

# ENGSCI 711

## QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(...and other dynamical systems)

Oliver Maclaren

oliver.maclaren@auckland.ac.nz

## MODULE OVERVIEW

### 3. Introduction to bifurcation theory [4 lectures/tutorials]

Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations.  
Bifurcation diagrams.

### 4. Centre manifold theory and putting it all together

[4 lectures/tutorials]

Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle: approximately decoupling non-hyperbolic systems.

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## MODULE OVERVIEW

Qualitative analysis of differential equations (Oliver Maclaren) [~16-17 lectures/tutorials]

### 1. Basic concepts [3 lectures/tutorials]

Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

### 2. Phase plane analysis, stability, linearisation and classification [5-6 lectures/tutorials]

General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling hyperbolic systems.

## LECTURE 1 - 'BIG PICTURE'

Why? *What problem are we trying to solve?*

- Complex models (and the phenomena they represent) are *difficult to understand*
- Even 'simple' models can be difficult to understand

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## LECTURE 1 - COMPLEX MODELS

- No *closed-form* solutions
- *Brute-force* simulation doesn't necessarily help us *understand* our model (and the phenomenon we are modelling)
- All models are *wrong* (Box)
- Better to be *approximately right* than exactly wrong (Tukey)

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## EXAMPLE - LORENZ SYSTEM

'Time' simulation.

See 'lorenz.ode'

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## EXAMPLE - LORENZ SYSTEM

Can we *understand* this three-dimensional, three-parameter ODE?

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z\end{aligned}$$

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## HIDDEN COMPLEXITY AND HIDDEN SIMPLICITY

Despite having a small number of parameters and variables, the Lorenz system *exhibits complex behaviour* (even simple 1-D discrete maps can also exhibit such complexity)

However, by looking at it from *a different point of view* we can get some understanding of this system.

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## HIDDEN SIMPLICITY AND HIDDEN COMPLEXITY

Let's plot a so-called *phase-portrait*

See 'lorenz.ode'

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## WHAT'S THE UPSHOT?

Think:

- Emergent, qualitative features of interest
- Stability and robustness
- Classification, decoupling, separation
- Approximation and bounds

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## WHAT'S THE UPSHOT?

Our point of view here will be

- The '*qualitative*' analysis of ODEs (and 'dynamical systems' in general). We 'step back' from the full detail to enable a better overall picture
- Essentially a *geometric* perspective
- Possibly a *different way of thinking* than you are used to
- Both *hand calculation and computer-based methods* will be used; the goals are the same though

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## WHAT'S THE UPSHOT?



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## WAYS OF THINKING

- Part of this module is about 'cookbook' methods to add to your toolbox
- Part of it is to give you an introduction to the underlying mathematical/geometric viewpoint

I'll try to balance practical analysis methods with formal definitions and mathematical ideas!

- Hands-on experience is important: lots of *tutorials*

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## FURTHER READING/WATCHING

- Strogatz course on YouTube! <https://goo.gl/Kqus6G>
- Handout on helpful books/references

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## EXAMPLE ANALYSES

"What do you expect me to be able to do for the exam?"

Let's look at the exam from last year!

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## Qualitative Analysis of ODEs (& Other Dynamical Systems)

Plan : 4 Sections

1. o Basic overview / terminology
2. o Phase Plane & Linearisation
3. o Bifurcation Theory
4. o Centre Manifold Theory

- Slides :
- o Big Picture
  - o Technical Definitions
    - L for the mathematicians
    - L moral obligation to show you!
  - o Tidier

- Handwritten  
&  
Scanned :
- o Messier --- BUT:
  - o What you need to be  
able to do & how!

### Qualitative ?

- complex models e.g. 100s - 1000s of parameters, equations etc; nonlinear
- we can simulate for one set of parameters, then another etc etc

L Brute force quantitative simulation approach

TBUT : - inefficient/infeasible?  
(can be)

- lacking in insight?

L why do we get answer  
we get?

L under what  
conditions etc?

L how could we change  
the behaviour?

} nonlinearity makes  
difficult  
(not just  
'sum of  
parts')

→ we want more qualitative, approximate  
methods to complement eg direct  
simulation

→ we use a geometrical & topological  
perspective

→ apply to non-linear systems

## Examples

- Lorenz system



ODE system

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z\end{aligned}$$

in XPPaut:

```
init x=-7.5 y=-3.6 z=30
par r=27 s=10 b=2.66666
x'=s*(y-x)
y'=x*(r-z)-y
z'=x*y-b*z
```

→ see lab!

how? see later!

