Tutorial 3: Bifurcation theory.

Selected solutions + Extra comments (see end)

1D, 1(c-d).

a) — c): see lecture notes.

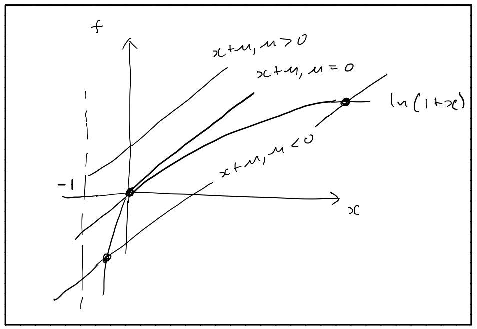
d). See later in this doc $\longrightarrow$

---

3. a). $\dot{x} = \underline{\mu + x} - \underline{\ln(1+x)} = f(x;\mu)$

FP: $f(x,\mu) = 0$

graphically or using computer: (Harder case)




Note!

$\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$

at $x=0$

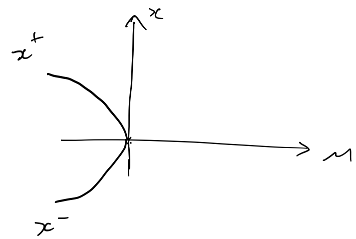
$\frac{d}{dx}(\ln(1+x)) = 1$

for $x>0$, $<1$.

Expect: no sol$^n$ for $\mu>0$
1 sol$^n$ for $\mu=0$
2 sol$^n$ for $\mu<0$

$\Big\}$ saddle-node

Diagram without stability




---

Stability

$f(x;\mu) = \mu + x - \ln(1+x)$

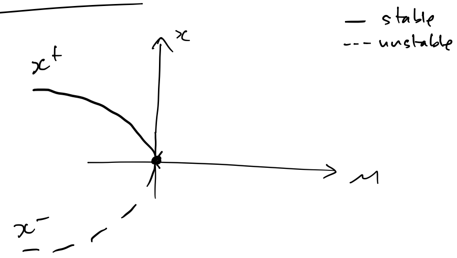
$Df(x;\mu) = 1 - \frac{1}{1+x}$

$>0$ if $1 > \frac{1}{1+x}$, ie $1+x>1$, ie $x>0$ — unstable

$<0$ if $1 < \frac{1}{1+x}$, ie $x<0$ — stable

---

Diagram with stability



3e). $\dot{x} = x + \frac{\mu x}{1+x^2}$ (another harder-ish case)

1. FP. $f(x;\mu) = 0$

$$x + \frac{\mu x}{1+x^2} = 0$$

$$\iff \frac{x(1+x^2+\mu x)}{1+x^2} = 0$$

$$\iff x(1+x^2+\mu x) = 0 \quad \& \quad 1+x^2 \neq 0$$

$$\Rightarrow \underline{x=0} \quad \text{or} \quad \underline{1+x^2+\mu x = 0}$$

always exists $\qquad x = \pm\sqrt{-\mu-1}$

---

Introduce $\lambda = -\mu$ for convenience.

$$\Rightarrow x = \pm\sqrt{\lambda - 1}$$

$\lambda < 1 \Rightarrow$ no real sol$^n$ $(+x=0)$

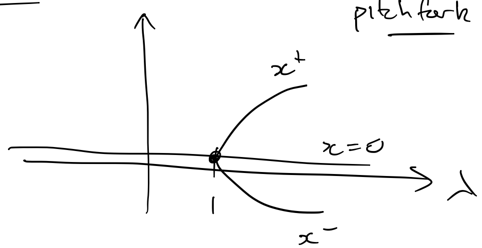
$\lambda = 1 \Rightarrow$ one real sol$^n$ $(+x=0)$

$\lambda > 1 \Rightarrow$ two real sol$^n$ $(+x=0)$

---

Diagram so far

pitchfork



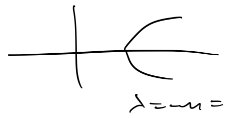

Stability

$$f = x + \frac{\mu x}{1+x^2}$$

$$Df = 1 + \frac{(1+x^2)\mu - \mu x \cdot 2x}{(1+x^2)^2} \quad \text{(quotient rule)}$$

$$= 1 + \frac{\mu + x^2(\mu - 2\mu)}{(1+x^2)^2}$$

$$= 1 + \frac{\mu(1-x^2)}{(1+x^2)^2}$$




Convenient points

$x = 0$

$Df = 1 + \mu = 1 - \lambda$

$> 0$ if $\lambda < 1$

$< 0$ if $\lambda > 1$

So:

