ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures/tutorials]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURES 8 AND 9

- Quick recap of Hopf bifurcation
- Intro. to centre manifold theory (geometry of reductions and bifurcation)

Note: important but somewhat difficult material - just want to introduce you to the general ideas!

For more - see Glendinning, Kuznetsov (1998) or Wiggins (1990,2003) Perko (2001), Strogatz notes, handouts.

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

Near a non-hyperbolic fixed point/bifurcation all the good stuff happens 'on' the centre manifold

- the (ordinary/solution) stability of a non-hyperbolic fixed point for (fixed parameter values) is determined here!
- all our bifurcations (saddle-node, Hopf etc) happen here as parameter values are varied.

BASIC MOTIVATION

This leads us to ask

- How do we *reduce* to a centre manifold at a non-hyperbolic fixed point?
- How do we determine *ordinary/asymptotic stability* at a non-hyperbolic fixed point?
- Why does this work?
- How do we track a centre manifold near a non-hyperbolic - i.e. structurally unstable - point *under parameter variations*. What happens on this manifold? Why is all the good stuff here?

SETTING ONE: 'CRITICAL CASE'

Suppose we have

$$\dot{x} = f(x; \mu) = f(x; 0) = f(x)$$

with a bifurcation (non-hyperbolic point/eigenvalue with zero real part) at $\mu = 0$.

We will start by considering the dynamics near this non-hyperbolic fixed point occuring at $\mu = 0$ and *keeping μ fixed* at 0.

Later we will consider the *effect of varying μ by small amounts* (to track bifurcations).

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Consider $\dot{x} = f(x)$ having a non-hyperbolic fixed point at $x = 0$.

Assume that there are n^+ eigenvalues (counting repeated cases) with Re $\lambda > 0$, n^0 eigenvalues with Re $\lambda = 0$, and n^- eigenvalues with Re $\lambda < 0$.

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Then there is a locally defined smooth n^0-dimensional invariant manifold $W^c_{loc}(0)$ that is tangent to the (linear) centre eigenspace E^c.

Moreover,there is a neighborhood U of $x_0 = 0$, such that if $\phi(x, t) \in U$ for all $t \geq 0$ (≤ 0) then $\phi(x, t) \to W^c_{loc}(0)$ for $t \to \infty$ $(t \to -\infty)$.

CENTRE MANIFOLD THEOREM

Example picture.

CENTRE MANIFOLD THEOREM - WHY/WHAT?

The solutions on the centre *eigenspace* are 'frozen' - neither growing nor decaying. The solutions on the centre *manifold* are slowly varying.

We can thus think of the eigenvalue = 0 case as defining the 'local' (linearised) steady-state behaviour (recall PDEs!).

So these can be considered 'infinitely slow' - 'frozen' - flows relative to the exponential behaviour on the other eigenspaces!

In 'slow-fast' systems the 'slow manifold' is a type of centre manifold.

CENTRE MANIFOLD THEOREM - WHY/WHAT?

More sketches.

CENTRE MANIFOLD - UNIQUENESS?

The centre manifold is unique to all orders of its Taylor expansion.

That is, center manifolds are *not quite unique but differ only by exponentially small functions* of the distance from the fixed point (think: 'faster scales').

CENTRE MANIFOLD - FAST/SLOW INTERPRETATION

Given our observations so far,

We expect all stable flows to *converge much faster* (exponentially) to the centre manifold *near a non-hyperbolic fixed point*, and then travel 'along' this manifold on a *slower* time scale.

Hence we have a sort of 'fast-slow' decoupling (reduction). We can formalise this via the following.

CENTRE MANIFOLD THEOREM - REDUCTION

If $(x, y, z) \in E^c \times E^s \times E^u$ are *coordinates in terms of the system's eigenbasis (diagonalised)* representation then we first write our system as

$$\dot{u} = Bu + g(u, v)$$
$$\dot{v} = Cv + h(u, v)$$

where $u \in \mathbb{R}^{n_0}$ are our centre manifold variables and $v \in \mathbb{R}^{n^+ + n^-}$ are our (locally) exponentially growing/decaying solutions.

CENTRE MANIFOLD THEOREM - REDUCTION

Note

- The matrices B and C have *eigenvalues with zero and non-zero real-part* respectively, while
- The *functions g and h represent the higher-order* (at least quadratic) terms (since B and C represent the linear dynamics) and we will assume they have Taylor expansions (which clearly start from quadratic order)

CENTRE MANIFOLD THEOREM - REDUCTION

The key point is to write the equations so that they are *linearly decoupled* according to the *sign of the real part of the eigenvalues*

(this corresponds to putting the system in 'normal form')

CENTRE MANIFOLD THEOREM - REDUCTION

We will assume (just like for the stable/unstable manifold case) that we can locally represent the centre manifold by a smooth curve, i.e.

$$W^c_{loc} = \{\ (u, v) \mid v = V(u)\}$$

CENTRE MANIFOLD THEOREM - REDUCTION

Putting all this together leads to the following *Reduction Principle*:

Near a non-hyperbolic fixed point our system (written in its eigenbasis/diagonalised form) is *locally* topologically equivalent to the system

$$\begin{aligned} \dot{u} &= Bu + g(u, V(u)) \\ \dot{v} &= Cv \end{aligned}$$

CENTRE MANIFOLD THEOREM - REDUCTION

Note that these reduced, local dynamics are now *uncoupled* and the dynamics in v are linear, 'fast' and essentially 'trivial'!

