. can be deformed into one another).

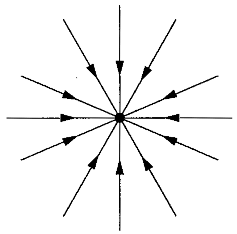

Figure 1: Star node - repeated eigenvalues but distinct eigenvectors. The eigenspace is thus the whole plane.

Fact

Given an $n \times n$ symmetric real matrix $A = A^T$ (or *Hermitian* when dealing with complex matrices) then there are n standard eigenvectors and these are distinct and linearly independent. Furthermore, the eigenvalues are all real, though not necessarily distinct.

Fact

For a non-symmetric $n \times n$ matrix with repeated eigenvalues we can always find an orthogonal basis of *generalised eigenvectors* (sometimes called principle vectors). A generalised eigenvector x of rank m is defined as a non-zero solution to

$$(A - \lambda I)^m x = 0$$

where $m \geq 1$ and

$$(A - \lambda I)^n x \neq 0$$

for $n < m$. That is, it satisfies the usual equation, except $(A - \lambda I)$ is raised to the power of m. The second part of the definition just makes sure x doesn't satisfy the equations of lower rank and so the rank well-defined (i.e. it is the smallest power m satisfying the above equation).

Fact

A general *block diagonal* form of a matrix is:

$$\begin{pmatrix} A_1 & ... & ... \\ ... & A_2 & ... \\ ... & ... & A_3 \end{pmatrix}$$

where each A_i can be any size and internal structure, and any '...' entries are taken to be zero. When this represents a system of linear differential equations, this form is sufficient to *decouple* the system into *self-contained subsystems that can be solved separately.*

Fact

Generalised eigenvectors can be used to put A into Jordan normal form (see below). For square matrices this gives a particular type of *block diagonal* structure. It is a *generalisation of matrix diagonalisation, and is always possible.* It can be thought of as 'as close as possible to diagonal'. This is a preview of the Jordan normal form:

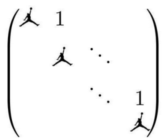

Figure 2: Jordan blocked (stolen from the internet...).

The following illustrates the structure in more detail, including the form of the *Jordan blocks*:

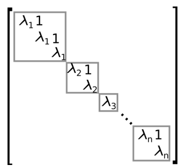

Figure 3: Jordan normal form with Jordan blocks illustrated (stolen from Wikipedia...). The off-diagonal ones occur when the corresponding eigenvalue is repeated, and the size of the block to the number of times the eigenvalue is repeated.

Fact

If x is a generalised eigenvector of rank m then $(A - \lambda I)^n x$ for $n < m$ is a generalised eigenvector of A of rank $m - n$. For example, if $m = 2$ and $n = 1$ then $(A - \lambda I)x$ is a generalised eigenvector of A.

Fact

From the above we see that, given a generalised eigenvectors of A of rank m, we can form a so-called chain of generalised eigenvectors $\{x_m, x_{m-1}, ...x_1\}$ of the form $x_m, (A - \lambda I)x_m, ..., (A - \lambda I)^{m-1}x_m$.

As an example, if we have a 2x2 system and a rank 2 eigenvector x_2, we form the *lower* rank eigenvector x_1 via

$$x_1 = (A - \lambda I)x_2$$

Implication for two-dimensional differential equations Suppose the eigenvalue of a two-dimensional (planar) system is repeated and there is *only one standard eigenvector* and hence one generalised eigenvector. As you might remember, in such cases we mutliply our solutions by t to get a linearly independent basis for our general solution.

Similarly, it can be shown (see e.g. Glendinning) that if e_1 is the standard eigenvector and e_2 is the generalised eigenvector, then the solution can be considered a linear supposition of exponential motions on e_1 and $te_1 + e_2$. Note that as $t \to \infty$, the vector $te_1 + e_2$ points more and more towards e_1, as the first term dominates the second term. Thus the motion on $te_1 + e_2$ - and, as time increases, the full motion - *collapses into motion approximately along the single, standard eigenvector* e_1 (note though: the collapse onto the standard eigenvector is not *exponentially* quick as it would be with a true slow manifold).

