ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*) *Oliver Maclaren*
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LECTURE 14 BONUS TOPIC

Introduction to Chaos.

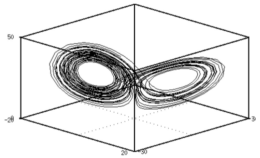
MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

EngSci 711 L14: Intro to chaos

- The Lorenz system
- Map dynamical systems 'induced' by ODEs
  - (discrete)

```
init x=-7.5  y=-3.6  z=30
par r=27  s=10  b=2.66666
x'=s*(y-x)
y'=x*(r-z)-y
z'=x*y-b*z
```



? ⟶ XPP!

Example Questions

→ Not examinable.

→ Some Refs: Strogatz ch. 9 – 12
Glendinning ch. 11 – 12.
Drazin ch 1, 3.
Ermentrout
⋮

What is chaos?

$\boxed{\text{No one fully accepted definition}}$

Key features (eg Strogatz ch. 9, Glendinning ch. 11)

> bounded, aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions

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NOT
- fixed point
- periodic orbit

} key possibilities in 2D ODEs
"no chaos in phase plane"
(for continuous ODEs)

Videos
- Strogatz (Lorenz)
- Water wheel (Lorenz)

Lorenz : (1963 "Deterministic non-periodic flow")

'simplified' model of atmosphere

$$\boxed{\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}}$$

via centre manifold reduction?

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

1. Introduction

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

These modes of behavior may all be observed in the familiar rotating-basin experiments, described by Fultz, *et al.* (1959) and Hide (1958). In these experiments, a cylindrical vessel containing water is rotated about its axis, and is heated near its rim and cooled near its center in a steady symmetrical fashion. Under certain conditions the resulting flow is as symmetric and steady as the heating which gives rise to it. Under different conditions a system of regularly spaced waves develops, and progresses at a uniform speed without changing its shape. Under still different conditions an irregular flow pattern forms, and moves and changes its shape in an irregular nonperiodic manner.

Lack of periodicity is very common in natural systems, and is one of the distinguishing features of turbulent flow. Because instantaneous turbulent flow patterns are so irregular, attention is often confined to the statistics of turbulence, which, in contrast to the details of turbulence, often behave in a regular well-organized manner. The short-range weather forecaster, however, is forced willy-nilly to predict the details of the large-scale turbulent eddies—the cyclones and anticyclones—which continually arrange themselves into new patterns. Thus there are occasions when more than the statistics of irregular flow are of very real concern.

In this study we shall work with systems of deterministic equations which are idealizations of hydrodynamical systems. We shall be interested principally in nonperiodic solutions, i.e., solutions which never repeat their past history exactly, and where all approximate repetitions are of finite duration. Thus we shall be involved with the ultimate behavior of the solutions, as opposed to the transient behavior associated with arbitrary initial conditions.

A closed hydrodynamical system of finite mass may ostensibly be treated mathematically as a finite collection of molecules—usually a very large finite collection—in which case the governing laws are expressible as a finite set of ordinary differential equations. These equations are generally highly intractable, and the set of molecules is usually approximated by a continuous distribution of mass. The governing laws are then expressed as a set of partial differential equations, containing such quantities as velocity, density, and pressure as dependent variables.

It is sometimes possible to obtain particular solutions of these equations analytically, especially when the solutions are periodic or invariant with time, and, indeed, much work has been devoted to obtaining such solutions by one scheme or another. Ordinarily, however, nonperiodic solutions cannot readily be determined except by numerical procedures. Such procedures involve replacing the continuous variables by a new finite set of functions of time, which may perhaps be the values of the continuous variables at a chosen grid of points, or the coefficients in the expansions of these variables in series of orthogonal functions. The governing laws then become a finite set of ordinary differential

[1] The research reported in this work has been sponsored by the Geophysics Research Directorate of the Air Force Cambridge Research Center, under Contract No. AF 19(604)-4969.

