

Engsci 741: Inverse Problems Assignment

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Due: **Monday 8th October 11.59 pm** (submit via Canvas).

Question 1

Consider the standard Tikhonov form of the regularised least squares problem

$$\min_x ||Ax - y||^2 + \lambda ||x||^2$$

where we will assume that A is $m \times n$ for $m < n$ and $\lambda > 0$ is the regularisation parameter.

- Rewrite this in the form of a new ‘augmented’ but ‘standard’ least squares problem.
- Hence explain why this form of regularisation can be thought of as helping us convert an underdetermined problem into an overdetermined problem.
- Justify the linear independence of the columns of the matrix in the resulting overdetermined problem.
- Use the above to derive an explicit expression for the ‘Tikhonov inverse’. You may state/use the normal equations for standard least squares problems without proof.
- What solutions does the Tikhonov inverse give in the two limiting cases a) $\lambda \rightarrow 0$ and b) $\lambda \rightarrow \infty$? Justify your answers.

Question 2

Here we consider the ‘inverse problem’ of polynomial regression for data $\{(x_i, y_i)\}$, $i = 1, \dots, m$. Suppose you have a forward model with three parameters:

$$y = \theta_1 + \theta_2 x + \theta_3 x^2$$

and four observations $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$.

- Set up the problem to be ‘solved’ as a matrix equation of the form $A\theta = y$. Explain why, however, you would not generally expect this problem to be solvable in the standard sense.
- Write down the general expressions for both the *model (parameter) resolution* operator R_θ and the *data resolution* operator R_d a) in terms of A and its generalised inverse A^+ **and** b) in terms of the matrices of singular vectors U_r, V_r of the (reduced) SVD of A .
- Which of the resolution operators do you expect to be the identity matrix in the present problem and which not? Why?
- Give a simple example of U_r, V_r that would be compatible with the above given information. (Justify your answer!)
- Using Python/Matlab etc, verify your answers to the question about which resolution operators are the identity for the case where the forward mapping is given by:

$$A = \begin{bmatrix} 1. & 1. & -0.5 \\ 1. & 5. & -12.5 \\ 1. & 7. & -24.5 \\ 1. & 13. & -84.5 \end{bmatrix}.$$

Question 3

[Note: your answers to this question should be given in the form of Python/Matlab etc code and output produced from this. You should include your code in your submission.]

This problem concerns the so-called source history reconstruction problem: we want to recover the time history of the concentration of a pollutant at a known source site from later measurements. That is, we want to recover a *boundary condition* given measurements of concentrations at a series of locations at a future time.

This is illustrated in the figure below (see also the Aster et al. ‘Inverse Problems and Parameter Estimation’ section handout on Canvas under readings).

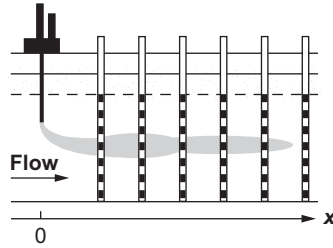


Figure 1: The source history recovery problem (from Aster et al. ‘Inverse Problems and Parameter Estimation’). We measure the concentration at the vertical observation wells at some fixed future time T , and want to use this to recover the time history of the pollutant source.

A simple forward model for this problem is the advection-diffusion equation

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - v \frac{\partial C}{\partial x}$$

with boundary conditions

$$C(0, t) = C_{\text{in}}(t)$$

and

$$C(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty$$

where D is the (known) diffusion coefficient, v is the (known) fluid velocity, and $C(x, t)$ is the pollutant concentration. Normally the boundary condition $C_{\text{in}}(t)$ would be taken as known, but here we will look to *invert* for it - i.e. solve an inverse problem!

It can be shown that the solution of the PDE above at time T can be expressed as

$$C(x, T) = \int_0^T k(x, T-t) C_{\text{in}}(t) dt$$

where the *kernel* k is given by

$$k(x, T-t) = \frac{x}{2\sqrt{\pi D(t-T)^3}} \exp\left(\frac{-[x-v(t-T)]^2}{4D(t-T)}\right).$$

Our goal is to recover the input history $C_{\text{in}}(t)$ at a fixed location, given a set of measurements $C(x, T)$ at a series of x locations taken at the same time T . Since the forward problem above takes the form of a (convolution) integral equation, we expect we will need to regularise the solution of the inverse problem.

