Engsci 711

Tutorial 3: Bifurcation theory

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Overview

The main purpose of this tutorial is to get some experience carrying out bifurcation analysis of various one- or two-dimensional systems. These include one-dimensional/co-dimension one problems, two-dimensional/co-dimension one problems and one-dimensional/co-dimension two problems. I put in a two-dimensional/co-dimension two problem for fun.

There are also some simple 'centre manifold reduction' problems - recall that the point of centre manifold reduction is to get from some 'full' system to the typical equations that we then analyse using bifurcation theory. Thus it is useful to get a feel for how/why this works, as in the 'real' world you would usually start from a full, more complicated model before identifying the bifurcations.

One-dimensional, co-dimension one problems

Problem 1

Draw the bifurcation diagrams of the following equations

$$
\begin{aligned}
(a) \quad & \dot{x} = \mu - x^2 \\
(b) \quad & \dot{x} = \mu x - x^2 \\
(c) \quad & \dot{x} = \mu x - x^3
\end{aligned}
$$

- What type of bifurcations is each of the above?

Problem 2

Find bifurcations in the following systems and sketch the bifurcation diagrams

$$
\begin{aligned}
(a) \quad & \dot{x} = \mu + x - \ln(1+x) \\
(b) \quad & \dot{x} = \mu x - \ln(1+x) \\
(c) \quad & \dot{x} = x(\mu - e^x) \\
(d) \quad & \dot{x} = \mu x - \sinh(x) \\
(e) \quad & \dot{x} = x + \frac{\mu x}{1+x^2}
\end{aligned}
$$

What kind of bifurcations do each correspond to?

Two-dimensional, co-dimension one (Hopf bifurcation) problems

Problem 1

Consider the biased van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

- Find the curves in (μ, a) space at which you expect a Hopf bifurcation to occur.

Problem 2

Consider the system

$$
\begin{aligned}
\dot{x} &= -y + \mu x + xy^2 \\
\dot{y} &= x + \mu y - x^2
\end{aligned}
$$

- For what parameter values do you expect a Hopf bifurcation to occur at the origin?
- Use XPPAut to explore the neighbourhood this bifurcation and determine whether the Hopf bifurcation is subcritical, supercritical or degenerate (periodic solution appears at bifurcation but disappears for all other parameter values).

Problem 3

Carry out the same steps as in the previous problem for the systems

$$
\begin{aligned}
\dot{x} &= y + \mu x \\
\dot{y} &= -x + \mu y - x^2 y
\end{aligned}
$$

and (Predator-prey model)

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a)$$

where in the latter case $x, y, a \geq 0$ and x, y represent, respectively, the prey population, predatory population

- In this latter case, given an interpretation to the terms in the model, e.g. reproduction, 'predation' (predator eating prey) etc. What do you think the a parameter represents?

Problem 4

(More difficult?). Carry out the same steps as in Problem 2 and the end of Problem 3 for the predator-prey model

$$\dot{x} = x(b - x - \frac{y}{1+x})$$
$$\dot{y} = y(\frac{x}{1+x} - ay)$$

One-dimensional, co-dimension two problems

Problem 1

Consider the following 'imperfect' bifurcation equations - i.e. standard bifurcation equations with 'external control' parameter μ plus an additional (uncontrolled, small) 'imperfection parameter' δ.

$$(a) \quad \dot{x} = \mu - x^2 + \delta$$
$$(b) \quad \dot{x} = \mu x - x^2 + \delta$$
$$(c) \quad \dot{x} = \mu x - x^3 + \delta$$

- Which bifurcations do these correspond to for $\delta = 0$.
- For each of $\delta >, =, < 0$, plot the usual bifurcation diagram vs μ.
- Which of the $\delta = 0$ bifurcation diagrams are 'structurally stable' (preserved) under small variations of δ?
- Summarise the bifurcation behaviour of each by drawing a picture in the (μ, δ) plane indicating lines/curves separating regions with different qualitative phase-space properties. Indicate in each region of the parameter space what the behaviour in the phase-space looks like (e.g. number of fixed points, stability etc).

