Engsci 711

Assignment 2

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Due: Friday 16 June (*online only!*)

Question 1

Consider the system

$$\begin{aligned}\dot{x} &= x^2y - x^5\\ \dot{y} &= -y + 2x^2\end{aligned}$$

where $x, y \in \mathbb{R}$.

- Verify that the origin is a fixed point of this system.
- Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$.
- Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not (yet) need to show any nearby trajectories.
- Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0,0)$ that is correct up to and including quartic order.
- Use the previous expression to determine the dominant dynamics on the centre manifold, again correct up to and including quartic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.
- Sketch the local phase portrait of the system near the equilibrium.

Question 2

Consider the system

$$\begin{aligned}\dot{x_1} &= x_1y - x_1x_2^2\\ \dot{x_2} &= x_2y - x_2x_1^2\\ \dot{y} &= -y + x_1^2 + x_2^2\end{aligned}$$

where $x_1, x_2, y \in \mathbb{R}$.

- Verify that the origin is a fixed point of this system.
- Find the Jacobian derivative - first as a function of x_1, x_2 and y and then evaluated at the origin $(0,0,0)$.

- Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively.

Hint: one of your spaces will be two-dimensional and will also correspond to a repeated root. In this particular case, however, you should be able to find two independent eigenvectors that span this space simply by sensibly choosing the values of two free variables occuring in your usual eigenvector equations. That is, you do not need to use generalised eigenvectors here.

- Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0,0,0)$ that is correct up to and including quadratic order.

Note: If your centre manifold is two-dimensional then you need to expand in two variables. An example of expanding an arbitrary two-variable function to quadratic order is

$$f(u,v) \approx a + bu + cv + du^2 + euv + fv^2$$

where $a, b, ..., f$ are constants. Note that a cross term like uv in the above is considered quadratic order.

- Use your expression for the centre manifold to determine the dynamics on the centre manifold, again correct up to and including quadratic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.
- Would you have gotten the same answer for this particular example if you used the linear centre manifold $E^c(0,0,0)$? Justify your answer.

Question 3

Carry out a bifurcation analysis for the system

$$\dot{x} = (x-1)(\mu - x - x^2)$$

Remember to say what types of bifurcation occur. Note - you do not need to use extended centre manifold theory for bifurcation questions!

Question 4 (Exam 2016)

Determine the equilibria and their stability, and hence find all bifurcations, in the following system

$$\dot{x} = (\lambda - b)x - ax^3$$

where $x \in \mathbb{R}$, $a, b > 0$ are fixed positive parameters and $\lambda \in \mathbb{R}$ is a controllable parameter. Sketch the bifurcation diagram.

(Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$, and $\lambda > b$ separately.)