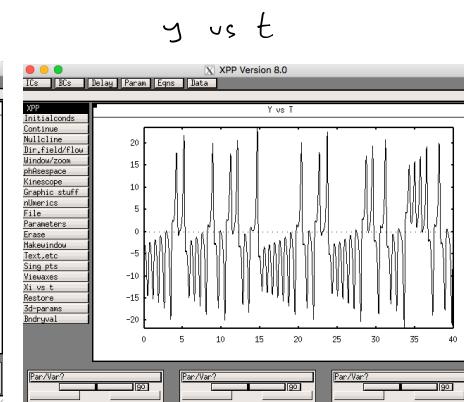
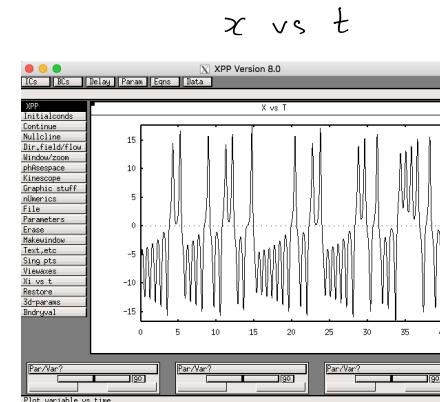
- simplified model of atmospheric convection (also arise as model of lasers, chemical reactions etc)

- Exhibits chaotic solutions!
- (even though 'simplified')

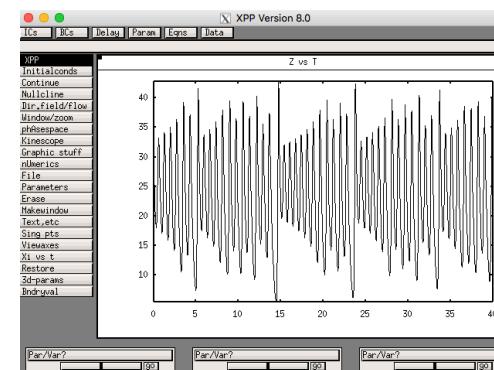


'Brute force' / direct time simulation

→ plot x,y,z vs time for fixed set of parameters



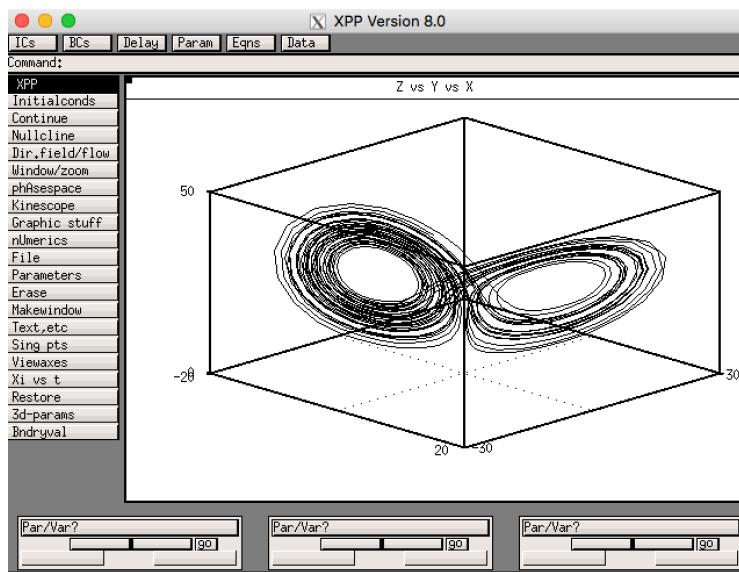
z vs t



→ What's going on?

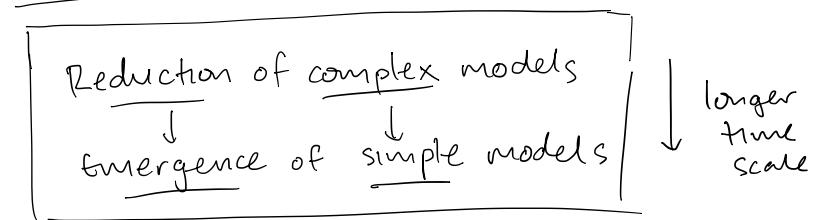
## 'Shift' perspective

- plot in so-called phase/ state space
- focus on the 'geometry'/topology of the problem
- where are trajectories attracted to?



