ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]
   Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.
2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]
   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling hyperbolic systems.

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.
4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.

LECTURE 1 - 'BIG PICTURE'

Why? *What problem are we trying to solve?*

- Complex models (and the phenomena they represent) are *difficult to understand*
- Even 'simple' models can be difficult to understand

LECTURE 1 - COMPLEX MODELS

- No *closed-form* solutions
- *Brute-force* simulation doesn't necessarily help us *understand* our model (and the phenomenon we are modelling)
- All models are *wrong* (Box)
- Better to be *approximately right* than exactly wrong (Tukey)

EXAMPLES

- Lorenz System
- Car and Trailer
- Emergent dynamics
  - Ecological catastrophe
  - Training deep neural nets

EXAMPLE - LORENZ SYSTEM

Can we *understand* this three-dimensional, three-parameter ODE?

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

EXAMPLE - LORENZ SYSTEM

'Time' simulation.

See 'lorenz.ode'

HIDDEN COMPLEXITY AND HIDDEN SIMPLICITY

Despite having a small number of parameters and variables, the Lorenz system *exhibits complex behaviour* (even simple 1-D discrete maps can also exhibit such complexity)

However, by looking at it from *a different point of view* we can get *some* understanding of this system.

HIDDEN SIMPLICITY AND HIDDEN COMPLEXITY

Let's plot a so-called *phase-portrait*

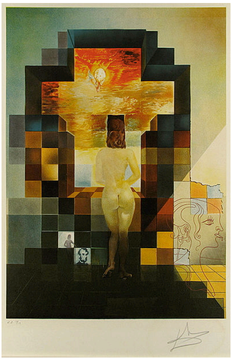
See 'lorenz.ode'

THE QUALITATIVE POINT OF VIEW

Rather than finding explicit, exact solutions, our point of view here will be

- The *'qualitative'* analysis of ODEs (and 'dynamical systems' in general). We step back from the full detail to enable a better overall picture
- Essentially a *geometric* perspective
- Possibly a *different way of thinking* than you are used to
- Both *hand calculation and computer-based methods* will be used; the goals are the same though

THE QUALITATIVE POINT OF VIEW



What can we say just by looking at 'pictures' in various ways?

WAYS OF THINKING

- Part of this module is about 'cookbook' methods to add to your toolbox
- Part of it is to give you an introduction to the underlying mathematical/geometric viewpoint

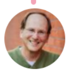
I'll *try* to balance practical analysis methods with formal definitions and mathematical ideas!

- Hands-on experience is important: make sure to do *tutorials*

HOW YOU SHOULD BE THINKING BY THE END OF THE COURSE

Car and trailer video https://bit.ly/2dJOXvw

Steven Strogatz comment:



Steven Strogatz @stevenstrogatz · Mar 5

Specifically, it is a subcritical Hopf bifurcation. After the weight is moved and the trailer is disturbed by the finger, the 1st small disturbance dies out, but the 2nd larger disturbance grows, indicating an unstable limit cycle in phase space surrounding a stable equilibrium.

EXAMINABLE EXPECTATIONS

"But what do you expect me to be able to do for the exam?"

Let's look at the exam from last year!

FURTHER READING/WATCHING

- Strogatz course on YouTube! https://goo.gl/Kqus6G
- Handout on helpful books/references

Eng Sci 711

Qualitative Analysis of ODEs
(& Other Dynamical Systems)

Plan : 4 Sections

1. ∘ Basic overview/terminology
2. ∘ Phase Plane & Linearisation
3. ∘ Bifurcation Theory
4. ∘ Centre Manifold Theory
   └ + intro to chaos ?

Slides : ∘ Big Picture
∘ Technical Definitions
  └ for the mathematicians
  └ moral obligation to show you!
∘ Tidier

Handwritten & Scanned : ∘ Messier .... BUT:
∘ What you need to be able to do & how!

Qualitative ?

- Complex models eg 100s – 1000s of parameters, equations etc ; non linear
- we can simulate for one set of parameters, then another etc etc
  └ Brute force quantitative simulation approach

$\boxed{\text{BUT}}$ (can be) : - inefficient/infeasible ?
- lacking in insight ?
  └ why do we get answer we get ?
  └ under what conditions etc ?
  └ how could we change the behaviour ?

nonlinearity makes difficult (not just 'sum of parts')

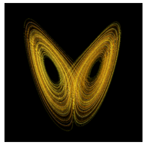
→ $\boxed{\text{we want more qualitative, approximate methods to compliment eg direct simulation}}$

→ $\boxed{\text{we use a geometrical \& topological perspective}}$

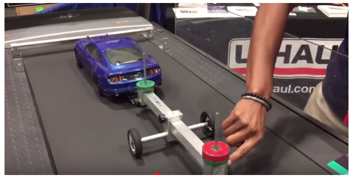
→ $\boxed{\text{apply to non-linear systems}}$

Some Examples : qualitative analysis of complex behaviour

Weather prediction { geometry of Lorenz system



Car + trailer dynamics { Stability depends on mass location



Deep Learning, Scale separation & 'emergent' dynamics { dynamics become slower & simpler in final stages.

