ENGSCI 213: MATHEMATICAL MODELLING 2SE

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

CURRENT (& LAST) PROBABILITY TOPIC

3. *Continuous probability models* [3 lectures]

   Continuous random variables and continuous probability distributions. Probability density functions and cumulative distribution functions. (Expectation and variance). Uniform, Exponential and Normal distributions.

LECTURE 8

- Example calcuations for simple continuous distributions
- Uniform and Exponential distributions

RECAP: THE CUMULATIVE DISTRIBUTION FUNCTION

The *cumulative distribution function* $F_X(x)$ for continuous variables is defined by

$$\boxed{F_X(x) := P(X \leq x) = P(-\infty \leq X \leq x)}$$

RECAP: THE PROBABILITY DENSITY FUNCTION

The *probability density function* can be characterised via its *relationship with the cumulative distribution function*:

$$P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f_X(x)dx$$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

Note $f_X(x)$ is *not the same as the mass function* from the discrete case! In fact...

RECAP: PROBABILITY DENSITY VS PROBABILITY MASS

For a small interval of length Δa, set $b = a + \Delta a$ and calculate

$$P(a < X < a + \Delta a) \approx f_X(a)\Delta a$$

This is the *closest analogue to the mass function at a*, which we used in the discrete case (think mass = density times volume).

RECAP: EXPECTATION FOR CONTINUOUS RANDOM VARIABLES

We get essentially the same result for *expectation* for continuous RVs as for discrete RVs:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

...and...

RECAP: VARIANCE FOR CONTINUOUS RANDOM VARIABLES

similarly for *variance* we get

$$Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 f_X(x)dx$$

RECAP: RELATIONSHIP BETWEEN VARIANCE AND MEAN

The *key relation between the variance and mean*

$$Var(X) = E(X^2) - [E(X)]^2$$

also holds again as for our discrete case.

EXAMPLES

Some more simple example calculations for continuous random variables.

UNIFORM DISTRIBUTION

We say X has a *Uniform distribution* on the interval $[a, b]$ if X is equally likely to fall anywhere in the interval $[a, b]$.

We write

$$\boxed{X \sim Uniform[a, b], \text{ or } X \sim U[a, b]}$$

Equivalently, $X \sim \mathrm{Uniform}(a, b)$, or $X \sim \mathrm{U}(a, b)$.

UNIFORM DISTRIBUTION - PROBABILITY DENSITY FUNCTION

If $X \sim \mathrm{U}[a, b]$, then

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

UNIFORM DISTRIBUTION - PROBABILITY DENSITY FUNCTION

Thus

$$F_X(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

UNIFORM DISTRIBUTION - CHECKING BASIC PROPERTIES

Let's check that these definitions make sense and are consistent.

UNIFORM DISTRIBUTION - MEAN AND VARIANCE

If $X \sim \mathrm{U}[a, b]$, then

$$\boxed{E(X) = \frac{a+b}{2}}$$

and

$$\boxed{Var(X) = \frac{(b-a)^2}{12}}$$

UNIFORM DISTRIBUTION - EXAMPLE

Simple example.

EXPONENTIAL DISTRIBUTION

The *Exponential distribution* can be related to the *Poisson process* - rather than count the number of events in a fixed time interval it describes

The length of time between events - i.e. the waiting time - in a Poisson process

Example.

EXPONENTIAL DISTRIBUTION

The exponential distribution has *one parameter*, λ (remember the Poisson process), which must be positive.

We write

$$\boxed{X \sim Exponential(\lambda), \text{ or } X \sim Exp(\lambda).}$$

EXPONENTIAL DISTRIBUTION - MEMORYLESSNESS

If $X \sim Exp(\lambda)$ then for any $s, t \geq 0$

$$P(X > (s + t) \mid X > t) = P(X > s)$$

So the process is *memoryless*:

the time of waiting an extra s units of time given you've waited t units is just the same as waiting s units of time - the t units of waiting gives no more information about when the next event will occur!

EXPONENTIAL DISTRIBUTION - PROBABILITY DENSITY FUNCTION AND DISTRIBUTION FUNCTION

If $X \sim \text{Exp}(\lambda)$, then

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

&

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-\lambda x} & \text{otherwise} \end{cases}$$

MEAN AND VARIANCE OF EXPONENTIAL DISTRIBUTION}

If $X \sim \mathrm{Exp}(\lambda)$, then

$$E(X) = \frac{1}{\lambda}$$

and

$$Var(X) = \frac{1}{\lambda^2}$$

EXAMPLES