ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*) *Oliver Maclaren*
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MODULE OVERVIEW

Qualitative analysis of differential equations

(*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]
   Basic concepts and formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.
2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]
   Stability and linearisation of nonlinear systems. General linear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.
4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 8: INTRODUCTION TO BIFURCATION THEORY

- Overview and motivation for bifurcation theory: first taste of non-hyperbolic phenomena

NON-HYPERBOLIC FIXED POINTS AND STRUCTURAL INSTABILITY

Recall

Hyperbolic fixed points are *robust* to parameter perturbations - they are *structurally stable* features.

> *Non-hyperbolic* fixed points, on the other hand, are the *sensitive* cases. They are *structurally unstable* features.

NON-HYPERBOLIC FIXED POINTS AND BIFURCATIONS

Here *bifurcations*, i.e.

> *Changes* in stability and/or number of solutions/periodic orbits

are *possible*. We hence analyse the neighbourhood of these cases.

DIMENSION?

Even in large systems we *often only have bifurcations occuring at a small number of parameter values at a time* - e.g. only one eigenvalue crossing the imaginary axis.

Relatedly, we typically find *a small number of slow modes* determine the main 'emergent' observable dynamics.

Centre manifold theory (which we will come back to) provides a way of first reducing to the lower-dimensional system and then of analysing the *dynamics* near any bifurcations in these systems.

ONE DIMENSIONAL, ONE PARAMETER SYSTEMS; LOCAL VS GLOBAL

Because of this *we will focus on one-state variable, one-parameter systems*, i.e. $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$, which are frequently encountered during bifurcations in larger systems.

Note: we can analyse changes (bifurcations) in either or both *local* and *global* qualitative features of the phase portrait.

We will mainly focus on local bifurcations.

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

As mentioned, we now consider systems of the form

$$\dot{x} = f(x; \mu)$$

where $x \in \mathbb{R}^n$ is the usual vector of state variables but we have explicitly included $\mu \in \mathbb{R}^m$, a vector of *problem parameters*. Generally we further restrict attention here to $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

Problem parameters define our model *structure*.

These define a *family* of models. They can also be thought of as specifying particular 'model perturbations' near some distinguished parameter value.

They can also be thought of as extra *'very-slowly-varying/frozen' state variables* summarising neglected processes or external/enviroment conditions (we'll come back to this in centre manifold theory).

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

Bifurcation theory is about what happens *when our choice of parameter value (model structure) matters crucially*.

Again, in these cases we say our model has *structural* (c.f. solution) instabilities - and these instabilities are called *bifurcations*.

BIFURCATION DIAGRAMS

A *bifurcation diagram* is sort of like a phase portrait but with a *parameter* ('very slow/frozen state variable') on one of the axes.

Since the parameter is taken as 'frozen' at each value in turn, we summarise the properties of the main system in terms of some (typically) *long-term/asymptotic property* (or properties) of interest e.g. the *locations of the equilibria*.

BIFURCATION DIAGRAMS

i.e.

A bifurcation diagram shows how system properties of interest like equilibria *depend on variations in a system parameter* (or parameters).

Today: simple examples.

KEY CASES

- Saddle-node/turning point/fold bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Hopf bifurcation (see later)

EngSci 711 L08 : Bifurcation Theory

Introduction / overview of bifurcation theory

→ a first proper encounter with non-hyperbolic fixed points

Examples

Question 5 (16 marks)

Consider the equation

$$\dot{u} = (u-2)(\lambda - u^2)$$

where $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a parameter that can vary.

(a) Determine the equilibria and their stability as λ varies.

(11 marks)

(b) Sketch the bifurcation diagram showing how the equilibria vary with λ. What types of bifurcations occur?

(5 marks)

Exam 2017

Question 6 (20 marks)

(a) Consider the equation

$$\dot{u} = (\lambda - b)u - au^3$$

where $u \in \mathbb{R}$, a and b are fixed positive constants and λ is a parameter that can vary.

(i). Determine the equilibria and their stability. Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$ and $\lambda > b$ separately.

(ii). Sketch the bifurcation diagram showing how the equilibria vary with λ. What sort of bifurcation is this?

(b) Consider the second-order equation

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a system parameter.

(i). Re-write the above equation as system of two first-order equations.

(ii). Determine a value of μ for which a Hopf bifurcation could potentially occur at the origin. Show all your working.

(iii). Sketch a bifurcation diagram for a typical supercritical Hopf bifurcation, along with associated typical phase portraits for parameter values before, at and after the critical parameter value. Note: your diagram for this part need not refer to the equation given.

