Tutorial 3: Bifurcation theory.

Selected solutions + Extra comments (see end)

1D, 1 Co-D.

a) — c): see lecture notes.

d). See later in this doc $\longrightarrow$

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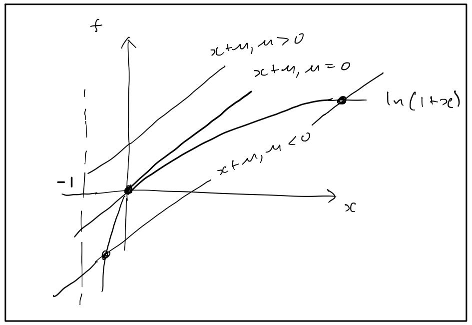
3. a). $\dot{x} = \underline{\mu + x} - \underline{\ln(1+x)} = f(x;\mu)$

FP: $f(x,\mu) = 0$

graphically or using computer: (Harder case)



Note!

$\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$

at $x=0$

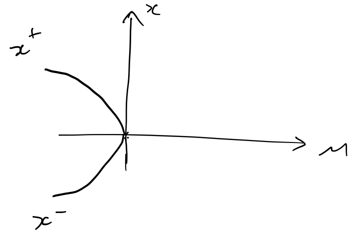
$\frac{d}{dx}(\ln(1+x)) = 1$

for $x > 0$, < 1.

Expect: no sol^n for $\mu > 0$
1 sol^n for $\mu = 0$
2 sol^n for $\mu < 0$

} saddle-node

Diagram without stability



Stability

$f(x;\mu) = \mu + x - \ln(1+x)$

$Df(x;\mu) = 1 - \frac{1}{1+x}$

> 0 { if $1 > \frac{1}{1+x}$, ie $1+x > 1$, ie $\underline{x > 0}$ } unstable

< 0 { if $1 < \frac{1}{1+x}$, ie $\underline{x < 0}$ } stable

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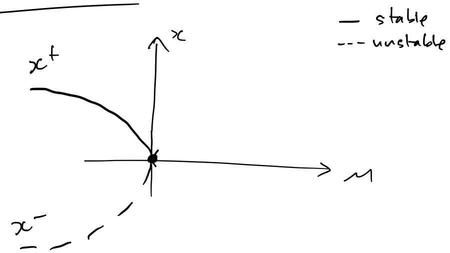
Diagram with stability



Note: Taylor series near bifurcation point. $(0,0)$

$$f(x;\mu) = \mu + x - \ln(1+x)$$

Taylor series in x & μ near $x=0, \mu=0$:

$$f(x;\mu) \approx \mu + \cancel{x} - \left[\cancel{x} - \tfrac{1}{2}x^2 + \ldots\right]$$

$$= \mu + \tfrac{1}{2}x^2 + \ldots$$

$\Rightarrow$ Looks like saddle node form

as expected

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* in general need to at least find these from full equations before doing local analysis ... but then often easy enough to just substitute in some values to full Df

↳ I want you to use this explicit approach

The Taylor series approach is common tho' and leads to Normal Form theory & various 'genericity conditions' in terms of $f_x, f_{xx}, f_\mu, f_{\mu\mu}, f_{x\mu}, \ldots$ etc.

→ requires more theory than above } implicit function theorem.
↳ which terms etc?

→ see eg Glendinning etc.

3 e). $$\dot{x} = x + \frac{\mu x}{1+x^2}$$ (another 'harder'-ish case)

1. FP. $f(x;\mu) = 0$

$$x + \frac{\mu x}{1+x^2} = 0$$

$$\Leftrightarrow \quad \frac{x(1+x^2+\mu x)}{1+x^2} = 0$$

$$\Leftrightarrow \quad x(1+x^2+\mu x) = 0 \quad \& \quad 1+x^2 \neq 0$$

$\Rightarrow$ $x = 0$ (always exists) or $1 + x^2 + \mu x = 0$, $x = \pm\sqrt{-\mu - 1}$

Introduce $\lambda = -\mu$ for convenience.

$$\Rightarrow x = \pm\sqrt{\lambda - 1}$$

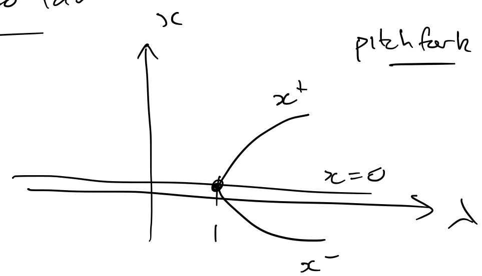
$\lambda < 1 \Rightarrow$ no real sol^n ($+x=0$)

