ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds. Bifurcation diagrams.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

LECTURE 3

Linear systems and classification

- Stability and classification of linear systems
- Two-dimensional (phase *plane*) linear systems in more detail
- Geometry of linear systems

RECALL - LINEAR STABILITY

Stability is *easy for linear systems*.

Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

RECALL - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues (of the linearisation) have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

TWO-DIMENSIONAL LINEAR SYSTEMS

Let's spend some time considering general *two-dimensional linear systems* in more detail:

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^2, A \in \mathbb{R}^{2\times 2}$, i.e.

$$\dot{x}_1 = ax_1 + bx_2$$
$$\dot{x}_2 = cx_1 + dx_2$$

and where a, b, c, d are parameters.

DIRECT SOLUTION

For *linear systems* we know to try solutions of the form

$$x(t) = e^{\lambda t} u$$

where u is a constant vector, here living in $\mathbb{R}^2$.

Substituting this into our equation gives the *eigenvalue problem*

$$Au = \lambda u$$

CLASSIFICATION

We know that for *non-trivial solutions* to our eigenvalue problem we need

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

Using $\operatorname{tr} A = a + d$ and $\det A = ad - bc$ we can write this as

$$\boxed{\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0}$$

CLASSIFICATION

This gives solutions

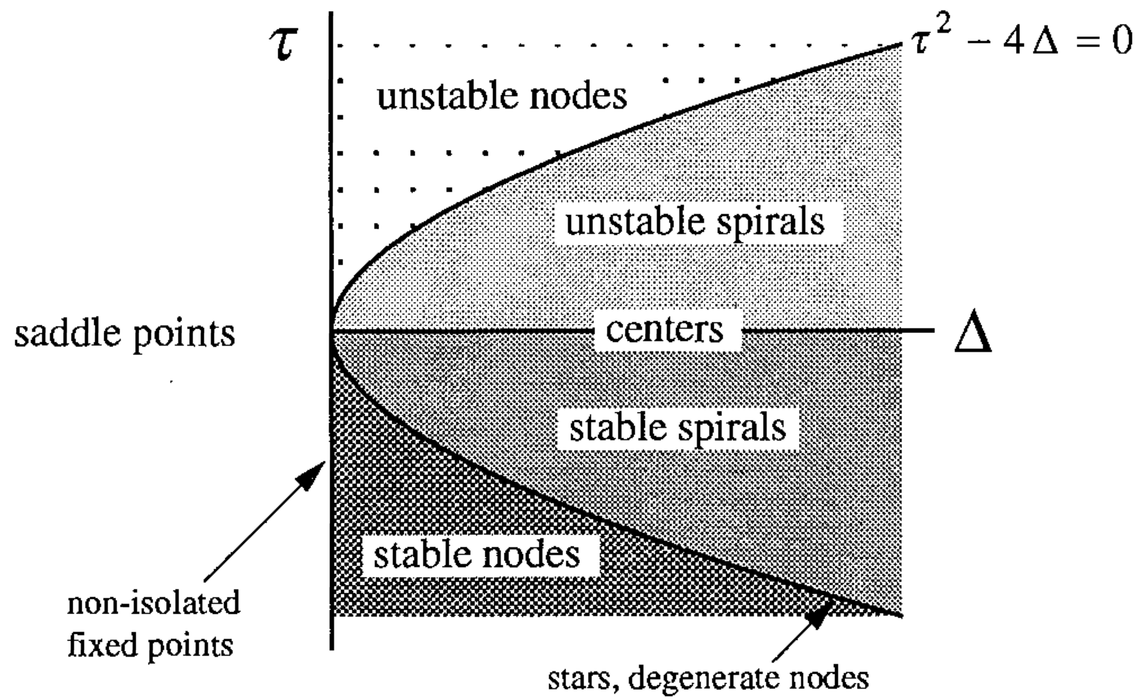
$$\lambda_1, \lambda_2 = \frac{1}{2}\left(\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4\det A}\right)$$

The solutions are either *a) real and distinct, b) real and equal, c) complex conjugate with non-zero real part or d) purely imaginary.*

Stability is simply determined by the *real part* of the eigenvalues.

