

# ENGSCI 711

## QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(...and other dynamical systems)

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## MODULE OVERVIEW

Qualitative analysis of differential equations (Oliver Maclaren) [[~16-17 lectures/tutorials](#)]

### 3. Introduction to bifurcation theory [[4 lectures/tutorials](#)]

Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations.  
Bifurcation diagrams.

## LECTURE 10: BIFURCATION THEORY CONTINUED...CONTINUED

Basic *one-parameter local bifurcation* involving *complex-valued* eigenvalues:

- Hopf bifurcation

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## HOPF BIFURCATION

The Hopf bifurcation occurs when a *pair of complex conjugate eigenvalues cross the imaginary axis together*.

Thus at the bifurcation a *a pair of complex conjugate eigenvalues become purely imaginary*

In contrast to before, we now have a non-zero imaginary component and hence have to deal with *oscillatory* components.

(Thus we need the system to have at least dimension two, though the *co-dimension* is still one.)

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## HOPF BIFURCATION

A Hopf bifurcation (for our purposes) is characterised by *a change in stability of a fixed point, along with the appearance or the disappearance of a periodic orbit at this fixed point.*

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## HOPF BIFURCATION THEOREM (OR NOT)

Instead we typically a) *find where a pair of complex eigenvalues become purely imaginary* and b) *directly verify that a periodic solution was created/destroyed* (as we pass through the bifurcation) via simulation (or analytical solution in simple cases).

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## HOPF BIFURCATION THEOREM (OR NOT)

There is a *Hopf bifurcation theorem* (See e.g. Glendinning) giving *conditions under which periodic solutions are created/destroyed* as a pair of complex eigenvalues pass through the imaginary axis (and the associated fixed point changes stability)

Unfortunately it is a bit tricky/ugly to verify the conditions for creation/destruction. Though the 'non-zero crossing speed - a *transversality condition* - is not so bad (see later).

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## HOPF BIFURCATION - CO-DIMENSION AGAIN

The Hopf bifurcation occurs in *two-dimensional systems* (or on a two-dimensional reduced/centre manifold of a larger system) BUT

The Hopf bifurcation essentially only *depends on varying one parameter, hence the co-dimension is one.*

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## HOPF BIFURCATION - CANONICAL EXAMPLE

$$\begin{aligned}\dot{x} &= -\omega y + x(\mu - (x^2 + y^2)) \\ \dot{y} &= \omega x + y(\mu - (x^2 + y^2))\end{aligned}$$

## HOPF BIFURCATION - CROSSING SPEED

As seen, *a necessary condition for a Hopf bifurcation is finding a pair of complex conjugate eigenvalues crossing the imaginary axis.*

We can also sometimes *verify that this crossing occurs with a non-zero speed* (another necessary condition)

So, we have eigenvalues  $\lambda = \pm i\omega$  at critical parameter value  
 $\mu = \mu_c$ .

and (for a non-degenerate bifurcation)  $\frac{\partial \lambda^r}{\partial \mu} \neq 0$  at  $\mu = \mu_c$ .

where  $\lambda^r$  is the real part of the eigenvalue.

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## HOPF BIFURCATION ANALYSIS

### Steps

- *Verify* we have a pair of complex conjugate eigenvalues crossing the imaginary axis and an associated change in stability of the fixed point.
- *Verify* (in this example by direct solution, in general via numerical methods) that a periodic orbit exists on one side of the bifurcation.

Note: If a direct solution is possible, then it is typically easiest to construct/verify in polar coordinates.

## HOPF BIFURCATION KEY FEATURES

- We call the bifurcation *supercritical* if the emerging/disappearing periodic orbit is stable.
- If it is unstable, we call it *subcritical*\*
- The *radius* of the limit cycles grow/shrink continuously from/to zero and proportional to  $\sqrt{\mu - \mu_c}$  near  $\mu_c$
- The *frequency* of the limit cycle is approximately  $\text{Im } \lambda$ , evaluated at  $\mu = \mu_c$

\*Similar terminology is used for other bifurcations (e.g. supercritical pitchfork - stable FP are born).

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## HOPF BIFURCATION KEY FEATURES

- The periodic orbit and the fixed point have *opposite stability for the parameter values that they both exist*
- I.e. *supercritical*: stable PO, unstable FP; *subcritical*: unstable PO, stable FP
- Note that it is also *difficult to manually check the theorem conditions* for whether the Hopf bifurcation is 'supercritical' or 'subcritical' (or degenerate).
- Again, it is easier to do it by simulation, direct construction, direct checking etc on a *case-by-case basis*.

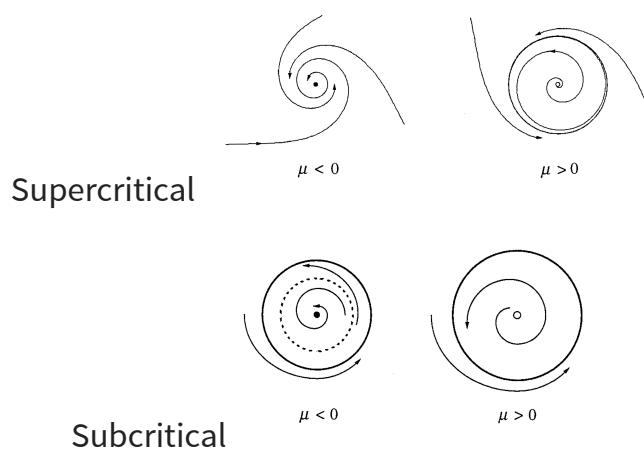
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## FURTHER EXAMPLES

- Saddle-node for periodic orbits
- Homoclinic

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## HOPF BIFURCATION PICTURES



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EngSci 711 L10 Bifurcation Theory Cont'd.

- o Hopf bifurcation :  $\text{Re}(\lambda_i) = 0, \text{Im}(\lambda_i) \neq 0$  at fixed point
  - Basic case
  - Application to chemical oscillator
- o Extras ('collisions' of periodic orbits, global features etc)
  - saddle-node bifurcation of periodic orbits
  - homoclinic bifurcations
- o More applications/extensions?
  - tutorial
  - assignment
  - recommended reading

Example Questions

Exam 2016

(b) Consider the second-order equation

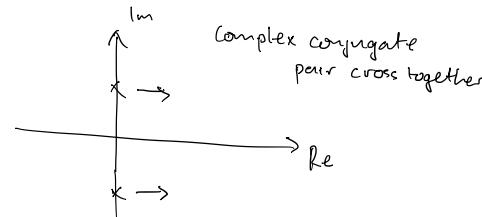
$$\ddot{x} + \mu \dot{x} + (x - x^3) = 0$$

where  $x \in \mathbb{R}$  and  $\mu \in \mathbb{R}$  is a system parameter.

- (i). Re-write the above equation as system of two first-order equations.
- (ii). Determine a value of  $\mu$  for which a Hopf bifurcation could potentially occur at the origin. Show all your working.
- (iii). Sketch a bifurcation diagram for a typical supercritical Hopf bifurcation, along with associated typical phase portraits for parameter values before, at and after the critical parameter value. Note: your diagram for this part need not refer to the equation given.

Hopf Bifurcation : Creation / destruction of periodic orbits via fixed points

- Two-dimensional state space } want complex eigenvalues
- Still one-parameter bifurcation



- o Purely imaginary ( $\text{Re}(\lambda) = 0$ ) at bifurcation } Key!

Theorem? There is a Hopf bifurcation theorem giving conditions for various possibilities

→ One key condition for a non-degenerate Hopf

$$\left. \frac{d}{du} (\text{Re}(\lambda(u))) \right|_{u=u_c} \neq 0$$

crosses imag. axis at non-zero speed at critical value  $u_c$

→ BUT in general, theorems

- |                                 |                          |
|---------------------------------|--------------------------|
| a) convoluted                   | } see e.g.<br>Abendmeyer |
| b) hardly ever used in practice | } _____                  |

Instead, I recommend

- find possible ( $\text{Re}(\lambda) = 0, \text{Im}(\lambda) \neq 0$ )
- verify directly by numerically investigating either side of bifurcation

in 'exam conditions'  
 just find possible,  
 maybe check crossing  
 speed

## Hopf Example

- 'simple' enough to do analytically
- 'Normal form' (generic model of Hopf).

$$\begin{aligned} \textcircled{1} \quad \dot{x} &= -\omega y + x(\mu - (x^2 + y^2)) \\ \dot{y} &= \omega x + y(\mu - (x^2 + y^2)) \end{aligned}$$

or

$$\begin{aligned} \textcircled{2} \quad \dot{r} &= r(\mu - r^2) \quad (\text{polar, for } r > 0)^* \\ \dot{\theta} &= \omega \end{aligned}$$

$\omega \neq 0$   
 $\mu$  bifurcation parameter

\* (coord. change?)

$$\textcircled{1} \rightarrow \textcircled{2}: \text{ either } \begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ r^2\dot{\theta} = x\dot{y} - y\dot{x} \end{cases}$$

$$\text{or } \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x+iy = re^{i\theta} \end{cases}$$

(see later)

Consider:

$$\dot{x} = -\omega y + x(\mu - (x^2 + y^2)) \quad \textcircled{1}$$

$$\dot{y} = \omega x + y(\mu - (x^2 + y^2)) \quad \textcircled{2}$$

$\omega \neq 0$ , fixed.

1. Fixed points

$$\textcircled{1} = 0 = -\omega y + x(\mu - (x^2 + y^2))$$

$$\textcircled{2} = 0 = \omega x + y(\mu - (x^2 + y^2))$$

hum... well: try  $(0, 0)$ .

$$\begin{aligned} 0 &= 0 + 0 & \checkmark \\ 0 &\approx 0, 0 & \checkmark \end{aligned}$$

Others?

