Engsci 213 Markov Chain Problems - Set 1

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Topics

So far we've covered Part 1 of 3 on Markov chains:

1. *Basic concepts* [2 lectures]
   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions. Diagrams of Markov chains.

Let's do some examples!

Motivations and basic definitions

- Write down the definition of a general discrete-time, discrete-space stochastic process
- Describe what it means for a stochastic process to have the Markov property in words
- Make up three examples of stochastic processes that you could model as Markov chains. For each case make sure to give the state space and the transition probabilities/matrix.

Transition probabilities and matrices

Problem one

Give the following transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

- Calculate the 2 and 3-step transition matrices

Problem two

Given the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Calculate the 2-step transition matrix $\mathbf{P}^2$

Initial and marginal distributions

Problem one

Consider the transition matrix from the previous question again, i.e.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now, suppose you start from an initial distribution of

$$\mu_0 = (1/2, 1/4, 1/4)$$

- Calculate the marginal distributions μ_1 and μ_2
- For μ_2 try calculating it via both $\mu_2 = \mu_0 \mathbf{P}^2$ and $\mu_2 = \mu_1 \mathbf{P}$
- Would you expect these to be the same? Why? Give the name of some property or equation etc that justifies your answer.

Problem two

Suppose you used the same transition matrix from the previous problem but now start from an initial distribution of

$$\mu_0 = (1/3, 1/3, 1/3)$$

- What is μ_1?
- What is μ_2?
- What is μ_n for any n?

Diagrams

Problem one

Suppose you model the process of passing notes between you and your group of five other friends in class. Give everyone a simple non-numerical alias (e.g. person 'A' etc).

- Write down your state space $\mathbb{X}$ in terms of your chosen labels.

Given the following transition matrix for the stochastic dynamics of note passing:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.75 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.25 & ? & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & ? \end{bmatrix}$$

- Fill in the two missing '?' values in the transition matrix
- Draw a state transition diagram using your chosen labels

Problem two

Make up a valid probability transition matrix.

- Justify why it is a valid probability transition matrix.
- Draw the state transition diagram for this matrix

Make up a different state transition diagram corresponding to a valid Markov chain.

- Justify why it is a valid Markov chain transition diagram.
- Write down the corresponding transition matrix.
- Calculate the two-step transition matrix.
- Draw the transition diagram for this two-step transition matrix.

Simulation challenges

Challenge one

Consider the coin-flip random walk we looked at in class, here with a state space of $\mathbb{X} = \{1, ...N\}$.

(Recall the process was - at each stage flip a coin with probability p of heads and $1-p$ of tails. If heads step right, tails step left, unless you hit the boundary of your state space, i.e. 1 or N, in which case you stop.)

- Write some code simulating this process directly for a given number of steps and given N
- Now, write some R code to generate a transition matrix for a given choice of N.
- Use the Markov evolution equation to evolve various initial conditions and compare to your original simulations
- Suppose you want the boundaries to be 'reflecting' instead of 'absorbing'. What would be a good transition matrix to use?
- Modify your code to allow this possibility.

Challenge two

Reconsider the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Write simulation code to update an initial distribution to new marginal distributions using this matrix.
- Try running your code for many steps from different initial distributions.
- What do you notice about the 'long-time' marginal distribution(s) that result?