MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

FIRST HALF (FIN)

First half (*Oliver Maclaren*) [15 lectures]

1. *Introduction and basic concepts* [3 lectures]

   PDEs, basic definitions. Modelling the diffusion (heat) equation and boundary conditions. Introduction to separation of variables.

2. *Expansions in orthogonal functions: Fourier series* [4 lectures]

   Orthogonality of functions/sets of functions and series expansions. Real trigonometric series. Convergence and sketching Fourier series. Complex Fourier series. Use in separation of variables.

3. *Sturm-Liouville eigenvalue problems* [4 lectures]

   Eigenvalue problems for function spaces: eigenvalues, eigenfunctions, Sturm-Liouville problems. Existence and orthogonality of solutions, eigenfunction expansions.

FIRST HALF (FIN)

4. *Separation of variables revisited, waves.* [~~3 lectures~~ 4 lectures]

   Derivation, separation of variables and travelling waves (D'Alembert's solution) for the wave equation. Derivation and seperation of variables for Laplace's equation. Separation of vars in several geometries.

LECTURE 15

Two-dimensional problems with 'circular symmetries'

- The Laplacian operator in polar coordinates
- Separation of variables for Laplace's equation in polar coordinates
- Separation of variables for the Wave equation in polar coordinates

COORDINATE INVARIANCE

Our *differential equation itself shouldn't depend on which coordinate system* (i.e. labelling method) we use as long as we know how to convert between them

- We are looking at the *same 'underlying objects'*, just giving them different labels

WHY DIFFERENT COORDINATES THEN?

On the other hand, *in some cases the natural geometry of a specific problem might mean some coordinate systems are better suited than others*. E.g.

- *Boundary conditions* specified along the surface of a sphere/edge of circle
- *Special symmetries/invariances*: maybe the heat distribution/wave spread is radially symmetric (derivatives vanish in certain directions)

Q: why exactly do these mean a different choice of coordinate system might help?

POLAR COORDINATES

Introduce r and θ using

$$x = rcos(\theta)$$
$$y = rsin(\theta)$$

Then...chain rule...(easy but messy)...

THE LAPLACIAN IN POLAR COORDINATES

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\boxed{= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Note: 'looks' (labelled) different but *represents the same object*

EXAMPLE:

Laplace's equation in a *two-dimensional disk* (we can consider this a limiting case of an *'annulus'*)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- In these problems θ ranges from $-\pi$ to π.
- $(r, -\pi)$ and (r, π) correspond to the same point in the plane.

So...

EXAMPLE

...we need to impose *periodic boundary conditions*

$$u(r,-\pi) = u(r,\pi), \qquad u_\theta(r,-\pi) = u_\theta(r,\pi).$$

- Alternatively we can allow θ to take any real value but require u to be a *periodic function of θ with period 2π*.

We also have

$$u(a,\theta) = f(\theta), \qquad u(r \to 0, \theta) \text{ is finite, or } \qquad u(b,\theta) = g(\theta)$$

EXAMPLE

Solution for disk

NOTE: EULER-CAUCHY EQUATION

The (second-order) *Euler-Cauchy equation* is an equation of the form

$$r^2 \frac{dR}{dr} + r\frac{dR}{dr} - \lambda R = 0$$

It has solutions of the form

$$R(r) = \begin{cases} Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}}, & \text{if } \lambda \neq 0 \\ C + Dln(r), & \text{if } \lambda = 0 \end{cases}$$

EXAMPLE

Outline: The *vibrating circular 'drumhead'* (see Tang 6.3.1!)

$$w_{tt} = c^2 \nabla^2 w \quad \text{for } r < a, t > 0$$
$$w(a, \theta, t) = 0, \quad w(r \to 0, \theta) \text{ is finite}$$
$$w(r, \theta, 0) = f(r, \theta), \quad w_t(r, \theta, 0) = g(r, \theta),$$

(also: w is periodic in θ with period 2π).

EXAMPLE

Solution sketch

(See 6.3.1 for full working)