→ Nope, since

$$\textcircled{1}: y = \frac{x}{\omega}(\mu - (x^2 + y^2))$$

$$\rightarrow \textcircled{2}: \omega x + \frac{x}{\omega}[\mu - (x^2 + y^2)]^2 = 0$$

$$x \left[ \omega + \frac{1}{\omega} [\ ]^2 \right] = 0 \quad \left. \begin{array}{l} x=0 \Rightarrow y=0 \\ \text{ie get } (0,0) \text{ soln.} \end{array} \right\}$$

→ Want  $x \neq 0$  ('others')

But  $\omega + \frac{1}{\omega} [\ ]^2 = 0 \Rightarrow \omega^2 + [\ ]^2 = 0$   
which can't happen  
if  $\omega \neq 0$  i.e.  
 $\omega^2 \neq 0$

So:  $(0, 0)$  only fixed point.

2. Calc. Jacobian & eval. at FP.

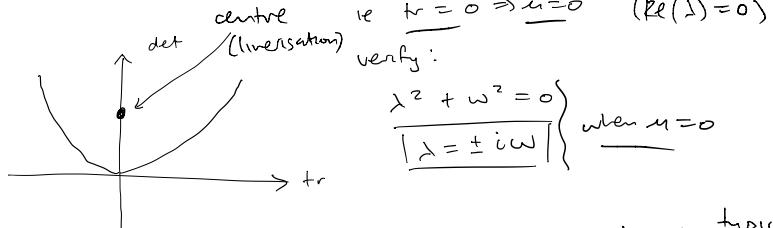
$$Df(0,0) = \begin{pmatrix} m & -\omega \\ \omega & m \end{pmatrix} \quad \begin{array}{l} \text{Verify the} \\ \text{long way!} \\ \hookrightarrow Df(x,y) \text{ first} \end{array}$$

$$\text{tr} = 2m$$

$$\det = m^2 + \omega^2 > 0$$

$$\lambda^2 - 2m\lambda + (m^2 + \omega^2) = 0 \Leftrightarrow \frac{(\lambda - m)^2 + \omega^2 = 0}{\Rightarrow |\lambda = m \pm i\omega|}$$

3. Find when pure imaginary ( $\text{Re}(\lambda) = 0 \Rightarrow \text{non-hyperbolic}$ )



$$\text{Note } Df(0,0)|_{m=0} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \left. \begin{array}{l} \text{typical} \\ \text{matrix} \end{array} \right\}$$

$$\text{so: } m=0 \Rightarrow \lambda = \pm i\omega, \text{ pure imaginary}$$

$\Rightarrow$  Possible Hopf bifurcation at  $m=0$ .

4. Verify crossing speed  $\neq 0$

$$\boxed{\lambda = m \pm i\omega} \text{ in general} \quad \begin{array}{l} \text{transversality} \\ \Rightarrow \text{Re}(\lambda) = m, \frac{d\text{Re}(\lambda)}{dm} = 1 \neq 0 \end{array} \quad \left. \begin{array}{l} \text{non-zero} \\ \text{crossing} \\ \text{speed.} \\ \rightarrow \text{non-degenerate} \end{array} \right\}$$

5. Analyse either side of bifurcation (here  $m \geq 0$ )

$\rightarrow$  In this case we are in normal form

& can analyse analytically what happens for  $m \neq 0$  reasonably well.

$\rightarrow$  In general, verify numerically!

So... let's do a more detailed analysis [for  $m$  either side of zero.]

Clever in polar coord. (fun!):

$$\left[ \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \\ x + iy = r e^{i\theta}, \quad x^2 + y^2 = r^2 \end{array} \right]$$

*[if I ask, I'll probably guide through steps --]*

$$\frac{d(re^{i\theta})}{dt} = ix + iy$$

$$e^{i\theta} \left[ \frac{dr}{dt} + ir \frac{d\theta}{dt} \right] = \omega \underbrace{(-y + ix)}_{-i(x-y)} + \underbrace{(m - x^2 - y^2)(x + iy)}_{r^2 e^{i\theta}}$$

$$-i(ix - y) = xc + iy = re^{i\theta}$$

$$\Rightarrow ix - y = \frac{re^{i\theta}}{-i} = \frac{i r e^{i\theta}}{-i^2} = i r e^{i\theta}$$

so:

~~$$e^{i\theta} [i + ir\dot{\theta}] = e^{i\theta} ir\omega + (m - r^2)r e^{i\theta}$$~~

$$i + ir\dot{\theta} = r(m - r^2) + ir\omega$$

$$\text{equate real & imaginary parts} \quad \left\{ \begin{array}{l} \dot{r} = r(m - r^2) \\ r\dot{\theta} = r\omega \Rightarrow \dot{\theta} = \omega \text{ if } r > 0. \end{array} \right.$$

$\dot{\theta}$  undefined for  $r=0$

$\Rightarrow$  take  $\dot{\theta}(r=0) = 0$ , say.

$$\Rightarrow \boxed{\begin{array}{l} \dot{r} = r(m - r^2) \\ \dot{\theta} = \omega \end{array}}$$

... phenf!

Key point:

$$\begin{cases} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = \omega \end{cases}$$

← 'like' pitchfork!

(but  $r > 0$  & have  $\theta \text{ eqn}$ ).

where  $\begin{cases} r > 0 \text{ (radius)} \\ \omega \neq 0, \text{ fixed} \\ \mu \text{ control parameter} \end{cases}$

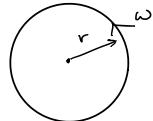
Properties:

→  $r$  &  $\theta$  decoupled

→ ' $r=0$ ' (origin) fixed point ( $\dot{r} \text{ eqn}$  OK for  $r=0$ ,  
the 'θ & θ̇ undefined at  $r=0$ )  
→ as before.

→  $r^2 = \mu, \omega \neq 0 \Rightarrow \dot{r} = 0, \dot{\theta} \neq 0 ?$

⇒ circles!



(if  $\mu > 0$ )

expect  $r \sim \sqrt{\mu}$  oscillations  
(near bifurcation)



Consider  $\mu$  variations

Note: if equation has eigenvalue  $\mu$  at origin (why?)

$\mu > 0 \Rightarrow$  unstable  
 $\mu < 0 \Rightarrow$  stable. } fixed point  
changes stability

Let's do cases directly

$\mu > 0$

If.  $\mu = r^2 \Rightarrow$  circles (oscillations)

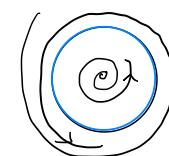
•  $\mu < r^2$  i.e.  $r^2 > \mu$

$$\Rightarrow \dot{r} = r(\mu - r^2) < 0$$

•  $\mu > r^2$  i.e.  $r^2 < \mu$

$$\Rightarrow \dot{r} > 0$$

So:



- spiral out from origin to circle  
- spiral in from outside to circle

} ie this orbit is a stable object

$\mu = 0$

radius 'collapses' to origin

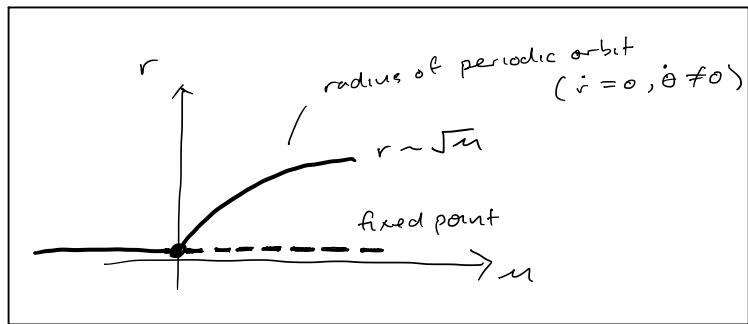
⇒ pure oscillation 'at origin'  
→ centre!

$\mu < 0$

⇒ no periodic orbit since

$$r^2 \neq \mu < 0$$

Bifurcation diagram : plot key features



Note :

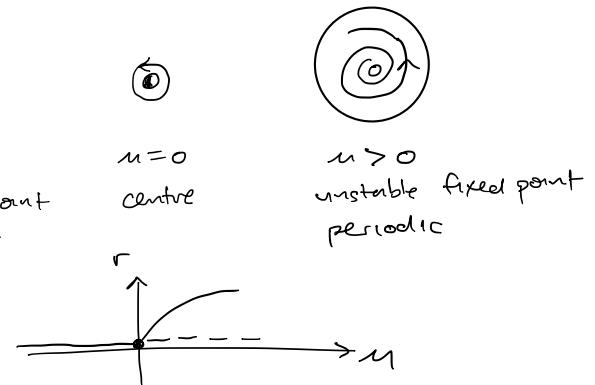
- Fixed point loses stability
- 'Gives birth to' a periodic orbit
- Like pitchfork but  $\begin{cases} r > 0 \\ \theta \neq 0 \end{cases}$

↳ same terminology:

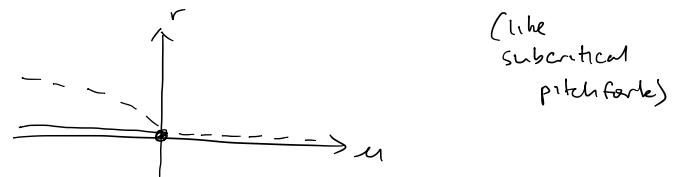
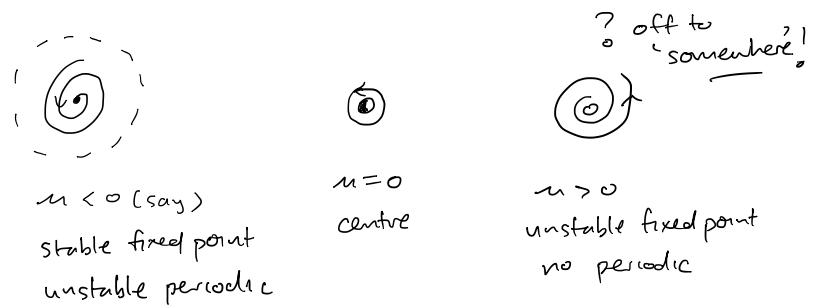
- supercritical (periodic stable)
- subcritical (periodic unstable)

Super vs Sub?

