

# ENGSCI 741

## INVERSE PROBLEMS

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## MODULE OVERVIEW

### 3. Applications [2-3 lectures/tutorials]

Deblurring, numerical differentiation, parameter estimation, tomography, remote sensing...(we'll see!)

1

3

## MODULE OVERVIEW

Inverse Problems (*Oliver Maclarens*) [~9 lectures/tutorials]

### 1. Basic concepts [3 lectures]

Forward vs inverse problems. Well-posed vs ill-posed problems. Algebra of inverse problems (generalised inverses etc). Regularisation and trade-offs.

### 2. More regularisation [3-4 lectures/tutorials]

Higher-order Tikhonov regularisation, truncated singular value decompositions, iterative regularisation. Statistical view of inverse problems?

## LECTURE 6: ITERATIVE METHODS

Topics:

- Features of nonlinear and/or large linear inverse problems
  - Benefits of/need for iterative methods
- Brief overview of iterative minimisation for regularised problems
- Iteration plus early stopping as a regularisation method
  - Semi-convergence and singular limits

2

4

## Eng Sci 741 : Lecture 6.

### Iterative methods

For nonlinear &/or large ,

need :

- Iterative minimisation algorithms
- Iterative regularisation

↳ main focus.

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### Motivation I. Solving regularised equations.

- Nonlinear problems :  $\min \|F(x) - y\|^2 + \lambda \|x\|^2$

↳ nonlinear optimisation → Iterative methods ✓

- Large linear systems → large sparse systems

factorisations like  
SVD, QR etc  
often denser  
than original  
matrix

↓  
require lots of  
memory ;)

methods that  
only require  
ability to  
compute  
'actions'  
e.g. matrix  
times vector

↓  
Iterative  
methods ✓

Problem ? regularisation parameter search:  $\lambda$  in range

→ iteration takes more function evaluations than single solve

↳ doing iteration for each  $\lambda$  is expensive

Can we combine  $\lambda$  search & iteration?

Motivation II As regularisation methods!

Key idea: early stopping rules for iterative opt.

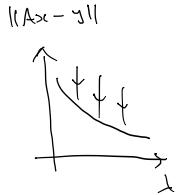
Stopping rule plays role of regularisation

- too soon: bad fit/underfit/biased
- too late: overfit/noise/too unstable

Advantage:

- rather than solve e.g.  $\|Ax - y\|$

for  $\lambda$  in range:  
best  $x_c(\lambda)$  via iterative



- generate:

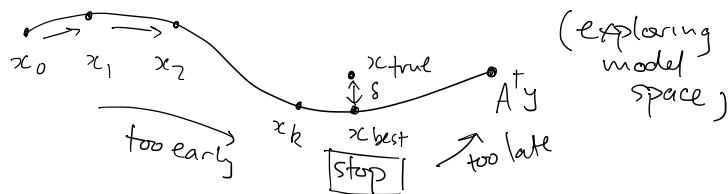
sequence  $x_k, k = 0, 1, \dots$   
where each is can be considered  
a regularised estimate &  
 $x_k \rightarrow x^+ = A^T y$  as  $k \rightarrow \infty$

## Iterative regularisation

Want:

- initial iterates close to regularised / simple solns
- better in middle stages, i.e. close to 'true' goal
- 'diverge' to exact but noise amplifying later

} controlled descent  
+  
stopping rule



(Note: continuation, i.e. use previous  $x(\lambda_n)$  as guess for  $x(\lambda_{n+1})$  is closely related!)

This idea is sometimes called  
'semi-convergence'

## Disadvantage:

- 'implicit' rather than explicit
- hard to analyse!

↳ But, really, true of nonlinear problems in general!

## Regularisation schemes

	Tikhonov	TSVD	iterative
parameter:	$\lambda$	$P$	$k$ (iteration number to step.)
	$\lambda \parallel Lx \parallel^2$	(singular values)	
choice rule:	<ul style="list-style-type: none"> <li>- L-curve / Pareto</li> <li>- discrepancy principle</li> <li>- cross-validation etc</li> </ul>	<ul style="list-style-type: none"> <li>- Pareto fit trade-off</li> <li>- Picard plot</li> <li>etc</li> </ul>	<ul style="list-style-type: none"> <li>- Pareto fit stopping rule</li> <li>(good enough)</li> <li>etc</li> </ul>

In general: start with 'simple' / biased model that doesn't fit so well, allow to become more complex / variable but not too complex.

