Red

exp. small terms?

where

$$\psi_\beta(x) = \begin{cases} \beta \exp\left(\frac{1}{x}\right) & \text{for } x < 0, \\ 0 & \text{for } x \geq 0, \end{cases}$$

(see **Fig. 5.4**(a)). The system

$$\begin{cases} \dot{x} = -y - x(x^2 + y^2), \\ \dot{y} = x - y(x^2 + y^2), \\ \dot{z} = -z, \end{cases}$$

has an equilibrium $(x, y, z) = (0, 0, 0)$ with $\lambda_{1,2} = \pm i$, $\lambda_3 = -1$ (Hopf case). There is a family of two-dimensional center manifolds in the system given by

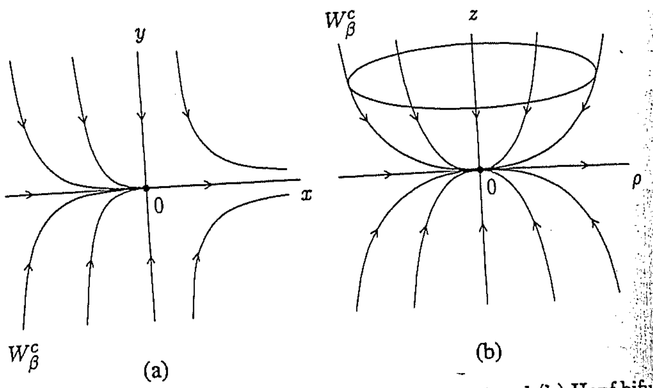


Fig. 5.4. Nonuniqueness of the center manifold at (a) fold and (b) Hopf bifurcations.

$$W^c_\beta(0) = \{(x, y, z) \ : \ z = \phi_\beta(x, y)\},$$

where

$$\phi_\beta(x, y) = \begin{cases} \beta \exp\left(-\frac{1}{2(x^2+y^2)}\right) & \text{for } x^2 + y^2 > 0, \\ 0 & \text{for } x = y = 0, \end{cases}$$

(see **Fig. 5.4**(b)). As we shall see, this nonuniqueness is actually irrelevan for applications.

(3) A center manifold W^c has the same *finite* smoothness as f (if $f \in C$ with finite k, W^c is also a C^k manifold) in some neighborhood U of However, as $k \to \infty$ the neighborhood U may shrink, thus resulting in the nonexistence of a C^∞ manifold W^c for some C^∞ systems (see Exercise 1).

To characterize dynamics near a nonhyperbolic equilibrium $x_0 = 0$ more explicitly, write (5.1) in an $\boxed{\textit{eigenbasis}}$ formed by all (generalized) eigenve ... if the corresponding eigenvalues

In eigenbasis (key is — uncoupled — E & C eigenvalues)

complex). Collecting critical and noncritical components, we can then rewrite (5.1) as

$$\begin{cases} \dot{u} = Bu + g(u, v), \\ \dot{v} = Cv + h(u, v), \end{cases} \tag{5.2}$$

center

where $u \in \mathbb{R}^{n_0}$, $v \in \mathbb{R}^{n_+ + n_-}$, B is an $n_0 \times n_0$ matrix with all its n_0 eigenvalues on the imaginary axis, while C is an $(n_+ + n_-) \times (n_+ + n_-)$ matrix with no eigenvalue on the imaginary axis[1]. Functions g and h have Taylor expansions starting with at least quadratic terms. A center manifold W^c of system (5.2) can be locally represented as the graph of a smooth function:

$$W^c = \{(u, v) : v = V(u)\}$$

(see **Fig. 5.5**). Here $V : \mathbb{R}^{n_0} \to \mathbb{R}^{n_+ + n_-}$, and due to the tangent property of W^c, $V(u) = O(\|u\|^2)$.

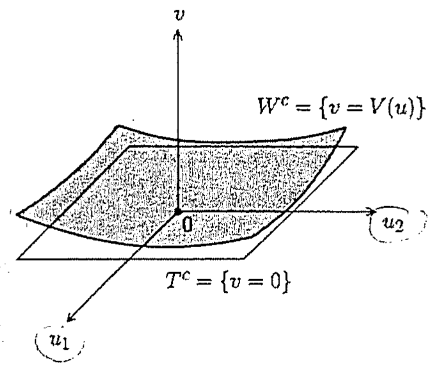


Fig. 5.5. Center manifold as the graph of a function $v = V(u)$.

