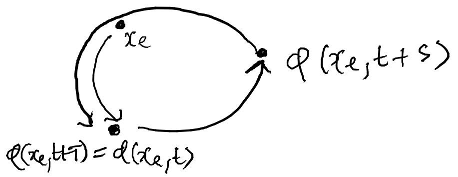
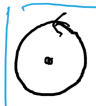
Engsci 711   Lecture 6   Examples

Periodic points



Ruling in/out orbits?

Key point: every P.O. contains at least one FP.



Out
- winding number → see Strogatz &/or Glendinning
- Dulac / divergence $\nabla\cdot f = (f_1)_x + (f_2)_y$ divergence of $f = (f_1, f_2)^T$ | $\nabla\cdot(fg)$ Dulac (weighted divergence).

[for proof see Strogatz / Glendinning]

⇒ Dulac examples

Dulac/Divergence (Rule out)

Examples:

- Simple ($g=1$)

Consider
$$\dot x = x + x^3 - 2y = f_1(x,y)$$
$$\dot y = -3x + y^5 = f_2(x,y)$$

$$\nabla\cdot f = \nabla\cdot(f_1, f_2) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1 + x^2 + 4y^4 > 0 \quad \forall x,y$$

⇒ no periodic orbits in whole x-y plane by Dulac/Div

- Harder (need clever choice* of g)   Ex 5.9 Glendinning.

consider
$$\dot x = x(A - a_1 x + b_1 y)$$
$$\dot y = y(B - a_2 y + b_2 x)$$
$a_i > 0 \ \forall i$

Magic: $g(x,y) = \frac{1}{xy}$

$$\Rightarrow f(x,y)g(x,y) = \left(\frac{1}{y}(A - a_1 x + b_1 y), \frac{1}{x}(B - a_2 y + b_2 x)\right)^T$$

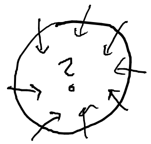
$$\nabla\cdot(fg) = -\frac{a_1}{y} - \frac{a_2}{x} < 0 \quad \text{for } x,y > 0$$ ⇒

by Dulac

* : typical g choices [try these first]
$$\begin{cases} 1 & \leftarrow \text{ie just } \nabla\cdot f \text{ divergence test.} \\ \frac{1}{x^a \cdot y^b} \\ e^{ax} \\ e^{ay} \end{cases}$$

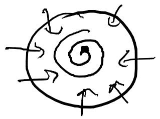
→

Poincare–Bendixson



trapping region 1 : limits: either FP or periodic orbit

↳ Problem : could be spiral



[also! remember that if there is a P.O. then there must be a FP inside.]

↳ Solution: find unstable FP & 'remove'

P.O.

Donut region with small area around FP removed

Trapping region v.2

Strogatz trapping region example (see attached)

→ read!

→ relevant for assignment

→ can look at in tutorial

→ other examples in tutorial

For now: nullclines →

Nullclines

— key for phase portraits : help find FP, Local flow directions, Trapping regions etc

$$\dot{x} = f_1(x,y) \qquad \dot{x} = 0 : \boxed{x\text{-nullcline}: f_1(x,y)=0}$$

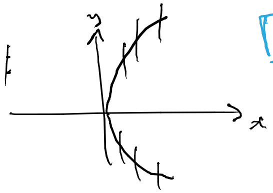
$$\dot{y} = f_2(x,y) \qquad \dot{y} = 0 : \boxed{y\text{-nullcline}: f_2(x,y)=0}$$

Example

$$\dot{x} = x - y^2$$
$$\dot{y} = y(a-x) \qquad a > 1$$

$\boxed{\dot{x}=0}$ $\boxed{x = y^2}$

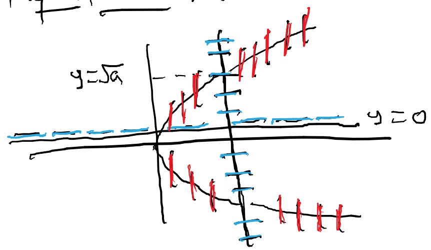
vertical flow



Step 1: curves and horizontal/vertical dir.

$\boxed{\dot{y}=0}$ $\boxed{y=0}$ &/OR $\boxed{x=a}$ } two lines, same nullcline.

horizontal flow



⇒

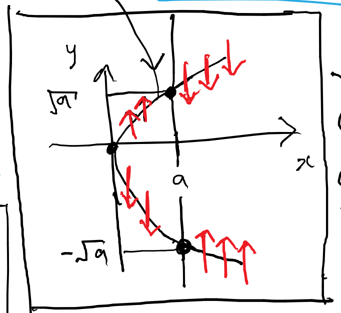
Step 2: Sign of flow on nullclines by subst. into other equation.