Analysis?

7.11? Iterative methods as dynamical systems ..

→ Singular limit:  $A^T y$  [nonhyperbolic?]

↳ goal 'in principle'

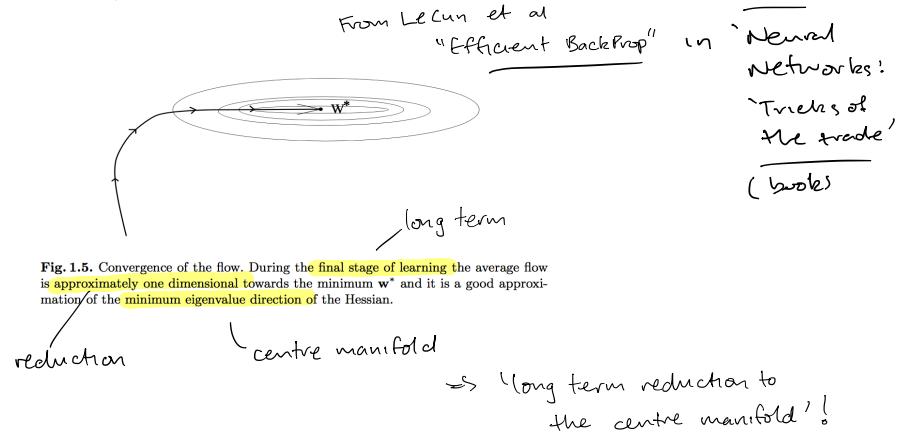
↳ unstable! all models / data wrong

→ Want to approach, but not too closely

→ design a controlled approach to flow on centre manifold?

eg →

even related to 'hot topics' like  
deep learning!



CBMM Memo No. 073

January 30, 2018

### Theory of Deep Learning III: the non-overfitting puzzle

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**Abstract:** A main puzzle of deep networks revolves around the apparent absence of overfitting intended as robustness of the expected error against overparametrization, despite the large capacity demonstrated by zero training error on randomly labeled data.

In this note, we show that the dynamics associated to gradient descent minimization of nonlinear networks is topologically equivalent, near the asymptotically stable minima of the empirical error, to a gradient system in a quadratic potential with a degenerate (for square loss) or almost degenerate (for logistic or crossentropy loss) Hessian. The proposition depends on the qualitative theory of dynamical systems and is supported by numerical results. The result extends to deep nonlinear networks two key properties of gradient descent for linear networks, that have been recently recognized (*1*) to provide a form of implicit regularization:

qualitative theory of dynamical systems

centre manifold theory

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. Thus the first step is to linearize the system, which means considering the Jacobian of  $F$  or equivalently the Hessian of  $L$  at  $w^*$ , that is

$$H_{ij} = \frac{\partial^2 L}{\partial w_i \partial w_j}$$

We obtain

$$\dot{w} = -HW,$$

(2)

where the matrix  $H$ , which has only real eigenvalues (since it is symmetric), defines in our case (by hypothesis we do not consider unstable critical points) two main subspaces:

- the stable subspace spanned by eigenvectors corresponding to negative eigenvalues
- the center subspace corresponding to zero eigenvalues

The center manifold existence theorem (*16*) states then that if  $F$  has  $r$  derivatives (as in the case of deep polynomial networks) then at every equilibrium  $w^*$  there is a  $C^{r-1}$  stable manifold and a  $C^{r-1}$  center manifold which is sometimes called *slow manifold*. The center manifold emergence theorem says that there is a neighborhood of  $w^*$  such that all solutions from the neighborhood tend exponentially fast to a solution in the center manifold. In general properties of the center manifold in the center manifold depend on the nonlinearities of  $F$ . We assume that the center manifold is responsible in our case reflecting empirical results in training networks. Of course, the dynamics associated with the center manifold may be non trivial, especially in the case of SGD. We simply assume for now that perturbations of it will not grow and trigger instability.

## First: Iterative minimisation

- Quick review/recap.

see  $\begin{cases} \text{Nocedal & Wright 'Numerical Optimisation'} \\ \text{Boyd & Vandenberghe 'Convex Optimisation'} \end{cases}$

for more.