Naive analysis

1. Find fixed points
2. Determine stability as a function of r

1.

$$\begin{aligned} \sigma(y-x) &= 0 \quad (1) \\ rx - y - xz &= 0 \quad (2) \\ xy - bz &= 0 \quad (3) \end{aligned}$$

(1): $y = x$

$\rightarrow$ (2): $rx - x - xz = 0$

$x(r-1-z) = 0$

$x = 0$ or $z = r-1$

$\left.\begin{aligned} &x = 0 \ \& \ (3) \ \& \ (1) \\ &z = 0 \ \& \ y = 0 \end{aligned}\right\}$ always exists

$z = r-1$ & (3) & (1)

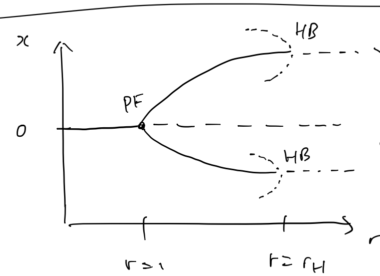
$x^2 - b(r-1) = 0$

$\left.\begin{aligned} x &= \pm\sqrt{b(r-1)} \\ y &= \pm\sqrt{b(r-1)} \\ z &= r-1 \end{aligned}\right\}$ two sol if $r>1$ $\Big\}$ pitchfork bifurcation

exercise!

also : Hopf

$\Big\}$ end up with 3 unstable fixed points



no stable objects ??!

Just like

Lorenz :

→ noticed that for some parameter ranges :

- no stable fixed points
- no stable periodic orbits

} no stable 'usual suspects'

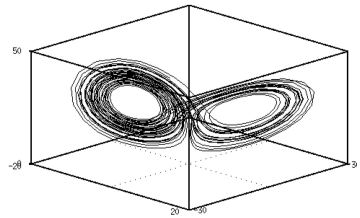
$\boxed{\text{BUT}}$

- trajectories remain in bounded region & attracted to a set of zero volume
  - ↳ ie 'smaller dimension' than full space.

- what is this 'object' that 'attracts' the solutions ?
- What do these solutions look like ?

→

The famous Lorenz 'strange attractor' :



→ trajectories do 'a few' loops around one (unstable) fixed point then 'jump' to loop around other } 3 (unstable) fixed points incl. origin

→ 'effectively' impossible to predict how many loops before a jump despite being deterministic } or is it ?!

↳ Lorenz map...

→ the attractor 'looks like' two surfaces 'merging' together

↳ actually "infinite complex of surfaces, each extremely close to one or other of two surfaces" (Lorenz)

↳ is a fractal : looks the same no matter how many times you zoom in !

→ Took until 1999 to rigorously prove it 'really exists'

↳ Warwick Tucker

→ use computational approach based on interval arithmetic

→ PhD Thesis : 'The Lorenz attractor exists'

→ article (1999) same title.

Fractal structure ?

(Strange : fractal)

- Self-similar / infinitely repeating.

See Strogatz 11 – Fractals
12 – Strange attractors.

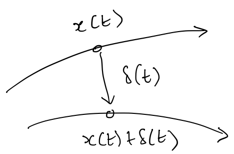
Exponential divergence of trajectories

- Consider the XPP file from Ermentrout
'Simulating, analyzing & animating dynamical systems'

(see Canvas)

- see also the video links

→ Liapunov exponent : (eg Strogatz 9.3)



$\|\delta_0\| = \epsilon$, eg 10^{-15}

Find :

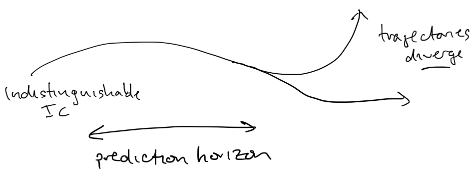
$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$

λ is called Liapunov exponent

$\lambda > 0 \Rightarrow$ exponential separation, even of 'indistinguishable' ICs

→ quickly becomes unpredictable

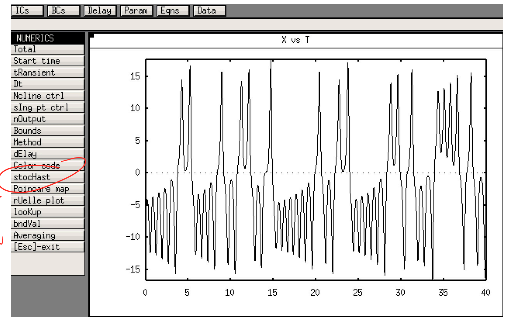
a hallmark of chaos



can't predict anymore (from original IC)

