Engsci 711

Linear Algebra supplement

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Note: you don't need to know this material for the exam (I don't think!).

Other the other hand, some *might* be useful for assignments and for e.g. understanding the full range of possible behaviour in linear systems.

The main material of interest is knowing how to use (generalised) eigenvectors to change coordinates to get a system into a form ready for a centre manifold reduction.

Transformation of variables

From basic linear algebra. See the facts that follow.

Suppose you have a vector x whose components are (x_1, x_2) in a given basis. Here we will assume the basis is given by $\{(1,0),(0,1)\}$ i.e. the standard basis for $\mathbb{R}^2$.

Then, given a new desired basis (coordinate axes) $\{a, b\}$ with coordinates specified relative to the original basis (standard $\mathbb{R}^2$ basis) (a_1, a_2) and (b_1, b_2), for a and b respectively, we can find the coordinates of x relative to the new coordinate axes, u_1, u_2, using

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

i.e. the columns of the transformation matrix are the vectors of the desired new coordinate system (as expressed relative to the original basis).

To motivate this, consider the invariant vector equation

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + u_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which expresses the *same* vector x relative to two different bases : $\{(1,0),(0,1)\}$ and $\{(a_1, a_2),(b_1, b_2)\}$, respectively.

So, to change to coordinates u_1, u_2 relative to an eigenbasis with eigenvectors e^s and e^u say, use

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1^s & e_1^u \\ e_2^s & e_2^u \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where e_1^s, e_2^s are the components of e^s relative to the original coordinate system (etc).

This gives two equations to solve for x_1, x_2 in terms of u_1, u_2 - we can then e.g. derive a new differential equation in u variables by replacing the x variables (in the differential equations for x) by u variables.

Easiest way to check you understand this is to try it! See handwritten examples from Lectures 9 and 10.

Eigenvalues, eigenvectors and all that

Here we recall some basic facts from linear algebra. Some will be stated in full generality, i.e. for $\mathbb{R}^n$, some will be stated for $\mathbb{R}^2$ or $\mathbb{R}^3$. This is because I'm lazy. For more detail see e.g. Glendinning or Kuznetsov (or more generally, any book on linear algebra - I recommend the ones by Strang).

Fact

An $n \times n$ matrix always has n eigenvalues, some of these may be repeated however. The number of times an eigenvalue is repeated is called the multiplicity of that eigenvalue (e.g. the number of times it is a repeated root of the characteristic equation).

Fact

If the eigenvalues of a matrix A are all distinct then the associated eigenvectors are linearly independent (but not necessarily orthogonal). Moreover these are unique up to a scale factor. These eigenvectors can be used to diagonalise A (see below).

Fact

Square matrices with distinct eigenvalues are always diagonalisable. Some matrices with non-distinct eigenvalues are diagonalisable, however - a simple example is the identity matrix.

Fact

Diagonalisability and invertibility are distinct concepts: e.g. you can diagonalise some non-invertible matrices. You can also invert some non-diagonalisable matrices.

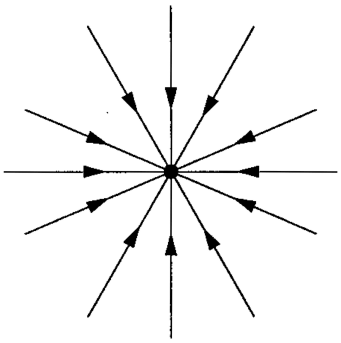
Fact

Given a repeated eigenvalue you can sometimes find multiple corresponding standard eigenvectors in the usual way. You just get multiple independent solutions to $(A - \lambda I)x = 0$. This is not guaranteed in general, however (see next result for a condition).

Implication for two-dimensional differential equations

If the eigenvalue of a two-dimensional (planar) system is repeated and it *happens to have distinct standard eigenvectors*, then *every* vector in the plane is an eigenvector.

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5). This is called a 'star node' (see figure below). These are on the margin between being a spiral and being a standard node. So they are *sort of* structurally unstable cases in the sense of being boundary cases. On the other hand the actual *existence and stability* of the fixed point is *unchanged* (even though it may change from a spiral to a node, for example, both will have the *same stability*) so they do *still count as hyperbolic fixed points* (another way to see this is to note that a stable spiral and a stable node are topologically equivalent, i.e. can be deformed into one another).



Star node - repeated eigenvalues but distinct eigenvectors. The eigenspace is thus the whole plane.

Fact

Given an $n \times n$ symmetric real matrix $A = A^T$ (or *Hermitian* when dealing with complex matrices) then there are n standard eigenvectors and these are distinct and linearly independent. Furthermore, the eigenvalues are all real, though not necessarily distinct.

Fact

For a non-symmetric $n \times n$ matrix with repeated eigenvalues we can always find an orthogonal basis of *generalised eigenvectors* (sometimes called principle vectors). A generalised eigenvector x of rank m is defined as a non-zero solution to

$$(A - \lambda I)^m x = 0$$

where $m \geq 1$ and

$$(A - \lambda I)^n x \neq 0$$

for $n < m$. That is, it satisfies the usual equation, except $(A - \lambda I)$ is raised to the power of m. The second part of the definition just makes sure x doesn't satisfy the equations of lower rank and so the rank well-defined (i.e. it is the smallest power m satisfying the above equation).

