ENGSCI 213: MATHEMATICAL MODELLING 2SE

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MODULE OVERVIEW

Introduction to Probability (*Oliver Maclaren*) [9 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts of probability. Sets and subsets, sample spaces and events. Probability and counting, conditional probability, independence, Bayes' theorem. Random variables. Simple data structures for probability calculations.

2. *Discrete probability models* [3 lectures]

   Discrete random variables and discrete probability distributions. Probability mass functions and cumulative distribution functions. Binomial and Poisson distributions.

3. *Continuous probability models* [3 lectures]

   Continuous random variables and continuous probability distributions. Probability density functions and cumulative distribution functions. Exponential and Normal distributions.

WHAT IS 'PROBABILITY'? WALK INTO A BAR...

- *Mathematician* - probability is about measuring the sizes subsets of a given set. It is a formal theory with axioms.
- *Software engineer/computer scientist* - probability is about the relative occurance of elements in multiset (bag) data structures or special types of dictionaries.
- *Everyday scientist/engineer* - probability is about quantifying uncertainty and odds or modelling variability.
- *Philosopher* - probability has something to do with inductive inference.

WHAT IS 'PROBABILITY'?

Informal definition

A probability is a number between 0 and 1 representing how 'likely' an 'event' is to happen.

WHAT IS 'PROBABILITY'

Informal classical probability definition (Laplace)

> *The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible.*

WHAT IS 'PROBABILITY'?

Informal frequentist definition

The probability of an event is the limit of its relative frequency in a large number of (possibly hypothetical) trials carried out under the same conditions.

WHAT IS 'PROBABILITY'?

Informal (subjective) Bayesian definition

The probability of an event is a measure of my degree of belief that an event will occur. You could measure it by asking me to bet on whether the event will occur or not.

WHAT SHALL WE DO?

Let's construct a *mathematical model* that works for a *wide range* of 'uncertain' situations/procedures/algorithms etc.

We will call these situations random experiments (or uncertain or stochastic experiments) i.e. the outcomes of running the experiment are not completely predictable.

...examples...

KOLMOGOROV - HERE ARE THE MODEL INGREDIENTS!

Formal probability theory consists of three ingredients

1. A *sample space S* - for the *set* of all *possible outcomes* of an uncertain experiment
2. A *collection Σ of events E* - each event is a *subset* of the sample space S, i.e. $E \subseteq S$, and an *element* of the collection of events Σ, i.e. $E \in \Sigma$
3. A *probability function P* (or 'measure') - this assigns a *probability* $P(E)$ to each event E (subset of the sample space)

CONDITIONS?

Our ingredients each come with certain conditions that must be satisfied. We'll first look at *sample spaces* and *events* and come back to probability functions next time.

INGREDIENT ONE: SAMPLE SPACES

The sample space is the set of all possible outcomes of an uncertain experiment

Each point of the sample space - i.e. each outcome - is also called a sample point and must be *listed once and only once*.

...examples...

EVENTS

An event E is a subset of the sample space, i.e. $E \subseteq S$.

Our probability model will include a *collection* of possible *events* (subsets) and associated probabilities.

Notice that an event is a *(sub)set of outcomes* (sample points/elements of sample space).

The *null (empty) event* $\emptyset$ and *sample space* S are *always included* in our collection of events since they are both subsets of S.

...examples...

EVENTS AS DEFINED BY PARTIAL INFORMATION CONSTRAINTS

You can think of events as defined in two steps

1. Start from the *full sample space* S
2. *Add constraints* on the sample space to represent additional *(usually partial) information describing that event* to get...

$$\boxed{E := \{s \in S \mid s \text{ satisfies the constraints}\}}$$

...examples...

OCCURRANCE OF EVENTS

We say that an event occurs in an experiment if, when we observe the outcome $s \in S$ of that experiment, the outcome is an element of that event, i.e. $s \in E$

...examples...

MANIPULATING AND VISUALISING EVENTS (SETS)

Venn diagrams are useful for *visualising* relationships between (a small number of) events

...examples...

COMBINING AND MANIPULATING EVENTS - UNIONS, INTERSECTIONS AND COMPLEMENTS

Given events A, B which are events on the same sample space S, i.e. $A, B \subseteq S$, we can define

- The *union* of events A and B written $A \cup B$
- The *intersection* of events A and B, written $A \cap B$
- The *complement* of A (say) written as $\overline{A}$

These are defined by...

COMBINING AND MANIPULATING EVENTS - UNION

$$\boxed{A \cup B := \{s \in S \mid s \in A \ OR \ s \in B\}}$$

where 'OR' is taken in its *logical/inclusive* meaning.

...examples...

COMBINING AND MANIPULATING EVENTS - INTERSECTION

$$\boxed{A \cap B := \{s \in S \mid s \in A \text{ AND } s \in B\}}$$

...examples...

COMBINING AND MANIPULATING EVENTS - COMPLEMENT

$$\boxed{\overline{A} := \{s \in S \mid s \notin A\}}$$

...examples...

DECOMPOSING EVENTS - MUTUALLY EXCLUSIVE EVENTS

The events $A_1, A_2, \ldots, A_k$ are called mutually exclusive (mutually disjoint) if the intersection of every pair is the empty event, i.e.

$$A_i \cap A_j = \varnothing \text{ for all } i \neq j$$

...examples...

DECOMPOSING EVENTS - PARTITIONS

A partition is a collection of events $A_1, A_2, \ldots, A_k$ which are mutually exclusive and whose union is the whole sample space, i.e.

$$\bigcup_{i=1}^{k} A_i = A_1 \cup A_2 \cup \ldots \cup A_k = S$$

...examples...

HOMEWORK/CHALLENGES

Try to write your own glossary for the terms introduced today without looking at the slides. Include any conditions we've introduced so far for each concept.

Think about what sort of data structures you would need to represent probability calculations. Code some up!

Come up with some simple coin flipping/dice rolling experiments and try to work out the sample spaces and various events.

Try some hand calculations and compare them to results from your code.