$\lambda = 1$ $\qquad x=0$

$\rightarrow$

Next consider $x^+$ & $x^-$.

$$Df(x,\lambda) = 1 - \lambda\frac{(1-x^2)}{(1+x^2)^2}$$

$x^+$ & $x^-$ have <u>same stability</u>

($x^2$ only in $Df$).

convenient:

$$x^+ = \sqrt{\lambda - 1}$$

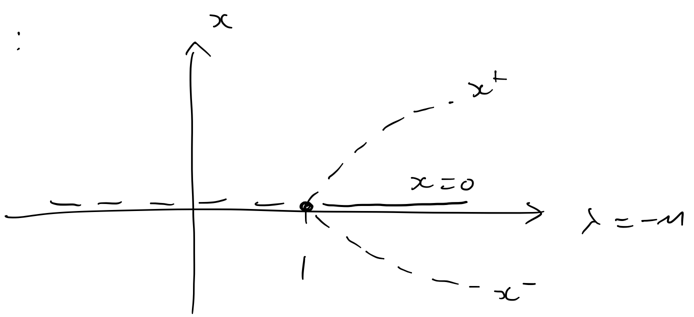
set $\underline{x^+ = 1} \Rightarrow \underline{\lambda = 2}$

$$Df = 1 - \cancel{2}$$

$= 1 > 0$ unstable.

---

<u>So</u>:




<u>pitch</u>fork at $\lambda = 1$

ie $\mu = -1$.

---

## <u>Two-dim, 1-co-dim.</u>

1. $\dot{x} = -y + \mu x + xy^2$

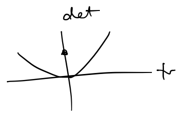
   $\dot{y} = x + \mu y - x^2$

<u>FP</u>: $(0,0) \Rightarrow \dot{x} = \dot{y} = 0$ ✓

$$\underline{Df}(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} = A.$$

$tr = 2\mu$

$det = \mu^2 + 1$




expect pure imag./Hopf

for $tr = 0 \Rightarrow \mu = 0$.

---

<u>charac.</u> equation.

$$\lambda^2 - 2\mu\lambda + \mu^2 + 1 = 0$$

$$= (\lambda - \mu)^2 + 1 = 0$$

$\underline{\lambda = \mu \pm i}$ $\left\{ \begin{array}{l} \mu < 0 \Rightarrow \underline{\text{stable}} \text{ spiral} \\ \mu = 0 \Rightarrow \text{pure im.} \\ \mu > 0 \Rightarrow \underline{\text{unstable}} \text{ spiral} \end{array} \right.$

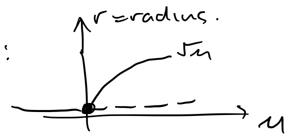
<u>Crossing</u> speed

$$\left.\frac{d(Re(\lambda))}{d\mu}\right|_{\mu=0} = 1 \neq 0$$

$\Rightarrow$ non-degenerate.

<u>Assume</u> supercritical (should use computer to check):


ie:



## 1d. More complicated example

$$\dot{x} = \mu x + x^3 - x^5$$

EP. $x(\mu + x^2 - x^4) = 0$

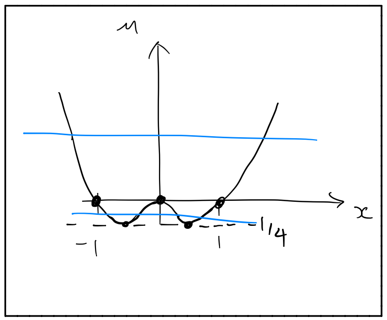
$x = 0$ or

$\mu + x^2 - x^4 = 0$.