Exam 2016

Motivation

- So far we have focused mainly on hyperbolic solutions (mainly fixed points, also a little on periodic solutions) of a single system
- A key feature of these solutions is that they are 'robust' features of a system & its 'neighbouring' systems → they are 'persistent'

This is crucial for the key themes so far:

- stability of solutions (esp. fixed points) to linear & nonlinear systems

&

- the relationship between the properties of solutions to a nonlinear system & its linearisation

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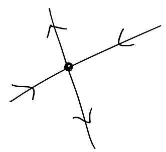
Eg if we analyse the stability of a fixed point of the linearised version of a nonlinear system, do the results still apply to the nonlinear system?

→ is there a corresponding fixed point?
→ does it have the same stability?

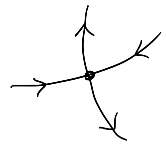
Yes... if hyperbolic



Fixed point in linearised system

connect :

⟺



Fixed point in nonlinear system

Structural stability

These ideas clearly concern 'stability' in a sense, but are more subtle than just 'solution' stability for a fixed/single model

→ The question concerns $\boxed{\text{structural stability}}$

↳ instead of replacing/perturbing a solution to get a 'nearby' solution & asking if it returns, we ask:

if I replace a model/system by a 'nearby' model/system, do I get equivalent results?

(eg same # of fixed points, periodic orbits, same stability...)

One version of this is replacing a nonlinear system by its linearisation, as discussed.

More generally we can consider the relation between

$$\boxed{\dot{x} = f(x) \quad \& \quad \dot{x} = f(x) + \epsilon\, g(x)}$$

(and corresponding fixed points, periodic orbits etc).

(eg $f(x)$ could be the linear part)

In practice, a common version of this is to consider a family of models with parameter μ

$$\boxed{\dot{x} = f(x;\mu)}$$

eg $\quad \dot{x} = x^2 - \mu$

→ each choice of μ gives a new model for x dynamics

→ in general, small variations in μ give small variations in the system properties

$\boxed{\text{BUT NOT ALWAYS!}}$

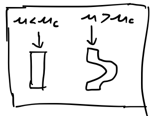
When? Hyperbolic!

As you might have guessed, we need to consider non-hyperbolic solutions (fixed points mainly) in more detail, & in systems with 'external' parameters, $\dot{x} = f(x;\mu)$

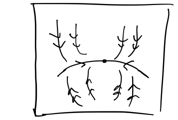
These are important for at least two reasons

1. ∘ The possibility of bifurcations: dramatic changes in solution existence &/or stability as an 'external' parameter is varied past a threshold/boundary

   → eg a beam suddenly buckling as its load is slowly increased past a certain limit



2. ∘ The possibility of dimension reduction: the 'unusual' solutions nevertheless tend to dominate/attract the long-term/emergent dynamics



   → recall how fixed points/singularities dominate the long-term dynamics, despite 'most' points not being fixed points!

Analogy:

a 'typical' data point is not an outlier, but a 'typical' dataset contains at least one outlier, & these can have a large influence: eg calculating a mean

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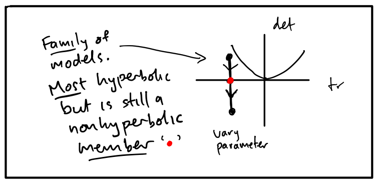
Here: data point ⟷ particular model

dataset ⟷ family or collection of models indexed by parameters



We will focus on the first of these here — the possibility of bifurcation — and return to the second when we look at centre manifold theory.

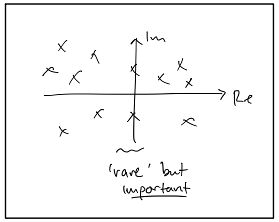
An important connection exists, however:

High dimensional, typically regular system	$\xrightarrow[\text{near 'irregular' point}]{\text{Centre manifold reduction}}$	Low dimensional system with potential for bifurcation

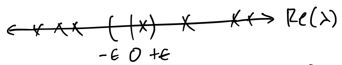
This 'explains' why the models we look at in this section are often one-dimensional

→ all the 'action' occurs on a lower-dimensional 'centre manifold'

↳ eigenvalues only cross imaginary axis (ie $Re(\lambda)=0$) one or two (etc) at a time



Note: Later we will see that we only really need a good separation or 'spectral gap' between 'small' & 'large' eigenvalues, ie



to carry out a centre manifold reduction.

So.... what are we actually doing then?