Descent methods for minimising a

function  $f(x)$ ,  $x \in \mathbb{R}^n$ ,  $f(x) \in \mathbb{R}$

are all based on constructing a sequence

$[x_0, x_1, \dots, x_k, \dots]$

such that

$$f(x_{k+1}) < f(x_k)$$

unless  $x_k$  is optimal / a stopping condition is reached.

→ even non monotonic methods are usually descent methods for subsequences e.g.

$$f(x_{k+m}) < f(x_k), m \geq 1 \text{ etc.}$$

## Descent method flavours

Two basic flavours:

1. line search given  $x_k$

↳ choose a search direction  $s_k$  (vector)

$$\underset{\alpha > 0}{\text{solve}} \min_{\alpha} f(x_k + \alpha s_k)$$

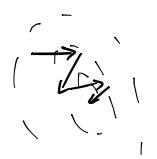
(scalar optimisation prob.)

2. trust region given  $x_k$ .

↳ replace  $f$  by 'surrogate' model  $m_k$  that we 'trust' to represent  $f$  inside a given trust region

$$\underset{s}{\text{solve}} \min m_k(x_k + s)$$

for all  $s$  inside trust region vectors



1. Choose direction, identify distance

2. Choose distance, identify direction.

### Common search methods

- Nelder-Mead
  - Gradient/steepest descent
  - Newton
  - Quasi-Newton
  - Conjugate gradient
- } derivative-free
- } derivative-based  
(at least first, often second)

etc

### Trade-offs:

- computational expensive  
(derivative computation etc)
- exploration vs exploitation

↓  
look for  
better in  
new/wider  
neighbourhood

↓  
make use of  
what you  
have!

- robustness
- etc

### Tools:

- Matlab:
- fminsearch (Nelder-Mead)
  - optimisation toolbox

- Python:
- scipy.optimize library
  - sklearn
  - etc.

### Iterative regularisation

#### Recall:

Want: - initial iterates close to regularised / 'simple' solns

- better in middle stages, close to 'true' goal

- 'diverge' to exact but noise amplifying later

### Linear example.

Full/  
initial:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

start simple'

→ think overdetermined (effectively)

→ approach  $A^+$  from over → under det.

So

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \stackrel{\mathbb{R}^m}{=} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \stackrel{\mathbb{R}^n}{=}$$

→ Can find a left  
inverse  
(if col indep.)

### Least squares & Normal equations ...

But first !

## Zeros & Fixed Points

'Theory of everything' (LeCun)

$$\boxed{F(x) = 0} \text{ for some } F, x$$

(everything is a zero of some function)

Alternative:

$$\boxed{T(x) = x} \text{ for some } T, x.$$

(everything is a fixed point of some function)

Relationship

If eg

$$\boxed{T(x) := \alpha F(x) + x}, \quad \alpha \neq 0$$

Then

$$\boxed{\begin{aligned} F(x) &= 0 \\ \Leftrightarrow \\ T(x) &= x \end{aligned}}$$

Fixed point iterations : (define goal as fixed point of some map.)

Define:

$$\boxed{x_{k+1} = T(x_k)}$$

If  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$

Then get  $\underline{x^* = T(x^*)}$  & hence solves  $F(x) = 0$

Contraction mappings

The above works if eg

$$\|Tx - Ty\| \leq \alpha \|x - y\| \text{ for all } x, y$$
  
&  $0 < \alpha < 1$

More detail: functional analysis  
& dynamical systems

Difficult/interesting area for eg  
nonlinear  $T$ !

## Normal equations

- $A^T A x = A^T y$
- $F(x) = A^T y - A^T A x = 0$
- $T(x) = \alpha F(x) + x = xc$

Define:

$$\begin{aligned} x_{k+1} &= T(x_k) \\ &= x_k + \alpha F(x_k) \\ &= x_k + \alpha A^T(y - Ax_k) \end{aligned}$$

→ 'Landweber iteration'

Note:

$$\nabla_x \|y - Ax\|^2 = -2A^T(y - Ax)$$

⇒ gradient descent!

$$x_{k+1} = x_k - \alpha \nabla_x \phi(x_k)$$

$$\text{for } \phi = \|y - Ax\|^2$$

## Landweber iteration

$$\boxed{\begin{aligned} x_{k+1} &= x_k + \alpha A^T(y - Ax_k) \\ &\quad \& \\ 0 < \alpha &< 2/\sigma_1^2 \end{aligned}}$$

Stop when e.g

- $\|Ax_k - y\| \approx \epsilon$  = measure of noise level
- 'Flattens' off (monitor)

Note: only need matrix-vector products

→ no matrix factorisations

→ efficient for large sparse systems.