Easier to think about $\mu$ vs $x$:

ie $\mu = x^4 - x^2 = x^2(x^2 - 1)$

Shape




Solutions: (for $\mu = x^4 - x^2$)

$\mu > 0 \Rightarrow$ two sol$^n$s

$\mu = 0 \Rightarrow$ three sol$^n$s

$-1/4 < \mu < 0 \Rightarrow$ four sol$^n$s

$\mu = -1/4 \Rightarrow$ two sol$^n$s

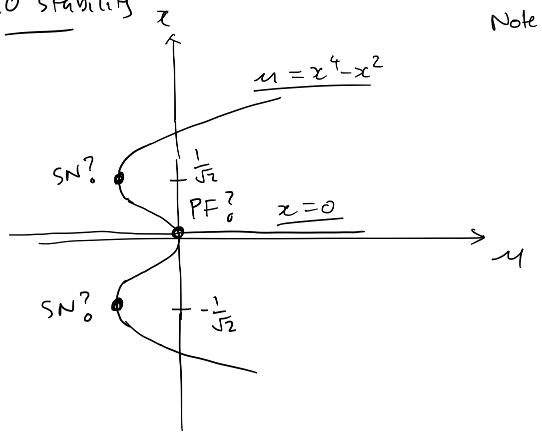
$\mu < -1/4 \Rightarrow$ zero sol$^n$s.

minima: $\frac{d\mu}{dx} = 4x^3 - 2x = 0 \Rightarrow 2x(2x^2 - 1) = 0$

ie $x = 0 \Rightarrow \mu = 0$

$x = \pm 1/\sqrt{2} \Rightarrow \mu = -1/4$

w/o stability




Note: symmetric in $x \to -x$.

$\to$ deformed pitchfork at origin

$\hookrightarrow$ expected since small higher power modification of $\mu x + x^3 - (x^5)$

expect higher powers to drop out.

$\to$ other bifurcations appear to be saddle-nodes.

Stability

$Df = \mu + 3x^2 - 5x^4$

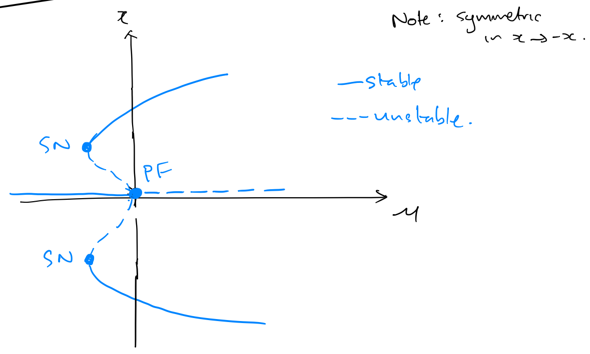
$Df(x=0) = \mu$

if $\mu > 0 \Rightarrow Df > 0$ unstable

$\mu < 0 \Rightarrow Df < 0$ stable.

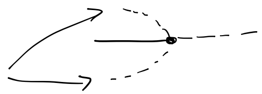
$$Df(\mu = x^4 - x^2) = x^4 - x^2 + 3x^2 - 5x^4$$
$$= 2x^2 - 4x^4$$
$$= 2x^2(1 - 2x^2) \begin{cases} > 0 & \text{if } x^2 < \frac{1}{2} \quad \text{unstable} \\ < 0 & \text{if } x^2 > \frac{1}{2} \quad \text{stable} \end{cases}$$

## Final diagram




Notes:

- The pitchfork is subcritical: new sol$^{n}$s are unstable



- The diagram below illustrates hysteresis & jumps can occur here.

→ consider increasing $\mu$ from $\mu^*$ up to B, $\mu^* \in (A, B)$

→ the system will follow stable branches.

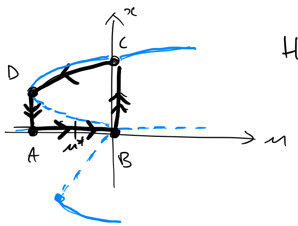
→ on $A \to B$, $x = 0$ is stable.

→ when $\mu > 0$ the system jumps to $x^+$ or $x^-$ (say $x^+$).

→ when $\mu$ is decreased back from its B value, $x$ stays at $x \neq 0$, rather than $x = 0$, until D.

→ Thus we have a source of irreversibility/hysteresis: to return to $(x, \mu) = (0, \mu^*)$, $\mu^*$ in $(A, B)$, we have to first decrease $\mu$ all the way back past A (& $\mu^*$) & then back up to $\mu^*$.

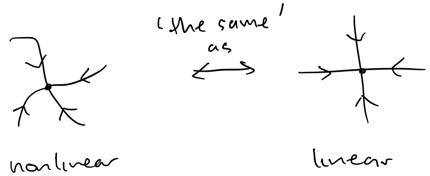
→ ie once we go past B, we can't just decrease $\mu$ back to $\mu^*$ & get $x = 0$.




