10. Using the result of the previous question, find the approximate amplitude of the periodic orbits for the equations

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

and

$$\ddot{x} + \epsilon(x^2\dot{x}^2 - 2) + 4x = 0,$$

$0 < \epsilon \ll 1$, and verify that these periodic orbits are stable.

11. Consider the equation

$$\ddot{x} + \dot{x} = -\epsilon(x^2 - x), \; 0 < \epsilon \ll 1.$$

(The dot on the second term is not a misprint.) Using the method of multiple scales show that

$$x_0(\tau, T) = A(T) + B(T)e^{-\tau}$$

and identify any resonant terms at order ϵ. Show that the non-resonance condition is

$$A_T = A - A^2$$

and describe the asymptotic behaviour of solutions.

8

Bifurcation theory I: stationary points

Bifurcation theory describes the way that topological features of a flow (properties such as the number of stationary points and periodic orbits) vary as one or more parameters are varied. There are many approaches to the problem of understanding the possible changes which occur in differential equations, ranging from a straightforward analytic description to a topological classification of all possible behaviours which may occur under an arbitrary, but small, perturbation of the system. In this chapter we aim to describe some of the simple techniques used to describe bifurcations of stationary points, adopting a heuristic approach rather than a rigorous mathematical treatment. Section 8.1 on the Centre Manifold Theorem follows the graduate textbook by Guckenheimer and Holmes (1983) quite closely. This book has become one of the standard introductions to nonlinear equations, and the next five chapters are littered with references to it. A great deal more of the detailed mathematical justification for the results described here and their implications can be found there. Our approach remains at the level of understanding systems and being able to deduce properties of examples.

The fundamental observation for stationary points of flows is that if the stationary point is hyperbolic, i.e. the eigenvalues of the linearized flow at the stationary point all have non-zero real parts, then the local behaviour of the flow is completely determined by the linearized flow (at least up to homeomorphism; see Chapter 4 and the Stable Manifold Theorem). Furthermore, small perturbations of the equation will also have a hyperbolic stationary point of the same type (Section 4.3). Hence bifurcations of stationary points can only occur at parameter values for which a stationary point is non-hyperbolic. This gives an easy criterion for detecting bifurcations: simply find parameter values for which the linearized flow near a stationary point has a zero or purely imaginary eigenvalue. One of the most important techniques for studying such bifurcations is based on the non-hyperbolic equivalent of the Stable Man-

Parameters ...

ifold Theorem, called the Centre Manifold Theorem. This generalizes the idea of the centre manifold for linear systems to nonlinear systems.

In this chapter we begin by outlining how to find the centre manifold for nonlinear systems and then go on to show how this can be used to derive the dynamics of systems near a system with a non-hyperbolic stationary point. We then describe the simple bifurcations that can arise in systems depending upon a single real parameter. Before describing this process in detail we can give a flavour of the type of manipulation that will be involved. Suppose that we have a system of equations on the real line ($x \in \mathbf{R}$) which depend upon a real parameter, μ. Thus each value of μ defines a differential equation and we are interested in the way that qualitative features of the solutions vary as μ takes different values. We have already determined that problems will arise when the system has a non-hyperbolic stationary point, so suppose that

$$\dot{x} = f(x, \mu) \tag{8.1}$$

where $f(0,0) = \frac{\partial f}{\partial x}(0,0) = 0$. Then the origin is a stationary point if $\mu = 0$ and since the Jacobian matrix vanishes it is non-hyperbolic. Now, expanding $f(x,\mu)$ as a Taylor series in some neighbourhood of $(x,\mu) = (0,0)$ we obtain

$$\dot{x} = A(\mu) + B(\mu)x + C(\mu)x^2 + \dots \tag{8.2}$$

where

$$A(\mu) = f_\mu \mu + \tfrac{1}{2} f_{\mu\mu}\mu^2 + \dots, \tag{8.3}$$

$$B(\mu) = f_{\mu x}\mu + \tfrac{1}{2} f_{\mu\mu x}\mu^2 + \dots \tag{8.4}$$

and

$$C(\mu) = f_{xx} + \tfrac{1}{2} f_{\mu x x}\mu + \dots. \tag{8.5}$$

Subscripts in these equations denote partial differentiation with respect to the relevant variable and all derivatives are evaluated at $(x,\mu) = (0,0)$. Much of bifurcation theory is simply about determining the stationary points of such systems (as functions of μ), i.e. looking for solutions of

