ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations for parameter dependent systems. Bifurcation diagrams.

LECTURE 9: BIFURCATION THEORY CONTINUED

Basic *one-parameter local bifurcations* involving *real-valued* eigenvalues:

- Saddle-node/turning point/fold bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation

More complicated cases:

- Multiple one-parameter local bifurcations in one system

RECALL: PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

As mentioned, we now consider systems of the form

$$\dot{x} = f(x; \mu)$$

where $x \in \mathbb{R}^n$ is the usual vector of state variables but we have explicitly included $\mu \in \mathbb{R}^m$, a vector of *problem parameters*. Generally we further restrict attention here to $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$
.

RECALL: PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

Problem parameters define our model *structure*, and hence a *family* of models.

Parameters can also be thought of as

- Specifying particular *model perturbations* near some distinguished parameter value.
- Extra *very-slowly-varying/frozen state variables* summarising neglected processes or external/enviroment conditions (we'll come back to this in centre manifold theory).

ONE-PARAMETER BIFURCATION?

Here we are considering systems with the *minimum number of parameters we need to vary to get this type of bifurcation*. (A *related* term is 'co-dimension').

Note: our original system may be higher-dimensional, but

1. The bifurcation typically occurs in lower dimensions and
2. Is determined by a small number of parameters (low 'co-dimension') - e.g. one eigenvalue crosses the imaginary axis (the real part changes sign, hence stability).

RECALL: BIFURCATION DIAGRAMS

A bifurcation diagram shows how system properties of interest like equilibria *depend on variations in a system parameter* (or parameters).

A *bifurcation diagram* is like a phase portrait but with a *parameter* ('very slow/frozen state variable') on one of the axes.

BASIC ONE-PARAMETER LOCAL BIFURCATIONS

- Saddle-node/turning point/fold bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Hopf bifurcation (next lecture)

Note: mathematically, the saddle-node and Hopf bifurcations are the key 'generic', co-dimension one local bifurcations and are basic mechanisms of 'creating' and 'destroying' fixed points and periodic orbits, respectively. The transcritical and pitchfork bifurcations can occur in physical systems with certain symmetries etc, but are actually themselves structurally unstable to *general* perturbations!

EngSci 711 L09 Bifurcation Theory Cont'd.

Examples of one parameter local bifurcations

- saddle-node/turning point
- transcritical
- pitchfork

Examples where multiple of these occur in one system

Example Questions

2016 Exam

(a) Consider the equation

$$\dot{u} = (\lambda - b)u - au^3$$

where $u \in \mathbb{R}$, a and b are fixed positive constants and λ is a parameter that can vary.

(i). Determine the equilibria and their stability. Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$ and $\lambda > b$ separately.

(ii). Sketch the bifurcation diagram showing how the equilibria vary with λ. What sort of bifurcation is this?

2017 Exam

Question 5 (16 marks)

Consider the equation

$$\dot{u} = (u - 2)(\lambda - u^2)$$

where $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a parameter that can vary.

(a) Determine the equilibria and their stability as λ varies.

(11 marks)

(b) Sketch the bifurcation diagram showing how the equilibria vary with λ. What types of bifurcations occur?

(5 marks)

'Generic' one-parameter bifurcations

- The saddle-node & Hopf bifurcations are the two 'generic'/'typical' local (cf global) one-parameter bifurcations:
  - $\boxed{\text{Saddle-node}}$: real eigenvalue equal to zero
  - $\boxed{\text{Hopf}}$: complex eigenvalue with zero real-part

- However, just like centres can be more common in conservative systems, the transcritical & pitchfork bifurcations can occur instead of the saddle-node, depending on the symmetries/structure of the system under study:
  - $\boxed{\text{Transcritical}}$: real eigenvalue equal to zero
  - $\boxed{\text{Pitchfork}}$: real eigenvalue equal to zero

These cases can be characterised geometrically/topologically according to various higher derivatives etc but we will just look at more directly

Real applications?

Many applications to biology/physiology

→ see eg Keener & Sneyd 'Mathematical Physiology' (Book):

Calcium oscillations

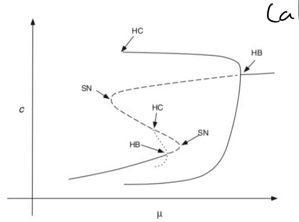


Figure 7.27 Typical bifurcation diagram of the model of Ca^{2+} oscillations in Exercise 10 (not drawn to scale). HB denotes a Hopf bifurcation, SN denotes a saddle-node bifurcation, HC denotes a homoclinic bifurcation.

