If $G_{\mu x}G_{xxx} < 0$, then there exist three stationary points near $x = 0$ if $\mu < 0$ (the outer pair are stable and the inner one is unstable if $G_{xxx} < 0$) and one stationary point near $x = 0$ if $\mu > 0$ (stable if $G_{xxx} < 0$). Stability properties are reversed if $G_{xxx} > 0$.

The bifurcation is said to be *supercritical* if the bifurcating pair of stationary points is stable, otherwise the bifurcation is *subcritical*. These various possibilities are sketched in Figure 8.8.

8.6 An example

In this section we shall go through the calculation of the centre manifold and determination of the type of bifurcation by projecting the flow (locally) onto the extended centre manifold in full detail. For our example we take the two-dimensional differential equation

$$\dot{x} = (1+\mu)x - 4y + x^2 - 2xy$$
$$\dot{y} = 2x - 4\mu y - y^2 - x^2$$

which has a stationary point at the origin for all real values of μ. The Jacobian matrix evaluated at $(0,0)$ is

$$\begin{pmatrix} (1+\mu) & -4 \\ 2 & -4\mu \end{pmatrix}$$

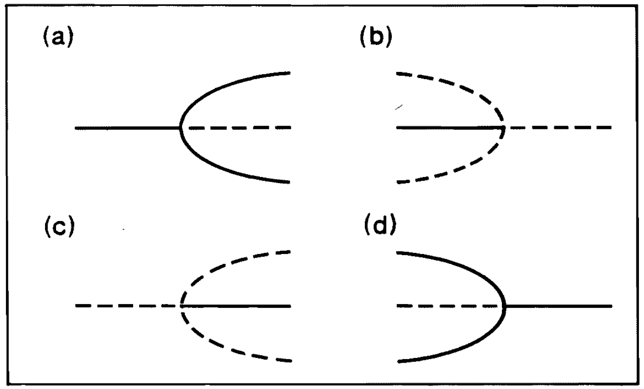

Fig. 8.8 Pitchfork bifurcations: (a) supercritical, $G_{\mu x} > 0$, $G_{xxx} < 0$; (b) subcritical, $G_{\mu x} > 0$, $G_{xxx} > 0$; (c) subcritical, $G_{\mu x} < 0$, $G_{xxx} > 0$; (d) supercritical, $G_{\mu x} < 0$, $G_{xxx} < 0$.

with characteristic equation

$$s^2 - (1-3\mu)s - 4\mu(1+\mu) + 8 = 0.$$

The origin is non-hyperbolic for values of μ at which $s = 0$ or $s = i\omega$ is a solution of the characteristic equation. Setting $s = 0$ we see that the origin has a simple zero eigenvalue if $-4\mu(1+\mu) + 8 = 0$, i.e. if $\mu = -2$ or $\mu = 1$. Similarly, setting $s = i\omega$ we find that the origin is non-hyperbolic if $\mu = \frac{1}{3}$ with a pair of purely imaginary eigenvalues. We will ignore this case until the next section and concentrate on the case $\mu = 1$: what sort of bifurcation occurs as μ passes through one? We begin the calculation by changing coordinates so that the linear part of the flow at the origin when $\mu = 1$ is in canonical form. Setting $\mu = 1$ in the Jacobian matrix we find that the linear part of the equation at the origin is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so the linear flow has eigenvalues 0 (as we knew already) and -2. The corresponding eigenvectors are $e_0 = (2,1)^T$ and $e_{-2} = (1,1)^T$. Now let P be the matrix whose columns are the eigenvectors,

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} P = P \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence if we set $\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$ we obtain the linear part of the equation for u and v as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Furthermore, $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and so

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u + v \\ u + v \end{pmatrix}.$$

Going back to the original equations and rewriting them in terms of the new variables u and v using these equations gives ($\dot{u} = \dot{x} - \dot{y}$, $\dot{v} = -\dot{x} + 2\dot{y}$)

$$\dot{u} = (-1+\mu)(6u + 5v) - 3u^2 - 4uv - v^2$$
$$\dot{v} = 10(1-\mu)u + (7 - 9\mu)v - 10u^2 - 10uv - 3v^2.$$