ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams. Geometry of bifurcations - invariant manifolds.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

LECTURE 7

More examples of bifurcations:

- Hopf bifurcation
- Two parameter bifurcations

BIFURCATIONS - 'CO-DIMENSION'

For our purposes, we define the *co-dimension* of a bifurcation type as the *minimum number of parameters* we need to vary to get this type of bifurcation.

Note: our original system may be higher-dimensional, but

1. the bifurcation typically occurs in lower dimensions and
2. is determined by a small number of parameters (low co-dimension) - e.g. one eigenvalue crosses the imaginary axis (the real part changes sign, hence stability).

RECALL: BIFURCATIONS

We have seen three *co-dimension one* bifurcations:

- saddle-node/turning point/fold bifurcation
- transcritical bifurcation
- pitchfork bifurcation

There is one more - which is *also co-dimension one* but is slightly more complicated - the *Hopf bifurcation*.

These four give all the possible co-dimension one bifurcations.

HOPF BIFURCATION

The Hopf bifurcation occurs when a *pair of complex conjugate eigenvalues cross the imaginary axis together*.

In contrast to before, we now have a non-zero imaginary component and hence have to deal with oscillatory components.

A Hopf bifurcation (for our purposes) is characterised by *a change in stability of a fixed point, along with the appearance or the disappearance of a periodic orbit at this fixed point*.

HOPF BIFURCATION THEOREM (OR NOT)

There is a *Hopf bifurcation theorem* (See e.g. Glendinning) giving *conditions under which periodic solutions are created/destroyed* as a pair of complex eigenvalues pass through the imaginary axis (and the associated fixed point changes stability)

Unfortunately it is a bit tricky/ugly to verify the conditions for creation/destruction.

HOPF BIFURCATION THEOREM (OR NOT)

Instead we typically a) *find where a pair of complex eigenvalues become purely imaginary* and b) *directly verify that a periodic solution was created/destroyed* (as we pass through the bifurcation) via simulation (or analytical solution in simple cases).

HOPF BIFURCATION - CO-DIMENSION AGAIN

The Hopf bifurcation occurs in two-dimensional systems (or on a two-dimensional reduced/centre manifold of a larger system) BUT

The Hopf bifurcation essentially only *depends on varying one parameter, hence the co-dimension is one*.

HOPF BIFURCATION - CANONICAL EXAMPLE

$$\dot{x} = -\omega y + x(\mu - (x^2 + y^2))$$
$$\dot{y} = \omega x + y(\mu - (x^2 + y^2))$$

HOPF BIFURCATION ANALYSIS

Steps

- Verify we have a pair of complex conjugate eigenvalues crossing the imaginary axis and an associated change in stability of the fixed point.
- Verify (in this example by direct solution, in general via numerical methods) that a periodic orbit exists on one side of the bifurcation.

Note: If possible, then a direct solution is typically easiest to construct/verify in polar coordinates.

HOPF BIFURCATION - CROSSING SPEED

As seen, *a necessary condition for a Hopf bifurcation is finding a pair of complex conjugate eigenvalues crossing the imaginary axis*.

We can also sometimes *verify that this crossing occurs with a non-zero speed* (another necessary condition)

So, we have eigenvalues $\lambda = \pm i\omega$ at critical parameter value $\mu = \mu_c$.

and (for a non-degenerate bifurcation) $\frac{\partial \lambda^r}{\partial \mu} \neq 0$ at $\mu = \mu_c$.

where λ^r is the real part of the eigenvalue.

