MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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EXAMPLE CLASSIFICATIONS REVISITED (FROM LECTURE 1)

$$Lu = u_t - ku_{xx} = 0$$

with k constant.

$$L(u + w) =?$$
$$L(au) =?$$

$$Lu = uu_t + cu_x + 2txu = sin(tx) = f$$

with c constant.

$$L(u + w) =?$$
$$L(au) =?$$

NOTATION FOR NONLINEAR OPERATORS REVISITED (FROM LECTURE 1)

Note that the second case is *nonlinear*.

For general nonlinear operators we *won't be able to factorise* L *so nicely*.

We have to be content with describing L via its *action on* u, e.g. something like $L : u \to u^2 u_{tt}$ (say).

LECTURE 2

Four ways of deriving PDEs

Localisation

Example: The Heat Equation

The extra ingredient: constitutive equations

HOW DO YOU DERIVE A PDE?

I can think of (at least) *four* ways...

...they are equivalent for simple problems but allow different generalisations...

GENERAL PRINCIPLES: THE FOURFOLD WAY?

1. Integral balances
2. Differential balances
3. Weak forms/virtual balances
4. Variational/extremum principles

We will mainly look at derivations using the first two approaches, but ...*

* ...the second two are very important in *physics* (e.g. classical, continuum and quantum mechanics problems) as well as for constructing *numerical* schemes (weighted residuals, FEM)

GENERAL BALANCE SCHEME

{ Change in the total amount of a quantity in a given region over a given time interval }