We can think of the forward problem above as a mapping of a *series of measurements in time at a fixed x location* to a *series of measurements in space at a fixed future time*.

If we discretise space x into a grid of m (mid-)points x_i and t into a grid of n (mid-)points t_j , and we use the simple numerical integration rule

$$\int_0^T k(x, T-t) C_{\text{in}}(t) dt \approx \sum_{j=1}^n k(x_i, T-t_j) C_{\text{in}}(t_j) \Delta t$$

where Δt is the time interval between t_j grid points, then the forward problem becomes

$$Ka = b$$

where $K(i, j) = k(x_i, T-t_j) \Delta t$, a is the (time) vector of source concentrations, i.e. $C_{\text{in}}(t_j)$, of length n and b is the (spatial) vector of concentrations at a known time T , i.e. $C(x_i, T)$, with length m .

- Implement a function (or set of functions) in Python/Matlab etc that returns the above linear forward operator K given problem inputs x_{\min} , x_{\max} , t_{\min} , t_{\max} , m , n , T , D and v .
- Verify that an input source history of $C_{\text{in}}(t) = 3 + \sin(2\pi t/n) \sin(0.01t)$ (see left panel in figure below) gives, when operated on by your forward model for the settings given below, output source measurements at T that look like those shown in the right panel of the figure below.

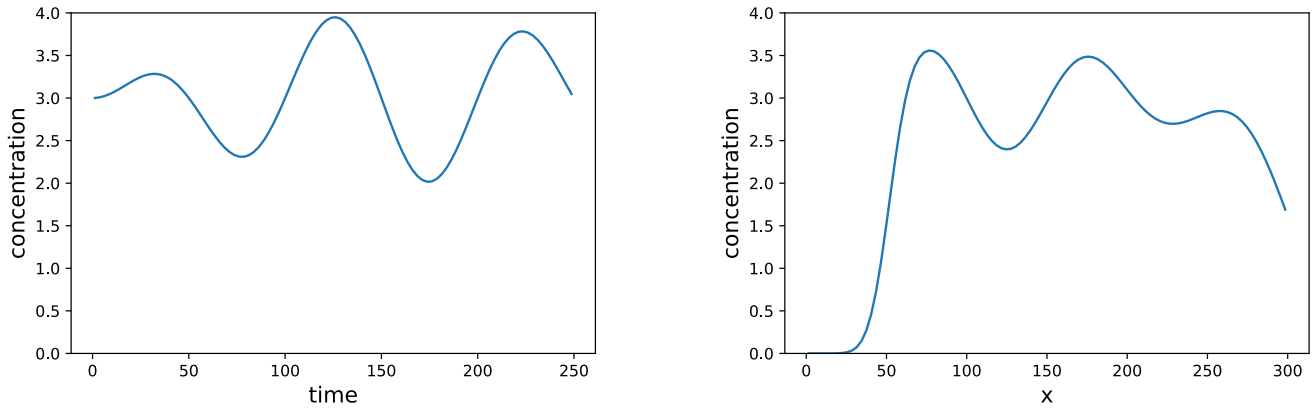


Figure 2: (L) Input signal. (R) Output from forward model.

Problem input settings: $x_{\min} = 0$, $x_{\max} = 300$, $t_{\min} = 0$, $t_{\max} = 250$, $m = 100$, $n = 100$, $D = 1$, $v = 1$, $T = 300$. We will assume everything is non-dimensionalised or expressed in appropriate units...

- Now create some synthetic data for an inverse problem by taking your output source measurement vector at T and adding a small amount of noise to each observation, e.g. add a realisation of normal noise to each output value using a normal distribution with mean 0.0 and standard deviation 0.01.

We can now attempt to recover an estimate of the original input source history given these measurements.

- First, show that the naive (unregularised) least squares solution produces bad results even though the noise is small.
- Next, solve the inverse problem properly, using a regularisation scheme of your choice, to recover an estimate of the original input source history. Produce a tradeoff curve for your choice of regularisation scheme (e.g. an L-curve) and use this to plot what you expect to be a) an underfitting solution, b) a good solution and c) an overfitting solution. (Note: you probably won't be able to *exactly* recover the truth - that's OK!). Then compare your results determined from the parameter choice method to the actual truth that we used to generate the data.