Empirical observations

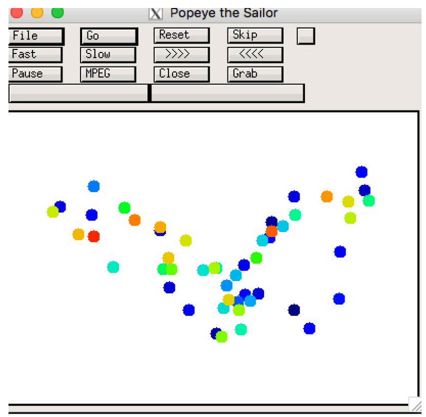
Liapunov exponent > 0



Liapunov

→ eg ≃ 0·83
('true' ≃ 0·9)

50 ICs separated by ϵ → diverge!

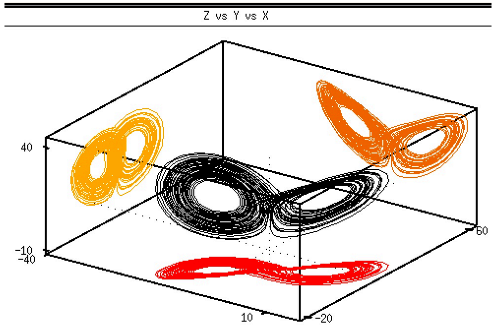


How can we analyse the Lorenz system?

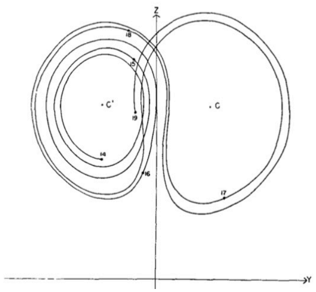
Search for a 'simpler picture'

→ Projections

→ Poincaré sections & maps



Lorenz 1963:



Returning to Fig. 2, we find that the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again.

It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit. A suitable feature of this sort is the maximum value of Z, which occurs when a circuit is nearly completed. Table 2 has again been prepared by the computer, and shows the values of X, Y, and Z at only those iterations N for which Z has a relative maximum. The succession of circuits about C and C' is indicated by the succession of positive and negative values of X and Y.

→

Lorenz 1963: surprise! (Expect 2D, not 1D!)

$$\boxed{z_{n+1} = f(z_n)}$$

[max value of 'circuit' determines next max value of circuit.]

→ hidden predictability.

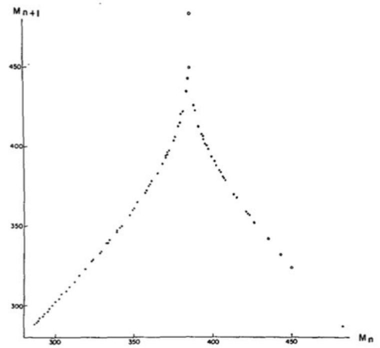


FIG. 4. Corresponding values of relative maximum of Z (abscissa) and subsequent relative maximum of Z (ordinate) occurring during the first 6000 iterations.

'Lorenz map'

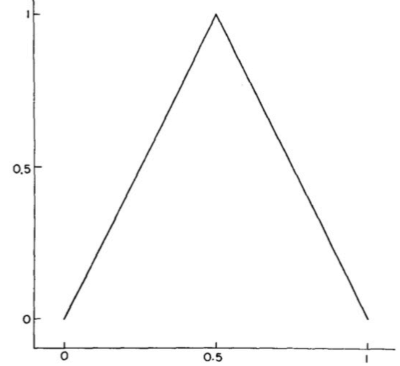


FIG. 5. The function $M_{n+1}=2M_n$ if $M_n<\frac{1}{2}$, $M_{n+1}=2-2M_n$ if $M_n>\frac{1}{2}$, serving as an idealization of the locus of points in Fig. 4.

'Tent map'

(simple model of above).