$$0 = A(\mu) + B(\mu)x + C(\mu)x^2 + \dots \tag{8.6}$$

in a neighbourhood of $(x,\mu) = (0,0)$. The number of solutions and their stability varies according to whether certain partial derivatives or combinations of partial derivatives vanish and their sign if they do not vanish. Hence bifurcation theory (in one dimension) is really about being able to solve (8.6).

8.1 Centre manifolds

In some of the examples in the previous chapter we have seen that pairs of stationary points can come together and disappear as a parameter is varied. This is an example of a bifurcation. More precisely, a bifurcation value of a parameter μ is a value at which the qualitative nature of the flow changes.

Example 8.1

Consider the simple equation

$$\dot{x} = \mu - x^2.$$

For $\mu < 0$ there are no stationary points, whereas for $\mu > 0$ there are two, one at $x_+ = \sqrt{\mu}$ and the other at $x_- = -\sqrt{\mu}$. The linearized flow is given by the (1×1) Jacobian matrix, $-2x$, so the stationary point in $x < 0$ is unstable and the stationary point in $x > 0$ is stable. An important change in the behaviour of the system clearly occurs as μ passes through zero: two stationary points are created, one stable and the other unstable. This is an example of a saddlenode bifurcation, and the bifurcation value of μ is $\mu = 0$. At this value of the parameter, $\dot{x} = -x^2$, so $x = 0$ is a stationary point of the flow, but the linear flow vanishes. In other words, if $\mu = 0$ there is a non-hyperbolic stationary point at $x = 0$. It is often useful to illustrate bifurcations by plotting the position of stationary points as a function of parameter as shown in Figure 8.1. The stability of solutions is indicated by solid lines for stable solutions and dotted lines for unstable solutions. Such pictures are referred to as *bifurcation diagrams*.

Recall that in Chapter 4 we defined the stable and unstable manifolds of a hyperbolic stationary point and, in Chapter 3, the linear centre manifold for the linear system $\dot{z} = Lz$, $z \in \mathbf{R}^n$ to be the space spanned by the generalized eigenvectors of L corresponding to eigenvalues λ with $\mathrm{Re}(\lambda) = 0$. To understand local bifurcations of stationary points we need the nonlinear equivalent of $E^c(0)$. This is given by the following theorem (Carr, 1981, Hirsch, Pugh and Shub, 1977, Kelley, 1967, ...).

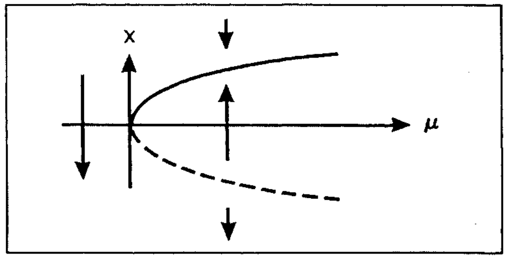

Fig. 8.1 Position (x) against parameter (μ) for the saddlenode bifurcation.

(8.1) THEOREM (CENTRE MANIFOLD THEOREM)

Let $f \in C^r(\mathbf{R}^n)$ with $f(0) = 0$. Divide the eigenvalues, λ, of $Df(0)$ into three sets, σ_u, σ_s and σ_c, where $\lambda \in \sigma_u$ if $\mathrm{Re}(\lambda) > 0$, $\lambda \in \sigma_s$ if $\mathrm{Re}(\lambda) < 0$ and $\lambda \in \sigma_c$ if $\mathrm{Re}(\lambda) = 0$. Let E^u, E^s and E^c be the corresponding generalized eigenspaces. Then there exist C^r unstable and stable manifolds (W^u and W^s) tangential to E^u and E^s respectively at $x = 0$ and a C^{r-1} centre manifold, W^c, tangential to E^c at $x = 0$. All are invariant, but W^c is not necessarily unique.

