EngSci 711 Centre Manifold: Implications & Applications.

Examples

Jordan Normal Form.

Short version → generalisation of diagonalisation that handles repeated eigenvalues & generalised eigenvectors

→ if eigenvalues are distinct then it reduces to a diagonal matrix

Longer version → see Linear Algebra handout.

Example. (Based on Exercise 8.3 in Glendinning).

$$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$
→ $(0,0)$ is a FP.

1. $Df(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ [linearise about FP 0,0]

2. $\left.\begin{matrix} tr = -2 \\ det = 0 \end{matrix}\right\}$ $\lambda^2 - tr\cdot\lambda + det = 0$ [eigenvalues]
$$\Leftrightarrow$$
$$\lambda^2 + 2\lambda = 0$$
$$\lambda(\lambda+2) = 0$$
$$\lambda_1 = 0,\ \lambda_2 = -2$$

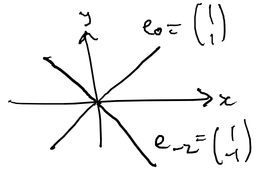
3. $\lambda = 0$ [eigenvectors]
$$\begin{pmatrix} -1-0 & 1 \\ 1 & -1-0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow -u_1 + u_2 = 0 \Rightarrow e_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -2$
$$\begin{pmatrix} -1+2 & 1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow e_{-2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

⇒

→ new step!

we have



we want to change coord to $\{e_0, e_{-2}\}$ basis

(from $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ currently).

use
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} | & | \\ e_0 & e_{-2} \\ | & | \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

↑ ↑ columns are eig. vectors. ↑ new coords

eig. vectors are LI so matrix is invertible.

ie
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

---

or
$$\begin{aligned} x &= u+v \\ y &= u-v \end{aligned} \quad\Leftrightarrow\quad \begin{aligned} u &= \frac{x+y}{2} \\ v &= \frac{x-y}{2} \end{aligned}$$

⇒

so $\dot{u} = \frac{1}{2}(\dot{x}+\dot{y})$

$\dot{v} = \frac{1}{2}(\dot{x}-\dot{y})$

- Linearised system: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

linearised → ie $\boxed{\begin{aligned}\dot{x} &= y-x\\ \dot{y} &= x-y\end{aligned}}$ $\begin{aligned}&= (u-v)-(u+v) = -2v\\ &= (u+v)-(u-v) = 2v\end{aligned}$

so

$\dot{u} = \frac{1}{2}(\dot{x}+\dot{y}) = \frac{1}{2}[-2v+2v] = 0$

$\dot{v} = \frac{1}{2}(\dot{x}-\dot{y}) = \frac{1}{2}[-2v-2v] = -\frac{2}{2}[v+v] = -2v$

---

so the linearised system is

$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \iff \boxed{\begin{aligned}\dot{u} &= 0\\ \dot{v} &= -2v\end{aligned}}$ } decoupled!

↑

Diagonal with eigenvalues on diagonals. } Jordan Normal Form.

- Nonlinear system. [use same linear transformation of vars]

$\dot{x} = y-x-x^2 = (u-v)-(u+v)-(u+v)^2$

$= -2v \underbrace{-u^2-2uv-v^2}$ } — extra nonlinear terms.

$\dot{y} = x-y-y^2 = 2v - (u-v)^2$

$= 2v \underbrace{-u^2+2uv-v^2}$ }

so $\dot{u} = \frac{1}{2}(\dot{x}+\dot{y}) = 0 + \frac{1}{2}[-2u^2-2v^2] = -[u^2+v^2]$

$\dot{v} = \frac{1}{2}(\dot{x}-\dot{y}) = -2v + \frac{1}{2}[-2uv-2uv] = -2v[1+u]$

---

ie $\boxed{\begin{aligned}\dot{u} &= \underbrace{0}_{\text{linear}} - (u^2+v^2)\\ \dot{v} &= -2v \underbrace{-2uv}_{\text{nonlinear}}\end{aligned}}$ ← slow. ← fast

---

Consider

$\boxed{\begin{aligned}\dot{x} &= y-x-x^2\\ \dot{y} &= (1+\mu)x-y-y^2\end{aligned}}$ (Glendinning Exercise 8.3)

Note $\boxed{\mu=0}$ gives previous system.

⇒ thus had a non-hyperbolic fixed point fixed point at $(x,y)=(0,0)$.

Hence the extended systems

$\boxed{\begin{aligned}\dot{x} &= y-x-x^2\\ \dot{y} &= (1+\mu)x-y-y^2\\ \dot{\mu} &= 0\end{aligned}}$ ← extra μx term

note: nonlinear since μ is a state var here.

has a non-hyperbolic fixed point at $\boxed{(x,y,\mu)=(0,0,0)}$!

