Engsci 721

Problem Set 2:

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

Problem 1

Consider the standard Tikhonov form of the regularised least squares problem

$$\min_x ||Ax - y||^2 + \lambda ||x||^2$$

Suppose you instead wanted to obtain solutions that emphasise 'smoothness' rather than small norm $||x||$. Show how to formulate an optimisation problem to do this. Define and give expressions for any new operators you introduce.

Problem 2

Next suppose that you wanted to obtain solutions that preserve *sharp* transitions or features in a signal or image.

- Formulate an appropriate regularised optimisation problem to do this.
- Briefly explain why your formulation is appropriate for this task.
- Sketch typical solutions that you might obtain from this approach compared to an approach emphasising smooth solutions.

Problem 3

Contrast the problem that must be solved to obtain eigevalues/eigenvectors of a matrix A to that which must be solved to obtain singular values/singular vectors.

Problem 4

Write the collection of solutions to the above problem in matrix form and hence derive the singular value decomposition of a matrix A. Interpret this geometrically.

Write down both the full and reduced forms of the SVD.

Problem 5

Give an explicit form of the generalised inverse of A in terms of its SVD component matrices. Verify that it is a left/right inverse when it should be.

Problem 6

Suppose you have a general linear forward model, A, with four observations (i.e. number of rows $m = 4$) and three parameters (i.e. number of columns $n = 3$). You may assume the columns are linearly independent.

- Write down the general expressions for the *model resolution* operator and the *data resolution* operator, respectively, using matrices appearing in the SVD of A. State which spaces they map between.
- Which of these do you expect to be the identity in this case and which not? Why?
- Give a simple example of U_r, V_r compatible with the basic problem description above (hint: if in doubt, use standard bases!).
- Next, check your answer to the question about which resolution operator is expected to be the identity (up to numerical rounding...) on the case of polynomial regression for projectile motion with observations of the form $\{(t_i, y_i)\}, i = 1, 2, 3, 4$. In particular, suppose that $t_1 = 1, t_2 = 3, t_3 = 5, t_4 = 13$ and hence the forward operator is:

$$A = \begin{bmatrix} 1. & 1. & -0.5 \\ 1. & 3. & -4.5 \\ 1. & 5. & -12.5 \\ 1. & 13. & -84.5 \end{bmatrix}.$$

This corresponds to a model with four observations and three parameters (the coefficients of constant, linear and quadratic terms, respectively). You can use Python/Matlab etc to compute the (reduced) SVD e.g. np.linalg.svd(A, full_matrices=False).

Hint: you should find that the (reduced) matrices of left and right singular vectors are given by:

$$U_r = \begin{bmatrix} -0.00769051 & 0.29107872 & 0.89685522 \\ -0.05693158 & 0.58920415 & 0.1049587 \\ -0.15174676 & 0.73868339 & -0.42646699 \\ -0.98674848 & -0.14986157 & 0.05253846 \end{bmatrix}, \; V_r = \begin{bmatrix} -0.01388544 & 0.35601275 & 0.93437793 \\ -0.15886432 & 0.92181289 & -0.3535861 \\ 0.98720278 & 0.15334901 & -0.04375794 \end{bmatrix}$$

respectively.

Problem 7

What would you expect the singular spectrum of the forward operator appearing in a typical ill-posed problem to look like?

Problem 8

Explain how to use truncated SVD to obtain regularised solutions.

Problem 9

If you've reached this point, you should be in a position to answer all of the 'Check your understanding' questions in the 'Inverse Problems Supplement I' on Canvas. Try them!

Problem 10

Reformulate the least squares normal equations

$$A^T A x = A^T y$$

as a fixed point problem and show how to derive a simple iterative solution scheme. Hint: if you get stuck, use a 'theory of everything' or two.

Problem 11

In reference to the problem above, explain the concept of 'iterative regularisation' (and/or 'semi-convergence') and how you might use it to obtain a 'good' solution to an ill-posed inverse problem. Make sure to describe what you'd expect/want the solutions to look like a) at the very begininning of iteration, b) in the 'middle' phase of iteration, and c) in the very last phase of iteration.

Problem 12

Solve the inverse problems...problems from the 2018 EngSci 741 Exam and the 2021 EngSci 721 Exam (see Canvas).