MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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RECALL - ADJOINT OPERATORS

The *adjoint* of an operator L operating on functions in some function space is the unique operator L^* operating on that same function space such that

$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

for all u, v in that function space. *We include in L and L^* the appropriate boundary conditions - possibly different for each - so as to satisfy this relation.*

RECALL - SELF-ADJOINT OPERATORS

The basic definition of a *self-adjoint* operator is

$$\langle Lu, v\rangle = \langle u, Lv\rangle$$

For all u, v in the function space of interest - *now we include, as part of L, the requirement that u and v satisfy the same boundary conditions*

RECALL - FORMAL ADJOINT

For the *formal operator* (no BC)

$$A = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

The *formal adjoint* of A is

$$A^* = a_2(x)\frac{d^2}{dx^2} + (2a_2(x)' - a_1(x))\frac{d}{dx} +$$
$$(a_2(x)'' - a_1(x)' + a_0(x))$$

(This is motivated by Green's identity - see supplement for more details)

RECALL - GREEN'S FORMULA

Green's formula/identity can be written as

$$\langle Au, v\rangle - \langle u, A^* v\rangle = J(u, v)|_a^b$$

where

$$J(u, v) = a_2(vu' - uv') + (a_1 - a_2')uv$$

The full adjoint is defined by determining boundary conditions for v from $J(u, v)|_a^b = 0$ and the BC B_1, B_2 for u, giving $L^* = (A^*, B_1^*, B_2^*)$.

RECALL - SELF-ADJOINT OPERATORS

If $J(u, v)|_a^b = 0$ implies that *u and v satisfy the same boundary conditions* then the formal operator plus BC satisfy the basic definition of a *self-adjoint* operator

$$\langle Lu, v\rangle = \langle u, Lv\rangle$$

where $L = (A, B_1, B_2) = L^* = (A^*, B_1^*, B_2^*)$ includes boundary conditions.

LECTURE 11

Typical adjoint example

Combining separation of variables and SL theory

TYPICAL EXAMPLE

Find the adjoint operator $L^* = (A^*, B_1^*, B_2^*)$for L given by

$$A = \frac{d^2}{dx^2} + \frac{d}{dx} + 1$$

and

$$u(0) = 0 \, , \, u'(0) = 0$$

for u defined on $[0, \pi]$ and having continuous second derivatives. (Use $\omega(x) = 1$ and a real-valued-function inner product). Is L^* self-adjoint? Why/why not?

EXERCISE

Make some changes to the original problem so that the operator so-defined is self-adjoint (make sure to demonstrate explicitly that your new problem defines a self-adjoint operator).

HEAT EQUATION REVISITED

Example: find the solution to the problem

$$
\begin{aligned}
&u_{xx} = u_t, && 0 < x < 1, t > 0,\\
&u(0,t) - 2u_x(0,t) = 5, && t > 0,\\
&u(1,t) = 35, && t > 0,\\
&u(x,0) = f(x), && 0 < x < 1,
\end{aligned}
$$

using *Separation of Variables*. Use *Sturm-Liouville theory* to justify your reasoning about eigenvalues/separation constants.

HOMEWORK

Go over the supplements (coming soon!)

Finish assignment

Make sure you understand how SL theory relates to separation of variables