ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

1. *Basic concepts, stability and linearisation* [4 lectures/tutorials]
   Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest: fixed points, periodic orbits etc. Overview of basic analysis procedure including linearisation, connecting stability of nonlinear systems and stability of linearised systems. Computer-based analysis.
2. *Phase plane analysis and geometry of hyperbolic systems* [5 lectures/tutorials]
   Analysis of two-dimensional linear and nonlinear systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling for general linear and nonlinear hyperbolic systems. Connecting geometry of nonlinear and linearised hyperbolic systems.

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations for parameter dependent systems. Bifurcation diagrams.
4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.
5. *Bonus topic?* [1 lectures if time]
   Introduction to mathematics of chaos…

LECTURE 11:

- Intro. to *centre manifold theory* (geometry of non-hyperbolic systems)
- Application to *asymptotic stability* of non-hyperbolic systems
- Preview of *reduction/emergence principle*

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

Most of the first part of the course dealt with *hyperbolic* fixed points.

We could decide if our system was asymptotically stable just by looking at the *linearisation*.

This is *not the case for non-hyperbolic systems* - linearisation is too ‘rough’ to handle these sensitive cases: here we *need to look at the higher-order terms*.

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

In particular, we can use *centre manifold theory* to:

- Help *reduce* complex dynamic models to ‘emergent’ simpler, approximate dynamic models
- Gain a deeper understanding of *bifurcation theory*
- Analyse *fast/slow systems* (‘geometric’ view of singular perturbation theory)

‘all the good stuff happens on the centre manifold’

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

Bifurcation theory gave us a first taste of analysing non-hyperbolic systems. This was essentially a ‘static’ analysis - no dynamics.

Now we look in more detail at the *geometry of non-hyperbolic systems* - we extend our stable/unstable manifold analysis to include a *centre manifold*.

This allows us to analyse the *dynamics* near non-hyperbolic fixed points.

CENTRE SUBSPACE? SLOW SUBSPACE?

The centre *subspace* (linear manifold) $E^c(0)$ is just the eigenspace corresponding to the *eigenvalues with real part zero*.

This works the *same way* as for $E^s(0)$ and $E^u(0)$.

If the eigenvalues are *exactly zero* - i.e. the *imaginary part is also zero* - then we call the centre subspace a *slow subspace*.

CENTRE MANIFOLD? SLOW MANIFOLD?

The centre/slow *manifold* $W^c(0)$ is just the *nonlinear correction* to the linear subspace $E^c(0)$.

Again, this is *just like* how $W^s(0)$ and $W^u(0)$ correct $E^s(0)$ and $E^u(0)$.

(The key difference is how we can further use $W^c(0)$ to do interesting things)

CENTRE MANIFOLD? SLOW MANIFOLD?

Now we want to calculate the *nonlinear correction*.

We will see that, *in contrast to hyperbolic fixed points*, these corrections can be very important - e.g. in determining *asymptotic stability*.

EXAMPLE (KUZNETSOV EXAMPLE 5.1)

Consider the system:

$$\frac{dx}{dt} = xy + x^3,$$

$$\frac{dy}{dt} = -y - 2x^2$$

Let's first do the usual *linear analysis*.

CENTRE MANIFOLD CALCULATION: BASIC PROCEDURE

The *basic procedure* is *essentially the same* as for the stable/unstable manifolds:

- *Assume* a functional relationship to describe the manifold
  - Here we use: fast $=$ f (slow), i.e. non-centre $=$ f (centre)
- *Substitute* in
- Use *chain rule*
- *Equate coefficients* of a power series
- Use *tangency* to $E^c(0)$

(we will see later why the centre manifold has some very useful additional properties though, and how to identify fast/slow variables carefully).

EXAMPLE (KUZNETSOV EXAMPLE 5.1) CONTINUED

$$\frac{dx}{dt} = xy + x^3,$$
$$\frac{dy}{dt} = -y - 2x^2$$

ASYMPTOTIC STABILITY OF A NON-HYPERBOLIC FIXED POINT

Note:

Using our linear eigenspace to determine the asymptotic stability our our non-hyperbolic fixed point gives the incorrect answer.

