ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*) *Oliver Maclaren*
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LECTURE 13 CENTRE MANIFOLD THEORY

- Extension to systems with parameters
  - Application to bifurcation theory
  - Application to fast/slow systems (further reading…)

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

Note: the material on *extended* centre manifolds that *follows next is not on the exam*, but may be relevant to the assignment/tutorial.

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

We surely want to consider systems where *some eigenvalues are much smaller* than the others but *not exactly zero*.

We also want to analyse the *dynamics in systems with parameter-dependent bifurcations*.

Both of these cases can handled by constructing an *extended* centre manifold which *includes the parameter(s)* of interest.

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

The key trick is simple: treat the *parameter* of interest as a *(super slow!) centre state variable*. i.e. rewrite a system like

$$\dot{x} = f(x; \mu)$$

as

$$\begin{aligned} \dot{x} &= f(x, \mu) \\ \dot{\mu} &= 0 \end{aligned}$$

where μ is *now a state variable*. Note: this means that in the second system terms like e.g. μx in f are now considered

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

How is this relevant to *bifurcation* theory? Suppose we have a *non-hyperbolic fixed point* $x = 0$ when $\mu = 0$. From Wiggins (2003):

> *…the [extended] center manifold exists for all μ in a sufficiently small neighborhood of $\mu = 0$…[but as we know] it is possible for solutions to be created or destroyed by perturbing nonhyperbolic fixed points…*

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

> *…Thus, since the invariant center manifold exists in a sufficiently small neighborhood in both x and μ of $(x, \mu) = (0, 0)$, all bifurcating solutions will be contained in the lower dimensional [extended] center manifold.*

i.e.

'…all the action is on the centre manifold…'

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

The easiest way to understand this is via an example.

Transcritical bifucation example: *bifurcation analysis the long way* (using centre manifold theory).

CENTRE MANIFOLD THEOREM - APPLICATION ENZYME REACTIONS, LASERS ETC

Usually approached via 'quasi-steady' or 'quasi-equilibrium' assumptions

We can justify and improve via centre manifold theory!

See further readings!

EngSci 711 L13 Centre manifold theory cont'd ... cont'd

Extended centre manifolds.
dealing with/making use of
parameters in centre manifold theory

- Application to bifurcation theory
- Application to systems with small param. (comments/readings)
  ↳ + comparison to singular perturbation theory.

$\boxed{\text{Key: parameters as 'super slow' variables}}$

Example Questions

- Not examinable [maybe assignmentable ...]
- See this lecture & tutorial for examples.
- See eg laser physics papers I posted to Canvas

Systems depending on parameters

Now we look at how to apply centre manifold theory to systems depending on parameters, ie

$$\boxed{\dot{x} = f(x\,;\mu)}$$

state vector

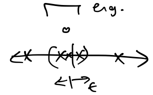
vector of 'external' or 'control' parameters

Why? This is useful for at least two reasons:

1. To connect centre manifold theory & bifurcation theory
   ↳ we will derive the simple, low-dimensional systems in which bifurcations occur by reducing larger systems to just the 'critical' (slow/centre) variables
   } low dimensional bifurcations are 'contained' in larger systems
   → near a bifurcation, 'all the action' occurs on the centre manifold

2. Parameters allow us to 'widen' the scope of application of centre manifold theory

'close to zero' eig.



   ↳ exactly zero eigenvalues are rare, but widely separated eigenvalues, with some $\approx \epsilon$ are much more common
   → can use small parameters to represent this
   → creates a centre manifold in ϵ that we can use
   → alternative/complement to 'singular perturbation theory'

Motivating example 1.

Recall the example from lecture 12 (Based on Glendinning Exercise 8.3)

$$\boxed{\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2 \end{aligned}}$$

We saw that this has one zero eigenvalue, & hence a one-dimensional centre manifold $\Big\}$ is nonhyperbolic

(we had to change coord. to identify/separate the fast/slow dynamics though!)