- a clearer structure 'emerges' as time goes on
- the dynamics are attracted to the 'strange' butterfly-like shape

## Other examples: Reduction & Emergence



Eg

- How might we get a 'simplified' model like the Lorenz system from a more complicated model (eg Navier-Stokes)?

- Turns out this model can actually be obtained using centre manifold reduction theory

↳ we will cover this theory towards the end of this part!

- Again, focus on the geometry/topology of the long term dynamics

even related to 'hot topics' like  
deep learning!

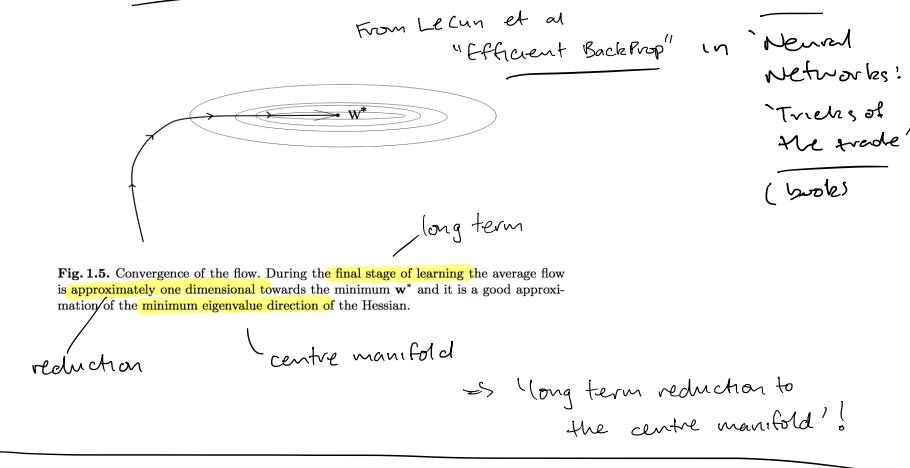


Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum  $w^*$  and it is a good approximation of the minimum eigenvalue direction of the Hessian.

CBMM Memo No. 073

January 30, 2018

### Theory of Deep Learning III: the non-overfitting puzzle

T. Poggio<sup>†</sup>, K. Kawaguchi<sup>†‡</sup>, Q. Liao<sup>†</sup>, B. Miranda<sup>†</sup>, L. Rosasco<sup>†</sup>  
X. Boix<sup>†</sup>, J. Hidary<sup>†‡</sup>, H. Mhaskar<sup>‡</sup>,  
<sup>†</sup>Center for Brains, Minds and Machines, MIT  
<sup>‡</sup>CSAIL, MIT  
<sup>††</sup>Alphabet (Google) X  
<sup>‡‡</sup>Claremont Graduate University

**Abstract:** A main puzzle of deep networks revolves around the apparent absence of overfitting intended as robustness of the expected error against overparametrization, despite the large capacity demonstrated by zero training error on randomly labeled data.

In this note, we show that the dynamics associated to gradient descent minimization of nonlinear networks is topologically equivalent, near the asymptotically stable minima of the empirical error, to a gradient system in a quadratic potential with a degenerate (for square loss) or almost degenerate (for logistic or crossentropy loss) Hessian. The proposition depends on the qualitative theory of dynamical systems and is supported by numerical results. The result extends to deep nonlinear networks two key properties of gradient descent for linear networks, that have been recently recognized (*I*) to provide a form of implicit regularization:

qualitative  
theory of  
dynamical  
systems

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. Thus the first step is to linearize the system, which means considering the Jacobian of  $F$  or equivalently the Hessian of  $L$  at  $W^*$ , that is

$$H_{ij} = \frac{\partial^2 L}{\partial w_i \partial w_j} \quad (2)$$

We obtain

$$\dot{W} = -HW, \quad (3)$$

where the matrix  $H$ , which has only real eigenvalues (since it is symmetric), defines in our case (by hypothesis we do not consider unstable critical points) two main subspaces:

- the stable subspace spanned by eigenvectors corresponding to negative eigenvalues
- the center subspace corresponding to zero eigenvalues

centre manifold  
theory

The center manifold existence theorem (*II*) states then that if  $F$  has  $r$  derivatives (as in the case of deep polynomial networks) then at every equilibrium  $W^*$  there is a  $C^r$  stable manifold and a  $C^{r-1}$  center manifold which is sometimes called *slow manifold*. The center manifold emergence theorem says that there is a neighborhood of  $W^*$  such that all solutions from the neighborhood tend exponentially fast to a solution in the center manifold in general properties of  $W$ . We note further the center manifold is not visible in our case reflecting empirical results in training networks. Of course, the dynamics associated with the center manifold may be non-trivial, especially in the case of SGD. We simply assume for now that perturbations of it will not grow and trigger instability.

