MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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RECAP COMPLEX FOURIER SERIES

See Lecture 6 supplement (details of derivation from real series, full working for example problem).

LECTURE 7: APPLICATIONS OF FOURIER SERIES

The need for even and odd extensions for non-periodic functions defined on finite/one-sided intervals

Return to separation of variables solution

RETURN TO THE HEAT EQUATION

After solving our heat equation using separation of variables and using our boundary conditions we found *we needed to solve the initial condition constraint*

$$\sum_{n=1}^{\infty} A_n sin(n\pi x) = g(x)$$

on the domain $[0, 1]$. Here we are choosing a sin series *based on the PDE and BC*

If we were *just given $g(x)$* what sort of expansion would we choose? Since it is only *defined on $[0, 1]$ is even/odd/neither/can't say?*

EVEN AND ODD EXTENSIONS

In order to connect a *general function* defined only on $[0, l]$ to our results from Fourier series expansions we need to *choose an appropriate even/odd periodic extension* in two steps

1. Extend our definition from $[0, l]$ to $[-l, l]$
2. Extend our definition from $[-l, l]$ to a periodic function over $\mathbb{R}$

EVEN AND ODD EXTENSIONS

The *odd extension* of f is defined by

$$f_{odd}(x) = \begin{cases} f(x), & x \in [0, l] \\ -f(-x), & x \in [-l, 0) \end{cases}$$

The *even extension* of f is defined by

$$f_{even}(x) = \begin{cases} f(x), & x \in [0, l] \\ f(-x), & x \in [-l, 0) \end{cases}$$

Note the domains of x!

PERIODIC EXTENSION

Step 2 - the *periodic extension* is done in the usual way, but we need to be *careful of end points*. For $f : [-l, l] \to \mathbb{R}$ we define the periodic extension as

$$f_{per}(x + 2nl) = f(x)$$

for all $x \in [-l, l)$ and all $n \in \mathbb{Z}$.

Note the domains of x!

EXAMPLES

Sketch the *odd extension* and *even extension* of the functions

$$f(x) = x, x \in [0, 2]$$

$$g(x) = 1, x \in [0, 1]$$

Then plot the *periodic extension* for both the odd and even extensions of these functions

FOURIER SERIES

We can calculate the Fourier series for the *extended* functions in the usual way.

Since the extensions are an 'artifice' to help construct a Fourier series, can we *re-write our extended Fourier series on our original domain using only our original function definition?*

LET'S TRY!

FOURIER SERIES FOR ODD EXTENSION

We get that the Fourier series of f_{odd}, also called the half-range sine (HRS) expansion of f, is

$$\mathrm{FS}\, f_{odd} = \mathrm{FS}_{HRS}\, f = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n = 1, 2, \ldots$$

Spot the differences!

FOURIER SERIES FOR EVEN EXTENSION

Similarly, the half-range cosine (HRS) expansion of f is

$$\mathrm{FS}\, f_{even} = \mathrm{FS}_{HRC}\, f = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where...

$$\cdots$$

$$a_0 = \frac{1}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \; n = 1, 2, \ldots$$

Spot the differences!

HOW DO WE CHOOSE?

We saw for the heat equation the *PDE + BC motivated a* $\sin$ *series*. We will discuss this more in the next module...

...but also remember for *general functions*

GIBBS PHENOMENON

- Any Fourier series of a function with a *jump discontinuity* will have a persistent 9% (of the jump) *overshoot near the discontinuity* as $N \to \infty$.
- At *fixed* x the Fourier series will converge according to the convergence theorem as N increases, but the *overshoot persists and moves towards the discontinuity*.

and...

CONVERGENCE RATES OF COEFFICIENTS

...

- A piecewise continuous function has Fourier coefficients that decay as $1/n$.
- A continuous function with discontinuous first derivative has Fourier coefficients that decay as $1/n^2$.

In general: a continuous periodic function whose *first k derivatives are all continuous* but whose *$k + 1$ derivative is discontinuous* will have Fourier coefficients that decay at a rate of $1/n^{k+2}$.

CHOOSE SMOOTHNESS IF POSSIBLE!

(note the connection to matching boundary conditions in the PDE case)

COMPLETE SOLUTION OF HEAT EQUATION USING SEPARATION OF VARIABLES AND FOURIER SERIES

FORMULATION

$$u_t = u_{xx}$$

for $0 < x < 1$, where here u is temperature (not energy!)

- $u(0,t) = u_1 \quad$ for $\; 0 < t < \infty$
- $u(1,t) = u_2 \quad$ for $\; 0 < t < \infty$

for known constants u_1, u_2 ($= 10, 50$ say). Note these are *non-homogeneous* BC.

- $u(x,0) = g(x) \quad$ for $\; 0 < x < 1$ and $g(x)$ known ($= 100$, say).

SOLUTION

HOMEWORK

Go over the various exercises from today

Try to summarise the key points from this module

Ask me about any questions you have

Tutorial 2! Assignment 1!