- supercritical : stable periodic appears/disappears via fixed point



- Subcritical : unstable periodic orbit destabilises or stable fixed point  
→ dramatic!



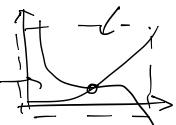
## Application

Recall Chlorine - Dioxide - Iodine - Malonic Acid  
(CDIMA)  
reaction

model (strogatz 8-3 - attached organ)

$$\begin{aligned}\dot{x} &= a - xc - \frac{4xy}{1+x^2} \\ \dot{y} &= bx \left(1 - \frac{y}{1+x^2}\right)\end{aligned}$$

→ we found trapping region:



→ need to verify/check when fixed point is unstable

→ turns out this is a Hopf bifurcation case

↪ look for when pure imaginary at fixed point

Details: attached

+ XPPAUT demonstration.

## Extras

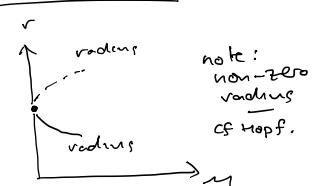
(typically analyse periodic orbits/global etc via maps)

### Saddle-node periodic

'collision' of / 'birth of' pair of stable & unstable periodic orbits

↪ like SN but periodic orbits instead of fixed points

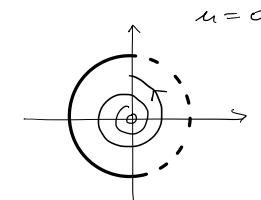
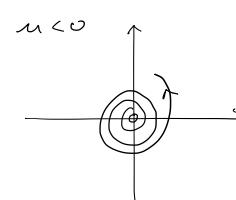
### Typical bif. diagram



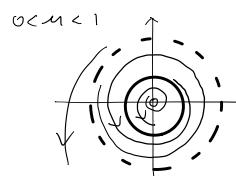
Example (see eg Perko 4.5, example 2)

eg

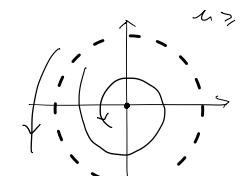
$$\begin{cases} \dot{r} = -r [m - (r^2 - 1)^2] \\ \dot{\theta} = \omega \end{cases}$$



stable & unstable periodic orbits 'appear/disappear' together 'out of nowhere' → away from the fixed point



unstable & stable periodic orbits separate



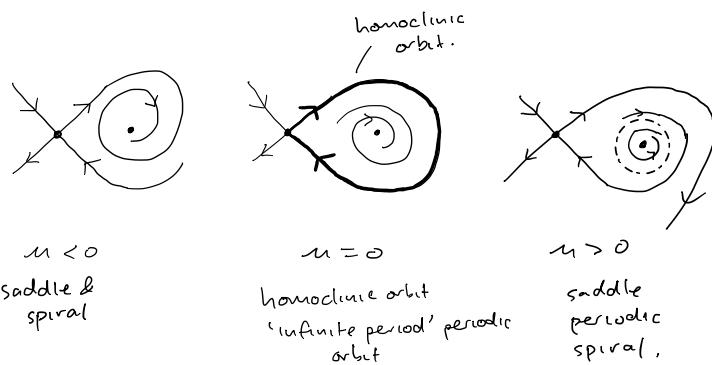
stable periodic disappears at origin via Hopf. (local to fixed point)

# Glendinning

- Extras

Homoclinic bifurcation (truly 'global')

- unstable & stable manifolds 'collide'
- ('large amplitude') ( $r \neq 0$ ) periodic orbit appears/disappears
- ie periodic orbit 'collides' with saddle point's manifolds



Example (Strogatz section 8.4)

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases}$$

→ see attached.

even more: see attached

## (8.6) THEOREM (HOPF BIFURCATION THEOREM)

Suppose that

$$\dot{x} = f(x, y, \mu), \quad \dot{y} = g(x, y, \mu)$$

with  $f(0, 0, \mu) = g(0, 0, \mu) = 0$  and that the Jacobian matrix  $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$  evaluated at the origin when  $\mu = 0$  is

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

for some  $\omega \neq 0$ . Then if

$$f_{\mu x} + g_{\mu y} \neq 0$$

and

$$a \neq 0$$

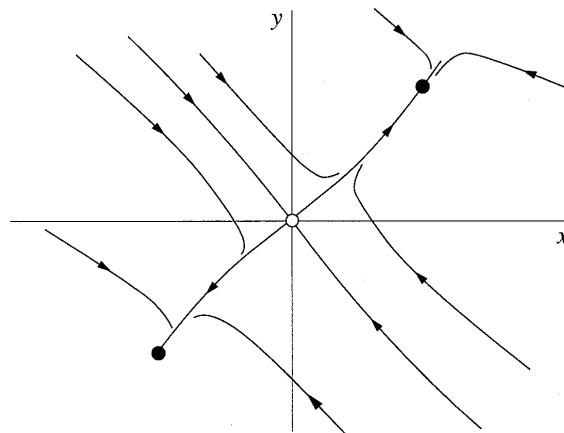
where  $a$  is a constant defined below, a curve of periodic solutions bifurcates from the origin into  $\mu < 0$  if  $a(f_{\mu x} + g_{\mu y}) > 0$  or  $\mu > 0$  if  $a(f_{\mu x} + g_{\mu y}) < 0$ . The origin is stable for  $\mu > 0$  (resp.  $\mu < 0$ ) and unstable for  $\mu < 0$  (resp.  $\mu > 0$ ) if  $f_{\mu x} + g_{\mu y} < 0$  (resp.  $> 0$ ) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of  $\mu = 0$  for which the periodic solutions exist. The amplitude of the periodic orbits grows like  $|\mu|^{\frac{1}{2}}$  whilst their periods tend to  $\frac{2\pi}{\omega}$  as  $|\mu|$  tends to zero. The bifurcation is supercritical if the bifurcating periodic orbits are stable, otherwise it is subcritical.

The genericity condition,  $a \neq 0$ , is the usual sort of condition for a bifurcation, involving the partial derivatives of the vector field  $(f(x, y, \mu), g(x, y, \mu))$  evaluated at  $(0, 0, 0)$ . More explicitly

$$\begin{aligned} a = & \frac{1}{16}(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) \\ & + \frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \end{aligned} \quad (8.44)$$

This horrible expression will be the cause of not a little pain at the end of this section. However, the most important part of the theorem, the existence of bifurcating periodic solutions, is not too hard to prove and the pain referred to above comes from the need to do lots of nasty algebraic manipulation rather than any sophisticated conceptual trickery. As promised we begin with an outline of the strategy of the proof.

zero eigenvalue becomes slightly positive. This information implies the phase portrait shown in Figure 8.1.7.



**Figure 8.1.7**

Note that because of the approximations we've made, this picture is only valid *locally* in both parameter and phase space—if we're not near the origin and if  $\mu$  is not close to  $\mu_c$ , all bets are off. ■

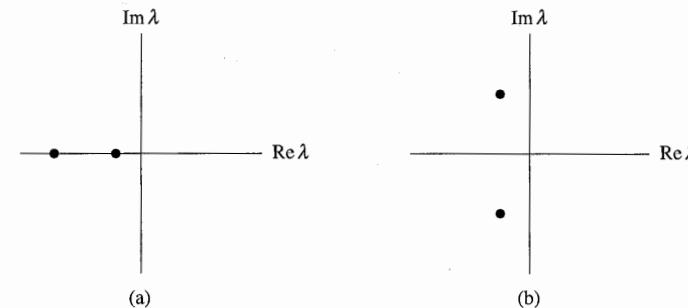
In all of the examples above, the bifurcation occurs when  $\Delta = 0$ , or equivalently, when one of the eigenvalues equals zero. More generally, the saddle-node, transcritical, and pitchfork bifurcations are all examples of *zero-eigenvalue bifurcations*. (There are other examples, but these are the most common.) Such bifurcations always involve the collision of two or more fixed points.

In the next section we'll consider a fundamentally new kind of bifurcation, one that has no counterpart in one-dimensional systems. It provides a way for a fixed point to lose stability without colliding with any other fixed points.

## 8.2 Hopf Bifurcations

Suppose a two-dimensional system has a stable fixed point. What are all the possible ways it could lose stability as a parameter  $\mu$  varies? The eigenvalues of the Jacobian are the key. If the fixed point is stable, the eigenvalues  $\lambda_1, \lambda_2$  must both lie in the left half-plane  $\text{Re } \lambda < 0$ . Since the  $\lambda$ 's satisfy a quadratic equation with real coefficients, there are two possible pictures: either the eigenvalues are both real and negative (Figure 8.2.1a) or they are complex conjugates (Figure 8.2.1b). To

destabilize the fixed point, we need one or both of the eigenvalues to cross into the right half-plane as  $\mu$  varies.

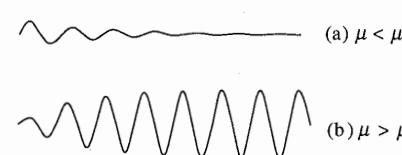


**Figure 8.2.1**

In Section 8.1 we explored the cases in which a real eigenvalue passes through  $\lambda = 0$ . These were just our old friends from Chapter 3, namely the saddle-node, transcritical, and pitchfork bifurcations. Now we consider the other possible scenario, in which two complex conjugate eigenvalues simultaneously cross the imaginary axis into the right half-plane.