A simpler example (illustrates same idea)  
as previous 'deep learning' slide

Consider the system:

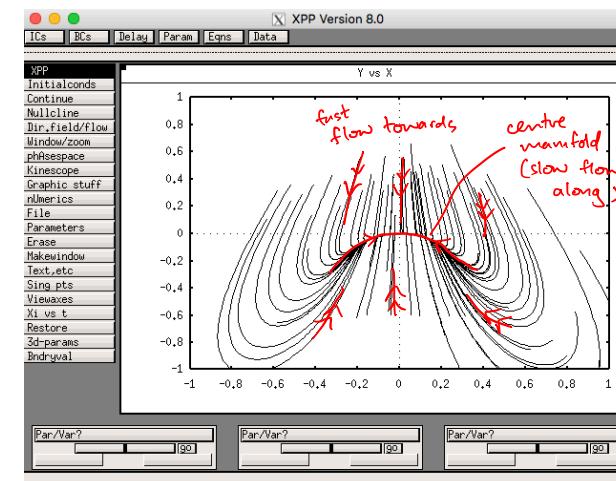
$$\begin{aligned} \frac{dx}{dt} &= xy + x^3, \\ \frac{dy}{dt} &= -y - 2x^2 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{nonlinear}$$

( Turns out the usual linear analysis doesn't work - we'll look at where & why this happens )

emergent, attracting 'manifold' / curved surface/ subspace

→ fast motion to manifold

→ slow motion along manifold



## Upshot

- emergent (long time) simple (reduced) descriptions of complex systems
- approximate, qualitative & geometric approach to complement detailed simulation methods
- stability, instability & emergence for nonlinear systems



(Dali)

- what you see depends on the scale  
→ a qualitatively different picture 'emerges' on the coarser scale.

- 'But what do I need to be able to do for the exam?'



Exam Qs 2017 - not too bad!

(most did very well)

### SECTION B

#### Question 4 (16 marks)

Consider the system

$$\begin{aligned}\dot{x} &= 2xy + x^3 \\ \dot{y} &= -y - x^2\end{aligned}$$

where  $x, y \in \mathbb{R}$ .

- (a) Verify that the origin is a fixed point of this system. (1 mark)  
 (b) Find the Jacobian derivative - first as a function of  $x$  and  $y$  and then evaluated at the origin  $(0, 0)$ . (2 marks)  
 (c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces,  $E^s, E^u$  and  $E^c$  respectively. Sketch the eigenspaces in the  $(x, y)$  plane. You do not need to show any nearby trajectories. (3 marks)  
 (d) Use a power series expansion to calculate an expression for the centre manifold  $W_{loc}(0, 0)$  that is correct up to and including cubic order. (8 marks)  
 (e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable. (2 marks)

solve  $\text{RMS} = 0$

- linearise

- calc eigenvalues of a matrix

use a power series approx & sub into centre manifold equation (new)

#### Question 5 (16 marks)

Consider the equation

$$\dot{u} = (u - 2)(\lambda - u^2)$$

where  $u \in \mathbb{R}$  and  $\lambda \in \mathbb{R}$  is a parameter that can vary.

- (a) Determine the equilibria and their stability as  $\lambda$  varies. (11 marks)  
 (b) Sketch the bifurcation diagram showing how the equilibria vary with  $\lambda$ . What types of bifurcations occur? (5 marks)

solve  $\text{RMS} = 0$  as function of  $\lambda$

draw a picture & label

#### Question 6 (18 marks)

Consider the system

$$\begin{aligned}\dot{x} &= x^2 + y^2 - 2 \\ \dot{y} &= x - 1\end{aligned}$$

where  $x, y \in \mathbb{R}$ .

- (a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors. (6 marks)  
 (b) Write down the equations for the  $x$ - and  $y$ -nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch. (10 marks)  
 (c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough. (2 marks)

solve  $\text{RMS} = 0$

solve  $\text{RMS} = 0$  one at a time

draw some pictures.

