ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*)

Oliver Maclaren

oliver.maclaren@auckland.ac.nz

LECTURE 7: RETURN OF THE MAP

More on 'global' analysis. Especially:

- Nullclines
- Periodic orbits in the plane: ruling out/in; Trapping regions
- Brief intro to periodic orbits in higher dimensions via return maps

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~17-18 lectures/tutorials]

2. *Phase plane analysis and geometry of hyperbolic systems*
   [5 lectures/tutorials]

   Analysis of two-dimensional linear and nonlinear systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling for linear and nonlinear hyperbolic systems. Connecting geometry of nonlinear and linearised hyperbolic systems.

RECALL: NULLCLINES

Given a system of equations for $x \in \mathbb{R}^n$ with components $\dot{x}_i = f_i(x)$, the jth *nullcline* is where

$$\dot{x}_j = f_j(x) = 0$$

The flow is *perpendicular* to the x_j-axis along the associated curves/surfaces (usually of dim $n-1$). Note: there *may be multiple lines/curves for one nullcline!*

Q: What are points where *all* nullclines intersect called?

Nullclines are a very useful part of sketching phase-plane portraits!

PERIOIC ORBITS IN THE PLANE

A key question for a given nonlinear ODE system is *whether it admits closed curve solutions* - i.e. *periodic orbits* (oscillations).

How can you *rule then out?* How can you *rule them 'in'?*

The results that follow *typically apply only in the plane* - 'no chaos in the plane'.

(We *can* make progress on understanding higher-dimensional problems, it's just much harder to have general theorems on what is/isn't going to happen.)

RECALL: PERIOIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t+T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t+s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

NONLINEAR PLANAR SYSTEM

Note that for the next few slides we are looking at *planar nonlinear systems*

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$

RULING OUT PERIODIC ORBITS

The *(Poincare) index/winding number* is a (topological) invariant of closed curves in the plane.

We won't go into it (see p. 126-129 Glendinning, p. 174-180 Strogatz (1994) if interested) but note that it can be used to show (among other things)

Inside any closed orbit in the plane there *must be at least one fixed point*.

Example 6.8.5 Strogatz (1994).

RULING OUT PERIODIC ORBITS

Recall the *divergence theorem* (with a weight g):

Suppose Γ is a simple (doesn't cross itself) closed curve with outward normal n enclosing a region R and f and g are continuously differentiable functions $\mathbb{R}^2 \to \mathbb{R}^2$ then

$$\int_\Gamma g(n \cdot f)dl = \int \int_R \nabla \cdot (gf)dxdy$$

where $gf := g(x, y)f(x, y)$.

RULING IN PERIODIC ORBITS - THE POINCARE-BENDIXSON THEOREM

The *Poincare-Bendixson theorem* allows one to establish the *existence of a periodic orbit*. It also establishes that there is *'no chaos in the plane'* (for smooth ODEs).

> Let D be a closed and bounded domain in the plane and suppose there are *no stationary solutions in D*. Then, if the orbit $\phi(x_0, t)$ *begins in/enters D and does not leave D for all time*, then the *orbit is either closed or spirals toward a closed orbit* as $t \to \infty$

RULING OUT PERIODIC ORBITS

This can be used to formulate *Dulac's criterion* (see Glendinning 5.6):

> If there exists a g (refer previous slide) such that $\nabla \cdot (gf)$ is continuous and *has one sign throughout* a simply connected domain D then there are *no closed orbits lying entirely in D*.

If we take $g = 1$ then this is often called the *divergence test*.

Example (Glendinning 5.9).

THE POINCARE-BENDIXSON THEOREM - TRAPPING REGIONS

The standard trick to finding an appropriate region is to construct a *trapping region* R - a closed connected subset such that the vector field *points 'inwards' everywhere on the boundary*.

This implies (proof not shown!) that *all orbits are confined* to R (i.e. once in don't leave).

If we can construct an R *without a fixed point inside* then there exists a closed (i.e. periodic) orbit.

- Strogatz (1994) Example 7.3.2 (see handout).

PERIODIC ORBITS IN HIGHER DIMENSIONS

Periodic orbits in high dimensions can be (very!) complicated.

We will hence try to *introduce a 'simpler' object to study* which can help us *understand periodic orbits in quite general/complicated systems*.

This leads to our first encounter with another type of dynamical system - *discrete maps*.

Here these will arise as a *tool to study (e.g. periodic orbits in) ODEs*; note that they can arise as interesting models in their own right.

RETURN MAPS (NOT EXAMINABLE)

Given a nonlinear ODE system $\dot{x} = f(x)$.

A *Poincare section* is P_Σ, is a *transverse section of the trajectories* of an ODE system, which is nowhere tangential to any trajectory.

We can *label* the points in (time) *order of their intersection* with P_Σ, giving $x_0, x_1, ...x_n, ...$.

RETURN MAPS (NOT EXAMINABLE)

The Poincare section defines a (discrete!) *Poincare/return map*

$$x_{n+1} = F(x_n)$$

There is a corresponding *theory of stability/instability/bifurcation for discrete maps*. This can be used to *deduce properties* of e.g. equilibria and periodic orbits in the *original ODE system*.

EngSci 711 Lecture 07 : More global features & tools

- Nuclclines
- Existence of Periodic orbits
  - Poincare winding index.
  - Dulac Criterion
  - Poincare-Bendixson theorem
    - Trapping regions

Application : chemical oscillations

Examples

→ In notes →

Other tools for nonlinear systems in the plane

↳ nullclines !

$\dot{x} = 0$ } x-nullcline

$\dot{y} = 0$ } y-nullcline.

Eg (Exam 2018)

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 - y - 1$$
$$\dot{y} = (x - 2)y$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all three of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(9 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields *on* the nullclines in your sketch.

(8 marks)

(c) The nullclines above separate the phase-plane into eight regions, each of which has one of four different qualitative flow directions: $\boxed{\nearrow}$, $\boxed{\searrow}$, $\boxed{\swarrow}$, $\boxed{\nwarrow}$. Add boxed arrows such as these to your sketch to indicate the direction field in each region of phase space (hint: use the information you determined in the previous part of this question - you do not need to do any further calculations).