### Supercritical Hopf Bifurcation

Suppose we have a physical system that settles down to equilibrium through exponentially damped oscillations. In other words, small disturbances decay after “ringing” for a while (Figure 8.2.2a). Now suppose that the decay rate depends on a control parameter  $\mu$ . If the decay becomes slower and slower and finally changes to *growth* at a critical value  $\mu_c$ , the equilibrium state will lose stability. In many cases the resulting motion is a small-amplitude, sinusoidal, limit cycle oscillation about the former steady state (Figure 8.2.2b). Then we say that the system has undergone a *supercritical Hopf bifurcation*.



**Figure 8.2.2**

In terms of the flow in phase space, a supercritical Hopf bifurcation occurs when a stable spiral changes into an unstable spiral surrounded by a small, nearly elliptical limit cycle. Hopf bifurcations can occur in phase spaces of any dimension  $n \geq 2$ , but as in the rest of this chapter, we'll restrict ourselves to two dimensions.

A simple example of a supercritical Hopf bifurcation is given by the following system:

$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + br^2.\end{aligned}$$

There are three parameters:  $\mu$  controls the stability of the fixed point at the origin,  $\omega$  gives the frequency of infinitesimal oscillations, and  $b$  determines the dependence of frequency on amplitude for larger amplitude oscillations.

Figure 8.2.3 plots the phase portraits for  $\mu$  above and below the bifurcation. When  $\mu < 0$  the origin  $r = 0$  is a stable spiral whose sense of rotation depends on the sign of  $\omega$ . For  $\mu = 0$  the origin is still a stable spiral, though a very weak one: the decay is only algebraically fast. (This case was shown in Figure 6.3.2. Recall that the linearization wrongly predicts a center at the origin.) Finally, for  $\mu > 0$  there is an unstable spiral at the origin and a stable circular limit cycle at  $r = \sqrt{\mu}$ .

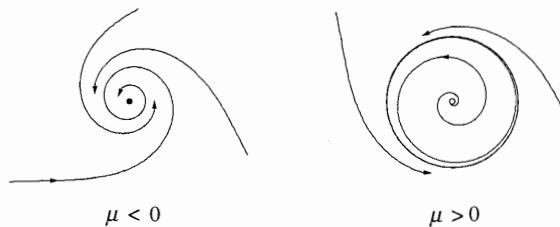


Figure 8.2.3

To see how the eigenvalues behave during the bifurcation, we rewrite the system in Cartesian coordinates; this makes it easier to find the Jacobian. We write  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then

$$\begin{aligned}\dot{x} &= \dot{r} \cos \theta - r \dot{\theta} \sin \theta \\ &= (\mu r - r^3) \cos \theta - r(\omega + br^2) \sin \theta \\ &= (\mu - [x^2 + y^2])x - (\omega + b[x^2 + y^2])y \\ &= \mu x - \omega y + \text{cubic terms}\end{aligned}$$

and similarly

$$\dot{y} = \omega x + \mu y + \text{cubic terms.}$$

So the Jacobian at the origin is

$$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix},$$

which has eigenvalues

$$\lambda = \mu \pm i\omega.$$

As expected, the eigenvalues cross the imaginary axis from left to right as  $\mu$  increases from negative to positive values.

### Rules of Thumb

Our idealized case illustrates two rules that hold *generically* for supercritical Hopf bifurcations:

1. The size of the limit cycle grows continuously from zero, and increases proportional to  $\sqrt{\mu - \mu_c}$ , for  $\mu$  close to  $\mu_c$ .
2. The frequency of the limit cycle is given approximately by  $\omega = \text{Im } \lambda$ , evaluated at  $\mu = \mu_c$ . This formula is exact at the birth of the limit cycle, and correct within  $O(\mu - \mu_c)$  for  $\mu$  close to  $\mu_c$ . The period is therefore  $T = (2\pi/\text{Im } \lambda) + O(\mu - \mu_c)$ .

But our idealized example also has some artifactual properties. First, in Hopf bifurcations encountered in practice, the limit cycle is elliptical, not circular, and its shape becomes distorted as  $\mu$  moves away from the bifurcation point. Our example is only typical topologically, not geometrically. Second, in our idealized case the eigenvalues move on horizontal lines as  $\mu$  varies, i.e.,  $\text{Im } \lambda$  is strictly independent of  $\mu$ . Normally, the eigenvalues would follow a curvy path and cross the imaginary axis with nonzero slope (Figure 8.2.4).

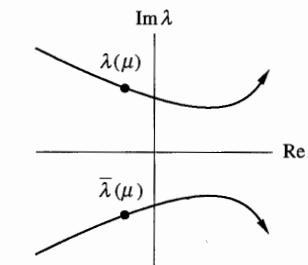


Figure 8.2.4

### Subcritical Hopf Bifurcation

Like pitchfork bifurcations, Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation, the trajectories must *jump* to a distant attractor, which may be a fixed point, another limit cycle, infinity, or—in

three and higher dimensions—a chaotic attractor. We'll see a concrete example of this last, most interesting case when we study the Lorenz equations (Chapter 9).

But for now, consider the two-dimensional example

$$\begin{aligned}\dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2.\end{aligned}$$

The important difference from the earlier supercritical case is that the cubic term  $r^3$  is now *destabilizing*; it helps to drive trajectories away from the origin.

The phase portraits are shown in Figure 8.2.5. For  $\mu < 0$  there are two attractors, a stable limit cycle and a stable fixed point at the origin. Between them lies an unstable cycle, shown as a dashed curve in Figure 8.2.5; it's the player to watch in this scenario. As  $\mu$  increases, the unstable cycle tightens like a noose around the fixed point. A **subcritical Hopf bifurcation** occurs at  $\mu = 0$ , where the unstable cycle shrinks to zero amplitude and engulfs the origin, rendering it unstable. For  $\mu > 0$ , the large-amplitude limit cycle is suddenly the only attractor in town. Solutions that used to remain near the origin are now forced to grow into large-amplitude oscillations.

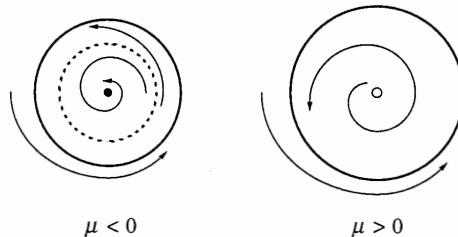


Figure 8.2.5

Note that the system exhibits *hysteresis*: once large-amplitude oscillations have begun, they cannot be turned off by bringing  $\mu$  back to zero. In fact, the large oscillations will persist until  $\mu = -1/4$  where the stable and unstable cycles collide and annihilate. This destruction of the large-amplitude cycle occurs via another type of bifurcation, to be discussed in Section 8.4.

Subcritical Hopf bifurcations occur in the dynamics of nerve cells (Rinzel and Ermentrout 1989), in aeroelastic flutter and other vibrations of airplane wings (Dowell and Ilgamova 1988, Thompson and Stewart 1986), and in instabilities of fluid flows (Drazin and Reid 1981).

### Subcritical, Supercritical, or Degenerate Bifurcation?

Given that a Hopf bifurcation occurs, how can we tell if it's sub- or supercritical? The linearization doesn't provide a distinction: in both cases, a pair of eigen-

values moves from the left to the right half-plane.

An analytical criterion exists, but it can be difficult to use (see Exercises 8.2.12–15 for some tractable cases). A quick and dirty approach is to use the computer. If a small, attracting limit cycle appears immediately after the fixed point goes unstable, and if its amplitude shrinks back to zero as the parameter is reversed, the bifurcation is supercritical; otherwise, it's probably subcritical, in which case the nearest attractor might be far from the fixed point, and the system may exhibit hysteresis as the parameter is reversed. Of course, computer experiments are not proofs and you should check the numerics carefully before making any firm conclusions.

Finally, you should also be aware of a **degenerate Hopf bifurcation**. An example is given by the damped pendulum  $\ddot{x} + \mu\dot{x} + \sin x = 0$ . As we change the damping  $\mu$  from positive to negative, the fixed point at the origin changes from a stable to an unstable spiral. However at  $\mu = 0$  we do *not* have a true Hopf bifurcation because there are no limit cycles on either side of the bifurcation. Instead, at  $\mu = 0$  we have a continuous band of closed orbits surrounding the origin. These are not limit cycles! (Recall that a limit cycle is an *isolated* closed orbit.)

This degenerate case typically arises when a nonconservative system suddenly becomes conservative at the bifurcation point. Then the fixed point becomes a nonlinear center, rather than the weak spiral required by a Hopf bifurcation. See Exercise 8.2.11 for another example.

### EXAMPLE 8.2.1:

Consider the system  $\dot{x} = \mu x - y + xy^2$ ,  $\dot{y} = x + \mu y + y^3$ . Show that a Hopf bifurcation occurs at the origin as  $\mu$  varies. Is the bifurcation subcritical, supercritical, or degenerate?

**Solution:** The Jacobian at the origin is  $A = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$ , which has  $\tau = 2\mu$ ,  $\Delta = \mu^2 + 1 > 0$ , and  $\lambda = \mu \pm i$ . Hence, as  $\mu$  increases through zero, the origin changes from a stable spiral to an unstable spiral. This suggests that some kind of Hopf bifurcation takes place at  $\mu = 0$ .

To decide whether the bifurcation is subcritical, supercritical, or degenerate, we use simple reasoning and numerical integration. If we transform the system to polar coordinates, we find that

$$\dot{r} = \mu r + ry^2,$$

as you should check. Hence  $\dot{r} \geq \mu r$ . This implies that for  $\mu > 0$ ,  $r(t)$  grows at least

as fast as  $r_0 e^{\mu t}$ . In other words, all trajectories are repelled out to infinity! So there are certainly no closed orbits for  $\mu > 0$ . In particular, the unstable spiral is not surrounded by a stable limit cycle; hence the bifurcation cannot be supercritical.