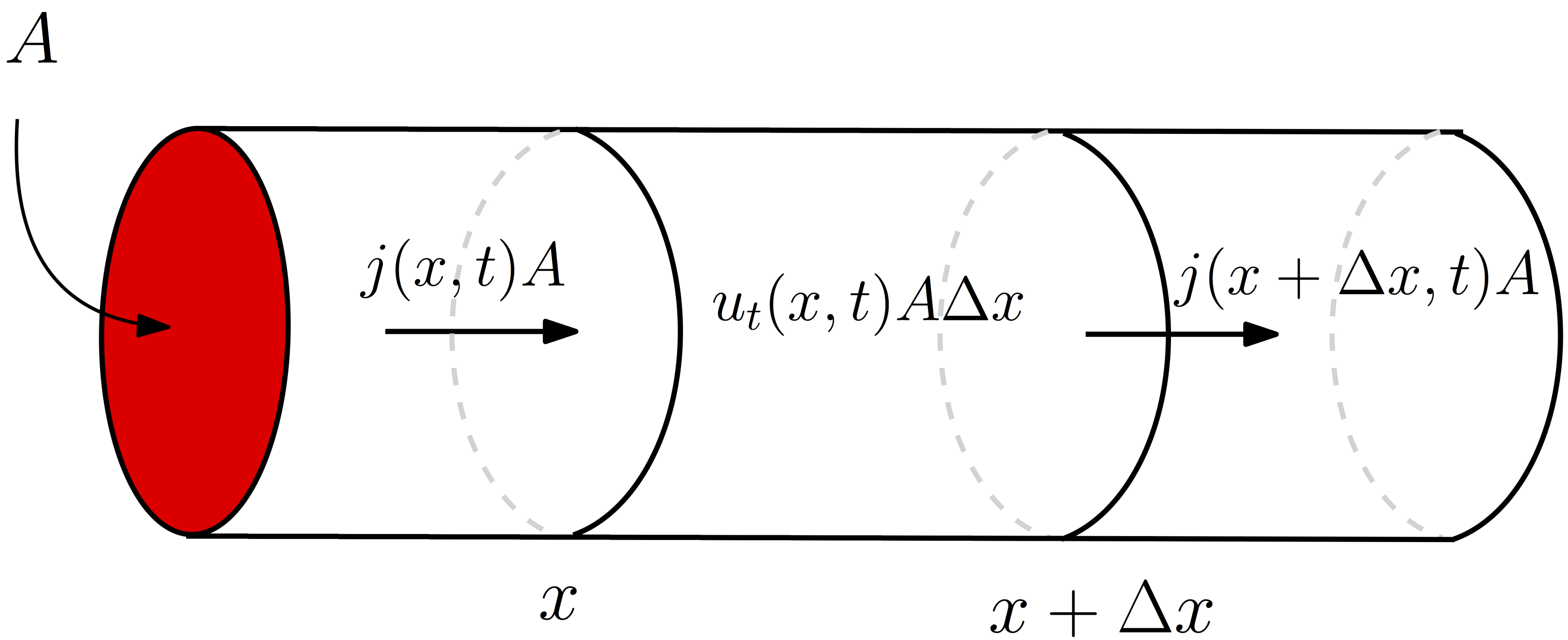
=

{ Net transport of the quantity into the region through its boundaries during the time interval }

+

{ Net production of the quantity within the region during the time interval }

A TYPICAL PICTURE



BUT FIRST, LET'S TRY CONVERTING THE WORDS TO EQUATIONS!

(Pictures and words are both very important for modelling/problem solving...but so are equations!)

GENERAL BALANCE SCHEME AGAIN

{ Change in the total amount of a quantity in a given region over a given time interval }

$$=$$

{ Net transport of the quantity into the region through its boundaries during the time interval }

$$+$$

{ Net production of the quantity within the region during the time interval }

INTEGRAL VERSION OF THE BALANCE SCHEME

$$\int_a^{a+\Delta a} u(x, t+\Delta t)dx - \int_a^{a+\Delta a} u(x,t)dx$$

$$=$$

$$\int_t^{t+\Delta t} j(a,t)dt - \int_t^{t+\Delta t} j(a+\Delta a, t)dt$$

$$+$$

$$\int_t^{t+\Delta t}\int_a^{a+\Delta a} f(x,t)dxdt$$

where Δt and Δa are arbitray *finite* (and independent) 1-D* displacements in time/space.

* Exercise: try to generalise this to multiple space dimensions. At least Google it!

PER UNIT WHAT?

In this case

- $u(x, t)$ is a *stored* amount of substance *per unit length**
- $j(x, t)$ is a *transported* amount of substance *per unit time* (*flux*)
- $f(x, t)$ is a *source/added amount* of substance *per unit length, per unit time*

* Or per unit volume, mass etc in general, depending on the specific problem. This is usually called a *density*, e.g. mass density, energy density etc

DIFFERENTIAL-INTEGRAL VERSION OF THE BALANCE SCHEME

$$\frac{d}{dt}\int_{a}^{a+\Delta a} u(x,t)dx = j(a,t) - j(a+\Delta a,t) + \int_{a}^{a+\Delta a} f(x,t)dx$$

Rate of change in the total amount = Net rate of transport + Net rate of production

DIFFERENTIAL-DIFFERENTIAL VERSION OF THE BALANCE SCHEME?

One way of further 'localising' our equation to a point in time and a point in space is using

$$\frac{d}{dt}\int_a^{a+\Delta a} u(x,t)dx \approx \frac{d}{dt}(u(a,t)\Delta a) = \frac{\partial u(a,t)}{\partial t}\Delta a$$

$$j(a,t) - j(a+\Delta a,t) \approx -\frac{\partial}{\partial x}j(a,t)\Delta a$$

$$\int_a^{a+\Delta a} f(x,t)dx \approx f(a,t)\Delta a$$

...which are valid to order $O((\Delta a)^2)$...

DIFFERENTIAL VERSION OF THE BALANCE SCHEME

...and, in the limit, leads to

$$\frac{\partial u(a,t)}{\partial t} = -\frac{\partial j(a,t)}{\partial x} + f(a,t)$$

or,

more compactly

$$u_t + j_x = f$$

This can be also taken as a basic modelling starting point, *as long as the required requlatity conditions* are kept in mind.

LOCALISATION DETOUR

The question of the validity of the passage from the differential-integral form to the fully localised version arises suprisingly often in *real applications*.

Difficulties often arise when the required *regularity assumptions break down* (e.g. shock waves).

