MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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REMINDER: DEFINITION OF CLASSICAL/TRIGONOMETRIC FOURIER SERIES

Given a function $f : [-l, l] \rightarrow \mathbb{R}$, the trigonometric (classical Fourier) series* is defined as

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

where ...

$*$: We will see that a *Generalised Fourier series* includes expansions in other orthogonal sets of functions besides trigonometric functions.

REMINDER: DEFINITION OF CLASSICAL/TRIGONOMETRIC FOURIER SERIES

$$a_0 := \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$a_n := \frac{1}{l}\int_{-l}^{l} f(x)cos(\frac{n\pi x}{l})dx$$

$$b_n := \frac{1}{l}\int_{-l}^{l} f(x)sin(\frac{n\pi x}{l})dx$$

$n = 1, 2, \ldots$ are called the *Fourier coefficients*.

REMINDER: CONVERGENCE THEOREM

Let f be a periodic function with fundamental period $2l$ such that f and f' are both piecewise continuous (i.e. f is piecewise smooth) on $[-l, l]$.

Then the Fourier series FS f of f converges to

- $f(x)$ at each point x at which f is continuous, and to
- the mean value $(f(x^+) + f(x^-))/2$ at every point x at which f is discontinuous, where $f(x^+)$ and $f(x^-)$ are the right- and left-hand (i.e. one-sided) limits, respectively.

REMINDER: PROPERTIES OF EVEN AND ODD FUNCTIONS

even + even = even

odd + odd = odd

odd x odd = even

even x even = even

even x odd = odd

REMINDER: PROPERTIES OF EVEN AND ODD FUNCTIONS

$$\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)$$

if f is *even*

and

$$\int_{-a}^{a} f(x) = 0$$

if f is *odd*

LECTURE 5

Computing and sketching Fourier series

Convergence of Fourier series for continuous functions

Gibbs phenomenon and convergence near discontinuities

SOME PERIODIC PROPERTIES OF TRIGONOMETRIC EXPANSIONS

Consider FS f =

$$a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

Note that each term is periodic and (in general) the *smallest common period* shared by terms in the series is $2l$. This expression is defined over all $\mathbb{R}$ even if f is not.

The functions $\{1, cos(\frac{n\pi x}{l}), sin(\frac{n\pi x}{l})\}, n = 1, 2...$ are orthogonal over $[-l, l]$.

ORTHOGONALITY OF $\sin$ REVISITED

Recall:

$$\int_0^1 sin(m\pi x)sin(n\pi x)dx = \begin{cases} 0, & \text{if } m \neq n \\ 1/2, & \text{if } m = n \end{cases}$$

But from the previous slide shouldn't these be orthogonal over $[-l, l] = [-1, 1]$?

A: Yes but use 'odd x odd = even' to consider $[0, 1]$. We will come back to different domains later but for now will focus on $[-l, l]$ as in the previous slide.

LET'S DO SOME PROPER EXAMPLES!

EXAMPLE 1

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} -x, & \text{on } (-\pi, 0] \\ x, & \text{on } (0, \pi] \end{cases}$$

and

$$f(x + 2\pi) = f(x)$$

for all x, i.e., f is periodic with period 2π.

Let's...

EXAMPLE 1

...

- Sketch the *graph of f* and sketch the function to which the Fourier series FS f *converges*.
- Calculate the *Fourier series of f* and plot the *sum of the first few terms* (partial sum).

SKETCH

CALCULATION

PLOT OF PARTIAL SUMS USING MUPAD

CONVERGENCE RATE FOR CONTINUOUS FUNCTIONS

Note that *convergence is very fast* in this example - there is almost no visible difference between the graphs for 21 terms and 61 terms.

This is typical of the Fourier series for a *continuous function*:

- Fourier coefficients a_n and b_n die out at least as fast as $1/n^2$
- partial sums of Fourier series therefore converge relatively fast
- there is no 'Gibbs phenomenon'

WAIT, WHAT'S THE GIBBS PHENOMENON?

Let's see!

EXAMPLE 2: SQUARE WAVE

Let f be the function defined by

$$f(x) = \begin{cases} 2, & \text{on } (-\pi, 0] \\ 0, & \text{on } (0, \pi] \end{cases}$$

and

$$f(x + 2\pi) = f(x)$$

i.e., f is periodic with period 2π

Let's...

EXAMPLE 2: SQUARE WAVE

...

- Sketch the *graph of f* and sketch the function to which the Fourier series $\mathrm{FS}\,f$ *converges*.
- Calculate the *Fourier series of f* and plot the *sum of the first few terms* (partial sum).

SKETCH

CALCULATION

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GIBBS PHENOMENON

We have seen that as $N \to \infty$ the sum of the first N terms of the Fourier series seems to converge to the function f wherever f is continuous. This is *as expected from the convergence theorem*.

However, we see that *all partial sums* of the Fourier series have an *overshoot near the discontinuity*. The overshoot *gets closer to the discontinuity as N increases, but the size of the overshoot does not decrease*.

GIBBS PHENOMENON

This is known as the *Gibbs phenomenon*:

- Any Fourier series of a function with a *jump discontinuity* will have a persistent 9% (of the jump) *overshoot near the discontinuity* as $N \to \infty$.
- At *fixed* x the Fourier series will converge according to the convergence theorem as N increases, but the *overshoot persists and moves towards the discontinuity*.

CONVERGENCE RATES OF COEFFICIENTS

- A piecewise continuous function has Fourier coefficients that decay as $1/n$.
- A continuous function with discontinuous first derivative has Fourier coefficients that decay as $1/n^2$.

In general: a continuous periodic function whose *first k derivatives are all continuous* but whose *$k + 1$ derivative is discontinuous* will have Fourier coefficients that decay at a rate of $1/n^{k+2}$.

HOMEWORK

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

- $f(x) = -x \text{ on } (-1, 1]$
- $f(x+2) = f(x)$ for all x, i.e., f is periodic with period 2.
- Sketch the graph of f and sketch the function to which the Fourier series of f converges.
- Compute the Fourier series of f and plot the sum of the first few terms.
- Check you understand how the Gibbs phenomenon is consistent with the convergence theorem