HOPF BIFURCATION KEY FEATURES

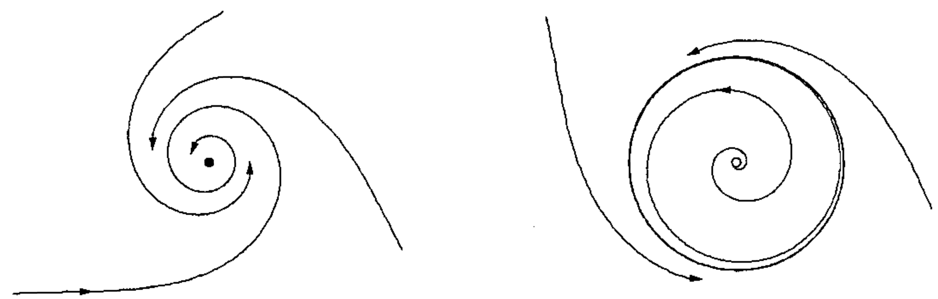
- We call the bifurcation *supercritical* if the emerging/disappearing periodic orbit is stable.
- If it is unstable, we call it *subcritical**
- The *radius* of the limit cycles grow/shrink continuously from/to zero and proportional to $\sqrt{\mu - \mu_c}$ near μ_c
- The *frequency* of the limit cycle is approximately Im λ, evaluated at $\mu = \mu_c$

* Similar terminology is used for other bifurcations (e.g. supercritical pitchfork - stable FP are born).

HOPF BIFURCATION KEY FEATURES

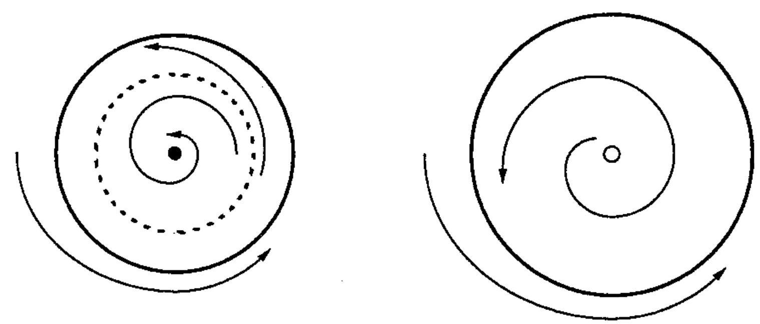
- The periodic orbit and the fixed point have *opposite stability for the parameter values that they both exist*
- I.e. *supercritical*: stable PO, unstable FP; *subcritical*: unstable PO, stable FP
- Note that it is also *difficult to manually check the theorem conditions* for whether the Hopf bifurcation is 'supercritical' or 'subcritical' (or degenerate).
- Again, it is easier to do it by simulation, direct construction, direct checking etc on a *case-by-case basis*.

HOPF BIFURCATION PICTURES



$\mu < 0$ $\mu > 0$

Supercritical



$\mu < 0$ $\mu > 0$

Subcritical

TWO PARAMETERS

What happens when we have bifurcations *depending on more than one parameter?*

Let's consider a system depending on two parameters and see if we can find a *co-dimension two bifurcation!*

See Strogatz (1994) Section 3.6.

TWO PARAMETERS - IMPERFECTIONS

We've noted that varying parameters can be a good way to see how 'structurally' stable a given system is, i.e. 'perturb' the model structure or the 'external environment' etc.

We can then e.g. plot a bifurcation diagram and see how our model results depend on these 'external' assumptions.

But - what if we perturb this again?! Is our bifurcation diagram itself structurally stable??

TWO PARAMETERS - IMPERFECTIONS

We can set up a model with e.g. one *external environment* parameter (controllable) and one *imperfection parameter* (not-controllable).

We will look at such models in more details in a tutorial/assignment (probably) but, for now, the point is that we are interested in a two-parameter (co-dimension two) bifurcation problem e.g.

$$\dot{x} = \lambda + \mu x - x^3$$

where λ is an imperfection parameter ($\lambda = 0$ gives us a 'symmetric' pitchfork bifurcation).

DIAGRAM TYPES

To analyse this system we can do either or both of

- Fix a value of one of the parameters, plot the bifurcation diagram in the other parameter; choose a new value of the fixed parameter, repeat.
- Plot a diagram purely in parameter space (e.g. two parameters gives plane) showing curves of parameters along which bifurcations occur and indicate typical properties of the phase-space for that parameter combination.

See Strogatz (1994) 3.6 and tutorial.