(1 mark)

} note !

Nullclines important for qualitative flow

- divide into regions of flow directions
- also for periodic orbits & 'trapping regions'

→

Nullclines : Example

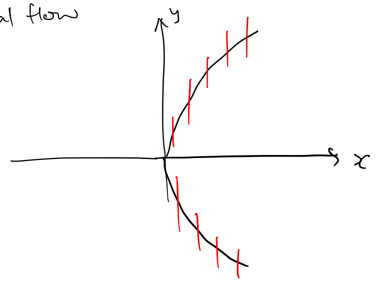
$$\dot{x} = x - y^2$$
$$\dot{y} = y(a - x), \quad a > 1.$$

$\dot{x} = 0 \Rightarrow x = y^2$

$\therefore$ vertical flow

Step 1: curves & orientation

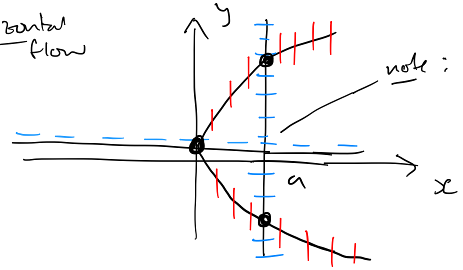
$\rightarrow$ not direction yet.



$\dot{y} = 0 \Rightarrow y = 0$ or $x = a$

$\therefore$ horizontal flow

$\bullet$: Fixed points = intersection of nullclines



note: intersection of same nullcline

$\Rightarrow$ not Fixed point!

$\rightarrow$

Step 2: Sign of flow on nullcline.

$\rightarrow$ substitute into other eqⁿ!

TIP: expect/can only change direction at fixed point

---

ie $\underbrace{++++\ldots\, 0}_{}\, ------$

continuous change from +ve to −ve.

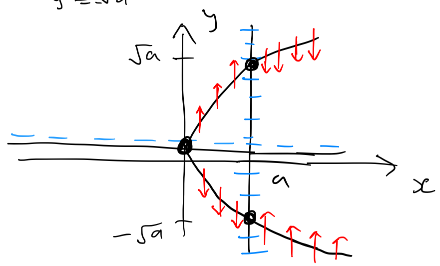
---

when $\dot{x} = 0 \Rightarrow x = y^2$

$\boxed{\updownarrow \quad \dot{x} = 0}$

$\dot{y} = ? \Rightarrow \dot{y} = y(a - x)$
$= y(a - y^2)$

if $x = a$
$y = \pm\sqrt{a}$



Cases:

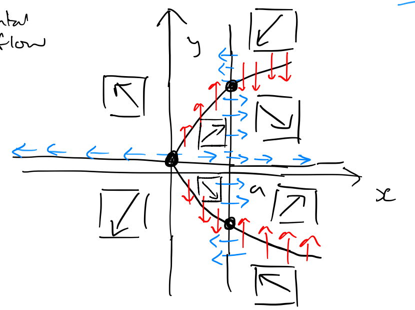
$y > 0$
$\bullet\ y > \sqrt{a} \Rightarrow y^2 > a$
$\Rightarrow \dot{y} < 0$
$\bullet\ y < \sqrt{a} \Rightarrow y^2 < a$
$\Rightarrow \dot{y} > 0$
$y < 0$
$\Rightarrow$ opposite.

Cont'd

$\dot y = 0 \Rightarrow y = 0$ or $x = a$

∴ horizontal flow

$\dot y = 0$ ↔



∘ partition into $\dot x, \dot y$ combos

∘ can only switch dir. at nullclines

∘ when $y = 0$

$$\dot x = x \begin{cases} > 0 & x > 0 \\ < 0 & x < 0 \end{cases}$$

∘ when $x = a$

$$\dot x = a - y^2 \begin{cases} > 0 & \text{if } y^2 < a \text{ ie} \\ < 0 & \text{if } y^2 > a \end{cases}$$

Again: flow only changes dir at nullclines! } Key.

$\Rightarrow$ $y = 0$ case actually enough

(why?)

Exercise: combine & sketch! (step 3).

Exercises:

(Exam 2017)

Question 6 (18 marks)

Consider the system

$$\dot x = x^2 + y^2 - 2$$
$$\dot y = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

(Exam 2018)

Question 6 (18 marks)

Consider the system

$$\dot x = x^2 - y - 1$$
$$\dot y = (x - 2)y$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all three of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(9 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields *on* the nullclines in your sketch.

(8 marks)

(c) The nullclines above separate the phase-plane into eight regions, each of which has one of four different qualitative flow directions: $\boxed{\nearrow}, \boxed{\searrow}, \boxed{\swarrow}, \boxed{\nwarrow}$. Add boxed arrows such as these to your sketch to indicate the direction field in each region of phase space (hint: use the information you determined in the previous part of this question - you do not need to do any further calculations).

(1 mark)

Existence of Periodic orbits : 3 Key Methods for phase Plane

- Ruling out Periodic orbits in the Plane :
  1. Poincare winding index. → topological idea
  2. Dulac Criterion → divergence/Green's theorem
- Ruling in Periodic orbits in the Plane
  3. Poincare-Bendixson theorem
     ↳ using trapping regions!

Unfortunately: need luck/cleverness.

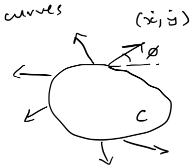
→ let's look at ruling out first....

Ruling Out

1. Winding index :

invariants of curves

$\boxed{\text{topological idea}\quad \text{Strogatz 6.8}\quad \text{Glendinning 5.5}}$



→ draw any closed curve in phase not passing through a fixed point (excludes homoclinic/heteroclinic)
  ↳ doesn't have to be a trajectory!

→ look at the angle the vector field makes as you travel around the curve

→ must return to same angle (why?)

→ number of rotations of vector field in 'travelling' around closed curve : winding number
  ↳ an integer (why?)

↺ counterclockwise

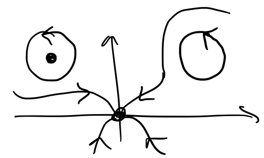
→ allows us to count number of fixed points inside closed curves! (Theorems!)

