ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling hyperbolic systems.

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle: approximately decoupling non-hyperbolic systems.

LECTURE 1 - 'BIG PICTURE'

Why? *What problem are we trying to solve?*

- Complex models (and the phenomena they represent) are *difficult to understand*
- Even 'simple' models can be difficult to understand

LECTURE 1 - COMPLEX MODELS

- No *closed-form* solutions
- *Brute-force* simulation doesn't necessarily help us *understand* our model (and the phenomenon we are modelling)
- All models are *wrong* (Box)
- Better to be *approximately right* than exactly wrong (Tukey)

EXAMPLE - LORENZ SYSTEM

Can we *understand* this three-dimensional, three-parameter ODE?

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

EXAMPLE - LORENZ SYSTEM

'Time' simulation.

See 'lorenz.ode'

HIDDEN COMPLEXITY AND HIDDEN SIMPLICITY

Despite having a small number of parameters and variables, the Lorenz system *exhibits complex behaviour* (even simple 1-D discrete maps can also exhibit such complexity)

However, by looking at it from *a different point of view* we can get *some* understanding of this system.

HIDDEN SIMPLICITY AND HIDDEN COMPLEXITY

Let's plot a so-called *phase-portrait*

See 'lorenz.ode'

WHAT'S THE UPSHOT?

Our point of view here will be

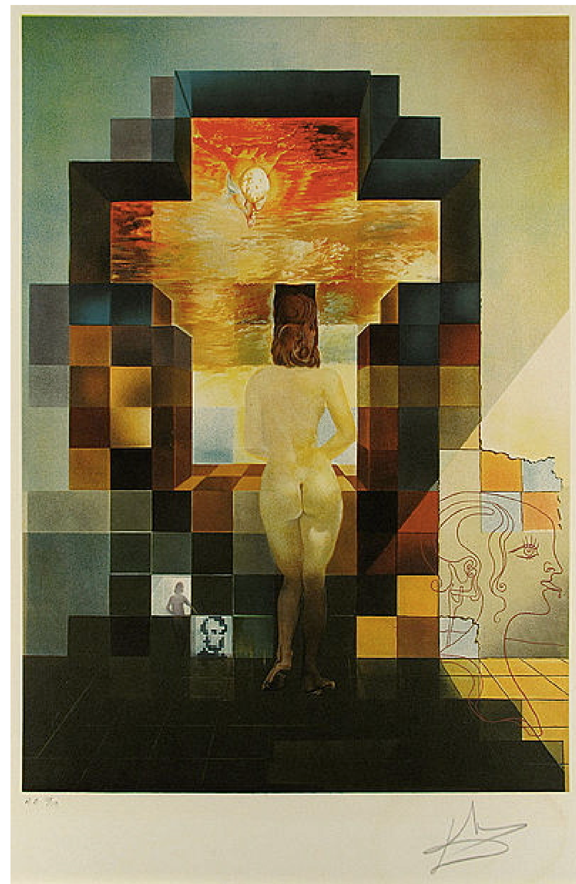
- The *'qualitative'* analysis of ODEs (and 'dynamical systems' in general). We 'step back' from the full detail to enable a better overall picture
- Essentially a *geometric* perspective
- Possibly a *different way of thinking* than you are used to
- Both *hand calculation and computer-based methods* will be used; the goals are the same though

WHAT'S THE UPSHOT?

Think:

- Emergent, qualitative features of interest
- Stability and robustness
- Classification, decoupling, separation
- Approximation and bounds

WHAT'S THE UPSHOT?



WAYS OF THINKING

- Part of this module is about 'cookbook' methods to add to your toolbox
- Part of it is to give you an introduction to the underlying mathematical/geometric viewpoint

I'll *try* to balance practical analysis methods with formal definitions and mathematical ideas!

- Hands-on experience is important: lots of *tutorials*

EXAMPLE ANALYSES

"What do you expect me to be able to do for the exam?"

Let's look at the exam from last year!

SOME CONCEPTS/DEFINITIONS: WHAT IS A 'DYNAMICAL SYSTEM'?

Informally, a *dynamical system* is a mathematical model of a *process which evolve in time*.

There are *three key ingredients*: a set or interval of *'times'*, possible *'states'* of a 'system' at any given time and an *'evolution rule'* or law governing how the system transititions between these states.

WHAT IS A 'DYNAMICAL SYSTEM'?

Examples are everywhere - ODEs, PDEs, difference equations/maps, stochastic processes even iterative computer algorithms and constructive mathematical proofs.

The *'dynamic' point of view* complements the (also important) *'static' point of view* - e.g. limiting processes vs the actual limits.

ORDINARY DIFFERENTIAL EQUATIONS

We will first look at ODE systems in the form

$$\dot{x} = f(x, t; \mu)$$

where $x \in \mathbb{R}^n$ is a vector of *state variables*, $t \in \mathbb{R}$ is the *independent variable* (usually time), $\mu \in \mathbb{R}^m$ is a vector of *problem parameters*.

(For now we often suppress the dependence on problem parameters - but see bifurcation theory!)

ORDINARY DIFFERENTIAL EQUATIONS

We have one equation in f for each entry in the state vector x e.g.

$$x = (x_1, x_2, \ldots)^T$$

$$f = (f_1, f_2, \ldots)^T$$

If there is no dependence on t then we say the system is *autonomous* (we can always add a new dependent variable to track time dependence). We will focus on these in this course.

EXAMPLE

A differential equation example.

FURTHER READING/WATCHING

- Strogatz course on YouTube! https://goo.gl/Kqus6G
- Handout on helpful books/references