ENGSCI 213: MATHEMATICAL MODELLING 2SE

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

MODULE OVERVIEW

Introduction to Probability (*Oliver Maclaren*) [9 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts of probability. Sets and subsets, sample spaces and events. Probability and counting, conditional probability, independence, Bayes' theorem. Random variables. Simple data structures for probability calculations.

2. *Discrete probability models* [3 lectures]

   Discrete random variables and discrete probability distributions. Probability mass functions and cumulative distribution functions. Binomial and Poisson distributions.

3. *Continuous probability models* [3 lectures]

   Continuous random variables and continuous probability distributions. Probability density functions and cumulative distribution functions. Exponential and Normal distributions.

NEW TOPIC

2. *Discrete probability models* [3 lectures]

Discrete random variables and discrete probability distributions. Probability mass functions and cumulative distribution functions. Binomial and Poisson distributions.

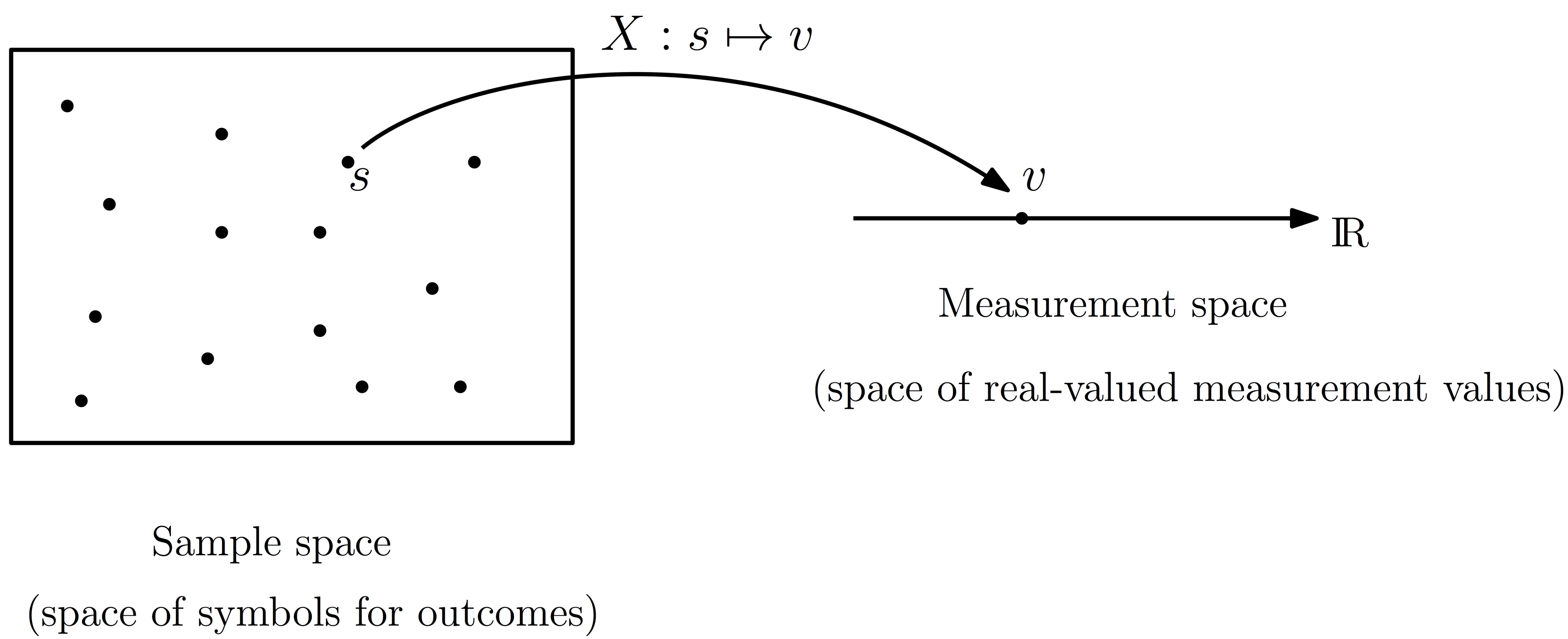
LECTURE 4

Recap of discrete random variables

Sequences of random variables and random processes

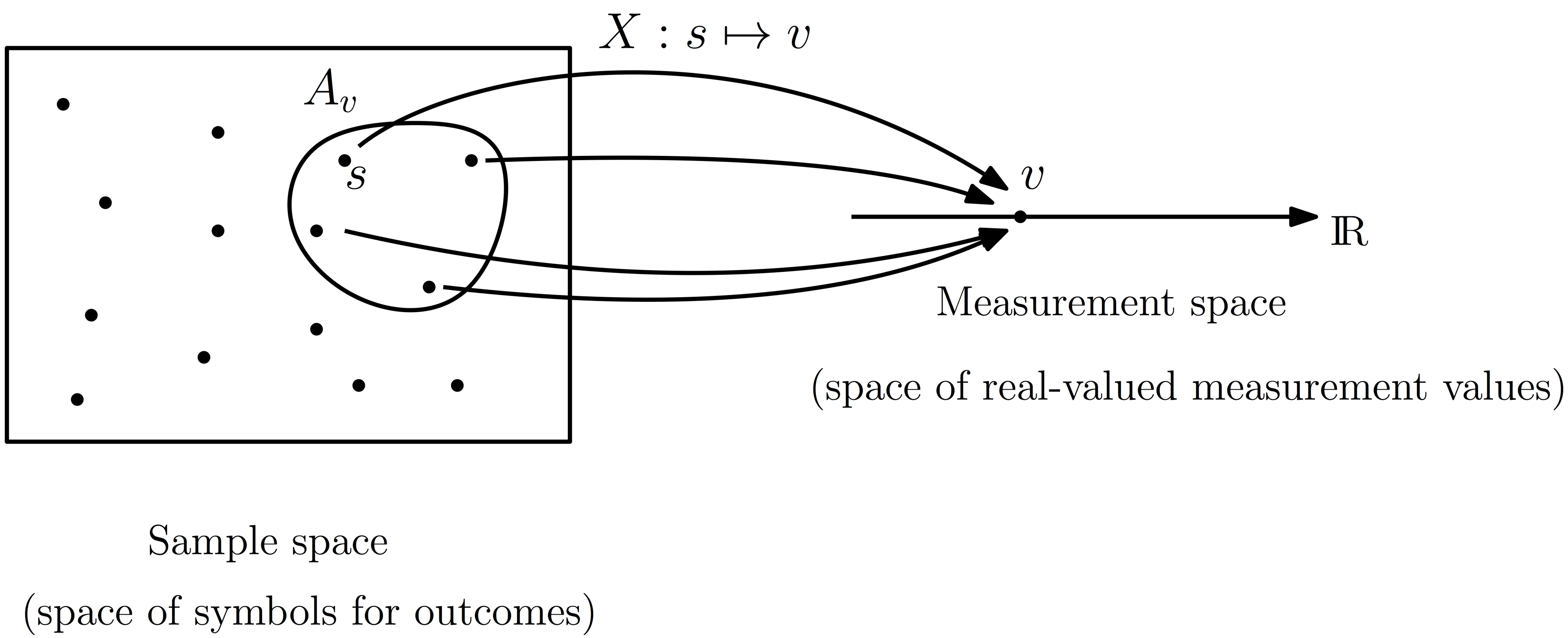
Sequences of Bernoulli trials and the Binomial distribution