Consider next the parameter dependent system:

$$\boxed{\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= (1+\mu)x - y - y^2 \end{aligned}}$$

For $\mu = 0$ this reduces to our previous system.

for $\mu \neq 0$ this will no longer be non-hyperbolic

→ thus we expect to pass through a bifurcation point as we vary μ through zero.

→ we also hope/expect that we only need to consider the 'slow' variable equation $\Big\}$

↳ we need to justify for $\mu \neq 0$!

Steps.

1. FP. → just look at $(0,0)$ here

→ $\dot{x}(0) = \dot{y}(0) = 0$ ✓

2. $Df(0,0) = \begin{pmatrix} -1 & 1 \\ 1+\mu & -1 \end{pmatrix}$

$tr = -2$

$det = 1 - (1+\mu) = -\mu$

Note: $det \neq 0$ if $\mu \neq 0$

$\left.\begin{aligned} \lambda_1 &= -1 - \sqrt{1+\mu} \\ \lambda_2 &= -1 + \sqrt{1+\mu} \end{aligned}\right\}$ $\lambda_2 \neq 0$ if $\mu \neq 0$.

Problem: no centre manifold for $\mu \neq 0$

→ no reduction!

> Q: does a centre manifold still exist for $\mu \neq 0$?
> can we still reduce to it for $\mu \neq 0$
>
> A: yes!

Key trick

Treat μ as a state var

→ consider extended system:

$$\boxed{\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= (1+\mu)x - y - y^2 \\ \dot{\mu} &= 0 \end{aligned}}$$

} note: 'super slow'
('frozen' to all orders)

How is this useful?

→ terms like μx are non-linear ← !

→ we can consider neighbourhood of $\mu = 0$

↳ CMT gives neighbourhood of nonhyperb.

1. Fixed point?

$(x, y, \mu) = (0,0,0)$ ✓

(actually μ can be anything...)

2. $Df(0,0,0) = \begin{pmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [0] \end{pmatrix}$

eigenvalues = eigenvalues of diagonal blocks

→ tr(upper) $= -2$, det(upper) $= 0$

⇒ $\lambda_1 = 0,\ \lambda_2 = -2$

lower ⇒ $\lambda_3 = 0$

Recap so far:

2D x, y system → no reduction

3D x, y, μ system → reduction to 2D

is this useful?

→ yes! one of remaining 2D will just be $\dot{\mu} = 0$ ⇒ trivial

→ other will be slow var

→ fast var can be eliminated.

} really 2D → 1D via 3D → 2D.

Let's see

First we transform coordinates so that our system is 'linearly separated' into 'fast/slow' subsystems

} see lecture 12 for details: → $u = \frac{1}{2}(x+y)$ → $v = \frac{1}{2}(x-y)$

Get:

$$\boxed{\begin{aligned} \dot{u} &= -(u^2+v^2) + \tfrac{1}{2}\mu(u+v) \\ \dot{v} &= -2v - 2uv - \tfrac{1}{2}\mu(u+v) \\ \dot{\mu} &= 0 \end{aligned}}$$

Note: $\mu = 0$ gives same as L12.

Form:

$$\begin{pmatrix}\dot{u}\\ \dot{\mu}\\ \dot{v}\end{pmatrix} = \overset{C}{\underset{F}{\begin{pmatrix} \begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & [-2]\end{pmatrix}}}\begin{pmatrix}u\\ \mu\\ v\end{pmatrix} + \begin{pmatrix}-(u^2+v^2)+\frac{1}{2}\mu(u+v)\\ 0\\ -2uv-\frac{1}{2}\mu(u+v)\end{pmatrix}$$

C : 'centre/slow' block : <u>zero eigenvalues</u>

F : 'fast' block : <u>non-zero</u>.

(note: here <u>stable</u>).

→ in appropriate form for <u>centre manifold reduction</u>

$$\boxed{\begin{array}{l}\text{slow vars are } u, \mu\\ \text{fast var is } v.\end{array}}$$

→ use same approach:

fast = f(slow)

ie $\boxed{v = f(u, \mu)}$ "<u>slaving principle</u>"

→ fast become 'slaved' by slow.

Reduction

Consider the <u>linear</u> part

$$A = \begin{pmatrix}0&0&0\\0&0&0\\0&0&-2\end{pmatrix}$$

E^c.