The proof of the theorem is similar in style to the proof of the Stable Manifold Theorem, but we will not give it here. For our purposes it is enough to know that the centre manifold exists and in the rest of this section we shall see how it can be used and constructed for simple examples. The non-uniqueness of the centre manifold may appear to be a little bizarre at first sight. It is, however, not as dangerous as it sounds, and reflects the possibility of adding exponentially small terms without changing the property of the manifold. Intuitively, it reflects the fact that a number of trajectories can do essentially the same thing tangential to E^c, and any one of these is a suitable choice for 'the' centre manifold. This is shown explicitly in Example 8.2.

The Centre Manifold Theorem implies that at a bifurcation point, where a stationary point is non-hyperbolic ($\sigma_c \neq \emptyset$), the system can be written locally in coordinates $(x, y, z) \in W^c \times W^s \times W^u$ on the invariant manifolds as

$$\dot{x} = g(x) \tag{8.7a}$$

$$\dot{y} = -By \tag{8.7b}$$

$$\dot{z} = Cz \tag{8.7c}$$

where B and C are positive definite matrices. The motion on W^s is unequivocally towards the stationary point and the motion on W^u is unequivocally away from the stationary point, so the local behaviour can be understood by solving (8.7a). In order to do this we must find a way to calculate the function $g(x)$.

For simplicity we shall consider the situation when $\sigma_u = \emptyset$, so the equations can be written (in coordinates x in the direction of E^c and y in the direction of E^s) as

$$\dot{x} = Ax + f_1(x, y) \tag{8.8a}$$

$$\dot{y} = -By + f_2(x, y) \tag{8.8b}$$

where the eigenvalues of A all have zero real parts, the eigenvalues of B all have strictly positive real parts and the functions f_i, $i = 1, 2$, represent nonlinear terms, so both f_i and their first derivatives with respect to the x and y variables vanish at $(x, y) = (0, 0)$. Since W^c is tangential to $E^c = \{(x, y) | y = 0\}$ it can be represented locally as the graph of a function of x, so

$$W^c = \{(x, y) | y = h(x),\ h(0) = 0,\ Dh(0) = 0\}$$

where $h : U \to \mathbf{R}^s$ is defined on some neighbourhood U of the origin in $\mathbf{R}^c$ as illustrated in Figure 8.2 and Dh is the Jacobian matrix of h. Thus, on W^c the flow is approximated (projecting onto the x directions) by

$$\boxed{\dot{x} = Ax + f_1(x, h(x)).} \tag{8.9}$$

This, then, is the equation for $g(x)$ we have been looking for. All that remains to do is to find the function $h(x)$, which can be done in precisely the same way as the stable and unstable manifolds were approached in Chapter 4. Assume that $h(x)$ can be expanded locally as a power series in x, building in the conditions that $h(0) = 0$ and $Dh(0) = 0$, i.e. that the manifold passes through the origin tangential to the linear centre manifold, E^c, at $x = 0$. Then equate powers of x from the two forms of $\dot{y}$ obtained by differentiating the equation $y = h(x)$ and from the definition of $\dot{y}$ in (8.8b).

On the centre manifold $y = h(x)$ and so $\dot{y} = Dh(x)\dot{x}$, where $Dh(x)$ is the $s \times c$ Jacobian matrix of h and $\dot{x}$ is a c dimensional vector. Substituting for $\dot{x}$ from (8.8a) gives

$$\dot{y} = Dh(x)[Ax + f_1(x, h(x))]. \tag{8.10}$$

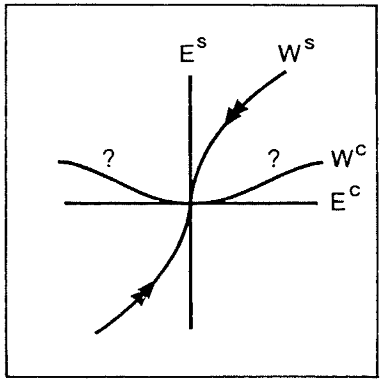

Fig. 8.2 Linear and nonlinear stable and centre manifolds.

