ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds. Bifurcation diagrams.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

LECTURE 4

Nonlinear systems - analysing local dynamics near hyperbolic fixed points

- Linearisation and hyperbolic fixed points again
- Geometry: stable/unstable manifolds and comparison to linear case
- Perturbation expansions for nonlinear stable/unstable manifolds

RECALL: LINEARISATION

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component gives

$$\dot{u}_i = f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2)$$

i.e.

$$\dot{u}_i = \frac{\partial f_i}{\partial x_j}(x_e)u_i \equiv [Df(x_e)]_{ij}u_j$$

or simply $u = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

EXAMPLE

We previously considered (in class and the tutorial) Example 2.8 from Glendinning.

Quick recap?

NOTE

Note that, in the x, y plane, the *nullclines* are the curves defined by either $\dot{x} = 0$ or $\dot{y} = 0$, i.e. where the flow is either *purely vertical or purely horizontal*.

This can help with *sketching, finding closed orbits and finding fixed points* (note fixed points are given by the intersection of the nullclines).

RECALL: HARTMAN-GROBMAN

We have previously considered how the *existence and stability of hyperbolic fixed points* are preserved during linearisation.

We now want consider the *differences in local dynamics between a nonlinear system and its linearisation in more detail*.

We'll look at how to do this using the *stable manifold theorem* and then using series expansions to approximate local stable/unstable manifolds.

GEOMETRY: STABLE AND UNSTABLE MANIFOLDS

We previously defined the *stable and unstable manifolds for linear systems*. (For non-hyperbolic there is also a centre manifold)

Now we want to give the definitions for *nonlinear hyperbolic fixed points*. First, recall the linear case...

RECALL - STABLE MANIFOLD IN LINEAR SYSTEMS

Suppose $x = 0 \in \mathbb{R}^n$ is a stationary solution to the linear system $\dot{x} = Ax$.

The *stable/unstable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^s(0)/E^u(0)$ and is the *span of the eigenvectors/generalised eigenvectors* corresponding to the eigenvalues of A with *real, negative/positive part*.

i.e. in 2D systems it will typically be either a line or plane.

SIDE NOTE: WHAT'S A MANIFOLD?

For our purposes it suffices to think of a *manifold* embedded in $\mathbb{R}^n$ as a subset of $\mathbb{R}^n$, each point of which satisfies m constraints.

This means that, given regularity conditions, a manifold is an $(n - m)$-dimensional object embedded in n dimensional space.

Example: a circle is a 1-dimensional manifold which is embedded in $\mathbb{R}^2$ and satisfies one constraint. So, the unit circle can be thought of as e.g. $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$.

SIDE NOTE: WHAT'S A MANIFOLD?

A key property is that locally an $(n-m)$-dimensional manifold embedded in n-dimensional space *'looks like' a small 'open ball' of dimension $\mathbb{R}^{(n-m)}$*.

E.g. a curve embedded in $\mathbb{R}^2$ can be considered as 'pieced together' from small segments of $\mathbb{R}$, a sphere in $\mathbb{R}^3$ can be considered as 'pieced together' from small 'patches' of $\mathbb{R}^2$ etc.

The more general definition throws away the 'background' space and works with the 'intrinsic' $(n-m)$-dimensional object itself.

GEOMETRY: STABLE MANIFOLD (LOCAL)

Given some neighbourhood U of a stationary point x, the *local stable manifold* on U for a nonlinear system $W^s_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to \infty, \phi(y,t) \in U \text{ for all } t \geq 0\}$$

Picture?

GEOMETRY: UNSTABLE MANIFOLD (LOCAL)

Similarly, given some neighbourhood U of a stationary point x, the *local unstable manifold* on U for a nonlinear system $W^u_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to -\infty, \phi(y,t) \in U \text{ for all } t \leq 0\}$$

Picture?

GLOBAL MANIFOLDS

Note that if we want *global* versions then we can *'glue' together* all the flows starting at points in the local stable/unstable manifolds. That is,

$$W^s(0) = \bigcup_{t \geq 0} \phi(W^s_{loc}(0), t)$$

$$W^u(0) = \bigcup_{t \leq 0} \phi(W^u_{loc}(0), t)$$

STABLE MANIFOLD THEOREM

What's the *connection between these linear and nonlinear stable/unstable manifolds?* We have the following theorem (for local manifolds).

Suppose the origin is a *hyperbolic fixed point* for $\dot{x} = f(x)$ in $\mathbb{R}^n$ and that $E^s(0)$ and $E^u(0)$ are the stable and unstable manifolds of the linearised system $\dot{x} = Df(0)x$.

Then...

STABLE MANIFOLD THEOREM

…there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the same dimension as $E^s(0)$ and $E^u(0)$, respectively, and which are (respectively) tangent to E^s and E^u at the origin.

These manifolds are equally smooth/unsmooth as the original function f.

STABLE MANIFOLD THEOREM

Picture?

POWER SERIES EXPANSIONS IN TWO-DIMENSIONS

Even on small neighbourhoods U of our fixed points, *our manifolds are no-longer straight lines* (or hyperplanes etc in higher dims) as in the linear case - they are *curves* (or surfaces in higher-dimensions).

We can, however, use the information from the previous theorem to (try to) *compute local expressions for these curves*.

POWER SERIES EXPANSIONS FOR ONE-DIMENSIONAL MANIFOLDS

Assume a stable/unstable manifold of interest can be described by a curve $x = g(y)$ (or $y = h(x)$).

We can try to approximate this by a *local series expansion* of the form

$$g(y) = \sum_{n=0}^{\infty} a_n y^n$$

EXAMPLES

Example 4.2 from Glendinning.

Tutorial sheet/assignment coming soon!