The exponential of a matrix has very similar properties to the exponential of a real number. The following exercises establish some of these properties.

(3.2) EXERCISE

*(For the pure minded.) Given a matrix $A$ (i.e. a linear map of $\mathbf{R}^n$ to itself) define the norm of $A$, $||A||$ by*

$$||A|| = \sup_{v\in\mathbf{R}^n,\ v\neq 0} \frac{|Av|}{|v|}.$$

*Show that*

i) $0 \leq ||A|| < \infty$ *[Hint: note that $A(\lambda v) = \lambda Av$ for all $\lambda \in \mathbf{R}$ and so $|A(\lambda v)|/|\lambda v| = |Av|/|v|$, which implies that we can take $v$ to be on the unit ball in $\mathbf{R}^n$, which is compact.]*

ii)

$$||\lambda A|| = |\lambda|.||A||, \text{ for } \lambda \in \mathbf{R}$$

$$||A+B|| \leq ||A|| + ||B||$$

$$||AB|| \leq ||A||.||B||$$

$$||A^k|| \leq ||A||^k.$$

*Let $(A_k)$ be a set of linear maps of $\mathbf{R}^n$. Then a series $\sum_0^\infty A_k$ is absolutely convergent if $\sum_0^\infty ||A_k||$ is absolutely convergent. If $\sum A_k$ is absolutely convergent then the sum exists and addition, multiplication and differentiation can be done term by term. Show that $e^A$ is absolutely convergent and $e^{(s+t)A} = e^{sA}e^{tA}$ for all real numbers $s$ and $t$. If $A$ and $B$ commute (i.e. $AB = BA$) show that $e^{A+B} = e^Ae^B$. What is $e^{A+B}$ in terms of $e^A$ and $e^B$ if $A$ and $B$ do not commute? [Hint: define $C = AB - BA$.]*

With these basic properties of the exponential we can prove the existence and uniqueness of solutions to autonomous linear differential equations.

(3.3) THEOREM

*Let $A$ be an $n \times n$ matrix with constant coefficients and $x \in \mathbf{R}^n$. Then the unique solution $x(t)$ of $\dot{x} = Ax$ with $x(0) = x_0$ is $x(t) = e^{tA}x_0$.*



*Proof*: First note that

$$\frac{d}{dt}e^{tA} = \sum_{k=0}^{\infty} \frac{d}{dt}\left(\frac{t^kA^k}{k!}\right) = \sum_{k=1}^{\infty}\left(\frac{t^{k-1}A^k}{(k-1)!}\right) = Ae^{tA}$$

where we have used the result quoted in the second exercise above to differentiate the sum term by term. Hence

$$\frac{d}{dt}x(t) = \frac{d}{dt}e^{tA}x_0 = Ae^{tA}x_0 = Ax(t)$$

and so $e^{tA}x_0$ is a solution of the equation $\dot{x} = Ax$ with $x(0) = x_0$. To prove uniqueness, suppose that $y(t)$ is another solution to the differential equation with the same initial value, $y(0) = x_0$. Set $z(t) = e^{-tA}y(t)$, then

$$\dot{z} = -Ae^{-tA}y(t) + e^{-tA}\dot{y} = -Ae^{-tA}y(t) + Ae^{-tA}y(t) = 0$$

i.e. $z(t)$ is constant. But $z(0) = y(0) = x_0$ and so $z(t) = x_0$. Now, from the definition of $z(t)$, this implies that $y(t) = e^{tA}x_0 = x(t)$, and hence solutions are unique.

This result shows that we can find the solutions of linear differential equations by exponentiating matrices, but it does not tell us *how* to calculate with exponentials of matrices. If we want to be able to write down solutions explicitly we must learn how to take the exponential of a matrix. If $A$ is diagonal, $A = \text{diag}(a_1, \ldots, a_n)$, then it should be obvious that $e^{tA} = \text{diag}(e^{a_1t}, \ldots, e^{a_nt})$ and solutions in component form are $x_i(t) = e^{a_it}x_{0i}$, but for more general matrices the solutions are not so obvious. To deal with this problem we need a little more linear algebra and the idea of a Jordan normal form.