We consider simple, low-dimensional model families of the form:

$$\boxed{\dot{x} = f(x\,;\mu)}$$

change in state var $\dot{x}\in\mathbb{R}^n$ — state var $x\in\mathbb{R}^n$ — parameter vector $\mu\in\mathbb{R}^p$

Such as

- $\dot{x} = a - x^2$
- $\dot{x} = ax(1-\frac{b}{a}x) = ax - bx^2$
- $\dot{x} = ax(1-\frac{b}{a}x^2) = ax - bx^3$

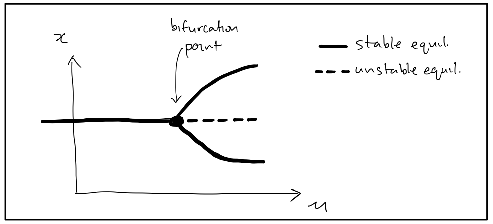
where a & b are parameters (ie $\mu = \binom{a}{b}$ etc)

These are 'generic' → can be found 'within' many large systems ('normal forms')

We want to consider what happens to the fixed points of these systems as the parameters vary

→

This leads to the idea of a **bifurcation diagram**, where we plot special solutions like fixed points etc &/or other properties of interest of a parameterised system vs a parameter (an 'interest' or 'control' parameter)



Bifurcation diagram for $\dot{x} = f(x;\mu)$

Q: why is the labelled point a 'bifurcation' point?

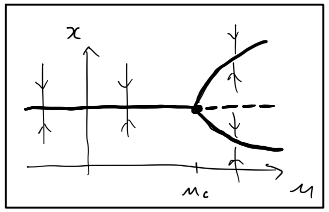
A: there is a change in stability &/or existence of fixed points of the system at this point

→ The system is expected to be non-hyperbolic at this point!

Note: since the parameter μ can be considered as a 'super slow'/'frozen' extra state variable, you can also view the above as a phase plane portrait for the extended system

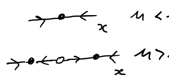
$$\boxed{\begin{aligned}\dot{x} &= f(x;\mu)\\ \dot{\mu} &= 0\end{aligned}}$$

ie



'extended' phase portrait.

[usual 1D phase portraits of x for each μ:



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Bifurcation diagrams are usually computed numerically using continuation → see end of this lecture

We will take a more simple-minded approach, best illustrated via examples

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So ... Consider

$$\boxed{\dot{x} = a - x^2}, \quad x \in \mathbb{R} \quad \text{(one-dimensional state space)}$$

where 'a' is a parameter that can be varied.

Q: How do we analyse?

A: Pretty much the same way!

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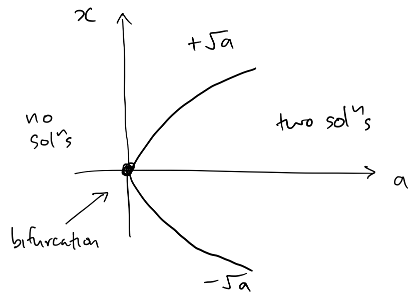
1. Fixed points

$$a - x^2 = 0$$

$$\Leftrightarrow \underline{x^2 = a} \Rightarrow \begin{cases} a < 0, & \text{no sol}^n \\ a = 0, & \text{one sol}^n \\ \underbrace{a > 0}_{\llcorner x = \pm\sqrt{a}}, & \text{two sol}^n \end{cases}$$

2. Sketch diagram of fixed points (etc) without stability



→

3. Consider stability

Find Jacobian!

$$Df = -2x \begin{cases} < 0, \text{ if } x > 0 \\ = 0, \text{ if } x = 0 \\ > 0, \text{ if } x < 0 \end{cases}$$

$\Rightarrow$ upper sol^n $(x>0)$ is stable, $Df = \lambda < 0$

$\Rightarrow$ lower sol^n $(x<0)$ is unstable, $\underbrace{Df = \lambda > 0}$

$\Rightarrow$ bif. for $x = 0$

1D system
Df = eigenvalue!

