MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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RECAP EXERCISE

Given the trigonometric (classical Fourier) series*

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

try to...

*: We will see that a *Generalised Fourier series* includes expansions in other orthogonal sets of functions besides trigonometric functions.

RECAP EXERCISE

...derive the expressions for the *Fourier coefficients* using orthogonality*

$$a_0 := \frac{1}{2l} \int_{-l}^{l} f(x)dx$$

$$a_n := \frac{1}{l} \int_{-l}^{l} f(x)cos(\frac{n\pi x}{l})dx$$

$$b_n := \frac{1}{l} \int_{-l}^{l} f(x)sin(\frac{n\pi x}{l})dx \; n = 1, 2, \ldots$$

*: Hint: you may also assume that sums and integrals can be exchanged.

LECTURE 6 GENERALISATIONS, SPECIAL CASES AND ALTERNATIVE FORMULATION

A note on Generalised Fourier series

Complex Fourier series

Even and odd extensions of non-periodic functions

A NOTE ON GENERALISED FOURIER SERIES

Our Fourier coefficient expressions are sometimes called the *Euler formulas** and are a special case of the general expression for the coefficients

$$c_n = \frac{1}{< f_n, f_n >} < f, f_n >$$

of a *Generalised Fourier series* expansion of a function f in terms of a *complete orthogonal system* of functions $\{f_n\}$:

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$*$: Note: Euler has a lot of things named after him!

COMPLETE?

Here *complete* means

$$\forall n(< f, f_n >= 0) \implies f = 0$$

which (roughly) means that our collection of functions is sufficiently large that *only the zero function is orthogonal to them all*.

This in turn implies we can make our 'error' for each finite approximation $\to 0$ and thus the *infinite series converges** .

* The convergence can be proved to be optimal in the least-squares sense. Taking an incomplete but 'good enough' set of orthogonal functions is the basis for a number of numerical/approximate methods (e.g. Galerkin/Finite Element Methods).

BACK TO BASICS

We now return to *classical* (trigonometric) Fourier series and different *representations* of these.

COMPLEX FOURIER SERIES

We can use Euler's (other!) formula to write our Fourier series in terms of *complex exponentials*

$$e^{iy} = cos(y) + isin(y)$$
$$e^{-iy} = cos(y) - isin(y)$$

The result is...

COMPLEX FOURIER SERIES

$$\mathrm{FS}\, f = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l}$$

for $n \in \mathbb{Z}$

Note that the c_n and exponentials in this expression are complex but *the series has a real-valued sum if f is real valued*

valued.

DERIVATION FROM REAL FORM

The formulae for the complex Fourier series are easy (enough) to derive from the expressions for the real Fourier series.

LET'S HAVE A GO!

EXAMPLE

Compute the complex Fourier series of the square-wave from last lecture

$$f(x) = \begin{cases} 2, & \text{on } (-\pi, 0] \\ 0, & \text{on } (0, \pi] \end{cases}$$

and

$$f(x + 2\pi) = f(x)$$

Verify that it gives the same answer! Remember yet another (another!) Euler formula (identity): $e^{i\pi} = -1$.

EVEN AND ODD EXTENSIONS

What if our function is only defined over $[0, l]$, say?

Is it even/odd/neither/can't say?

EVEN AND ODD EXTENSIONS

We need to *choose an appropriate even/odd periodic extension* in two steps

1. Extend our definition from $[0, l]$ to $[-l, l]$
2. Extend our definition from $[-l, l]$ to a periodic function over $\mathbb{R}$

Number 2 (the *periodic extension*) is done in the usual way: take $f(x+2l) = f(x)$ for all x.

EVEN AND ODD EXTENSIONS

The *odd extension* of f is defined by

$$f_{odd}(x) = \begin{cases} f(x), & x \in [0, l] \\ -f(-x), & x \in [-l, 0] \end{cases}$$

The *even extension* of f is defined by

$$f_{even}(x) = \begin{cases} f(x), & x \in [0, l] \\ f(-x), & x \in [-l, 0] \end{cases}$$

EXAMPLE

Sketch the *odd extension* and *even extension* of the function

$$f(x) = x, x \in [0, 2]$$

Then plot the *periodic extension* of each of these functions

FOURIER SERIES

We can calculate the Fourier series for the *extended* functions in the usual way.

Since the extensions are an 'artifice' to help construct a Fourier series, can we *re-write our extended Fourier series on our original domain using only our original function definition?*

LET'S TRY!

FOURIER SERIES FOR ODD EXTENSION

We get that the Fourier series of f_{odd}, also called the half-range sine (HRS) expansion of f, is

$$\mathrm{FS}\, f_{odd} = \mathrm{FS}_{HRS}\, f = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n = 1, 2, \ldots$$

Spot the differences!

FOURIER SERIES FOR EVEN EXTENSION

Similarly, the half-range cosine (HRS) expansion of f is

$$\mathrm{FS}\, f_{even} = \mathrm{FS}_{HRC}\, f = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where...

...

$$a_0 = \frac{1}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \ n = 1, 2, \ldots$$

Spot the differences!

HOW DO WE CHOOSE?

We will discuss this more in the next module…but also remember

GIBBS PHENOMENON

- Any Fourier series of a function with a *jump discontinuity* will have a persistent 9% (of the jump) *overshoot near the discontinuity* as $N \to \infty$.
- At *fixed* x the Fourier series will converge according to the convergence theorem as N increases, but the *overshoot persists and moves towards the discontinuity*.

and...

CONVERGENCE RATES OF COEFFICIENTS

...

- A piecewise continuous function has Fourier coefficients that decay as $1/n$.
- A continuous function with discontinuous first derivative has Fourier coefficients that decay as $1/n^2$.

In general: a continuous periodic function whose *first k derivatives are all continuous* but whose *$k + 1$ derivative is discontinuous* will have Fourier coefficients that decay at a rate of $1/n^{k+2}$.

CHOOSE SMOOTHNESS IF POSSIBLE!

HOMEWORK

Go over the various exercises from today

Revise separation of variables (lectures/tutorial)