ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds. Bifurcation diagrams.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

ORGANISATIONAL

- Assignment 2 (from Mike) due ~~in class on Wednesday 27th April~~ today!

LECTURE 2 - 'BIG PICTURE P.II'

- A few more definitions and key features of interest
- Overview of basic procedures for analysing dynamical systems

BASIC ANALYSIS PROCEDURE

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).
- Sketch/compute the *phase portrait*.

BASIC ANALYSIS PROCEDURE

...we can then go on to

- find more 'global' features such as *periodic orbits*
- analyse *bifurcations* (loss of stability)
- use *perturbation methods* to construct approximate solutions

etc.

TERMINOLOGY

We've used terms like *equilibria, stability, Jacobian, eigenvalues, bifurcations* etc.

We'll formally introduce/recap these (and some others) today and then get on with analysing some equations!

INTERESTING FEATURES - EQUILIBRIA

A point x_e is an *equilibrium solution/fixed point/stationary point* iff

$$\phi(x_e, t) = x_e$$

for all t. Equivalently it is a *zero of the vector field* (RHS)

$$f(x_e) = 0$$

Example

MORE FEATURES - PERIODIC POINTS AND PERIODIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t + T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t + s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

MORE FEATURES - LIMIT SETS

Other useful definitions include the following (invariant!) sets:

The *ω-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in forward time*.

Formally it consists of elements y such that there exists a sequence (t_n) with $t_n \rightarrow \infty$ and $\phi(x, t_n) \rightarrow y$ as $n \rightarrow \infty$.

MORE FEATURES - LIMIT SETS

The *α-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in backwards time* (exercise: write down the formal definition!)

Note that the points in these sets *don't have to lie on the orbits* through x - they are *limit* points for a reason!

Examples of limit sets

STABILITY OF SOLUTIONS

There are various general formal definitions of *stability* for solutions.

These can be defined for both equilibria as well as more complicated objects like periodic orbits.

We will just give the *definitions for equilibria* (points) for now.

STABILITY OF SOLUTIONS

- A point x is *Lyapunov stable* iff for all $\epsilon > 0$ there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| < \epsilon$ for all $t \geq 0$.
- A point x is *quasi-asymptotically stable* (attracting) iff there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| \to 0$ as $t \to 0$.
- A point is *asymptotically stable* iff it is both Lyapunov stable and quasi-asymptotically stable. If it is just Lyapunov stable then it is *neutrally stable* (i.e. is just bounded).

WHAT DO THESE MEAN?

In pictures!

STABILITY OF LINEAR SYSTEMS

Stability is *easy for linear systems*.

Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

(This can be proved by constructing a so-called Lyapunov function - ask me if interested)

LINEARISATION AND LOCAL STABILITY ANALYSIS

We have seen that stability for linear systems is easy. We will analyse and classify linear systems in more detail soon.

This will be useful because, as mentioned, our first steps in analysing nonlinear systems will usually be through local *linearisation* about steady-states/equilibria.

We need to know the *connection between linear and nonlinear stability*, however!

LINEAR AND NONLINEAR STABILITY - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues of the linearisation have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

LINEAR AND NONLINEAR STABILITY CONNECTED

The *Hartman-Grobman theorem* states that the local properties *near a hyperbolic fixed point* of a *nonlinear* system are *topologically equivalent* to those of the *linearisation*:

> *A hyperbolic fixed point persists* in the change from nonlinear to linear systems (though its location may shift slightly) and its *stability properties are also preserved*

(We will also see the similar 'stable manifold theorem' later)

LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component gives

$$\dot{u}_i = f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2)$$

i.e.

$$\dot{u}_i = \frac{\partial f_i}{\partial x_j}(x_e)u_i \equiv [Df(x_e)]_{ij}u_j$$

or simply $u = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

EXAMPLE

Let's see an example!

BIFURCATIONS AND STRUCTURAL INSTABILITY

Recall: fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

When these occur our *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities* - i.e. small variations in problem parameters can have a large effect on the qualtiative/topological features of our phase space.

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

BIFURCATIONS AND STRUCTURAL INSTABILITY

These instabilities are called *bifurcations*.

We will return to this topic in more detail in later lectures.
First a teaser.

A simple one-dimensional example.

HOMEWORK

Readings - see Canvas

Downloading and running XPPAut - see Canvas