## 3.2 Normal forms

Consider a simple change of coordinates $x = Py$ where $P$ is an $n \times n$ invertible matrix ($\det P \neq 0$). Then $\dot{x} = Ax$ implies that

$$\begin{aligned}\dot{y} &= P^{-1}\dot{x} = P^{-1}Ax \\ &= P^{-1}APy = \Lambda y \end{aligned} \tag{3.8}$$

where $\Lambda = P^{-1}AP$ and the initial value $x(0) = x_0$ is transformed to $y(0) = P^{-1}x_0 = y_0$. In these new coordinates the solution is $y(t) = e^{t\Lambda}y_0$, and so transforming back to the $x$ coordinates

$$x(t) = Py(t) = Pe^{t\Lambda}y_0 = Pe^{t\Lambda}P^{-1}x_0. \tag{3.9}$$

Comparing this with the known solution $e^{tA}x_0$ we see that

$$e^{tA} = Pe^{t\Lambda}P^{-1}. \tag{3.10}$$

The strategy in this section is to choose $P$ in such a way that $\Lambda$ takes a particularly simple form which will allow us to calculate $e^{t\Lambda}$ and hence $e^{tA}$ with a minimum of fuss.

We have already seen that it is easy to exponentiate diagonal matrices. So suppose that $A$ has $n$ distinct real eigenvalues, $\lambda_1, \ldots, \lambda_n$, with associated eigenvectors $e_i$, so $Ae_i = \lambda_i e_i$, $1 \leq i \leq n$. Let $P = [e_1, \ldots, e_n]$, the matrix with the eigenvectors of $A$ as columns. Then, since the eigenvectors of distinct real eigenvalues are real and independent, det $P \neq 0$ and

$$\begin{aligned} AP &= [Ae_1, \ldots, Ae_n] = [\lambda_1 e_1, \ldots, \lambda_n e_n] \\ &= [e_1, \ldots, e_n]\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \end{aligned}$$

and so if $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ we have $AP = P\Lambda$ or $\Lambda = P^{-1}AP$. Hence, with this choice of $P$ we can bring the differential equation into the form

$$\dot{y} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)y, \tag{3.11}$$

which is, of course, particularly easy to solve. Note in particular that if $y_0 = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$, then $y(t) = (0, 0, \ldots, 0, e^{\lambda_i t}, 0, \ldots, 0)$. Thus each coordinate axis is invariant under the flow: solutions starting on a coordinate axis remain on that coordinate axis for all time. Translating back to the $x$ coordinates the coordinate axis corresponds to the eigenvector $e_i$, so each eigendirection is invariant.

*Example 3.1*

Consider the matrix

$$A = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalue equation is $(s+2)(s-2) = 0$ and so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Solving for the eigenvectors we find

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

and so

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}.$$



Now, for a general invertible $2 \times 2$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and so

$$P^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix}.$$

We leave it as an exercise to verify that $P^{-1}AP = \mathrm{diag}(-2, 2)$ and hence

$$e^{tA} = P\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{pmatrix}P^{-1} = \begin{pmatrix} e^{-2t} & (e^{2t} - e^{-2t})/4 \\ 0 & e^{2t} \end{pmatrix}.$$

Hence the solution to $\dot{x} = Ax$ with $x(0) = (a, b)^T$ is

$$\begin{pmatrix} ae^{-2t} + b(e^{2t} - e^{-2t})/4 \\ be^{2t} \end{pmatrix}.$$

Now suppose that $A$ is a $2 \times 2$ matrix with a pair of complex conjugate eigenvalues, $\rho \pm i\omega$. Then we claim that there is a real invertible matrix, $P$, such that

$$P^{-1}AP = \Lambda = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}. \tag{3.12}$$

This is a convenient form since $\Lambda = D + C$ where

$$D = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \text{ and so } e^{tD} = \begin{pmatrix} e^{\rho t} & 0 \\ 0 & e^{\rho t} \end{pmatrix}$$

and $C = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, so

$$C^{2n} = \begin{pmatrix} \omega^{2n} & 0 \\ 0 & \omega^{2n} \end{pmatrix} \text{ and } C^{2n+1} = \begin{pmatrix} 0 & -\omega^{2n+1} \\ \omega^{2n+1} & 0 \end{pmatrix},$$

which gives, using the series definitions of sine and cosine,

$$e^{tC} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \tag{3.13}$$

Since $D$ and $C$ commute, $e^{t\Lambda} = e^{tD}e^{tC}$, and so

$$e^{t\Lambda} = e^{\rho t}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \tag{3.14}$$