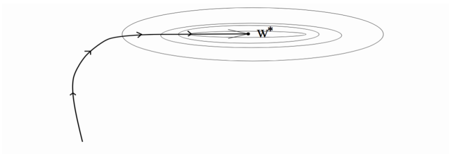
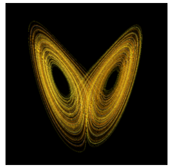


Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum w* and it is a good approximation of the minimum eigenvalue direction of the Hessian.

Examples



- Lorenz system

ODE system

$$\frac{dx}{dt} = \sigma(y-x),$$
$$\frac{dy}{dt} = x(\rho-z)-y,$$
$$\frac{dz}{dt} = xy-\beta z$$

in XPPAut :

```
init x=-7.5  y=-3.6  z=30
par r=27  s=10  b=2.66666
x'=s*(y-x)
y'=x*(r-z)-y
z'=x*y-b*z
```

↳ see lab !

how? see later!

- Simplified model of atmospheric convection (also arise as model of lasers, chemical reactions etc.)

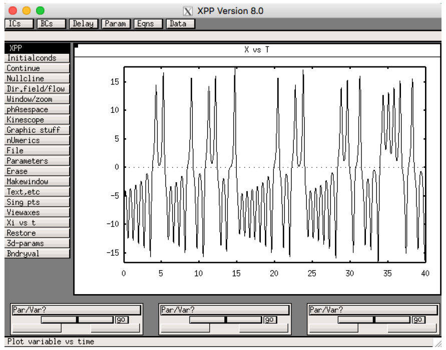
- Exhibits chaotic solutions ! (even though 'simplified')

⟶

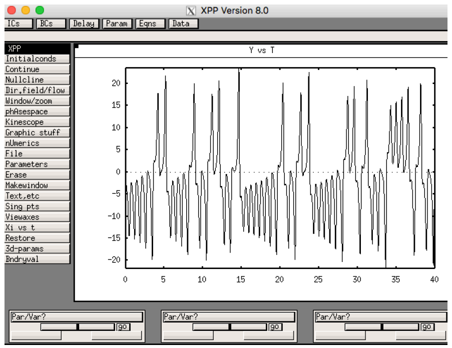
'Brute force' / direct time simulation

→ plot x, y, z vs time for fixed set of parameters

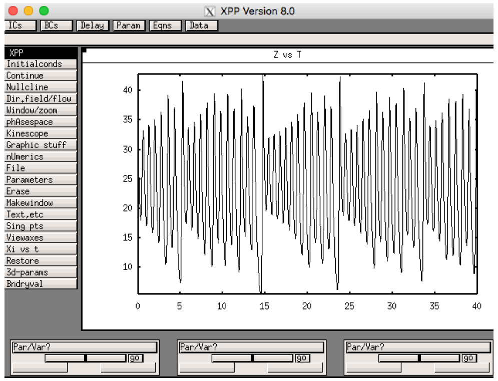
x vs t



y vs t



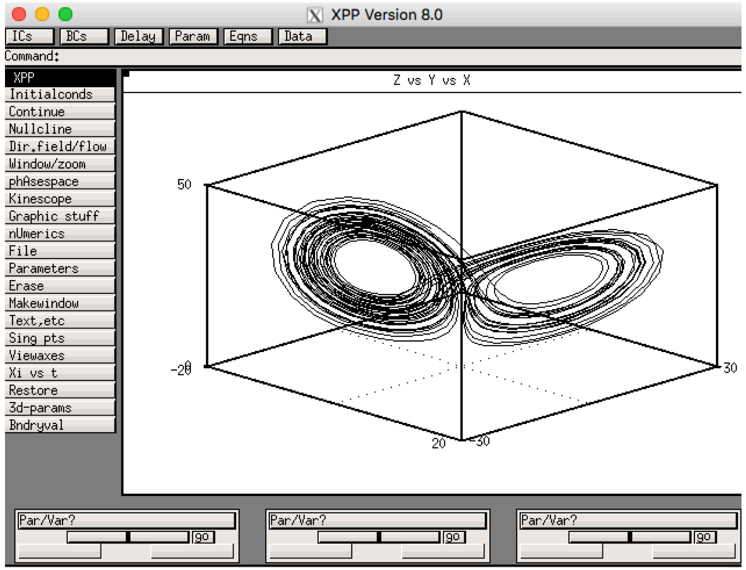
z vs t



→ what's going on?