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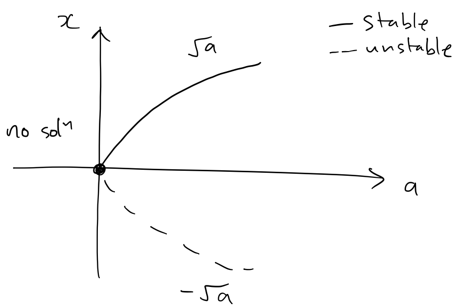
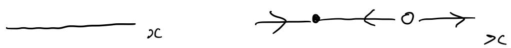
4. Combine to give final bifurcation diagram



Note: as series of 1D phase portraits:



$a < 0$ $\qquad$ $a > 0$

Key bifurcations of fixed points

[Note: in each case, reversed is same

eg  same as]

Terminology:

- 'co-dimension one': get bifurcation by varying one parameter
- 'normal forms': standard forms of equations giving bifurcations

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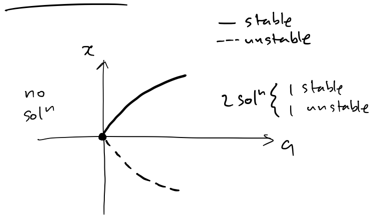
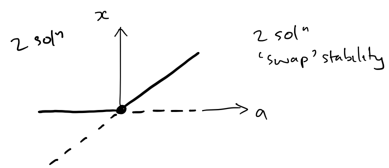
- Saddle node (or turning point, fold)



Normal form:

$$\boxed{\dot{x} = a - x^2}$$

- Transcritical

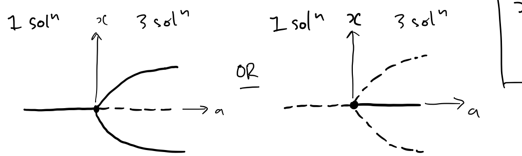


Normal form:

$$\boxed{\begin{aligned} \dot{x} &= ax\left(1 - \tfrac{b}{a}x\right) \\ &= ax - bx^2 \end{aligned}}$$

(logistic)

- Pitchfork



new branches stable ('supercritical') $\qquad$ new branches unstable ('subcritical')

Normal form:

$$\boxed{\begin{aligned} \dot{x} &= ax\left(1 - \tfrac{b}{a}x^2\right) \\ &= ax - bx^3 \end{aligned}}$$

+ Hopf (see later $\rightarrow$ 'birth' of periodic orbits)

Note: 'net stability' is conserved! ('matter & antimatter')

Extra

[details not examinable]

Continuation of regular fixed points

- In essence we are just trying to find how solutions to

$$\boxed{f(x;\mu)=0}$$

depend on the parameter (or a subset of them)

- $\boxed{\text{Parameter continuation}}$ is based on the idea that at 'regular' fixed points – ie hyperbolic fixed points – we can 'continue' a solution found for $\mu=\mu_0$ to a 'nearby' solution for $\mu=\mu_0+\delta$ } Implicit function theorem

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Consider

$f(x_0;\mu_0)=0$ where x_0 & μ_0 are given

we want to find $x=x_0+\epsilon$, for $\mu=\mu_0+\delta$ &

$$f(x;\mu)=0 \iff f(x_0+\epsilon;\mu_0+\delta)=0$$

Expanding gives

$$f(x_0+\epsilon;\mu_0+\delta)=0=\underbrace{f(x_0;\mu_0)}_{=0}+\overbrace{\left.\frac{\partial f}{\partial x}\right|_{x_0,\mu_0}}^{Df}\cdot\epsilon+\left.\frac{\partial f}{\partial \mu}\right|_{x_0,\mu_0}\cdot\delta+\dots$$

Jacobian / linearisation in x.

which gives approximately

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,\mu_0}\cdot(x-x_0)=-\left.\frac{\partial f}{\partial \mu}\right|_{x_0,\mu_0}\cdot(\mu-\mu_0)+\dots$$

ie

$$\boxed{x=x_0-\left(\frac{\partial f}{\partial x}\right)^{-1}\frac{\partial f}{\partial \mu}(\mu-\mu_0)}+\dots$$

as long as $Df=\frac{\partial f}{\partial x}$ is invertible → no zero eigenvalues!

Extra

[details not examinable]

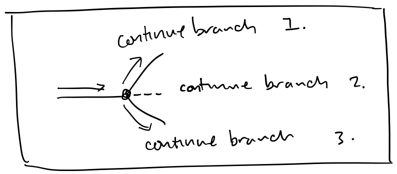
Continuation at 'irregular' or 'singular' fixed points

- When the linear system is not invertible (eg has zero eigenvalue) there may be multiple or no solutions

- What happens here depends on the various higher derivatives f_{xx}, f_{xa}, f_{aa} etc

- These can be used to give defining equations for different types of bifurcation in terms of these higher derivatives

↳ these tell us how to detect different types of bifurcation & to continue along particular solution branches if they exist



As mentioned, in practice we will use a more simple-minded approach:

- find solutions to $f(x;\mu)$ & sketch
- find Df
- find where $Df(x;\mu)=0$ } possible bifurcation
- determine stability along branches using Df, noting that changes in stability can only happen at bifurcation points

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