Engsci 711 L3 – Examples

stability of solutions – general

See • [Strogatz 1994] end of section 5.1 & Exercise 5.1.10

• [Glendinning 1994] section 2.1 esp. Fig 2.1

Stability

Lyapunov (or Liapounov):

• "points that start nearby stay nearby"

$B(x,\delta)$ — Ball of all points y st. $|x-y| < \delta$

$\Longrightarrow$

$B(x,\epsilon)$ — Ball of all points within ϵ of x

$\varphi(y,t)$ — flow from $y \in B(x,\delta)$

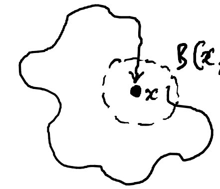
$|\varphi(y,t) - \varphi(x,t)| < \epsilon$ — traj. starting in $B(x,\delta)$ stay in $B(x,\epsilon)$

"Give me a target tolerance on trajectories ϵ & I'll give you a tolerance δ on starting points"

Stability

Quasi-asymptotic

"start nearby, eventually return (arb. closely) to"



$B(x,\delta) = \{ y \mid |x-y| < \delta \}$

if $|x-y| < \delta$ ($y \in B(x,\delta)$)

then $\underbrace{\varphi(y,t)}_{\text{flow from } y} \xrightarrow{t\to\infty} \underbrace{\varphi(x,t)}_{\text{flow from } x} = x$ for fixed point

(general def. can be used for more complicated objects than FP)

Asymptotic:

• "start nearby, stay nearby & return to"

$\boxed{\text{Lyapunov \& Quasi}}$

Stability of linear systems

→ find eigenvalues λ

→ check real part $\mathrm{Re}(\lambda)$

(complex part relates to oscillations)

Eg

- $\lambda_1 = 1$, $\lambda_2 = -2$ ⇒ unstable
- $\lambda_1 = -1$, $\lambda_2 = -2$ ⇒ stable
- $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$ ⇒ stable

[$\lambda = 0$ ⇒ marginal stability ⇒ need to analyse more carefully!

(we'll do lots more examples)

Linearisation of nonlinear systems

→ see Lecture 1 handout for linearisation procedure

Q: is stability of linearised system representative of stability of nonlinear system?

A: Yes, if the linear system is definitely stable/unstable
⇒ ie no marginal $\mathrm{Re}(\lambda) = 0$ cases

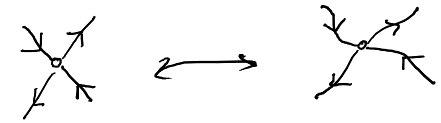
} Hartman-Grobman theorem

→ call cases where all $\mathrm{Re}(\lambda) \neq 0$ hyperbolic

→ if exists even one $\mathrm{Re}(\lambda) = 0$ then non-hyperbolic

"Topological equivalence"?

[see 'normal form' theory → nonlinear coord. transf.]



"smooth deformation"

→ just a stretched/deformed version of same thing

↳ same number of fixed points

↳ fixed points have same stability

etc.

Example analysis

- see L1 handout.
- also:

$$\begin{aligned} \dot{x} &= x(y-1) = f_1(x_1, x_2) \quad (1) \\ \dot{y} &= 3x - 2y + x^2 - 2y^2 = f_2(x_1, x_2) \quad (2) \end{aligned}$$

(x = x_1, y = x_2)

⇒ Roots? (Fixed points)

(1) = 0 →

- $x = 0$ & $-2y - 2y^2 = 0$ ⇒ $-2y(1+y) = 0$ ((2)=0)
  ⇒ $y = 0$ or $y = -1$
- $y = 1$ & $3x - 2 + x^2 - 2 = 0$ ((2)=0)
  $(x-1)(x+4) = 0$
  ⇒ $x = 1$ or $x = -4$

$$\boxed{\text{So: } \{(0,0), (0,-1), (1,1), (-4,1)\}}$$

cont'd

→ Linearisation

$$Df_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} y-1 , & x \\ 3+2x , & -2-4y \end{pmatrix}$$

at $(0,0)$ FP:

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}.$$

eigenvalues?

$\det(A-\lambda I) = 0$ (long way....)

$$\begin{vmatrix} -1-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(-2-\lambda) - 3\times 0 = 0$$

$$(1+\lambda)(2+\lambda) = 0$$

$$\lambda_1 = -1 , \lambda_2 = -2$$

$Re(\lambda_i) < 0$ for $i = 1,2$

so stable

Bifurcation teaser

consider the two systems

topologically equiv

① $\dot{x} = -x$

&

② $\dot{x} = -x^3$

both have FP $x=0$

both FP can be shown to be stable.

solutions start near, return

BUT

topologically different

①′ $\dot{x} = -x + \epsilon x = -x(1-\epsilon)$

②′ $\dot{x} = -x^3 + \epsilon x = -x(x^2-\epsilon)$

1 sol.

3 sol.

different number of FP

topological change under small change in system

small change to system: $f(x) \to f(x) + \epsilon \cdot g(x)$

Q: is ① hyperbolic?
is ② hyperbolic?