Problem I (1D - codim one).

(a) $\dot{x} = \mu - x^2 = f(x;\mu)$

FP: $f(x;\mu) = 0$

$$x^2 = \mu$$

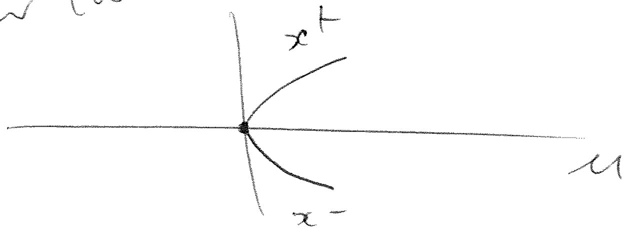
$$x = \pm\sqrt{\mu}$$

If $\mu < 0 \Rightarrow$ no real sol.

$\mu = 0 \rightarrow$ one real sol $x = 0$

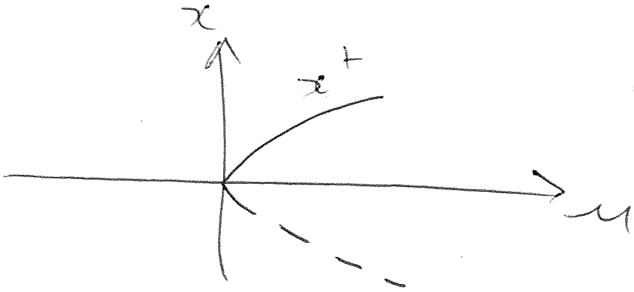
$\mu > 0 \Rightarrow$ two real sol $\begin{cases} x^+ = +\sqrt{\mu} \\ x^- = -\sqrt{\mu} \end{cases}$

So far (without stability).



Stability: $Df = -2x = \begin{cases} < 0 \text{ if } x > 0 \text{ ie } x^+ \\ > 0 \text{ if } x < 0 \text{ ie } x^- \end{cases}$

so



(saddlenode/ turning point etc)

Problem 7 (1D - codim one)

(b) exercise!

(c). $\dot{x} = f(x, \mu) = \mu x - x^3$

<u>Roots</u> $x(\mu - x^2) = 0$

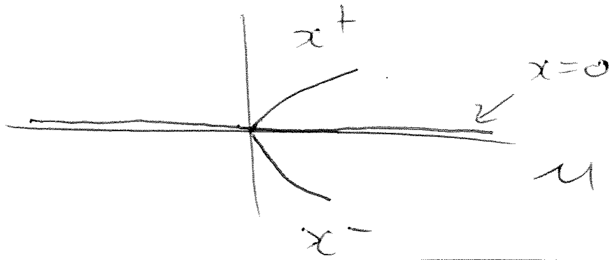
(1) $x = 0 \to$ for all μ.

(2) $x^2 = \mu \to x = \pm\sqrt{\mu}$ $\begin{cases} \text{no real } \mu < 0 \\ \text{one real } \mu = 0 \\ \text{two real } \mu > 0 \end{cases}$

$\hookrightarrow x^+ = +\sqrt{\mu}$

$x^- = -\sqrt{\mu}$

so far (without stability)



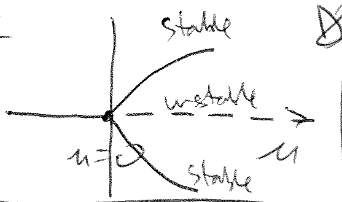
(looks like pitchfork)

<u>Stability</u>

$Df = \mu - 3x^2$. $\boxed{x=0}$ $Df(x=0) = \mu$ $\begin{cases} \text{stable } \mu < 0 \\ \text{unstable } \mu > 0 \end{cases}$

$\boxed{\text{Trick for } x^+/x^-}$ use $\boxed{\text{convenient points}}$ $\boxed{+}$ $\boxed{\text{'only change at bif.'}}$

So



$\boxed{x^+ = \sqrt{\mu}}$ use $x^+ = 1 \Rightarrow \mu|_{(x^+=1)} = 1$.

$Df(x^+ = 1, \mu|_{x^+} = 1) = 1 - 3 = -2 \Rightarrow$ stable.

Similarly $\boxed{x^- = -\sqrt{\mu}}$ & $x^- = -1 \Rightarrow \mu|_{(x^-=-1)} = 1$ (sym).