See 'The Weird and Beautiful World of Fluids' at http://www.wired.com/2011/06/weird-world-of-fluids/

LOCALISATION DETOUR

Some basic *theorems** used to analyse this more carefully are:

1. The fundamental theorem of calculus/divergence theorem/generalised Stokes' theorem
2. The Leibniz–Reynolds transport theorem, and
3. The fundamental lemma of the calculus of variations/localisation theorem

* *We (might) come across these in more detail in a tutorial/assignment/exam! (1) is about relating surface integrals to volume integrals, (2) is about integrating over time-varying spatial regions and (3) is about relating a vanishing integral to a vanishing integrand.*

LOCALISATION DETOUR

Alternatively, we can

- Use the *integral form* as our basic equation, or
- *Re-interpret the PDE* itself in a more general manner

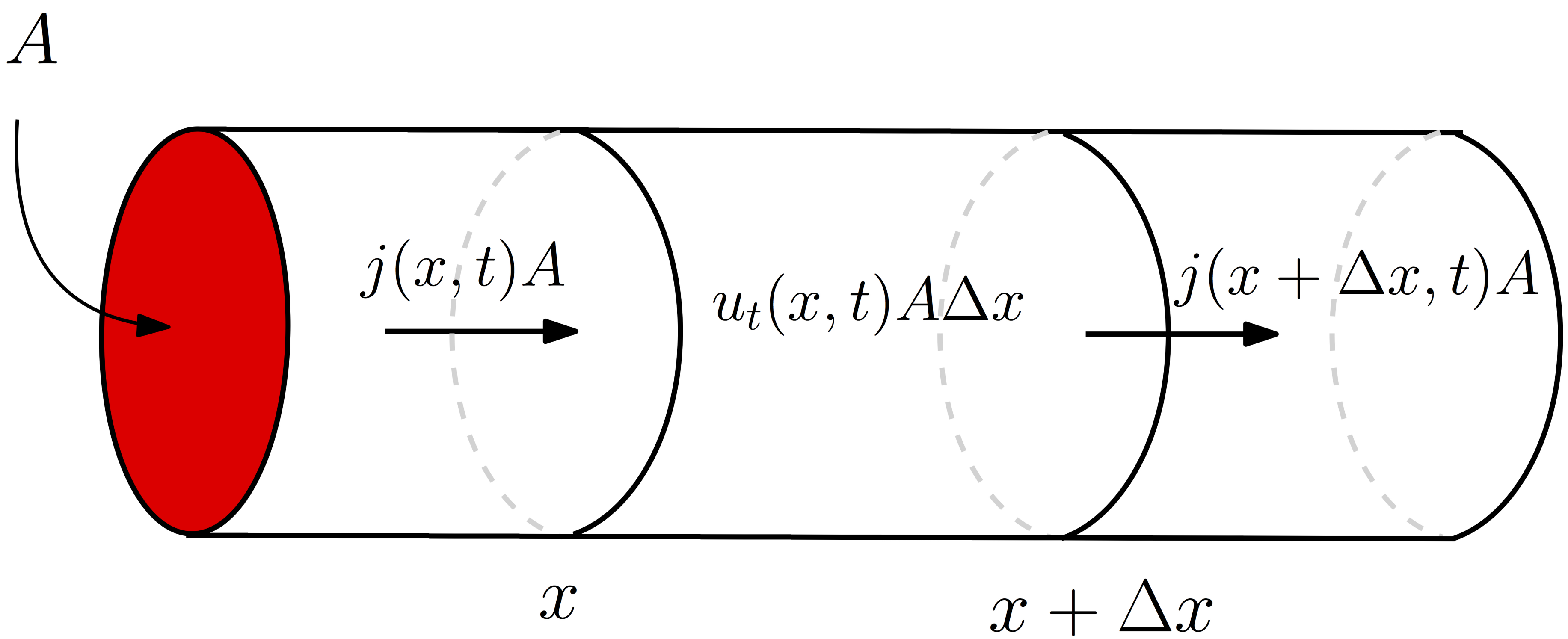
These ideas lead to the notions of

- Weak solution
- Distributional solution

There are a many *types of generalised solution* * ...you will encounter some in the second-half of the course!

