ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations

(*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 3: STABILITY AND LINEARISATION

- Stability of solutions for general systems
- Stability of solutions for linear systems
- Linearisation of nonlinear systems
- Connecting stability of linearised systems to stability of nonlinear systems
- Sneak peak at structural stability

STABILITY OF SOLUTIONS

There are various *general* formal definitions of *stability* for solutions.

These can be defined for both equilibria as well as more complicated objects like periodic orbits. They also apply for both linear and nonlinear systems.

We will just give the *definitions for equilibria* (points) for now.

STABILITY OF SOLUTIONS

- A point x is *Lyapunov stable* iff for all $\epsilon > 0$ there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| < \epsilon$ for all $t \geq 0$.
- A point x is *quasi-asymptotically stable* (attracting) iff there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| \to 0$ as $t \to \infty$.
- A point is *asymptotically stable* iff it is both Lyapunov stable and quasi-asymptotically stable. If it is just Lyapunov stable then it is *neutrally stable* (i.e. is just bounded).

WHAT DO THESE MEAN?

In pictures!

STABILITY OF LINEAR SYSTEMS

These general definitions can be hard to check in general but are *easy to check for linear systems*.

> Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

(This can be proven by constructing a so-called Lyapunov function - ask me if interested/see further reading)

LINEARISATION AND LOCAL STABILITY ANALYSIS

We have seen that stability for linear systems is easy. We will analyse and classify linear systems in more detail soon.

This will be useful because, as mentioned, our first steps in analysing *nonlinear* systems will usually be through local *linearisation* about steady-states/equilibria.

We will also need to know the *connection between linear and nonlinear stability*!

LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component gives

$$\dot{u}_i = f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2) \text{ i.e.}$$

$$\dot{u}_i = \frac{\partial f_i}{\partial x_j}(x_e)u_i \equiv [Df(x_e)]_{ij}u_j$$

or simply $u = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

LINEAR AND NONLINEAR STABILITY - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues of the linearisation have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

LINEAR AND NONLINEAR STABILITY CONNECTED

The *Hartman-Grobman theorem* states that the local properties *near a hyperbolic fixed point* of a *nonlinear* system are *topologically equivalent* to those of the *linearisation*:

> *A hyperbolic fixed point persists* in the change from nonlinear to linear systems (though its location may shift slightly) and its *stability properties are also preserved*

(We will also see the similar 'stable manifold theorem' later)

EXAMPLE

Let's see an example stability analysis!

BIFURCATIONS AND STRUCTURAL INSTABILITY

Recall: fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

When these occur our *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities* - i.e. small variations in problem parameters can have a large effect on the qualtiative/topological features of our phase space.

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

BIFURCATIONS AND STRUCTURAL INSTABILITY

These instabilities are called *bifurcations*.

We will return to this topic in more detail in later lectures.
First a teaser.

Example.

EXERCISE

Complete (as much as you can of) the stability analysis examples from the Lecture 1 handout (based on past exams).