## Landweber iteration

Can show in linear case iterates have form

$$x_k = \underbrace{V F_k}_{\substack{\text{model} \\ \text{space} \\ \text{basis}}} \underbrace{\Sigma^{-1} U^T y}_{\substack{\text{coefficients} \\ = \\ \text{Filter factors} \times \text{usual SVD}}}$$

$$F_k = \begin{bmatrix} f_1^k & \dots & f_n^k \end{bmatrix}$$

$$f_i^k = 1 - (1 - \alpha \sigma_i^2)^k$$

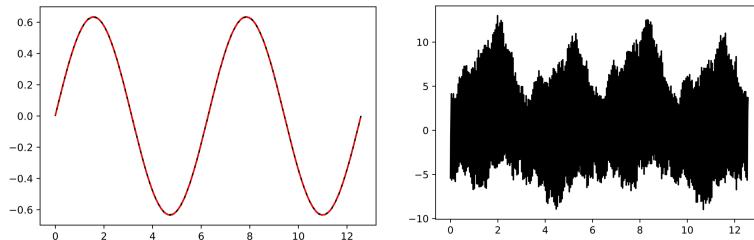
I.e. 'damps out singular values'  
similar to Tikhonov / TSVD

## Iterative regularisation : Example.

Deconvolution again!

slightly noisy  $Ax = b$

Naive  $A^T y$ :



## Landweber Iteration (Naive!)

```
niter = 1000
xs = np.zeros((niter, len(x)))
for i in range(0, niter-1):
    # update rule
    xs[i+1, :] = xs[i, :] + A.T @ (y_noisy - A @ xs[i, :])
```

Notes:

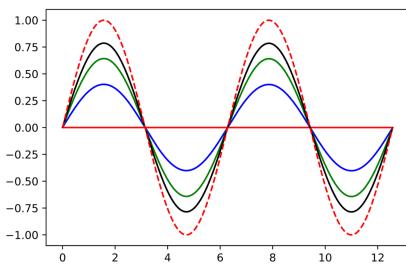
- just do 1000 & choose earlier stopping after (fast here so ok)
- largest  $\sigma_i$ ,  $\sigma_1$ , is 1 here  
 $\Rightarrow 0 < \alpha < 2$

I used  $\alpha = 1$   
in iterations.

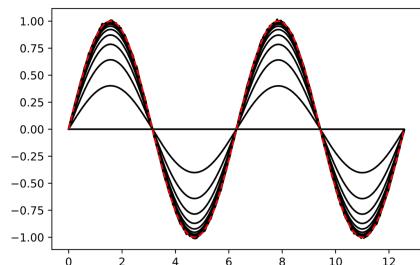
```
from scipy.sparse.linalg import svds
U, s, VT = svds(A, k=1)
s1 = s
print(s1)
[ 1.]
```

## Iterative regularisation : results

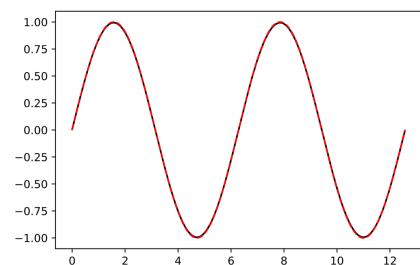
Solutions :



First few iterations



All iterations



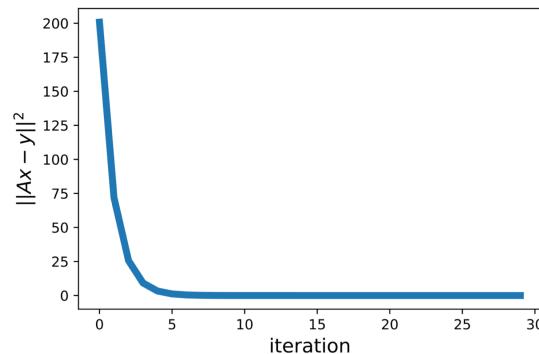
10<sup>th</sup> iteration.