* See e.g. [1] www.math.ucla.edu/~tao/preprints/generalized_solutions.pdf and [2] http://www.math.umn.edu/~garrett/m/fun/dangerous_and_illegal.pdf

HEAT EQUATION



Uniform rod with constant cross-section A

HEAT EQUATION - BALANCE LAW

Balance of what? Energy!

- Stored/internal energy $u(x, t)$ at x, t
- Heat flux (rate of energy transfer due to heat flow) $j(x, t)$ at x, t. Using these in the balance law gives*, for small Δx,

$$u_t(x, t)A\Delta x = j(x, t)A - j(x + \Delta x)A$$

And, as $\Delta x \to 0$ we get

$$\to u_t = -j_x$$

* Note here we are working per unit volume but since A is uniform it cancels and we may as well be working per unit length.

CLOSURE: CONSTITUTIVE EQUATIONS

Our *general* conservation scheme is *generally* applicable: energy is conserved

How do we distinguish between say a material that is a *conductor* vs. one that is an *insulator*?

The answer is we need *substance-specific constitutive equations*

This will also give us a *closed set of equations* (loosely, the same number of equations as unknowns)

CLOSURE/CONSTITUTIVE ASSUMPTIONS

1. Internal energy is a linear increasing function of temperature Θ for all x, t

$$u(x,t) = u(\Theta(x,t))$$

and

$$u(\Theta + \Delta\Theta) \approx u(\Theta) + C\Delta\Theta$$

Where C is the *heat capacity** ...

* *Units*/dimensions are a crucial way to check your equations make sense. If u is measured in Joules/Metre J/m (since *here* u is internal energy *per unit length of bar*) and Θ in Kelvin (K) then here C must have units $J/(Km)$. We can similarly define u and hence C per unit volume or per unit mass, depending on the problem of interest.

…E.g. for changes in time:

$$u(\Theta(x, t + \Delta t)) \approx u(\Theta(x, t)) + C\,(\Theta(x, t + \Delta t) - \Theta(x, t))$$

So

$$u_t \approx C\Theta_t$$

CLOSURE/CONSTITUTIVE ASSUMPTIONS

2. Heat flows from hot to cold: down *temperature gradients* Θ_x (Fourier*)

$$j = -k\frac{\partial \Theta}{\partial x}$$

where k is the thermal conductivity** (and will be assumed constant in this case)

* This can also be justified based on the 2nd law of thermodynamics, which in the present context can be thought of as a *general constraint on constitutive equations/closure assumptions*.

** Exercise: What are units and/or dimensions of k?

CLOSURE

So for the particular assumptions given we find the dependencies

$$u_t = u_t(\Theta_t)$$

and

$$j_x = j_x(\Theta_{xx})$$

...

CLOSURE

...substituting these into the energy conservation equation gives the Heat/Diffusion Equation for the spatiotemporal variation in *temperature*

$$\frac{\partial \Theta}{\partial t} = \frac{k}{C} \frac{\partial^2 \Theta}{\partial x^2}$$

i.e.

$$\Theta_t - D\Theta_{xx} = 0$$

where $D = \frac{k}{C}$ is the diffusion coefficeint*

* Exercise again: What are units and/or dimensions of D?

WAIT, CLOSURE? INITIAL/BOUNDARY CONDITIONS

Our final (*balance + constitutive*) differential equation now describes what happens *internally** to our space-time modelling domain

$$(x, t) \in \Omega \times T \subset \mathbb{R}^n \times \mathbb{R}^+$$

We also want to 'close' our *overall domain* of interest; we need to specify the *boundaries* of this domain.

