ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations

(*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]
   Basic concepts and formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.
2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]
   Stability and linearisation of nonlinear systems. General linear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.
4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 3: STABILITY AND LINEARISATION

- Stability of solutions for general systems
- Stability of solutions for linear systems
- Linearisation of nonlinear systems
- Connecting stability of linearised systems to stability of nonlinear systems
- Sneak peak at structural stability

STABILITY OF SOLUTIONS

There are various *general* formal definitions of *stability* for solutions.

These can be defined for both equilibria as well as more complicated objects like periodic orbits. They also apply for both linear and nonlinear systems.

We will just give the *definitions for equilibria* (points) for now.

- Recall how to find equilibria.

STABILITY OF SOLUTIONS

- A point x is *Lyapunov stable* iff for all $\epsilon > 0$ there exists a δ such that if $|x - y| < \delta$ then $|\phi(x,t) - \phi(y,t)| < \epsilon$ for all $t \geq 0$.
- A point x is *quasi-asymptotically stable* (attracting) iff there exists a δ such that if $|x - y| < \delta$ then $|\phi(x,t) - \phi(y,t)| \to 0$ as $t \to \infty$.
- A point is *asymptotically stable* iff it is both Lyapunov stable and quasi-asymptotically stable. If it is just Lyapunov stable then it is *neutrally stable* (i.e. is just bounded).

WHAT DO THESE MEAN?

Pictures!

STABILITY OF LINEAR SYSTEMS

These general definitions can be hard to check in general but are *easy to check for linear systems*.

> Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

(This can be proven by constructing a so-called Lyapunov function - ask me if interested/see further reading)

LINEARISATION AND LOCAL STABILITY ANALYSIS

We have seen that stability for linear systems is easy. We will analyse and classify linear systems in more detail soon.

This will be useful because, as mentioned, our first steps in analysing *nonlinear* systems will usually be through local *linearisation* about steady-states/equilibria.

We will also need to know the *connection between linear and nonlinear stability*!

LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component gives

$$\dot{u}_i = f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2) \text{ i.e.}$$

$$\dot{u}_i = \frac{\partial f_i}{\partial x_j}(x_e)u_i \equiv [Df(x_e)]_{ij}u_j$$

or simply $u = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

LINEAR AND NONLINEAR STABILITY - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues of the linearisation have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

LINEAR AND NONLINEAR STABILITY CONNECTED

The *Hartman-Grobman theorem* states that the local properties *near a hyperbolic fixed point* of a *nonlinear* system are *topologically equivalent* to those of the *linearisation*:

> *A hyperbolic fixed point persists* in the change from nonlinear to linear systems (though its location may shift slightly) and its *stability properties are also preserved*

(We will also see the similar 'stable manifold theorem' later)

BIFURCATIONS AND STRUCTURAL INSTABILITY

Recall: fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

When these occur our *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities* - i.e. small variations in problem parameters can have a large effect on the qualtiative/topological features of our phase space.

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

BIFURCATIONS AND STRUCTURAL INSTABILITY

These instabilities are called *bifurcations*.

We will return to this topic in more detail in later lectures.

EngSci 711 L03.

- Stability & Linearisation

Goals:
- understand the various concepts of stability in general
- understand when a linear stability analysis gives a 'faithful' picture of a nonlinear system
- be able to carry out a linear stability analysis of a nonlinear system

Examples: Key skills

Consider the system

$$\dot{x} = y(2x - y)$$
$$\dot{y} = x^2 - y$$

2016 Exam

(a) Find the two fixed points of this system. Show your working. You do not need to classify these.

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin (0,0).

} today

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

Consider the system

2017 Exam

$$\dot{x} = 2xy + x^3$$
$$\dot{y} = -y - x^2$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system. (1 mark)

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin (0,0). (2 marks)

} today

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories. (3 marks)

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

2016 Exam

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

(a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$.

} today

Example

Consider the nonlinear system from the computer tutorial:

$$\boxed{\begin{aligned} \dot{x} &= x(y-1) \quad (1) \\ \dot{y} &= 3x - 2y + x^2 - 2y^2 \quad (2) \end{aligned}}$$

- We're interested in long-term behaviour like fixed points (equilibria) & their stability
- what does this mean?

→ First let's find the fixed points } first step in most analyses!

