

ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(...and other dynamical systems)

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MODULE OVERVIEW

3. Introduction to bifurcation theory [4 lectures/tutorials]

Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. Centre manifold theory and putting it all together

[4 lectures/tutorials]

Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

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MODULE OVERVIEW

Qualitative analysis of differential equations

(Oliver Maclaren) [~16-17 lectures/tutorials]

1. Basic concepts [3 lectures/tutorials]

Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. Phase plane analysis, stability, linearisation and classification [5-6 lectures/tutorials]

General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

LECTURE 3: STABILITY AND LINEARISATION

- Stability of solutions for general systems
- Stability of solutions for linear systems
- Linearisation of nonlinear systems
- Connecting stability of linearised systems to stability of nonlinear systems
- Sneak peak at structural stability

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STABILITY OF SOLUTIONS

There are various *general* formal definitions of *stability* for solutions.

These can be defined for both equilibria as well as more complicated objects like periodic orbits. They also apply for both linear and nonlinear systems.

We will just give the *definitions for equilibria* (points) for now.

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WHAT DO THESE MEAN?

In pictures!

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STABILITY OF SOLUTIONS

- A point x is *Lyapunov stable* iff for all $\epsilon > 0$ there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| < \epsilon$ for all $t \geq 0$.
- A point x is *quasi-asymptotically stable* (attracting) iff there exists a δ such that if $|x - y| < \delta$ then $|\phi(x, t) - \phi(y, t)| \rightarrow 0$ as $t \rightarrow \infty$.
- A point is *asymptotically stable* iff it is both Lyapunov stable and quasi-asymptotically stable. If it is just Lyapunov stable then it is *neutrally stable* (i.e. is just bounded).

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STABILITY OF LINEAR SYSTEMS

These general definitions can be hard to check in general but are *easy to check for linear systems*.

Given a *linear* system of the form $\dot{x} = Ax$ where A is an $n \times n$ matrix then, if all the *eigenvalues* of A have *negative real part*, the origin $x = 0$ is *asymptotically stable*.

(This can be proven by constructing a so-called Lyapunov function - ask me if interested/see further reading)

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LINEARISATION AND LOCAL STABILITY ANALYSIS

We have seen that stability for linear systems is easy. We will analyse and classify linear systems in more detail soon.

This will be useful because, as mentioned, our first steps in analysing *nonlinear* systems will usually be through local *linearisation* about steady-states/equilibria.

We will also need to know the *connection between linear and nonlinear stability!*

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LINEARISATION PROCEDURE AND THE JACOBIAN DERIVATIVE

Let x_e be a stationary point of the nonlinear ODE (vector field) $\dot{x} = f(x)$, i.e. $f(x_e) = 0$. Letting $u = x - x_e$ and expanding in each component gives

$$\begin{aligned}\dot{u}_i &= f_i(x_e) + \frac{\partial f_i}{\partial x_j}(x_e)u_j + O(|u|^2) \text{ i.e.} \\ \dot{u}_i &= \frac{\partial f_i}{\partial x_j}(x_e)u_i \equiv [Df(x_e)]_{ij}u_j\end{aligned}$$

or simply $\dot{u} = Df(x_e)u$, where Df is called the Jacobian matrix/derivative.

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LINEAR AND NONLINEAR STABILITY - HYPERBOLIC FIXED POINTS

Fixed points for which all the *eigenvalues of the linearisation have non-zero real part* (i.e. don't lie on the imaginary axis) are called *hyperbolic*. These are the *robust* cases.

Non-hyperbolic points have zero real part and thus are *marginal* or 'sensitive' 'cases between 'true stability' and 'true instability'.

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LINEAR AND NONLINEAR STABILITY CONNECTED

The *Hartman-Grobman theorem* states that the local properties *near a hyperbolic fixed point* of a *nonlinear* system are *topologically equivalent* to those of the *linearisation*:

A *hyperbolic fixed point persists* in the change from nonlinear to linear systems (though its location may shift slightly) and its *stability properties are also preserved*

(We will also see the similar 'stable manifold theorem' later)

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BIFURCATIONS AND STRUCTURAL INSTABILITY

Recall: fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

When these occur our *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities* - i.e. small variations in problem parameters can have a large effect on the qualitative/topological features of our phase space.

