ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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LECTURE 2 - TERMINOLOGY & PAINTING PICTURES

- Some terminology: definitions, key features of interest etc
- Phase portraits

i.e. a quick tour of the dynamical systems *palette* that we use to 'paint' pictures of system behaviour, called *phase portraits*

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

1. *Basic concepts, stability and linearisation* [4 lectures/tutorials]
   Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest: fixed points, periodic orbits etc. Overview of basic analysis procedure including linearisation, connecting stability of nonlinear systems and stability of linearised systems. Computer-based analysis.

RECALL: ORDINARY DIFFERENTIAL EQUATIONS

Our main focus is on a very familiar type of dynamical system - systems of ordinary differential equations of the form:

$$\dot{x} = f(x, t; \mu)$$

where $x \in \mathbb{R}^n$ is a vector of *state variables*, $t \in \mathbb{R}$ is the *independent variable* (usually time), $\mu \in \mathbb{R}^m$ is a vector of *problem parameters*.

(For now we often suppress the dependence on problem parameters - but see bifurcation theory!)

RECALL: ORDINARY DIFFERENTIAL EQUATIONS

We have one equation in f for each entry in the state vector x e.g.

$$x = (x_1, x_2, \ldots)^T$$

$$f = (f_1, f_2, \ldots)^T$$

If there is no dependence on t then we say the system is *autonomous* (we can always add a new dependent variable to track time dependence). We will focus on these in this course.

TERMINOLOGY: SOLUTIONS AND INTEGRAL CURVES

A *solution* x_s (or trajectory) is a *function* assigning a state vector to each time in a given time interval and which satisfies the ODE, i.e. $x_s : T \subset \mathbb{R} \to \mathbb{R}^n$, where $\dot{x}_s(t) = f(x_s(t))$.

An *integral curve* is the *graph* of a solution, including the time dimension, i.e. the *set of points*

$$\{(x, t) \mid t \in T \text{ and } x(t) \text{ defines a solution}\}$$

TERMINOLOGY: FLOW FUNCTIONS

The *flow function* ϕ is a convenient way to combine our description of *solutions and their dependence on initial conditions*.

We write

$$\phi(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

where for fixed x_0, $\phi(x_0, t)$ gives the solution to the differential equation at time t which starts from an initial value (at $t = 0$) equal to x_0.

TERMINOLOGY: FLOW FUNCTIONS

So, for any t, and for $\phi(x, 0) = x$, we have

$$\frac{d}{dt}\phi(x, t) = f(\phi(x, t))$$

and

$$\phi(x, t + s) = \phi(\phi(x, t), s) = \phi(\phi(x, s), t) = \phi(x, s + t)$$

(for any 'allowed' t, s). We talk about 'flows' when we want to emphasise the dependence on initial conditions (rather than just time)

TERMINOLOGY: ORBITS

Orbits are the geometric objects in the phase space that are generated by solutions/flows.

In terms of the flow function an orbit beginning at x_0 can be described by $\{\phi(x_0, t) \mid t \in T\}$.

Usually we take $T = \mathbb{R}$ and hence consider all solutions passing through x_0 (and both forwards and backwards in time *if* invertible!).

These can be labelled with a 'time direction' but are otherwise essentially 'static' (geometric) objects.

EXISTENCE AND UNIQUENESS?

We won't go into this, but for a sufficiently smooth system written in our standard form and given appropriate initial conditions *there exists a unique solution*.

Key: the solution curves/trajectories (for autonomous systems) *do not intersect in phase space*. We always know exactly where to go next!

TERMINOLOGY: VECTOR FIELD

The solutions/orbits are *tangent to the 'velocity vector'* $(\dot{x}_1, \dot{x}_2, ...)^T$ defined by the ODE $\dot{x} = f(x)$ at each point in the state space (and at each time).

We often call $f(x)$ the *vector field* of the ODE. The *direction* (relative velocity of components) can be determined by dividing through by one (non-zero) component i.e.

$$\frac{\dot{x}_k}{\dot{x}_1} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, ..., x_n)}{f_1(x_1, ..., x_n)}$$

(Now how this relates the 'static' and 'dynamic' objects).

FEATURES OF INTEREST

We will look at (and define!) various *'interesting features'* of our equations in phase space e.g.

- Stationary/fixed/equilibrium points
- Periodic orbits

and try to analyse their properties such as *stability* under different types of 'perturbations' - both 'within' a model (*solution* stability) and 'externally' (*structural* stability) to a model.