Hysteresis loop.

## Comments on bifurcation theory & 'qualitative' analysis

Recall in analysing dynamics we were interested in connecting




The idea is that the dynamics are 'qualitatively the same' near hyperbolic fixed points

→ they are 'the same' in the key/interesting respects
- ∟ # fixed points
- ∟ stability

→ we don't care about the rest of the exact 'quantitative' details!

Formally, 'the same' usually means:

$\boxed{\text{can continuously transform one to other \& back. Called a 'Homeomorphism'}}$

Informally: equivalent up to 'stretching' etc.

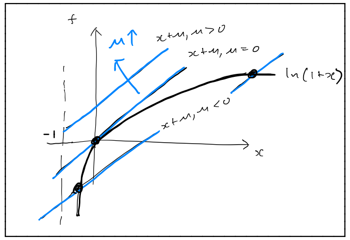
Similarly in bifurcation theory, i.e. near non-hyperbolic fixed points, we focused on the key 'qualitative' properties:

- number of fixed points
- stability

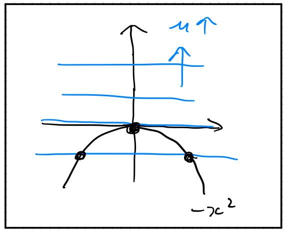
& how these change as a parameter varies

Two bifurcations are 'qualitatively the same' if they 'change' in the 'same way'.

Eg we saw more complicated cases give rise to the same types of bifurcation as in simple examples:




'the same' as ⟷




FP: $x+\mu-\ln(1+x)=0$

i.e.

$y=x+\mu$ intersects $y=\ln(1+x)$

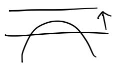
FP: $\mu+x^2=0$

i.e.

$y=\mu$ intersects $y=-x^2$

---

The 'key property' is 'looking like':



$\uparrow \mu$ } hence 'turning point' lol!

i.e. the appearance/disappearance of fixed points $0 \Longleftrightarrow 2$ FP.

Similarly, FP: $x(\mu+x^2-x^4)=0$

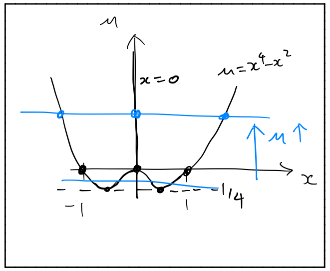
$\Rightarrow$

$x=0$

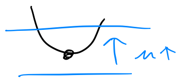
or

$\mu=x^4-x^2$

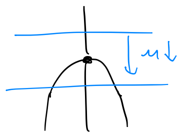
$=x^2(1-x^2)$




Has local regions that 'look like':




SN ($\times 2$ by symmetry in $x$).




PF.

---

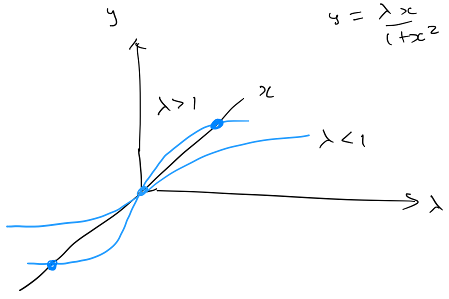
Exercise: sketch similar diagrams for

$$\boxed{f = x - \frac{\lambda x}{1+x^2}}$$

Hint: split into two functions of $x$ one depending on $\lambda$, one independent of $\lambda$, & consider $\lambda$ variations.

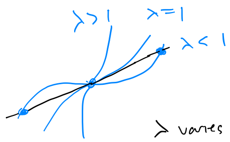
# Partial solution

$$\left.\begin{array}{l} y = x \\ y = \dfrac{\lambda x}{1+x^2} \end{array}\right\} \text{FP} \quad x = \frac{\lambda x}{1+x^2}$$




'Like'

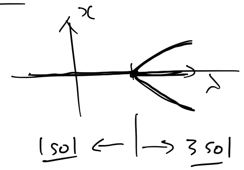
$x - \lambda x^3 = 0$ :




λ varies

$1 \longleftrightarrow 3$ fixed points.

ie pitchfork:




[ Could also deduce from Taylor series ]

---

**Upshot**: geometrically, the 'same' things are happening in the more complicated systems as happen in the 'simple' systems.