(Figs from Lorenz 1963)

Lorenz map

Higher 'iterates' (XPP via Ermentrout):

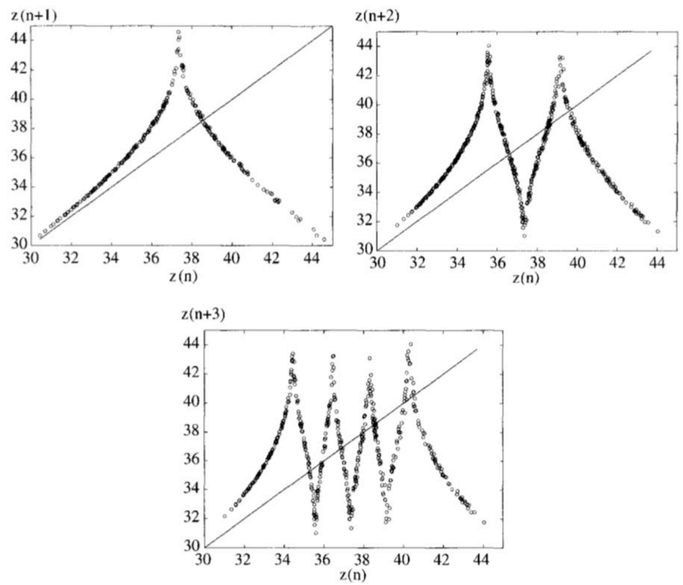


Figure 4.13. *Poincaré map for the Lorenz attractor looking at successive values of the maximum of z. Period 1, 2, and 3 points are illustrated.*

$$z(n+1) = f(z(n))$$
$$z(n+2) = f(z(n+1)) = f^2(z(n))$$
$$\vdots$$
$$z(n+m) = f^m(z(n))$$

Lesson:

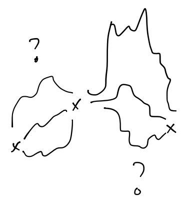
- We can analyse complex orbits via 'simpler' maps 'induced' by 'projecting', taking intersections etc

We need to learn how to analyse discrete dynamical systems ('maps')

- These systems are usually discrete time maps

$$z(n+1) = f(z(n))$$

→ lack of continuity allows very complex behaviour in even 'simple' 1D systems



- They fall into important qualitative classes

↳ eg
  - 1D unimodal
    - quadratic max. (logistic)
    - non-quad. max (tent)
  - 1D multimodal

another example →

Rössler System

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

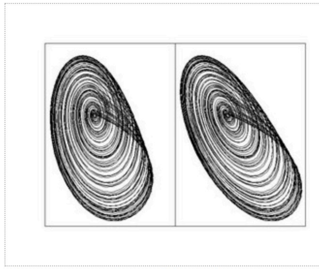


Figure 1: Stereoscopic view of the Rössler attractor.
Parameter values: a=0.432, b=2 and c=4.

Images:
scholarpedia article on Rössler attractor, written by.... Rössler!

→ also exhibits a strange attractor

→ even simpler than Lorenz system (eg only one nonlinear term)

→ its Poincaré/Lorenz map looks remarkably like the logistic map (see soon...)

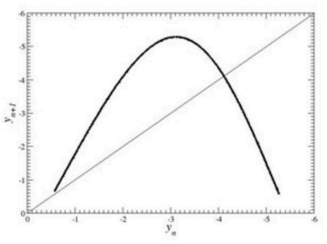


Figure 5: First-return map of the Rössler attractor.
Parameter values: a=0.432, b=2 and c=4.

Maps (discrete dynamical systems)

$$\boxed{x_{n+1} = F(x_n)} \quad , \; x \in \mathbb{R}^m$$

→ analysis is very similar! small differences.

eg. FP: $x^* = F(x^*)$

- Linearisation

$$x_n = x^* + \xi_n$$

$$\Rightarrow x_{n+1} = \xi_{n+1} + x^* = F(x^* + \xi_n)$$

$$\Rightarrow \xi_{n+1} + \cancel{x^*} = \cancel{F(x^*)} + DF(x^*)\xi_n + O(\|\xi_n\|^2)$$

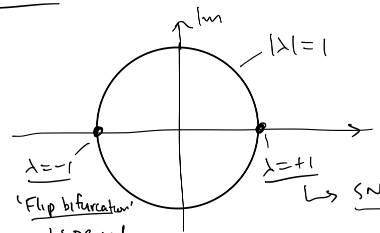
$$\Rightarrow \xi_{n+1} = DF(x^*)\xi_n$$

- Stability

Eigenvalues $|\lambda| < 1 \Rightarrow$ stable/hyperbolic
$|\lambda| = 1 \Rightarrow$ bifurcations/non-hyp.

'multipliers'

note: compare to 1, not zero!



$\lambda = -1$
'Flip bifurcation'
↳ period doubling bifurcation
↳ a route to chaos!

