ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 5

Geometry

- Geometry of linear systems
- Geometry of nonlinear systems
- Connecting the geometry of linear and nonlinear systems

GEOMETRY OF LINEAR SYSTEMS

Let's return to general linear systems in $\mathbb{R}^n$ for a moment to give the following geometric definitions of three key *invariant manifolds/subspaces* for linear systems.

The *flow in the full phase space* is then given by a *linear supposition of motion on these three subspaces*.

These subspaces *also have nonlinear counterparts* (but we will need to consider some aspects more carefully, e.g. superposition fails)

TERMINOLOGY: LINEAR SUBSPACES AND SPANS

A linear subspace of $\mathbb{R}^n$ is a subset E of $\mathbb{R}^n$ which contains the *zero vector* $0 \in \mathbb{R}^n$ and which is *closed* under *vector addition* $u + v$, where $u, v \in \mathbb{R}^n$, and *scalar multiplication* cu, where $c \in \mathbb{R}, u \in \mathbb{R}^n$.

The *span* of a set of vectors is the set generated by *all linear combinations* of those vectors i.e.
$$\text{span}\{u, v, ...\} = \{x \in \mathbb{R}^n \mid x = au + bv + ..., a, b \in \mathbb{R}, u, v... \in \mathbb{R}^n\}$$

TERMINOLOGY: WHAT'S A MANIFOLD?

For our purposes it suffices to think of a *manifold* embedded in $\mathbb{R}^n$ as a subset of $\mathbb{R}^n$, each point of which satisfies $m \leq n$ *constraints*.

This means that, given regularity conditions, a manifold is an $(n - m)$-dimensional object embedded in n dimensional space.

Example: a circle is a 1-dimensional manifold which is embedded in $\mathbb{R}^2$ and satisfies one constraint. So, the unit circle can be thought of as e.g. $\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$.

TERMINOLOGY: WHAT'S A MANIFOLD?

A key property is that locally an $(n-m)$-dimensional manifold embedded in n-dimensional space *'looks like' a small 'open ball' of dimension $\mathbb{R}^{(n-m)}$*.

E.g. a curve embedded in $\mathbb{R}^2$ can be considered as 'pieced together' from small segments of $\mathbb{R}$, a sphere in $\mathbb{R}^3$ can be considered as 'pieced together' from small 'patches' of $\mathbb{R}^2$

The more general definition throws away the 'background' space and works with the 'intrinsic' $(n-m)$-dimensional object itself.

GEOMETRY OF LINEAR SYSTEMS - STABLE MANIFOLD

Suppose $x \in \mathbb{R}^n$ is a stationary solution to the linear system $\dot{x} = Ax$.

The *stable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^s(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, negative part*.

GEOMETRY OF LINEAR SYSTEMS - UNSTABLE MANIFOLD

Similarly:

The *unstable manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^u(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *real, positive part*.

GEOMETRY OF LINEAR SYSTEMS

Finally:

The *centre manifold* (or subspace/generalised eigenspace) of the origin is then denoted by $E^c(0)$ and is the *span* of the eigenvectors/generalised eigenvectors corresponding to the eigenvalues of A with *zero real part*.

RECALL: HARTMAN-GROBMAN

We have previously considered how the *existence and stability of hyperbolic fixed points* are preserved during linearisation.

We now want consider the *differences in local dynamics between a nonlinear system and its linearisation in more detail*.

We'll look at how to do this using the *stable manifold theorem* and then using series expansions to approximate local stable/unstable manifolds.

GEOMETRY: STABLE AND UNSTABLE MANIFOLDS

Above we defined the *stable and unstable manifolds for linear systems*. (For non-hyperbolic there is also a centre manifold)

Now we want to give the definitions for *nonlinear hyperbolic fixed points*.

GEOMETRY: STABLE MANIFOLD (LOCAL)

Given some neighbourhood U of a stationary point x, the *local stable manifold* on U for a nonlinear system $W^s_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to \infty, \phi(y,t) \in U \text{ for all } t \geq 0\}$$

Picture?

GEOMETRY: UNSTABLE MANIFOLD (LOCAL)

Similarly, given some neighbourhood U of a stationary point x, the *local unstable manifold* on U for a nonlinear system $W^u_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to -\infty, \phi(y,t) \in U \text{ for all } t \leq 0\}$$

Picture?

GLOBAL MANIFOLDS

Note that if we want *global* versions then we can *'glue' together* all the flows starting at points in the local stable/unstable manifolds. That is,

$$W^s(0) = \bigcup_{t \geq 0} \phi(W^s_{loc}(0), t)$$

$$W^u(0) = \bigcup_{t \leq 0} \phi(W^u_{loc}(0), t)$$

STABLE MANIFOLD THEOREM

What's the *connection between these linear and nonlinear stable/unstable manifolds?* We have the following theorem (for local manifolds).

Suppose the origin is a *hyperbolic fixed point* for $\dot{x} = f(x)$ in $\mathbb{R}^n$ and that $E^s(0)$ and $E^u(0)$ are the stable and unstable manifolds of the linearised system $\dot{x} = Df(0)x$.

Then...

STABLE MANIFOLD THEOREM

…there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the same dimension as $E^s(0)$ and $E^u(0)$, respectively, and which are (respectively) tangent to E^s and E^u at the origin.

These manifolds are equally smooth/unsmooth as the original function f.

STABLE MANIFOLD THEOREM

Picture?

POWER SERIES EXPANSIONS IN TWO-DIMENSIONS

Even on small neighbourhoods U of our fixed points, *our manifolds are no-longer straight lines* (or hyperplanes etc in higher dims) as in the linear case - they are *curves* (or surfaces in higher-dimensions).

We can, however, use the information from the previous theorem to (try to) *compute local expressions for these curves*.

POWER SERIES EXPANSIONS FOR ONE-DIMENSIONAL MANIFOLDS

Assume a stable/unstable manifold of interest can be described by a curve $y = h(x)$ (or $x = g(y)$).

We can try to approximate this by a *local series expansion* of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

POWER SERIES: STEPS

- *Assume* the manifold can be described by $y = h(x)$ (or the other way around).
- *Substitute* $y = h(x)$ into our x and y equations to give $\dot{x} = f_1(x, h(x))$ and $\dot{y} = f_2(x, h(x))$.
- Use $y = h(x)$ *again*, along with the *chain rule* for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dh}{dx}\dot{x}$.

POWER SERIES: STEPS

- Use the above relationships along with an assumed *power series* expansion such as $h(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain *two expressions* in x for $\dot{y}$
- *Equate* powers of x to determine the unknown coefficients.
- Make sure to use the fact that the stable/unstable manifold *passes through* the fixed point and *is tangent* to the linearised stable/unstable manifold to determine the *first two terms* of the series.

EXAMPLES

Example 4.2 from Glendinning.

Tutorial sheet/assignment coming soon!