

ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(...and other dynamical systems)

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LECTURE 3: STABILITY AND LINEARISATION

- Stability of solutions for general systems
- Stability of solutions for linear systems
- Linearisation of nonlinear systems
- Connecting stability of linearised systems to stability of nonlinear systems
- Sneak peak at structural stability

MODULE OVERVIEW

Qualitative analysis of differential equations (Oliver Maclaren)

[~17-18 lectures/tutorials]

1. Basic concepts, stability and linearisation [4 lectures/tutorials]

Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest: fixed points, periodic orbits etc. Overview of basic analysis procedure including linearisation, connecting stability of nonlinear systems and stability of linearised systems. Computer-based analysis.

STABILITY OF SOLUTIONS

There are various *general* formal definitions of *stability* for solutions.

These can be defined for both equilibria as well as more complicated objects like periodic orbits. They also apply for both linear and nonlinear systems.

We will just give the *definitions for equilibria* (points) for now.

- Recall how to find equilibria.

STABILITY OF SOLUTIONS

- A point x is *Lyapunov stable* iff for all $\epsilon > 0$ there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| < \epsilon$ for all $t \geq 0$.
- A point x is *quasi-asymptotically stable* (attracting) iff there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| \rightarrow 0$ as $t \rightarrow \infty$.
- A point is *asymptotically stable* iff it is both Lyapunov stable and quasi-asymptotically stable. If it is just Lyapunov stable then it is *neutrally stable* (i.e. is just bounded).

WHAT DO THESE MEAN?

Pictures!

STABILITY OF LINEAR SYSTEMS

These general definitions can be hard to check in general but are *easy to check for linear systems*.

Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

(This can be proven by constructing a so-called Lyapunov function - ask me if interested/see further reading)

LINEARISATION AND LOCAL STABILITY ANALYSIS

We have seen that stability for linear systems is easy. We will analyse and classify linear systems in more detail soon.

This will be useful because, as mentioned, our first steps in analysing *nonlinear* systems will usually be through local *linearisation* about steady-states/equilibria.

We will also need to know the *connection between linear and nonlinear stability*!

LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component (using the Einstein summation convention) gives

$$\dot{u}_i = f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2)$$

i.e....

LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

$$\dot{u}_i \approx \frac{\partial f_i}{\partial x_j}(x_e)u_j \equiv [Df(x_e)]_{ij}u_j$$

or simply $u = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

LINEAR AND NONLINEAR STABILITY - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues of the linearisation have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

LINEAR AND NONLINEAR STABILITY CONNECTED

The *Hartman-Grobman theorem* states that the local properties *near a hyperbolic fixed point* of a *nonlinear* system are *topologically equivalent* to those of the *linearisation*:

A hyperbolic fixed point persists in the change from nonlinear to linear systems (though its location may shift slightly) and its *stability properties are also preserved*

(We will also see the similar 'stable manifold theorem' later)

BIFURCATIONS AND STRUCTURAL INSTABILITY

Recall: fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

When these occur our *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities* - i.e. small variations in problem parameters can have a large effect on the qualitative/topological features of our phase space.

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

BIFURCATIONS AND STRUCTURAL INSTABILITY

These instabilities are called *bifurcations*.

We will return to this topic in more detail in later lectures.

Zero eigenvalues also lead to *scale separation* and decomposition into *slow* and *fast* solution components. We will look at this in *centre manifold theory*.

Eng Sci 711 L03.

• Stability & Linearisation

- Goals :
- understand the various concepts of stability in general
 - Understand when a linear stability analysis gives a 'faithful' picture of a nonlinear system
 - be able to carry out a linear stability analysis of a nonlinear system

Examples: Key skills

Consider the system

$$\begin{aligned}\dot{x} &= y(2x - y) \\ \dot{y} &= x^2 - y\end{aligned}$$

- (a) Find the two fixed points of this system. Show your working. You do not need to classify these.

- (b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$.

- (c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x,y) plane. You do not need to show any nearby trajectories.

2016 Exam

} today

Consider the system

$$\begin{aligned}\dot{x} &= 2xy + x^3 \\ \dot{y} &= -y - x^2\end{aligned}$$

where $x, y \in \mathbb{R}$.

- (a) Verify that the origin is a fixed point of this system. (1 mark)

- (b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$. (2 marks)

- (c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x,y) plane. You do not need to show any nearby trajectories. (3 marks)

2017 Exam

} today

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- (a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$.

2016 Exam

} today

Example

Consider the nonlinear system (see the computer tutorial)

$$\begin{cases} \dot{x} = x(y-1) \\ \dot{y} = 3x - 2y + x^2 - 2y^2 \end{cases} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

- We're interested in long-term behaviour like fixed points (equilibria) & their stability

- What does this mean?

→ First let's find the fixed points } first step in most analyses!