← still linear change of coord.

In terms of u,v,μ coord this is

$\boxed{\begin{aligned}\dot{u} &= 0-(u^2+v^2)+\tfrac{1}{2}\mu(u+v)\\ \dot{v} &= -2v-2uv-\tfrac{1}{2}\mu(u+v)\\ \dot{\mu} &= 0\end{aligned}}$

from extra μx term (nonlinear): $\boxed{\begin{aligned}\dot{u} &= \tfrac{1}{2}(\dot{x}+\dot{y})\\ \dot{v} &= \tfrac{1}{2}(\dot{x}-\dot{y})\end{aligned}}$

⇒ $\dot{u} = \text{old} + \frac{1}{2}\mu x$

$\dot{v} = \text{old} - \frac{1}{2}\mu x$

& $\mu x = \mu(u+v)$

Note: CM is two-dimensional here

→ both u & μ are centre/slow vars

[μ is 'super slow' since higher order terms also zero]

Hence assume

$v = g(u,\mu) = \underbrace{a+bu+c\mu+du^2+eu\mu+f\mu^2+\dots}_{\text{two-variable Taylor series}}$

→

First consider the linear part of the extended sys.

$$\begin{pmatrix} \dot{u} \\ \dot{\mu} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ \mu \\ v \end{pmatrix}$$

$\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

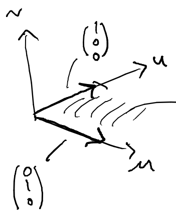
$\Rightarrow e^3 = 0$

e^1 & e^2 both free

$\Rightarrow$ whole plane

$\Rightarrow$ choose $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as orthog. basis.

$E^c = \{(u, \mu, v) \mid v = 0\}$



Centre subspace is u, μ plane.

so

$v = g(u, \mu)$ is tangent to u, μ plane at $(0,0)$

ie $g(0,0) = 0 \Rightarrow a = 0$

$\frac{\partial g}{\partial u}(0,0) = b = 0$

$\frac{\partial g}{\partial \mu}(0,0) = c = 0$ } in power series expansion.

so $v = g(u, \mu) = du^2 + eu\mu + f\mu^2 + \ldots$

use chain rule.

$$\dot{v}(u, \mu, g(u,v)) = \frac{\partial v}{\partial u}\dot{u} + \frac{\partial v}{\partial \mu}\cancel{\dot{\mu}}^{\,0} \text{ } \} \text{ note!}$$

$\Rightarrow$

Now

① $\dot{v} = -2v - 2uv - \frac{1}{2}\mu(u+v)$

$\rightarrow$ subs. $v = du^2 + eu \cdot \mu + f\mu^2$

$\Rightarrow \dot{v} = -2du^2 - 2eu \cdot \mu - 2f\mu^2 - \frac{1}{2}\mu u$

$+ O(\|\cdot\|^3)$

└ ie uv, μv terms

$$\Rightarrow \boxed{\dot{v} = -2du^2 - 2\left[e + \tfrac{1}{4}\right] \cdot u \cdot \mu - 2f\mu^2 + \ldots} \text{ ①}'$$

② $\frac{\partial v}{\partial u} = 2du + e\mu = O(\|\ \|^1)$

$\dot{u} = -(u^2 + v^2) + \frac{1}{2}\mu[u + v]$

$= O(\|\ \|^2)$ so...

$$\Rightarrow \boxed{\frac{\partial v}{\partial u}\dot{u} = O(\|\ \|^3)} \text{ ②}'$$

①$'$ = ②$'$ $\Rightarrow d = 0$, $e = -\frac{1}{4}$, $f = 0$

so $v = -\frac{1}{4}\mu u = O(\|\ \|^2)$

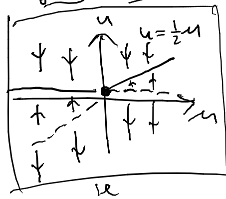
& $W^c = \{(u, v, \mu) \mid v = -\frac{\mu u}{4}\}$

&
$$\boxed{\begin{cases} \dot{u} = -(u^2 + O(\|\ \|^4)) + \frac{1}{2}\mu u = \frac{1}{2}u[\mu - 2u] \\ \dot{\mu} = 0 \end{cases}}$$

super slow } either – treat as CM again
– fix at constant & solve $\dot{u}$ to equilibrium

$$\boxed{\begin{aligned} \dot{u} &= \tfrac{1}{2}u[\mu - 2u] \\ \mu &= \text{frozen param} \end{aligned}}$$

↳ get: transcritical.

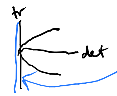


ie

phase portrait of non-hyperbolic system = Bifurcation diagram!

phew....

classif. diagram.



recall: non-isolated fixed points!