We care about result :

- If the closed curve is an orbit, ie is a periodic orbit then must contain at least one fixed point.

see Strogatz/Glendinning for more.

- Eg : is this possible:



?

Ruling out

2. Dulac / Divergence : (uses Divergence / Green's theorem)

$\nabla\cdot(gf)$ has one sign in D $\Rightarrow$ no closed orbits entirely in D

$\nabla\cdot f = (f_1)_x + (f_2)_y$ divergence

$\nabla\cdot(gf) = (gf_1)_x + (gf_2)_y$ weighted divergence (g is scalar weighting function)

$\rightarrow$ can choose g : hard part!

try: $g = \begin{cases} 1 \\ 1/xy, \; 1/x^a y^b \\ e^{ax}, e^{ay} \quad \text{etc.} \end{cases}$

- Eg. $\dot{x} = x + x^3 - 2y = f_1(x,y)$
  $\dot{y} = -3x + y^5 = f_2(x,y)$

$$\nabla\cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1 + 2x^2 + 5y^4 > 0 \quad \forall x, y$$

$\Rightarrow$ no periodic orbits in x-y plane.

- Eg $\dot{x} = x(A - a_1 x + b_1 y)$
  $\dot{y} = y(B - a_2 y + b_2 y)$ $\quad a_i, b_i > 0$

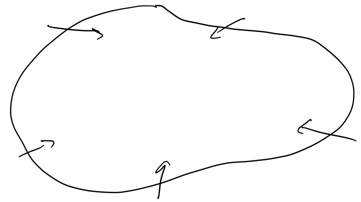
Try $\frac{1}{xy}$ (magic!):

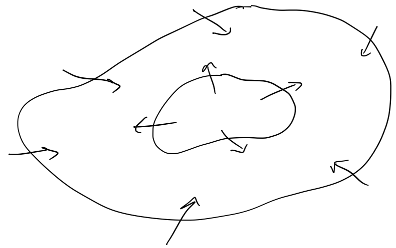
$$gf = \left(\frac{1}{y}(A - a_1 x + b_1 y), \frac{1}{x}(B - a_2 y + b_2 y)\right) = (gf_1, gf_2)$$

$\nabla\cdot(gf) = \ldots$ exercise! Show > 0 if $x > 0$ & $y > 0$

Ruling in? Key tool: Trapping regions

'compact' regions ('closed & bounded') such that flow is inwards.





Result : flow points inward to 'bounded' region.

$\Rightarrow$ Q: what happens?

$\Rightarrow$ In 2D, have Poincare-Bendixson Theorem

intuition: fixed point $\boxed{\text{or}}$ periodic ... almost.
No chaos tho'!

$\rightarrow$ 'No chaos in the phase plane'

$\rightarrow$ also tells us about existence of periodic orbits.

Ruling In

Intuition: Flow in enclosed region $\Big\}$ → fixed point, → period orbit, → homo/hetero clinic

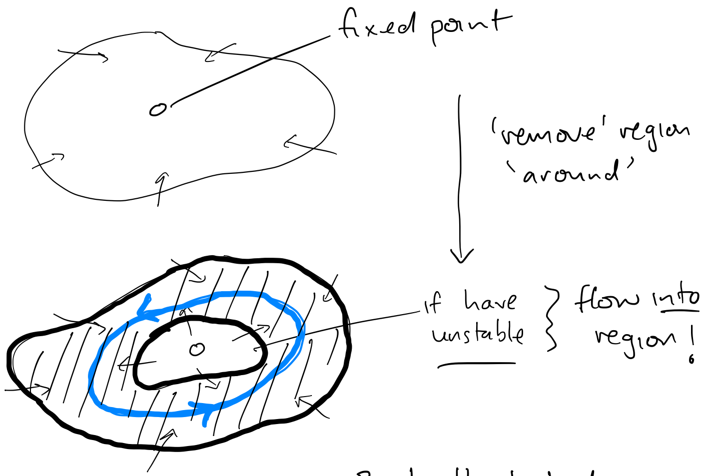
Poincare-Bendixson:

$\boxed{\text{If have region that flow begins in/enters \& does not leave \& there are no fixed points inside Then flow} \to \text{periodic orbit}}$

But: don't all periodic orbits surround at least one fixed point?

Consider:

Trapping Regions!



→ Best illustrated via examples

Oscillating Chemical Reactions

- Glycolysis
- BZ
- Brusselator

Strogatz Examples
- 7.3.2
- 8.3.1 + exercise 8.3.1.

See also: tutorial

Glycolysis (Sel'kov 1968 model)

$$\boxed{\begin{aligned}\dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y\end{aligned}}$$

BZ (attempt to recreate simple 'Krebs cycle' in lab
- Belousov 1959
- Zhabotinsky 1968)

Experimental!
Was controversial!

Brusselator (from 'Brussels'!)

general model

$$\boxed{\begin{aligned}\dot{x} &= 1 - (b+1)x + ax^2 y \\ \dot{y} &= bx - ax^2 y\end{aligned}}$$

Example (Strogatz 8.3.1)

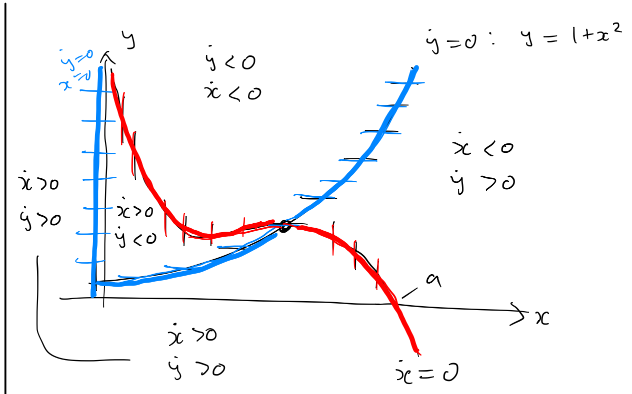
Simplified model of CDIMA reaction

→ Chlorine-Dioxide-Iodine-Malonic Acid

→ simplified using quasi-steady/quasi-equil. arguments (see later)

$$\boxed{\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \end{aligned}}$$

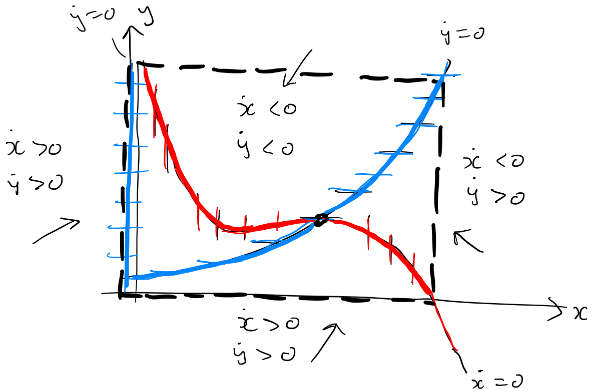
Consider:



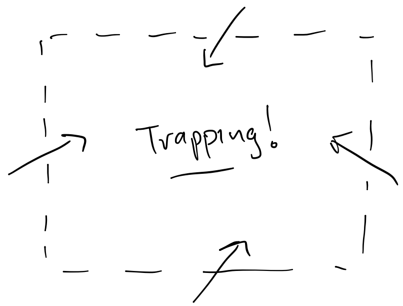
Idea

Cross $\dot{y}=0$
⇒ switch $\dot{y}$ sign
Cross $\dot{x}=0$
⇒ switch $\dot{x}$ sign

(should verify in each case though).



ie



Are we done?

Not quite: our trapping region has a fixed point inside!

Need: remove 'small' surrounding region near fixed point

verify/see when fixed point is unstable

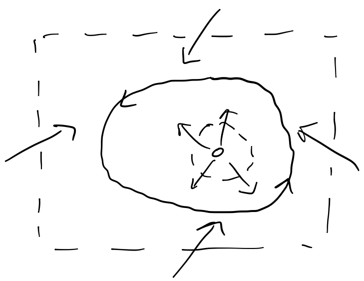
→ Find Jacobian, evaluate at fixed point

→ Find when $Re(\lambda_i) > 0$ for both

→ Details: Strogatz (attached)

Tutorial Q!.

Result:



Poincare-Bendixson applies!

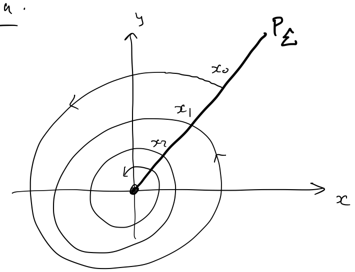
Analysis of Periodic orbits in higher dimensions:

Return Maps (see eg Glendinning 6.2, Strogatz 8.7)

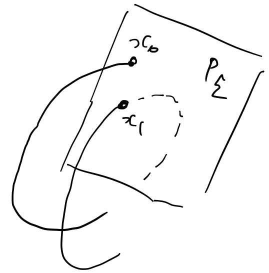
Not examinable (I guess...maybe assignment?)

→ useful to be aware of

Idea:



Higher dim:



Define a 'section' at look at 'return map' (Poincare!)

$\boxed{x_{n+1} = F(x_n)}$ discrete! (can be complicated)

defined by ... return of flow to section

Periodic orbit satisfies

$x = F(x)$ (or $x = F^p(x)$ etc)

ie is a fixed point of return map

→ stability theory for discrete maps! } analogous theory

↳ conclusions about continuous orbit!

Strogatz Ch. 7

Solution: A hunch tells us to pick $g = 1/xy$. Then

$$
\begin{aligned}
\nabla\cdot(g\dot{\mathbf{x}}) &= \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y}) \\
&= \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right) \\
&= -1/y \\
&< 0.
\end{aligned}
$$

Since the region $x, y > 0$ is simply connected and g and $\mathbf{f}$ satisfy the required smoothness conditions, Dulac's criterion implies there are no closed orbits in the positive quadrant. ■

EXAMPLE 7.2.5:

Show that the system $\dot{x} = y$, $\dot{y} = -x - y + x^2 + y^2$ has no closed orbits.

Solution: Let $g = e^{-2x}$. Then $\nabla\cdot(g\dot{\mathbf{x}}) = -2e^{-2x}y + e^{-2x}(-1+2y) = -e^{-2x} < 0$. By Dulac's criterion, there are no closed orbits. ■

7.3 Poincaré–Bendixson Theorem

Now that we know how to rule out closed orbits, we turn to the opposite task: finding methods to *establish that closed orbits exist* in particular systems. The following theorem is one of the few results in this direction. It is also one of the key theoretical results in nonlinear dynamics, because it implies that chaos can't occur in the phase plane, as discussed briefly at the end of this section.

Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

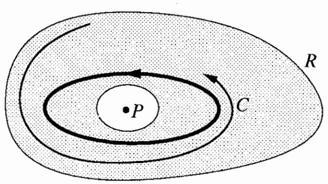


Figure 7.3.1

The proof of this theorem is subtle, and requires some advanced ideas from topology. For details, see Perko (1991), Coddington and Levinson (1955), Hurewicz (1958), or Cesari (1963).

In Figure 7.3.1, we have drawn R as a ring-shaped region because any closed orbit must encircle a fixed point (P in Figure 7.3.1) and no fixed points are allowed in R.

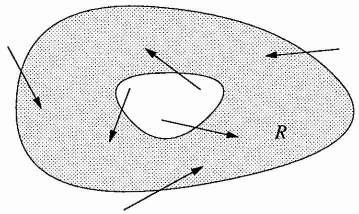


Figure 7.3.2

When applying the Poincaré–Bendixson theorem, it's easy to satisfy conditions (1)–(3); condition (4) is the tough one. How can we be sure that a confined trajectory C exists? The standard trick is to construct a ***trapping region*** R, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of R (Figure 7.3.2). Then *all* trajectories in R are confined. If we can also arrange that there are no fixed points in R, then the Poincaré–Bendixson theorem ensures that R contains a closed orbit.

The Poincaré–Bendixson theorem can be difficult to apply in practice. One convenient case occurs when the system has a simple representation in polar coordinates, as in the following example.

