Engsci 711

Tutorial 2: Full phase plane analysis

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Overview

The purpose of this tutorial is to give you some practice analysing (mainly) two-dimensional systems by hand from woah to go.

Tips and tricks

Analysis procedure

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).

- Sketch/compute the *phase portrait*.

Tips for phase portrait sketching

- Start drawing locally near individual fixed points
- Check for any obvious invariant axes, lines/curves
- Draw nullclines and any other helpful/obvious flow directions (e.g. trapping regions)
- Think about whether periodic orbits might exist anywhere
- Think about various ways the local flows might connect up or be extended more globally.

Calculating manifolds

Consider a system such as $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$. We usually use our information (give or take a swap of x, y variables etc) to construct series expansions for stable/unstable manifolds as follows

- Assume the manifold can be described by a functional relationship such as $y = h(x)$.
- Substitute $y = h(x)$ into our x and y equations to give $\dot{x} = f_1(x, h(x))$ and $\dot{y} = f_2(x, h(x))$.
- Use the functional relation again, along with the chain rule for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dh}{dx}\dot{x}$.
- Use the above relationships along with an assumed power series expansion such as $h(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain two polynomial expressions in x (say) for $\dot{y}$ involving the unknown coefficients of the power series. Equate powers of x to determine the coefficients.
- You will need to use the information that the stable/unstable manifold passes through the fixed point and is tangent to the linearised stable/unstable manifold to determine the first two terms of the series. These will not be zero in general (but should be known)!

Exercises

- Find the fixed points of the following equations, determine their stability and sketch their phase portraits

$$
\begin{aligned}
(a)&\quad \dot{x} = -2x - y, \quad \dot{y} = x + x^3\\
(b)&\quad \dot{x} = x + y - 2x^2, \quad \dot{y} = -2x + y + 3y^2\\
(c)&\quad \ddot{x} + \sin x = 0
\end{aligned}
$$

Approximating stable/unstable manifolds

Short

- Complete the example from the lecture, i.e. find the stable manifold $W^s_{loc}(0)$ for

$$
\begin{aligned}
\dot{x} &= x\\
\dot{y} &= -y + x^2
\end{aligned}
$$

- Consider

$$\dot{x} = 2x + y^2$$
$$\dot{y} = -y$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ to all orders.

Long

(Warning: somewhat long and tedious - but worth attempting. Can also use symbolic math in Matlab/Python/Wolfram alpha to help...).

Consider the system

$$x' = -2x - 3y - x^2$$
$$y' = x + 2y + xy - 3y^2$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ up to (i.e. including) cubic order.

Periodic orbits, trapping regions etc.

1. Work through Strogatz (1994) Example 7.3.2 on constructing a trapping region (see the Lecture 6 handout).
2. See if you can determine whether the following systems have any periodic orbits.

a)

$$\dot{x} = y$$
$$\dot{y} = (x^2 + 1)y - x^5$$

b)

$$\dot{x} = y$$
$$\dot{y} = y^2 + x^2 + 1$$

c)

$$\dot{x} = 1 + x^2 + y^2$$
$$\dot{y} = (x - 1)^2 + 4$$

3. Consider the system

$$x' = x - y - x(x^2 + 2y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

- Re-write the system in polar coordinates (r, θ) where $x = r\cos\theta, y = r\sin(\theta)$. Hint: use the identities $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.
- Determine a region bounded by two circles (i.e. annulus shaped), each of which are centred at the origin, such that the flow is radially outward on the inner circle and radially inward on the outer circle.
- Show that there is a periodic orbit in this region, i.e. for $r_{inner} \leq \sqrt{x^2 + y^2} \leq r_{outer}$.
- Use XPP (or Matlab or Python etc) to plot the periodic orbit (and some neighbouring orbits) in the phase plane.