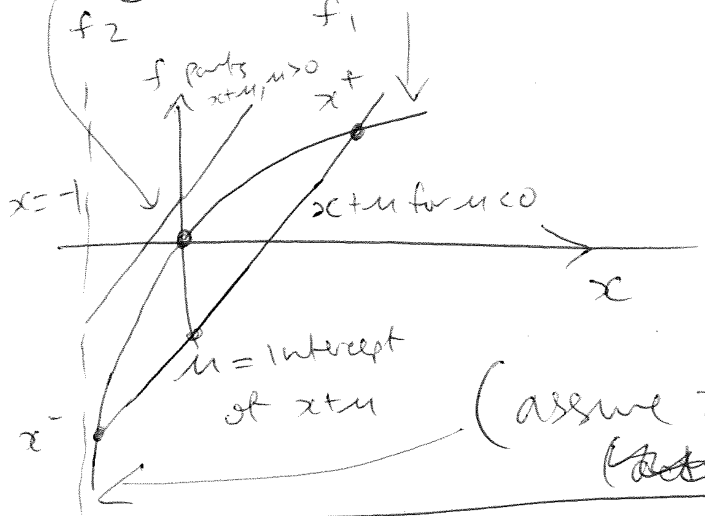
$\Rightarrow Df(x^- = -1, \mu|_{x^-} = 1) = -2 \Rightarrow$ stable.

1D-Continuum 1 (Tut 3)

(2a) $\dot{x} = \mu + x - \ln(1+x) = f(x;\mu)$

FP: $f(x;\mu) = 0$

$$\underbrace{\mu + x}_{f_2} - \underbrace{\ln(1+x)}_{f_1} = 0$$



$\ln(1+x)$
$= \ln 1$ for $x=0$
$= 0$

$\ln 0 \to -\infty$
$x=-1$

(assume $x > -1$ here)

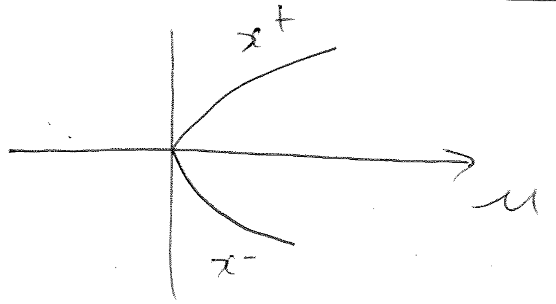
~~(actually $x<0$)~~

graphically → expect no solution for $\mu>0$
one soln for $\mu=0$
two soln for $\mu<0$
} turning point

↳ x^+ & x^-.

(or using computer)

So expect

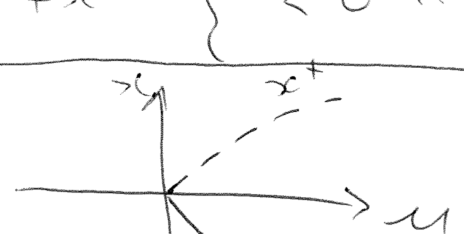


(without stability)

$\mu=0$

Now stability $Df = 1 - \dfrac{1}{1+x}$ $\begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$

So



Turning Point Bifurcation.

1D - codim 1

Problem 2(e) (Tutorial 3)

$$\dot{x} = x + \frac{\mu x}{1+x^2} = f(x;\mu)$$

① Equilibria $f(x;\mu) = 0$

$$x + \frac{\mu x}{1+x^2} = 0$$

$$\Leftrightarrow (1+x^2+\mu)\cdot x = 0 \qquad (1+x^2 \neq 0)$$

Roots $x = 0$ (exists for all μ)

&/OR $1+x^2+\mu = 0$

$$\Rightarrow x = \pm\sqrt{-\mu-1}$$

use $\lambda = -\mu$ for convenience

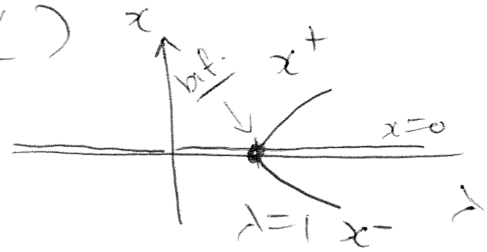
$$\Rightarrow x = \pm\sqrt{\lambda-1}$$

Cases
- no real roots $\lambda < 1$
- one real root $\lambda = 1 \Rightarrow x = 0$
- two real roots $\lambda > 1$

$\Rightarrow x_+ = \sqrt{\lambda-1}$

& $x_- = -\sqrt{\lambda-1}$

(ignore stability & consider so far)



looks like pitchfork

now, consider stability

Res

0 | Change in stability at bifurcation points | (2).

| only | → branches meet / solutions born etc.

→ quick trick → evaluate Df for | convenient points | either side | of ~~separation~~ bifurcations.

(Note: Glendinning gives theorems stated in terms of derivatives at bif.
→ we could use, but will just use direct evaluations of Df as above).

$$f = x + \frac{\mu x}{1+x^2}$$

$$Df = 1 + \frac{(1+x^2)\mu - \mu x \cdot 2x}{(1+x^2)^2} \quad \text{(quotient rule)}.$$

$$= 1 + \frac{\mu + x^2(\mu - 2\mu)}{(1+x^2)^2}$$

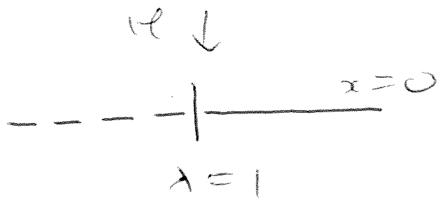
$$= 1 + \frac{\mu - \mu x^2}{(1+x^2)^2} = 1 + \frac{\mu(1-x^2)}{(1+x^2)^2}$$

consider $Df(x=0;\mu) = 1+\mu = 1-\lambda = Df(x=0;\lambda)$

$\Rightarrow Df(x=0) > 0 \quad \lambda < 1$ unstable

$< 0 \quad \lambda > 1$ stable.