The idea to carry out this reduction is essentially to *set the fast v dynamics to (quasi-) steady state (in the full nonlinear v equation!) and then focus on the slowly vaying u dynamics on the centre manifold*.

CENTRE MANIFOLD THEOREM - REDUCTION

Note: we need to *use the full v equation to find the (quasi-) steady state* to find the region to 'localise' about. This corresponds to finding out *where the non-hyperbolic fixed point is*.

i.e. only when we are *near* this fixed point do the v dynamics become linear and trivial!

CENTRE MANIFOLD THEORY - HOW TO APPLY

We find the centre manifold in the same (long) way we found the stable/unstable manifolds and then 'project' the flow onto this near our non-hyperbolic fixed poing (in accordance with the Reduction Principle).

This allows us to consider the 'slow' and '(weakly) nonlinear' dynamics on the centre manifold - i.e. which side of the linearised zero eigenvalue system is the full system on - and hence *determine the (solution/asymptotic) stability at this point*.

EXAMPLE

Consider (Kuznetsov, 2004, Example 5.1)

$$\dot{x} = xy + x^3$$
$$\dot{y} = -y - 2x^2$$

EXAMPLE

Analysis.

Note: a *quick trick* for getting a first approximation to the centre manifold is to *set the (full, nonlinear) equation for the fast variables to (quasi-) steady state*. This gives the first term(s) of centre manifold! Easy. Though *we need to do things the long way for the next order terms*.

QUICK ORIENTATION

We always come back to two questions of interest

- How do we determine *asymptotic stability of solutions* at fixed parameter values?
- How do we determine the *structural stability* of our conclusions as parameter values are varied?

QUICK ORIENTATION

We have seen that these are intimately related. In the first case we now know that for hyperbolic fixed points we can linearise, while *for non-hyperbolic fixed points we need to go to the 'next order' to decide (asymptotic) stability.*

We also know that *hyperbolic fixed points are structurally stable while non-hyperbolic fixed points are not*.

QUICK ORIENTATION

This leads to our next key question:

Can we follow the *structurally-unstable* non-hyperbolic dynamics *as parameters are varied*? Answer: yes, using *extended* centre manifolds.

CENTRE MANIFOLD THEOREM - VARIATIONS IN PARAMETERS

We have shown how to 'go to the next order' to analyse *asymptotic stability of solutions for parameters fixed* exactly at values giving non-hyperbolic equilibria.

We know, however, that *these dynamics are not guaranteed to be preserved*

Basic idea - we can use an *extended centre manifold to analyse this system*. *Bifurcations* happen here.

CENTRE MANIFOLD THEOREM - VARIATIONS IN PARAMETERS

We will just outline the idea here (you will fill in the details in the assignment!)

We return to

$$\dot{x} = f(x; \mu)$$

CENTRE MANIFOLD THEOREM - VARIATIONS IN PARAMETERS

Now we re-write this as an equivalent system defined by

$$\dot{x} = f(x; \mu)$$
$$\dot{\mu} = 0$$

Now μ is a *'completely frozen' (or 'super slow') centre manifold variable* - even its higher order terms are zero.

CENTRE MANIFOLD THEOREM - VARIATIONS IN PARAMETERS

Linear vs. nonlinear.

Note: a term like μu is linear in the original system where μ is just a parameter but *is considered nonlinear in the extended system* since it is a product of state variables.

CENTRE MANIFOLD THEOREM - VARIATIONS IN PARAMETERS

Since μ is now considered as an additional centre manifold (and dependent) variable (or set of variables), we can *simply apply centre manifold theory to this extended system*.

This guarantees a centre manifold exists for both x near the fixed point *and for parameters near the critical parameter*.

If $\mu \in \mathbb{R}$ is a single parameter then the centre manifold has dimension $n^0 + 1$, i.e. one more dimension.

On the other hand, the dynamics are trivial: μ = constant.

CENTRE MANIFOLD REDUCTION - UPSHOT I.

Our bifurcation diagrams in the $u - \mu$ plane are *actually phase-portraits of the extended centre manifold.*

Focusing on the *equilibria u^* in this extended system corresponds to a further centre manifold reduction* where the parameters are now the (super) 'slow' variables relative to the 'fast(er)' centre manifold variables u (see next slide).

CENTRE MANIFOLD REDUCTION - UPSHOT II.

Since the *parameter dynamics are trivial and 'super slow'* we can also consider the sub-manifold defined by the *equilibria u^* of u for each parameter value*

> Note: this is essentially just a *further centre manifold reduction!*

i.e. set fast(er) dynamics = steady-state, solve centre equation. Here (super slow) centre equation is the parameter equation and its solution is just μ = constant. The 'fast(er)' dynamics are now the u dynamics!

CENTRE MANIFOLD REDUCTION - UPSHOT III.

- The overall extended centre manifold exists and is locally exponentially attracting for variations of the parameters (Reduction 1)
- This generates nearby equlibrium (not strictly centre) invariant (and locally exponentially attracting/repelling) manifolds for each parameter value (Reduction 2).
- At the critical parameter value this invariant manifold matches the extended centre manifold (correspondence between Reductions 1 and 2)

CENTRE MANIFOLD THEOREM - UPSHOT IIII.

- The dynamics still might be weird after using centre manifold theory (no guarantee fixed points are preserved etc)
- BUT, all the weird stuff happens on the extended centre manifold/on the invariant manifold passing through the extended centre manifold.

Picture and example.