$\lambda = +1$
↳ SN, transcritical or pitchfork bif.

Other properties

- **Iterates**

Consider the iterates:

$$x_{n+1} = F(x_n)$$

$$x_{n+2} = F(x_{n+1}) = F(F(x_n)) = F^2(x_n) \qquad F^2 = F \circ F \text{ (composite)}$$

$$\vdots$$

$$\boxed{x_{n+k} = F^k(x_n)} \qquad \text{'kth iterate'}$$

- **Periodic points**

$x = x_p$ is a periodic point of period k

if $F^k(x_p) = x_p$

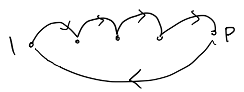
But $F^r(x_p) \neq x_p$ for $r = 1, \dots, k-1$ } repeats at k, not before.

- **Periodic cycle**

$C = \{x_1, x_2, \dots, x_p\}$

where $x_{n+1} = F(x_n)$ for $n = 1, \dots p-1$

& $x_1 = F(x_p)$



- **Centre manifolds etc**

↳ also exist/apply!

↳ beyond scope here...

Stability of cycles & fixed points of higher iterates

Consider the second iterate:

$$x_{n+2} = G(x_n) = F^2(x_n) = F(F(x_n)) = F(x_{n+1})$$

& a fixed point of this

$$x_* = G(x_*) \iff \text{defines a period-2 cycle if } x_* \neq F(x_*)$$

Stability is given by the (eig. of the) linearisation of G

$$DG(x_*).$$

Now the chain rule gives

$$\frac{\partial G_i(x_j)}{\partial x_j} = \frac{\partial F_i}{\partial x_j}(F_k(x_j))$$

$$= \sum_{k=1}^{m} \left[\frac{\partial F_i}{\partial x_k}\right]_{F(x)} \left[\frac{\partial F_k}{\partial x_j}\right]_x$$

Moral:

$$\underbrace{DG = DF(F(x)) \cdot DF(x)}_{\text{multiply Jacobian of } F \text{ } n \text{ times for an } n\text{-cycle, each evaluated at one point on cycle.}} \qquad \leftarrow \text{for 2-cycle.}$$

ie

$$\boxed{\text{Overall multiplier} = \text{product of multipliers}}$$

eg for (x_1, x_2, x_3) 3 cycle get

$$DG = DF(x_3)DF(x_2)DF(x_1) \qquad \text{etc.}$$

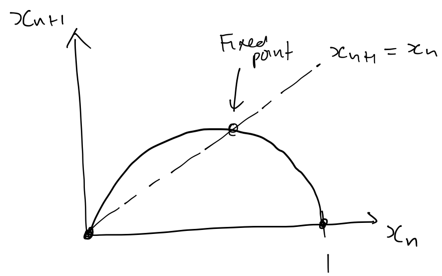
Analysis of the logistic map : route to chaos

Logistic map : arises in many contexts

- 'induced' by Rossler ODE system
- model of discrete population dynamics (Robert May 1976)

One-dimensional, unimodal, quadratic

$$x_{n+1} = \mu \cdot x_n (1 - x_n) = f(x_n ; \mu) \ , \ \mu \geq 0$$



Fixed points

1. $x^* = \mu x^* (1 - x^*)$

   $x^* = 0$ ✓ always call x_0

---

2. $\cancel{x^*} = \mu \cdot \cancel{x^*} (1 - x^*)$, $x^* \neq 0$

   $1 = \mu (1 - x^*)$

   $\Rightarrow x^* = 1 - \frac{1}{\mu} = \frac{\mu - 1}{\mu}$ for $\mu > 0$, call x_*

Stability

$Df = \mu(1 - 2x) = \lambda$

$\lambda_0 = \lambda(x = 0) = \mu$ $\begin{cases} |\lambda| > 1 & \text{if } \mu > 1 \Rightarrow \text{unstable} \\ |\lambda| < 1 & \text{if } \mu < 1 \Rightarrow \text{stable.} \end{cases}$

$(\mu > 0)$.