TIP: expect change of sign at equilibria
(+++...0...---)
continuous change from +ve to -ve goes through zero

when $\dot{x}=0$
$x=y^2$

other → & $\dot{y}=y(a-y^2)$

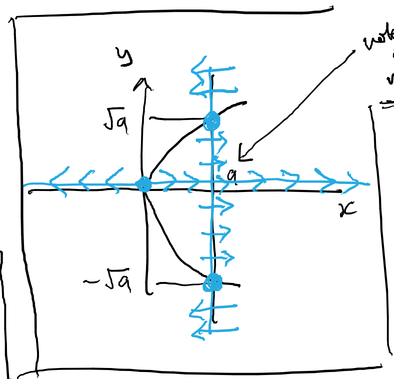
Cases
- if $y>0$ & $y^2<a$ then $\dot{y}>0$
- $y>0$ & $y^2>a$ then $\dot{y}<0$
- if $y<0$ & $y^2<a$ then $\dot{y}<0$
- $y<0$ & $y^2>a$ then $\dot{y}>0$



note 'reflection'
→ verify by subs. $y\to -y$ in original equations.

when $\dot{y}=0$
$y=0$ &/or $x=a$

Cases
- $y=0$: $\dot{x}=x$; >0 if $x>0$, <0 if $x<0$
- $x=a$: $\dot{x}=a-y^2$; >0 if $y^2<a$, <0 if $y^2>a$

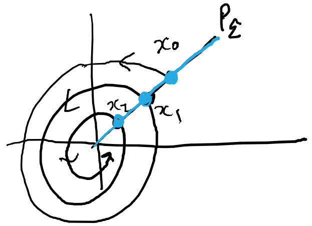


note: intersection of same nullcline
⇒ not an equilibrium point.

Exercise: continue!

Return Maps

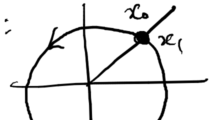
• **Picture** (examinable?)



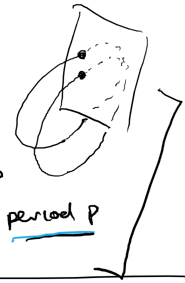
defines $x_{n+1}=F(x_n)$ } implicitly defined via system & P_Σ

[see eg Example 6.3 Glendinning]

Note:



Periodic orbit satisfies
$x=F(x)$
or $x=F^P(x)$
& $x\neq F^k(x)$ for $1\le k<P$
for least period P



• **Example** (not examinable)

Suppose $x_{n+1}=F(x_n)=1-x_n^2$ (map)

FP: $x=1-x^2$
$x^2+x-1=0$
$x=\begin{cases}\frac{-1+\sqrt{5}}{2}\\ \frac{-1-\sqrt{5}}{2}\end{cases}$ (2 period-1 fixed points.)

$Df=-2x$ ← linearised map
stability depends on $|\lambda|<1$ etc ← not 0

etc. See Example 6.1 Glendinning.

Strogatz 1994

soning is required.) By a similar argument, the flow is inward on the outer circle if $r_{\max} = 1.001\sqrt{1+\mu}$.

Therefore a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. ■

The estimates used in Example 7.3.1 are conservative. In fact, the closed orbit can exist even if $\mu \geq 1$. Figure 7.3,3 shows a computer-generated phase portrait of (1) for $\mu = 1$. In Exercise 7.3.8, you're asked to explore what happens for larger μ, and in particular, whether there's a critical μ beyond which the closed orbit disappears. It's also possible to obtain some analytical insight about the closed orbit for small μ (Exercise 7.3.9).

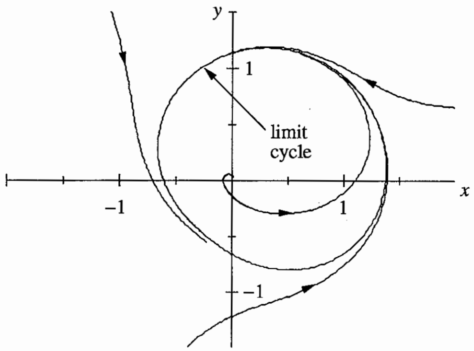


Figure 7.3.3

When polar coordinates are inconvenient, we may still be able to find an appropriate trapping region by examining the system's nullclines, as in the next example.

EXAMPLE 7.3.2:

In the fundamental biochemical process called ***glycolysis***, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an *oscillatory* fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. For reviews, see Chance et al. (1973) or Goldbeter (1980).