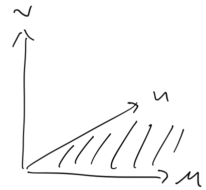
<u>Diagonal</u> → axes are eigenvectors

<u>or</u> explictly $Au = \lambda u$

$$\underline{\lambda = 0} \Rightarrow \begin{pmatrix}0&0&0\\0&0&0\\0&0&-2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\Rightarrow u_3 = 0$, u_1, u_2 <u>free</u>.

choose $\underbrace{u_{(1)} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \; u_{(2)} = \begin{pmatrix}0\\1\\0\end{pmatrix}}_{u,\mu \text{ plane}}$.



$$v = V(u,\mu) = a + bu + c\mu + du^2 + eu\mu + f\mu^2$$

CMT: $V(u,\mu)$ <u>tangent</u> to <u>u,μ plane</u> & goes through (0,0).

$$\Rightarrow \underbrace{V(0,0) = 0, \quad \frac{\partial V}{\partial u}(0,0) = 0, \quad \frac{\partial V}{\partial \mu}(0,0) = 0}_{a=0, \quad b=0, \quad c=0}$$

→

Reduction cont'd.

$$v = V(u, \mu) = du^2 + eu\mu + f\mu^2$$

1. chain rule

$$\dot v = \frac{\partial V}{\partial u}\dot u + \frac{\partial V}{\partial \mu}\overset{\dot\mu = 0}{\cancel{\dot \mu}} = (2du + e\mu)\left(O(\|\cdot\|^2)\right)$$

$$= 0 + O(\|\cdot\|^3)$$

Interlude: $O(\|\cdot\|^3)$?, where $\|\ \|$ → 'size' ie norm of vector/object.

$\cdot$ = whatever

"order size cubed"

eg $x = (x_1, x_2)$ ← vector/multivariate

$O(\|x\|^3)$ includes $\left\{ \begin{array}{l} x_1^3 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ x_2^3 \end{array} \right.$

→.

Reduction

2. subst.

$$\dot v(u, \mu, V(u,\mu)) =$$

$$-2(du^2 + eu\mu + f\mu^2) - \overset{O(\|\cdot\|^3)}{\cancel{2u(\ \ )}}$$

$$-\tfrac{1}{2}\mu u + O(\|\cdot\|^3)$$

① = ②

$$\Rightarrow \quad 0 = -2du^2 - \left(2e + \tfrac{1}{2}\right)\mu u - 2f\mu^2$$

$$\Rightarrow \quad e = -\tfrac{1}{4}, \quad d = 0, \quad f = 0$$

$$\boxed{v = -\tfrac{1}{4}\mu u \quad + \text{ h.o.t.}}$$

& $\boxed{W^c = \left\{ (u, \mu, v) \mid v = -\tfrac{1}{4}\mu u \right\}}$

Dynamics: $\dot u = -\left(u^2 + O(\|\cdot\|^4)\right) + \tfrac{1}{2}\mu u + O(\|\cdot\|^3)$

$$= -u^2 + \tfrac{1}{2}\mu u = \frac{u}{2}(\mu - 2u)$$

& $\dot \mu = 0$. "super slow"

don't forget!

→

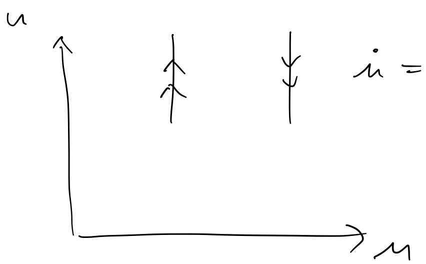
Bifurcation theory

$$\boxed{\begin{aligned} \dot{u} &= \tfrac{\mu}{2}(\mu - 2u) \\ \dot{\mu} &= 0 \end{aligned}} \quad \Bigg\} \text{ centre manifold dynamics}$$

Either – downgrade μ back to 'parameter'
↳ plot FP of u as function of μ

– think of μ as 'super slow' with decoupled dynamics & argue $u \to u_c, \mu = \mu_0$ 'emerges' via time-scale separation. } not linearly separated as in usual CM reduction tho'!

