Engsci 711

Tutorial 4: Centre manifold theory

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Overview

The purpose of this tutorial is to get some experience with carrying out centre manifold analysis. The most relevant parts are the basic centre manifold reductions (no parameters etc).

I have, however, put a few more complicated examples in involving extended centre manifolds (i.e. involving parameters). This is for interest/completeness rather than being examinable…

Basic centre manifold reduction problems

Problem 1

Consider the worked example from Lecture 9 (Kuznetsov, 2004, Example 5.1)

$$\begin{aligned}\dot{x} &= xy + x^3 \\ \dot{y} &= -y - 2x^2\end{aligned}$$

- Confirm the origin $(0, 0)$ is an equilibrium point
- Find the Jacobian derivative Df and evaluate it at the origin
- Find the eigenvalues and associated linear subspaces E^s, E^u, E^c.
- Is the system expressed relative to the eigenbasis? That is, are the eigen-directions parallel to the x and y axis?
- If they are, proceed. If they aren't, define a linear transformation converting x, y to new variables u, v such that u, v are coordinates in the eigenbasis (hint: here we are fine; for other cases, you might need to change coordinates)).
- Assume that (relative to the eigendirections) the centre manifold W^c_{loc} can be expressed as a curve expressing the non-centre coordinate(s) in terms of the centre coordinate(s), $y = V(x)$.
- Assume this has a Taylor series expansion; what can you say about the zeroth and first-order terms?

- Use the same process as outlined in the previous tutorial and the lectures to determine the terms of the Taylor series for the centre manifold.

- Use this expression for W^c_{loc} and the Reduction Principle from the lectures to determine the (approximate) dynamics on the centre manifold (i.e. substitute your expression for $y(x)$ into the $\dot{x}$ equation!).

- Are these dynamics stable or unstable? Are the dynamics of the full system stable or unstable? Sketch the local dynamics near the origin.

- What are the relative rates of the dynamics on the centre and the stable/unstable manifolds near the origin (look at the equations you have!).

- Instead of substituting W^c_{loc} into the $\dot{x}$ equation, try substituting the equation defining E^c into the $\dot{x}$ equation. What do you notice?

Problem 2

Carry out the same process for the example from Lecture 10

$$\begin{aligned}\dot{x} &= y - x - x^2\\ \dot{y} &= x - y - y^2\end{aligned}$$

Note that for this problem it is (probably) easiest to transform variables to be relative to the eigendirections (after you find them), though it is technically possible to do the analysis without transforming.

Problem 3

Carry out the same process for

$$\begin{aligned}\dot{x} &= -2x + y - x^2\\ \dot{y} &= x(y - x)\end{aligned}$$

Problem 4

Carry out the same process for

$$\begin{aligned}\dot{x} &= y - 3x^2 + xy\\ \dot{y} &= -3y + y^2 + x^2\end{aligned}$$

Extended centre manifold reduction problems

Extended lecture manifold reduction - guided example (see also Lecture 10!)

The purpose of this question is to understand how we can get to bifurcation theory via centre manifold theory using the idea of an 'extended' centre manifold. This is the same example as 'covered' in Lecture 10, repeated here for convenience.

Really, everything works just the same as in normal centre manifold theory, *we just 'upgrade' the parameter to a (still trivial!) state variable.*

Don't worry, I'll guide you through!

Consider the system of equations

$$\begin{aligned}\dot{x} &= y - x - x^2, \\ \dot{y} &= \mu x - y - y^2\end{aligned}$$

- Find a value $\mu = \mu_c$, for which the origin $(0, 0)$ is non-hyperbolic.
- Linearise the system about the origin $(0, 0)$ and with μ fixed at μ_c.
- Determine a linear change of coordinates $(x, y) \to (u, v)$ that puts the linearised system into diagonal form.
- Define a new parameter $\lambda = \mu - \mu_c$ which is zero at the non-hyperbolic point. Write the *full, nonlinear* system in terms of your u, v above and your new parameter λ.

Now comes the key - yet simple - step.

- 'Upgrade' the parameter λ to a state variable. This means we take the u, v equations and add the trivial equation $\dot{\lambda} = 0$.

Note that, since we are in diagonal form this corresponds to adding another *centre* variable (eigenvalue = 0). In this case it is 'super slow' since *both* linear and nonlinear parts are zero (the other centre variable will have 'zero-eigenvalue' linear dynamics but non-trivial higher-order dynamics, so can be thought of as 'slowly varying').

You should now have a system of the form

$$\begin{aligned}\dot{u} &= ... \\ \dot{v} &= ... \\ \dot{\lambda} &= 0\end{aligned}$$

Note that, while for the λ-as-parameter system a term like λu is linear, when λ is considered as a state variable a term like this is considered *nonlinear*!

We should now have a diagonalised system where the (extended) centre manifold component is two-dimensional (check you understand why). Suppose u and λ are your centre manifold variables. Our centre manifold will be tangent to the (λ, u) plane at $(u, v, \lambda) = (0, 0, 0)$.

We can now proceed as normal in centre manifold theory

- Assume that the two-dimensional centre manifold is described by a restriction of three-dimensional (u, v, λ) space by one constraint $v = h(\lambda, u)$, and that h can be approximated using a two-variable Taylor series expansion. This takes the form

$$v = a + b\lambda + cu + d\lambda^2 + e\lambda u + fu^2$$

where a, b, c, d, e, f are constants.

- What are a, b, c? You should be able to write these down instantly.
- Now use the usual procedure for finding the other coefficients. That is, use

$$\dot{v} = \frac{\partial h}{\partial \lambda}\dot{\lambda} + \frac{\partial h}{\partial u}\dot{u}$$

and substitute in what you know about $\dot{v}, \dot{\lambda}, \dot{u}$.

- Equate coefficients to determine d, e, f.
- Now, use the Reduction Principle to determine the dynamics on the extended (u, λ) centre manifold. That is, substitute your expression into the u equation (the λ equation remains trivial). Your answer should consist of writing down two differential equations.

Note: we have effectively obtained a one-dimensional bifurcation problem (as expected)! To see, note that since the λ dynamics are trivial, we can effectively downgrade λ back to a control parameter. That is, we fix it to different values and solve the u equation for each of these.

This can be considered as either a u vs λ phase-portrait OR a u vs λ bifurcation diagram (particularly when we just plot the equilibria of u). The point of 'upgrading' it temporarily was to derive the bifurcation diagram from the centre manifold phase portrait. Interestingly, reducing it back to a parameter can be thought of as an *additional* centre manifold reduction with λ as the (super) slow variable.

So we can now carry out the last step.

- Draw a bifurcation plot/(u, λ) phase portrait. What sort of bifurcation is this?

Extended lecture manifold reduction - training wheels come off!

Carry out the above analysis for (Glendinning Example 8.6)

$$\begin{aligned}\dot{x} &= (1 + \mu)x - 4y + x^2 - 2xy,\\ \dot{y} &= 2x - 4\mu y - y^2 - x^2\end{aligned}$$