$\lambda = 1 \Rightarrow$ one real sol^n ($+x=0$)

$\lambda > 1 \Rightarrow$ two real sol^n ($+x=0$)

Diagram so far

pitchfork



Stability [long/explicit way]

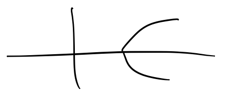
$$f = x + \frac{\mu x}{1+x^2}$$

$$Df = 1 + \frac{(1+x^2)\mu - \mu x \cdot 2x}{(1+x^2)^2} \quad \text{(quotient rule)}$$

$$= 1 + \frac{\mu + x^2(\mu - 2\mu)}{(1+x^2)^2}$$

$$= 1 + \frac{\mu(1-x^2)}{(1+x^2)^2}$$

Convenient points



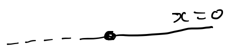
$x = 0$

$Df = 1 + \mu = 1 - \lambda$

> 0 if $\lambda < 1$

< 0 if $\lambda > 1$

So:



$\rightarrow$

Next consider x^+ & x^-.

$$Df(x,\lambda) = 1 - \frac{\lambda(1-x^2)}{(1+x^2)^2}$$

x^+ & x^- have same stability

(x^2 only in Df).

convenient:

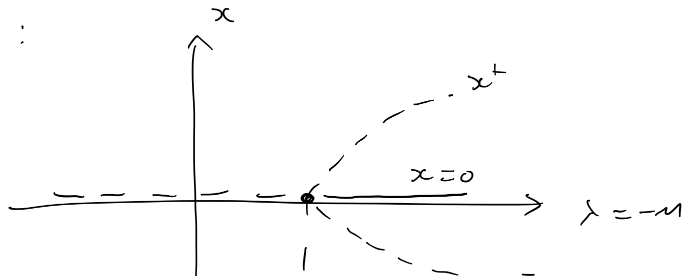
$x^+ = \sqrt{\lambda - 1}$

set $x^+ = 1 \Rightarrow \lambda = 2$

$Df = 1 - \cancel{2}$

$= 1 > 0$ unstable.

So:



pitchfork at $\lambda = 1$

ie $\mu = -1$.

Note: Taylor series near bifurcation point (x_c, μ_c)

(↳ after find where!)

$$f = x + \mu \frac{x}{1+x^2}$$

$$= x + (-1+\alpha)x\left(1 - x^2 + \cdots\right)$$

$$\simeq \alpha x + x^3 \quad \Big\} \text{ same form as pitchfork.}$$

near $x = 0, \mu_c = -1$

& where: $\mu = -1 + \alpha$

BUT: I want you to do in longer/intuitive/ more explicit way, ie actually checking Df etc, or at least graphical arguments

↳ Taylor series approach would require more rigorous development that can get ugly quickly (lots of different derivatives req.)

↳ see eg Glendinning.

Two-dim, 1-co-dim.

1. $\dot{x} = -y + \mu x + x y^2$

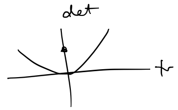
$\dot{y} = x + \mu y - x^2$

FP: $(0,0) \Rightarrow \dot{x} = \dot{y} = 0$ ✓

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} = A.$$

$\mathrm{tr} = 2\mu$

$\det = \mu^2 + 1$



expect pure imag./Hopf for $\mathrm{tr} = 0 \Rightarrow \mu = 0$.

charac. equation.

$$\lambda^2 - 2\mu\lambda + \mu^2 + 1 = 0$$

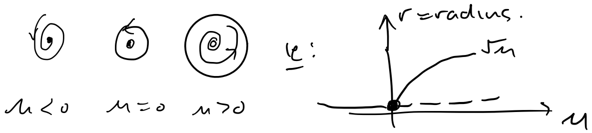
$$= (\lambda - \mu)^2 + 1 = 0$$

$$\lambda = \mu \pm i \quad \begin{cases} \mu < 0 \Rightarrow \text{stable spiral} \\ \mu = 0 \Rightarrow \text{pure im.} \\ \mu > 0 \Rightarrow \text{unstable spiral} \end{cases}$$

Crossing speed $\left.\frac{d(\mathrm{Re}(\lambda))}{d\mu}\right|_{\mu=0} = 1 \neq 0$

→ non-degenerate.