Pattern : underfit  $\rightarrow$  good fit  $\rightarrow$  overfit  
 i.e. 'semi-convergence'

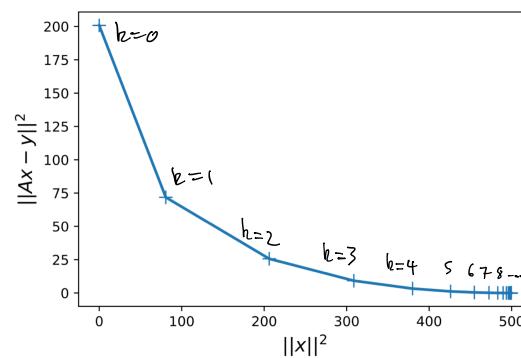
(not as  
obv here  
 $\rightarrow$  see noise  $\Rightarrow$ )

## Iterative regularisation : results

Convergence



Fit vs Iteration

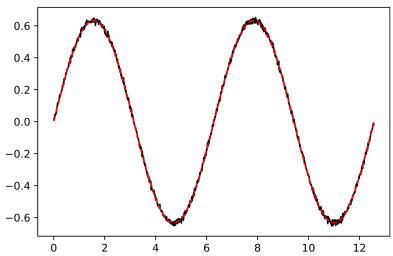


Lcurve

Note: 'slowing down' (slow manifold!)  
 $\rightarrow$  smaller changes per it.

## Iterative regularisation: Noiser

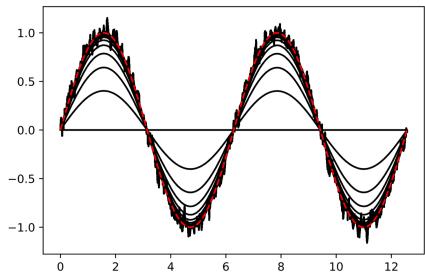
Noise  $\uparrow \times 10$



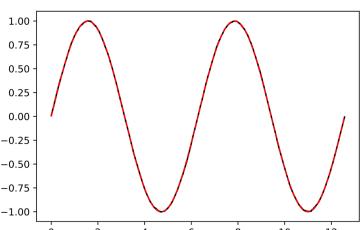
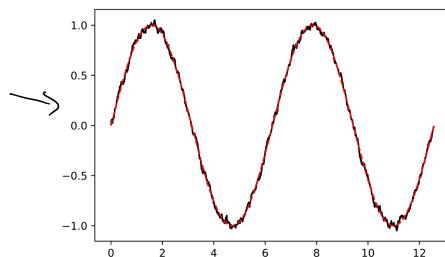
data.

$Ax + \epsilon$

all inversions (1000)



$k = 300$ . Noise ampl.



$k = 30$  still good!

overfitting noise

## Iterative regularisation: more sophisticated

→ Landweber iteration is basically  
first-order gradient descent

→ just as in optimisation, we can  
choose a search/update direction  
based on higher-order information

↳ more derivatives, "larger"  
local neighbourhood

General

$$x_{k+1} = x_k + f_k(x_k, y)$$

expansion based  
on deriv. at  
 $x_k$ .  
 $f$  →  $f_k$ .  
 $x_k$  →  $x_{k+1}$ .

where  $f_k$  can take into account  
higher-derivatives etc.

[\* Think: Taylor series for  $x(k+1) = T(x(k))$ ]

Trade-off: don't want it to  
optimise too well!

→ want to avoid  $x^+ = A^T y$  limit!

## Iterative regularisation

Aster et al. (Chapter 6) Example 6.3 (see attached)

→ Deblurring 2D image (200x200 pixels)

→ with & without explicit  
regularisation

→ use conjugate gradient  
least squares

→ Blurring operator:

◦ 40,000 x 40,000

◦ very sparse

↳ SVD produces denser  
matrices & requires  
a lot of storage

↳ CGLS works well

& only requires  
access to matrix-vector  
products

→ CGLS w/out explicit regularisation  
→ doesn't matter if only iterative!  
→ much cheaper!

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Note that this algorithm only requires one multiplication of  $\mathbf{G}\mathbf{p}_k$  and one multiplication of  $\mathbf{G}^T\mathbf{s}_{k+1}$  per iteration. We never explicitly compute  $\mathbf{G}^T\mathbf{G}$ , which might require considerable time, and which might have far more nonzero elements than  $\mathbf{G}$  itself.

