ENGSCI 213: MATHEMATICAL MODELLING 2SE

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

MODULE OVERVIEW

Introduction to Probability (*Oliver Maclaren*) [9 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts of probability. Sets and subsets, sample spaces and events. Probability and counting, conditional probability, independence, Bayes' theorem. Random variables. Simple data structures for probability calculations.

2. *Discrete probability models* [3 lectures]

   Discrete random variables and discrete probability distributions. Probability mass functions and cumulative distribution functions. Binomial and Poisson distributions.

3. *Continuous probability models* [3 lectures]

   Continuous random variables and continuous probability distributions. Probability density functions and cumulative distribution functions. Exponential and Normal distributions.

RECAP: KOLMOGOROV

Formal probability theory consists of three ingredients

1. A *sample space S* - for the *set* of all *possible outcomes* of an uncertain experiment
2. A *collection Σ of events E* - each event is a *subset* of the sample space S, i.e. $E \subseteq S$, and an *element* of the collection of events Σ, i.e. $E \in \Sigma$
3. A *probability function P* (or 'measure') - this assigns a *probability* $P(E)$ to each event E (subset of the sample space)

RECAP: EVENTS AS DEFINED BY PARTIAL INFORMATION

outcome ↔ the 'individual' defined by the most-detailed description

event ↔ the set (of individuals) determined by a partial description

sample space ↔ the set (of individuals) determined by the least-detailed description

NOTE: PROBABILITY AS AN INTERMEDIATE-LEVEL DESCRIPTION

What should I include in the sample space? Err on the side of *noting down as much as you think is relevant at first*.

> *We can always drop some - and/or include more - 'background' information!*

i.e. we work at an *intermediate* level of description. Everything we don't note down is effectively set to an arbitrary value and *assumed irrelevant (for now)*.

LECTURE 2

The third model ingredient: probability functions

Conditional probability

Applications and interpretations

FORMAL MODEL INGREDIENT THREE: PROBABILITY FUNCTIONS

Our model is

$$(S, \Sigma, P)$$

- S is our underlying set of outcomes
- Σ is our set of events
- P is our 'probability function'

We want to talk about this last object today.

BASIC REQUIREMENTS

'Within' a probability *model* we have a probability *function* which assigns probability *numbers* (to events):

- Any probability is a number between 0 and 1.
- All possible outcomes together must have probability 1.
- If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities.

PROBABILITY AXIOMS (KOLMOGOROV)

This leads to

The probability $P(A)$ of any event A satisfies $0 \leq P(A) \leq 1$

If S is the sample space in a probability model, then $P(S) = 1$

If A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$

CONDITIONAL PROBABILITY (KOLMOGOROV)

Conditional probability is then defined by

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) > 0}$$

But isn't all probability 'conditional on something'??

CONDITIONAL PROBABILITY AS THE BASIC CONCEPT

Jaynes (for example) starts from conditional probability and emphasises that the *sum and product rules* (for any events A, B, C) are fundamental

$$\boxed{P(A|C) + P(\overline{A}|C) = 1}$$

$$\boxed{\begin{aligned} P(A \cap B|C) &= P(A|B \cap C)P(B|C) \\ &= P(B|A \cap C)P(A|C) \end{aligned}}$$

CONSEQUENCES OF THE SUM AND PRODUCT RULES

Some useful consequences of the sum and product rules include the *extended sum rule*:

> *If A and B are mutually exclusive events then $P(A \cup B|C) = P(A|C) + P(B|C)$*

(or

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

for arbitrary events) as well as...

CONSEQUENCES OF THE SUM AND PRODUCT RULES

...the *partition theorem*:

If $B_1, \ldots, B_k$ *form a partition of* C *then*

$$P(A|C) = \sum_{i=1}^{k} P(A \cap B_i|C) \text{ i.e.}$$

$$P(A|C) = \sum_{i=1}^{k} P(A|B_i \cap C)P(B_i|C)$$

CONSEQUENCES OF THE SUM AND PRODUCT RULES

Bayes' theorem is a simple consequence of conditional probability but is conceptually important as it allows us to *'invert'* a given conditioning

$$\boxed{P(B|A \cap C) = \frac{P(A|B \cap C)P(B|C)}{P(A|C)}}$$

or, if $B_1, \ldots, B_k$ form a *partition* of C then

$$P(B_j|A \cap C) = \frac{P(A|B_j \cap C)P(B_j|C)}{\sum_{i=1}^{k} P(A|B_i \cap C)P(B_i|C)}$$

CONSEQUENCES OF THE SUM AND PRODUCT RULES

$$P(\varnothing) = 0$$
$$P(\overline{A}) = 1 - P(A)$$

STATISTICAL INDEPENDENCE

We can define *statistical independence* of A relative to B (given C) using either

$$\boxed{P(A|B \cap C) = P(A|C)}$$

or (equivalently)

$$\boxed{P(A \cap B|C) = P(A|C)P(B|C)}$$

DAILY REMINDER

We always operate 'within' a probability model with boundaries given by the 'background assumptions' or 'context'.

RELATION TO KOLMOGOROV APPROACH

- We can think of our 'unconditional' Kolmogorov axioms for a probability *function* as *implicitly conditioned* on $C = S$
- Or we can note that Kolmogorov defines a probability *model* as consisting of three parts (S, Σ, P). Changing one part - e.g. S or Σ - changes our model!
- In particular, Kolmogorov's definition of *conditional* probability $P(A|B)$ is a way of *changing context* from the original sample space S to the new sample space B and then assuming everything works as usual *within the new context*.

USING AND INTERPRETING PROBABILITY FUNCTIONS

Example: discrete probability distribution on a discrete sample space

USING AND INTERPRETING PROBABILITY FUNCTIONS

Example: equally likely outcomes and counting

USING AND INTERPRETING PROBABILITY FUNCTIONS

Example: tables of survery counts, conditional probability and 'conditional counting' with probability trees.

TABLE

Results from survey asking respondents which they value more in a car: ease of parking or style/prestige

	Younger than 40	**Older than 40**	**Total**
Prestige more important than parking	79	51	130
Prestige less important than parking	71	99	170
Total	150	150	300

USING AND INTERPRETING PROBABILITY FUNCTIONS

Example: Bayes' theorem and solving 'inverse problems'

HOMEWORK/CHALLENGES

Finish the example problems

The Monty Hall problem!