Question 4 (20 marks)

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

(a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$.

Suppose your linearised system is now written in the form

$$\dot{\eta} = A\eta$$

or, explictly,

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$ are constants.

(b) By seeking solutions of the form $\eta(t) = e^{\lambda t}u$, where $u \in \mathbb{R}^2$ is a constant vector, derive an appropriate eigenvalue problem that you could use to classify the fixed point of the linear system.

(c) Write down the general form of the characteristic equation and associated solution for the eigenvalue problem in terms of the trace, tr$A$, and determinant, det$A$, of the matrix A. What are the trace and determinant of A in terms of a, b, c, d?

(d) Draw a diagram with axes corresponding to tr$A$ and det$A$ that shows how to classify the fixed point of the linear system. Notes: you should include stability properties and typical names of fixed point types. You should include centres but do not need to include other boundary cases. You do not need to draw phase portraits on this diagram.

(e) Sketch a phase portrait for

(i). A case where the (local/linearised) phase portrait for x^* would be structurally stable.

(ii). A case where the (local/linearised) phase portrait for x^* would be structurally unstable.

Question 5 (20 marks)

Consider the system

$$\begin{aligned} \dot{x} &= y(2x - y) \\ \dot{y} &= x^2 - y \end{aligned}$$

(a) Find the two fixed points of this system. Show your working. You do not need to classify these.

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$.

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0,0)$ that is correct up to and including cubic order.

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.

Question 6 (20 marks)

(a) Consider the equation

$$\dot{u} = (\lambda - b)u - au^3$$

where $u \in \mathbb{R}$, a and b are fixed positive constants and λ is a parameter that can vary.

(i). Determine the equilibria and their stability. Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$ and $\lambda > b$ separately.

(ii). Sketch the bifurcation diagram showing how the equilibria vary with λ. What sort of bifurcation is this?

(b) Consider the second-order equation

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a system parameter.

(i). Re-write the above equation as system of two first-order equations.

(ii). Determine a value of μ for which a Hopf bifurcation could potentially occur at the origin. Show all your working.

(iii). Sketch a bifurcation diagram for a typical supercritical Hopf bifurcation, along with associated typical phase portraits for parameter values before, at and after the critical parameter value. Note: your diagram for this part need not refer to the equation given.