Procedure: try solving in steps as follows:

(easiest) $\rightarrow (1) = 0 \Rightarrow x = 0 \text{ or } y = 1$

o $x = 0$ & $(2) = 0$: ← (harder one)
 $-2y - 2y^2 = 0 \Rightarrow -2y(1+y) = 0$
 $\Rightarrow y = 0 \text{ or } y = -1$
so: $(0,0)$ or $(0,-1)$

o $y = 1$ & $(2) = 0$:

$$3x - 2 + x^2 - 2 = 0$$

$$\Leftrightarrow x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

$$\Rightarrow x = -4 \text{ or } x = 1$$

$$\text{so: } (-4,1) \text{ or } (1,1)$$

⇒ Solutions: $\left\{ (0,0), (0,-1), (1,1), (-4,1) \right\}$ | (fixed points)

So we see that eg

$(x, y) = (0, 0)$, i.e. the origin, is a fixed point

$$\boxed{f(0, 0) = 0 \text{ or } \varphi((0, 0), t) = (0, 0)} \\ \text{& } \dot{x} = \dot{y} = 0$$

i.e. don't go anywhere if start exactly there

What does it mean for this to be a stable fixed point?

→ relates to starting nearby, not just exactly there

↳ all models are wrong/imperfect etc

We want a general definition, valid for nonlinear systems

→ we will then relate this to the more familiar linear definition & see under what conditions they give the same answer

↳ often but not always

Stability: general definitions

There are at least three 'general' concepts of stability

→ can be applied to general 'objects'
(invariant sets like equilibria, periodic orbits, etc.)

→ we'll look at stability of equilibria

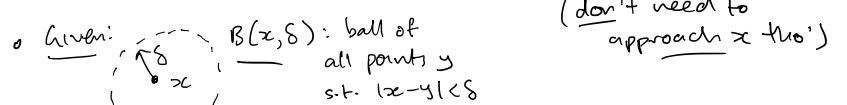
1. o Lyapunov (or Liapounov)

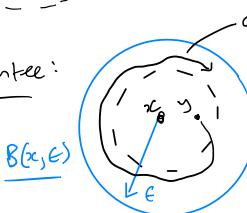
2. o Quasi-asymptotic

3. o Asymptotic (Lyapunov & Quasi.)

goal:
intuitive
understanding
of these.

1. Lyapunov: "points that start nearby, stay nearby"
('bounded')

o Given:  $B(x, S)$: ball of all points y s.t. $|x - y| < S$
(don't need to approach x tho')

o Guarantee:  $\varphi(y, t)$ for $y \in B(x, \epsilon)$
 $|\varphi(y, t) - \varphi(x, t)| < \epsilon$
 $|\varphi(y, t) - x| < \epsilon$ if x fixed point
since $\varphi(x, t) = x$ w.t.b.

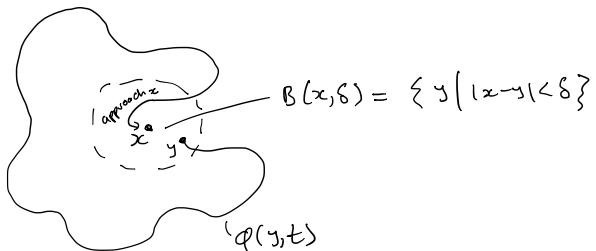
"give me a target tolerance on future trajectories ϵ & I'll give you a starting tolerance S that guarantees you'll meet this target"

Math: $\forall \epsilon > 0 \exists S > 0$ s.t.

if $y \in B(x, S)$ then $\varphi(y, t) \in B(x, \epsilon)$

2. Quasi-asymptotic (attracting)

"start nearby, eventually return/approach
arbitrarily close to"



Math:

If $y \in B(x, \delta)$ (i.e. $|x-y| < \delta$)

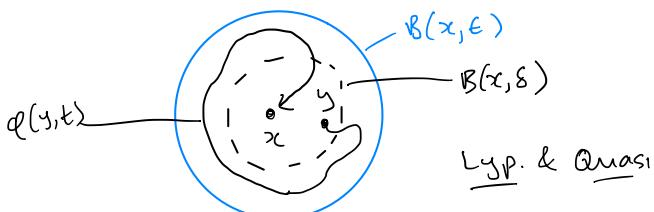
then $q(y, t) \xrightarrow{t \rightarrow \infty} q(x, t) = x$

if fixed point

(can use general defn
 for more complex
 objects)

3. Asymptotic: Lyapunov & Quasi-asymptotic

"start nearby, stay nearby [] return/approach to"



→ our goal is usually
asymptotic stability

Problem?

- These definitions are general
 but therefore: hard to apply/check!

