ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]
   Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

LECTURE 2 - TERMINOLOGY & PAINTING PICTURES

- Some terminology: definitions, key features of interest etc
- Phase portraits

i.e. a quick tour of the dynamical systems *palette* that we use to 'paint' pictures of system behaviour, called *phase portraits*

SO...WHAT IS A 'DYNAMICAL SYSTEM'?

Informally, a *dynamical system* is a mathematical model of a *process which evolve in time*.

There are *three key ingredients*: a set or interval of *'times'*, possible *'states'* of a 'system' at any given time and an *'evolution rule'* or law governing how the system transititions between these states.

SO...WHAT IS A 'DYNAMICAL SYSTEM'?

Examples are everywhere - ODEs, PDEs, difference equations/maps, stochastic processes even iterative computer algorithms and constructive mathematical proofs.

Our aim is to relate the *'dynamic' point of view* to the *'static' point of view* by looking for *invariant and/or limiting features in state space*

TERMINOLOGY: STATE/PHASE SPACE

In practice the *'state'* is defined by *'contains everything we need to know to get from the current state to the next state'*. E.g. position and momentum for classical mechanics.

The *state space/phase space* is...the 'space' of all states - usually (embedded in) $\mathbb{R}^n$, for real-valued differential equations.

(More general phase spaces include the circle, torus etc, depending on the 'problem structure')

ORDINARY DIFFERENTIAL EQUATIONS

Our main focus is on a very familiar type of dynamical system - systems of ordinary differential equations of the form:

$$\dot{x} = f(x, t; \mu)$$

where $x \in \mathbb{R}^n$ is a vector of *state variables*, $t \in \mathbb{R}$ is the *independent variable* (usually time), $\mu \in \mathbb{R}^m$ is a vector of *problem parameters*.

(For now we often suppress the dependence on problem parameters - but see bifurcation theory!)

ORDINARY DIFFERENTIAL EQUATIONS

We have one equation in f for each entry in the state vector x e.g.

$$x = (x_1, x_2, \ldots)^T$$

$$f = (f_1, f_2, \ldots)^T$$

If there is no dependence on t then we say the system is *autonomous* (we can always add a new dependent variable to track time dependence). We will focus on these in this course.

TERMINOLOGY: SOLUTIONS AND INTEGRAL CURVES

A *solution* x_s (or trajectory) is a *function* assigning a state vector to each time in a given time interval and which satisfies the ODE, i.e. $x_s : T \subset \mathbb{R} \to \mathbb{R}^n$, where $\dot{x_s}(t) = f(x_s(t))$.

An *integral curve* is the *graph* of a solution, including the time dimension, i.e. the *set of points*

$$\{(x, t) \mid t \in T \text{ and } x(t) \text{ defines a solution}\}$$

TERMINOLOGY: FLOW FUNCTIONS

The *flow function* ϕ is a convenient way to combine our description of *solutions and their dependence on initial conditions*.

We write $\phi(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

where for fixed x_0, $\phi(x_0, t)$ gives the solution to the differential equation at time t which starts from an initial value (at $t = 0$) equal to x_0.

TERMINOLOGY: FLOW FUNCTIONS

So, for any t such that $\phi(x, 0) = x$ we have

$$\frac{d}{dt}\phi(x, t) = f(\phi(x, t))$$

and

$$\phi(x, t + s) = \phi(\phi(x, t), s) = \phi(\phi(x, s), t) = \phi(x, s + t)$$

(for any 'allowed' t, s). We talk about 'flows' when we want to emphasise the dependence on initial conditions (rather than just time)

TERMINOLOGY: ORBITS

Orbits are the geometric objects in the phase space that are generated by solutions/flows.

In terms of the flow function an orbit beginning at x_0 can be described by $\{\phi(x_0, t) \mid t \in T\}$.

Usually we take $T = \mathbb{R}$ and hence consider all solutions passing through x_0 (and both forwards and backwards in time *if* invertible!).

These can be labelled with a 'time direction' but are otherwise somewhat 'static' (geometric) objects.

TERMINOLOGY: VECTOR FIELD

The solutions/orbits are *tangent to the 'velocity vector'* $(\dot{x_1}, \dot{x_2}, ...)^T$ defined by the ODE $\dot{x} = f(x)$ at each point in the state space (and at each time).

We often call $f(x)$ the *vector field* of the ODE. The *direction* (relative velocity of components) can be determined by dividing through by one (non-zero) component i.e.

$$\frac{\dot{x_k}}{\dot{x_1}} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, ..., x_n)}{f_1(x_1, ..., x_n)}$$

(Now how this relates the 'static' and 'dynamic' objects).

EXISTENCE AND UNIQUENESS?

We won't go into this, but for a sufficiently smooth system written in our standard form and given appropriate initial conditions *there exists a unique solution*.

Upshot: the solution curves/trajectories (for autonomous systems) *do not intersect in phase space*. We always know exactly where to go next!

TERMINOLOGY: PHASE PORTRAITS

We will summarise these key features in *phase portraits* of a given system.

A phase portrait is a 'picture' of the phase space in which we further *partition it according to orbit/solution 'types' or behaviour* in different regions.

FEATURES OF INTEREST

We will look at (and define!) various *'interesting features'* of our equations in phase space e.g.

- Stationary/fixed/equilibrium points
- Periodic orbits

and try to analyse their properties such as *stability* under different types of 'perturbations' - both 'within' a model (*solution* stability) and 'externally' (*structural* stability) to a model.

INTERESTING FEATURES - EQUILIBRIA

A point x_e is an *equilibrium solution/fixed point/stationary point/singular point* iff

$$\phi(x_e, t) = x_e$$

for all t. Equivalently it is a *zero of the vector field* (RHS)

$$f(x_e) = 0$$

INTERESTING FEATURES - NULLCLINES

Given a system of equations for $x \in \mathbb{R}^n$ with components $\dot{x}_i = f_i(x)$, the jth *nullcline* is where

$$\dot{x}_j = f_j(x) = 0$$

The flow is *perpendicular* to the x_j-axis along the associated curves/surfaces (usually of dim $n - 1$). Note: there *may be multiple lines/curves for one nullcline!*

Q: What are points where *all* nullclines intersect called?

Nullclines are a very useful part of sketching phase-plane portraits!

MORE FEATURES - PERIODIC POINTS AND PERIODIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t + T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t + s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

MORE FEATURES: INVARIANT SETS

A set of points in the phase space M is called *invariant under the flow* if for all $x \in M$ we have

$$\phi(x, t) \in M$$

for all t. That is, every point in M leads to another point in M - once in, we never leave!

MORE FEATURES - LIMIT SETS

Other useful definitions include the following (invariant!) sets:

The *ω-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in forward time*.

Formally it consists of elements y such that there exists a sequence (t_n) with $t_n \to \infty$ and $\phi(x, t_n) \to y$ as $n \to \infty$.

MORE FEATURES - LIMIT SETS

The *α-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in backwards time* (exercise: write down the formal definition!)

Note that the points in these sets *don't have to lie on the orbits* through x - they are *limit* points for a reason!

EngSci 711 L02.

- Terminology & 'Painting Pictures'

  Goals: • know what I'm talking about – equilibria, nullclines, phase portraits etc.

  • write ODEs in state-space form

Example questions:

- Exam 2016:

(b) Consider the second-order equation

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a system parameter.

(i). Re-write the above equation as system of two first-order equations.

- Exam 2017: (Notice the terminology in particular, as well as steps of analysis)

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

Painting pictures

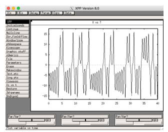
- Our goal is going to be to take a 'dynamical system' that 'evolves in time'

  – eg a system of ODEs

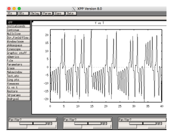
  & 'paint a geometric/qualitative picture' of this system in state space (phase space)

- eg Lorenz system: from L1.

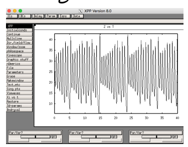
x vs t



y vs t

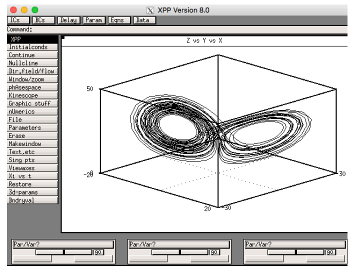


z vs t



State space/Phase space

x vs y vs z



a 'picture'

Specifically:

Analysis Procedure (basic)

- Find fixed points
- Analyse (linear) stability
- Classify fixed points
- Find other 'global' features
  - L eg periodic orbits

'portrait' {
- Sketch these + interesting trajectories }
- Analyse cases of marginal stability
- Use perturbation methods to construct approximate sol^ns

etc.

→ First, we should probably establish some formal terminology

(not about memorising
→ so we are on same page!

What is a 'dynamical system'?

→ something that 'evolves in time' from 'state to state'

ingredients:
- sequence of 'times'
- possible 'states'
- evolution/update rule

→ This can be quite abstract!

- eg 'time' can be any natural ordering

Examples: ODEs
PDEs
⋮
Computer programs
Constructive proofs
⋮

Dynamic ⟷ Static:

dyn. ↓ static	'when things change, look for what stays the same & what "limits" the process tends to'
static ↓ dyn.	'Given some (implicitly defined) "static" object, think about how to define a procedure or sequence that can construct/converge to this object'

State & State space / Phase space

- Informally, the system 'state' is defined as 'everything you need to know to get from the current state to the next state using the update rule' — enough to define a first order in time, deterministic evolution rule

  → a bit circular!

  → best understood via examples etc.

- the state space/phase space is the set of all possible states

  eg $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$

Examples:

- In classical mechanics, knowing the position & momentum at time t is enough to tell you the position & momentum at time $t+dt$, using Newton's law

- In quantum mechanics you need to know the wave function

  → although this defines a probability distribution over observables, the wave function itself evolves deterministically (state ≠ observable)

  ↳ (Schrodinger eqn.)

ODEs

We'll focus on systems of ODEs in the form

$$\underbrace{\dot{x}}_{\text{first order in time}} = f(\underbrace{x}_{\text{state vector, } x\in\mathbb{R}^n}, \underbrace{t}_{\text{time}\in\mathbb{R}} ; \underbrace{\mu}_{\text{problem parameters}})$$

ie

$$\underset{\text{vector}}{\dot{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(\overbrace{x_1, x_2, \ldots}^{x \text{ vector}}) \\ f_2(\ldots) \\ \vdots \\ f_n(\ldots) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \underset{\text{vector}}{f}$$

- parameter?
  - ↳ held fixed during evolution
  - ↳ eg rate constant, diffusion constant etc
  - ↳ later, we'll sometimes imagine as super slow state vars, ie $\dot{\mu} = 0 \Rightarrow \mu = \text{const.}$
  - ↳ for now we'll most ignore explicit dep. on param.

- time dependence?
  - ↳ when f independent of t, called autonomous system
  - ↳ can always intro $x_{n+1} = t$ & $\dot{x}_{n+1} = 1$ to convert n dim non-autonomous to $n+1$ dim autonomous system

→ hence we will assume autonomous!

ODE example

→ converting to state-space form (should be familiar!)

Q.

(b) Consider the second-order equation

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a system parameter.

(i). Re-write the above equation as system of two first-order equations.

Exam 2016.

} ie state-space form.

A.

Let $\boxed{\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} = \dot{x}_1 \end{aligned}}$

define new state vars: x & x derivatives up to one less than highest derivative in ODE.

So $\underbrace{\dot{x}_2 = \ddot{x}}_{\text{highest order derivative}} = \underbrace{-\mu\dot{x} - (x - x^3)}_{\text{rest of ODE}}$

$\Rightarrow \dot{x}_2 = \underbrace{-\mu x_2 - (x_1 - x_1^3)}_{\text{in terms of new vars}}$

& $\dot{x}_1 = x_2$ by definition (just relates derivatives/definitions)

$$\Rightarrow \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 - (x_1 - x_1^3) \end{aligned}} \quad \begin{aligned} &\checkmark = f_1(x_1, x_2) \\ &\checkmark = f_2(x_1, x_2) \end{aligned}$$

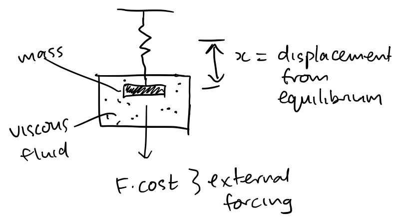
↑ system of first order

Another example

Given

$$\underbrace{m\frac{d^2x}{dt^2}}_{\text{mass x accel}} + \underbrace{b\frac{dx}{dt}}_{\text{damping force}} + \underbrace{kx}_{\text{spring force}} = \underbrace{F\cos t}_{\text{external force}}$$

} Newton's second law



state space?

Let $x_1 = x$, $x_2 = \dot{x}$

$\Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{m}\left[-b\dot{x} - kx + F\cos t\right]$

$= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m}\cos t = \underbrace{f(x_1, x_2, t)}_{\text{non-auton.}}$

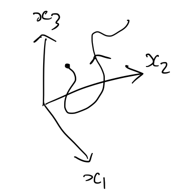
Next, Let $x_3 = t$

$\Rightarrow \dot{x}_3 = 1$

So $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{cases} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m}\cos t \\ 1 \end{cases}$

ie $\frac{d}{dt}(\text{state}) = \text{evolution rule}$

3D state space:



Solutions, integral curves, orbits & vector fields

(variations on a theme; often lazy about distinction...)

- **Solution**: a function $\boxed{t \mapsto \text{state}}$
  (also called: • trajectory • phase curve etc.)

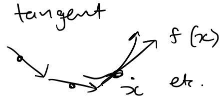
- **Integral curve**: a graph/set of the solⁿ function incl. time $\boxed{\{(x,t) \mid t \in T \,\&\, x(t) \text{ is sol}^n\}}$ — incl. time

- **Orbit**: a set of points in state space generated by a solⁿ $\boxed{\{x \mid t \in T \,\&\, x(t) \text{ a sol}^n\}}$ — don't incl. time explicitly (usually use all times to generate 'static' object)

- **Vector field**: the RHS $f(x)$, → the direction (relative velocities) can be determined via eg
  (also called: direction fields)

$$\boxed{\frac{\dot{x}_k}{\dot{x}_1} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, \ldots x_n)}{f_1(x_1, \ldots x_n)}} \text{ for } \dot{x}_1 \neq 0$$

think: little arrows to which local flow is tangent



Example (Based on Wiggins 2003, example 0.0.1)

Consider
$$\dot{u} = v, \quad \dot{v} = -u, \quad (u,v) \in \mathbb{R}^1 \times \mathbb{R}^1$$

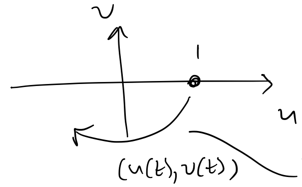
- Find the solution, integral curve & orbit, all passing through $(u,v) = (1,0)$
- Sketch the vector field in the u–v plane

- **Solution**: function $t \to (u(t), v(t))$

  Here: $(u(t), v(t)) = (\cos t, -\sin t)$

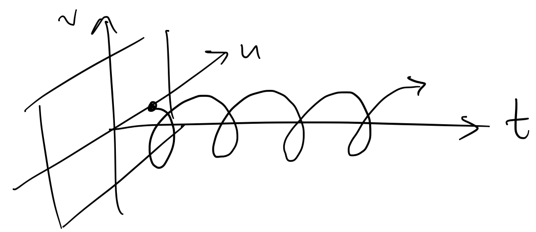
  [verify: $\dot{u} = -\sin t = v(t)$ ✓
  $\dot{v} = -\cos t = -u(t)$ ✓]

ie



often called 'trajectory' when plotted like this

- **Integral curve**: set $\{(u,v,t) \mid u(t) = \cos t, v(t) = -\sin(t), \forall t\}$

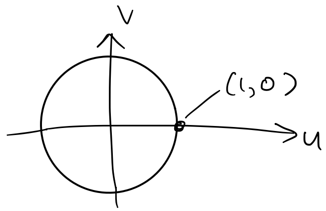


Example cont'd

Orbit: set of points in (u,v) space generated by solutions

→ convenient representation:

$$\{(u,v) \mid u^2+v^2=1\}$$



---

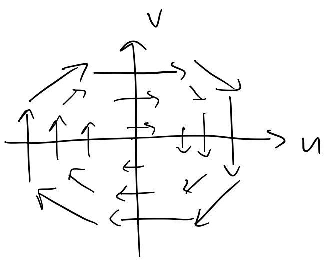
vector field $(\dot u = v, \dot v = -u)$

note: $\dfrac{\dot v}{\dot u} = \dfrac{dv}{du}$ = slope of v vs u

$= \dfrac{-u}{v}$ (for $v \neq 0$)

⇒ plug in some points

(0,1)
(1,0)
(0,-1)
(-1,0)
(1,1)
etc.



$u \uparrow \Rightarrow \dot v$ more neg.
$v \uparrow \Rightarrow \dot u$ more pos.

Flow function φ

When we want to emphasise dependence of sol^n on an initial condition x_0

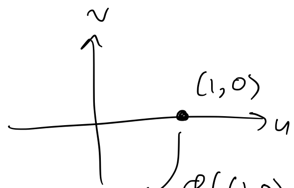
$$\varphi(x,t) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$$

$\mathbb{R}^n$ ↑ initial state; $\mathbb{R}$ ↑ time interval; $\mathbb{R}^n$ ↑ new state after time interval

} think: evolution operator, advancing $x_0 \longrightarrow x_{new}$, $t=0$ → at t

eg if $\varphi(x,0) = x_0$

then $\varphi(x_0,t)$ = new state at time t after starting from $t=0$ at x_0.



$\varphi((1,0),t)$ in previous example

Formal properties (just for fun! :)

- $\dfrac{d}{dt}\varphi(x,t) = f(\varphi(x,t))\ \forall t$ (ie satisfies ODE)
- $\varphi(x,t+s) = \varphi(\varphi(x,t),s) = \varphi(\varphi(x,s),t) = \varphi(x,s+t)$

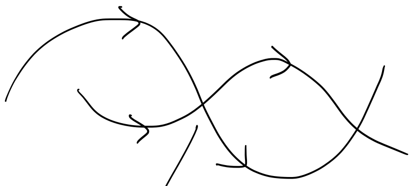
  (can evolve bit by bit)

Existence & uniqueness

"always know where to go next"

→ no intersecting/crossing trajectories (unless stop at fixed point)

NO!



can't cross & keep going!

→ helpful in phase-plane to 'trap' flows

Phase Portraits : a depiction of key 'interesting' features in phase (state) space

→ try to 'partition' phase space into different regions of behaviour

→ identify fixed points, periodic orbits etc & analyse their stability

→ sketch local trajectories & direction fields

Eg (not quite ready to answer yet)

- Exam 2017 :

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

Equilibria & Nullclines

- Fixed point/equilibrium point etc:

$$\underbrace{\phi(x_e, t) = x_e}_{\text{don't go anywhere!}} \quad \forall t$$

OR

$$f(x_e) = 0$$ } this is how we find (set RHS $=0$).

Usually best to solve for in steps:

Question 6 (18 marks) (Exam 2017)

Consider the system

$$\dot{x} = x^2 + y^2 - 2 \quad (1)$$
$$\dot{y} = x - 1 \quad (2)$$

where $x, y \in \mathbb{R}$.

Step 1. set easiest to zero.

(2) $= 0 \Rightarrow x = 1$

Step 2 plug each sol^n (only one here) into other eq^n(s). etc.

$x = 1$

(1): $1 + y^2 - 2 = 0$

$\Rightarrow y = \pm 1$

Sol^{ns}: $\{(1,1), (1,-1)\}$

Nullclines

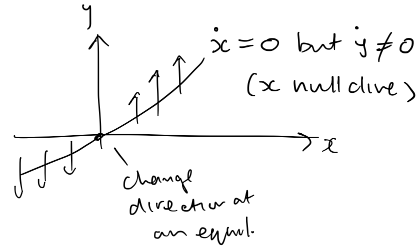
- Nullcline: zero components of f

ie if $\dot{x}_i = f_i(x)$, $x_i \in \mathbb{R}^1$

then jth nullcline is

$$\dot{x}_j = \boxed{f_j(x) = 0}$$ just the j^{th} component is zero

eg



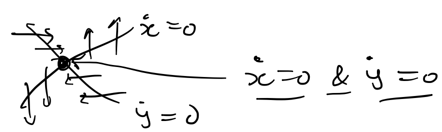
} will cover in detail later.

Exercise: try Exam 2017 example.

- Note: relation to fixed points

a fixed point is where $f_j(x) = 0 \ \forall j$

$\Rightarrow$ where all nullclines intersect



Periodic point

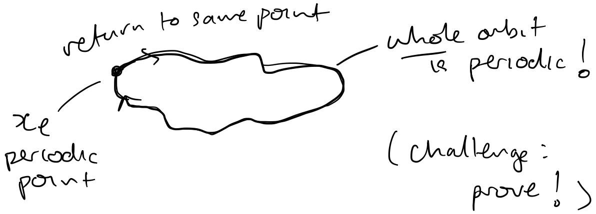
- $\boxed{\varphi(x_e, t+T) = \varphi(x_e, t)}$ & $\varphi(x_e, t+s) \neq \varphi(x_e, t)$, $\underline{0 < s < T}$ } T is smallest period

---

Periodic orbit

$$\{ \varphi(x_e, t) \mid t \in \mathbb{R} \}$$

where x_e is periodic



Invariant Sets

- generalisation of fixed points & periodic orbits

"once in, don't leave" (eg 'trapping region')

If $x \in M$ then $\varphi(x, t) \in M$ $\forall t$.

(eg fixed point $\varphi(x^*, t) = x^*$ $\forall t$
$\Rightarrow \{x^*\}$ is invariant)

Limit Sets

- Another type of invariant set & generalisation of fixed point/periodic orbit ideas

---

ω-limit set of a point x

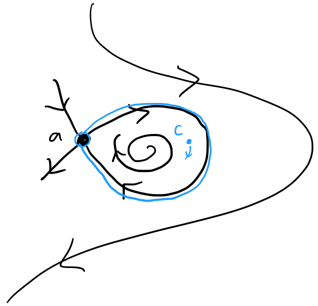
→ 'forward limit set'

→ all points towards which flow from x tends as $t \to \infty$

↳ don't have to lie on trajectory: are limit points!

---

eg.



$\omega(c)$?

- flow approaches 'a' arbitrarily closely
- also approaches the orbit connecting 'a' to itself ('homoclinic orbit')

$\Rightarrow \omega(c) = \{$all points in blue$\}$
(not incl. c of course.)

$\Rightarrow \underline{\omega(c) = \{a\} \cup \{\text{homoclinic orbit}\}}$

α-limit set of x

→ same idea but in 'backwards time'!