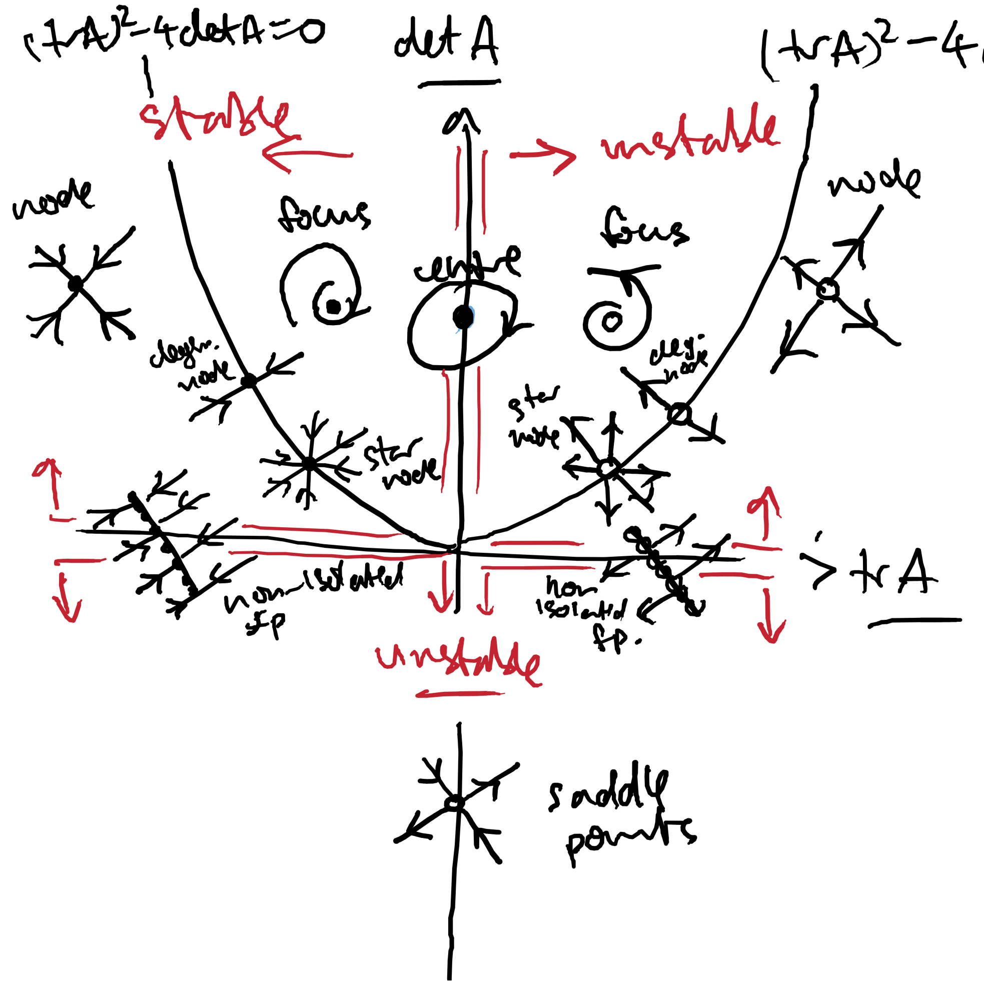
A diagram to help us organise things...

CLASSIFICATION DIAGRAM



(Strogatz 1994, 5.2.8)

CLASSIFICATION DIAGRAM



[Based (badly) on Drazin 1992, 6.4]

CLASSIFICATION NOTES

Typical/robust: nodes, spirals/foci, saddles

Borderline: stars, improper/degenerate nodes, non-isolated fixed points, centres

In particular: *centres may easily change stability* under perturbations of the model - i.e. they are *structurally unstable*.

DETAILED EXAMPLES

Consider (see Strogatz 1994, example 5.2.1)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x - 2y \end{aligned}$$

DETAILED EXAMPLES

Consider (see Strogatz 1994, example 5.3.1 - cautious lovers)

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

i.e.

$$\dot{R} = aR + bJ$$
$$\dot{R} = bR + aJ$$

DETAILED EXAMPLES

Other typical phase portraits

(Strogatz 1994, examples 5.2.3,5.2.4,5.2.5)

Homework: Access an edition of Strogatz and have a read of the chapter on linear systems (in particular look at the examples)

GEOMETRY OF LINEAR SYSTEMS

Let's return to general linear systems in $\mathbb{R}^n$ for a moment to give the following geometric definitions of three key *invariant manifolds/subspaces* for linear systems.

The *flow in the full phase space* is then given by a *linear supposition of motion on these three subspaces*.

We will also see later that these subspaces *also have nonlinear counterparts* (but we will need to consider the flow more carefully since superposition fails)

GEOMETRY OF LINEAR SYSTEMS - STABLE MANIFOLD

Suppose $x \in \mathbb{R}^n$ is a stationary solution to the linear system

$$\dot{x} = Ax.$$

The *stable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^s(0)$ and is the span of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, negative part*.

GEOMETRY OF LINEAR SYSTEMS - UNSTABLE MANIFOLD

Similarly:

The *unstable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^u(0)$ and is the span of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, positive part*.

GEOMETRY OF LINEAR SYSTEMS

Finally:

The *centre manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^c(0)$ and is the span of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *zero real part*.