We now have to prove this claim, i.e. define a matrix $P$ such that $\Lambda = P^{-1}AP$ given a matrix $A$ with a complex conjugate pair of eigenvalues $\rho \pm i\omega$. As in the previous case we will define $P$ using the eigenvectors of $A$. Suppose that $A$ is a (real) $2 \times 2$ matrix with a pair of complex conjugate eigenvalues $\rho \pm i\omega$. Then there is a (complex) eigenvector $z = (z_1, z_2)^T$ such that $Az = (\rho + i\omega)z$. Consider the real matrix whose columns are made up of the imaginary and real parts of $z$; then, since $A$ is real,

$$\begin{aligned} A[\mathrm{Im}(z),\ \mathrm{Re}(z)] &= [\mathrm{Im}((\rho+i\omega)z),\ \mathrm{Re}((\rho+i\omega)z)] \\ &= [\rho\mathrm{Im}(z) + \omega\mathrm{Re}(z),\ \rho\mathrm{Re}(z) - \omega\mathrm{Im}(z)]. \end{aligned}$$

It is now easy to verify that this equals

$$[\mathrm{Im}(z),\ \mathrm{Re}(z)]\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \tag{3.15}$$

and hence $\Lambda = P^{-1}AP$ where $P = [\mathrm{Im}(z),\ \mathrm{Re}(z)]$.

*Example 3.2*

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}.$$

The eigenvalue equation is $s^2 - 2s + 2 = 0$ and so the eigenvalues are $1 \pm i$. An eigenvector corresponding to the eigenvalue $1 + i$ is $z = (1, -1+i)^T$ and so

$$P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Once again we leave it as an exercise to verify that $P^{-1}AP = \Lambda$ where

$$\Lambda = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad e^{t\Lambda} = e^t \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Hence $e^{tA} = Pe^{t\Lambda}P^{-1}$, i.e.

$$\begin{aligned} e^{tA} &= e^t \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{pmatrix}. \end{aligned}$$

Thus the solution of $\dot{x} = Ax$ with $x(0) = (a, b)^T$ is

$$\begin{pmatrix} ae^t(\cos t + \sin t) + be^t \sin t \\ -2ae^t \sin t + be^t(\cos t - \sin t) \end{pmatrix}.$$

This argument generalizes to higher dimension in the obvious way (described below), which enables us to use a change of coordinate to bring any differential equation $\dot{x} = Ax$, where $A$ has distinct eigenvalues into normal form $\dot{y} = \Lambda y$ where $\Lambda = \mathrm{diag}(B_1, \ldots, B_m)$. The blocks $(B_i)$ are given by $B_i = \lambda_i$ if the $i^{th}$ eigenvalue is real and the $2 \times 2$ matrix



$$B_i = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$$

if the corresponding eigenvalues are a complex conjugate pair, $\rho \pm i\omega$.

(3.4) THEOREM

*Let $A$ be a real $n \times n$ matrix with $k$ distinct real eigenvalues $\lambda_1, \ldots, \lambda_k$ and $m = \frac{1}{2}(n-k)$ pairs of distinct complex conjugate eigenvalues $\rho_1 \pm i\omega_1, \ldots, \rho_m \pm i\omega_m$. Then there exists an invertible matrix $P$ such that*

$$P^{-1}AP = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k, B_1, \ldots, B_m)$$

*where the $B_i$, $1 \le i \le m$, are $2 \times 2$ blocks, $B_i = \begin{pmatrix} \rho_i & -\omega_i \\ \omega_i & \rho_i \end{pmatrix}$. Furthermore, $e^{tA} = Pe^{t\Lambda}P^{-1}$ and*

$$e^{t\Lambda} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_k t}, e^{tB_1}, \ldots, e^{tB_m})$$

*where*

$$e^{tB_i} = e^{\rho_i t}\begin{pmatrix} \cos\omega_i t & -\sin\omega_i t \\ \sin\omega_i t & \cos\omega_i t \end{pmatrix}.$$

*Proof*: Let $(e_i)$, $1 \le i \le k$, be the real eigenvectors associated with the real eigenvalues $\lambda_i$ and $(z_j)$, $1 \le j \le m$, be the (complex) eigenvectors associated with the eigenvalues $\rho_j + i\omega_j$. Set $P$ to be the matrix whose first $k$ columns are the eigenvectors $e_1, \ldots, e_k$ and the remaining $n-k$ columns are the imaginary and real parts of the eigenvectors $z_j$, i.e.

$$P = [e_1,\ \ldots\ , e_k,\ \mathrm{Im}(z_1),\ \mathrm{Re}(z_1),\ \ldots\ , \mathrm{Im}(z_m),\ \mathrm{Re}(z_m)].$$

Since the eigenvalues are distinct the eigenvectors are independent and so $\det P \neq 0$ and by the arguments rehearsed above

$$AP = P\Lambda.$$

The rest of the theorem is just a restatement of results which have already been proved above.