Could the bifurcation be degenerate? That would require that the origin be a nonlinear center when  $\mu = 0$ . But  $\dot{r}$  is strictly positive away from the  $x$ -axis, so closed orbits are still impossible.

By process of elimination, we expect that the bifurcation is *subcritical*. This is confirmed by Figure 8.2.6, which is a computer-generated phase portrait for  $\mu = -0.2$ .

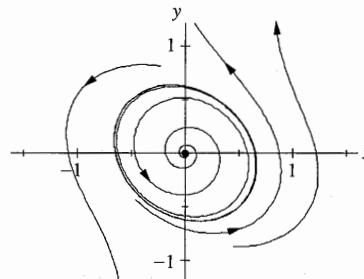


Figure 8.2.6

Note that an *unstable* limit cycle surrounds the stable fixed point, just as we expect in a subcritical bifurcation. Furthermore, the cycle is nearly elliptical and surrounds a gently winding spiral—these are typical features of *either* kind of Hopf bifurcation. ■

### 8.3 Oscillating Chemical Reactions

For an application of Hopf bifurcations, we now consider a class of experimental systems known as *chemical oscillators*. These systems are remarkable, both for their spectacular behavior and for the story behind their discovery. After presenting this background information, we analyze a simple model proposed recently for oscillations in the chlorine dioxide–iodine–malonic acid reaction. The definitive reference on chemical oscillations is the book edited by Field and Burger (1985). See also Epstein et al. (1983), Winfree (1987b) and Murray (1989).

#### **Belousov's "Supposedly Discovered Discovery"**

In the early 1950s the Russian biochemist Boris Belousov was trying to create a test tube caricature of the Krebs cycle, a metabolic process that occurs in living

cells. When he mixed citric acid and bromate ions in a solution of sulfuric acid, and in the presence of a cerium catalyst, he observed to his astonishment that the mixture became yellow, then faded to colorless after about a minute, then returned to yellow a minute later, then became colorless again, and continued to oscillate dozens of times before finally reaching equilibrium after about an hour.

Today it comes as no surprise that chemical reactions can oscillate spontaneously—such reactions have become a standard demonstration in chemistry classes, and you may have seen one yourself. (For recipes, see Winfree (1980).) But in Belousov's day, his discovery was so radical that he couldn't get his work published. It was thought that all solutions of chemical reagents must go *monotonically* to equilibrium, because of the laws of thermodynamics. Belousov's paper was rejected by one journal after another. According to Winfree (1987b, p.161), one editor even added a snide remark about Belousov's “supposedly discovered discovery” to the rejection letter.

Belousov finally managed to publish a brief abstract in the obscure proceedings of a Russian medical meeting (Belousov 1959), although his colleagues weren't aware of it until years later. Nevertheless, word of his amazing reaction circulated among Moscow chemists in the late 1950s, and in 1961 a graduate student named Zhabotinsky was assigned by his adviser to look into it. Zhabotinsky confirmed that Belousov was right all along, and brought this work to light at an international conference in Prague in 1968, one of the few times that Western and Soviet scientists were allowed to meet. At that time there was a great deal of interest in biological and biochemical oscillations (Chance et al. 1973) and the BZ reaction, as it came to be called, was seen as a manageable model of those more complex systems.

The analogy to biology turned out to be surprisingly close: Zaikin and Zhabotinsky (1970) and Winfree (1972) observed beautiful propagating *waves* of oxidation in thin unstirred layers of BZ reagent, and found that these waves annihilate upon collision, just like waves of excitation in neural or cardiac tissue. The waves always take the shape of expanding concentric rings or spirals (Color plate 1). Spiral waves are now recognized to be a ubiquitous feature of chemical, biological, and physical excitable media; in particular, spiral waves and their three-dimensional analogs, “scroll waves” (Front cover illustration) appear to be implicated in certain cardiac arrhythmias, a problem of great medical importance (Winfree 1987b).

Boris Belousov would be pleased to see what he started.

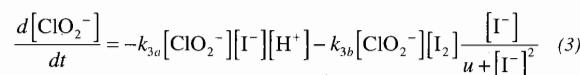
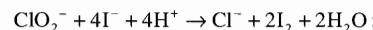
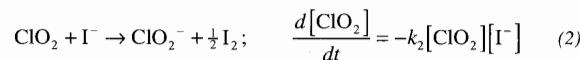
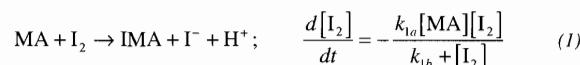
In 1980, he and Zhabotinsky were awarded the Lenin Prize, the Soviet Union's highest medal, for their pioneering work on oscillating reactions. Unfortunately, Belousov had passed away ten years earlier.

For more about the history of the BZ reaction, see Winfree (1984, 1987b). An English translation of Belousov's original paper from 1951 appears in Field and Burger (1985).

## Chlorine Dioxide–Iodine–Malonic Acid Reaction

The mechanisms of chemical oscillations can be very complex. The BZ reaction is thought to involve more than twenty elementary reaction steps, but luckily many of them equilibrate rapidly—this allows the kinetics to be reduced to as few as three differential equations. See Tyson (1985) for this reduced system and its analysis.

In a similar spirit, Lengyel et al. (1990) have proposed and analyzed a particularly elegant model of another oscillating reaction, the chlorine dioxide–iodine–malonic acid ( $\text{ClO}_2$ – $\text{I}_2$ –MA) reaction. Their experiments show that the following three reactions and empirical rate laws capture the behavior of the system:



Typical values of the concentrations and kinetic parameters are given in Lengyel et al. (1990) and Lengyel and Epstein (1991).

Numerical integrations of (1)–(3) show that the model exhibits oscillations that closely resemble those observed experimentally. However this model is still too complicated to handle analytically. To simplify it, Lengyel et al. (1990) use a result found in their simulations: Three of the reactants (MA,  $\text{I}_2$ , and  $\text{ClO}_2$ ) vary much more slowly than the intermediates  $\text{I}^-$  and  $\text{ClO}_2^-$ , which change by several orders of magnitude during an oscillation period. By approximating the concentrations of the slow reactants as *constants* and making other reasonable simplifications, they reduce the system to a two-variable model. (Of course, since this approximation neglects the slow consumption of the reactants, the model will be unable to account for the eventual approach to equilibrium.) After suitable nondimensionalization, the model becomes

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \quad (4)$$

$$\dot{y} = bx \left(1 - \frac{y}{1+x^2}\right) \quad (5)$$

where  $x$  and  $y$  are the dimensionless concentrations of  $\text{I}^-$  and  $\text{ClO}_2^-$ . The parameters  $a, b > 0$  depend on the empirical rate constants and on the concentrations assumed for the slow reactants.

We begin the analysis of (4), (5) by constructing a trapping region and applying the Poincaré–Bendixson theorem. Then we'll show that the chemical oscillations arise from a supercritical Hopf bifurcation.

### EXAMPLE 8.3.1:

Prove that the system (4), (5) has a closed orbit in the positive quadrant  $x, y > 0$  if  $a$  and  $b$  satisfy certain constraints, to be determined.

*Solution:* As in Example 7.3.2, the nullclines help us to construct a trapping region. Equation (4) shows that  $\dot{x} = 0$  on the curve

$$y = \frac{(a-x)(1+x^2)}{4x} \quad (6)$$

and (5) shows that  $\dot{y} = 0$  on the  $y$ -axis and on the parabola  $y = 1 + x^2$ . These nullclines are sketched in Figure 8.3.1, along with some representative vectors.

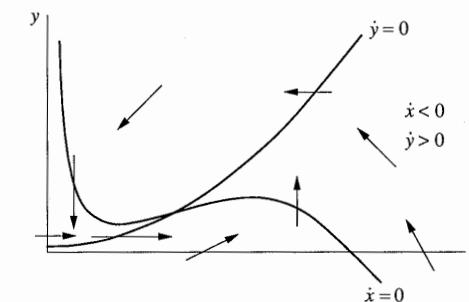
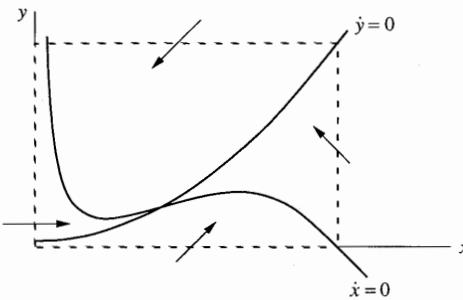


Figure 8.3.1

(We've taken some pedagogical license with Figure 8.3.1; the curvature of the nullcline (6) has been exaggerated to highlight its shape, and to give us more room to draw the vectors.)

Now consider the dashed box shown in Figure 8.3.2. It's a trapping region because all the vectors on the boundary point into the box.



**Figure 8.3.2**

We can't apply the Poincaré–Bendixson theorem yet, because there's a fixed point

$$x^* = a/5, \quad y^* = 1 + (x^*)^2 = 1 + (a/5)^2$$

inside the box at the intersection of the nullclines. But now we argue as in Example 7.3.3: if the fixed point turns out to be a *repeller*, we *can* apply the Poincaré–Bendixson theorem to the “punctured” box obtained by removing the fixed point.

