ENGSCI 213: MATHEMATICAL MODELLING 2SE

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

CURRENT TOPIC

2. *Discrete probability models* [3 lectures]

   Discrete random variables and discrete probability distributions. Probability mass functions and cumulative distribution functions. (Expectation and variance). Binomial and Poisson distributions.

[last lecture on discrete probability models]

LECTURE 6

- Expectation and variance recap and application to the Binomial distribution
- Recap of basic ideas of random/stochastic processes
- The Poisson process/distribution

RECAP: BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION

The probability (mass) function for a *Binomial* random variable is given by

$$P(X = x) = f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

RECAP: EXPECTATION

The expectation or 'mean' of a discrete random variable X is written $E(X)$ and defined by

$$E(X) := \sum_{x \in S_X} xP(X = x)$$

$E(X)$ is also often denoted by μ_X and called the 'mean'. It is a *property of the whole/'true'/'population' distribution*. It represents what value of X you expect to get 'on average'.

RECAP: VARIANCE AND STANDARD DEVIATION

A measure of the *'variabiilty'* of X values can be also defined as a type of expectation

The *variance* of a discrete random variable X is defined by

$$\boxed{\begin{aligned} Var(X) &:= E[(x - E[x])^2] \\ &= \sum_{x \in S_X} (x - E(X))^2 P(X = x) \end{aligned}}$$

(think: 'mean square difference'; note: we can also define 'higher' moments/averages)

RECAP: VARIANCE AND STANDARD DEVIATION

The standard deviation is given by

$$SD(X) = \sqrt{Var}(X)$$

We often write $\mathrm{SD}(X)$ as σ_X.

MEAN, VARIANCE AND STANDARD DEVIATION FOR THE BINOMIAL DISTRIBUTION

Let $X \sim$ Binomial(n, p). Then

$$\boxed{\begin{gathered} \mu_X = E(X) = np \\ \sigma^2 = Var(X) = np(1 - p) \end{gathered}}$$

Proof.

REMINDER: RANDOM SEQUENCES AND RANDOM PROCESSES

Recall that a *sequence of random variables* is how we capture the idea of a *'random process'* - i.e. a sequence of random *sub-experiments* combining to give an *overall experiment*

> *A random (or stochastic) process is a sequence of random variables, one per 'trial'. This sequence of trials may occur in discrete steps e.g.* $(X_1, X_2, X_3, \ldots)$ *or continuous steps e.g.* $\{X_t \mid t \in T\}$ *.*

REMINDER: RANDOM SEQUENCES AND RANDOM PROCESSES

Note that if each random variable X_t, corresponding to a given 'instant' t of *continuous time*, has a *discrete* set of possible outcomes then we are dealing with a *'discrete outcome (or discrete state), continuous time'* process.

> *We still count this as a 'discrete probability model' because we have a discrete random variable for each trial.*

REMINDER: SEQUENCE OF BERNOULLI TRIALS

(Discrete state, discrete time)

- a *sequence* of independent Bernoulli trials ('coin tosses') each with outcome $\{0, 1\}$ giving an *overall* random experiment generating outcomes like $(0, 0, 1, 1, 0, 0, 0, 1 \ldots)$.
- We can call this a *'Bernoulli process'*
- We *summarised* the complete results from a 'Bernoulli process' into a *single 'Binomial random variable'*.

THE POISSON PROCESS AND THE POISSON DISTRIBUTION: MOTIVATION

The Poisson process is a continuous-time, discrete outcome process. It is like a special limit of a sequence of Bernoulli trials.

The Poisson distribution summarises the overall results of this process (over a given time interval). Hence it is like a special limit of the Binomial distribution.

THE POISSON PROCESS AND THE POISSON DISTRIBUTION: MOTIVATION

In particular we imagine having some *fixed time interval*, after which we will *'summarise'* the results and

- *Lots of Bernoulli trials* - 'continuously-occurring' sequence of trials *within* the time interval
- *Low probability* of success for each *individual trial within* the time interval
- *Fixed 'average'* number of outcomes *per overall time interval* (overall summary) ($\approx np$ when related to the Binomial distribution)

THE POISSON PROCESS AND THE POISSON DISTRIBUTION: MOTIVATION

'Within' the time interval we call the continuously-occurring sequence of trials a Poisson process.

At the 'end' of each interval we call the summary the Poisson distribution' for that interval.

Motivating examples.

THE POISSON DISTRIBUTION: DEFINITION

The Poisson distribution for a Poisson random variable X has *one parameter*, λ (think $\approx np$ for Binomial), which in a Poisson process equals the *average rate at which individual events occur.* We write

$$\boxed{X \sim Poisson(\lambda) \text{ or } X \sim Po(\lambda)}$$

where

$$\boxed{\begin{gathered} P(X = x) = f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda} \\ \text{for } x = 0, 1, 2, \ldots \end{gathered}}$$

THE POISSON DISTRIBUTION

So, roughly speaking,

The Poisson distribution counts the number of (individually rare) events occurring over a given time (or space) interval, when individual events occur independently and at a constant average rate.

It represents a summary of a *'purely or completely random' process* over a given interval.

Examples.

THE POISSION DISTRIBUTION

Assumptions

- events (trials) are *independent*
- events occur at *constant average rate* over a given time interval
- events *do not occur simultaneously*

Note also: λ has units of 'number of successes *per given time interval*'.

POISSON DISTRIBUTION FOR OTHER INTERVALS

Since the Poisson distribution summarises a *'completely random'* process we can summarise this process over *any other interval*, given the rate over one interval, in the *same form with a scaled parameter*

> *If λ_0 is the average rate of events per unit time interval then $X \sim Poisson(\lambda_0 t)$ is the Poisson distribution for the time interval of t units.*

More examples.

THE POISSION DISTRIBUTION

Additional properties of a Poisson random variable X.

$$\boxed{\begin{aligned} \mu_X &= E(X) = \lambda \\ \sigma^2 &= Var(X) = \lambda \end{aligned}}$$

NOTES

Tutorial tomorrow - would you rather do lab or tutorial sheet?

End of discrete probability models (for now) - on to continuous models!