But on the centre manifold (8.8b) implies that

$$\dot{y} = -Bh(x) + f_2(x, h(x)). \tag{8.11}$$

Comparing these two equations and including the conditions on $h(x)$ yields the problem

$$Dh(x)[Ax + f_1(x, h(x))] = -Bh(x) + f_2(x, h(x)) \tag{8.12}$$

$$h(0) = 0, \quad Dh(0) = 0. \tag{8.13}$$

These equations can be solved, at least in principle, to arbitrary order in x by posing a series solution and equating coefficients at each order. This is probably best illustrated by a simple example.

Example 8.2

Consider the equations

$$\dot{x} = xy$$
$$\dot{y} = -y - x^2.$$

The origin is a non-hyperbolic stationary point, and the linearization about the origin is already in normal form, so $E^c(0) = \{(x,y)|y=0\}$ and $E^s(0) = \{(x,y)|x=0\}$. The nonlinear centre manifold will be (at least locally) the graph of a function $y = h(x)$ which passes through the origin tangential to the linear centre manifold (so $h(0) = h'(0) = 0$). This suggests that we try a solution of the form

$$y = h(x) = ax^2 + bx^3 + cx^4 + dx^5 + O(x^6).$$

Substituting for y this gives

$$\dot{y} = h'(x)\dot{x} = h'(x)xy = xh(x)h'(x),$$

and using the trial solution we find

$$\begin{aligned}\dot{y} &= x(ax^2 + bx^3 + \ldots)(2ax + 3bx^2 + \ldots)\\ &= 2a^2x^4 + 5abx^5 + O(x^6).\end{aligned}$$

On the other hand, $\dot{y} = -y - x^2$ so

$$\begin{aligned}\dot{y} &= -h(x) - x^2\\ &= -(a+1)x^2 - bx^3 - cx^4 - dx^5 + O(x^6).\end{aligned}$$

Now, equating coefficients at order x^2 gives $(a+1) = 0$; the cubic terms obviously give $b = 0$ and the quartic terms give $2a^2 = -c$, so

$$a = -1; \quad b = 0; \quad c = -2; \quad d = 0;$$

giving an approximation to the nonlinear centre manifold of the form

$$y = h(x) = -x^2 - 2x^4 + O(x^6).$$

The motion on the centre manifold (which is tangential to the x-axis at $x = 0$) is therefore given by the equation

$$\dot{x} = xh(x) = -x(x^2 + 2x^4 + O(x^6)) \quad \simeq -x^3$$ (note: can see by inspection

and so this shows that the motion on the centre manifold is (at least locally) towards the origin. Since the y-axis is (approximately) the stable manifold of the origin we see that the non-hyperbolic stationary point is in fact a nonlinear sink (Fig. 8.3). Figure 8.3 also illustrates the point about the non-uniqueness of the centre manifold made earlier. Any one of the trajectories which tends to the origin tangential to

$$y = -x^2 - 2x^4 + O(x^6)$$

can be chosen to be 'the' centre manifold. For each such trajectory, the the distance between the derived manifold (defined by a power series) in a suffiently small neighbourhood of the origin is exponentially small, and so they all have the same power series expansion about the origin. Hence any of them can be chosen as a centre manifold.

Remark: The lowest order approximation to the centre manifold for this example could have been obtained by observing that on the centre manifold $\dot{y}$ is approximately zero and so y is approximately $-x^2$. This sort of observation can often act as a useful way of checking that an algebraic slip has not been made in the calculation of the centre manifold.

y → equilibrium

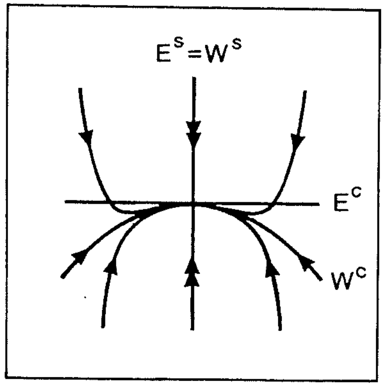

Fig. 8.3 Phase portrait for Example 8.2.