Theorem 5.2 (Reduction Principle) *System* (5.2) *is locally topologically equivalent near the origin to the system*

$$\begin{cases} \dot{u} = Bu + g(u, V(u)), \\ \dot{v} = Cv. \end{cases} \tag{5.3}$$

If there is more than one center manifold, then all the resulting systems (5.3) *with different V are locally smoothly equivalent.* $\square$

Notice that the equations for u and v are uncoupled in (5.3). The first equation is the *restriction* of (5.2) to its center manifold. Thus, the dynamics of the structurally unstable system (5.2) are essentially determined by this

[1] Actually, any basis in the noncritical eigenspace is allowed. In other words, the matrix C may not be in real canonical (Jordan) form.

restriction since the second equation in (5.3) is linear and has exponentially decaying/growing solutions. For example, if $u = 0$ is the asymptotically stable equilibrium of the restriction and all eigenvalues of C have negative real part, then $(u, v) = (0, 0)$ is the asymptotically stable equilibrium of (5.2). Clearly, the dynamics on the center manifold are determined not only by the linear but also by the *nonlinear* terms of (5.2).

Example 5.1 (Failure of the tangent approximation) Consider the planar system

$$\begin{cases} \dot x = xy + x^3, \\ \dot y = -y - 2x^2. \end{cases} \tag{5.4}$$

There is an equilibrium at $(x, y) = (0, 0)$. Is it stable or unstable? The Jacobian matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

has eigenvalues $\boxed{\lambda_1 = 0,}$ $\lambda_2 = -1$. Thus, system (5.4) is written in the form (5.2) and has a one-dimensional center manifold W^c represented by a scalar function

$$y = V(x).$$

$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$-y = 0$

$\boxed{x = 0}$

Let us find the quadratic term in the Taylor expansion of this function:

$$V(x) = \tfrac{1}{2} w x^2 + \cdots .$$

The unknown coefficient w can be found by expressing $\dot y$ as

$$\dot y = \frac{\partial V}{\partial x}\dot x = (wx + \cdots)\dot x = wx^2 y + wx^4 + \cdots = w\left(\tfrac{1}{2}w + 1\right)x^4 + \cdots$$

or, alternatively, as

$$\dot y = -y - 2x^2 = -\left(\tfrac{1}{2}w + 2\right)x^2 + \cdots .$$

Therefore, $w + 4 = 0$ and

$$w = -4.$$

Thus, the center manifold has the following quadratic approximation:

$$V(x) = -2x^2 + O(x^3),$$

and the restriction of (5.4) to its center manifold is given by

$$\dot x = xV(x) + x^3 = -2x^3 + x^3 + O(x^4) = -x^3 + O(x^4).$$

W^c vs T^c

Therefore, the origin is *stable* and the phase portrait of the system near the equilibrium is as sketched in **Fig. 5.6**. By restriction of (5.4) onto its critical eigenspace $\boxed{y = 0,}$ one gets

$$\dot x = x^3$$

$E^c = \{(x, y) \mid y = 0\}$

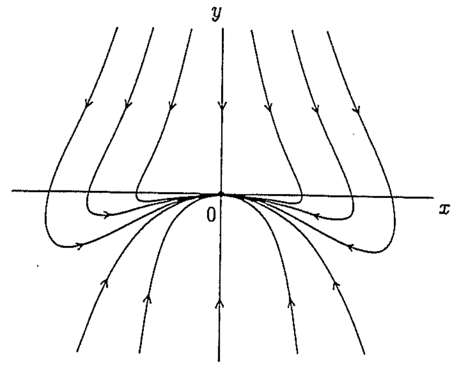
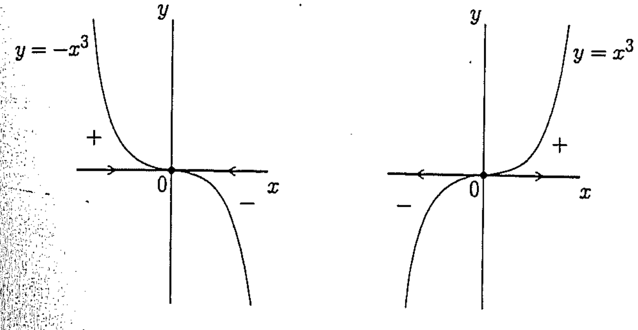


Fig. 5.6. Phase portrait of (5.4): The origin is stable.

W^c requires nonlinear terms



(a) (b)

Fig. 5.7. Restricted equations: (a) to the center manifold W^c; (b) to the tangent line T^c.

This equation has an *unstable* point at the origin and thus gives the wrong answer to the stability question. **Fig. 5.7** compares the equations restricted to W^c and T^c. ◇

The second equation in (5.3) can be replaced by the equations of a *standard saddle*:

$$\begin{cases} \dot v = -v, \\ \dot w = \ \ w, \end{cases} \tag{5.5}$$

with $(v, w) \in \mathbb{R}^{n_-} \times \mathbb{R}^{n_+}$. Therefore, the Reduction Principle can be expressed neatly in the following way: *Near a nonhyperbolic equilibrium the system is locally topologically equivalent to the suspension of its restriction to the center manifold by the standard saddle.*