Procedure: try solving in steps as follows:

(easiest one) → (1) $=0 \Rightarrow x = 0$ $\boxed{\text{or}}$ $y = 1$

- $x = 0$ & (2) $= 0$: ← (harder one)

$-2y - 2y^2 = 0 \Rightarrow -2y(1+y) = 0$

$\Rightarrow y = 0$ $\boxed{\text{or}}$ $y = -1$

so: $(0,0)$ or $(0,-1)$

- $y = 1$ & (2) $= 0$:

$3x - 2 + x^2 - 2 = 0$

$\Leftrightarrow x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$\Rightarrow x = -4$ $\boxed{\text{or}}$ $x = 1$

so: $(-4, 1)$ or $(1, 1)$

$\Rightarrow$ Solutions: $\boxed{\{(0,0), (0,-1), (1,1), (-4,1)\}}$ (fixed points)

So we see that eg

$(x,y) = (0,0)$, ie the <u>origin</u>, is
a fixed <u>point</u>

$$\boxed{\begin{array}{l} f(0,0)=0 \quad \text{or} \quad \varphi((0,0),t)=(0,0) \\ \& \ \dot{x} = \dot{y} = 0 \end{array}}$$

ie don't go anywhere if start
<u>exactly</u> there

What does it mean for this to be
a <u>stable</u> fixed point?

⇒ relates to starting <u>nearby</u>, not
just <u>exactly</u> there

↳ all models are wrong/imperfect
etc

We want a <u>general</u> definition, valid
for <u>nonlinear</u> systems

→ we will then relate this to
the more familiar <u>linear</u>
definition & see <u>under</u>
<u>what conditions</u> they give
the <u>same answer</u>

↳ often but <u>not always</u>

<u>Stability</u>: general definitions

There are at least <u>three</u> 'general' concepts
of stability

→ can be applied to general 'objects'
(invariant sets like equilibria, periodic
orbits etc)

→ we'll look at stability of <u>equilibria</u>

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1. ∘ Lyapunov (or Liapounov)

2. ∘ Quasi-asymptotic

3. ∘ Asymptotic (Lyapunov <u>&</u> Quasi.)

<u>goal</u>:
<u>intuitive</u>
understanding
of these.

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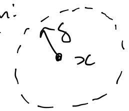
1. <u>Lyapunov</u>: "points that <u>start</u> <u>nearby</u>, <u>stay</u> <u>nearby</u>"
('<u>bounded</u>')

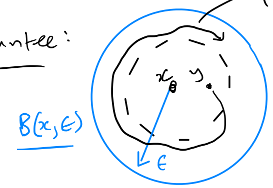
(<u>don't</u> need to
<u>approach</u> x tho')

∘ <u>Given</u>:



<u>B</u>(x,δ): ball of
all points y
s.t. $|x-y|<\delta$

∘ <u>Guarantee</u>:



$\varphi(y,t)$ for $y \in B(x,\delta)$

$|\varphi(y,t) - \varphi(x,t)| < \epsilon$

ie

$|\varphi(y,t) - x| < \epsilon$ <u>if</u> x <u>fixed</u> <u>point</u>

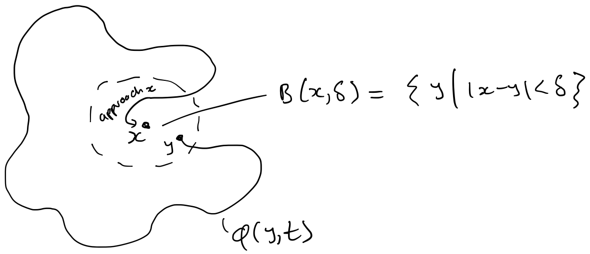
since
$\varphi(x,t) = x$
$\forall t$.

"Give me a <u>target</u> tolerance on <u>future</u> <u>trajectories</u> ϵ
& I'll give you a <u>starting</u> <u>tolerace</u> δ that
<u>guarantees</u> you'll meet this <u>target</u>"

<u>Math</u>: $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.
<u>if</u> $y \in B(x,\delta)$ <u>then</u> $\varphi(y,t) \in B(x,\epsilon)$

2. Quasi-asymptotic (attracting)

"start nearby, eventually return/approach (arbitrarily close) to"



Math:

if $y \in B(x,\delta)$ (ie $|x-y|<\delta$)

then $\varphi(y,t) \xrightarrow{t\to\infty} \varphi(x,t) = x$

if fixed point
(can use general def^n for more complex objects)

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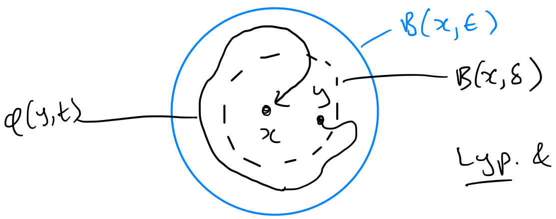
3. Asymptotic: Lyapunov $\boxed{\&}$ Quasi-asymptotic

"Start nearby, stay nearby $\boxed{\&}$ return/approach to"



Lyp. & Quasi

$\Rightarrow$ our goal is usually asymptotic stability

Problem?

