EngSci 711 Lecture 1 Dynamical Systems Handout

OTM. Examples.

Exam 2016.

- Q4a, b.
- Q5a, b
- Q6 a_i (part)

} We can pretty much do these already!

linearisation, roots, rewriting systems of equations

4a, b.

	components	vector
Given	$\dot{x}_1 = f_1(x_1, x_2)$ $\dot{x}_2 = f_2(x_1, x_2)$	$x = f(x)$ $x \in \mathbb{R}^2$

& a fixed point x^*,

write $x = x^* + \eta$ ← new variable (small perturbation about fixed point)

& find a linearised system for η.

→

Firstly: what's a 'fixed point'?

A → It's where the derivatives are zero (ie set RHS = 0)

$\Rightarrow f_1(x_1^*, x_2^*) = 0$ & $f_2(x_1^*, x_2^*) = 0$ } component form.

OR

$f(x) = 0$ } vector form

How will we use this info to get a linearisation?

↳ $\boxed{\text{Taylor series}}$ using $x^* + \eta$

← the workhorse of applied maths!

[note: full vector]

TS: $\dot{x}_i = f_i(x) = f_i(x^* + \eta)$ [vector components]

$\approx \underbrace{f_i(x^*)}_{=0} + \eta_j \frac{\partial f_i}{\partial x_j}(x^*)$ [sum over j]

since fixed point!

⇒

$$\dot{x}_i \approx f_i(x^*) + \eta_j \frac{\partial f_i}{\partial x_j}(x^*)$$

$$= \eta_j \frac{\partial f_i}{\partial x_j}(x^*) \quad \text{at FP.}$$

$$= A\eta \quad \text{where } A_{ij} = \frac{\partial f_i \leftarrow \text{row}}{\partial x_j \leftarrow \text{column}}$$

"Jacobian derivative"
↳ also write as 'Df'
ie 'derivative of f'

Goal $\dot{\eta} = A\eta$
?

Note: $[\sum_j A_{ij}\eta_j$: is matrix multiplic$]$

Note $x = x^* + \eta$
↑
fixed vector (time indep.)

so $\dot{x} = 0 + \dot{\eta}$
$= \dot{\eta}$

$$\Rightarrow \boxed{\dot{\eta} = A\eta} \quad \square$$

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b).

$\eta = e^{\lambda t} u$

$\dot{\eta} = \lambda e^{\lambda t} u$ [note: u is const. in t]

$$\Rightarrow \dot{\eta} = A\eta \equiv \lambda \cancel{e^{\lambda t}} u = \cancel{e^{\lambda t}} Au$$

so we get

$$\boxed{Au = \lambda u}$$

$\Rightarrow$ eigenvalue problem. $\square$

5a. $f_1(x,y) = y(2x - y) = 0$ for FP

$f_2(x,y) = x^2 - y = 0$ for FP.

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(1) $y(2x-y) = 0$

(2) $x^2 - y = 0$.

Divide & conquer.

(1) : $y = 0$ or $y = 2x$.

(2) : $y = x^2$

comb.

- $y=0$ & $y = x^2 \Rightarrow x = 0$
- $y = 2x$ & $y = x^2 \Rightarrow 2\cancel{x} = x^{\cancel{2}} \Rightarrow x = 2$
  $\Rightarrow y = 4$

so $\boxed{(0,0) \text{ \& } (2,4)}$ are the FP.

5b).

Goal is 'Df' ie

$$\frac{\partial f_i \leftarrow \text{row}}{\partial x_j \leftarrow \text{col}}.$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Here

$$Df(x,y) = \begin{bmatrix} 2y & 2x-2y \\ 2x & -1 \end{bmatrix}$$

& at $(0,0)$

$$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Q6 b.

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

$x_1 = x$

$x_2 = \dot{x} = \dot{x}_1$

want $\dot{x}_1 = f_1(x_1, x_2)$

$\dot{x}_2 = f_2(x_1, x_2)$.

so $\dot{x}_2 = \ddot{x} = -\mu\dot{x} - (x - x^3)$

$$= -\mu\dot{x}_1 - (x_1 - x_1^3)$$

$$= -\mu x_2 - x_1(1 - x_1^2) \checkmark$$

$$\Rightarrow \dot{x}_1 = x_2 = f_1(x_1, x_2) \checkmark$$

$$\dot{x}_2 = -x_1(1-x_1^2) - \mu x_2 = f_2(x_1, x_2) \checkmark$$