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

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BIFURCATIONS AND STRUCTURAL INSTABILITY

These instabilities are called *bifurcations*.

We will return to this topic in more detail in later lectures.

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Stability & Linearisation

- Goals :
- understand the various concepts of stability in general
 - understand when a linear stability analysis gives a 'faithful' picture of a nonlinear system
 - be able to carry out a linear stability analysis of a nonlinear system

Examples: Key skills

Consider the system

$$\begin{aligned}\dot{x} &= y(2x - y) \\ \dot{y} &= x^2 - y\end{aligned}$$

2016 Exam

- (a) Find the two fixed points of this system. Show your working. You do not need to classify these.

- (b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$.

- (c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x,y) plane. You do not need to show any nearby trajectories.

} today

Consider the system

$$\begin{aligned}\dot{x} &= 2xy + x^3 \\ \dot{y} &= -y - x^2\end{aligned}$$

2017 Exam

where $x, y \in \mathbb{R}$.

- (a) Verify that the origin is a fixed point of this system.

(1 mark)

- (b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0,0)$.

(2 marks)

- (c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x,y) plane. You do not need to show any nearby trajectories.

(3 marks)

} today

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

2016 Exam

- (a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$.

} today

Example

Consider the nonlinear system from the computer tutorial:

$$\begin{cases} \dot{x} = x(y-1) \\ \dot{y} = 3x - 2y + x^2 - 2y^2 \end{cases} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

- o We're interested in long-term behaviour like fixed points (equilibria) & their stability

- o what does this mean?

→ First let's find the fixed points } first step in most analyses!

Procedure: try solving in steps as follows:

$$\begin{aligned} \text{(easiest)} \rightarrow (1) &= 0 \Rightarrow x = 0 \quad \text{or} \quad y = 1 \\ \circ x = 0 \quad &\& (2) = 0: \leftarrow (\text{harder one}) \\ -2y - 2y^2 &= 0 \Rightarrow -2y(1+y) = 0 \\ \Rightarrow y = 0 \quad &\text{or} \quad y = -1 \\ \text{so: } (0,0) \quad &\text{or} \quad (0,-1) \end{aligned}$$

- o $y = 1$ & $(2) = 0$:

$$3x - 2 + x^2 - 2 = 0$$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

$$\Rightarrow x = -4 \quad \text{or} \quad x = 1$$

$$\text{so: } (-4,1) \quad \text{or} \quad (1,1)$$

⇒ Solutions: $\{(0,0), (0,-1), (1,1), (-4,1)\}$ | (fixed points)

So we see that eg

$(x, y) = (0, 0)$, i.e. the origin, is
a fixed point

$$\boxed{f(0, 0) = 0 \text{ or } \varphi((0, 0), t) = (0, 0)} \\ & \& x = y = 0$$

ie don't go anywhere if start
exactly there

What does it mean for this to be
a stable fixed point?

→ relates to starting nearby, not
just exactly there

↳ all models are wrong/imperfect
etc

We want a general definition, valid
for nonlinear systems

→ we will then relate this to
the more familiar linear
definition & see under
what conditions they give
the same answer

↳ often but not always

Stability: general definitions

There are at least three 'general' concepts
of stability

- can be applied to general 'objects'
(invariant sets like equilibria, periodic
orbits etc)
- we'll look at stability of equilibria

1. o Lyapunov (or Liapounov)

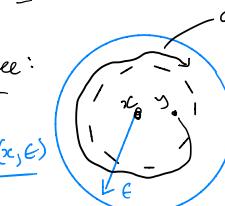
2. o Quasi-asymptotic

3. o Asymptotic (Lyapunov & Quasi.)

goal:
intuitive
understanding
of these.