* i.e. for all open subsets of our domain of interest

LET'S TRY TO FORMALISE THIS A LITTLE

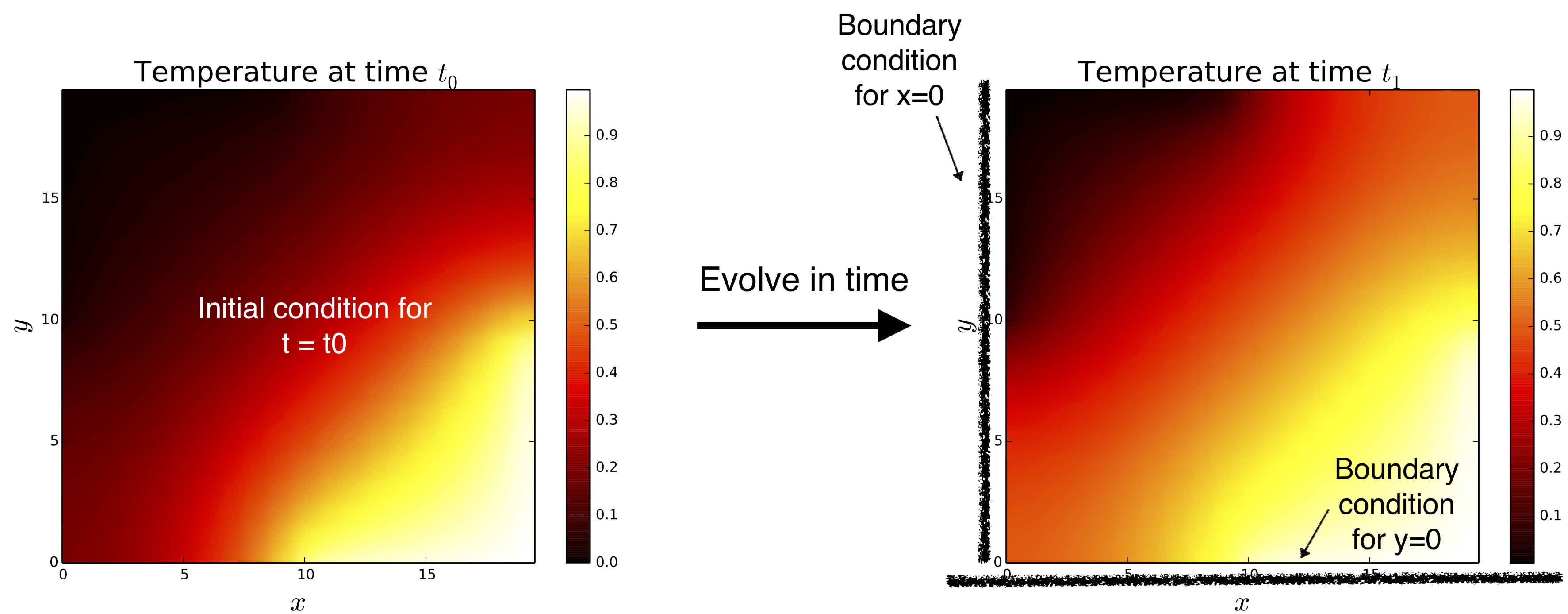
We can think of the boundary of our overall domain as something like

$$\partial\left(\Omega \times T\right) = \partial\Omega \times T + \Omega \times \partial T$$