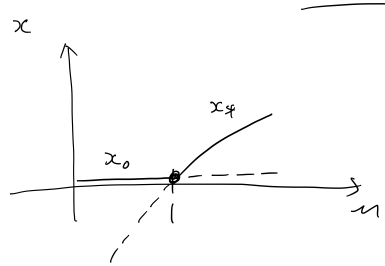
$$\lambda_* = \lambda\left(x = 1 - \frac{1}{\mu}\right) = \mu\left(1 - 2 - \frac{2}{\mu}\right)$$

$$= -\mu\left(1 + \frac{2}{\mu}\right)$$

$$= \underline{2 - \mu}$$

For $\mu = 1$, $\lambda_0 = \lambda_* = 1$, $x_* = 1 - \frac{1}{1} = 0 = x_0$

$\Rightarrow$ fixed points 'collide' & exchange stabilities $\rightarrow$ Transcritical



so far

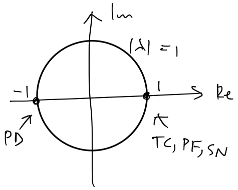
$\rightarrow$

Flip / Periodic doubling

Consider:

$x_* = 1 - \frac{1}{\mu}$

$\lambda_* = 2 - \mu$



For $\mu = 3$ get $\lambda_* = -1$, $x_* = 1 - \frac{1}{3} = 2/3$

→ loses stability as multiplier crosses -1 (cf $+1$)

→ a 'flip' bifurcation

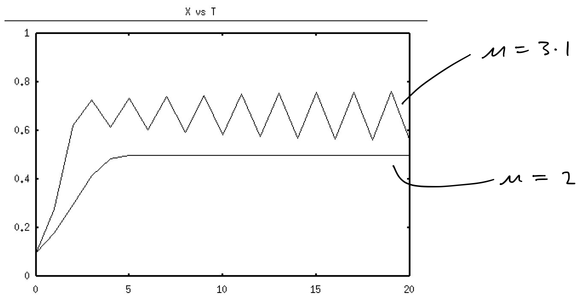
↳ new type of bifurcation!

What happens?

In short:

- x_* loses stability to a period-2 cycle
- 'Period doubling'

Time series plots:



→ Expect new period-2 cycle for $\mu \geq 3$ →

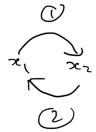
Period-2 solution for $\mu \geq 3$

Let's try to find the period-2 solⁿ. } cycle = fixed point of higher-d system.

Need (x_1, x_2), $x_1 \neq x_2$

s.t. $x_2 = F(x_1)$ ①

$x_1 = F(x_2)$ ②



$x_2 = \mu x_1(1 - x_1) = \mu x_1 - \mu x_1^2$

$x_1 = \mu x_2(1 - x_2) = \mu x_2 - \mu x_2^2$

Solve (should really use a computer alg. program...)

1. $x_2 - x_1 = \mu(x_1 - x_2) - \mu(x_1^2 - x_2^2)$

ie $x_2 - x_1 = -\mu(x_2 - x_1) + \mu(x_2 - x_1)(x_1 + x_2)$

$x_2 \neq x_1$ by def.

$\Rightarrow 1 = -\mu + \mu(x_1 + x_2)$

$\Rightarrow \boxed{x_1 + x_2 = \frac{1+\mu}{\mu}}$ Ⓐ

2. $\cancel{x_1 x_2} = \mu^2 \cancel{x_1 x_2}(1 - x_1)(1 - x_2)$

$(1 - x_1)(1 - x_2) = \frac{1}{\mu^2}$

$1 - \underbrace{(x_1 + x_2)}_{= \frac{1+\mu}{\mu}} + x_1 x_2 = \frac{1}{\mu^2} \Rightarrow \boxed{x_1 x_2 = \frac{\mu + 1}{\mu^2}}$ Ⓑ

$\left(x_1 x_2 = \frac{1}{\mu^2} + \frac{1+\mu}{\mu} - 1 = \frac{1 + \mu + \cancel{\mu^2} - \cancel{\mu^2}}{\mu^2} \right)$

→

Algebra cont'd.

$$x_1 + x_2 = \frac{1+\mu}{\mu}, \qquad x_1 x_2 = \frac{\mu+1}{\mu^2}$$

equivalent conditions

Recall: tr & det & quadratic

$\Rightarrow$ x_1 & x_2 satisfy

$$x^2 - (x_1+x_2)x + x_1 x_2 = 0$$

$$\Rightarrow \quad x^2 - \left(\frac{1+\mu}{\mu}\right)x + \frac{\mu+1}{\mu^2} = 0$$

$$\rightarrow \quad x_{1,2} = \frac{(1+\mu)}{2\mu} \pm \frac{1}{2\mu}\sqrt{(1+\mu)(\mu-3)}$$

Note: real sol^{n}s for $\mu > 3$, as expected!

Multiplier for cycle:

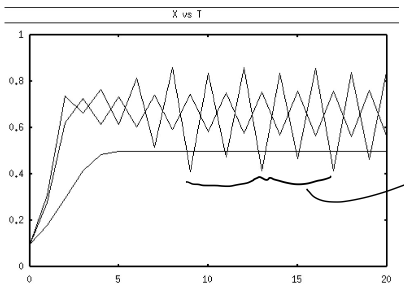
$$\lambda_{p2} = Df(x_2)\,Df(x_1), \quad Df = \mu - 2\mu x$$

$$= -\mu^2 + 2\mu + 4 \qquad , \ \lambda_{p2} \downarrow \text{ as } \mu \uparrow$$

when $\mu = 3$, $\lambda_{p2} = -9+6+4 = 1$ – appears via pitchfork in 2D system

$\mu > 3$, $\lambda_{p2} < 1$ – stable

$\mu = 1+\sqrt{6} \approx 3.449$, $\lambda = -1$ – undergoes another period doubling!

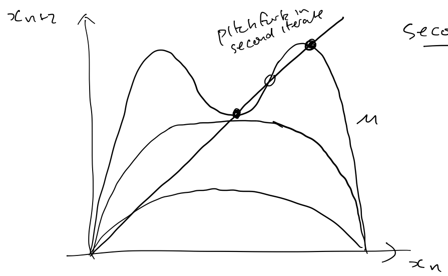


period-4 cycle for $\mu = 3.45$

Geometry of period doubling.

$1 \rightarrow 2$ cycle

(need two sol^{n}s to $x_{n+2} = x_n$)

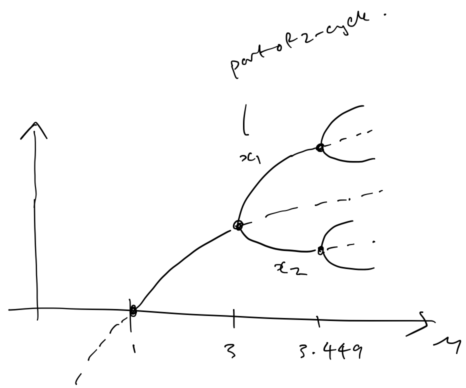


Second iterate map: quartic $\sim x^4$

Period doubling cascade

As indicated, this process continues indefinitely

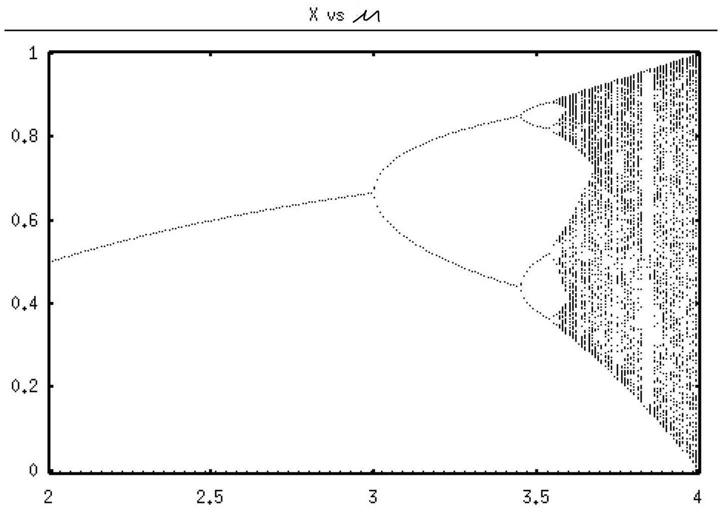
$1 \rightarrow 2 \rightarrow 4 \rightarrow$



Bifurcation diagram

This period doubling sequence provides a route to chaos

↳ get 'infinite period' ie chaotic solutions



(actually: orbit diagram; only showing stable sol^{n}s).