All that remains is to see under what conditions (if any) the fixed point is a repeller. The Jacobian at  $(x^*, y^*)$  is

$$\frac{1}{1+(x^*)^2} \begin{pmatrix} 3(x^*)^2 - 5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}.$$

(We've used the relation  $y^* = 1 + (x^*)^2$  to simplify some of the entries in the Jacobian.) The determinant and trace are given by

$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0, \quad \tau = \frac{3(x^*)^2 - 5 - bx^*}{1+(x^*)^2}.$$

We're in luck—since  $\Delta > 0$ , the fixed point is never a saddle. Hence  $(x^*, y^*)$  is a repeller if  $\tau > 0$ , i.e., if

$$b < b_c \equiv 3a/5 - 25/a. \quad (7)$$

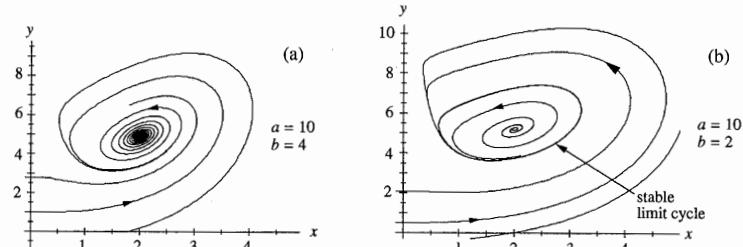
When (7) holds, the Poincaré–Bendixson theorem implies the existence of a closed orbit somewhere in the punctured box. ■

### EXAMPLE 8.3.2:

Using numerical integration, show that a Hopf bifurcation occurs at  $b = b_c$  and

decide whether the bifurcation is sub- or supercritical.

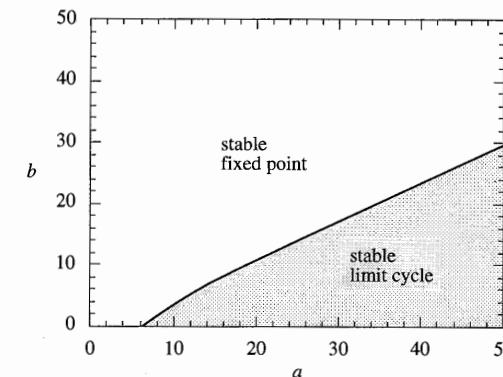
*Solution:* The analytical results above show that as  $b$  decreases through  $b_c$ , the fixed point changes from a stable spiral to an unstable spiral; this is the signature of a Hopf bifurcation. Figure 8.3.3 plots two typical phase portraits. (Here we have chosen  $a = 10$ ; then (7) implies  $b_c = 3.5$ .) When  $b > b_c$ , all trajectories spiral into the stable fixed point (Figure 8.3.3a), while for  $b < b_c$  they are attracted to a stable limit cycle (Figure 8.3.3b).



**Figure 8.3.3**

Hence the bifurcation is *supercritical*—after the fixed point loses stability, it is surrounded by a stable limit cycle. Moreover, by plotting phase portraits as  $b \rightarrow b_c$  from below, we could confirm that the limit cycle shrinks continuously to a point, as required. ■

Our results are summarized in the stability diagram in Figure 8.3.4. The boundary between the two regions is given by the Hopf bifurcation locus  $b = 3a/5 - 25/a$ .



**Figure 8.3.4**

**EXAMPLE 8.3.3:**

Approximate the period of the limit cycle for  $b$  slightly less than  $b_c$ .

*Solution:* The frequency is approximated by the imaginary part of the eigenvalues at the bifurcation. As usual, the eigenvalues satisfy  $\lambda^2 - \tau\lambda + \Delta = 0$ . Since  $\tau = 0$  and  $\Delta > 0$  at  $b = b_c$ , we find

$$\lambda = \pm i\sqrt{\Delta}.$$

But at  $b_c$ ,

$$\Delta = \frac{5b_c x^*}{1+(x^*)^2} = \frac{5\left(\frac{3a}{5} - \frac{25}{a}\right)\left(\frac{a}{5}\right)}{1+(a/5)^2} = \frac{15a^2 - 625}{a^2 + 25}.$$

Hence  $\omega \approx \Delta^{1/2} = [(15a^2 - 625)/(a^2 + 25)]^{1/2}$  and therefore

$$\begin{aligned} T &= 2\pi/\omega \\ &= 2\pi[(a^2 + 25)/(15a^2 - 625)]^{1/2}. \end{aligned}$$

A graph of  $T(a)$  is shown in Figure 8.3.5. As  $a \rightarrow \infty$ ,  $T \rightarrow 2\pi/\sqrt{15} \approx 1.63$ . ■

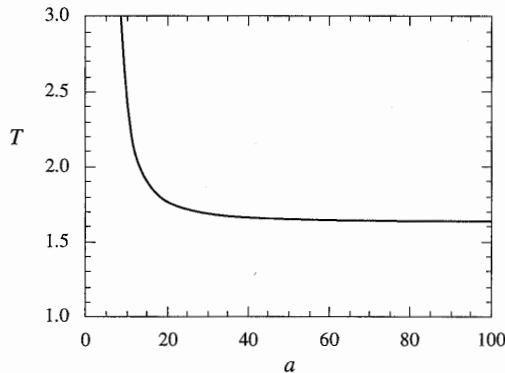


Figure 8.3.5

## 8.4 Global Bifurcations of Cycles

In two-dimensional systems, there are four common ways in which limit cycles are created or destroyed. The Hopf bifurcation is the most famous, but the other three deserve their day in the sun. They are harder to detect because they involve large

regions of the phase plane rather than just the neighborhood of a single fixed point. Hence they are called *global bifurcations*. In this section we offer some prototypical examples of global bifurcations, and then compare them to one another and to the Hopf bifurcation. A few of their scientific applications are discussed in Sections 8.5 and 8.6 and in the exercises.

### Saddle-node Bifurcation of Cycles

A bifurcation in which two limit cycles coalesce and annihilate is called a *fold* or *saddle-node bifurcation of cycles*, by analogy with the related bifurcation of fixed points. An example occurs in the system

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2 \end{aligned}$$

studied in Section 8.2. There we were interested in the subcritical Hopf bifurcation at  $\mu = 0$ ; now we concentrate on the dynamics for  $\mu < 0$ .

It is helpful to regard the radial equation  $\dot{r} = \mu r + r^3 - r^5$  as a one-dimensional system. As you should check, this system undergoes a saddle-node bifurcation of fixed points at  $\mu_c = -1/4$ . Now returning to the two-dimensional system, these fixed points correspond to circular *limit cycles*. Figure 8.4.1 plots the “radial phase portraits” and the corresponding behavior in the phase plane.

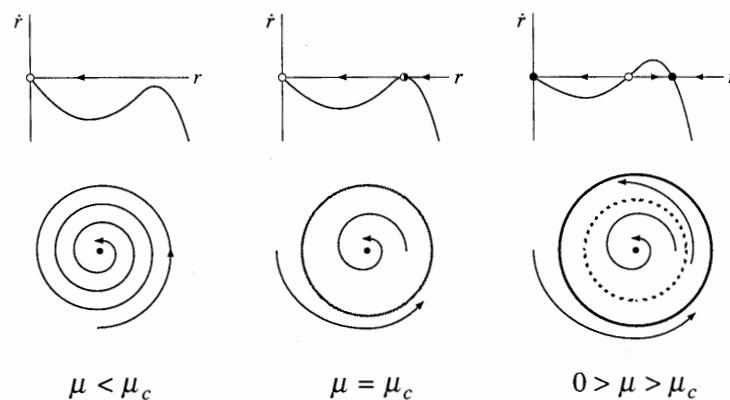


Figure 8.4.1

At  $\mu_c$  a half-stable cycle is born out of the clear blue sky. As  $\mu$  increases it splits into a pair of limit cycles, one stable, one unstable. Viewed in the other direction, a stable and unstable cycle collide and disappear as  $\mu$  decreases through  $\mu_c$ . Notice that the origin remains stable throughout; it does not participate in this bifurcation.

For future reference, note that at birth the cycle has  $O(1)$  amplitude, in contrast to the Hopf bifurcation, where the limit cycle has small amplitude proportional to  $(\mu - \mu_c)^{1/2}$ .

### Infinite-period Bifurcation

Consider the system

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \mu - \sin \theta\end{aligned}$$

where  $\mu \geq 0$ . This system combines two one-dimensional systems that we have studied previously in Chapters 3 and 4. In the radial direction, all trajectories (except  $r^* = 0$ ) approach the unit circle monotonically as  $t \rightarrow \infty$ . In the angular direction, the motion is everywhere counterclockwise if  $\mu > 1$ , whereas there are two invariant rays defined by  $\sin \theta = \mu$  if  $\mu < 1$ . Hence as  $\mu$  decreases through  $\mu_c = 1$ , the phase portraits change as in Figure 8.4.2.

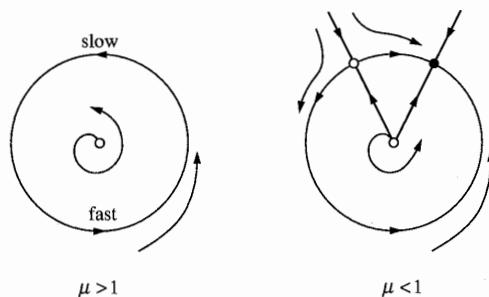


Figure 8.4.2

As  $\mu$  decreases, the limit cycle  $r = 1$  develops a bottleneck at  $\theta = \pi/2$  that becomes increasingly severe as  $\mu \rightarrow 1^+$ . The oscillation period lengthens and finally becomes infinite at  $\mu_c = 1$ , when a fixed point appears on the circle; hence the term **infinite-period bifurcation**. For  $\mu < 1$ , the fixed point splits into a saddle and a node.

As the bifurcation is approached, the amplitude of the oscillation stays  $O(1)$  but the period increases like  $(\mu - \mu_c)^{-1/2}$ , for the reasons discussed in Section 4.3.

### Homoclinic Bifurcation

In this scenario, part of a limit cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic or-

bit. This is another kind of infinite-period bifurcation; to avoid confusion, we'll call it a *saddle-loop* or *homoclinic bifurcation*.

