MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

NEXT MODULE

3. *Sturm-Liouville eigenvalue problems* [4 lectures]

   Eigenvalue problems for function spaces: eigenvalues, eigenfunctions, Sturm-Liouville problems. Existence and orthogonality of solutions, eigenfunction expansions.

LECTURE 8: INTRODUCTION TO STURM-LIOUVILLE PROBLEMS

COMPLETE ORTHOGONAL BASES

We have seen that piecewise smooth functions $f(x)$ can be represented as an infinite sum of cosines and sines, and that cosines and sines are mutually orthogonal, i.e. the set

$$\{1, cos\frac{\pi x}{l}, cos\frac{2\pi x}{l}, \ldots, sin\frac{\pi x}{l}, sin\frac{2\pi x}{l}, \ldots, \}$$

forms a *complete orthogonal basis for the space of piecewise smooth functions* (see Lectures 4 and 6 for definitions).

HOW MIGHT THESE ARISE?

We saw that these *arose naturally in the context of solving the ODE for the spatial component* $X(x)$ of our separated solution along with associated BC.

We have touched on the fact that there are other possible orthogonal bases for function spaces. *Maybe we can find others through similar means?*

SIDE NOTE - WHY ARE WE FOCUSING ON BASES FOR THE *SPACE* PART?

Basically: the idea of a system *'state'* is useful. This is usually how *dynamical systems* are formulated.

Informally the state is *all the information required at a single moment in time* to evolve the system one small time step into the future.

Remember how we evolved a solution from an initial condition (spatial field) to a future spatial state.

STURM-LIOUVILLE PROBLEMS

A (regular) Sturm-Liouville problem is a combination of a linear homogeneous second-order *ODE* for $y(x)$

$$(p(x)y')' + q(x)y + \lambda\omega(x)y = 0, \, x \in (a, b)$$

and homogeneous *boundary conditions* of the form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

where...

STURM-LIOUVILLE PROBLEMS

...

- a and b are *finite*,
- $q, \omega\ p$ and p' are *continuous* functions on $x \in [a, b]$,
- $p(x) > 0$ and $\omega(x) > 0$ on $[a, b]$, i.e. are *positive*
- λ is a *constant* (and is a free parameter, i.e., not specified/is to be determined)
- α_1 and α_2 are *not both zero*, β_1 and β_2 are *not both zero* and
- $a, b, p(x), q(x), \omega(x), \alpha_1, \alpha_2, \beta_1, \beta_2$ are *all real*.

(we can also consider *singular* cases where these fail to hold)

OPERATOR NOTATION

The Sturm-Liouville problem can be written compactly in *operator notation* as

$$Ay := -\frac{1}{\omega(x)}[(p(x)y')' + q(x)y] = \lambda y$$

subject to

$$B_1 y(a) := \alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$B_2 y(b) := \beta_1 y(b) + \beta_2 y'(b) = 0$$

The combination $\{Ay, B_1 y(a), B_2 y(b)\}$ is sometimes (even more) compactly denoted by Ly, i.e. L *includes the BC*.

EIGENVALUES AND EIGENFUNCTIONS

We are interested in finding values of λ for which a Sturm-Liouville problem (SLP) has a *non-trivial solution*.

A value of λ for which there is a non-trivial solution is called an *eigenvalue* of the SLP and the corresponding non-trivial solution is called an *eigenfunction* of the SLP.

Example 1: Show that the BVP

$$y'' + \lambda y = 0, \quad 0 < x < L$$
$$y'(0) = 0, \ \ y(L) = 0$$

is a SLP. Find its eigenvalues and eigenfunctions.

WHERE DOES THIS COME FROM? RECALL:

- If A is *symmetric* (or *Hermitian* in the complex case) matrix then the *eigenvalues are real* and the *eigenvectors* corresponding to distinct eigenvalues are *orthogonal*.
- If an $n \times n$ matrix A is symmetric (or Hermitian in the complex case) then its eigenvectors *form an orthogonal basis* for $\mathbb{R}^n$ (or $\mathbb{C}^n$ in the complex case).

We will see that Sturm-Liouville problems are *infinite-dimensional analogues* of finite-dimensional eigenvalue problems for symmetric/Hermitian matrices.

WEIGHTED INNER PRODUCT

Here we will use the *inner product* $\langle f, g \rangle$ between two (real) functions f and g defined by

$$\langle f, g \rangle := \int_a^b f(x)g(x)\omega(x)dx$$

where $\omega(x)$ is the *weight function from the SLP* of interest.

Then, if $\langle f, g \rangle = 0$ we say f and g are *orthogonal* (as before)

STURM-LIOUVILLE THEOREM

Let λ_n and $\phi_n(x)$ be any eigenvalue and corresponding eigenfunction of the Sturm-Liouville problem defined earlier.

Then...

STURM-LIOUVILLE THEOREM

- The eigenvalues are all *real*.
- The eigenvalues are *simple*, i.e., to each eigenvalue there corresponds just one linearly independent eigenfunction.
- There are *infinitely many eigenvalues*, and they can be *ordered* so that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.
- Eigenfunctions corresponding to different eigenvalues are *orthogonal*, i.e., if $\lambda_n \neq \lambda_n$ then $\langle \phi_n, \phi_m \rangle = 0$.

and...

STURM-LIOUVILLE THEOREM

... Let f be *piecewise smooth* on $[a, b]$. Then if $a_n = \langle f, \phi_n \rangle / \langle \phi_n, \phi_n \rangle$ the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

converges to $(f(x+) + f(x^-))/2$ at each point $x \in (a, b)$.

FOURIER SERIES

The theorem tells us:

- The eigenfunctions of a SLP defined on $[a, b]$ *form an orthogonal basis* for the vector space PS$[a, b]$, the set of piecewise smooth functions defined on $[a, b]$.
- At each point x at which f is continuous,
$$f(x) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(x)$$
- The Fourier series we have seen so far are thus special cases of this type of eigenfunction expansion (see also Lecture 6 - Note on generalised Fourier series).

THEOREM: NON-NEGATIVE EIGENVALUES

If $q(x) \leq 0$ on $[a, b]$ and $[p(x)\phi_n(x)\phi_n'(x)]_a^b \leq 0$ for the eigenfunction $\phi_n(x)$, then λ_n is *non-negative*.

(We already know λ_n is real from the SL theorem).

Example 2: Use the theorem to show that the eigenvalues for the BVP

$$y'' + \lambda y = 0, \quad 0 < x < L$$

$$y(0) = 0, \; y(L) = 0$$

must be non-negative.

ANOTHER EXAMPLE

Example 3: Find the eigenvalues and eigenfunctions for the SLP

$$y'' + \lambda y = 0, \quad 0 < x < \pi$$

$$y(0) = 0, \; y'(\pi) = 0$$

and hence work out the eigenfunction expansion for the function $f(x) = 50$ for $x \in [0, \pi]$.

HOMEWORK

- What's the relation of our example eigenfunction expansions to the half-range sine series
- Using the eigenvalues and eigenfunctions from Example 3, work out the eigenfunction expansion for the function

$$g(x) = \begin{cases} 0, & 0 \le x \le \pi/2 \\ 50, & \pi/2 < x \le \pi. \end{cases}$$