= {spatial boundary for all time} + {time boundary for all space}

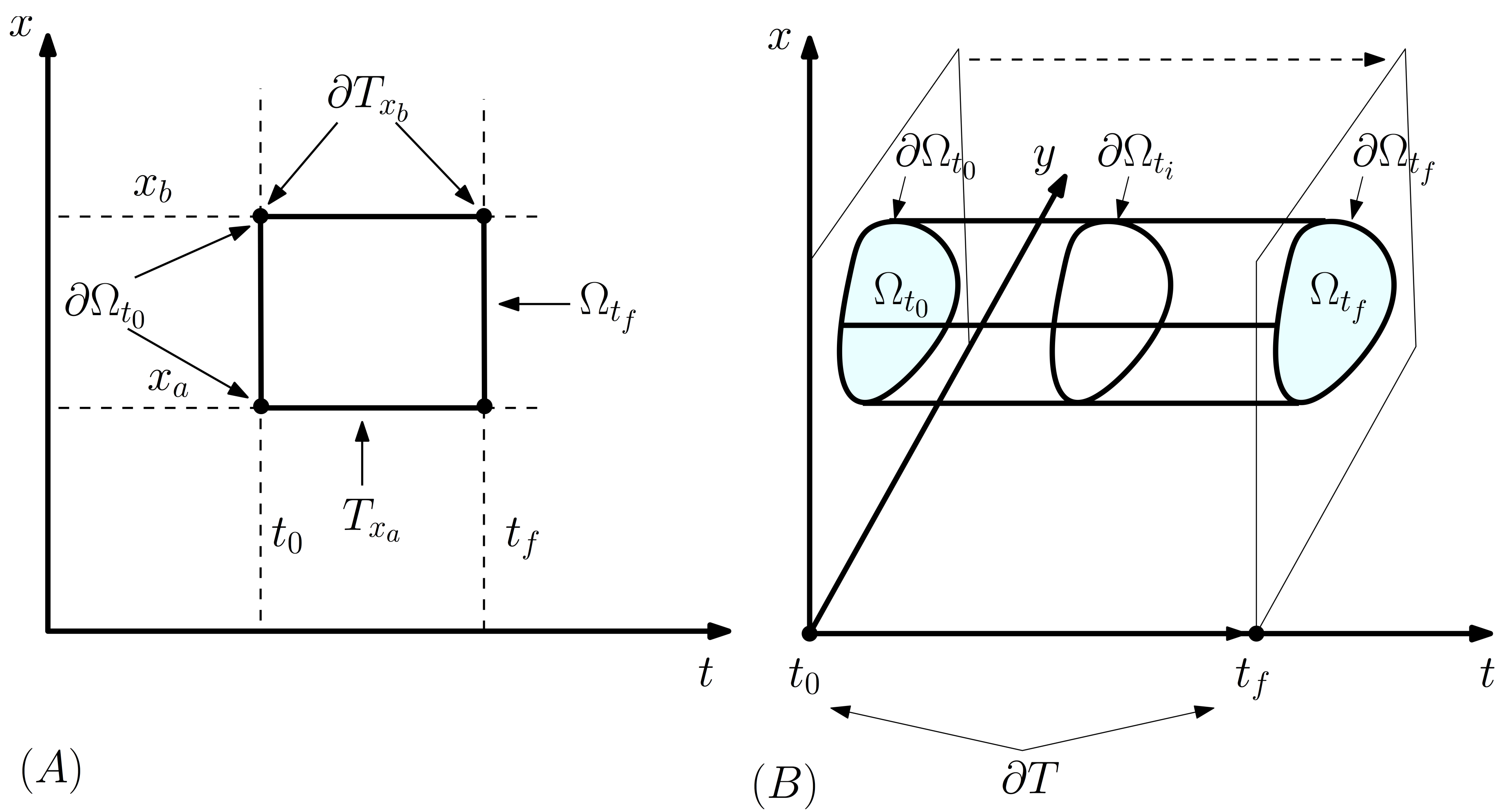
REMEMBER OUR SIMULATION

Recall:



Spatiotemporal temperature field $u(x, y, t)$

MORE FORMAL PICTURES



(A) IC/BC in 1 space and 1 time dimension; (B) IC/BC in 2 space and 1 time dimension

TYPES OF IC/BC

Generally speaking, a necessary condition for setting up a well-posed problem is

- The same number of IC as the highest-order time derivative
- The same number of BC as the highest-order spatial derivative

TYPES OF IC

Initial conditions are just arbitrary profiles over the whole spatial domain specified at the initial time, e.g.

$$u(x, t = 0) = f(x)$$

for

$$x_a \leq x \leq x_b$$

We might measure the profile at time zero and hope to predict it at some later time.