'Shift' perspective

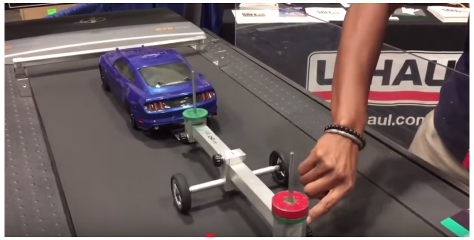
- plot in so-called phase/state space
- focus on the 'geometry'/topology of the problem
- where are trajectories attracted to?



- a clearer structure 'emerges' as time goes on
- the dynamics are attracted to the 'strange' butterfly-like shape

Examples: Stability of a trailer

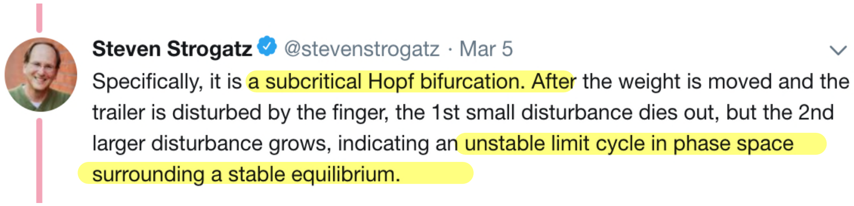
Video: see link in slides.



change mass distribution

↓

$\boxed{\text{unstable}}$: avoid!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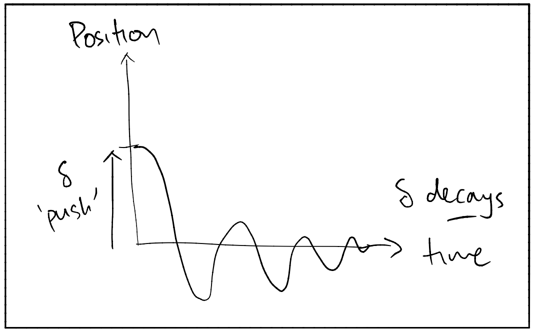
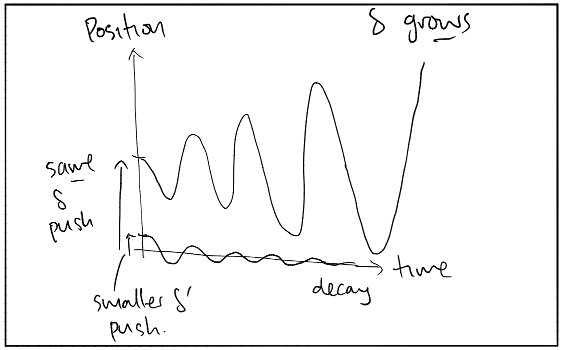
Examples: Stability of a trailer



mass closer to car

Change in stability as 'control parameter' varied.



mass further away from car

"**Bifurcation**"

→ Change in qualitative behaviour

$\boxed{\text{stable} \rightarrow \text{unstable}}$

as an external control parameter is varied

Examples: Reduction & Emergence near centre manifolds

Reduction of complex models/dynamics
↓
Emergence of simple models/dynamics

↓ longer time scale

Eg

- How might we get a 'simplified' model like the Lorenz system from a more complicated model (eg Navier-Stokes)?

- Turns out this model can actually be obtained using centre manifold reduction theory

  ↳ we will cover this theory towards the end of this part!

- Again, focus on the geometry/topology of the long term dynamics

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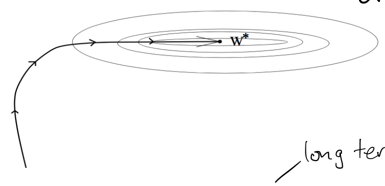
Examples: [Emergence of 'simpler' dynamics near 'centre manifolds']

Deep Learning!

From Le Cun et. al. "Efficient BackProp" in 'Neural Networks: 'Tricks of the trade' (book)



long term

Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum $\mathbf{w}^*$ and it is a good approximation of the minimum eigenvalue direction of the Hessian.

reduction

centre manifold

⇒ 'long term reduction to the centre manifold'!

CBMM Memo No. 073 January 30, 2018

Theory of Deep Learning III: the non-overfitting puzzle

by

T. Poggio†, K. Kawaguchi †⊤, Q. Liao†, B. Miranda†, L. Rosasco†
with
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†Center for Brains, Minds and Machines, MIT
⊤CSAIL, MIT
††Alphabet (Google) X
◇Claremont Graduate University

} explaining deep learning/non-over fitting using qualitative theory of ODEs, incl. centre manifolds.

Abstract: A main puzzle of deep networks revolves around the apparent absence of overfitting intended as robustness of the expected error against overparametrization, despite the large capacity demonstrated by zero training error on randomly labeled data.

