Engsci 711

Tutorial 3: Bifurcation theory

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Overview

The main purpose of this tutorial is to get some experience carrying out bifurcation analysis of various one- or two-dimensional systems. These include one-dimensional/co-dimension one problems and two-dimensional/co-dimension one problems.

I put in some one-dimensional/co-dimension two problems and a two-dimensional/co-dimension two problem for fun. *These are not examinable: I only expect you to be able to handle co-dimension one problems in the exam.*

One of the key tricks in more difficult problems is, when in doubt

- Find some *convenient points* on fixed point branches (points on branches of fixed points where you can easily evaluate their stability)

- Use the fact that branches of fixed points *only change stability at bifurcations*, so the *whole branch* (between bifurcation points) has the same stability as your convenient point.

This trick of using 'continuity of stability' is related to the numerical strategy of *parameter continuation* in bifurcation analysis.

One-dimensional, co-dimension one problems

Problem 1

Draw the bifurcation diagrams of the following equations

$$\begin{aligned} (a) \quad & \dot{x} = \mu - x^2 \\ (b) \quad & \dot{x} = \mu x - x^2 \\ (c) \quad & \dot{x} = \mu x - x^3 \end{aligned}$$

What types of bifurcations occur in each of the above?

Problem 2

Find all the bifurcations and sketch the bifurcation diagrams for the following systems

$$
\begin{aligned}
(a) \quad & \dot{x} = \mu + x - \ln(1+x) \\
(b) \quad & \dot{x} = \mu x - \ln(1+x) \\
(c) \quad & \dot{x} = x(\mu - e^x) \\
(d) \quad & \dot{x} = \mu x - \sinh(x) \\
(e) \quad & \dot{x} = x + \frac{\mu x}{1+x^2}
\end{aligned}
$$

What kind of bifurcations do each correspond to?

Problem 3 (Exam 2016)

Determine the equilibria and their stability, and hence find all bifurcations, in the following system

$$\dot{x} = (\lambda - b)x - ax^3$$

where $x \in \mathbb{R}$, $a, b > 0$ are fixed positive parameters and $\lambda \in \mathbb{R}$ is a controllable parameter. Sketch the bifurcation diagram.

(Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$, and $\lambda > b$ separately.)

Problem 4 (Multiple co-dimension-one bifurcations in one system)

Find all bifurcations and sketch the bifurcation diagram for the following system

$$\dot{x} = (x-1)(x^2 + 2ax - \mu)$$

where $x \in \mathbb{R}$, $a > 0$ is a fixed positive parameter and $\mu \in \mathbb{R}$ is a controllable parameter.

Two-dimensional, co-dimension one (Hopf bifurcation) problems

Problem 1

Consider the system

$$
\begin{aligned}
\dot{x} &= -y + \mu x + xy^2 \\
\dot{y} &= x + \mu y - x^2
\end{aligned}
$$

- For what parameter values do you expect a Hopf bifurcation to occur at the origin?

- Check the stability and type of the fixed point at the origin on either side of the bifurcation.

- Use XPPAut to explore the neighbourhood this bifurcation and determine whether the Hopf bifurcation is subcritical, supercritical or degenerate (periodic solution appears at bifurcation but disappears for all other parameter values).

Problem 2

Consider the biased van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

- Find the curves in (μ, a) space at which you expect a Hopf bifurcation to occur.

Problem 3

Carry out the same steps as in the previous problem for the systems

$$\begin{aligned}
\dot{x} &= y + \mu x \\
\dot{y} &= -x + \mu y - x^2 y
\end{aligned}$$

and (Predator-prey model)

$$\begin{aligned}
\dot{x} &= x(x(1-x) - y) \\
\dot{y} &= y(x - a)
\end{aligned}$$

where in the latter case $x, y, a \geq 0$ and x, y represent, respectively, the prey population, predatory population

- In this latter case, given an interpretation to the terms in the model, e.g. reproduction, 'predation' (predator eating prey) etc. What do you think the a parameter represents?

Problem 4

(More difficult?). Carry out the same steps as in Problem 2 and the end of Problem 3 for the predator-prey model

$$\begin{aligned}
\dot{x} &= x(b - x - \frac{y}{1+x}) \\
\dot{y} &= y(\frac{x}{1+x} - ay)
\end{aligned}$$

One-dimensional, co-dimension two problems

Problem 1

Consider the following 'imperfect' bifurcation equations - i.e. standard bifurcation equations with 'external control' parameter μ plus an additional (uncontrolled, small) 'imperfection parameter' δ.

$$
\begin{aligned}
(a) &\quad \dot{x} = \mu - x^2 + \delta \\
(b) &\quad \dot{x} = \mu x - x^2 + \delta \\
(c) &\quad \dot{x} = \mu x - x^3 + \delta
\end{aligned}
$$

- Which bifurcations do these correspond to for $\delta = 0$.
- For each of $\delta >, =, < 0$, plot the usual bifurcation diagram vs μ.
- Which of the $\delta = 0$ bifurcation diagrams are 'structurally stable' (preserved) under small variations of δ?
- Summarise the bifurcation behaviour of each by drawing a picture in the (μ, δ) plane indicating lines/curves separating regions with different qualitative phase-space properties. Indicate in each region of the parameter space what the behaviour in the phase-space looks like (e.g. number of fixed points, stability etc).

Two-dimensional, co-dimension two problems

For the brave. The Bogdanov-Takens bifurcation. See e.g. the scholarpedia article by Guckenheimer and Kuznetsov at http://www.scholarpedia.org/article/Bogdanov-Takens_bifurcation or Kuznetsov's book (Elements of Applied Bifurcation Theory -see reading list on Canvas).

Consider the system

$$
\begin{aligned}
\dot{x} &= \lambda - \mu x + y^2 + xy \\
\dot{y} &= x
\end{aligned}
$$

- See if you can find a saddle-node bifurcation
- See if you can find a (possible) Hopf-bifurcation
- Draw a bifurcation picture in the two-dimensional parameter space (λ, μ). That is, try to find where in the (λ, μ) plane (i.e. curves/lines) there are bifurcations and indicate in each parameter region what the corresponding typical behaviour in the phase-plane looks like.