A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters. Construct a trapping region for this system.

Solution: First we find the nullclines. The first equation shows that $\dot{x} = 0$ on the curve $y = x/(a + x^2)$ and the second equation shows that $\dot{y} = 0$ on the curve $y = b/(a + x^2)$. These nullclines are sketched in Figure 7.3.4, along with some representative vectors.

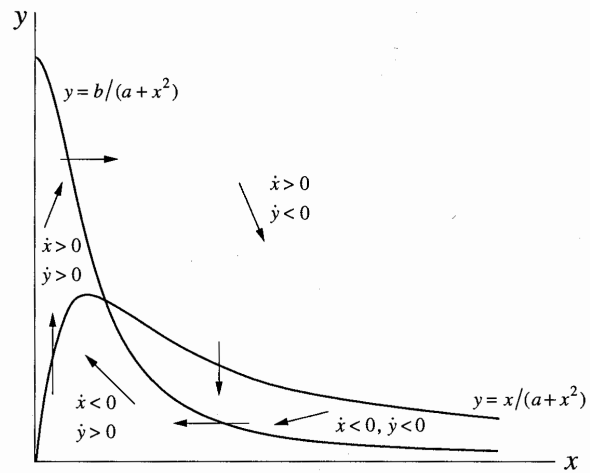


Figure 7.3.4

How did we know how to sketch these vectors? By definition, the arrows are vertical on the $\dot{x} = 0$ nullcline, and horizontal on the $\dot{y} = 0$ nullcline. The direction of flow is determined by the signs of $\dot{x}$ and $\dot{y}$. For instance, in the region above both nullclines, the governing equations imply $\dot{x} > 0$ and $\dot{y} < 0$, so the arrows point down and to the right, as shown in Figure 7.3.4.

Now consider the region bounded by the dashed line shown in Figure 7.3.5. *We claim that it's a trapping region.* To verify this, we have to show that all the vectors on the boundary point into the box. On the horizontal and vertical sides, there's no problem: the claim follows from Figure 7.3.4. The tricky part of the construction is the diagonal line of slope −1 extending from the point $(b, b/a)$ to the nullcline $y = x/(a + x^2)$. Where did this come from?

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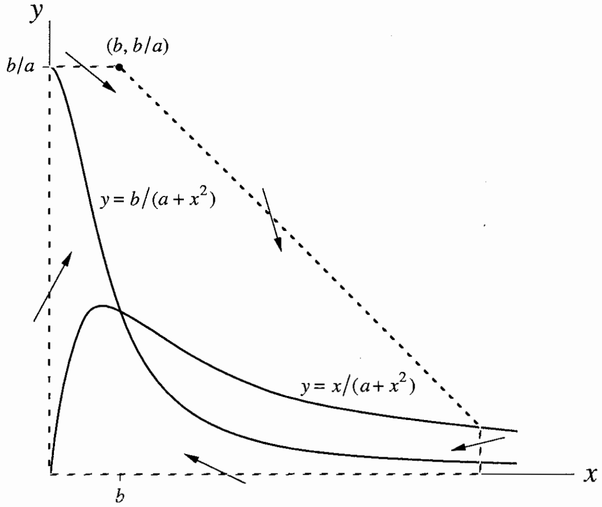

Figure 7.3.5

To get the right intuition, consider $\dot{x}$ and $\dot{y}$ in the limit of very large x. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\dot{y}/\dot{x} = dy/dx \approx -1$ along trajectories. Hence the vector field at large x is roughly parallel to the diagonal line. This suggests that in a more precise calculation, we should compare the sizes of $\dot{x}$ and $-\dot{y}$, for some sufficiently large x.

In particular, consider $\dot{x} - (-\dot{y})$. We find

$$\begin{aligned}\dot{x} - (-\dot{y}) &= -x + ay + x^2 y + (b - ay - x^2 y) \\ &= b - x.\end{aligned}$$

Hence

$$-\dot{y} > \dot{x} \text{ if } x > b.$$

This inequality implies that the vector field points inward on the diagonal line in Figure 7.3.5, because dy/dx is more negative than -1, and therefore the vectors are steeper than the diagonal line. Thus the region is a trapping region, as claimed. ■

Can we conclude that there is a closed orbit inside the trapping region? No! There is a fixed point in the region (at the intersection of the nullclines), and so the conditions of the Poincaré–Bendixson theorem are not satisfied. But if this fixed point is a *repeller*, then we *can* prove the existence of a closed orbit by considering

the modified "punctured" region shown in Figure 7.3.6. (The hole is infinitesimal, but drawn larger for clarity.)