In this note, we show that the dynamics associated to gradient descent minimization of nonlinear networks is topologically equivalent, near the asymptotically stable minima of the empirical error, to a gradient system in a quadratic potential with a degenerate (for square loss) or almost degenerate (for logistic or crossentropy loss) Hessian. The proposition depends on the qualitative theory of dynamical systems and is supported by numerical results. The result extends to deep nonlinear networks two key properties of gradient descent for linear networks, that have been recently recognized (*1*) to provide a form of implicit regularization:

qualitative theory of dynamical systems

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. Thus the first step is to linearize the system, which means considering the Jacobian of F or equivalently the Hessian of L at W^*, that is

$$\mathbf{H}_{ij} = \frac{\partial^2 L}{\partial w_i \partial w_j} \tag{2}$$

.

We obtain

$$\dot{W} = -HW, \tag{3}$$

where the matrix H, which has only real eigenvalues (since it is symmetric), defines in our case (by hypothesis we do not consider unstable critical points) two main subspaces:

- the stable subspace spanned by eigenvectors corresponding to negative eigenvalues
- the center subspace corresponding to zero eigenvalues.

centre manifold theory

The center manifold existence theorem (*16*) states then that if F has r derivatives (as in the case of deep polynomial networks) then at every equilibrium W^* there is a $\mathbf{C}^r$ stable manifold and a $\mathbf{C}^{r-1}$ center manifold which is sometimes called *slow manifold*. The center manifold emergence theorem says that there is a neighborhood of W^* such that all solutions from the neighborhood tend exponentially fast to a solution in the center manifold. In general, properties of the solutions in the center manifold depends on the nonlinear parts of F. We assume that the center manifold is not unstable in our case, reflecting empirical results in training networks. Of course, the dynamics associated with the center manifold may be non trivial, especially in the case of SGD. We simply assume for now that perturbations of it will not grow and trigger instability.

A simpler example (illustrates same idea as previous 'deep learning' slide)

Consider the system:

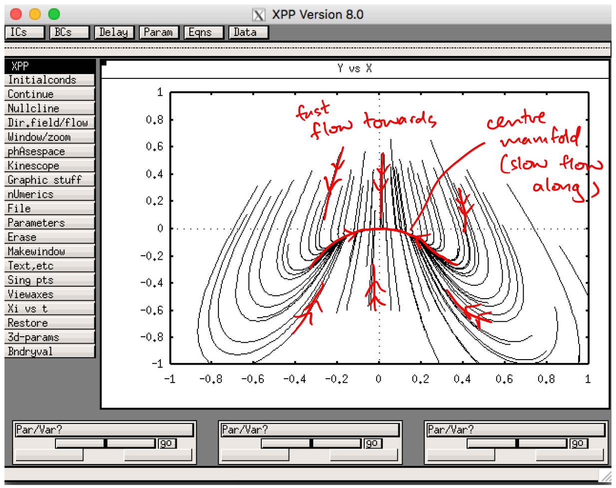
$$\left.\begin{aligned}\frac{dx}{dt} &= xy + x^3,\\ \frac{dy}{dt} &= -y - 2x^2\end{aligned}\right\}\ \text{nonlinear}$$

(Turns out the usual linear analysis doesn't work — we'll look at when & why this happens)

Emergent, attracting 'manifold' / curved surface/subspace

→ fast motion to manifold

→ slow motion along manifold



Qualitative View

- emergent (long time) simple (reduced) descriptions of complex systems
- approximate, qualitative & geometric approach to complement detailed simulation methods
- stability, instability & emergence for nonlinear systems

(Dali)
→ what you see depends on the scale
→ a qualitatively different picture 'emerges' on the coarser scale.

- 'But what do I need to be able to do for the exam?'

→

Exam Qs 2017 — Not too bad! (Most did very well)

SECTION B

Question 4 (16 marks)

Consider the system

$$\dot{x} = 2xy + x^3$$
$$\dot{y} = -y - x^2$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system.

(1 mark)

— solve RHS = 0

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0, 0)$.

(2 marks)

— linearise

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

(3 marks)

— calc eigenvalues of a matrix

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0, 0)$ that is correct up to and including cubic order.

(8 marks)

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.

(2 marks)

use a power series approx & sub into centre manifold equation (new)

Question 5 (16 marks)

Consider the equation

$$\dot{u} = (u - 2)(\lambda - u^2)$$

where $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a parameter that can vary.

(a) Determine the equilibria and their stability as λ varies.

(11 marks)

— solve RHS = 0 as function of λ

(b) Sketch the bifurcation diagram showing how the equilibria vary with λ. What types of bifurcations occur?

(5 marks)

— draw a picture & label

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

— solve RHS = 0

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

— solve RHS = 0 one at a time

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

— draw some pictures.