RECAP: RANDOM VARIABLES AS FUNCTIONS FOR LABELLING *OUTCOMES*



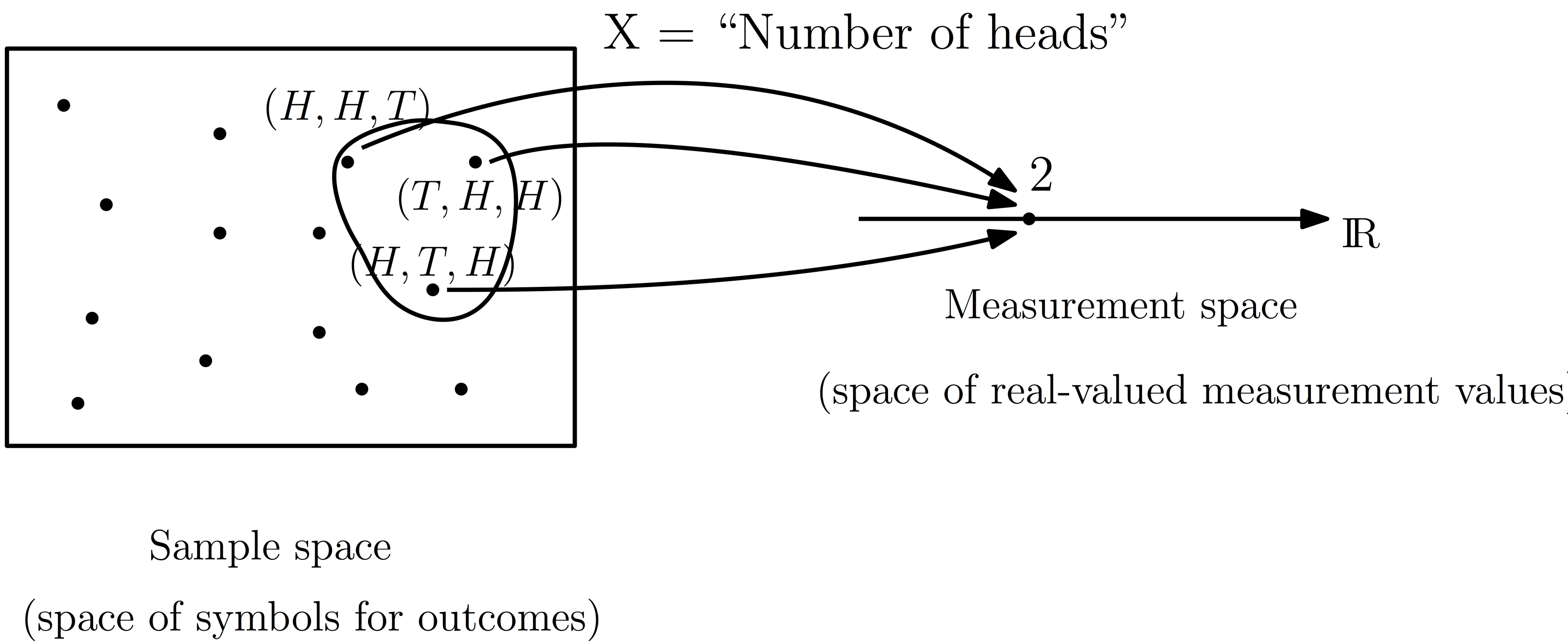
RECAP: RANDOM VARIABLES AS FUNCTIONS FOR LABELLING *EVENTS*

$$\boxed{A_v = \{s | X(s) = v\} = X^{-1}(\{v\})}$$



RECAP: EXAMPLE

$$A_2 = \{s | \text{"Number of heads"} = 2\} = X^{-1}(\{2\})$$



SAMPLE SPACE?

Since a random variable simply maps our original outcomes to new numerical outcomes, it also has its *own (new) sample space*. E.g. flip a coin once. Original sample space:

$$S = \{H, T\}$$

Define the random variable Y mapping $H \mapsto 1$ and $T \mapsto 0$. Then the sample space for Y is simply

$$S_Y = \{1, 0\}$$

FULL INFO OR SUMMARY?

The random variable may *preserve* the full info of each $s \in S$ (as in previous slide) or *reduce it to a summary* in terms of properties of each s (these properties thus define events)

Reductions will *change the probability allocation* in general.

Example.

RECAP: DEFINITIONS

Discrete random variables: the (original) sample space is a discrete set

Continuous random variables: the (original) sample space is a continuous set

RECAP: PROBABILITY FOR RANDOM VARIABLES

The probability of a random variable taking on a value is just the *probability of the associated event in the original sample space*

RECAP: PROBABILITY FOR RANDOM VARIABLES: DISCRETE CASE

So for *discrete* RVs we use

$$P(X = x) = P(A_x) = P(\{s \mid X(s) = x\})$$

Represents the probability of the random variable taking on any particular value. $P(X = x)$ is sometimes called the *probability 'mass' function*.

For a discrete RV this gives enough info to build up the probability for other events. We often *start from knowledge of $P(X = x)$.*

RANDOM VARIABLES, PARTITIONS AND PROBABILITY DISTRIBUTIONS

A discrete *probability distribution* of a random variable X is simply a *tabulation* (or list) of the probability associated to each possible value x of X e.g. the set of pairs

$$\{(x, P(X = x)) \mid x \in S_X\}$$

We also often use the notation $f_X(x) := P(X = x)$ for the probability function.

Example

EVENTS ON THE RANDOM VARIABLE SAMPLE SPACE

Since S_X is also a sample space, we can consider events defined by *set* of values of the RV. E.g.

$$P(a \leq X \leq b) = P(A_{[a,b]}) = P(\{s \mid X(s) \in [a,b]\})$$