We need to calculate the *nonlinear* centre manifold to *correctly determine asymptotic stability* of non-hyperbolic fixed points (note: we still need justify this somewhat!).

OTHER NOTES

Somewhat in contrast to the stable/unstable manifold cases, for the centre manifold theory we *usually first carefully separate out the fast linear and slow linear dynamics* before calculating the nonlinear centre manifold.

This allows us to *just focus on the emergent centre manifold dynamics*

We will first preview the theorem that justifies what we did.

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Consider $\dot{x} = f(x)$ having a *non-hyperbolic fixed point* at $x = 0$, where $x \in \mathbb{R}^n$.

Assume that there are n^+ eigenvalues (counting repeated cases) with Re $\lambda > 0$, n^0 eigenvalues with Re $\lambda = 0$, and n^- eigenvalues with Re $\lambda < 0$.

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Then there is a locally defined smooth n^0-dimensional invariant manifold $W^c_{loc}(0)$ that is tangent to the (linear) centre eigenspace E^c.

Moreover, there is a neighborhood U of $x_0 = 0$, such that if $\phi(x,t) \in U$ for all $t \geq 0$ (≤ 0) then $\phi(x,t) \to W^c_{loc}(0)$ for $t \to \infty$ ($t \to -\infty$).

CENTRE MANIFOLD THEOREM - WHY/WHAT?

The solutions on the centre *eigenspace* are *'frozen'* - neither growing nor decaying. The solutions on the centre *manifold* are *slowly varying*.

We can thus think of the eigenvalue = 0 case as defining the *linearised steady-state* behaviour of the full system.

The linearised dynamics are 'infinitely slow' flows relative to the *exponential behaviour on the other eigenspaces*.

The *nonlinear* dynamics can *vary slowly* (e.g. 'quasi-steady states'). This is *often our 'emergent' timescale of interest!*

CENTRE MANIFOLD - UNIQUENESS?

The centre manifold is unique to all orders of its Taylor expansion.

That is, center manifolds are *not quite unique but differ only by exponentially small functions* of the distance from the fixed point (think: 'faster scales').

CENTRE MANIFOLD THEOREM - RESTRICTION AND REDUCTION

Usually we are interested in equilibria where $n^+ = 0$, i.e. where *all eigenvalues are negative or zero*.

Thus the dynamics are *exponentially attracted to the centre manifold*. (see simulation example).

This justifies our use of the *restriction to the centre manifold* for determining asymptotic stability.

It also naturally leads to a *reduction/emergence principle* based on centre manifold theory (next lecture).

EngSci 711 L11 Centre manifold theory:

Dynamics* near non-hyperbolic fixed points:

- Introduction to centre manifolds
- application to asymptotic stability of non-hyperbolic fixed points
- preview of reduction/emergence principle & decoupling

(* cf bifurcation theory: mostly 'static' analysis)

Example Questions

2016:

Question 5 (20 marks)

Consider the system

$$\begin{aligned} \dot{x} &= y(2x - y) \\ \dot{y} &= x^2 - y \end{aligned}$$

(a) Find the two fixed points of this system. Show your working. You do not need to classify these.

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin (0,0).

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0,0)$ that is correct up to and including cubic order.

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.

2017:

Question 4 (16 marks)

Consider the system

$$\begin{aligned} \dot{x} &= 2xy + x^3 \\ \dot{y} &= -y - x^2 \end{aligned}$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system. (1 mark)

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0, 0)$. (2 marks)

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories. (3 marks)

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0, 0)$ that is correct up to and including cubic order. (8 marks)

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable. (2 marks)

2018:

Question 4 (16 marks)

Consider the system

$$\begin{aligned} \dot{x} &= -x + 2xy + y^3 \\ \dot{y} &= -2xy - y^3 \end{aligned}$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system. (1 mark)

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0, 0)$. (2 marks)

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. You do not need to sketch anything. (3 marks)

(d) Use a power series expansion, expressing the 'fast' variable in terms of the 'slow' variable, to calculate an expression for the centre manifold $W^c_{loc}(0, 0)$ that is correct up to and including cubic order. (8 marks)

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable. (2 marks)

Motivating examples: consider the dynamics near the origin of:

1.
$$\begin{aligned} \dot{x} &= xy + x^3 \\ \dot{y} &= -y - 2x^2 \end{aligned}$$

2.
$$\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2 \end{aligned}$$

3.
$$\begin{aligned} \dot{x}_1 &= x_1 y - x_1 x_2^2 \\ \dot{x}_2 &= x_2 y - x_2 x_1^2 \\ \dot{y} &= -y + x_1^2 + x_2^2 \end{aligned}$$

4.
$$\begin{aligned} \dot{x} &= -xy \\ \dot{y} &= -2y + z + x^2 \\ \dot{z} &= y - z + x^2 \end{aligned}$$

5.
$$\begin{aligned} \dot{x} &= \mu x - xy \\ \dot{y} &= -y + x^2 \\ \dot{\mu} &= 0 \end{aligned}$$

6.
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -xz \\ \dot{z} &= -z + x^2 + xy \end{aligned}$$

Each of these has:

- one or two 'slow' vars

&

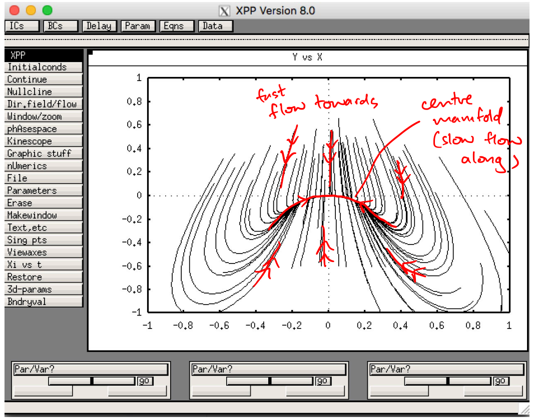
- one or two 'fast' vars.

ie

→ each has a centre manifold (near the origin here)

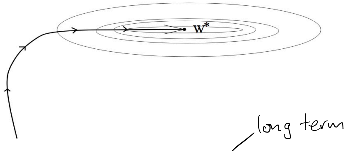
Let's look at 1.

XPP:



Note: similar phenomena are observed when training large/deep neural networks:

From LeCun et al "Efficient BackProp" in 'Neural Networks: Tricks of the trade' (book)



long term

Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum $\mathbf{w}^*$ and it is a good approximation of the minimum eigenvalue direction of the Hessian.

reduction

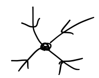
centre manifold

→ 'long term reduction to the centre manifold'!

The basic idea of centre manifold theory is that near a non-hyperbolic fixed point, the dynamics are exponentially attracted/repelled to the non-decaying – slow &/or oscillating – variables with $Re(\lambda)=0$.

→ For purely real eigenvalues, these are also called slow variables: they are 'frozen' at the linear scale, slowly varying at nonlinear scale } main focus here.

In many ways, this is a simple generalisation of the local decay to/growth away from fixed points



long-time dynamics } → fixed points (etc)

Instead, however, we have a slow manifold as our attracting/repelling object.

→ At the linear order we have a line/plane etc of non-separated fixed points.

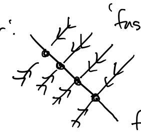
↳ The motion is zero/frozen on this linear scale

→ At the nonlinear order we have a curve/surface, passing through a true fixed point,

↳ The motion is nonzero on this nonlinear scale

↳ Can be more complicated than usual (bcf. etc).

Linear:

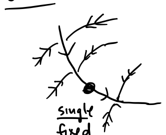


'fast'

frozen motion

line of fixed points

Nonlinear



single fixed point

'slow' motion along manifold

} sometimes get full manifold of fixed points in nonlinear → see later?

Some implications

- If we want to decide the 'true' nonlinear asymptotic stability of a nonhyperbolic fixed point we need to consider the nonlinear terms: does the motion 'creep' towards or away from it?

- More constructively, we can carry out model reduction near non-hyperbolic fixed points
  - └ we have a separation of scales:
    - 'fast' motion to manifold
    - 'slow' motion along manifold

$$\boxed{\text{If the 'slow' variables correspond to our macroscopic observation scale of interest, then we can 'coarse-grain out' many fast variables \& just focus on the emergent slow motion}}$$

→ eg why we can study continuum mechanics without knowing string theory/particle physics!

→ however, the emergent slow motion can still be /is necessarily quite complicated! } bifurcations etc.

