ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

RETURN OF THE MAP

- Periodic orbits in the plane: ruling out/in; nullclines, trapping regions
- Intro to periodic orbits in higher dimensions via return maps

PERIOIC ORBITS IN THE PLANE

A key question for a given nonlinear ODE system is *whether it admits closed curve solutions* - i.e. *periodic orbits* (oscillations).

How can you *rule then out?* How can you *rule them 'in'?*

The results that follow *typically apply only in the plane* - 'no chaos in the plane'.

(We *can* make progress on understanding higher-dimensional problems, it's just much harder to have general theorems on what is/isn't going to happen.)

NONLINEAR PLANAR SYSTEM

Note that for the next few slides we are looking at *planar nonlinear systems*

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$

RECALL: PERIOIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t + T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t + s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

RULING OUT PERIODIC ORBITS

The *(Poincare) index/winding number* is a (topological) invariant of closed curves in the plane.

We won't go into it (see p. 126-129 Glendinning, p. 174-180 Strogatz (1994) if interested) but note that it can be used to show (among other things)

Inside any closed orbit in the plane there *must be at least one fixed point*.

Example 6.8.5 Strogatz (1994).

RULING OUT PERIODIC ORBITS

Recall the *divergence theorem* (with a weight g):

Suppose Γ is a simple (doesn't cross itself) closed curve with outward normal n enclosing a region R and f and g are continuously differentiable functions $\mathbb{R}^2 \to \mathbb{R}^2$ then

$$\int_\Gamma g(n \cdot f) dl = \int \int_R \nabla \cdot (gf) dx dy$$

where $gf := g(x, y) f(x, y)$.

RULING OUT PERIODIC ORBITS

This can be used to formulate *Dulac's criterion* (see Glendinning 5.6):

> If there exists a g (refer previous slide) such that $\nabla \cdot (gf)$ is continuous and *has one sign throughout* a simply connected domain D then there are *no closed orbits lying entirely in D.*

If we take $g = 1$ then this is often called the *divergence test*.

Example (Glendinning 5.9).

RULING IN PERIODIC ORBITS - THE POINCARE-BENDIXSON THEOREM

The *Poincare-Bendixson theorem* allows one to establish the *existence of a periodic orbit*. It also establishes that there is *'no chaos in the plane'* (for smooth ODEs).

Let D be a closed and bounded domain in the plane and suppose there are *no stationary solutions in D*. Then, if the orbit $\phi(x_0, t)$ *begins in/enters D and does not leave D for all time*, then the *orbit is either closed or spirals toward a closed orbit* as $t \to \infty$

THE POINCARE-BENDIXSON THEOREM - TRAPPING REGIONS

The standard trick to finding an appropriate region is to construct a *trapping region* R - a closed connected subset such that the vector field *points 'inwards' everywhere on the boundary*.

This implies (proof not shown!) that *all orbits are confined* to R (i.e. once in don't leave).

If we can construct an R *without a fixed point inside* then there exists a closed (i.e. periodic) orbit.

- Strogatz (1994) Example 7.3.2 (see handout).

RECALL: NULLCLINES

Nullclines are a very useful part of sketching phase-plane portraits and constructing trapping regions!

Given a system of equations for $x \in \mathbb{R}^n$ with components $\dot{x}_i = f_i(x)$, the jth *nullcline* is where

$$\dot{x}_j = f_j(x) = 0$$

The flow is *perpendicular* to the x_j-axis along the associated curves/surfaces (usually of dim $n - 1$). Note: there *may be multiple lines/curves for one nullcline!*

Q: What are points where *all* nullclines intersect called?

PERIODIC ORBITS IN HIGHER DIMENSIONS

Periodic orbits in high dimensions can be (very!) complicated.

We will hence try to *introduce a 'simpler' object to study* which can help us *understand periodic orbits in quite general/complicated systems*.

This leads to our first encounter with another type of dynamical system - *discrete maps*.

Here these will arise as a *tool to study (e.g. periodic orbits in) ODEs*; note that they can arise as interesting models in their own right.

RETURN MAPS

Given a nonlinear ODE system $\dot{x} = f(x)$.

A *Poincare section* is P_Σ, is a *transverse section of the trajectories* of an ODE system, which is nowhere tangential to any trajectory.

We can *label* the points in (time) *order of their intersection* with P_Σ, giving $x_0, x_1, ...x_n, ...$

Picture.

RETURN MAPS

The Poincare section defines a (discrete!) *Poincare/return map*

$$x_{n+1} = F(x_n)$$

There is a corresponding *theory of stability/instability/bifurcation for discrete maps*. This can be used to *deduce properties* of e.g. equilibria and periodic orbits in the *original ODE system*.

Example (not examinable).