- On the other hand, linear systems have an easy criterion:

given $\dot{x} = Ax$, $x \in \mathbb{R}^n$, A $n \times n$ matrix
 the origin $x=0$ is asymptotically stable if all of the eigenvalues of A have negative real part

Eg $\lambda_1 = 1, \lambda_2 = -2 \rightarrow$ unstable

$\lambda_1 = -1, \lambda_2 = -2 \rightarrow$ stable

$\lambda_1 = -1+i, \lambda_2 = -1-i \rightarrow$ stable

$\lambda_1 = 0, \lambda_2 = -1 \rightarrow$ marginal (see later)

→ it would be nice if we could just 'linearise' our nonlinear systems & use this instead!

↳ we can, under certain conditions!

→ first: linearisation

Linearisation (at a fixed point)

General procedure: derivation

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- (a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$. 'small perturbation' from x^*

exam

2016

1. Fixed point: $f(x^*) = 0$ vector form

or $f_i(x^*) = 0, \forall i$ component form \leftarrow easier

2. Taylor series for $x = x^* + \eta$ ($\eta = x - x^*$)
in component form:

work-horse
of
applied
math!

$$\begin{aligned}\dot{x}_i &= f_i(x) = f_i(x^* + \eta) \quad \text{full vector} \\ &\approx f_i(x^*) + \eta_j \frac{\partial f_i}{\partial x_j}(x^*) \quad \text{(sum over } j\text{)} \\ &= 0 \quad \text{since fixed point}\end{aligned}$$

$$\Rightarrow \dot{x}_i \approx \eta_j \frac{\partial f_i}{\partial x_j}(x^*) \quad \text{eval. at fixed point}$$

$\downarrow \quad \downarrow \quad \downarrow$
vector vector matrix

$\left(\eta_j \frac{\partial f_i}{\partial x_j} \text{ means } \sum_j \eta_j \frac{\partial f_i}{\partial x_j} \right)$
i.e. sum over repeated indices

Have $\dot{x}_i = \frac{\partial f_i}{\partial x_j} \eta_j$

want: $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j$

\Rightarrow Note $x = x^* + \eta$

\uparrow
fixed vector (time independent)

$$\Rightarrow \dot{x} = \dot{\eta} \quad (\dot{x}^* = 0)$$

so $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j$ (sum over j)

i.e. $\dot{\eta} = A\eta$ (note: $A\eta$ is just matrix multiplication)

\Rightarrow Linear system in new 'perturbation' variable $\eta = x - x^*$

Here $A = 'Df'$ the Jacobian derivative of f

\rightarrow generalises derivative to vector-valued functions of vector variables, like f

\rightarrow get a matrix, with components:

$$(Df)_{ij} = Df_{ij} = \frac{\partial f_i}{\partial x_j} \quad \begin{array}{l} \text{since vector valued} \\ \text{since vector variable} \end{array}$$

\rightarrow

Huh?

→ Concrete example!

Let's linearise our previous system:

$$\begin{cases} \dot{x} = x(y-1) &= f_1(x, y) \\ \dot{y} = 3x - 2y + x^2 - 2y^2 &= f_2(x, y) \end{cases} \quad \begin{aligned} &= f_1(x_1, x_2) \\ &= f_2(x_1, x_2) \end{aligned}$$

& then evaluate at $(0, 0)$.
 (here $x_1 = x$
 $x_2 = y$)

First, find $Df(x, y) = Df(x_1, x_2)$

(at any x, y first. Then at eg $(0, 0)$
 or other fixed point)

i.e. $Df_{ij} = \left(\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right) = \left(\begin{array}{cc} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{array} \right)$

$$= \left(\begin{array}{cc} y-1 & x \\ 3+2x & -2-4y \end{array} \right)$$

Next, evaluate at $(0, 0)$ (for example) -

$$Df(0, 0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

Be able to do these!

Eigenvalues?

$$Df(0, 0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix} = A$$

Long way: (for now) — see later for 'shortcuts'
 $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -1 - \lambda & 0 \\ 3 & -2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1 - \lambda)(-2 - \lambda) - 3 \times 0 = 0$$

$$\Rightarrow +(\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1, \lambda_2 = -2} \quad \begin{array}{l} \text{linearised} \\ \text{system is} \\ \text{stable} \end{array}$$

$$(Re(\lambda_i) < 0 \forall i)$$

But:

→ Is this linear stability analysis applicable to the original nonlinear system?

→ When? Why?