Other approaches to the same sort of ideas:
- singular perturbation theory
- quasi-steady state/adiabatic approximations
- intermediate asymptotics
- scaling analysis
- averaging & homogenisation.

} with Richard? other courses?

Centre manifold theory is { modern, systematic, geometric, unifying

but these other methods are important complements!

The question is how to get to the emergent slow motion!

Consider ①:

$$\dot{x} = xy + x^3$$
$$\dot{y} = -y - 2x^2$$

Try the usual process.

1. Look at $(0,0)$. FP?

$$\dot{x}(0,0) = 0 \checkmark$$
$$\dot{y}(0,0) = 0 \checkmark$$

2. Linearise.

$$Df = \begin{pmatrix} y + 3x^2 & x \\ -4x & -1 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Eigenvalues

$$\lambda = 0, -1$$

centre/slow　　stable/fast!

→

Zero eigenvalue -- what do we do?.

Let's continue.

4. Find eigenvectors

$$A = Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(A - \lambda I)u = 0$$

$\lambda = -1$.

$$\begin{pmatrix} 0+1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow u_1 = 0, u_2$ free

$$e_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \& \quad E^s = \{(x,y) \mid x = 0\}$$

$\lambda = 0$

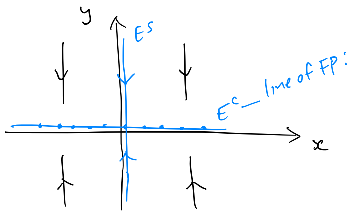
How? Usual way! Can still have eigenspace for 'frozen' motion.

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow u_2 = 0, u_1 =$ free

$$e_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad E^c = \{(x,y) \mid y = 0\}$$

So: (linear picture)



ie $\begin{bmatrix} \dot{x} = 0 \\ \dot{y} = -y \end{bmatrix}$

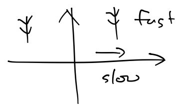
What about the nonlinear picture? Want W^c!

→ Same procedure as for W^s & W^u but with some caveats

↳ we need to follow a 'reduction procedure' to properly decouple fast & slow parts

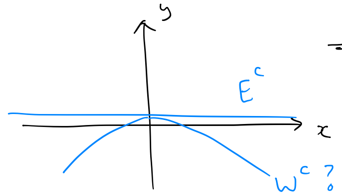
↳ we'll look at this in more detail soon

↳ for now, our system here is already in an appropriate form



[orthogonal in linear parts]

Expansion type?



$\Rightarrow W^c: \; y = h(x)$

ie $\boxed{\text{fast} = f(\text{slow})}$

Steps: same as before, using $\boxed{\text{fast} = f(\text{slow})}$ ← Key

$y = h(x)$
$$\begin{cases} y = \cancel{a_0} + \cancel{a_1 x} + a_2 x^2 + a_3 x^3 \\ \frac{dy}{dx} = \cancel{a_1} + 2a_2 x + 3a_3 x^2 \end{cases}$$

Tangent to x-axis:

$a_0 = 0$

$a_1 = 0$.

ODEs along $h(x)$.

$$\dot{x}(x, h(x)) = x(a_2 x^2 + a_3 x^3) + x^3$$
$$= (1 + a_2)x^3 + \cancel{a_3 x^4}$$

$$\dot{y}(x, h(x)) = -a_2 x^2 - a_3 x^3 - 2x^2$$
$$= -(2 + a_2)x^2 - a_3 x^3$$

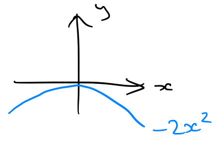
Chain along $h(x)$.

$$\dot{y} = \frac{dy}{dx}\dot{x}$$
$$= [2a_2 x + 3a_3 x^2][(1 + a_2)x^3]$$
$$= 0 + O(x^4)$$
$$= -(2 + a_2)x^2 - a_3 x^3$$

$$\Rightarrow \boxed{\begin{aligned} a_2 &= -2 \\ a_3 &= 0 \end{aligned}} \Rightarrow$$

So $y = h(x) = -2x^2 + \dots$

& $W^c = \{(x, y) \mid y = -2x^2\}$



as expected from XPP!