INTERESTING FEATURES - EQUILIBRIA

A point x_e is an *equilibrium solution/fixed point/stationary point/singular point* iff

$$\phi(x_e, t) = x_e$$

for all t. Equivalently it is a *zero of the vector field* (RHS)

$$f(x_e) = 0$$

INTERESTING FEATURES - NULLCLINES

Given a system of equations for $x \in \mathbb{R}^n$ with components $\dot{x}_i = f_i(x)$, the jth *nullcline* is where

$$\dot{x}_j = f_j(x) = 0$$

The flow is *perpendicular* to the x_j-axis along the associated curves/surfaces (usually of dim $n-1$). Note: there *may be multiple lines/curves for one nullcline!*

Q: What are points where *all* nullclines intersect called?

Nullclines are a very useful part of sketching phase-plane portraits!

MORE FEATURES - PERIODIC POINTS AND PERIODIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t+T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t+s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

MORE FEATURES: INVARIANT SETS

A set of points in the phase space M is called *invariant under the flow* if for all $x \in M$ we have

$$\phi(x, t) \in M$$

for all t. That is, every point in M leads to another point in M - once in, we never leave!

MORE FEATURES - LIMIT SETS

Other useful definitions include the following (invariant!) sets:

The *ω-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in forward time*.

Formally it consists of elements y such that there exists a sequence (t_n) with $t_n \to \infty$ and $\phi(x, t_n) \to y$ as $n \to \infty$.

TERMINOLOGY: PHASE PORTRAITS

We will summarise the various key features we've seen in *phase portraits* of a given system.

A phase portrait is a 'picture' of the phase space in which we further *partition it according to orbit/solution 'types' or behaviour* in different regions.

MORE FEATURES - LIMIT SETS

The *α-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in backwards time* (exercise: write down the formal definition!)

Note that the points in these sets *don't have to lie on the orbits* through x - they are *limit* points for a reason!

EngSci 711 L02.

- Terminology & 'Painting Pictures'

  [Goals]: • know what I'm talking about –
  ↳ equilibria, nullclines, phase portraits etc.

Example questions:

- Exam 2017: (Notice the terminology in particular, as well as steps of analysis)

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors. (6 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch. (10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough. (2 marks)

Painting pictures

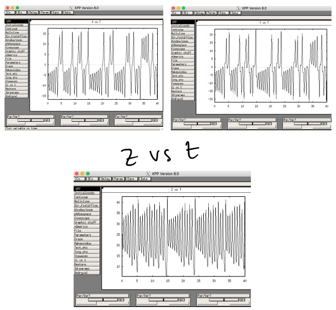
- Our goal is going to be to take a 'dynamical system' that 'evolves in time'
  ↳ eg a [system of ODEs]

  & 'paint a geometric/qualitative picture' of this system in state space (phase space)

- eg Lorenz system: from L1.

x vs t y vs t

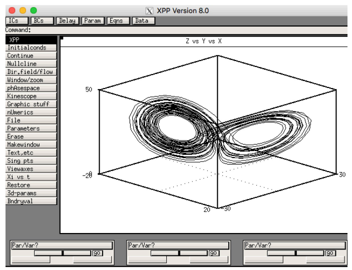
z vs t



→

State space / Phase space

x vs y vs z



a 'picture'

→ First, we should probably establish some formal terminology

(not about memorising → so we are on same page!)

Recall:

We'll focus on systems of ODEs in the form

$$\dot{x} = f(x;\mu)$$ [autonomous: no t in RHS]

$\dot{x}$ ↑ first order in time

x ↑ state vector, $x \in \mathbb{R}^n$

μ ← problem parameters } for bifurcation theory

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ie
$$\underset{\uparrow\ \text{vector}}{\dot{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(\overbrace{x_1, x_2, \ldots}^{x \text{ vector}}) \\ f_2(\ldots\ldots) \\ \vdots \\ f_n(\ldots) \end{pmatrix} = \underset{\uparrow\ \text{vector}}{\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}} = f(x)$$

- parameter?
  - L held fixed during evolution
  - L eg rate constant, diffusion constant etc
  - L later, we'll sometimes imagine as super slow state vars, ie $\dot{\mu} = 0 \Rightarrow \mu = \text{const.}$
  - L for now we'll most ignore explicit dep. on param.

Solutions, integral curves, orbits & vector fields

(variations on a theme; often lazy about distinction...)

- Solution: a function $\boxed{t \mapsto \text{state}}$
  - L also called: • trajectory • phase curve etc.