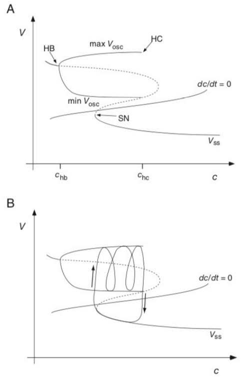


Figure 9.4 A: Sketch of the bifurcation diagram of the simplified Chay–Keizer β cell model, with c as the bifurcation parameter. V_{ss} denotes the curve of steady states of V as a function of c. A solid line indicates a stable steady state; a dashed line indicates an unstable steady state. The two branches of V_{osc} denote the maximum and minimum of V over one oscillatory cycle. HB denotes a Hopf bifurcation, HC denotes a homoclinic bifurcation, and SN denotes a saddle-node bifurcation. B: A burst cycle projected on the (V, c) plane. (Adapted from Rinzel and Lee, 1986, Fig. 3.)

pancreatic β cells

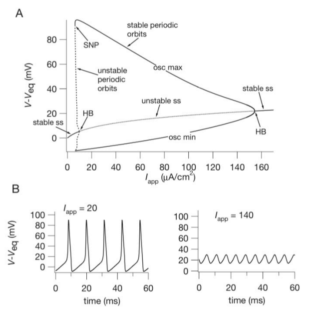


Figure 5.7 A: Bifurcation diagram of the Hodgkin–Huxley equations, with the applied current, I_{app} as the bifurcation parameter. HB denotes a Hopf bifurcation, SNP denotes a saddle-node of periodics bifurcation, osc max and osc min denote, respectively, the maximum and minimum of an oscillation, and ss denotes a steady state. Solid lines denote stable branches, dashed or dotted lines denote unstable branches. B: Sample oscillations at two different values of I_{app}.

Hodgkin–Huxley model of action potentials in neurons

+ many other complex systems!

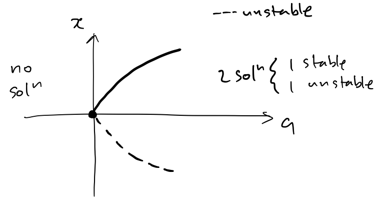
→ lasers
→ statistical mechanics of magnetic/spin etc systems
→ neural networks
→ etc

See eg
- Strogatz
- Drazin
- Tutorial/assignment?

Overview

∘ Saddle node (or turning point, fold)

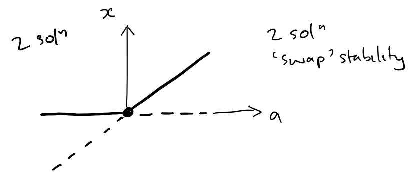
— stable
--- unstable



Normal form:

$$\boxed{\dot{x} = a - x^2}$$

∘ Transcritical

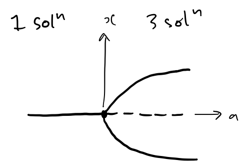


Normal form:

$$\boxed{\begin{aligned}\dot{x} &= ax\left(1-\frac{b}{a}x\right)\\ &= ax - bx^2\end{aligned}}$$

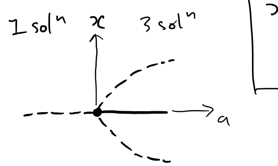
(logistic)

∘ Pitchfork



new branches stable ('supercritical')

OR



new branches unstable ('subcritical')

Normal form:

$$\boxed{\begin{aligned}\dot{x} &= ax\left(1-\frac{b}{a}x^2\right)\\ &= ax - bx^3\end{aligned}}$$

Notes: ∘ in each case above we vary the 'lowest' order term, ie a.

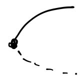
∟ called 'control' or 'bifurcation' parameter

∟ only need: $a-x^2$, $ax-x^2$, $a-x^3$

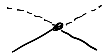
∘ the 'net stability' is conserved either side of the bifurcation.

Analysis details

- $\boxed{\text{Saddle-node}}$: see yesterday!



- $\boxed{\text{Transcritical}}$



$$\dot{x} = ax - bx^2 \quad , x \in \mathbb{R}$$

assume $b > 0$ & fixed.

$a \in \mathbb{R}$ & can be varied.