EXAMPLE 7.3.1:

Consider the system

$$
\begin{aligned}
\dot{r} &= r(1-r^2) + \mu r\cos\theta \\
\dot{\theta} &= 1.
\end{aligned}
\tag{1}
$$

When $\mu = 0$, there's a stable limit cycle at $r = 1$, as discussed in Example 7.1.1. Show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

Solution: We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$, such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{\min} \le r \le r_{\max}$ will be our desired trapping region. Note that there are no fixed points in the annulus since $\dot{\theta} > 0$; hence if $r_{\min}$ and $r_{\max}$ can be found, the Poincaré–Bendixson theorem will imply the existence of a closed orbit.

To find $r_{\min}$, we require $\dot{r} = r(1-r^2) + \mu r\cos\theta > 0$ for all θ. Since $\cos\theta \ge -1$, a sufficient condition for $r_{\min}$ is $1 - r^2 - \mu > 0$. Hence any $r_{\min} < \sqrt{1-\mu}$ will work, as long as $\mu < 1$ so that the square root makes sense. We should choose $r_{\min}$ as large as possible, to hem in the limit cycle as tightly as we can. For instance, we could pick $r_{\min} = 0.999\sqrt{1-\mu}$. (Even $r_{\min} = \sqrt{1-\mu}$ works, but more careful rea-

Strogatz Ch. 7

soning is required.) By a similar argument, the flow is inward on the outer circle if $r_{\max} = 1.001\sqrt{1+\mu}$.

Therefore a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. ■

The estimates used in Example 7.3.1 are conservative. In fact, the closed orbit can exist even if $\mu \geq 1$. Figure 7.3,3 shows a computer-generated phase portrait of (1) for $\mu = 1$. In Exercise 7.3.8, you're asked to explore what happens for larger μ, and in particular, whether there's a critical μ beyond which the closed orbit disappears. It's also possible to obtain some analytical insight about the closed orbit for small μ (Exercise 7.3.9).

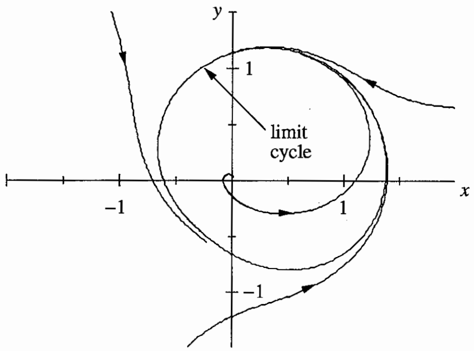


Figure 7.3.3

When polar coordinates are inconvenient, we may still be able to find an appropriate trapping region by examining the system's nullclines, as in the next example.

EXAMPLE 7.3.2:

In the fundamental biochemical process called ***glycolysis***, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an *oscillatory* fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. For reviews, see Chance et al. (1973) or Goldbeter (1980).

A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters. Construct a trapping region for this system.

Solution: First we find the nullclines. The first equation shows that $\dot{x} = 0$ on the curve $y = x/(a + x^2)$ and the second equation shows that $\dot{y} = 0$ on the curve $y = b/(a + x^2)$. These nullclines are sketched in Figure 7.3.4, along with some representative vectors.

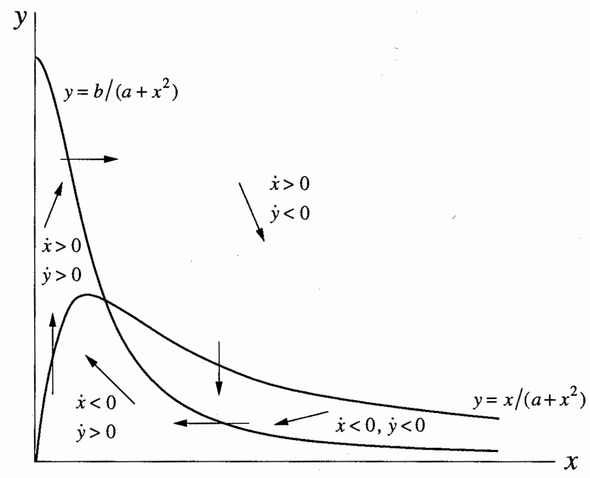


Figure 7.3.4

How did we know how to sketch these vectors? By definition, the arrows are vertical on the $\dot{x} = 0$ nullcline, and horizontal on the $\dot{y} = 0$ nullcline. The direction of flow is determined by the signs of $\dot{x}$ and $\dot{y}$. For instance, in the region above both nullclines, the governing equations imply $\dot{x} > 0$ and $\dot{y} < 0$, so the arrows point down and to the right, as shown in Figure 7.3.4.

Now consider the region bounded by the dashed line shown in Figure 7.3.5. *We claim that it's a trapping region.* To verify this, we have to show that all the vectors on the boundary point into the box. On the horizontal and vertical sides, there's no problem: the claim follows from Figure 7.3.4. The tricky part of the construction is the diagonal line of slope -1 extending from the point $(b, b/a)$ to the nullcline $y = x/(a + x^2)$. Where did this come from?

Strogatz Ch. 7

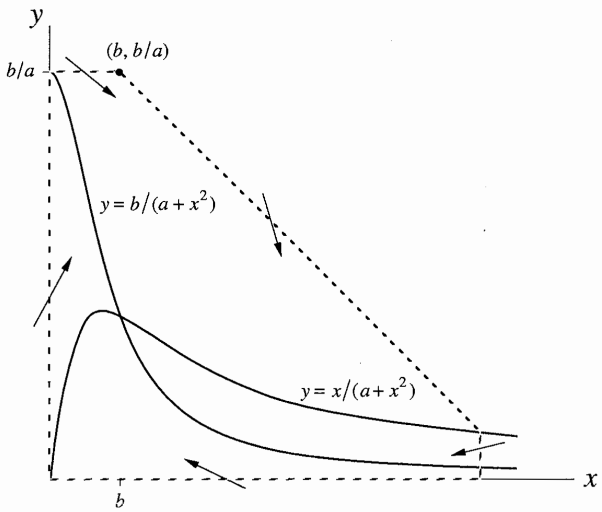


Figure 7.3.5

To get the right intuition, consider $\dot{x}$ and $\dot{y}$ in the limit of very large x. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\dot{y}/\dot{x} = dy/dx \approx -1$ along trajectories. Hence the vector field at large x is roughly parallel to the diagonal line. This suggests that in a more precise calculation, we should compare the sizes of $\dot{x}$ and $-\dot{y}$, for some sufficiently large x.

