Linear or nonlinear with respect to what?

Overview

One of the students noticed that this introduced some ambiguity into our classi cation problem and asked a great question. I think it illustrates a useful general point about terminology like linear vs nonlinear and how these terms can be misleading or ambiguous. So here's the question and my attempt at clarifying the ambiguity.

The question

It's my understanding that a PDE is linear if we **can** write it in the form $Lu = f(x,t)$, where L is a linear differential operator.

If we are given a PDE that looks like $Au = 0$ for some differential operator A and asked to show that the PDE is *nonlinear*, I can (probably) show that A is not a linear differential operator. However *this doesn't necessarily imply that you cannot rearrange the equation in such a way to make it linear*.

For example the operator A defined by

$$Au = (u^2+1)u_t + (u^2+1)u_{xx}$$

is not a linear differential operator. However the equation Au = 0 is the same as $u_t + u_{xx} = 0$, and the differential operator B defined by $Bu = u_t + u_{xx}$ is linear.

So (I believe I'm correct in saying this), the original *PDE* is linear, because it *can* be rewritten in this form Lu = f(x,t) for some linear differential operator L and function f(x,t).

My question is what sort of working are we expected to show, if we aim to prove the PDE Au=0 is not linear? For the purposes of the assignment does it suffice to prove that A is not linear?

My response
Here was my response and attempt to clarify (corrections/comments welcome!).

Great question!

As you've noticed there is some ambiguity when we move back and forward between talking about equations and operators. This is to be expected since a *function* (e.g. an operator) is a *different type of mathematical object* to an *equation***.**

For example the function $f : x \mapsto x^2$ is a different 'object' to the equation $x^2 = 0$.

You've correctly noticed that if we can write a differential *equation* as Lu = f where L is *some* linear *operator* then the differential equation is also called linear. Unfortunately, again as you've noticed, *this definition makes it hard to decide when an equation is nonlinear* as you may be able to write a *linear equation* in terms of a *nonlinear operator with the right choice of f*. This is because the negation of 'there exists' a linear operator is 'there doesn't exist a linear operator'.

So proving that an equation is *linear is easy* using the operator definition – we just find *any linear operator* that works.

On the other hand, proving that an equation is *nonlinear* is *harder using this definition* – it would require showing *all* operators for which Au = f are nonlinear.

This seems too hard to do directly, so let's reformulate it in an equivalent but easier-to-use way.

We want to keep our definitions of linear and nonlinear as close as possible for the two cases of *operators* and *equations***.**

So, how about:

Improved definitions
An ***operator*** acting on u is linear iff L(au+bv) = aL(u) + bL(v) for ***any u and v in the operator's domain*** and constants a, b.

and

Given an ***equation*** written in the from $Au = f$ for some operator A and forcing function f, the ***equation*** is linear iff $A(au+bv) = aA(u) + bA(v)$ for ***any two solutions u, v*** to the ***equation*** $Au = f$.

I think this de nition should cover your example (try it! Note that it is slightly subtle how this makes a difference! But, basically, we get to use the $f = 0$ in the equation case now).

Also note that:

*The **function definition** now explicitly talks about linearity with respect to how it operates on objects in its domain while the **equation definition** talks explicitly about behaviour with respect to solutions to that equation. This seems natural given the different 'nature' of 'functions' and 'equations'.*

Does that make sense?

Morally speaking

I think the broader lesson is that terms like linear/nonlinear are *relative to the specific mathematical representation chosen and how we interact with that representation. A 'system' is not really intrinsically linear or nonlinear, rather an 'action' (or function or operator or process) is linear or nonlinear with respect to a speci c set of 'objects' or 'measurements' or 'perturbations' or whatever. This needs to be made explicit for an unambiguous classification to be carried out.*

Generalisation

It has been pointed out that while 'chaos' is typically associated with (usually *finite-dimensional*) *nonlinear* systems, there are examples of *infinite-dimensional linear systems* that exhibit all the hallmarks of chaos – see e.g. 'Linear vs nonlinear and infinite vs finite: An interpretation of chaos' by Protopopescu for just one example. So, changing the underlying 'objects' used in the *representation* changes the classification as 'linear' or 'nonlinear' or, as Protopopescu states

> *Linear and nonlinear are somewhat interchangeable features, depending on scale and representation...chaotic behavior occurs... when we have to deal with infinite amounts of information at a finite level of operability. In this sense, even the most deterministic system will behave stochastically due to unavoidable and unknown truncations of information.*