↳ 'control' or 'bifurcation' parameter

1. Fixed points.

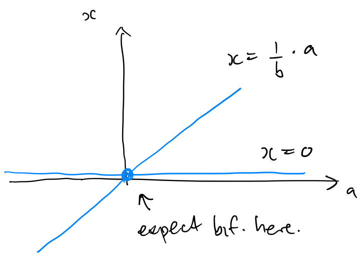
$$ax - bx^2 = 0$$

$$ax\left(1 - \frac{b}{a}x\right) = 0 \implies x = 0 \quad \text{or} \quad x = \frac{a}{b}$$

Note: always exist

→ looking for change in stability

2. Diagram without stability



Transcritical cont'd.

3. Stability.

$$Df = a - 2bx.$$

at $x = 0$

$$\lambda = Df(x=0) = a \begin{cases} > 0 \text{ (unstable) if } a > 0 \\ < 0 \text{ (stable) if } a < 0 \end{cases}$$

at $x = a/b$

$$\lambda = Df(x = a/b) = a - 2\cancel{b} \cdot \frac{a}{\cancel{b}} = -a$$

$\implies$ opposite!

so: exchange of stabilities at $a = 0$

ie transcritical.

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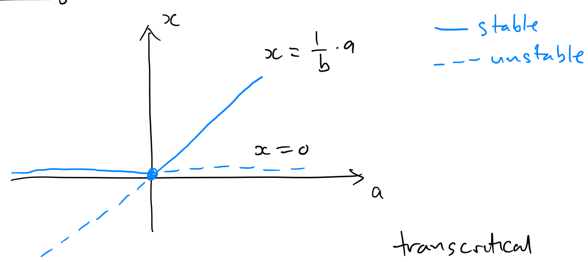
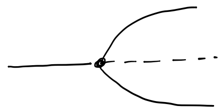
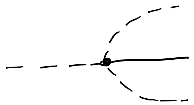
4. Final diagram



Analysis details





- $\boxed{\text{Pitchfork}}$

$$\dot{x} = ax - bx^3 \quad , x \in \mathbb{R}$$

assume $b > 0$ & fixed

$a \in \mathbb{R}$ & can be varied.

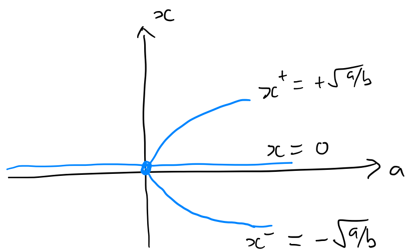
1. Fixed points

$$ax\left(1 - \frac{b}{a}x^2\right) = 0$$

$$\Rightarrow \underbrace{x = 0}_{\text{always exists}} \quad \text{or} \quad \underbrace{x = \pm\sqrt{a/b}}_{\text{if } a \geq 0}$$

$\rightarrow$ expect pitchfork

2. Diagram without stability



Pitchfork cont'd

3. Stability.

$$Df = a - 3bx^2$$

$$Df(x=0) = a \begin{cases} > 0 \text{ (unstable) if } a > 0 \\ < 0 \text{ (stable) if } a < 0 \end{cases}$$

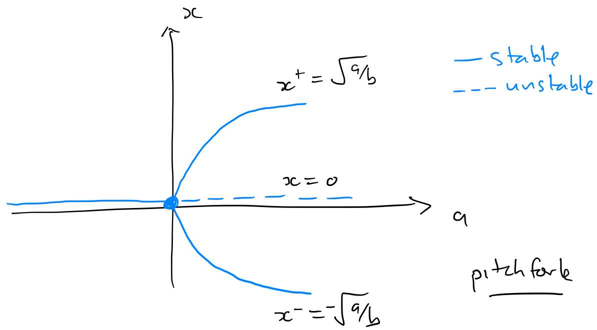
$$Df(\underbrace{x^2 = a/b}) = a - 3\cancel{b}\cdot\frac{a}{\cancel{b}} = -2a$$

note: Df only depends on $x^2 \Rightarrow x^+ \,\&\, x^-$ have same stability

$= \Big\{$ opposite to above!