BESSEL EQUATION AND BESSEL FUNCTIONS

The *Bessel functions* $J_n(z), Y_n(z)$ are *defined* via *Bessel's equation:*

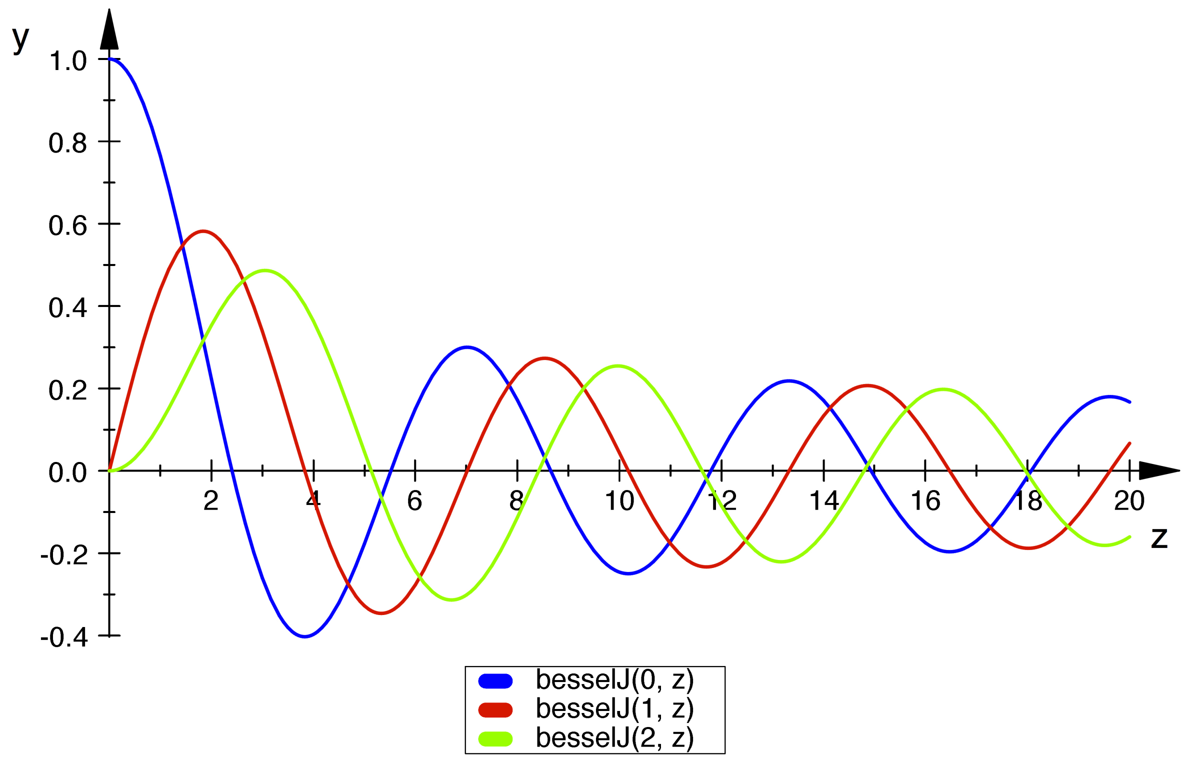
$$z^2 y''(z) + zy'(z) + (z^2 - n^2)y(z) = 0$$

They are *linearly independent* solutions of Bessel's equation. We can't express them in terms of simple functions - we have to compute them approximately.

$J_n(z)$ behaves nicely near $z = 0$ but $|Y_n(z)| \to \infty$ as $z \to 0$.

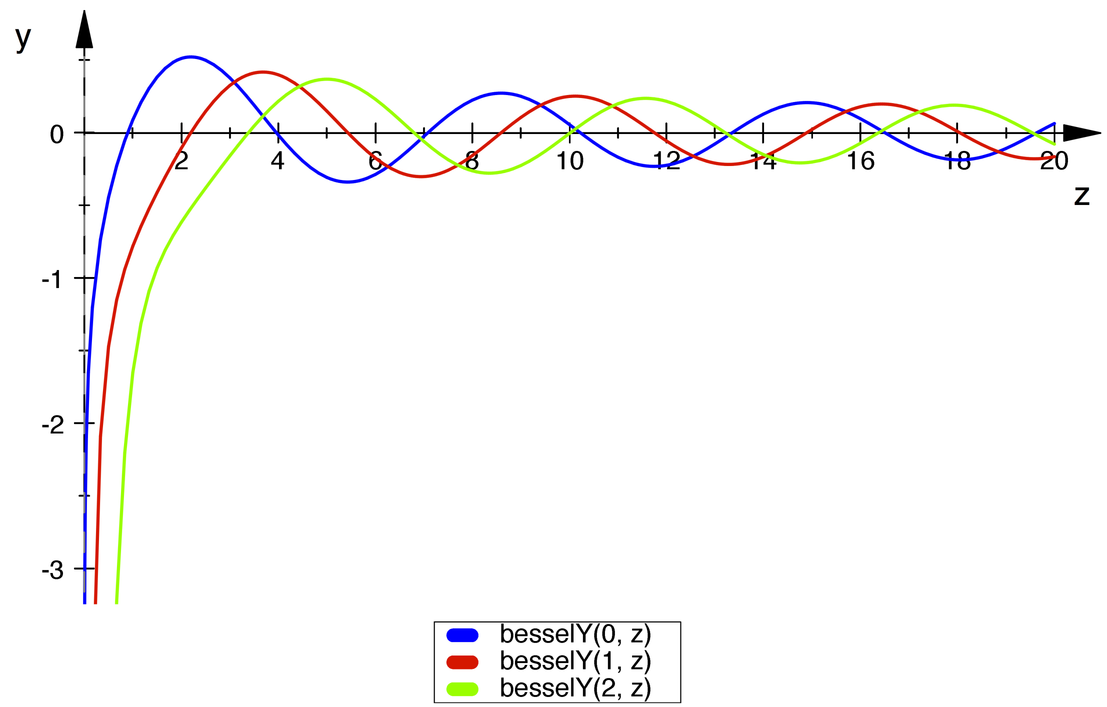
BESSEL FUNCTIONS: MUPAD

```
plot(besselJ(0,z),besselJ(1,z),besselJ(2,z), z=0..20, LegendVisible=TRUE)
```



BESSEL FUNCTIONS: MUPAD

```
plot(besselY(0,z),besselY(1,z),besselY(2,z), z=0..20, LegendVisible=TRUE)
```



ANNULUS OR DISK?

Note that the Y_n *diverge* as $z \to 0$.

- Exclude for disk
- Include for annulus!

BESSEL FUNCTIONS: ROOTS

Table 4.1. Zeros of the Bessel function

Number of zeros	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

HOMEWORK

- Go through the use of the chain rule for change of coordinates
- Tang Chapter 4 has lots of info on special functions (e.g. Bessel)
- Tang Chapter 6 has worked examples of separation of variables with non-cartesian coordinates
- Test! (Friday, in class). We'll do some examples and answer some questions on Monday and Tuesday.
- Assignment! (Wednesday)