Two-dimensional, co-dimension two problems

For the brave. The Bogdanov-Takens bifurcation. See e.g. the scholarpedia article by Guckenheimer and Kuznetsov at http://www.scholarpedia.org/article/Bogdanov-Takens_bifurcation or Kuznetsov's book (Elements of Applied Bifurcation Theory -see reading list on Canvas).

Consider the system

$$\dot{x} = \lambda - \mu x + y^2 + xy$$
$$\dot{y} = x$$

- See if you can find a saddle-node bifurcation
- See if you can find a (possible) Hopf-bifurcation
- Draw a bifurcation picture in the two-dimensional parameter space (λ, μ). That is, try to find where in the (λ, μ) plane (i.e. curves/lines) there are bifurcations and indicate in each parameter region what the corresponding typical behaviour in the phase-plane looks like.

Centre manifold reduction problems

Problem 1

Consider the example from the lecture (Kuznetsov, 2004, Example 5.1)

$$\dot{x} = xy + x^3$$
$$\dot{y} = -y - 2x^2$$

- Confirm the origin $(0, 0)$ is an equilibrium point
- Find the Jacobian derivative Df and evaluate it at the origin
- Find the eigenvalues and associated linear subspaces E^s, E^u, E^c.
- Is the system expressed relative to the eigenbasis? That is, are the eigen-directions parallel to the x and y axis?
- If they are, proceed. If they aren't, define a linear transformation converting x, y to new variables u, v such that u, v are coordinates in the eigenbasis (hint: here we are fine; for other cases, see the appendix below).
- Assume that (relative to the eigendirections) the centre manifold W^c_{loc} can be expressed as a curve expressing the non-centre coordinate(s) in terms of the centre coordinate(s), $y = V(x)$.

- Assume this has a Taylor series expansion; what can you say about the zeroth and first-order terms?

- Use the same process as outlined in the previous tutorial and the lectures to determine the terms of the Taylor series for the centre manifold.

- Use this expression for W^c_{loc} and the Reduction Principle from the lectures to determine the (approximate) dynamics on the centre manifold (i.e. substitute your expression for $y(x)$ into the $\dot{x}$ equation!).

- Are these dynamics stable or unstable? Are the dynamics of the full system stable or unstable? Sketch the local dynamics near the origin.

- What are the relative rates of the dynamics on the centre and the stable/unstable manifolds near the origin (look at the equations you have!).

- Instead of substituting W^c_{loc} into the $\dot{x}$ equation, try substituting the equation defining E^c into the $\dot{x}$ equation. What do you notice?

Problem 2

Carry out the same process for

$$\begin{aligned}\dot{x} &= -2x + y - x^2 \\ \dot{y} &= x(y - x)\end{aligned}$$

Problem 3

Carry out the same process for

$$\begin{aligned}\dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2\end{aligned}$$

Note that for this problem it is (probably) easiest to transform variables to be relative to the eigendirections (after you find them), though it is possible to do the analysis without transforming. See the appendix below for how to transform (to so-called Jordan canonical/normal form, known from linear algebra)

Appendix

Transformation of variables

From basic linear algebra.

Suppose you have a vector x whose components are (x_1, x_2) in a given basis. Here we will assume the basis is given by $\{(1,0),(0,1)\}$ i.e. the standard basis for $\mathbb{R}^2$.

Then, given a new desired basis (coordinate axes) $\{a, b\}$ with coordinates specified relative to the original basis (standard $\mathbb{R}^2$ basis) (a_1, a_2) and (b_1, b_2), for a and b respectively, we can find the coordinates of x relative to the new coordinate axes, u_1, u_2, using

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e. the columns of the transformation matrix are the vectors of the desired new coordinate system (as expressed relative to the original basis)

So, to change to coordinates u_1, u_2 relative to an eigenbasis with eigenvectors e^s and e^u say, use

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e_1^s & e_1^u \\ e_2^s & e_2^u \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where e_1^s, e_2^s are the components of e^s relative to the original coordinate system (etc).

This gives two equations to solve for x_1, x_2 in terms of u_1, u_2 - we can then e.g. derive a new differential equation in u variables by replacing the x variables (in the differential equations for x) by u variables.

Easiest way to check you understand this is to try it! See Problem 3 of the Centre Manifold Reduction set.