MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

Oliver Maclaren

oliver.maclaren@auckland.ac.nz

LECTURE 14

Laplace's equation

- Relation to e.g. (non-trivial) steady-state heat conduction problems
- Solving for rectangular domains using separation of variables
- Handling multiple non-homogeneous BC
- Solving for box domains (three spatial dimensions)

LAPLACE'S EQUATION

$$\nabla^2 u = 0$$

e.g.

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

Represents *equillibrium problems*. These are *pure PDE boundary-value problems* - no initial conditions (i.e. not an IBVP)

RECALL: HEAT EQUATION IN ONE SPACE DIMENSION

We have looked at the one spatial dimension heat equation in quite some detail

$$\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

HEAT EQUATION IN ONE, TWO OR THREE SPACE DIMENSIONS

In arbitrary spatial dimensions we have

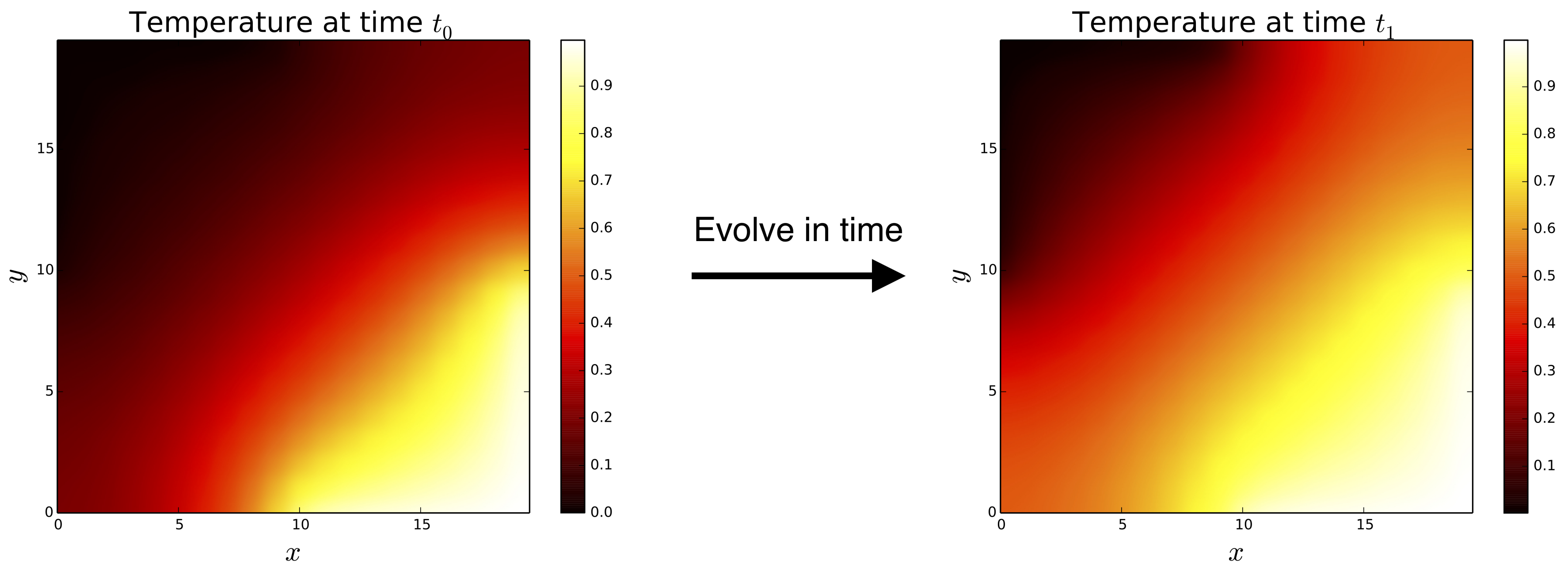
$$\frac{\partial u}{\partial t} - D\nabla^2 u = 0$$

Note: the *steady-state solutions of the heat equation* in ≥ 2 spatial dimensions give the *Laplace equation*. There are (lots) of other ways that Laplace's equation arises, however! (E.g. sum of forces = 0.)

RECALL: EXAMPLE SIMULATIONS IN ONE SPACE DIMENSION

See lecture 7 MuPad notebook. IC evolves (e.g. relaxes/decays) to the steady state.

RECALL: EXAMPLE SIMULATIONS IN TWO SPACE DIMENSIONS



Spatiotemporal temperature field $u(x, y, t)$. Here: want to find the steady state.

EXAMPLE: STEADY-STATE HEAT CONDUCTION IN TWO SPATIAL DIMENSIONS

Example: Steady state of heat conduction in a rectangular plate with non-zero temperature distribution on the boundary.

$$
\begin{aligned}
\text{PDE} \quad & u_{xx} + u_{yy} = 0, \\
\text{BC} \quad & u(0, y) = 0, \qquad u(a, y) = 0, \\
& u(x, 0) = 0, \qquad u(x, b) = f(x)
\end{aligned}
$$

EXAMPLE: STEADY-STATE HEAT CONDUCTION IN TWO SPATIAL DIMENSIONS

Note that *here we want at least one non-homogeneous BC* - want *non-trivial steady-state!* We can (very roughly!) imagine this as playing a similar role (i.e. giving non-zero solutions) of our IC in IBVPs. Illustration.

EXAMPLE: MULTIPLE NON-HOMOGENEOUS BOUNDARY CONDITIONS

What if we need to solve problems such as

$$
\begin{array}{rll}
\text{PDE} & u_{xx} + u_{yy} = 0, & \\
\text{BC} & u(0, y) = f_1(y), & u(a, y) = f_2(y), \\
 & u(x, 0) = f_3(x), & u(x, b) = f_4(x)
\end{array}
$$

The basic strategy is to *split the problem into a sum* (yay linearity!) of problems with *each having 'appropriate' (easy) BCs*.

DECOMPOSITION FOR MULTIPLE NON-HOMOGENEOUS BOUNDARY CONDITIONS

$$\begin{array}{rll} \text{PDE} & u_{xx} + u_{yy} = 0, & \\ \text{BC} & u(0, y) = f_1(y), & u(a, y) = f_2(y), \\ & u(x, 0) = f_3(x), & u(x, b) = f_4(x) \end{array}$$

EXAMPLE: LAPLACE'S EQUATION IN THREE-SPATIAL DIMENSIONS

Can we use separation of variables in *three dimensions*? Yup! Let's outline the process.

$$\begin{array}{rll} \text{PDE} & u_{xx} + u_{yy} + u_{zz} = 0, & \\ \text{BC} & u(0, y, z) = 0, & u(a, y, z) = 0, \\ & u(x, 0, z) = 0, & u(x, b, z) = 0, \\ & u(x, y, 0) = 0, & u(x, y, c) = f(x, y), \end{array}$$

HOMEWORK

Assignment 2!

Keep preparing for the test!