The CGLS algorithm has an important property that makes it particularly useful for ill-posed problems. It can be shown that, for exact arithmetic,  $\|\mathbf{m}_k\|_2$  increases monotonically and  $\|\mathbf{G}\mathbf{m}_k - \mathbf{d}\|_2$  decreases monotonically with increasing iterations [61, 65]. We can use the discrepancy principle together with this property to obtain a regularized solution. Simply stop the CGLS algorithm as soon as  $\|\mathbf{G}\mathbf{m}_k - \mathbf{d}\|_2 < \delta$ . In practice, this algorithm typically gives good solutions after a very small number of iterations.

An alternative way to use CGLS is to solve the Tikhonov regularization problem (4.4) by applying CGLS to

$$\min \left\| \begin{bmatrix} \mathbf{G} \\ \alpha \mathbf{L} \end{bmatrix} \mathbf{m} - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2^2. \quad (6.76)$$

For nonzero values of the regularization parameter  $\alpha$ , this least squares problem should be reasonably well-conditioned. By solving this problem for several values of  $\alpha$ , we can compute an L-curve. The disadvantage of this approach is that the number of CGLS iterations for each value of  $\alpha$  may be large, and we need to solve the problem for several values of  $\alpha$ . Thus the computational effort is far greater.

### Example 6.3

A commonly used mathematical model of image blurring involves the two-dimensional convolution of the true image  $I_{\text{true}}(x, y)$  with a **point spread function**,  $\Psi(u, v)$  [14]:

$$I_{\text{blurred}}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{\text{true}}(x-u, y-v) \Psi(u, v) du dv. \quad (6.77)$$

Here the point spread function shows how a point in the true image is altered in the blurred image. A point spread function that is commonly used to represent the blurring that occurs because an image is out of focus is the **Gaussian point spread function**

$$\Psi(u, v) = e^{\frac{u^2+v^2}{2\sigma^2}}. \quad (6.78)$$

Here the parameter  $\sigma$  controls the relative width of the point spread function. In practice, the blurred image and point spread function are discretized into pixels. In theory,  $\Psi$  is nonzero for all values of  $u$  and  $v$ . However, it becomes small quickly as  $u$  and  $v$  increase. If we set small values of  $\Psi$  to 0, then the  $\mathbf{G}$  matrix in the discretized problem will be sparse.

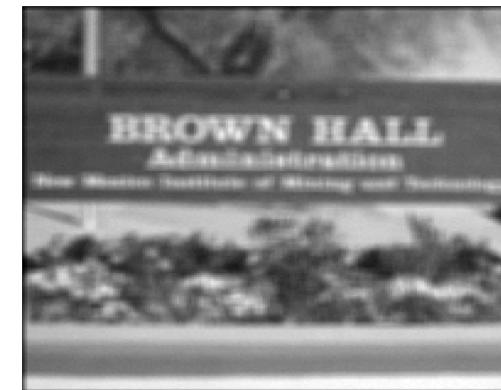


Figure 6.8 Blurred image.

Figure 6.8 shows an image that has been blurred and also has a small amount of added noise. This image is of size 200 pixels by 200 pixels, so the  $\mathbf{G}$  matrix for the blurring operator is of size 40,000 by 40,000. Fortunately, the blurring matrix  $\mathbf{G}$  is quite sparse, with less than 0.1% nonzero elements. The sparse matrix requires about 12 MB of storage. A dense matrix of this size would require about 13 GB of storage. Using the SVD approach to Tikhonov regularization would require far more storage than many current computers have. However, CGLS works quite well on this problem.

Figure 6.9 shows the L-curve for the solution of this problem by CGLS without explicit regularization and by CGLS with explicit regularization. The line with circles shows the solutions obtained by CGLS without explicit regularization. For the first 30 or so iterations of CGLS without explicit regularization,  $\|\mathbf{G}\mathbf{m} - \mathbf{d}\|_2$  decreases quickly. After that point, the improvement in misfit slows down, while  $\|\mathbf{m}\|_2$  increases rapidly.

Figure 6.10 shows the CGLS solution without explicit regularization after 30 iterations. The blurring has been greatly improved. Note that 30 iterations is far less than the size of the matrix ( $n = 40,000$ ). Unfortunately, further CGLS iterations do not significantly improve the image. In fact, noise builds up rapidly, both because of the accumulation of round-off errors and because the algorithm is converging slowly towards an unregularized least squares solution. Figure 6.11 shows the CGLS solution after 100 iterations. In this image the noise has been greatly amplified, with little or no improvement in the clarity of the image.