Again, these cases are on the margin of being a standard node and being a spiral. The solutions thus have a sort of *spiralling collapse onto the standard eigenvector.*

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5; see also figure below). Note: you can get the directions of motion (the 'spiralling' direction) by guessing a few interesting points and plugging them into the equations to find the flow direction.

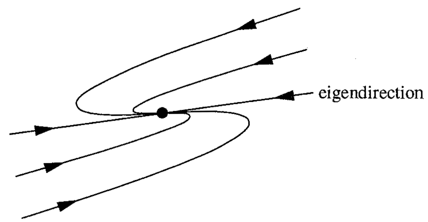

Figure 4: Degenerate node - repeated eigenvalues and only one standard eigenvector. As $t \to \infty$, the contribution of the generalised eigenvector disappears and the motion collapses to the single standard eigenvector.

Change of coordinates, diagonalisation and the Jordan normal form

Change of coordinates: eigenvector system

If we use eigenvectors in our coordinate system we get nice looking matrices.

Suppose you have a vector x whose components are (x_1, x_2) in a given basis. Here we will assume the basis is given by $\{(1,0)^T, (0,1)^T\}$ i.e. the standard basis for $\mathbb{R}^2$.

Then, given a new desired basis (coordinate axes) $\{a, b\}$ with coordinates specified relative to the original basis (standard $\mathbb{R}^2$ basis) (a_1, a_2) and (b_1, b_2), for a and b respectively, we can find the coordinates of x relative to the new coordinate axes, u_1, u_2, using

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

i.e. the columns of the transformation matrix are the vectors of the desired new coordinate system (as expressed relative to the original basis).

To motivate this, consider the invariant vector equation

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + u_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which expresses the *same* vector x relative to two different bases : $\{(1,0)^T, (0,1)^T\}$ and $\{(a_1, a_2)^T, (b_1, b_2)^T\}$, respectively.

So, to change to coordinates u_1, u_2 relative to an eigenbasis with eigenvectors e^s and e^u say, use

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1^s & e_1^u \\ e_2^s & e_2^u \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where e_1^s, e_2^s are the components of e^s relative to the original coordinate system (etc).

This gives two equations to solve for x_1, x_2 in terms of u_1, u_2 - we can then e.g. derive a new differential equation in u variables by replacing the x variables (in the differential equations for x) by u variables.

Easiest way to check you understand this is to try it! See handwritten examples from Lectures 11 and 12. The next two facts elaborate on this.

Fact

Suppose we are given a matrix A. In the case of distinct, real eigenvalues, we can define a change of coordinates from old variables x to new variables y as above, i.e.

$$x = Py$$

where the columns of P are distinct eigenvectors and P is invertible. In this case the matrix

$$\Lambda = P^{-1}AP$$

is diagonal (and hence the diagonal terms are the eigenvalues).

Fact

In the above case, a linear differential equation $\dot{x} = Ax$ gives

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy$$

i.e. the diagonalised linear system

$$\dot{y} = \Lambda y$$

The motion is thus a linear superposition of flows along with eigendirections, with rates given by the eigenvalues. This can be extended to complex eigenvalues, but I'm too lazy to write this out.

Fact

In the case of non-distinct real eigenvalues, we either still have distinct normal eigenvectors - in which case we proceed as normal - or we instead use our generalised eigenvectors in the transformation. This leads to the *Jordan normal form* pictured previously. In the 2x2 case we have

$$e_1 = (A - \lambda I)e_2$$

and so

$$Ae_2 = \lambda e_2 + e_1$$

whereas, obviously,

$$Ae_1 = \lambda e_1$$

which means

$$P^{-1}AP = [e_1, e_2]^{-1}A[e_1, e_2] = [e_1, e_2]^{-1}[\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2]^{-1}[e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

giving our Jordan normal form. Again, we can extend to the complex but I won't here.