If we are prepared to work with complex eigenvectors and matrices it is possible to diagonalize any matrix with distinct eigenvalues. To illustrate this suppose that $A$ is a matrix which satisfies the conditions of Theorem 3.4 above, and let $\rho_j \pm i\omega_j = \gamma_j$, $1 \le j \le m$. Then each complex eigenvalue $\gamma_j$ has a complex eigenvector $\epsilon_j$, and $\epsilon_j^*$ is an eigenvector of the complex conjugate eigenvalue $\gamma_j^*$. In this case the matrix of eigenvectors (with $e_i$ real and $\epsilon_j$ complex)

$$P = [e_1, \dots e_k, \epsilon_1, \epsilon_1^*, \dots, \epsilon_m, \epsilon_m^*]$$

diagonalizes $A$. So if $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_k, \gamma_1, \gamma_1^*, \dots, \gamma_m, \gamma_m^*)$ then

$$AP = P\Lambda. \tag{3.16}$$

In this case the differential equation $\dot{x} = \Lambda x$ must be interpreted with $x = (x_1, \dots, x_k, z_1, z_1^*, \dots, z_m, z_m^*)$ where the $x_i$ are real variables and the $z_i$ are complex variables. The differential equation for $z_j$ is then $\dot{z}_j = \gamma_j z_j$. We can regain the real form of this equation by setting $z_j = X + iY$ and $\gamma_j = \rho_j \pm i\omega_j$ in which case (equating real and imaginary parts)

$$\dot{X} = \rho_j X - \omega_j Y, \quad \dot{Y} = \omega_j X + \rho_j Y. \tag{3.17}$$

In some sections of this book it will be more convenient to work in complex notation.

Theorem 3.4 deals with the cases when $A$ has distinct roots, so we should now consider the possibility of multiple roots. Suppose that $A$ is a real $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_p$, $p \le n$. Then the characteristic polynomial of $A$ is

$$\prod_{k=1}^{p} (s - \lambda_k)^{n_k} \tag{3.18}$$

where $n_k \ge 1$ and $\sum_k n_k = n$. If $A$ has distinct eigenvalues then $p = n$ and $n_k = 1$, $1 \le k \le n$, but if $p < n$ then at least one of the $n_k$ must be greater than 1 and the characteristic polynomial has repeated roots. The number $n_k$ is called the multiplicity of the eigenvalue $\lambda_k$ and the generalized eigenspace of $\lambda_k$ is

$$E_k = \{x \in \mathbf{R}^n | (A - \lambda_k I)^{n_k} x = 0\}. \tag{3.19}$$

The dimension of $E_k$ is $n_k$ and so we can choose a set of basis vectors $(e_k^1, \dots, e_k^{n_k})$ of $E_k$ which we will refer to as generalized eigenvectors of $A$. To go through all the possible cases of repeated roots would take several chapters of linear algebra (see, for example, Hirsch and Smale (1976) or Arnold (1973)); here we shall stick to the possibilities which arise for differential equations in $\mathbf{R}^2$ and $\mathbf{R}^3$.



Consider first the case where $A$ is a $2 \times 2$ matrix with a repeated real eigenvalue $\lambda$, so the characteristic polynomial of $A$ is $(s - \lambda)^2 = 0$. Since $A$ satisfies its own characteristic equation this implies that

$$(A - \lambda I)^2 x = 0 \tag{3.20}$$

for all $x \in \mathbf{R}^2$. Now, either $(A - \lambda I)x = 0$ for all $x$ or there exists $e_2$ such that $(A - \lambda I)e_2 \ne 0$ and $e_2 \ne 0$. In the first case $A = \mathrm{diag}(\lambda, \lambda)$ for *any* choice of basis vectors $e_1$ and $e_2$, since $Ax = \lambda x$ for all $x \in \mathbf{R}^2$. In the second case define

$$e_1 = (A - \lambda I)e_2. \tag{3.21}$$

Then $(A - \lambda I)^2 e_2 = 0 = (A - \lambda I)e_1$ and so $Ae_1 = \lambda e_1$, whilst $Ae_2 = e_1 + \lambda e_2$ from the definition of $e_1$, (3.21). Hence

$$A[e_1, e_2] = [\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{3.22}$$

In other words, if $\lambda$ is a double eigenvalue of $A$ then there is a change of coordinates which brings $A$ into one of the two cases

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{3.23}$$

In both cases it is easy to solve the differential equation $\dot{x} = Ax$ in this choice of coordinate system and then translate back to find the solutions in the original coordinate system.