1. Lyapunov: "points that start nearby, stay nearby"
('bounded')

o Given:  $B(x, \delta)$: ball of all points y s.t. $|x - y| < \delta$
(don't need to approach x tho')

o Guarantee:

 $|q(y, t) - q(x, t)| < \epsilon$
 $|q(y, t) - x| < \epsilon$ if x fixed point
since $q(x, t) = x$ w.t.
/

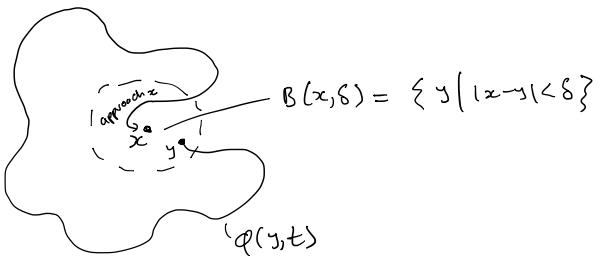
"Give me a target tolerance on future trajectories ϵ
& I'll give you a starting tolerance δ that
guarantees you'll meet this target"

Math: $\forall \epsilon > 0 \exists \delta > 0$ s.t.

if $y \in B(x, \delta)$ then $q(y, t) \in B(x, \epsilon)$

2. Quasi-asymptotic (attracting)

"start nearby, eventually return/approach
(arbitrarily close) to"



Math:

If $y \in B(x, \delta)$ (i.e. $|x-y| < \delta$)

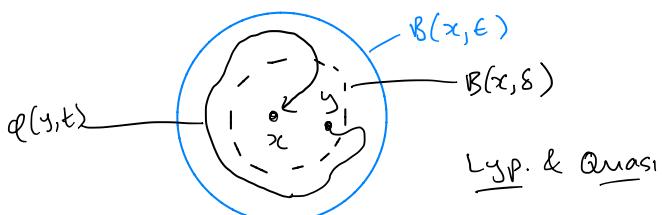
then $\varphi(y, t) \xrightarrow{t \rightarrow \infty} \varphi(x, t) = x$

if fixed point

(can use general def.
for more complex
objects)

3. Asymptotic: Lyapunov & Quasi-asymptotic

"start nearby, stay nearby \square return/approach to"



→ our goal is usually
asymptotic stability

Problem?

- These definitions are general
but therefore: hard to apply/check!

- On the other hand, linear systems
have an easy criterion:

given $\dot{x} = Ax$, $x \in \mathbb{R}^n$, A $n \times n$ matrix
the origin $x=0$ is asymptotically stable if all of the eigenvalues of A have negative real part

Eg $\lambda_1 = 1, \lambda_2 = -2 \rightarrow$ unstable

$\lambda_1 = -1, \lambda_2 = -2 \rightarrow$ stable

$\lambda_1 = -1+i, \lambda_2 = -1-i \rightarrow$ stable

$\lambda_1 = 0, \lambda_2 = -1 \rightarrow$ marginal (see later)

→ it would be nice if we could just
'linearise' our nonlinear systems
& use this instead!

↳ we can, under certain
conditions!

→ first: linearisation

Linearisation (at a fixed point)

General procedure : derivation

Consider an arbitrary nonlinear system of differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, i.e.

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- (a) Show how to linearise this system about a fixed point $x^* \in \mathbb{R}^2$ by introducing the new variable η where $x = x^* + \eta$. *'small perturbation' from x^**

exam

2016

1. Fixed point: $f(x^*) = 0$ { vector form }

or $f_i(x^*) = 0, \forall i$ { component form } \leftarrow easier

2. Taylor series for $x = x^* + \eta$ $(\eta = x - x^*)$
in component form:

work-horse
of
applied
math!

$$\begin{aligned}\dot{x}_i &= f_i(x) = f_i(x^* + \eta) \quad \text{full vector} \\ &\approx f_i(x^*) + \eta_j \frac{\partial f_i}{\partial x_j}(x^*) \quad (\text{sum over } j) \quad \text{system convention}\end{aligned}$$

$$\stackrel{=} 0 \quad \text{since fixed point} \quad \text{eval. at fixed point}$$

$$\Rightarrow \dot{x}_i \approx \eta_j \frac{\partial f_i}{\partial x_j}(x^*) \quad \rightarrow$$

vector vector matrix

$$\left(\text{if } \eta_j \frac{\partial f_i}{\partial x_j} \text{ means } \sum_j \eta_j \frac{\partial f_i}{\partial x_j} \right)$$