- These definitions are general
  but/therefore: hard to apply/check!

- On the other hand, linear systems have an easy criterion:

> given $\boxed{\dot{x} = Ax}$, $x \in \mathbb{R}^n$, A $n\times n$ matrix
> the origin $x=0$ is asymptotically stable if all of the eigenvalues of A have negative real part

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Eg
$\lambda_1 = 1,\ \lambda_2 = -2 \rightarrow$ unstable
$\lambda_1 = -1,\ \lambda_2 = -2 \rightarrow$ stable
$\lambda_1 = -1+i,\ \lambda_2 = -1-i \rightarrow$ stable
$\lambda_1 = 0,\ \lambda_2 = -1 \rightarrow$ marginal (see later)

$\rightarrow$ it would be nice if we could just 'linearise' our nonlinear systems & use this instead!

$\hookrightarrow$ we can, under certain conditions!

$\rightarrow$ first: linearisation

Linearisation (at a fixed point)

General procedure : derivation

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

(a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$. ← 'small perturbation' from x^*

Exam 2016

1. Fixed point : $f(x^*) = 0$ } vector form

or $f_j(x^*) = 0, \forall j$ } component form ← easier

2. Taylor series for $x = x^* + \eta$ in component form: $(\eta = x - x^*)$

work-horse of applied math!

full vector

$$\dot{x}_i = f_i(x) = f_i(x^* + \eta)$$

$$\approx \underbrace{f_i(x^*)}_{\substack{=0 \\ \text{since fixed} \\ \text{point}}} + \eta_j \frac{\partial f_i}{\partial x_j}(x^*)$$

(sum over j) Einstein convention *

eval. at fixed point

$$\Rightarrow \underbrace{\dot{x}_i}_{\text{vector}} \approx \underbrace{\eta_j}_{\text{vector}} \underbrace{\frac{\partial f_i}{\partial x_j}(x^*)}_{\text{matrix}}$$

→

(* $\eta_j \frac{\partial f_i}{\partial x_j}$ means $\sum_j \eta_j \frac{\partial f_i}{\partial x_j}$)

Have $\dot{x}_i = \frac{\partial f_i}{\partial x_j} \eta_j$

cf ↕

want: $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j$

⇒ Note $x = x^* + \eta$

↑ fixed vector (time independent)

$\Rightarrow \dot{x} = \dot{\eta} \quad (\dot{x}^* = 0)$

So $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j$ (sum over j)

ie $\boxed{\dot{\eta} = A\eta}$ (note: $\sum_j A_{ij}\eta_j$ is just matrix multiplication)

⇒ Linear system in new 'perturbation' variable $\eta = x - x^*$

Here A = 'Df' the Jacobian derivative of f

→ generalises derivative to vector-valued functions of vector variables, like f

→ get a matrix, with components.

$$\boxed{(Df)_{ij} = Df_{ij} = \frac{\partial f_i}{\partial x_j}}$$

since vector valued

since vector variable

→

Huh?

→ Concrete example! — Be able to do these!

Let's linearise our previous system:

$$\begin{aligned} \dot{x} &= x(y-1) &&= f_1(x,y) &&= f_1(x_1,x_2) \\ \dot{y} &= 3x - 2y + x^2 - 2y^2 &&= f_2(x,y) &&= f_2(x_1,x_2) \end{aligned}$$

& then evaluate at $(0,0)$.

(here $x_1 = x$, $x_2 = y$)

First, find $Df(x,y) = Df(x_1,x_2)$

(at any x, y first. Then at eg $(0,0)$ or other fixed point)

ie
$$\underset{\text{row}\ \ \text{column}}{Df_{ij}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y-1 & x \\ 3+2x & -2-4y \end{pmatrix}$$

Next, evaluate at $(0,0)$ (for example).

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

→

Eigenvalues?

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix} = A$$

Long way: (for now) — see later for 'shortcuts'

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(-2-\lambda) - 3\times 0 = 0$$

$$\Rightarrow +(1+\lambda)(2+\lambda) = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1, \ \lambda_2 = -2} \Big\} \text{ — linearised system is stable}$$

$$(\mathrm{Re}(\lambda_i) < 0 \ \forall i)$$

BUT!

→ Is this linear stability analysis applicable to the original nonlinear system?

→ when? why?