(3.5) EXERCISE

*Show that the solution to*

$$\dot{x} = \lambda x, \quad \dot{y} = \lambda y$$

*with* $(x(0), y(0)) = (x_0, y_0)$ *is*

$$x(t) = x_0 e^{\lambda t}, \quad y(t) = y_0 e^{\lambda t}$$

*and that the solution to*

$$\dot{x} = \lambda x + y, \quad \dot{y} = \lambda y$$

*subject to the same initial condition is*

$$x(t) = e^{\lambda t}(x_0 + y_0 t), \quad y(t) = y_0 e^{\lambda t}.$$

*Show from first principles that if* $\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ *then*

$$e^{t\Lambda} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

*Example 3.3*

Consider the matrix

$$A = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$$

with characteristic polynomial $s^2 - 4s + 4 = (s-2)^2 = 0$. So $\lambda = 2$ is a double eigenvalue. It is easy to see that $e_2 = (1, -1)^T$ does not satisfy the equation $(A - 2I)e_2 = 0$ and that $e_1 = (A - 2I)e_2 = (6, 6)^T$. Hence set

$$P = \begin{pmatrix} 6 & 1 \\ 6 & -1 \end{pmatrix} \quad \text{with} \quad P^{-1} = -\frac{1}{12}\begin{pmatrix} -1 & -1 \\ -6 & 6 \end{pmatrix}$$

so

$$P^{-1}AP = \Lambda = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad e^{t\Lambda} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

To find $e^{tA}$ note that

$$e^{tA} = Pe^{t\Lambda}P^{-1} = \begin{pmatrix} e^{2t}(1+3t) & -3te^{2t} \\ 3te^{2t} & e^{2t}(1-3t) \end{pmatrix}$$

which allows us to write down the solution to the differential equation $\dot{x} = Ax$ with initial condition $x(0) = (x_0, y_0)^T$ as

$$\begin{pmatrix} x_0e^{2t}(1+3t) - 3y_0te^{2t} \\ 3x_0te^{2t} + y_0e^{2t}(1-3t) \end{pmatrix}.$$

The case of three repeated real eigenvalues in $\mathbf{R}^3$ is similar, but there are three cases. First note that the characteristic polynomial is $(s-\lambda)^3 = 0$ and so

$$(A - \lambda I)^3 x = 0 \tag{3.24}$$

for all $x \in \mathbf{R}^n$. The three cases we need to consider are

i) there exists $e_3 \neq 0$ such that $(A - \lambda I)^2 e_3 \neq 0$;
ii) $(A - \lambda I)^2 x = 0$ for all $x$ but there exists $e_2 \neq 0$ such that $(A - \lambda I)e_2 \neq 0$; and
iii) $(A - \lambda I)x = 0$ for all $x$.



These three cases give rise (respectively) to the normal forms

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The last of these three possibilities is as straightforward as the diagonal case in $\mathbf{R}^2$ and so we will not dwell on it. In case (i) define

$$e_1 = (A - \lambda I)^2 e_3, \quad \text{and} \quad e_2 = (A - \lambda I)e_3.$$

Clearly $Ae_1 = \lambda e_1$ (as $(A - \lambda I)^3 x = 0$ for all $x$) and $Ae_3 = \lambda e_3 + e_2$ (from the definition of $e_2$). Furthermore, the definition of $e_1$ can be rewritten as $e_1 = (A - \lambda I)e_2$ and so $Ae_2 = \lambda e_2 + e_1$. Putting these three relationships together and forming the matrix $[e_1, e_2, e_3]$ as before gives

$$\begin{aligned} A[e_1, e_2, e_3] &= [\lambda e_1, e_1 + \lambda e_2, e_2 + \lambda e_3] \\ &= [e_1, e_2, e_3]\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \end{aligned}$$

and so the matrix $P = [e_1, e_2, e_3]$ gives the transformation for the matrix $A$ to have the required form.