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### Qualitative analysis of differential equations

*Oliver Maclaren*  
*oliver.maclaren@auckland.ac.nz*

### Some references specifically from the ‘dynamical systems’ perspective

*Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering by Steven H. Strogatz.*

- Great book, easy to read, lots of examples. A little on the wordy side. A math prof. once complained ‘it’s great but it doesn’t mention the centre manifold theorem’. I didn’t care at the time, though I have more sympathy for this complaint now. Often recommended as a supplement in more ‘advanced’ courses to give the ‘big picture’, also used in many introductory courses. We will mainly work at this sort of level, but introduce some material and definitions from the other books below.
- Two copies of the 1994 edition (the one I have!) are available in the main library. The 2005 Edition is available as online ebook through the library (you should be able to find it - if not, ask me).

*Simulating, analyzing, and animating dynamical systems : a guide to XPPAUT for researchers and students by Bard Ermentrout.*

- Bard wrote XPPAut, an interface to the AUTO and XPP dynamical systems software. Also known for his contributions to mathematical biology. Easy to read, great way to get some hands-on experience doing examples. If you wanted to use XPPAut for a real problem then this is a great place to start. My PhD supervisor used it to make many figures in his book ‘Mathematical Physiology’ and they don’t look terrible.
- Available from general library and to read online through the library access to SIAM.

*Stability, instability, and chaos : an introduction to the theory of nonlinear differential equations by Paul Glendinning*

- Nice coverage, all the key theorems. For the more mathematically-inclined but still quite ‘applied’. In the ‘Cambridge Texts in Applied Mathematics’ series. I used this when I took Maths 761 at Auckland, it really grew on me.

- Multiple copies available from main library, engineering library and short loan (since used for Maths 761 course?).

*Introduction to applied nonlinear dynamical systems and chaos by Stephen Wiggins.*

- Multiple copies/editions in both general and engineering library. Also available for free download through library access to Springer book. Another solid book, similar level to Glendinning. Probably more comprehensive.

*Nonlinear systems by P. G. Drazin.*

- Good broad coverage of both difference and differential equations. Has particularly good material on bifurcation theory. For the more mathematically-inclined. A bit less systematic than Glendinning but lots of good extra stuff in there. Also in the ‘Cambridge Texts in Applied Mathematics’ series. I used this when I TA’d a third-year applied mathematics course in the Oxford mathematics department.
- A copy in the main library and a copy in the engineering library.

*Differential equations and dynamical systems by Lawrence Perko*

- Good material on centre manifold theory, including simple examples!
- Available online through the main library (can download through Springer) and multiple hard copies in the general library.

*Model Emergent Dynamics in Complex Systems by A. J. Roberts*

- An attempt to present undergraduate mathematical modelling from the point of view of centre manifold theory and nonlinear coordinate transformations
- The theory is viewed as a way to derive emergent, simple models from more complex models (rather than just as a tool for analysing given models)
- Lots of interesting material and examples. Quite idiosyncratic presentation.
- Available online through the main library.

*Nonlinear systems by Khalil*

- Two copies, one in engineering one in general library. Focused on engineering applications but is a graduate-level text. Accessible material on centre manifold theory.

*Nonlinear oscillations, dynamical systems, and bifurcations of vector fields by John Guckenheimer and Philip Holmes.*

- Multiple copies available in general library and short loan (probably for Maths 761?). The classic, still really good, written by ‘legends’ of the

field. A little more advanced than the others. John Guckenheimer often visits/works with people in the Auckland math department, who are really strong on this stuff. If you like this area and wanna continue (e.g. do a Masters/PhD) then I can point you to them.

*Elements of applied bifurcation theory* by Y. A. Kuznetsov.

- Multiple copies in both engineering and main library. Similar level to Guckenheimer and Holmes but a surprisingly easy read for a fairly advanced book. Good material on ‘practical’ bifurcation analysis e.g. numerical methods. Yuri wrote (with others) ‘MatCont’ which is Matlab-based software for bifurcation analysis.

## Other references

Almost all books on ‘Advanced Engineering Mathematics’ or (often ‘Introduction to’) ‘Applied Mathematics’ will have good condensed material on qualitative analysis (esp. phase plane analysis) of differential equations. They often also have some brief material on bifurcation theory.

- In the ‘Advanced Engineering Mathematics’ category I quite like Greenberg
- In the ‘Applied Mathematics’ category I quite like Logan.

Both are available in the engineering library.

These sorts of references might often be easier to quickly learn the basics from than the more comprehensive books exclusively focused on dynamical systems.