Note: in this case, could have guessed to this order:

$\dot{x} = xy + x^3 \approx$ quadratically small near $x, y = 0, 0$

$\dot{y} = \underbrace{-y}_{\text{linear}} - 2x^2$

L linear: dominates/fast near $x, y = 0, 0$.

Intuition:

$\dot{y}$: moves fast, $\dot{y} \approx -y$, at first.

then

reaches 'equilibrium' $\dot{y} = 0$

ie $y = -2x^2$

$\boxed{\text{'quasi-steady state/quasi-equilibrium'}}$

So, often guess:

→ linear/fast dynamics rapidly approaches equilibrium

→ $\dot{y} \approx 0$ gives approximate slow manifold.

BUT: in general centre manifold theory guides how to do 'properly'

→ see assignment for naive vs centre manifold } can be different.

Back to dynamics

$W^c: \; y = -2x^2$

Consider

$\dot{x}|_{W^c}$ ie $\dot{x}(x, h(x))$ } nonlinear dynamics on nonlinear manifold

└ correct approach

$\dot{x}$ restricted to W^c

$= x(-2x^2) + x^3$

$= -x^3$

Note: $x > 0, \; \dot{x} < 0$
$x < 0, \; \dot{x} > 0$ } note: need to analyse like this, can't do linear!

stable (ie asymptotically stable)

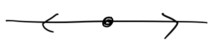
(correct)

Compare to linear

$\dot{x}|_{E^c} = x(0) + x^3$ } nonlinear dynamics on linear centre space

└ incorrect approach

$\dot{x}$ restricted to E^c



unstable! wrong conclusion!

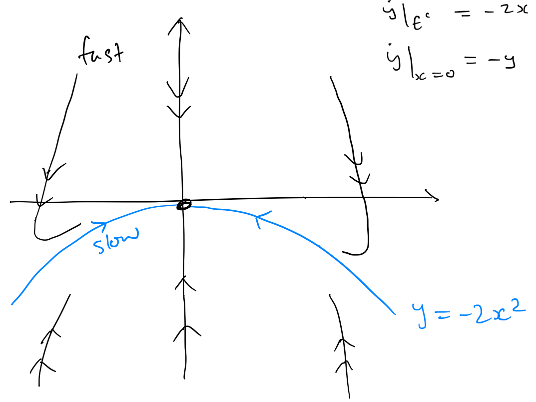
→

Picture

$\dot{y}|_{W^c} \approx 0$

$\dot{y}|_{E^c} = -2x^2$

$\dot{y}|_{x=0} = -y$



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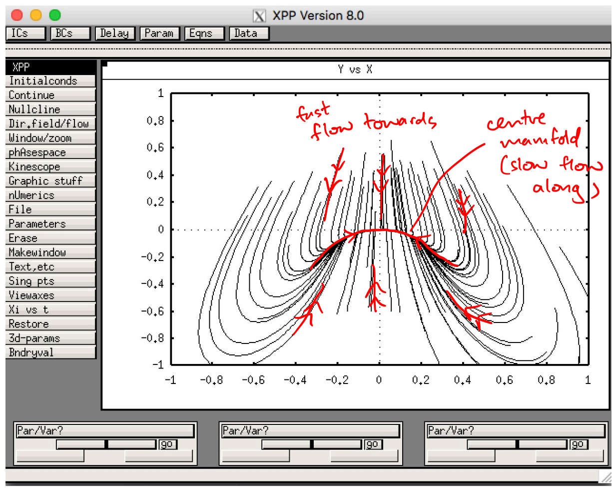
compare to:

XPP:



Complications / Extensions :

- identifying fast/slow variables/subsystems
- higher/lower dim manifolds in higher/lower dim systems

L12

- separation of scales without exact zeros
- parameters & bifurcation theory

L13

Tomorrow: consider eg

(2)
$$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$

(3)
$$\dot{x}_1 = x_1 y - x_1 x_2^2$$
$$\dot{x}_2 = x_2 y - x_2 x_1^2$$
$$\dot{y} = -y + x_1^2 + x_2^2$$

(4)
$$\dot{x} = -xy$$
$$\dot{y} = -2y + z + x^2$$
$$\dot{z} = y - z + x^2$$

Exercise: Try in XPP!