~~[illegible]~~



(2e)

Next, consider

$$Df(x,\lambda) = 1 - \lambda \frac{(1-x^2)}{(1+x^2)^2}$$

for $\boxed{x_+ = \sqrt{\lambda - 1}}$

$\boxed{\text{set}}$ $\boxed{x_+ = 1 > 0}$ (convenient point on upper branch).

$\Rightarrow \lambda = 2$

$\Rightarrow Df(1,2) = 1 - 2 \times \cancel{0}$

$= 1 > 0 \rightarrow$ unstable.

Now well $\boxed{x_- = -\sqrt{\lambda - 1}}$

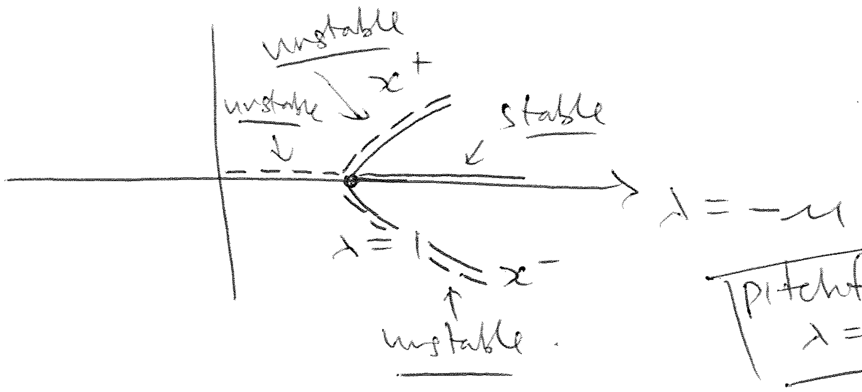
& $\boxed{\text{set}}$ $x_- = -1 < 0$ (convenient lower)

$\Rightarrow \lambda = 2$

$\Rightarrow Df(-1,2) = 1 > 0 \rightarrow$ also unstable

(expect same stability since pitchfork is symmetric).

So



$\boxed{\text{pitchfork at } \lambda = 1 \text{ i.e. } \mu = -1}$

Problem 2 (2D - Codim one Tut. 3)

①

$$\dot{x} = -y + \mu x + x y^2$$

$$\dot{y} = x + \mu - x^2$$

$$Df = \begin{pmatrix} \mu + y^2 & -1 + 2xy \\ 1 - 2x & 0 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix} = A$$

updated

$$\operatorname{tr} A = \mu$$

$$\det A = \cancel{2}\ 0 - (-1 \times 1) = 1$$

roots

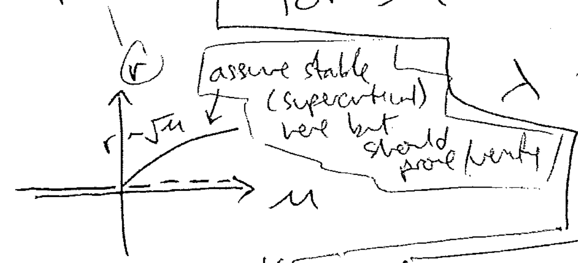
$$\lambda^2 - \mu\lambda + \cancel{2}^{\,1} = 0$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - \cancel{8}\,4}}{2}$$

$\boxed{\text{complex}}$ if $\mu^2 < 4$ ie $\boxed{-2 < \mu < 2}$

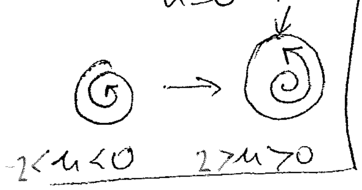
$\boxed{\text{pure imaginary}}$ for $\boxed{\mu = 0}$

polar



for $\mu = 0$ get

$$\lambda = \pm \frac{\sqrt{-\cancel{4}}}{2} = \pm \frac{\cancel{2}\cdot\sqrt{-\cancel{1}}}{\cancel{2}} \boxed{= \pm i} \boxed{= \pm \sqrt{\det A}\, i}$$



$\Rightarrow$ ie pure imaginary

& for

$\mu < 0 \Rightarrow$ stable spiral

$\mu > 0 \Rightarrow$ unstable spiral.

(at least for $\mu^2 < 4$ ie $|\mu| < 2$)

expect Hopf.