Regardless: effectively a one-dimensional (state) system, depending on one parameter



$\dot{\mu} = 0 \Rightarrow$ vertical flow

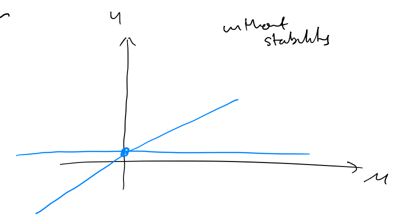
$$\boxed{\dot{u} = f(u;\mu)} \quad \longrightarrow$$

Bifurcation diagram?

FP. $\dot{u} = 0 \Rightarrow u = 0$ or $u = \mu/2$.

always exist → expect non hyperbolic
$\Rightarrow$ expect swap stability
$\Rightarrow$ Transcritical

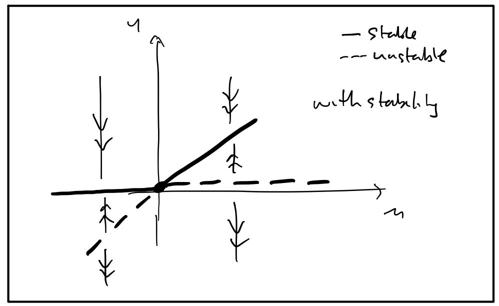
Diagram



Stability

$$Df = \frac{\mu}{2} - 2u = 2\left(\frac{\mu}{4} - u\right) \quad \Bigg| \quad \begin{aligned} &Df(u=0) \Rightarrow Df = \frac{\mu}{2} \begin{cases} <0 \text{ if } \mu<0 \\ >0 \text{ if } \mu>0 \end{cases} \\ &Df\left(u=\frac{\mu}{2}\right) \Rightarrow Df = -\frac{\mu}{2} \ \{ \text{ opposite.} \end{aligned}$$

Final diagram



Bifurcation diagram = extended centre manifold phase portrait!

Singular perturbation theory & quasi-steady states:

Enzyme kinetics: famous example

To the mathematician this hypothesis, known as the pseudo-steady state hypothesis (pssh), is somewhat scandalous. For clearly $dc/dt = 0$ in the strict sense at only one instant, and it is notorious that to say that c is small does not of itself assert anything about the smallness of dc/dt. To the biochemist the pssh is a valued method of simplifying equations, justified by the excellent agreement with experiment that it gives. In fact, it is related to the smallness of the initial ratio of enzyme to substrate concentration through the singular perturbation theory of differential equations. Although this observation is not new (see the

Heineken et al 1967

CMT →
is alternative
& more 'rigorous'

Centre Manifolds:

near non-hyp. (sep. of scales)

'quasi-steady state'

small number of slow vars. fast vars are 'slaved' to these.

The adiabatic-elimination (AE) procedure is used [1] to simplify the equations of motion close to bifurcation points, where the slowing-down of a few variables let the fast ones relax to their instantaneous equilibrium positions. This procedure is particularly successful in those physical systems where some variables undergo fast-relaxation processes. A particular care has to be taken, however, in these cases, first suitably rescaling the variables, and then examining their effective fastness. LUGIATO and co-workers have recently suggested [2] a series of conditions to be fulfilled in order to guarantee the correctness of AE, and showed some examples where it cannot be applied.

In this letter, following the guideline of the centre manifold theory [3], we develop a more general approach, and apply it to the homogeneously broadened resonant laser. The main idea is to look for a global expression of the invariant surface where the slow motion develops, thus overcoming the explicit dependence on the co-ordinates of the standard method. In this way we will be able to handle general cases where no component of the vector field vanishes on the manifold.

(Oppo & Politi 1986)

CMT 'more general'

Also: consider whole manifolds
→ 'geometric singular perturbation theory' (Fenichel 1979, Jones, 1995)

For more, see:

Carr (1981): Applications of Centre Manifold Theory

Roberts (2015): Model Emergent Dynamics in Complex Systems