Represents the probability of the random variable taking on a value in the interval $[a, b]$.

SEQUENCES OF RANDOM VARIABLES AND TRIALS

We can often think of the result from an *overall* experiment to consist of a sequence of sub-experiments - or *trials* - where each trial is 'of the same kind'.

If the outcome of trial n, for each $n = 1, 2, \ldots N$, can be described by a RV Y_n, then a collection

$$\{Y_n \mid n = 1, 2, \ldots N\}$$

defines a *random experiment given by sequence of random variables* (or a discrete-time stochastic/random process).

SEQUENCES OF RANDOM VARIABLES AND TRIALS

Here the overall random variable for the random experiment, Y, takes the form

$$Y = (Y_1, Y_2, Y_3, \ldots, Y_N)$$

Example

BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION

A *Bernoulli random variable* Y is a random variable which only takes 2 values, 0 and 1 say.

The probability (mass) function is

$$P(Y = y) = f_Y(y) = \begin{cases} p, & \text{if } y = 1 \\ 1 - p, & \text{if } y = 0 \end{cases}$$

BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION

A random experiment is called a *set of Bernoulli trials* if it consists of several trials such that

- Each trial has only *2 possible outcomes* (e.g. 1 or 0, often called "Success" or "Failure")
- The *probability of "Success" p is the *same* for all trials
- The trials are *independent* i.e. "Success in trial i" doesn't affect the chances of "Success" in any other trial.

EXAMPLE

Example

Note - each full outcome with x successes in n trials has probability $p^x(1-p)^{n-x}$

BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION

We define a *new* random variable X to *summarise* the results of a *set of Bernoulli trials*. This is called a *Binomial* random variable and reduces the full detail to just a count of the number of success.

$X \sim Bin(n, p)$ if X is the number of successes out of n independent Bernouilli trials

BINOMIAL DISTRIBUTION

The probability function is given by

$$P(X = x) = f_X(x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

This is simply (number outcomes with x successes)×(prob. of each outcome) since each full outcome with x successes in n trials has probability $p^x(1 - p)^{n-x}$

BINOMIAL DISTRIBUTION - EXAMPLES