TYPES OF BC

Boundary conditions apply at given spatial locations for all time; they are often given fancier names

Dirichlet conditions specify the *value* of a variable at a location, e.g.

$$u(x = x_a, t) = u_a(t)$$

These are externally imposed 'clamps' on the variable values - e.g. insulated $u_a = 0$ or perodic forcing $u_a(t) = sin(t)$

TYPES OF BC

(von) Neumann boundary conditions specify the *derivative* of a variable at a location, e.g.

$$u_x(x = x_a, t) = q(t)$$

Robin conditions specify a combination of the *value* of a variable and its *derivative*, e.g.

$$\alpha u(x = x_a, t) + \beta u_x(x = x_a, t) = g(t)$$

for arbitrary constants α, β

Mixed boundary conditions refers to having *different types* of BC at *different locations* (c.f. Robin)

TYPES OF BC

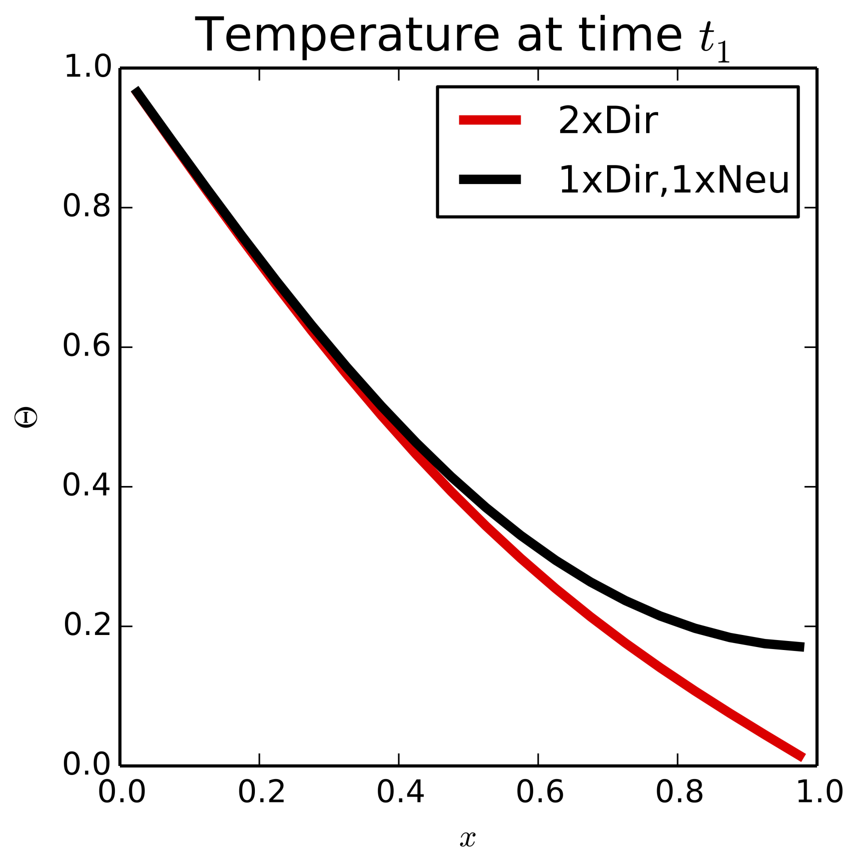
von Neumann and Dirichlet BCs are often derived from combined assumptions on the *fluxes & constitutive* equations for the fluxes/variables at the boundary.

E.g. [*insulated* ($j = 0$) & *Fourier's law* ($j \propto u_x$)] at $x = x_a$

implies

$$u_x(x = x_a, t) = 0$$

SIMULATIONS



$$\Theta(x=0,t)=1;\, \Theta(x=1,t)=0$$
$$\Theta(x=0,t)=1;\, \Theta_x(x=1,t)=0$$

~~SUMMARY~~ HOMEWORK!

- Read Chapter 5.4 of Tang on the heat equation
- Look up 'Newton's law of cooling'. What type of boundary condition would this be?
- What would the heat equation look like if we still used Fourier's law but now allowed the conductivity to be a function of x, i.e. $k(x)$?
- Extra prep: take a look at 5.1.2 then 5.5 of Tang (separation of variables - for next time). Or Google it/go to the library.