It is hard to find an analytically transparent example, so we resort to the computer. Consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \mu y + x - x^2 + xy.\end{aligned}$$

Figure 8.4.3 plots a series of phase portraits before, during, and after the bifurcation; only the important features are shown.

Numerically, the bifurcation is found to occur at  $\mu_c \approx -0.8645$ . For  $\mu < \mu_c$ , say  $\mu = -0.92$ , a stable limit cycle passes close to a saddle point at the origin (Figure 8.4.3a). As  $\mu$  increases to  $\mu_c$ , the limit cycle swells (Figure 8.4.3b) and bangs into the saddle, creating a homoclinic orbit (Figure 8.4.3c). Once  $\mu > \mu_c$ , the saddle connection breaks and the loop is destroyed (Figure 8.4.3d).

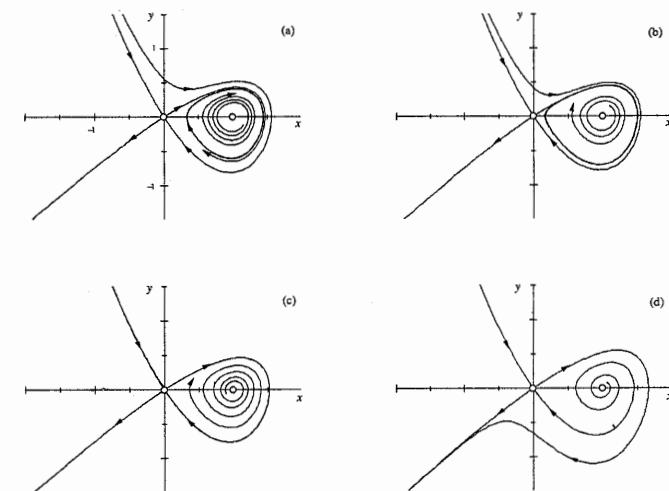


Figure 8.4.3

The key to this bifurcation is the behavior of the unstable manifold of the saddle. Look at the branch of the unstable manifold that leaves the origin to the northeast: after it loops around, it either hits the origin (Figure 8.4.3c) or veers off to one side or the other (Figures 8.4.3a, d).

**Example 2 (A Saddle-Node Bifurcation at a Nonhyperbolic Periodic Orbit).** Consider the planar system

$$\begin{aligned}\dot{x} &= -y - x[\mu - (r^2 - 1)^2] \\ \dot{y} &= x - y[\mu - (r^2 - 1)^2]\end{aligned}$$

which is of the form of equation (5) in Section 4.4. Writing this system in polar coordinates yields

$$\begin{aligned}\dot{r} &= -r[\mu - (r^2 - 1)^2] \\ \dot{\theta} &= 1.\end{aligned}$$

For  $\mu > 0$  there are two one-parameter families of periodic orbits

$$\Gamma_\mu^\pm: \gamma_\mu^\pm(t) = \sqrt{1 \pm \mu^{1/2}}(\cos t, \sin t)^T$$

with parameter  $\mu$ . Since the origin is unstable for  $0 < \mu < 1$ , the smaller limit cycle  $\Gamma_\mu^-$  is stable and the larger limit cycle  $\Gamma_\mu^+$  is unstable. For  $\mu = 0$  there is a semistable limit cycle  $\Gamma_0$  represented by  $\gamma_0(t) = (\cos t, \sin t)^T$ . The phase portraits for this system are shown in Figure 1 and the bifurcation diagram is shown in Figure 2. Note that there is a supercritical Hopf bifurcation at the origin at the bifurcation value  $\mu = 1$ .

In Example 2 the points  $r_\mu^\pm = \sqrt{1 \pm \mu^{1/2}}$  are fixed points of the Poincaré map  $P(r, \mu)$  of the periodic orbit  $\gamma_0(t)$  along any ray from the origin

$$\Sigma = \{x \in \mathbb{R}^2 \mid r > 0, \theta = \theta_0\};$$

i.e., we have  $d(\sqrt{1 \pm \mu^{1/2}}, \mu) = 0$  where  $d(r, \mu) = P(r, \mu) - r$  is the displacement function. The bifurcation diagram is given by the graph of the relation  $d(r, \mu) = 0$  in the  $(\mu, r)$ -plane. Using equation (2), we can compute the derivative of the Poincaré map at  $r_\mu^\pm = \sqrt{1 \pm \mu^{1/2}}$ :

$$DP(\sqrt{1 \pm \mu^{1/2}}, \mu) = e^{\pm 8\mu^{1/2}(1 \pm \mu^{1/2})\pi},$$

cf. Problem 2. We see that for  $0 < \mu < 1$ ,  $DP(\sqrt{1 - \mu^{1/2}}, \mu) < 1$  and  $DP(\sqrt{1 + \mu^{1/2}}, \mu) > 1$ ; the smaller limit cycle is stable and the larger limit cycle is unstable as illustrated in Figures 1 and 2. Furthermore, for  $\mu = 0$  we have  $DP(1, 0) = 1$ , i.e.,  $\gamma_0(t)$  is a nonhyperbolic periodic orbit with both of its characteristic exponents equal to zero.

**Remark 1.** In general, for planar systems, the bifurcation diagram is given by the graph of the relation  $d(s, \mu) = 0$  in the  $(\mu, s)$ -plane where

$$d(s, \mu) = P(s, \mu) - s$$

is the displacement function along a straight line  $\Sigma$  normal to the nonhyper-

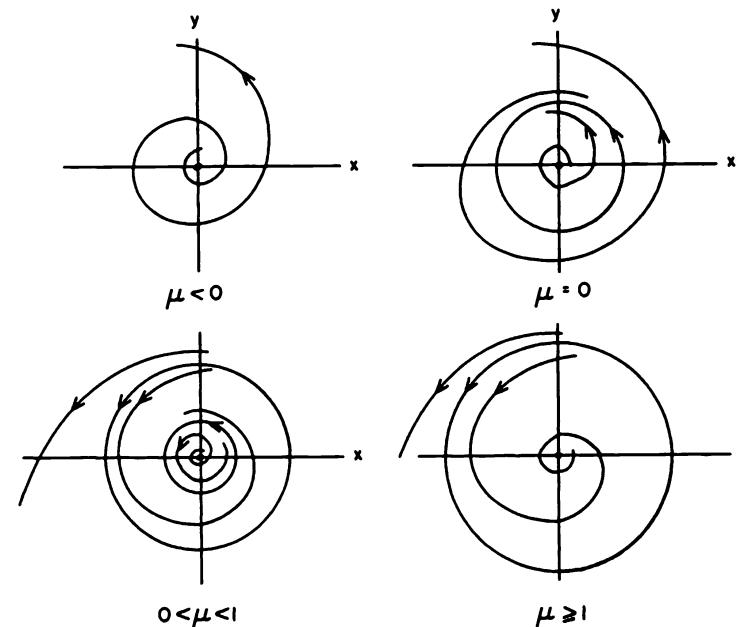


Figure 1. The phase portraits for the system in Example 2.

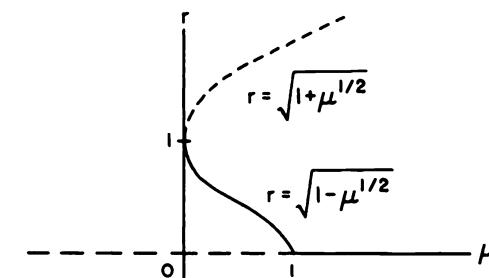


Figure 2. The bifurcation diagram for the saddle-node bifurcation at the nonhyperbolic periodic orbit  $\gamma_0(t)$  of the system in Example 2.

bolic periodic orbit  $\Gamma_0$  at  $x_0$ . We take  $s$  to be the signed distance along the straight line  $\Sigma$ , with  $s$  positive at points on the exterior of  $\Gamma_0$  and negative at points on the interior of  $\Gamma_0$ , as in Figure 3 in Section 3.4 of Chapter 3.

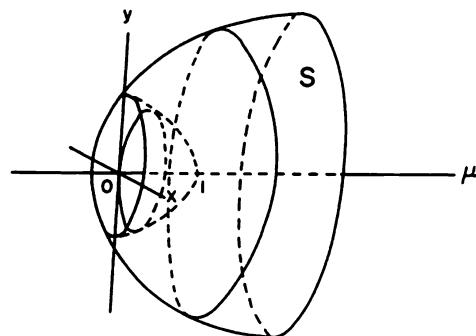


Figure 3. The one-parameter family of periodic orbits  $S$  of the system in Example 2.

In this context, it follows that for each fixed value of  $\mu$ , the values of  $s$  for which  $d(s, \mu) = 0$  define points  $x_j$  on  $\Sigma$  near the point  $x_0 \in \Gamma_0 \cap \Sigma$  through which the system (1) has periodic orbits  $\gamma_j(t) = \phi_t(x_j)$ . For example, in Figure 2, each vertical line  $\mu = \text{constant}$ , with  $0 < \mu < 1$ , intersects the curve  $d(r, \mu) = 0$  in two points  $(\mu, \sqrt{1 \pm \mu^{1/2}})$ ; and the system in Example 2 has periodic orbits  $\gamma_\mu^\pm(t)$  through the points  $(\sqrt{1 \pm \mu^{1/2}}, 0)$  on the  $x$ -axis in the phase plane. As in Figure 2 in Section 4.4, each one-parameter family of periodic orbits generates a surface  $S$  in  $\mathbb{R}^2 \times \mathbb{R}$ . For example, the periodic orbits of the system in Example 2 generate the surface  $S$  shown in Figure 3. Since in general there is only one surface generated at a saddle-node bifurcation at a nonhyperbolic periodic orbit, we regard the two one-parameter families of periodic orbits (with parameter  $\mu$ ) as belonging to one and the same family of periodic orbits. In this case, we can always define a new parameter  $\beta$  (such as the arc length along a path on the surface  $S$ ) so that  $\Gamma_{\mu(\beta)}$  defines a one-parameter family of periodic orbits with parameter  $\beta$ .