Feigenbaum Renormalisation: properties of the periodic doubling cascade

The values at which bifurcations appear to follow a geometric progression:

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \quad \& \quad \mu_n = \mu_\infty - A\delta^{-n} + \text{h.o.t.}$$

where $\boxed{\begin{array}{l}\mu_\infty = 3.5700\ldots \\ \delta = 4.6692\ldots\end{array}}$ ('universal' irrational numbers like π etc!)

Feigenbaum (1979) showed the scaling constants μ_∞, δ are 'universal' (...for all unimodal maps with locally quadratic maxima...)

used 'renormalisation group' (self-similarity etc)

[eg $g_k(x) = \lim_{n\to\infty} \alpha^n F^{(2n)}\left(\frac{x}{\alpha^n}, \mu_{n+k}\right)$

$g(x) = \lim_{k\to\infty} g_k(x)$

$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)$]

↳ rescaling operation.

see Strogatz, Drazin, Glendinning

ie $\boxed{\text{same scaling numbers show up in quite different systems}}$

↳ eg a 'generic route to chaos'

→ other 'universality classes' exist depending on qualitative properties

Again: classify into classes according to 'generic' properties

Analysis of tent map.

Challenge:

Look up the 'Tent Map' (Lorenz's simple model)

in eg Strogatz, Drazin or Glendinning

& carry out a similar analysis (they'll guide you through). } easier in many ways.

$\boxed{z_{n+1} = f(z_n)}$

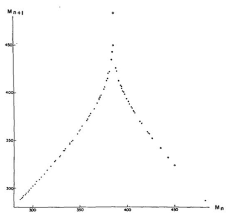


FIG. 4. Corresponding values of relative maximum of Z (abscissa) and subsequent relative maximum of Z (ordinate) occurring during the first 6000 iterations.

'Lorenz map'

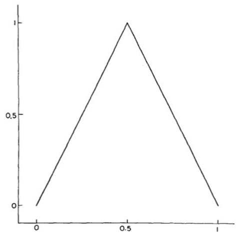


FIG. 5. The function $M_{n+1}=2M_n$ if $M_n<\frac{1}{2}$, $M_{n+1}=2-2M_n$ if $M_n>\frac{1}{2}$, serving as an idealization of the locus of points in Fig. 4.

'Tent map' (simple model of above).

(Figs from Lorenz 1963)

Note: is unimodal, but not quadratic/smooth

→ doesn't actually have a period doubling cascade

→ instead, has a 'sharp' transition to chaos.

Other things:

Robert May: blackboard problem for graduate students:
"what the Christ happens for $\mu > \mu_\infty = 3.57\ldots$?"

Sarkovskii's theorem.

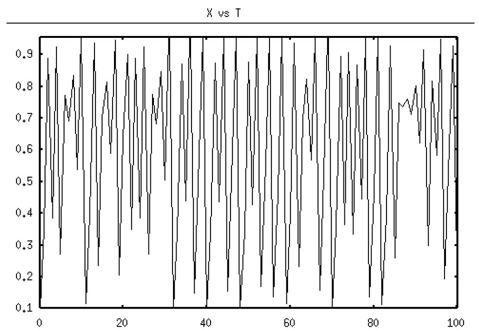
- 1D maps
- period three implies chaos!

↳ if there exists a period 3 cycle then it has an l-cycle for all positive integers l

↳ has a countable infinity of arbitrarily long cycles

↳ ie has 'chaotic' solutions

→ For the logistic map, $\mu = 3.841499$ gives (an unstable) period-3 cycle & hence exists 'chaotic' solutions



$\mu = 3.9$

Other things:

- Geometry/topology of strange attractors & chaos (in original ODE): } how/why of strange attractor

  - Smale's Horseshoe example

  → stretching & folding mechanism

  ↳ diverge yet stay bounded? } stretch: diverge; fold: mix together

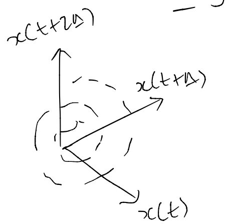
  → stable & unstable manifolds both play a role

- Attractor reconstruction (Takens 1981)

  - data analysis of time series $x(t)$

  → reconstruct full phase space dynamics from observations of single var over time! } topologically

  $$x(t) \longrightarrow \underbrace{(x(t), x(t+\Delta), x(t+2\Delta), \ldots)}_{\text{delay embedding}}$$



} topologically equivalent to full (x, y, z) etc dynamics*!

(Takens' theorem)

*in theory...