Assume supercritical (should use computer to check):



1d. More complicated example

$$\dot{x} = \mu x + x^3 - x^5$$

F.P. $\quad x(\mu + x^2 - x^4) = 0$

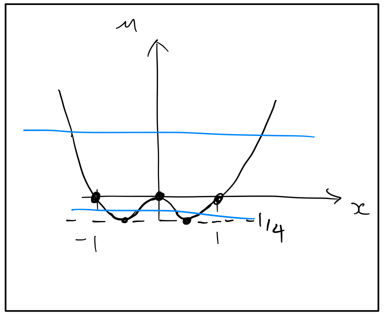
$x = 0$ or

$\mu + x^2 - x^4 = 0.$

Easier to think about μ vs x:

ie $\quad \mu = x^4 - x^2 = x^2(x^2 - 1)$

Shape



Solutions: (for $\mu = x^4 - x^2$)

$\mu > 0 \Rightarrow$ two sol^{n}s

$\mu = 0 \Rightarrow$ three sol^{n}s

$-1/4 < \mu < 0 \Rightarrow$ four sol^{n}s

$\mu = -1/4 \Rightarrow$ two sol^{n}s

$\mu < -1/4 \Rightarrow$ zero sol^{n}s.

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minima: $\dfrac{d\mu}{dx} = 4x^3 - 2x = 0 \Rightarrow 2x(2x^2 - 1) = 0$

ie $x = 0 \Rightarrow \mu = 0$

$x = \pm 1/\sqrt{2} \Rightarrow \mu = -1/4$

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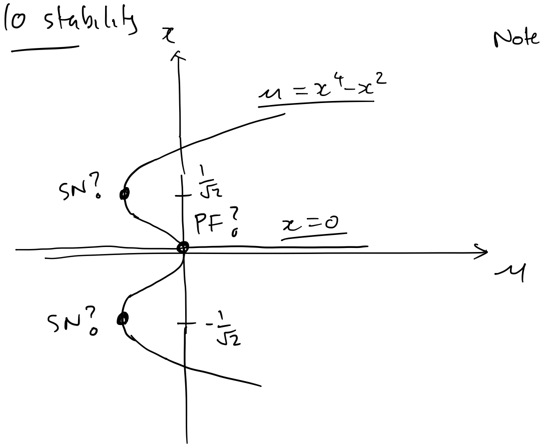
w/o stability

Note: symmetric in $x \to -x$.



$\to$ deformed pitchfork at origin

$\hookrightarrow$ expected since small higher power modification of $\mu x + x^3 - (x^5)$

expect higher powers to drop out near $x = 0$

$\to$ other bifurcations appear to be saddle-nodes.

Stability

$$Df = \mu + 3x^2 - 5x^4$$

$Df(x=0) = \mu$

if $\mu > 0 \Rightarrow Df > 0$ unstable

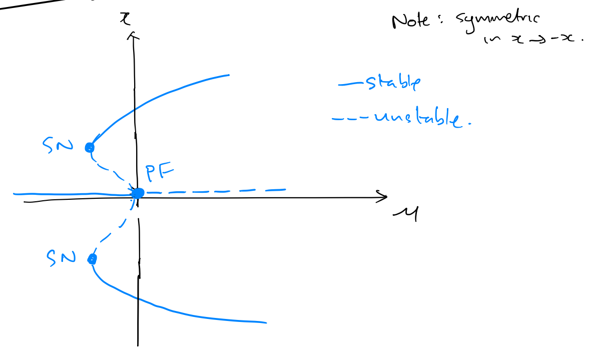
$\mu < 0 \Rightarrow Df < 0$ stable.

$$Df(\mu = x^4 - x^2) = x^4 - x^2 + 3x^2 - 5x^4$$

$$= 2x^2 - 4x^4$$

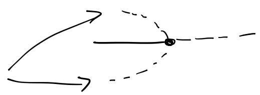
$$= 2x^2(1 - 2x^2) \left\{ \begin{array}{ll} > 0 \text{ if } x^2 < \frac{1}{2} & \text{unstable} \\ < 0 \text{ if } x^2 > \frac{1}{2} & \text{stable} \end{array} \right.$$

Final diagram

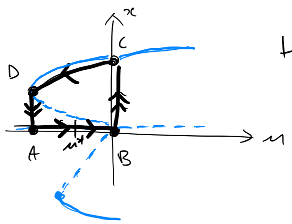


Notes:

- The pitchfork is subcritical: new sol^{ns} are unstable



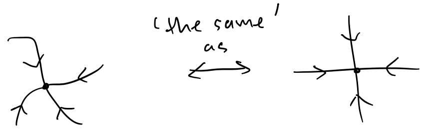
- The diagram below illustrates hysteresis & jumps can occur here.
  - $\to$ consider increasing μ from μ^* up to B, $\mu^* \in (A, B)$
  - $\to$ the system will follow stable branches.
  - $\to$ on $A \to B$, $x = 0$ is stable.
  - $\to$ when $\mu > 0$ the system jumps to x^+ or x^- (say x^+).
  - $\to$ when μ is decreased back from its B value, x stays at $x \neq 0$, rather than $x = 0$, until D.
  - $\to$ Thus we have a source of irreversibility / hysteresis: to return to $(x, \mu) = (0, \mu^*)$, μ^* in (A, B), we have to first decrease μ all the way back past A (& μ^*) & then back up to μ^*.
  - $\to$ ie once we go past B, we can't just decrease μ back to μ^* & get $x = 0$.