In particular, consider $\dot{x} - (-\dot{y})$. We find

$$\begin{aligned} \dot{x} - (-\dot{y}) &= -x + ay + x^2 y + (b - ay - x^2 y) \\ &= b - x. \end{aligned}$$

Hence

$$-\dot{y} > \dot{x} \text{ if } x > b.$$

This inequality implies that the vector field points inward on the diagonal line in Figure 7.3.5, because dy/dx is more negative than -1, and therefore the vectors are steeper than the diagonal line. Thus the region is a trapping region, as claimed. ■

Can we conclude that there is a closed orbit inside the trapping region? No! There is a fixed point in the region (at the intersection of the nullclines), and so the conditions of the Poincaré–Bendixson theorem are not satisfied. But if this fixed point is a *repeller*, then we *can* prove the existence of a closed orbit by considering

the modified "punctured" region shown in Figure 7.3.6. (The hole is infinitesimal, but drawn larger for clarity.)

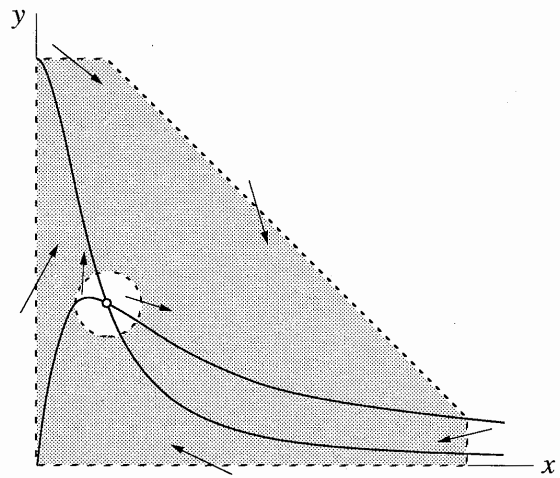


Figure 7.3.6

The repeller drives all neighboring trajectories into the shaded region, and since this region *is* free of fixed points, the Poincaré–Bendixson theorem applies.

Now we find conditions under which the fixed point is a repeller.

EXAMPLE 7.3.3:

Once again, consider the glycolytic oscillator $\dot{x} = -x + ay + x^2 y$, $\dot{y} = b - ay - x^2 y$ of Example 7.3.2. Prove that a closed orbit exists if a and b satisfy an appropriate condition, to be determined. (As before, $a, b > 0$.)

Solution: By the argument above, it suffices to find conditions under which the fixed point is a repeller, i.e., an unstable node or spiral. In general, the Jacobian is

$$A = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}.$$

After some algebra, we find that at the fixed point

$$x^* = b, \qquad y^* = \frac{b}{a + b^2},$$

the Jacobian has determinant $\Delta = a + b^2 > 0$ and trace

Strogatz Ch. 7

$$\tau = -\frac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}.$$

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when

$$b^2 = \tfrac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right).$$

This defines a curve in (a, b) space, as shown in Figure 7.3.7.

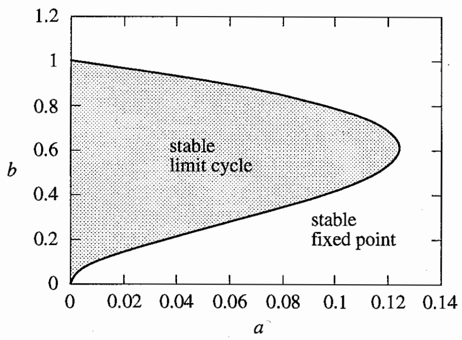


Figure 7.3.7

For parameters in the region corresponding to $\tau > 0$, we are guaranteed that the system has a closed orbit—numerical integration shows that it is actually a stable limit cycle. Figure 7.3.8 shows a computer-generated phase portrait for the typical case $a = 0.08$, $b = 0.6$. ■

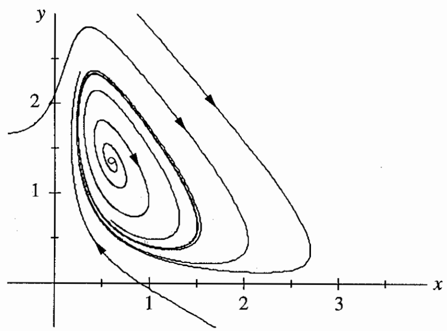


Figure 7.3.8

No Chaos in the Phase Plane

The Poincaré–Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n \geq 3$), the Poincaré–Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*, a fractal set on which the motion is aperiodic and sensitive to tiny changes in the initial conditions. This sensitivity makes the motion unpredictable in the long run. We are now face to face with *chaos*. We'll discuss this fascinating topic soon enough, but for now you should appreciate that the Poincaré–Bendixson theorem implies that chaos can never occur in the phase plane.

7.4 Liénard Systems

In the early days of nonlinear dynamics, say from about 1920 to 1950, there was a great deal of research on nonlinear oscillations. The work was initially motivated by the development of radio and vacuum tube technology, and later it took on a mathematical life of its own. It was found that many oscillating circuits could be modeled by second-order differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \qquad (1)$$

now known as ***Liénard's equation***. This equation is a generalization of the van der Pol oscillator $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ mentioned in Section 7.1. It can also be interpreted mechanically as the equation of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$.

Liénard's equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y. \end{aligned} \qquad (2)$$

The following theorem states that this system has a unique, stable limit cycle under appropriate hypotheses on f and g. For a proof, see Jordan and Smith (1987), Grimshaw (1990), or Perko (1991).

Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

Strogatz Ch. 8

as fast as $r_0 e^{\mu t}$. In other words, all trajectories are repelled out to infinity! So there are certainly no closed orbits for $\mu > 0$. In particular, the unstable spiral is not surrounded by a stable limit cycle; hence the bifurcation cannot be supercritical.

Could the bifurcation be degenerate? That would require that the origin be a nonlinear center when $\mu = 0$. But $\dot{r}$ is strictly positive away from the x-axis, so closed orbits are still impossible.

