MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

RECALL - STURM-LIOUVILLE PROBLEMS

The (regular) Sturm-Liouville problem can be written compactly in *operator notation* as

$$Ay := -\frac{1}{\omega(x)}\left[(p(x)y')' + q(x)y\right] = \lambda y$$

subject to

$$B_1 y(a) := \alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$B_2 y(b) := \beta_1 y(b) + \beta_2 y'(b) = 0$$

The combination $\{Ay, B_1 y(a), B_2 y(b)\}$ is sometimes (even more) compactly denoted by Ly, i.e. L *includes the BC*. The conditions are...

RECALL - STURM-LIOUVILLE PROBLEMS

- a and b are *finite*,
- $q, \omega\ p$ and p' are *continuous* functions on $x \in [a, b]$,
- $p(x) > 0$ and $\omega(x) > 0$ on $[a, b]$, i.e. are *positive*
- λ is a *constant* (and is a free parameter, i.e., not specified/is to be determined)
- α_1 and α_2 are *not both zero*, β_1 and β_2 are *not both zero* and
- $a, b, p(x), q(x), \omega(x), \alpha_1, \alpha_2, \beta_1, \beta_2$ are *all real*.

(we can also consider *singular* cases where these fail to hold)

RECALL - STURM-LIOUVILLE THEOREM

- The eigenvalues are all *real*.
- The eigenvalues are *simple*, i.e., to each eigenvalue there corresponds just one linearly independent eigenfunction.
- There are *infinitely many eigenvalues*, and they can be *ordered* so that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $\lambda_n \to \infty$ as $n \to \infty$.
- Eigenfunctions corresponding to different eigenvalues are *orthogonal*, i.e., if $\lambda_n \neq \lambda_n$ then $\langle \phi_n, \phi_m \rangle = 0$.

and...

STURM-LIOUVILLE THEOREM

... Let f be *piecewise smooth* on $[a, b]$. Then if $a_n = \langle f, \phi_n \rangle / \langle \phi_n, \phi_n \rangle$ the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

converges to $(f(x+) + f(x^-))/2$ at each point $x \in (a, b)$.

RECALL - THEOREM: NON-NEGATIVE EIGENVALUES?

If $q(x) \leq 0$ on $[a, b]$ and $[p(x)\phi_n(x)\phi_n'(x)]_a^b \leq 0$ for the eigenfunction $\phi_n(x)$, then λ_n is *non-negative*.

(We already know λ_n is real from the SL theorem).

LECTURE 9: STURM-LIOUVILLE EXAMPLES

Some (slightly more difficult) *examples*

Leftovers = homework

Questions so far?

EXAMPLE 1

Find the eigenvalues and eigenfunctions for the SLP

$$y'' + \lambda y = 0, \quad 0 < x < 1$$
$$y(0) = 0, \ \ y(1) + y'(1) = 0$$

Show how to work out the eigenfunction expansion for the function $f(x) = 50$ for $x \in [0, 1]$.

EXAMPLE 2

Find the eigenvalues and eigenfunctions for the SLP

$$y'' + \lambda y = 0, \quad -1 < x < 1$$

$$y'(-1) = 0, \; y'(1) = 0$$

and hence work out the eigenfunction expansion for the function

$$f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 50, & 0 < x \leq 1 \end{cases}$$

EXAMPLE 3

Find the eigenvalues and eigenfunctions for the SLP

$$y'' + \lambda y = 0, \quad 0 < x < 1$$
$$y(0) = 0, \; y(1) - y'(1) = 0$$

Show how to work out the eigenfunction expansion for the function $f(x) = 50$ for $x \in [0, 1]$.

EXAMPLE 4

Show that the DE

$$y'' + 5y' + \lambda y = 0, \quad y(0) = 0, \; y(1) = 0$$

is not of the usual SL form, then find an overall multiplying factor $\sigma(x)$ that will put the DE into SL form.

Hence expand the function $f(x) = -1$ for $x \in [0, 1]$ in eigenfunctions of the BVP.

NEXT WEEK

Sturm-Liouville theorem revisited