Hysteresis loop.

Comments on bifurcation theory & 'qualitative' analysis

Recall in analysing dynamics we were interested in connecting



The idea is that the dynamics are 'qualitatively the same' near hyperbolic fixed points

$\to$ they are 'the same' in the key/interesting respects

- ∟ fixed points
- ∟ stability

$\to$ we don't care about the rest of the exact 'quantitative' details!

Formally, 'the same' usually means:

can continuously transform one to other & back. Called a 'Homeomorphism'

Informally: equivalent up to 'stretching' etc.

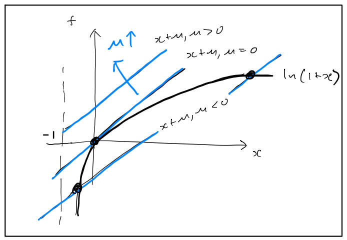
Similarly in bifurcation theory, ie near non-hyperbolic fixed points, we focused on the key 'qualitative' properties:

- number of fixed points
- stability

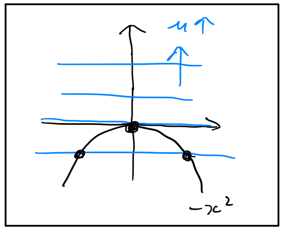
& how these change as a parameter varies

Two bifurcations are 'qualitatively the same' if they 'change' in the same way'.

Eg we saw more complicated cases give rise to the same types of bifurcation as in simple examples:



'the same' as $\longleftrightarrow$ [normal form theory]



FP: $x+\mu-\ln(1+x)=0$

ie

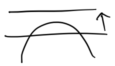
$y = x+\mu$
intersects
$y=\ln(1+x)$

FP: $\mu + x^2 = 0$

ie

$y = \mu$
intersects
$y = -x^2$

The 'key property' is 'looking like':



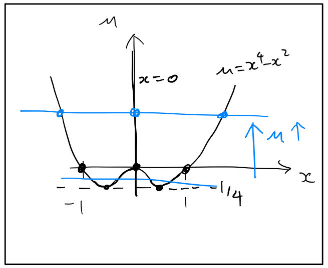
$\uparrow \mu$ } hence 'turning point' lol.!

ie the appearance/disappearance of fixed points $0 \Leftrightarrow 2$ FP.

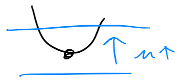
Similarly, FP: $x(\mu + x^2 - x^4) = 0$

$\Rightarrow$

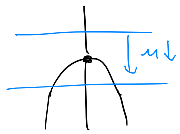
$x=0$
or
$\mu = x^4 - x^2$
$= x^2(1-x^2)$



Has local regions that 'look like':



SN ($\times 2$ by symmetry in x).



PF.

[can consider Taylor series of $f(x;\mu)$ near $f(x^*, \mu_c)$ for each x^*, μ_c].

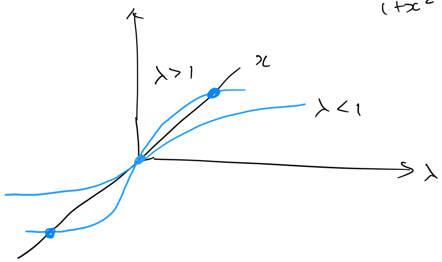
Exercise: sketch similar diagrams for

$$\boxed{f = x - \frac{\lambda x}{1+x^2}}$$

Hint: split into two functions of x one depending on λ, one independent of λ, & consider λ variations.

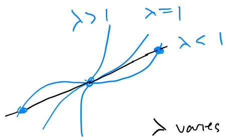
Partial solution

$$\left.\begin{array}{l} y = x \\ y = \dfrac{\lambda x}{1+x^2} \end{array}\right\} \text{FP } x = \frac{\lambda x}{1+x^2}$$



'Like'

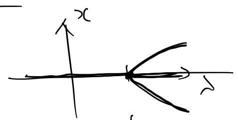
$x - \lambda x^3 = 0$:



λ varies

1 ⟷ 3 fixed points.

ie pitchfork:



1 sol ← | → 3 sol

[Could also deduce from Taylor series]

Lesson: geometrically, the 'same' things are happening in the more complicated systems as happen in the 'simple' systems.

→ rigorous: coordinate change that preserves key features. } normal form theory