By process of elimination, we expect that the bifurcation is *subcritical*. This is confirmed by Figure 8.2.6, which is a computer-generated phase portrait for $\mu = -0.2$.

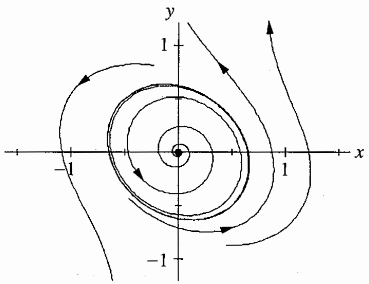


Figure 8.2.6

Note that an *unstable* limit cycle surrounds the stable fixed point, just as we expect in a subcritical bifurcation. Furthermore, the cycle is nearly elliptical and surrounds a gently winding spiral—these are typical features of *either* kind of Hopf bifurcation. ■

8.3 Oscillating Chemical Reactions

For an application of Hopf bifurcations, we now consider a class of experimental systems known as ***chemical oscillators***. These systems are remarkable, both for their spectacular behavior and for the story behind their discovery. After presenting this background information, we analyze a simple model proposed recently for oscillations in the chlorine dioxide–iodine–malonic acid reaction. The definitive reference on chemical oscillations is the book edited by Field and Burger (1985). See also Epstein et al. (1983), Winfree (1987b) and Murray (1989).

Belousov's "Supposedly Discovered Discovery"

In the early 1950s the Russian biochemist Boris Belousov was trying to create a test tube caricature of the Krebs cycle, a metabolic process that occurs in living

cells. When he mixed citric acid and bromate ions in a solution of sulfuric acid, and in the presence of a cerium catalyst, he observed to his astonishment that the mixture became yellow, then faded to colorless after about a minute, then returned to yellow a minute later, then became colorless again, and continued to oscillate dozens of times before finally reaching equilibrium after about an hour.

Today it comes as no surprise that chemical reactions can oscillate spontaneously—such reactions have become a standard demonstration in chemistry classes, and you may have seen one yourself. (For recipes, see Winfree (1980).) But in Belousov's day, his discovery was so radical that he couldn't get his work published. It was thought that all solutions of chemical reagents must go *monotonically* to equilibrium, because of the laws of thermodynamics. Belousov's paper was rejected by one journal after another. According to Winfree (1987b, p.161), one editor even added a snide remark about Belousov's "supposedly discovered discovery" to the rejection letter.

Belousov finally managed to publish a brief abstract in the obscure proceedings of a Russian medical meeting (Belousov 1959), although his colleagues weren't aware of it until years later. Nevertheless, word of his amazing reaction circulated among Moscow chemists in the late 1950s, and in 1961 a graduate student named Zhabotinsky was assigned by his adviser to look into it. Zhabotinsky confirmed that Belousov was right all along, and brought this work to light at an international conference in Prague in 1968, one of the few times that Western and Soviet scientists were allowed to meet. At that time there was a great deal of interest in biological and biochemical oscillations (Chance et al. 1973) and the BZ reaction, as it came to be called, was seen as a manageable model of those more complex systems.

The analogy to biology turned out to be surprisingly close: Zaikin and Zhabotinsky (1970) and Winfree (1972) observed beautiful propagating *waves* of oxidation in thin unstirred layers of BZ reagent, and found that these waves annihilate upon collision, just like waves of excitation in neural or cardiac tissue. The waves always take the shape of expanding concentric rings or spirals (Color plate 1). Spiral waves are now recognized to be a ubiquitous feature of chemical, biological, and physical excitable media; in particular, spiral waves and their three-dimensional analogs, "scroll waves" (Front cover illustration) appear to be implicated in certain cardiac arrhythmias, a problem of great medical importance (Winfree 1987b).

Boris Belousov would be pleased to see what he started.

In 1980, he and Zhabotinsky were awarded the Lenin Prize, the Soviet Union's highest medal, for their pioneering work on oscillating reactions. Unfortunately, Belousov had passed away ten years earlier.

For more about the history of the BZ reaction, see Winfree (1984, 1987b). An English translation of Belousov's original paper from 1951 appears in Field and Burger (1985).

Strogatz Ch. 8

Chlorine Dioxide-Iodine-Malonic Acid Reaction

The mechanisms of chemical oscillations can be very complex. The BZ reaction is thought to involve more than twenty elementary reaction steps, but luckily many of them equilibrate rapidly—this allows the kinetics to be reduced to as few as three differential equations. See Tyson (1985) for this reduced system and its analysis.

In a similar spirit, Lengyel et al. (1990) have proposed and analyzed a particularly elegant model of another oscillating reaction, the chlorine dioxide-iodine-malonic acid (ClO_2-I_2-MA) reaction. Their experiments show that the following three reactions and empirical rate laws capture the behavior of the system:

$$MA + I_2 \to IMA + I^- + H^+; \qquad \frac{d[I_2]}{dt} = -\frac{k_{1a}[MA][I_2]}{k_{1b}+[I_2]} \tag{1}$$

$$ClO_2 + I^- \to ClO_2^- + \tfrac{1}{2}I_2; \qquad \frac{d[ClO_2]}{dt} = -k_2[ClO_2][I^-] \tag{2}$$

$$ClO_2^- + 4I^- + 4H^+ \to Cl^- + 2I_2 + 2H_2O;$$

$$\frac{d[ClO_2^-]}{dt} = -k_{3a}[ClO_2^-][I^-][H^+] - k_{3b}[ClO_2^-][I_2]\frac{[I^-]}{u+[I^-]^2} \tag{3}$$

Typical values of the concentrations and kinetic parameters are given in Lengyel et al. (1990) and Lengyel and Epstein (1991).

Numerical integrations of (1)–(3) show that the model exhibits oscillations that closely resemble those observed experimentally. However this model is still too complicated to handle analytically. To simplify it, Lengyel et al. (1990) use a result found in their simulations: Three of the reactants (MA, I_2, and ClO_2) vary much more slowly than the intermediates I^- and ClO_2^-, which change by several orders of magnitude during an oscillation period. By approximating the concentrations of the slow reactants as *constants* and making other reasonable simplifications, they reduce the system to a two-variable model. (Of course, since this approximation neglects the slow consumption of the reactants, the model will be unable to account for the eventual approach to equilibrium.) After suitable nondimensionalization, the model becomes

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \tag{4}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right) \tag{5}$$

where x and y are the dimensionless concentrations of I^- and ClO_2^-. The parameters $a, b > 0$ depend on the empirical rate constants and on the concentrations assumed for the slow reactants.