8.2 Local bifurcations

The analysis of the preceding section allows us to determine the nature of the non-hyperbolic stationary point by looking at the motion on the centre manifold where the linear behaviour does not completely determine the flow in a neighbourhood of the stationary point. The next step towards understanding local bifurcations is to introduce parameters and extend the idea of the centre manifold into parameter space in such a way as to capture the behaviour of families of systems near bifurcation values of the parameter. To this end we consider differential equations which depend on one or more parameter, $\mu \in \mathbf{R}^m$, so the differential equation is $\dot{w} = f(w,\mu)$, where $w \in \mathbf{R}^n$ and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is a smooth function with $f(0,0) = 0$, so that the origin (in $\mathbf{R}^n$) is a stationary point of the flow. Now suppose that if $\mu = 0$ the linearized flow, $\dot{w} = Lw$ near $w = 0$ has some eigenvalues with zero real part. In this case we can choose coordinates (x,y,z) in the linear eigenspaces of L as in the previous section such that

$$E^c(0) = \{(x,y,z)|y = z = 0\},$$
$$E^u(0) = \{(x,y,z)|x = y = 0\},$$

and

$$E^s(0) = \{(x,y,z)|x = z = 0\}.$$

In these coordinates the differential equation $\dot{w} = f(w,\mu)$ becomes

$$\dot{x} = A(\mu)x + f_1(x,y,z;\mu) \tag{8.14a}$$
$$\dot{y} = -B(\mu)y + f_2(x,y,z;\mu) \tag{8.14b}$$
$$\dot{z} = C(\mu)z + f_3(x,y,z;\mu) \tag{8.14c}$$

where all the eigenvalues of $A(0)$ have zero real parts and there is some open neighbourhood of $\mu = 0$ for which the eigenvalues of both $B(\mu)$ and $C(\mu)$ have strictly positive real parts. (This means that we have assumed implicitly that the eigenvalues of $B(0)$ and $C(0)$ have real parts which are bounded away from zero. If n is finite this is not a problem, but in infinite dimensional problems this has to be excluded explicitly.) The functions $f_i(x,y,z;\mu)$ contain the nonlinear terms in x, y and z, so they vanish, together with their first derivatives with respect to the variables x, y and z at the bifurcation point $(x,y,z;\mu) = (0,0,0;0)$.

To describe the dynamics in a neighbourhood of $(0,0,0;0)$ we use the centre manifold on an extended system of equations: we want to include the μ variables as part of the centre manifold. To do this, add the trivial equations

$$\dot{\mu} = 0. \tag{8.14d}$$

On the face of it, this may appear to be a pretty useless thing to do. After all, this trivial equation has a very easy solution, μ equals a constant. The point is that by including these equations we obtain a centre manifold which stretches into our parameter space, so the centre manifold can be used on the extended system in such a way as to be valid for small $|x|$ *and* $|\mu|$. We can then 'solve' the trivial equation ($\dot{\mu} = 0$) without too much effort to get a simplified equation for the evolution of x which involves the parameters.

The extended system has a centre manifold of dimension dim $E^c(0) + m$ which is tangential to $E^c(0)$ and $\mu = 0$ at $(0,0,0;0)$. Thus, using precisely the same arguments as in the previous section, we can try to find an approximation to the centre manifold by solving for y and z as a graph over x and μ. So set

$$y = h_s(x,\mu) \tag{8.15}$$

and

$$z = h_u(x,\mu) \tag{8.16}$$

as the equation of the graph of the centre manifold and proceed as before. Substituting (8.15) and (8.16) into (8.14a) and (8.14d) we find

$$\dot{x} = A(\mu)x + f_1(x, h_s(x,\mu), h_u(x,\mu);\mu) \tag{8.17a}$$

$$\dot{\mu} = 0, \tag{8.17b}$$

valid for sufficiently small $|x|$ and $|\mu|$. As we have already remarked, the second of these two equations is not hard to solve, and so we are left with the (dim $E^c(0)$) equation

$$\dot{x} = A(\mu)x + f_1(x, h_s(x,\mu), h_u(x,\mu);\mu) = G(x,\mu) \tag{8.18}$$

which describes the local dynamics on the centre manifold for $|x|$ and $|\mu|$ sufficiently small. The remainder of this chapter is devoted to the study of such equations for particular choices of dim $E^c(0)$ and restrictions on higher derivatives of the function $G(x,\mu)$. This process will result in a sequence of bifurcation theorems, each saying that if $A(0)$ has a particular form and certain genericity (or non-degeneracy) conditions hold for $G(x,\mu)$ then particular changes in the dynamics of the family must occur as μ passes through zero.

