ENGSCI 213: MATHEMATICAL MODELLING 2SE

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MODULE OVERVIEW

Markov Processes (*Oliver Maclaren*) [6 lectures]

1. *Basic concepts* [2 lectures]

   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions. Diagrams of Markov chains.

2. *Properties of Markov chains* [2 lectures]

   Accessibile, recurrent, transient states. Communication of states. Stationary/invariant distributions and limiting/equilibrium distributions.

3. *Applications of Markov chains* [2 lectures]

   Modelling with Markov chains. Value calculations. Possible examples: random walks, branching processes, a hint of MCMC.

LECTURE 2

More on transition probabilities and matrices:

- n-step transitions probabilities and matrices.
- Chapman-Kolmogorov equation.

as well as:

- Initial and marginal distributions.
- Diagrams of Markov chains.

RECALL: TRANSITION PROBABILITIES AND TRANSITION MATRIX

For homogeneous Markov chains the *transition probabilities* are defined by

$$p_{ij} := P(X_{n+1} = j \mid X_n = i)$$

The matrix $\mathbb{P}$ with (i, j)th element $\mathbb{P}_{ij}$ equal to p_{ij} is called the *transition matrix*.

Note: i is 'from' and j is 'to'.

RECALL: EXAMPLE

Random walk

n-STEP TRANSITIONS

Suppose we are currently at state i at stage m, i.e. $X_m = i$.

We want to know the probabilities of being in each of the possible states j *after taking n more steps* ($n = 1$ gives our standard case).

These are called (unsurprisingly) *$n - step$ transition probabilities* and we denote them by

$$p_{ij}(n) = P(X_{m+n} = j \mid X_m = i)$$

NOTE: HOMOGENEITY

Note again that we will assume in general that we are dealing with *homogeneous* MCs so that *we don't need to remember which 'stage' we were at (just which state)*, and so

$$p_{ij}(n) = P(X_{m+n} = j \mid X_m = i) = P(X_n = j \mid X_0 = i)$$

etc. Hence why $p_{ij}(n)$ only depends on n and not m in our notation.

$\mathbf{n}$-STEP TRANSITION MATRICES

Just as for the single-step case we can arrange these in a matrix, the *n-step transition matrix*, here denoted by

$$\mathbb{P}_n$$

with the (i, j)th element of $\mathbb{P}_n$ equal to $p_{ij}(n)$.

Note again that i is 'from' and j is 'to'.

CHAPMAN-KOLMOGOROV EQUATIONS

The n-step transition probabilities satisfy

$$p_{ij}(m+n) = \sum_k p_{ik}(m) p_{kj}(n)$$

In terms of the transition matrices, *this is just matrix multiplication*:

$$\mathbb{P}_{m+n} = \mathbb{P}_m \mathbb{P}_n$$

INTUITIVE IDEA AND CONSEQUENCE

Intuitively, this means that to find the probability of getting from state i to state j after $n + m$ steps total steps we can *multiply the probabilities along all intermediate paths* having state k lying m steps after i and n steps before j.

Note also $\mathbb{P}_1 = \mathbb{P}, \mathbb{P}_2 = \mathbb{P}_{1+1} = \mathbb{P}\mathbb{P} = \mathbb{P}^2$ etc. So

$$\mathbb{P}_n = \mathbb{P}^n$$

i.e. the n-step transition matrix is obtained by *multiplying the single-step transition matrix by itself n times*.

EXAMPLE

Let's calculate an n-step transition matrix!

PROOF

Proof of the Chapman-Kolmogorov equations.

Let's prove the Chapman-Kolmogorov equations hold for Markov processes!

INITIAL DISTRIBUTIONS

Typically we want to *start from some initial state* and then 'evolve' our process forward.

Rather than starting from a definite state, however, suppose *we only know which state we start from with some probability* - i.e. *we start from an initial distribution over the states*.

We write this initial distribution as a (row) vector μ_0 with components

$$\mu_0(i) = P(X_0 = i)$$

INITIAL DISTRIBUTIONS

You might also think of this as telling you to *'randomly draw' a definite state* from this distribution, evolve the process forward from this state, *then repeat this by drawing a new state*, evolving etc.

Rather than having to individually draw starting states, *however, we can 'evolve' the whole initial distribution forward at the same time* (or at least write down the equations for this!)

'EVOVLING' AN INITIAL DISTRIBUTION TO A NEW MARGINAL DISTRIBUTION

First, note that *after 'evolving' (running) our process forward n stages we get a new probability distribution* over states i at stage n.

We call this the *marginal distribution* μ, which is a (row) vector with components

$$\mu_n(i) = P(X_n = i)$$

'EVOVLING' AN INITIAL DISTRIBUTION TO A NEW MARGINAL DISTRIBUTION

The *'evolution equation' connecting the marginal and initial distributions* is simply

$$\mu_n = \mu_0 \mathbb{P}^n$$

Proof and example.

DIAGRAMS OF MARKOV CHAINS

We can represent Markov chains *graphically in terms of state transition diagrams*.

We represent the *states by nodes* and the *transitions by directed edges*, along with the associated *transition probabilities* labelling the edges.

Example.