Fact

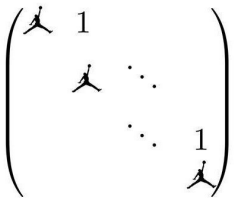
A general *block diagonal* form of a matrix is:

$$\begin{pmatrix} A_1 & ... & ... \\ ... & A_2 & ... \\ ... & ... & A_3 \end{pmatrix}$$

where each A_i can be any size and internal structure, and any '...' entries are taken to be zero. When this represents a system of linear differential equations, this form is sufficient to *decouple* the system into *self-contained subsystems that can be solved separately.*

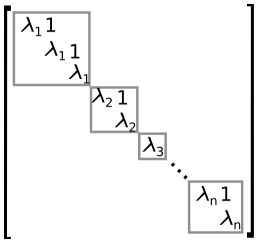
Fact

Generalised eigenvectors can be used to put A into Jordan normal form (see below). For square matrices this gives a particular type of *block diagonal* structure. It is a *generalisation of matrix diagonalisation, and is always possible*. It can be thought of as 'as close as possible to diagonal'. This is a preview of the Jordan normal form:



Jordan blocked (stolen from the internet…).

The following illustrates the structure in more detail, including the form of the *Jordan blocks*:



Jordan normal form with Jordan blocks illustrated (stolen from Wikipedia…). The off-diagonal ones occur when the corresponding eigenvalue is repeated, and the size of the block to the number of times the eigenvalue is repeated.

Fact

If x is a generalised eigenvector of rank m then $(A-\lambda I)^n x$ for $n < m$ is a generalised eigenvector of A of rank $m-n$. For example, if $m=2$ and $n=1$ then $(A-\lambda I)x$ is a generalised eigenvector of A.

Fact

From the above we see that, given a generalised eigenvectors of A of rank m, we can form a so-called chain of generalised eigenvectors $\{x_m, x_{m-1}, ...x_1\}$ of the form $x_m, (A-\lambda I)x_m, ..., (A-\lambda I)^{m-1}x_m$.

As an example, if we have a 2x2 system and a rank 2 eigenvector x_2, we form the *lower* rank eigenvector x_1 via

$$x_1 = (A-\lambda I)x_2$$

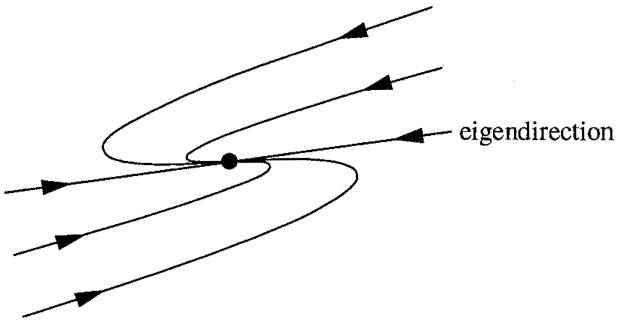
Implication for two-dimensional differential equations

Suppose the eigenvalue of a two-dimensional (planar) system is repeated and there is *only one standard eigenvector* and hence one generalised eigenvector. As you might remember, in such cases we mutliply our solutions by t to get a linearly independent basis for our general solution.

Similarly, it can be shown (see e.g. Glendinning) that if e_1 is the standard eigenvector and e_2 is the generalised eigenvector, then the solution can be considered a linear supposition of exponential motions on e_1 and te_1+e_2. Note that as $t \to \infty$, the vector $te_1 + e_2$ points more and more towards e_1, as the first term dominates the second term. Thus the motion on $te_1 + e_2$ - and, as time increases, the full motion - *collapses into motion approximately along the single, standard eigenvector* e_1 (note though: the collapse onto the standard eigenvector is not *exponentially* quick as it would be with a true slow manifold).

Again, these cases are on the margin of being a standard node and being a spiral. The solutions thus have a sort of *spiralling collapse onto the standard eigenvector.*

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5; see also figure below). Note: you can get the directions of motion (the 'spiralling' direction) by guessing a few interesting points and plugging them into the equations to find the flow direction.



Degenerate node - repeated eigenvalues and only one standard eigenvector. As $t \to \infty$, the contribution of the generalised eigenvector disappears and the motion collapses to the single standard eigenvector.

Diagonalisation and the Jordan normal form

If we use eigenvectors in our coordinate system we get nice looking matrices.

Fact

Suppose we are given a matrix A. In the case of distinct, real eigenvalues, there exists a change of coordinates from old variables x to new variables y expressed via

$$x = Py$$

where the columns of P are distinct eigenvectors and P is invertible, such that the matrix

$$\Lambda = P^{-1}AP$$

is diagonal. Furthermore, the diagonal terms are the eigenvalues.

Fact

In the above case, a linear differential equation $\dot{x} = Ax$ gives

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy$$

i.e. the diagonalised linear system

$$\dot{y} = \Lambda y$$

The motion is thus a linear superposition of flows along with eigendirections, with rates given by the eigenvalues. This can be extended to complex eigenvalues, but I'm too lazy to write this out.

Fact

In the case of non-distinct real eigenvalues, we either still have distinct normal eigenvectors - in which case we proceed as normal - or we instead use our generalised eigenvectors in the transformation. This leads to the *Jordan normal form* pictured previously. In the 2x2 case we have

$$e_1 = (A - \lambda I)e_2$$

and so

$$Ae_2 = \lambda e_2 + e_1$$

whereas, obviously,

$$Ae_1 = \lambda e_1$$

which means

$$P^{-1}AP = [e_1, e_2]^{-1}A[e_1, e_2] = [e_1, e_2]^{-1}[\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2]^{-1}[e_1, e_2]\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$

$$= \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$

giving our Jordan normal form. Again, we can extend to the complex but I won't here.