MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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RECALL - STURM-LIOUVILLE PROBLEMS

The (regular) Sturm-Liouville problem can be written compactly in *operator notation* as

$$Ay := -\frac{1}{\omega(x)}[(p(x)y')' + q(x)y] = \lambda y$$

subject to

$$B_1 y(a) := \alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$B_2 y(b) := \beta_1 y(b) + \beta_2 y'(b) = 0$$

The combination $(Ay, B_1 y(a), B_2 y(b))$ is sometimes (even more) compactly denoted by Ly, i.e. L *includes the BC*. The conditions are...

RECALL - STURM-LIOUVILLE PROBLEMS

- a and b are *finite*,
- $q, \omega\ p$ and p' are *continuous* functions on $x \in [a, b]$,
- $p(x) > 0$ and $\omega(x) > 0$ on $[a, b]$, i.e. are *positive*
- λ is a *constant* (and is a free parameter, i.e., not specified/is to be determined)
- α_1 and α_2 are *not both zero*, β_1 and β_2 are *not both zero* and
- $a, b, p(x), q(x), \omega(x), \alpha_1, \alpha_2, \beta_1, \beta_2$ are *all real*.

(we can also consider *singular* cases where these fail to hold)

RECALL - STURM-LIOUVILLE THEOREM

- The eigenvalues are all *real*.
- The eigenvalues are *simple*, i.e., to each eigenvalue there corresponds just one linearly independent eigenfunction.
- There are *infinitely many eigenvalues*, and they can be *ordered* so that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $\lambda_n \to \infty$ as $n \to \infty$.
- Eigenfunctions corresponding to different eigenvalues are *orthogonal*, i.e., if $\lambda_n \neq \lambda_n$ then $\langle \phi_n, \phi_m \rangle = 0$.

and...

RECALL - STURM-LIOUVILLE THEOREM

... Let f be *piecewise smooth* on $[a, b]$. Then if $a_n = \langle f, \phi_n \rangle / \langle \phi_n, \phi_n \rangle$ the series

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

converges to $(f(x+) + f(x^-))/2$ at each point $x \in (a, b)$.

RECALL - THEOREM: NON-NEGATIVE EIGENVALUES?

If $q(x) \leq 0$ on $[a, b]$ and $[p(x)\phi_n(x)\phi_n'(x)]_a^b \leq 0$ for the eigenfunction $\phi_n(x)$, then λ_n is *non-negative*.

(We already know λ_n is real from the SL theorem).

LECTURE 10: STURM-LIOUVILLE THEORY REVISITED

The adjoint of an operator and self-adjoint operators

Proof that SLPs define self-adjoint operators

Proof that eigenvalues of SLPs are real

Proof that eigenfunctions of SLPs are orthogonal

Proof that eigenvalues of SLPs are positive (under additional assumptions)

WEIGHTED INNER PRODUCT FOR COMPLEX FUNCTIONS

We can generalise the *inner product* $\langle f, g\rangle$ to the case of two *complex* functions f and g by defining

$$\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}\omega(x)dx$$

where $\omega(x)$ is the *weight function from the SLP* of interest and $\overline{g(x)}$ is the *complex conjugate* of $g(x)$.

Again, if $\langle f, g\rangle = 0$ we say f and g are *orthogonal* (as before). Now we have an inner product space defined over a complex scalar field.

THE ADJOINT OF AN OPERATOR

The *adjoint* of an operator is a *generalisation of the transpose* of a real matrix (or the conjugate transpose/Hermitian transpose of a complex matrix) to *infinite-dimensional* operators (e.g. differential operators).

ADJOINT OPERATORS

The *adjoint* of an operator L operating on functions in some function space is the unique operator L^* operating on that same function space such that

$$\langle Lu, v\rangle = \langle u, L^* v\rangle$$

for all u, v in that function space. *We include in L and L^* the appropriate boundary conditions (possibly different for each) so as to satisfy the relation*.

Note: we can also consider *formal adjoints* which relax the requirement on boundary conditions somewhat.

SELF-ADJOINT OPERATORS

Self-adjoint operators are a generalisation of *symmetric* real matrices (or Hermitian complex matrices) to *infinite-dimensional* operators (e.g. differential operators).

SELF-ADJOINT OPERATORS

The basic definition of a *self-adjoint* operator is

$$\langle Lu, v \rangle = \langle u, Lv \rangle$$

For all u, v in the function space of interest - *now we include the requirement that u and v satisfy the same boundary conditions*

LAGRANGE AND GREEN IDENTITIES

To *show that SLPs define self-adjoint operators* and to understand *adjoint boundary conditions* we need to recall (one of) the following basic identities.

Note: *I recommend Green's version* but will start from Lagrange's for 'fun'!

LAGRANGE AND GREEN IDENTITIES

Let A be the linear second-order ordinary differential operator

$$A = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

Then...

LAGRANGE'S IDENTITY

Lagrange's identity is

$$\overline{v}Au - u\overline{A^* v} = \frac{d}{dx}J(u, v)$$

where

$$J(u, v) = a_2(vu' - uv') + (a_1 - a_2')uv$$

(note - we haven't assumed real functions so require some complex conjugation in general) and...

FORMAL ADJOINT

$$A^* = a_2(x)\frac{d^2}{dx^2} + (2a_2(x)' - a_1(x))\frac{d}{dx} +$$
$$(a_2(x)'' - a_1(x)' + a_0(x))$$

is the *formal adjoint* of A.

FORMAL SELF-ADJOINTNESS

Note that for SL operators $a_2' = a_1$ and $a_2'' = a_1'$ and so $A^* = A$.

In this case we say that A is *formally self-adjoint*. We can't say it's properly self-adjoint, however, without considering the $J(u, v)$ (boundary) terms.

GREEN'S FORMULA

After integrating Lagrange's identity we get...*Green's formula/identity*

$$\int_a^b (\bar{v}Au - u\overline{A^* v})dx = J(u, v)|_a^b$$

i.e.

$$\langle Au, v \rangle - \langle u, A^* v \rangle = J(u, v)|_a^b$$

where

$$J(u, v) = a_2(vu' - uv') + (a_1 - a_2')uv$$

I recommend just starting from this (simpler) form!

ADJOINT BOUNDARY CONDITIONS

Comparing the previous result with the definition of the adjoint

$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

We see that we require, for *'full'* adjointness,

$$J(u, v)|_a^b = 0$$

This tells us how to find the *adjoint boundary conditions* B_1^*, B_2^* from the original boundary conditions B_1, B_2 that, combined with the *formal* adjoint operator A^*, give us the *full adjoint operator* $L^* = (A^*, B_1^*, B_2^*)$.

ADJOINT SUMMARY

If operator is *formally self-adjoint* - i.e. is SL with $a_2' = a_1$ - *and* the adjoint *boundary conditions are the same* (determined by solving $J(u, v)|_a^b = 0$) as the original boundary conditions then the operator is *self-adjoint*.

ADJOINT EXAMPLES

Examples - see supplement.

SLPS AS DEFINING SELF-ADJOINT OPERATORS

Sturm-Liouville problems *define self-adjoint operators* (note - the definition includes the boundary conditions!)

Proof.

THINGS TO PROVE

We've noted that self-adjoint operators are the analogues of symmetric/Hermitian matrices and have similar properties...let's prove some!

(See supplement for more details)

PROOFS!

Proof that eigenvalues of SLPs are *real.*

PROOFS!

Proof that eigenfunctions of SLPs are *orthogonal.*

PROOFS!

Proof that eigenvalues of SLPs are *positive if ...*

HOMEWORK

Go back over the analogous results for linear algebra

Make sure you (sort of) see how everything fits together

Practice integration by parts

Go over the examples from last lecture