- Integral curve: a graph/set of the solⁿ function incl. time $\boxed{\{(x,t) \mid t \in T \ \& \ x(t) \text{ is sol}^n\}}$
  - ↑ incl. time

- Orbit: a set of points in state space generated by a solⁿ $\boxed{\{x \mid t \in T \ \& \ x(t) \text{ a sol}^n\}}$
  - ↑ (don't incl. time explicitly (usually use all time to generate 'static' object)

- Vector field: the RHS $f(x)$,
  - (also called: direction fields)
  - → the direction (relative velocities) can be determined via eg

$$\boxed{\frac{\dot{x}_k}{\dot{x}_1} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, \ldots x_n)}{f_1(x_1, \ldots x_n)}} \text{ for } \dot{x}_1 \neq 0$$

think: little arrows to which local flow is tangent

Example (Based on Wiggins 2003, example 0.0.1)

Consider $\dot u = v$, $\dot v = -u$, $(u,v) \in \mathbb{R}^1 \times \mathbb{R}^1$

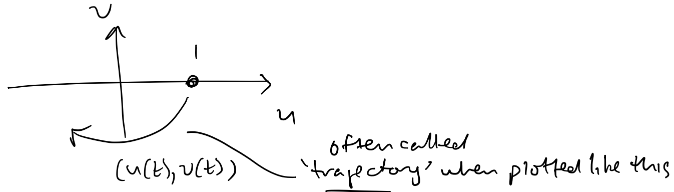
- Find the solution, integral curve & orbit, all passing through $(u,v)=(1,0)$
- Sketch the vector field in the u–v plane

- Solution: function $t \to (u(t), v(t))$

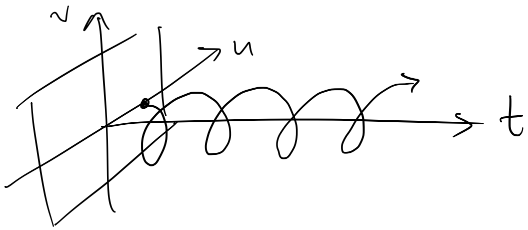
  Here: $(u(t), v(t)) = (\cos t, -\sin t)$

  [verify: $\dot u = -\sin t = v(t)$ ✓
  $\dot v = -\cos t = -u(t)$ ✓]

ie



- Integral curve: set $\{(u,v,t) \mid u(t)=\cos t, v(t) = -\sin(t), \forall t\}$

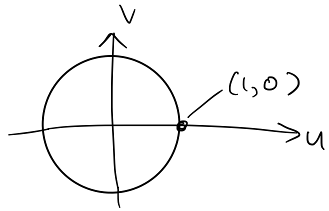


Example cont'd

Orbit: set of points in (u,v) space generated by solutions

→ convenient representation:

$$\{(u,v) \mid u^2+v^2=1\}$$



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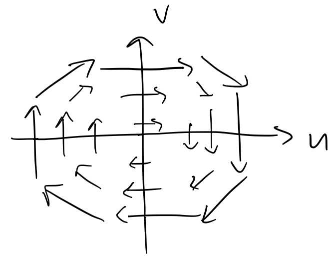
vector field ($\dot u = v, \dot v = -u$)

note: $\frac{\dot v}{\dot u} = \frac{dv}{du}$ = slope of v vs u

$= -\frac{u}{v}$ (for $v \neq 0$)

⇒ plug in some points

(0,1)
(1,0)
(0,−1)
(−1,0)
(1,1)
etc.



u↑ ⇒ $\dot v$ more neg.

v↑ ⇒ $\dot u$ more pos.

Flow function φ

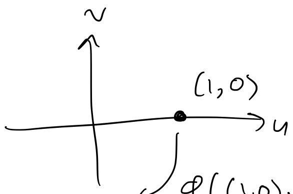
When we want to emphasise dependence of sol^n on an initial condition x_0

$$\varphi(x,t) : \underset{\substack{\uparrow \\ \text{initial} \\ \text{state}}}{\mathbb{R}^n} \times \underset{\substack{\uparrow \\ \text{time} \\ \text{interval}}}{\mathbb{R}} \longrightarrow \underset{\substack{\uparrow \\ \text{new} \\ \text{state after} \\ \text{time interval}}}{\mathbb{R}^n}$$

} think: evolution operator, advancing $\underset{t=0}{x_0} \longrightarrow \underset{\text{at } t}{x_{new}}$

eg if $\varphi(x,0) = x_0$

then $\varphi(x_0,t)$ = new state at time t after starting from $t=0$ at x_0.



$\varphi((1,0),t)$ in previous example

Formal properties (just for fun!)

- $\dfrac{d\,\varphi(x,t)}{dt} = f(\varphi(x,t)) \quad \forall t$ (ie satisfies ODE)
- $\varphi(x,t+s) = \varphi(\varphi(x,t),s) = \varphi(\varphi(x,s),t) = \varphi(x,s+t)$

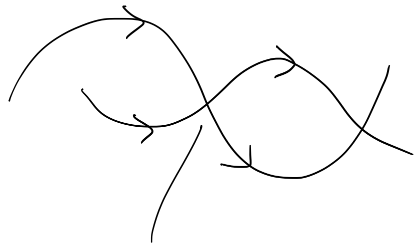
  (can evolve bit by bit)

Existence & uniqueness

"always know where to go next"

→ no intersecting/crossing trajectories (unless stop at fixed point)

NO!



can't cross & keep going!