$\rightarrow$ only exists for $a > 0$ so always stable

4. Final diagram



Terminology: supercritical if 'extra' solutions (branches) are stable

subcritical otherwise (unstable branches)

More 'complicated' variations

① $\dot{x} = x + \dfrac{\mu x}{1+x^2}$, $x \in \mathbb{R}$

② $\dot{x} = (\lambda - b)x - a x^3$

a, b fixed

λ can vary (control parameter)

③ $\dot{x} = (x-1)(x^2 + 2ax - \mu)$

$a > 0$ & fixed

$x, \mu \in \mathbb{R}$, μ can vary

Let's do ③! (① & ②: exercise/tutorial/...)

1. Fixed points (& existence)

$$(x-1)(x^2+2ax-\mu) = 0$$

$x = 1$ &/or $x^2 + 2ax - \mu = 0$

$x=1$: always a sol.

$x^2 + 2ax - \mu = 0$: $x = -a \pm \sqrt{a^2+\mu}$

sol^{n}s for $\mu \geq -a^2$

Note: when $\mu = -a^2$, $x = -a$

when $\mu = 0$, $x = 0$ or $-2a$

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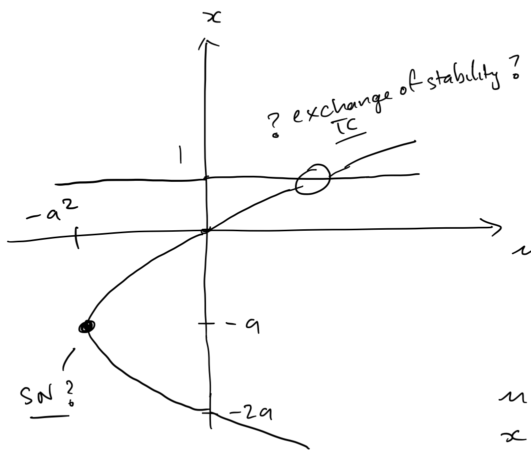
2. Diagram w/o stability



$\mu = -a^2$
$x = -a$

$\mu = 0$
$x = \begin{cases} 0 \\ -2a \end{cases}$

$\mu = -a$
$x =$

3. Difficult points

A ∘ Find intersection then stability either side

OR

B ∘ Find change of stability, verify intersection

→ let's try A.

$x = 1$ & $x = -a + \sqrt{a^2+\mu}$

→ solve for μ.

→

$x=1$ & $x=-a+\sqrt{a^2+\mu}$

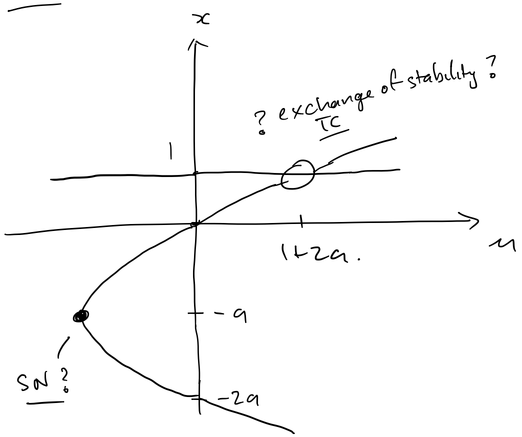
$$\Rightarrow \quad 1+a = \sqrt{a^2+\mu}$$

$$(1+a)^2 = a^2+\mu$$

$$1+2a+\cancel{a^2} = \cancel{a^2}+\mu$$

$$\Rightarrow \quad \underline{\mu = 1+2a}. \quad \Big\}\ \text{intersection point}$$

<u>So:</u>



$\Rightarrow$

4. <u>Stability</u>

$$f = (x-1)(x^2+2ax-\mu)$$

$$= x^3+2ax^2-\mu x$$
$$\quad -x^2-2ax+\mu$$

$$= x^3+(2a-1)x^2-(\mu+2a)x+\mu$$

$$\underline{Df} = 3x^2+2(2a-1)x-\mu-2a$$

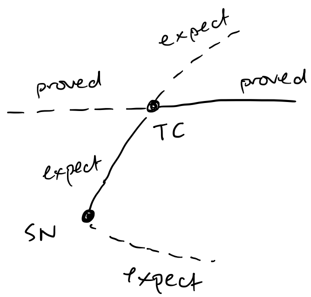
$$Df(\underline{x=1}) = 3+4a-2-\mu-2a$$

$$= 1+2a-\mu$$

$$\begin{cases} >0 & \text{if } \mu < 1+2a \\ <0 & \text{if } \mu > 1+2a \end{cases}$$

$\mu = 1+2a$ is intersection (from before).