Hyperbolic fixed points & the Hartman–Grobman theorem

Informal version:

- If the linearised system is definitely stable/unstable, ie $Re(\lambda_i) \neq 0 \ \forall i$, then the nonlinear system is too
- Fixed points for which $Re(\lambda_i) \neq 0 \ \forall i$ are called hyperbolic
- These are 'robust' in a number of ways:
  - persistence under linearisation (continue to exist)
  - persistence under small changes to the governing equations themselves (no bifurcations)
  - stability properties are preserved under linearisation

('structural stability' → see later (bifurcation theory))

Eg:

$\lambda_1 = 1, \lambda_2 = -2 \rightarrow$ hyp

$\lambda_1 = -1, \lambda_2 = -2 \rightarrow$ hyp.

$\lambda_1 = -1+i, \lambda_2 = -1-i \rightarrow$ hyp.

$\lambda_1 = 0, \lambda_2 = -1 \rightarrow$ non-hyp.

$\lambda_1 = i, \lambda_2 = -i \rightarrow$ non-hyp.

} need centre manifold theory

More formal: Hartman–Grobman theorem

"If $x = 0$, $x \in \mathbb{R}^n$, is a hyperbolic fixed point of $\dot{x} = f(x)$, then there is a continuous 1–1 map on some neighbourhood of x that maps orbits of the nonlinear flow to those of the linear flow & which preserves the direction of time"

hyperbolic — $Re(\lambda_i) \neq 0 \ \forall i$.

continuous 1–1 map } – 'homeomorphism'

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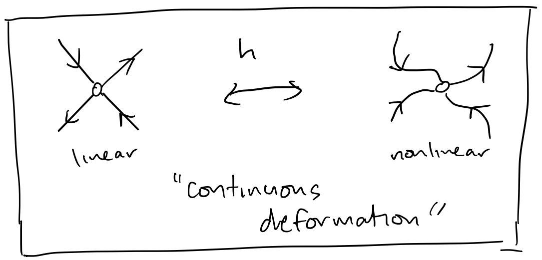
ie the linear & nonlinear systems are locally, topologically equivalent near a hyperbolic fixed point:



topologically conjugate:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

$h \circ f = g \circ h$

$f = h^{-1} \circ g \circ h$

→ h is a nonlinear change of coordinates*

→ same number of fixed points

→ fixed points have same stability etc

} just a 'stretched' version of 'same thing' → qualitatively equivalent

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* But usually 'near identity': $y = x + \text{nonlinear terms}$

→ Idea goes back to Poincare: 'normal form theory'

'linearization theorem' – H.G. Thm extends this

Bifurcations? (Preview of next section of course)

- we have looked at stability of solutions to a fixed system
- a related idea is stability of the systems/equations themselves
  ↳ "structural stability"

Important since: 'all models are wrong'!

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eg $\dot{x} = f(x)$

$\downarrow$

$\dot{x} = f(x) + \underbrace{\epsilon\, g(x)}$ small change to equations themselves

$\uparrow$ small epsilon

> Q: do these systems have 'similar' properties:
> → eg same number of fixed points with same stability?

similar to asking if linear system is 'like' nonlinear system!
→ 'ignoring higher order' terms

Not surprisingly, the answer is:

- yes if near hyperbolic fixed point
- no if near non-hyperbolic fixed point

→ see later.

More Examples

$$\boxed{\begin{aligned} \dot{x} &= y(2x-y) \quad (1) \\ \dot{y} &= x^2 - y \quad (2) \end{aligned}}$$

From: Exam 2016 Q5.

- Find fixed points
- Linearise
- Determine if origin is hyperbolic
- Determine if origin is stable

- Fixed Points

'Divide & conquer'

- $(1) = 0$: $\Rightarrow y(2x-y) = 0$
  $\Rightarrow y = 0$ OR $y = 2x$
  - $y = 0$ & $(2) = 0$:
    $x^2 = 0 \Rightarrow x = 0$
    So: $(0,0)$
  - $y = 2x$ & $(2) = 0$:
    $x^2 - 2x = 0 \Rightarrow x(x-2) = 0$
    $\Rightarrow x = 0$ OR $x = 2$
    So: $(0,0)$ (already have)
    or $(2,4)$

Solutions: $\boxed{\{(0,0),(2,4)\}}$ (Fixed points)

Linearisation : arbitrary point (x, y)

$$\dot{x} = y(2x - y) = 2xy - y^2 = f_1(x, y)$$
$$\dot{y} = x^2 - y = f_2(x, y)$$

want $\underline{Df}$, where $\underset{\text{row} \quad \text{col.}}{Df_{ij}} = \frac{\partial f_i}{\partial x_j}$

$$Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2y & 2x - 2y \\ 2x & -1 \end{bmatrix}$$ (valid for any (x, y))

At origin $(x, y) = (0, 0)$ (recall: is a fixed point)

$$Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda = 0, -1$ ⟹ non-hyperbolic!

⟹ marginal / unknown stability