8.3 The saddlenode bifurcation

Suppose that the equation $\dot{w} = f(w,\mu)$ has a non-hyperbolic stationary point (which we can take to be at the origin, $w = 0$) if $\mu = 0$. If the Jacobian matrix of the linear flow at $w = 0$ has a simple zero eigenvalue for $\mu = 0$ and all other eigenvalues lie off the imaginary axis, then dim $E^c(0) = 1$ and the equation on the centre manifold is

$$\dot{x} = G(x,\mu)$$

where $x \in \mathbf{R}$ and we can take $\mu \in \mathbf{R}$ so $G : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$. From the definition of G above we see that $G(0,0) = 0$ and $A(0) = G_x(0,0) = 0$, where the subscript denotes partial differentiation with respect to x. To understand the local behaviour of the flow for (x,μ) near $(0,0)$, expand G as a Taylor series in both the variables about $(x,\mu) = (0,0)$. Then, since $G(0,0) = G_x(0,0) = 0$,

$$\dot{x} = G_\mu\mu + \tfrac{1}{2}(G_{xx}x^2 + 2G_{x\mu}\mu x + G_{\mu\mu}\mu^2) + O(3). \tag{8.19}$$

All the partial derivatives are, of course, evaluated at $(x,\mu) = (0,0)$. This is a simple differential equation in one dimension which can be analyzed using the standard techniques developed over previous chapters,

valid for $|x|$ and $|\mu|$ sufficiently small. The first step in any such analysis is to determine the locus of stationary points of the equation. In this case it is quite possible to do this rigorously without any serious problem (use the Implicit Function Theorem). However, we shall leave this more mathematical approach until the end of this chapter and concentrate on developing asymptotic expansions for the locus of stationary points in the (x,μ) plane. We begin by rewriting the equation on the centre manifold as

$$\dot{x} = \sum_{k\geq 0} A_k(\mu)x^k \tag{8.20}$$

where

$$A_0(\mu) = G_\mu\mu + \tfrac{1}{2}G_{\mu\mu}\mu^2 + O(\mu^3),\quad A_1(\mu) = G_{\mu x}\mu + O(\mu^2),$$

$$A_2(\mu) = \tfrac{1}{2}G_{xx} + O(\mu)$$

and so on. Trying to solve the lowest order approximation, $A_0 + A_1 x \sim 0$ gives solutions $x \sim -A_0/A_1$ which is $O(1)$ or larger if $G_\mu \neq 0$, so this will not give us a local solution. Including the quadratic term, $A_0 + A_1x + A_2x^2 \sim 0$ gives solutions

$$x \sim \frac{-A_1 \pm \sqrt{A_1^2 - 4A_0A_2}}{2A_2} \tag{8.21}$$

and on substituting for the functions $A_i(\mu)$ we find the leading order solutions

$$x \sim \pm\sqrt{\frac{-2G_\mu\mu}{G_{xx}}}. \tag{8.22}$$

Hence if $G_\mu/G_{xx} > 0$ there is a pair of solutions near the origin if $\mu < 0$ and there are no solutions if $\mu > 0$. On the other hand, if $G_\mu/G_{xx} < 0$ there is a pair of solutions near the origin if $\mu > 0$, and no solutions exist if $\mu < 0$.

Suppose that $G_\mu/G_{xx} < 0$, so solutions bifurcate into $\mu > 0$. We pose the asymptotic series suggested by (8.22)

$$x \sim \sum_{n\geq 1} \alpha_n \mu^{\frac{n}{2}}$$

which we can now substitute for the full equation, (8.19), to determine the coefficients α_n to whatever order of accuracy is desired. For example, at order μ the equations are

$$G_\mu + \tfrac{1}{2}G_{xx}\alpha_1^2 = 0 \tag{8.23}$$