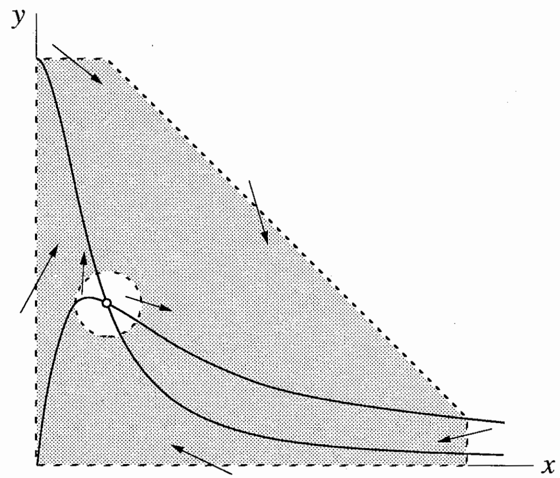

Figure 7.3.6

The repeller drives all neighboring trajectories into the shaded region, and since this region *is* free of fixed points, the Poincaré–Bendixson theorem applies.

Now we find conditions under which the fixed point is a repeller.

EXAMPLE 7.3.3:

Once again, consider the glycolytic oscillator $\dot{x} = -x + ay + x^2 y$, $\dot{y} = b - ay - x^2 y$ of Example 7.3.2. Prove that a closed orbit exists if a and b satisfy an appropriate condition, to be determined. (As before, $a, b > 0$.)

Solution: By the argument above, it suffices to find conditions under which the fixed point is a repeller, i.e., an unstable node or spiral. In general, the Jacobian is

$$A = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}.$$

After some algebra, we find that at the fixed point

$$x^* = b, \qquad y^* = \frac{b}{a + b^2},$$

the Jacobian has determinant $\Delta = a + b^2 > 0$ and trace

Strogatz 1994

$$\tau = -\frac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}.$$

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when

$$b^2 = \tfrac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right).$$

This defines a curve in (a,b) space, as shown in Figure 7.3.7.

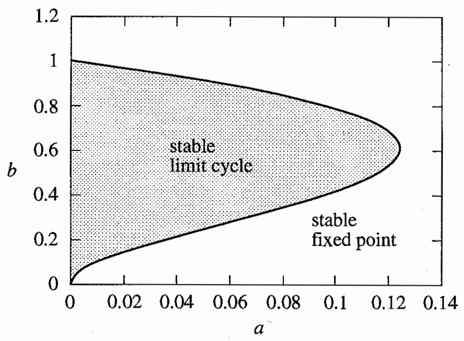

Figure 7.3.7

For parameters in the region corresponding to $\tau > 0$, we are guaranteed that the system has a closed orbit—numerical integration shows that it is actually a stable limit cycle. Figure 7.3.8 shows a computer-generated phase portrait for the typical case $a = 0.08$, $b = 0.6$. ■

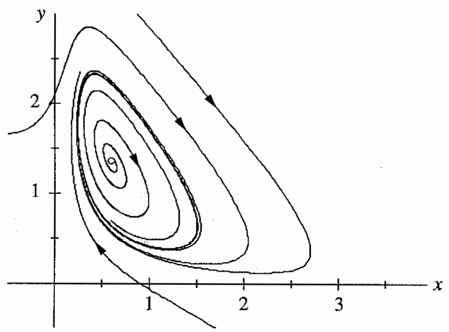

Figure 7.3.8

No Chaos in the Phase Plane

The Poincaré–Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n \geq 3$), the Poincaré–Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*, a fractal set on which the motion is aperiodic and sensitive to tiny changes in the initial conditions. This sensitivity makes the motion unpredictable in the long run. We are now face to face with *chaos*. We'll discuss this fascinating topic soon enough, but for now you should appreciate that the Poincaré–Bendixson theorem implies that chaos can never occur in the phase plane.

7.4 Liénard Systems

In the early days of nonlinear dynamics, say from about 1920 to 1950, there was a great deal of research on nonlinear oscillations. The work was initially motivated by the development of radio and vacuum tube technology, and later it took on a mathematical life of its own. It was found that many oscillating circuits could be modeled by second-order differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

now known as ***Liénard's equation***. This equation is a generalization of the van der Pol oscillator $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ mentioned in Section 7.1. It can also be interpreted mechanically as the equation of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$.

Liénard's equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y. \end{aligned} \tag{2}$$

The following theorem states that this system has a unique, stable limit cycle under appropriate hypotheses on f and g. For a proof, see Jordan and Smith (1987), Grimshaw (1990), or Perko (1991).

Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions: