Engsci 741

Problem Set 1: Basic Background

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Problem 1.

Consider a mathematical model of the form $F(x) = y$ where F is the 'forward' mapping, $x \in \mathbb{R}^n$ is a vector of length n, and y is a vector of length m. Suppose that F is a *linear* forward model, i.e.

$$F(ax_1 + bx_2) = aF(x_1) + bF(x_2)$$

for arbitrary vectors x_1, x_2 and scalars a, b. Show by construction that $F(x)$ can hence be written in the form

$$F(x) = Ax$$

where A is an $m \times n$ matrix. Hint: consider the standard basis for $\mathbb{R}^n$ and consider an arbitrary x written in terms of this basis. Then consider the action of F on x. Finally, recall the definition of matrix-vector multiplication in terms of linear combinations of columns to help you construct A.

Problem 2.

Consider the polynomial regression model

$$y = a + bx + cx^2 + dx^3$$

for given data $(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)$. Show that this can be expressed as a *linear* forward model. What is the model linear with respect to?

Problem 3.

Suppose you are working on a problem involving an intergral equation and real world data with your supervisor 'Richard'. He says to you that there exists a unique solution to the integral equation and hence there is no need for 'ugly regularisation methods and all that'. You go away and invert your integral equation to find the desired solution using the measured data but you get a terrible looking solution. You check your solution method using a known solution and noise-free data and everything seems fine. Briefly explain (e.g. to Richard!) what is likely going wrong and why regularisation might be necessary. Mention Hadamard to impress Richard.

Problem 4.

Write down matrices to represent discrete versions of the following operations

- Cumulative integration (summation)
- Differentiation (finite differencing)
- Local (window) averaging

Problem 5.

Give a simple constructive example illustrating the instability of differentiation. Hint: see the end of lecture 1!

Problem 6.

Suppose you have a 'tall' linear system with more rows than columns and that the columns are linearly independent. Explain how generalised inverses enable you to 'solve' these equations. What characteristics does the corresponding solution possess? Which of Hadamard's conditions do generalised inverses help with? Which don't they help with? What is another name for the generalised inverse in this context (hint: left/right something..).

Problem 7.

Conversely, suppose you have a 'wide' linear system with more columns than rows and that the rows are linearly independent. Explain how generalised inverses enable you to 'solve' these equations. What characteristics does the corresponding solution possess? Which of Hadamard's conditions do generalised inverses help with? Which don't they help with? (hint: left/right something..).

Problem 8.

What is the key algebraic property that generalised inverses of all types satisfy. Name one other that pseudo-inverses satisfy.

Problem 9.

Starting from the least squares problem for an overdetermined linear system

$$\min_x ||y - Ax||^2$$

derive the normal equations

$$A^T Ax = A^T y$$

using matrix calculus. You may use the 'three key rules' from lecture 2 without proof.

Problem 10.

Suppose you have an underdetermined linear system. Define a constrained optimisation problem to 'solve' this problem. How does this relate to the generalised inverse? For extra credit, use the theory of Lagrange multipliers to derive an explicit solution.

Problem 11.

Suppose that you have a linear problem of the form

$$Ax = y.$$

Consider the under- and over-determined cases respectively. You may suppose that right and/or left) inverses R and L exist as needed and as appropriate. Describe the effect of the operators

$$RA$$

and

$$AL$$

What are these operators called? What are their domains and ranges (i.e. what spaces do they map between)? State and prove a key property of each that they share with the identity matrix.

Problem 12.

Define and contrast the optimisation problems a) satisfied by the generlised inverse and b) used to define regularised solutions.