<u>So far:</u>



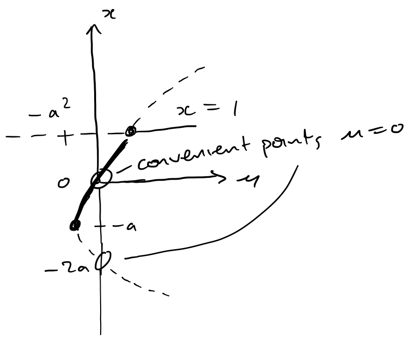
First : suspected SN at $\mu = -a^2,\ x = -a$.

$$\boxed{Df(x,\mu) = 3x^2 + 2(2a-1)x - \mu - 2a}$$

Key trick : evaluate at convenient points on branches

+

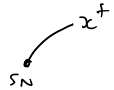
stability only changes at bifurcation } 'continuity of stability' away from bifurcation



$\mu = 0$ $\begin{cases} x^+ = 0 \\ x^- = -2a \end{cases}$ since $x^2+2ax-\cancel{\mu}^{\,0}=0 \Rightarrow x(x+2a)=0$

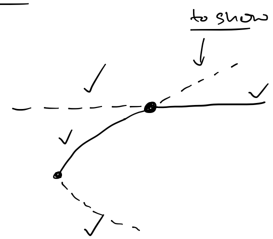
on branches

$Df(x^+=0, \mu=0) = -2a < 0$ since $a>0$ } stable ✓



$$\begin{aligned} Df(x^-=-2a, \mu=0) &= 3\times 4a^2 + 2(2a-1)(-2a) - 2a \\ &= 12a^2 + (-8a^2) + 4a - 2a \\ &= 4a^2 + 2a \\ &> 0 \end{aligned}$$ } unstable ✓

So far



Just need stability of x^+ for $x > 1$

x^+ satisfies $x^2 + 2ax - \mu = 0$

choose convenient : $x^+ = 2$. (easier to choose x here)

→ find μ

→ evaluate $Df(x,\mu)$.

μ : $4 + 4a - \mu = 0 \Rightarrow \mu = 4 + 4a$

$Df(x,\mu) = 3x^2 + 2(2a-1)x - \mu - 2a$

for $x = 2,\ \mu = 4 + 4a$

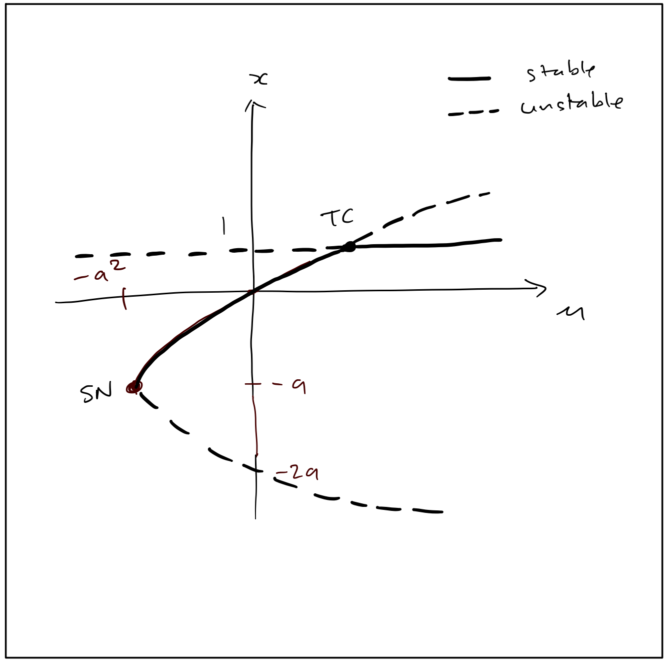
$$\begin{aligned} Df &= 3\times 4 + 8a - 4 - 4 - 4a - 2a \\ &= (12-8) + 2a \\ &= 4 + 2a > 0 \Rightarrow \text{unstable} \end{aligned}$$

(as expected)

→

5. Final diagram

Bifurcation diagram



Exercises

① $\dot{x} = x + \frac{\mu x}{1+x^2}$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is control param.

② $\dot{x} = (\lambda - b)x - ax^3$

a, b fixed

λ can vary (control parameter)

③ $\dot{x} = (x-2)(\lambda - x^2)$, $x \in \mathbb{R}$

$\lambda \in \mathbb{R}$ & is control param.