We begin the analysis of (4), (5) by constructing a trapping region and applying the Poincaré–Bendixson theorem. Then we'll show that the chemical oscillations arise from a supercritical Hopf bifurcation.

EXAMPLE 8.3.1:

Prove that the system (4), (5) has a closed orbit in the positive quadrant $x, y > 0$ if a and b satisfy certain constraints, to be determined.

Solution: As in Example 7.3.2, the nullclines help us to construct a trapping region. Equation (4) shows that $\dot{x} = 0$ on the curve

$$y = \frac{(a-x)(1+x^2)}{4x} \tag{6}$$

and (5) shows that $\dot{y} = 0$ on the y-axis and on the parabola $y = 1 + x^2$. These nullclines are sketched in Figure 8.3.1, along with some representative vectors.

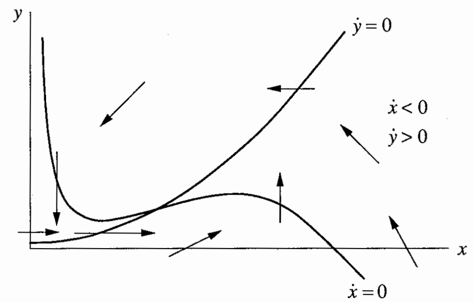


Figure 8.3.1

(We've taken some pedagogical license with Figure 8.3.1; the curvature of the nullcline (6) has been exaggerated to highlight its shape, and to give us more room to draw the vectors.)

Now consider the dashed box shown in Figure 8.3.2. It's a trapping region because all the vectors on the boundary point into the box.

Strogatz Ch. 8

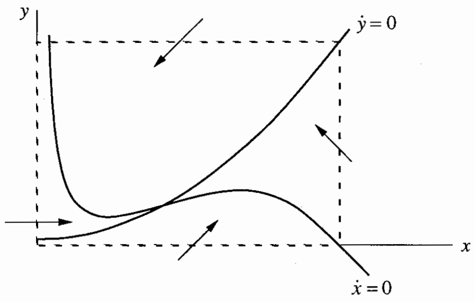


Figure 8.3.2

We can't apply the Poincaré–Bendixson theorem yet, because there's a fixed point

$$x^* = a/5, \qquad y^* = 1 + (x^*)^2 = 1 + (a/5)^2$$

inside the box at the intersection of the nullclines. But now we argue as in Example 7.3.3: if the fixed point turns out to be a *repeller*, we *can* apply the Poincaré–Bendixson theorem to the "punctured" box obtained by removing the fixed point.

All that remains is to see under what conditions (if any) the fixed point is a repeller. The Jacobian at (x^*, y^*) is

$$\frac{1}{1+(x^*)^2}\begin{pmatrix} 3(x^*)^2 - 5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}.$$

(We've used the relation $y^* = 1 + (x^*)^2$ to simplify some of the entries in the Jacobian.) The determinant and trace are given by

$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0, \qquad \tau = \frac{3(x^*)^2 - 5 - bx^*}{1+(x^*)^2}.$$

We're in luck—since $\Delta > 0$, the fixed point is never a saddle. Hence (x^*, y^*) is a repeller if $\tau > 0$, i.e., if

$$b < b_c \equiv 3a/5 - 25/a. \qquad (7)$$

When (7) holds, the Poincaré–Bendixson theorem implies the existence of a closed orbit somewhere in the punctured box. ■

EXAMPLE 8.3.2:

Using numerical integration, show that a Hopf bifurcation occurs at $b = b_c$ and

decide whether the bifurcation is sub- or supercritical.

Solution: The analytical results above show that as b decreases through b_c, the fixed point changes from a stable spiral to an unstable spiral; this is the signature of a Hopf bifurcation. Figure 8.3.3 plots two typical phase portraits. (Here we have chosen $a = 10$; then (7) implies $b_c = 3.5$.) When $b > b_c$, all trajectories spiral into the stable fixed point (Figure 8.3.3a), while for $b < b_c$ they are attracted to a stable limit cycle (Figure 8.3.3b).

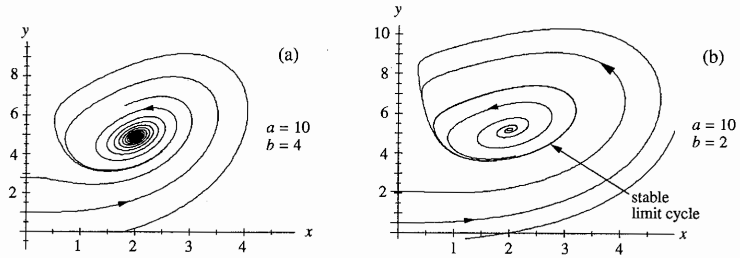


Figure 8.3.3

Hence the bifurcation is *supercritical*—after the fixed point loses stability, it is surrounded by a stable limit cycle. Moreover, by plotting phase portraits as $b \to b_c$ from below, we could confirm that the limit cycle shrinks continuously to a point, as required. ■

Our results are summarized in the stability diagram in Figure 8.3.4. The boundary between the two regions is given by the Hopf bifurcation locus $b = 3a/5 - 25/a$.

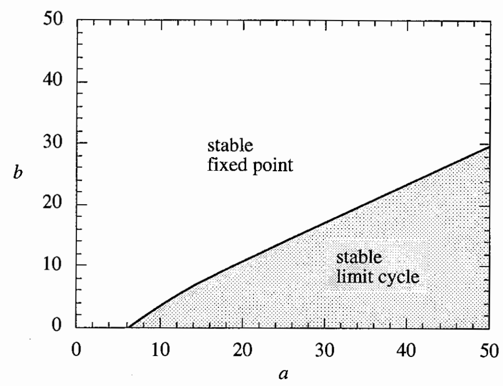


Figure 8.3.4