ENGSCI 213: MATHEMATICAL MODELLING 2SE

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

MODULE OVERVIEW

Markov Processes (*Oliver Maclaren*) [6 lectures]

1. *Basic concepts* [2 lectures]

   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions. Diagrams of Markov chains.

2. *Properties of Markov chains* [2 lectures]

   Accessibile, recurrent, transient states. Communication of states. Stationary/invariant distributions and limiting/equilibrium distributions.

3. *Applications of Markov chains* [2 lectures]

   Modelling with Markov chains. Value calculations. Possible examples: random walks, branching processes, a hint of MCMC.

LECTURE 3

Properties of Markov chains - what do we want to know about a given Markov chain?

States

- Accessibile, recurrent, transient, absorbing states.
- Communication of states; classes, trapping sets

Distributions

- Stationary/invariant distributions.
- Limiting/equilibrium distributions.

WHAT DO WE WANT TO KNOW?

- What are the possible dynamics?
  - Which *states* can we get to?
  - Which *states* attract' or 'trap' the process etc?
  - Which *distributions* does the process tend to/get 'trapped' in?
- What does this look like *graphically*?
- What does this look like in terms of *matrix calculations*?

NOTE: TIPS FOR ANALYSIS

For *states* - i.e. accessible, trapping etc - we typically use *diagrams*

For *distributions* - i.e. invariant, limiting etc - we typically use *matrix calculations*

COMMUNICATION AND ACCESSIBILITIY OF STATES

We say that a state *i communicates with j* if $p_{ij}(n) > 0$ for some n, and write $i \rightarrow j$.

(Also: i *reaches* j, j is *accessible* from i)

If $i \rightarrow j$ and $j \rightarrow i$ then we say that *i and j communicate*, and write $i \leftrightarrow j$.

This is easiest to see in a *state transition diagram*. Example.

COMMUNICATION AND ACCESSIBILITIY OF STATES

We can divide all the states up into *mutually disjoint classes*, i.e. $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup \ldots$

- *Each pair* of states *in a given class communicate* with each other but
- *No state in a given class* will form a *communicating pair* with any state *outside* their class (there may be one-way communication however!)

Example.

TRAPPING SETS, ABSORBING STATES

A set of states is called *trapping (or closed)* if once you enter that set of states you never leave.

A trapping set with a *single member* is called an *absorbing state*.

Example.

RECURRENT/TRANSIENT STATES

A state i is *recurrent* if

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1$$

Otherwise (i.e. the above probability < 1) we call it *transient*

That is, we will *return to recurrent states sometime in the future with probability one*.

If the *probability of returning to a state is less than one, the state is called transient*.

RECURRENT STATES VS INVARIANT DISTRIBUTIONS

We won't look at recurrence/transient etc of individual states or classes in detail in this course (can get technical - but *note we will consider communication, trapping, absorbing states*).

Instead we will consider similar *'invariance'* ideas at the ('higher') *level of (probability) distributions* over states.

STATIONARY (INVARIANT) DISTRIBUTIONS.

If, *once we start from or reach a distribution π*, we continue to *have this distribution forever* we call it an *invariant or stationary distribution*.

Hence a stationary distribution defined simply by $\pi = \pi\mathbb{P}$

Example.

STATIONARY (INVARIANT) DISTRIBUTIONS.

Note:

- This is a *property of a distribution under single-step update*.
- *Depends on starting distribution* - 'once start from/reach' is key.
- Given a candidate, easy to *check by matrix multiplication*.
- *Find by solving an eigenvalue problem*: $\pi\mathbb{P} = \lambda\pi$ with $\lambda = 1$.

Example.

LIMITING/EQUILIBRIUM DISTRIBUTIONS

If the *n-step transition matrix has a limit*

$$\mathbb{P}^n \to \begin{bmatrix} -- & \pi & -- \\ -- & \pi & -- \\ -- & \pi & -- \end{bmatrix}$$

as $n \to \infty$, i.e. rows tend to the same vector* π, we call this limit the *limiting n-step transition matrix*.

We call the distribution π, that the matrix *maps all initial states towards*, the *limiting distribution*.

LIMITING/EQUILIBRIUM DISTRIBUTIONS

Note:

- This is a *property of the n-step transition matrix* for large n.
- This concerns *long-time (not single-step!)* behaviour.
- Idea is that we *eventually reach the same distribution independently of starting distribution*.

Example.

NOTE: STATIONARY/INVARIANT VS. LIMITING/EQUILIBRIUM

Again, these are *not the same thing*.

All limiting distributions are invariant distributions but not all invariant distributions are limiting.

So: Just because has an invariant distribution doesn't mean it converges to this when starting from 'outside' this distribution.

Example.

WHAT DO I NEED TO KNOW???

There are theorems/conditions for when chains converge to invariant distributions (i.e. when an invariant distribution exists, is unique and is equal to the limiting distribution).

We won't worry about these too much!

In *applications* we will typically be sloppy and *assume that a stationary distribution = the limiting distribution* without verifying this.

WHAT DO I NEED TO KNOW???

On the other hand, *if explicitly asked, you should know how to:*

For invariant distributions

- *Check* that a distribution is invariant (stationary) under single-step transition matrix
- *Solve* a simple (2-D) eigenvalue problem to obtain an invariant distribution (row eigenvector)

and...

WHAT DO I NEED TO KNOW???

For limiting distributions

- *Know* what the n-step transition matrix looks like when a limiting distribution exists
- *Show* that a limiting n-step transition matrix maps all initial states to the limiting distribution

MORE EXAMPLES

More examples.