The second case is the most difficult to establish. We have that $(A - \lambda I)^2 x = 0$ and there exists $e_2$ such that $(A - \lambda I)e_2 \neq 0$. As in the two-dimensional case define $e_1 = (A - \lambda I)e_2$, so $Ae_1 = \lambda e_1$ and $Ae_2 = \lambda e_2 + e_1$. We now claim that there is another vector, $e_3$, which is independent of $e_1$ and $e_2$ and which also satisfies $Ae_3 = \lambda e_3$ (we shall not prove this claim, but it is easy to verify in examples). The matrix $P = [e_1, e_2, e_3]$ then produces the desired form.

(3.6) EXERCISE

*Show that if*

$$\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

*then*

$$e^{t\Lambda} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

In the case of complex repeated eigenvectors similar manipulations yield normal forms

$$\begin{pmatrix} \rho & -\omega & 1 & 0 \\ \omega & \rho & 0 & 1 \\ 0 & 0 & \rho & -\omega \\ 0 & 0 & \omega & \rho \end{pmatrix},$$

or the standard block on the diagonal with the unit matrix on the off-diagonal. In general (for matrices in $\mathbf{R}^n$) there is always a coordinate transformation to bring into normal forms which are blocks of the kind found above.

## 3.3 Invariant manifolds

The manipulations of the previous section show that given the system $\dot{x} = Ax$ we can do a simple change of coordinates to bring the equation into the normal form $\dot{y} = \Lambda y$. In the simplest case, where $A$ has distinct eigenvalues, the matrix $\Lambda$ is

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k, B_1, \ldots, B_m) \tag{3.25}$$

where $(\lambda_i)$ are the real eigenvalues and $B_j$ are the matrices

$$\begin{pmatrix} \rho_j & -\omega_j \\ \omega_j & \rho_j \end{pmatrix}$$

associated with the complex conjugate pairs of eigenvalues $\rho_j \pm i\omega_j$. In component form, with $y = (y_1, \ldots, y_k, w_1, \ldots w_{2m})$ the equation $\dot{y} = \Lambda y$ is therefore

$$\dot{y}_i = \lambda y_i, \quad 1 \leq i \leq k \tag{3.26}$$

and

$$\begin{pmatrix} \dot{w}_{2j-1} \\ \dot{w}_{2j} \end{pmatrix} = \begin{pmatrix} \rho_j & -\omega_j \\ \omega_j & \rho_j \end{pmatrix} \begin{pmatrix} w_{2j-1} \\ w_{2j} \end{pmatrix}, \quad 1 \leq j \leq m, \tag{3.27}$$

and since they are uncoupled each of these $k+m$ equations can be solved separately. It follows immediately that the real eigenspaces of $\Lambda$ are invariant, since if $y_0 = (0, \ldots, Y_0, \ldots, 0)$ then $y(t) = (0, \ldots, Y_0 e^{\lambda_i t}, \ldots, 0)$, and similarly the two-dimensional eigenspaces corresponding to the complex conjugate pair of eigenvalues $\rho_j \pm i\omega_j$ are also invariant. Returning to the original equation $\dot{x} = Ax$ this implies that the corresponding eigenspaces of $A$ are invariant. Since all the eigenvalues of $A$ are distinct, these eigenspaces are either one-dimensional or two-dimensional



depending on whether the corresponding eigenvalue is real or not. This proves the following theorem.

(3.7) THEOREM

*If the eigenvalues of the $n \times n$ real matrix $A$ are distinct then $\mathbf{R}^n$ decomposes into a direct sum of one-dimensional spaces and two-dimensional spaces. Each of these eigenspaces is invariant under the flow defined by $\dot{x} = Ax$.*

If the eigenvalues of $A$ are not distinct then the normal form of $A$ will contain blocks of matrices with 1s down the off-diagonal as described in the previous section. The corresponding generalized eigenspaces are, of course, also invariant but they may be of dimension greater than two.

*Example 3.4*

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

$A$ has 3 blocks: a one-dimensional eigenspace corresponding to the eigenvalue 2, a two-dimensional eigenspace corresponding to the multiple eigenvalue $-3$ and a two-dimensional eigenspace corresponding to the complex conjugate pair of eigenvalues $-2 \pm i$. To solve the differential equation $\dot{x} = Ax$ with $x(0) = (y_1, y_2, y_3, y_4, y_5)$ we can work independently in these three eigenspaces. In the first,

$$\dot{x}_1 = 2x_1, \quad x_1(0) = y_1$$

and so

$$x_1(t) = x_0 e^{2t}.$$

In the degenerate eigenspace

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}$$

and so

$$\begin{pmatrix} x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} y_2 e^{-3t} + y_3 t e^{-3t} \\ y_3 e^{-3t} \end{pmatrix}.$$