→ helpful in phase-plane to 'trap' flows

Equilibria & Nullclines

- Fixed point/equilibrium point etc:

$$\varphi(x_e, t) = x_e \quad \forall t$$

don't go anywhere!

or

$$f(x_e) = 0$$

} this is how we find (set RHS = 0).

Usually best to solve for in steps:

Question 6 (18 marks) (Exam 2017)

Consider the system

$$\dot{x} = x^2 + y^2 - 2 \quad (1)$$
$$\dot{y} = x - 1 \quad (2)$$

where $x, y \in \mathbb{R}$.

Find fixed points.

Step 1. set easiest to zero.

$(2) = 0 \Rightarrow x = 1$

Step 2 plug each sol^n (only one here) into other eq^n(s). etc.

$x = 1$

$(1): 1 + y^2 - 2 = 0$

$\Rightarrow y = \pm 1$

Sol^ns: $\{(1,1), (1,-1)\}$

Nullclines

- Nullcline: zero components of f

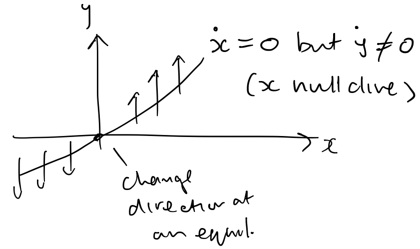
ie if $\dot{x}_i = f_i(x)$, $x_i \in \mathbb{R}^1$

then jth nullcline is

$$\dot{x}_j = \boxed{f_j(x) = 0}$$

just the jth component is zero

eg



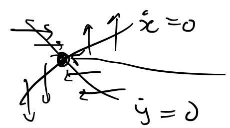
} will cover in detail later.

Exercise: try Exam 2017 example.

- Note: relation to fixed points

a fixed point is where $f_j(x) = 0 \ \forall j$

$\Rightarrow$ where all nullclines intersect



$\dot{x} = 0$ & $\dot{y} = 0$

Periodic point

- $\boxed{\varphi(x_e, t+T) = \varphi(x_e, t)}$ & $\varphi(x_e, t+s) \neq \varphi(x_e, t)$, $\underline{0 < s < T}$ } T is smallest period

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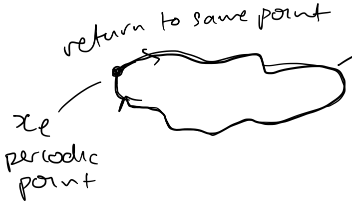
Periodic orbit

$$\{ \varphi(x_e, t) \mid t \in \mathbb{R} \}$$

where x_e is periodic



whole orbit is periodic!

(challenge: prove!)

Invariant Sets

- generalisation of fixed points & periodic orbits

"once in, don't leave" (eg 'trapping region')

If $x \in M$ then $\varphi(x, t) \in M \ \forall t$.

(eg fixed point $\varphi(x^*, t) = x^* \ \forall t$ $\Rightarrow \{x^*\}$ is invariant)

Limit Sets

- Another type of invariant set & generalisation of fixed point/periodic orbit ideas

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ω-limit set of a point x

→ 'forward limit set'

→ all points towards which flow from x tends as $t \to \infty$

└ don't have to lie on trajectory: are limit points!

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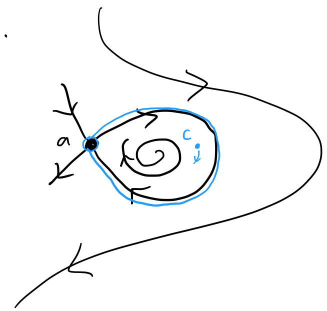
eg.



$\omega(c)$?

- flow approaches 'a' arbitrarily closely
- also approaches the orbit connecting 'a' to itself ('homoclinic orbit')

$\Rightarrow \omega(c) = \{$all points in blue$\}$ (not incl. c of course.)

$\Rightarrow \omega(c) = \{a\} \cup \{$homoclinic orbit$\}$

α-limit set of x

→ same idea but in 'backwards time'!

Phase Portraits : a depiction of key 'interesting' features in phase (state) space

→ try to 'partition' phase space into different regions of behaviour

→ identify fixed points, periodic orbits etc & analyse their stability

→ sketch local trajectories & direction fields

Eg (not quite ready to answer yet)

- Exam 2017 :

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(6 marks)

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(10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

Overview/preview of analysis procedure :

- Find fixed points
- Analyse (linear) stability of fixed points
- Classify fixed points
- Find other 'global' features
  - eg periodic orbits

'portrait' { - Sketch these + interesting trajectories }

bifurcation & centre manif. theory
- Analyse cases of marginal stability
- Use perturbation methods to construct approximate sol^ns & complete sketches