ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 7

- Introduction to bifurcation theory

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

We now return to systems of the form

$$\dot{x} = f(x; \mu)$$

where $x \in \mathbb{R}^n$ is the usual vector of state variables but we have explicitly included $\mu \in \mathbb{R}^m$, a vector of *problem parameters*.

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

Problem parameters define our model *structure*.

They can also be thought of as extra *(very) 'slowly-varying' state variables* summarising neglected processes or external/enviroment conditions (we'll come back to this in centre manifold theory).

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

Bifurcation theory is about what happens *when our choice of parameter value (model structure) matters crucially*.

In these cases we say our model has *structural* (c.f. solution) instabilities - and these instabilities are called *bifurcations*.

BIFURCATION DIAGRAMS

A *bifurcation diagram* is sort of like a phase portrait but with a *parameter* ('very slow state variable') on one of the axes.

Since the parameter is taken as 'frozen' at each value in turn, we summarise the properties of the main system in terms of some (typically) *long-term/asymptotic property* (or properties) of interest e.g. the *locations of the equilibria*.

BIFURCATION DIAGRAMS

i.e.

A bifurcation diagram shows how system properties of interest like equilibria *depend on variations in a system parameter* (or parameters).

Simple example.

RECALL: BIFURCATIONS AND STRUCTURAL INSTABILITY

When do these occur?

Hyperbolic fixed points are *robust* to parameter perturbations - they are *structurally stable* features.

Non-hyperbolic fixed points, on the other hand, are the *sensitive* cases. They are *structurally unstable* features.

Here *bifurcations* (changes in stability or number of solutions/periodic orbits) are *possible*. We hence analyse the neighbourhood of these cases.

DIMENSION?

Even in large systems we *often only have bifurcations occuring at a small number of parameter values at a time* - e.g. only one eigenvalue crossing the imaginary axis.

Relatedly, we typically find *a small number of slow modes* determine the main 'emergent' observable dynamics.

Centre manifold theory (which we will come back to) provides a way of first reducing to the lower-dimensional system and then of analysing the *dynamics* near any bifurcations in these systems.

ONE DIMENSIONAL, ONE PARAMETER SYSTEMS; LOCAL VS GLOBAL

Because of this *we will focus on one-state variable, one-parameter systems*, i.e. $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$, which are frequently encountered during bifurcations in larger systems.

Note: we can analyse changes (bifurcations) in either or both *local* and *global* qualitative features of the phase portrait.

We will mainly focus on local bifurcations.

KEY CASES

- Saddle-node/turning point/fold bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Hopf bifurcation (see later)

EXERCISES

See examples handout (will cover in tutorial - but best to try yourself).

Good exam practice!

EXTRA READING: THE CURSE OF INSTABILITY VS THE CURSE OF DIMENSIONALITY?

See Kuehn 'The curse of instability' (http://arxiv.org/abs/1505.04334)

> Curse? (Structural) *instabilities* cause an *increase in dimensionality*, substantially raise the analytical difficulty and are a strong indicator for multiscale dynamical complexity.

i.e. we often have to keep adding parts to our model until it becomes robust.

EXTRA READING: THE CURSE OF INSTABILITY VS THE CURSE OF DIMENSIONALITY?

Cure? *Separate* your system/data/model into a (hopefully) small(-er) set of *regimes divided by instabilities* in the underlying process. Then 'glue' these together to give a map of possible behaviour.

So-called “universal” or “generic” dynamical principles are based upon the absence or presence of certain instability classes.