**Example 3 (A Transcritical Bifurcation at a Nonhyperbolic Periodic Orbit).** Consider the planar system

$$\begin{aligned}\dot{x} &= -y - x(1 - r^2)(1 + \mu - r^2) \\ \dot{y} &= x - y(1 - r^2)(1 + \mu - r^2).\end{aligned}$$

In polar coordinates we have

$$\begin{aligned}\dot{r} &= -r(1 - r^2)(1 + \mu - r^2) \\ \dot{\theta} &= 1.\end{aligned}$$

For all  $\mu \in \mathbb{R}$ , this system has a one-parameter family of periodic orbits

represented by

$$\gamma_0(t) = (\cos t, \sin t)^T$$

and for  $\mu > -1$ , there is another one-parameter family of periodic orbits represented by

$$\gamma_\mu(t) = \sqrt{1 + \mu}(\cos t, \sin t)^T.$$

The bifurcation diagram, showing the transcritical bifurcation that occurs at the nonhyperbolic periodic orbit  $\gamma_0(t)$  at the bifurcation value  $\mu = 0$ , is shown in Figure 4. Note that a subcritical Hopf bifurcation occurs at the nonhyperbolic critical point at the origin at the bifurcation value  $\mu = -1$ .

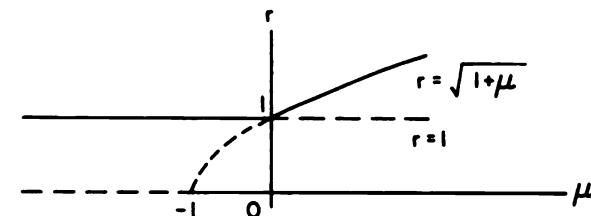


Figure 4. The bifurcation diagram for the transcritical bifurcation at the nonhyperbolic periodic orbit  $\gamma_0(t)$  of the system in Example 3.

In Example 3 the points  $r_\mu = \sqrt{1 + \mu}$  and  $r_\mu = 1$  are fixed points of the Poincaré map  $P(r, \mu)$  of the nonhyperbolic periodic orbit  $\Gamma_0$ ; i.e., we have

$$d(\sqrt{1 + \mu}, \mu) = 0$$

for all  $\mu > -1$  and

$$d(1, \mu) = 0$$

for all  $\mu \in \mathbb{R}$  where  $d(r, \mu) = P(r, \mu) - r$ . Furthermore, using equation (2), we can compute

$$DP(\sqrt{1 + \mu}, \mu) = e^{-4\mu(1+\mu)\pi}$$

and

$$DP(1, \mu) = e^{4\mu\pi},$$

cf. Problem 2. This determines the stability of the two families of periodic orbits as indicated in Figure 4. We see that  $DP(1, 0) = 1$ ; i.e., there is a nonhyperbolic periodic orbit  $\Gamma_0$  with both of its characteristic exponents equal to zero at the bifurcation value  $\mu = 0$ . In this example, there are two distinct surfaces of periodic orbits, a cylindrical surface and a parabolic surface, which intersect in the nonhyperbolic periodic orbit  $\Gamma_0$ ; cf. Problem 3.

**Example 4 (A Pitchfork Bifurcation at a Nonhyperbolic Periodic Orbit).** Consider the planar system

$$\begin{aligned}\dot{x} &= -y + x(1 - r^2)[\mu - (r^2 - 1)^2] \\ \dot{y} &= x + y(1 - r^2)[\mu - (r^2 - 1)^2].\end{aligned}$$

In polar coordinates we have

$$\begin{aligned}\dot{r} &= r(1 - r^2)[\mu - (r^2 - 1)^2] \\ \dot{\theta} &= 1.\end{aligned}$$

For all  $\mu \in \mathbb{R}$ , this system has a one-parameter family of periodic orbits represented by

$$\gamma_0(t) = (\cos t, \sin t)^T$$

and for  $\mu > 0$  there is another family (with two branches as in Remark 1) represented by

$$\gamma_\mu^\pm(t) = \sqrt{1 \pm \mu^{1/2}}(\cos t, \sin t)^T.$$

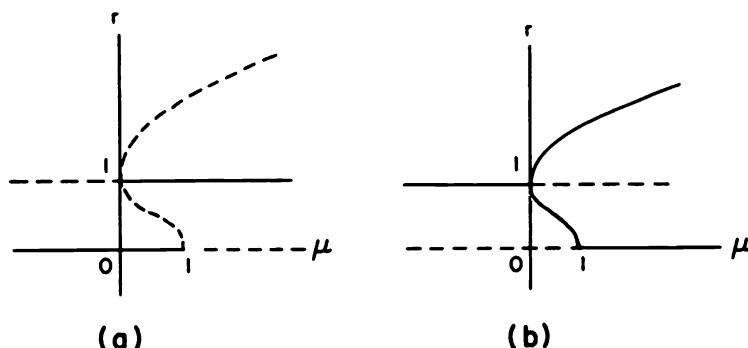
Using equation (2) we can compute

$$DP(\sqrt{1 \pm \mu^{1/2}}, \mu) = e^{4\mu(1 \pm \mu^{1/2})\pi}$$

and

$$DP(1, \mu) = e^{-4\pi\mu}.$$

This determines the stability of the two families of periodic orbits as indicated in Figure 5(a). We also see that  $DP(1, 0) = 1$ ; i.e., there is a



**Figure 5.** The bifurcation diagram for the pitchfork bifurcation at the nonhyperbolic periodic orbit  $\gamma_0(t)$  of the system (a) in Example 4 and (b) in Example 4 with  $t \rightarrow -t$ .

nonhyperbolic periodic orbit  $\Gamma_0$  with both of its characteristic exponents equal to zero at the bifurcation value  $\mu = 0$ . Note that a Hopf bifurcation occurs at the nonhyperbolic critical point at the origin at the bifurcation value  $\mu = 1$ . Also, note that if we reverse the sign of  $t$  in this example, i.e., let  $t \rightarrow -t$ , then we reverse the stability of the periodic orbits and we would have the bifurcation diagram with a pitchfork bifurcation shown in Figure 5(b).

Using the implicit function theorem, conditions can be given on the derivatives of the Poincaré map which imply the existence of a saddle-node, transcritical or pitchfork bifurcation at a nonhyperbolic periodic orbit of (1). We shall only give these conditions for planar systems. In the next theorem  $P(s, \mu)$  denotes the Poincaré map along a normal line  $\Sigma$  to a nonhyperbolic periodic orbit  $\Gamma_0$  at a bifurcation value  $\mu = \mu_0$  in (1). As in Section 4.2,  $D$  denotes the partial derivative of  $P(s, \mu)$  with respect to the spatial variable  $s$ .

**Theorem 1.** Suppose that  $f \in C^2(E \times J)$  where  $E$  is an open subset of  $\mathbb{R}^2$  and  $J \subset \mathbb{R}$  is an interval. Assume that for  $\mu = \mu_0$  the system (1) has a periodic orbit  $\Gamma_0 \subset E$  and that  $P(s, \mu)$  is the Poincaré map for  $\Gamma_0$  defined in a neighborhood  $N_\delta(0, \mu_0)$ . Then if  $P(0, \mu_0) = 0$ ,  $DP(0, \mu_0) = 1$ ,

$$D^2P(0, \mu_0) \neq 0 \quad \text{and} \quad P_\mu(0, \mu_0) \neq 0, \quad (3)$$

it follows that a saddle-node bifurcation occurs at the nonhyperbolic periodic orbit  $\Gamma_0$  at the bifurcation value  $\mu = \mu_0$ ; i.e., depending on the signs of the expressions in (3), there are no periodic orbits of (1) near  $\Gamma_0$  when  $\mu < \mu_0$  (or when  $\mu > \mu_0$ ) and there are two periodic orbits of (1) near  $\Gamma_0$  when  $\mu > \mu_0$  (or when  $\mu < \mu_0$ ). The two periodic orbits of (1) near  $\Gamma_0$  are hyperbolic and of the opposite stability.

If the conditions (3) are changed to

$$\begin{aligned}P_\mu(0, \mu_0) &= 0 \quad DP_\mu(0, \mu_0) \neq 0 \quad \text{and} \\ D^2P(0, \mu_0) &\neq 0,\end{aligned} \quad (4)$$

then a transcritical bifurcation occurs at the nonhyperbolic periodic orbit  $\Gamma_0$  at the bifurcation value  $\mu = \mu_0$ . And if the conditions (3) are changed to

$$\begin{aligned}P_\mu(0, \mu_0) &= 0, \quad DP_\mu(0, \mu_0) \neq 0 \\ D^2P(0, \mu_0) &= 0 \quad \text{and} \quad D^3P(0, \mu_0) \neq 0,\end{aligned} \quad (5)$$

then a pitchfork bifurcation occurs at the nonhyperbolic periodic orbit  $\Gamma_0$  at the bifurcation value  $\mu = \mu_0$ .

**Remark 2.** Under the conditions (3) in Theorem 1, the periodic orbit  $\Gamma_0$  is a multiple limit cycle of multiplicity  $m = 2$  and exactly two limit cycles bifurcate from the semi-stable limit cycle  $\Gamma_0$  as  $\mu$  varies from  $\mu_0$  in one sense or the other. In particular, if  $D^2P(0, \mu_0)$  and  $P_\mu(0, \mu_0)$  have opposite signs, then there are two limit cycles near  $\Gamma_0$  for all sufficiently small  $\mu - \mu_0 > 0$  and if  $D^2P(0, \mu_0)$  and  $P_\mu(0, \mu_0)$  have the same sign, then there are two limit cycles near  $\Gamma_0$  for all sufficiently small  $\mu_0 - \mu > 0$ .