We also computed CGLS solutions with explicit Tikhonov regularization for 22 values of  $\alpha$ . For each value of  $\alpha$ , 200 iterations of CGLS were performed. The resulting

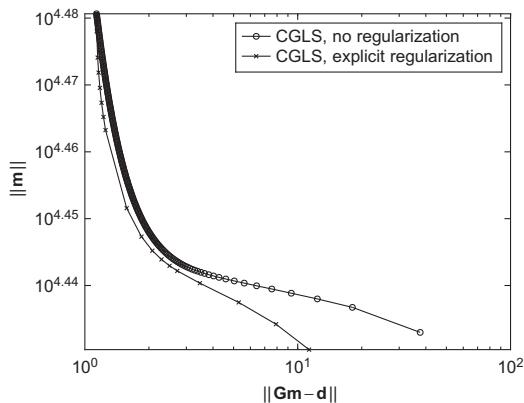


Figure 6.9 L-curves for CGLS deblurring.

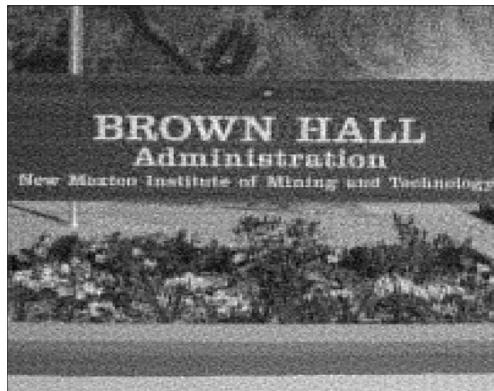


Figure 6.10 CGLS solution after 30 iterations, no explicit regularization.

L-curve is shown in Figure 6.9 with “x” markers for each regularized solution that was obtained. This L-curve is slightly better than the L-curve from the CGLS solution without explicit regularization in that the values of  $\|m\|_2$  and  $\|Gm - d\|_2$  are smaller. However, it required 40 times as much computational effort. The corner solution for  $\alpha = 7.0 \times 10^{-4}$  is shown in Figure 6.12. This solution is similar to the one shown in Figure 6.10.

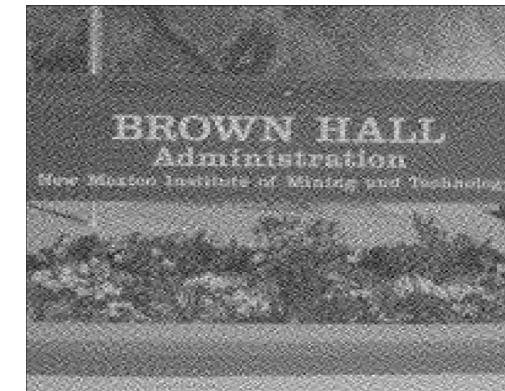
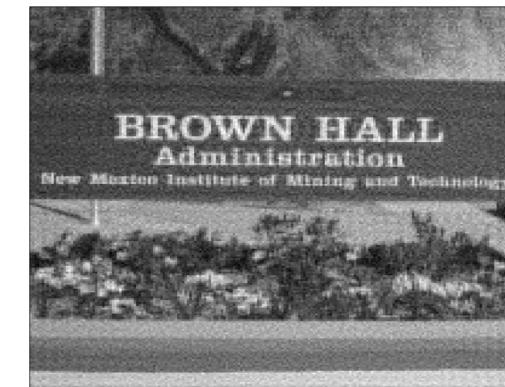


Figure 6.11 CGLS solution after 100 iterations, no explicit regularization.

Figure 6.12 CGLS solution, explicit regularization,  $\alpha = 7.0 \times 10^{-4}$ .

We have focused on the CGLS method in this section because it is particularly easy to derive and because of the implicit regularization effect of the CGLS method. However, many other iterative methods have been developed for large-scale least squares problems [5, 135]. An analysis of the implicit regularizing effects of these methods can be found in Hanke [61]. The LSQR method of Paige and Saunders has been very widely used in many geophysical applications [123, 124]. MATLAB has an efficient and robust implementation of LSQR in its `lsqr` routine.