Hyperbolic fixed points & the Hartman-Grobman theorem

Informal version:

- If the linearised system is definitely stable/unstable, i.e. $\operatorname{Re}(\lambda_i) \neq 0 \forall i$, then the nonlinear system is too
- Fixed points for which $\operatorname{Re}(\lambda) \neq 0 \forall i$ are called hyperbolic
- These are 'robust' in a number of ways:
 (continue to exist)
 - persistence under linearisation

Structural stability/
stability/
see later
Conjugacy theory

{ - persistence under small changes to
the governing equations
themselves (no bifurcations)

- stability properties are preserved
under linearisation

Eg:

$\lambda_1 = 1, \lambda_2 = -2$	\rightarrow	hyp.
$\lambda_1 = -1, \lambda_2 = -2$	\rightarrow	hyp.
$\lambda_1 = -1+i, \lambda_2 = -1-i$	\rightarrow	hyp.
$\lambda_1 = 0, \lambda_2 = -1$	\rightarrow	non-hyp.
$\lambda_1 = i, \lambda_2 = -i$	\rightarrow	non-hyp.

} need
centre
manifold
theory

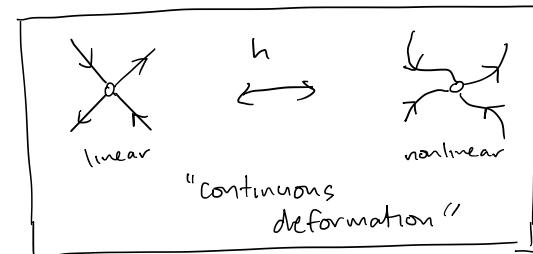
More formal: Hartman-Grobman theorem

"If $x=0, x \in \mathbb{R}^n$, is a hyperbolic fixed point of $\dot{x} = f(x)$, then there is a continuous 1-1 map on some neighbourhood of x that maps orbits of the nonlinear flow to those of the linear flow & which preserves the direction of time"

$\operatorname{Re}(\lambda_i) \neq 0 \forall i$

? - homeomorphism

i.e. the linear & nonlinear systems are locally, topologically equivalent near a hyperbolic fixed point:



topologically conjugate:

$$\begin{array}{ccc} f & & \\ x \xrightarrow{f} x & \xleftarrow{h} & y \xrightarrow{g} y \\ h \downarrow & & \downarrow h \\ Y \xrightarrow{g} Y & & \end{array}$$

$$h \circ f = g \circ h$$

$$f = h^{-1} \circ g \circ h$$

$\rightarrow h$ is a nonlinear change of coordinates*

\rightarrow same number of fixed points

\rightarrow fixed points have same stability etc

} just a 'stretched'
version of
'same thing'
 \rightarrow qualitatively
equivalent

* But usually 'near identity': $y = x + \text{nonlinear terms}$

\rightarrow Idea goes back to Poincaré: 'normal form theory'

(linearization theorem)
H.G. Hirsch
extends this

Bifurcations? (Preview of next section of course)

- we have looked at stability of solutions to a fixed system

- a related idea is stability of the systems/equations themselves
- } important since:
'all models are wrong'!
- \hookrightarrow "Structural stability"

eg $\dot{x} = f(x)$

$$\downarrow$$

$$\dot{x} = f(x) + \epsilon g(x)$$

↑ small epsilon
small change to equations themselves

Q: do these systems have 'similar' properties?
 → same number of fixed points with same stability?

Similar to asking if linear system is 'like' nonlinear system!
 → ignoring 'higher order' terms

Not surprisingly, the answer is :

- yes if near hyperbolic fixed point
- no if near non-hyperbolic fixed point

→ see later.

More Examples

$$\begin{cases} \dot{x} = y(2x-y) \\ \dot{y} = x^2 - y \end{cases}$$

From: Exam 2016 Q5.

- Find fixed points
- Linearise
- Determine if origin is hyperbolic
- Determine if origin is stable

Fixed Points

'Divide & conquer'

- ① $= 0: \Rightarrow y(2x-y) = 0$
 $\Rightarrow y=0$ OR $y=2x$

- $y=0$ & ② $= 0:$

$$x^2 = 0 \Rightarrow x = 0$$

so: $(0,0)$

- $y=2x$ & ② $= 0:$

$$x^2 - 2x = 0 \Rightarrow x(x-2) = 0$$

$$\Rightarrow x=0$$
 OR $x=2$

so: $(0,0)$ (already have)
 OR $(2,4)$

Solutions: $\boxed{\{(0,0), (2,4)\}}$ (Fixed points)

Linearisation: arbitrary point (x, y)

$$\dot{x} = y(2x - y) = 2xy - y^2 = f_1(x, y)$$

$$\dot{y} = x^2 - y = f_2(x, y)$$

want Df , where $Df_{ij} = \frac{\partial f_i}{\partial x_j}$
row col.

$$Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 2y \\ 2x & -1 \end{bmatrix} \quad \text{(valid for any } (x, y))$$

At origin $(x, y) = (0, 0)$ (recall: is a fixed point)

$$Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\lambda = 0, -1 \Rightarrow \text{non-hyperbolic!}$$

\Rightarrow marginal/unknown stability