Have $\dot{x}_i = \frac{\partial f_i}{\partial x_j} \eta_j$
 \Leftrightarrow

want: $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j$

\Rightarrow Note $x = x^* + \eta$
 \uparrow

fixed vector (time independent)

$$\Rightarrow \dot{x} = \dot{\eta} \quad (\dot{x}^* = 0)$$

so $\dot{\eta}_i = \frac{\partial f_i}{\partial x_j} \eta_j \quad (\text{sum over } j)$

i.e. $\dot{\eta} = A\eta$ | (note: $\sum A_{ij} \eta_j$ is just matrix multiplication)

\Rightarrow Linear system in new 'perturbation' variable $\eta = x - x^*$

Here $A = 'Df'$ the Jacobian derivative of f

\rightarrow generalises derivative to vector-valued functions of vector variables, like f

\rightarrow get a matrix, with components:

since vector valued

$$(Df)_{ij} = Df_{ij} = \frac{\partial f_i}{\partial x_j} \quad \rightarrow$$

since vector variable

Huh?

Be able to do these!

→ Concrete example!

Let's linearise our previous system:

$$\begin{cases} \dot{x} = x(y-1) \\ \dot{y} = 3x - 2y + x^2 - 2y^2 \end{cases} \quad \begin{aligned} &= f_1(x, y) \\ &= f_2(x, y) \end{aligned} \quad \begin{aligned} &= f_1(x_1, x_2) \\ &= f_2(x_1, x_2) \end{aligned}$$

& then evaluate at $(0, 0)$.
→ (here $x_1 = x$
 $x_2 = y$)

First, find $Df(x, y) = Df(x_1, x_2)$

(at any x, y first. Then at eg $(0, 0)$
or other fixed point)

i.e. $Df_{ij} = \left(\begin{array}{cc} \frac{\partial f_1}{\partial x_i} & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_i} & \frac{\partial f_2}{\partial x_j} \end{array} \right) = \left(\begin{array}{cc} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{array} \right)$

$$= \left(\begin{array}{cc} y-1 & x \\ 3+2x & -2-4y \end{array} \right)$$

Next, evaluate at $(0, 0)$ (for example) -

$$Df(0, 0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$$

Eigenvalues?

$$Df(0, 0) = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix} = A$$

Long way: (for now) → see later for 'shortcuts'

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(-2-\lambda) - 3 \times 0 = 0$$

$$\Rightarrow +(\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1, \lambda_2 = -2} \quad \begin{array}{l} \text{linearised} \\ \text{system is} \\ \text{stable} \end{array}$$

$$(Re(\lambda_i) < 0 \forall i)$$

But:

→ Is this linear stability
analysis applicable to
the original nonlinear system?

→ When? Why?

Hyperbolic fixed points & the Hartman-Grobman theorem

Informal version:

- If the linearised system is definitely stable/unstable, i.e. $\operatorname{Re}(\lambda_i) \neq 0 \forall i$, then the nonlinear system is too
- Fixed points for which $\operatorname{Re}(\lambda) \neq 0 \forall i$ are called hyperbolic
- These are 'robust' in a number of ways:
 (continue to exist)
 - persistence under linearisation

Structural stability/
 → see later
 (bifurcation theory)

{ - persistence under small changes to
the governing equations
themselves (no bifurcations)
 - stability properties are preserved
under linearisation

Eg:

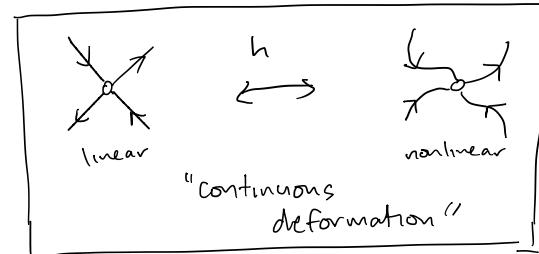
$\lambda_1 = 1, \lambda_2 = -2$	\rightarrow	hyp
$\lambda_1 = -1, \lambda_2 = -2$	\rightarrow	hyp.
$\lambda_1 = -1+i, \lambda_2 = -1-i$	\rightarrow	hyp.
$\lambda_1 = 0, \lambda_2 = -1$	\rightarrow	non-hyp.
$\lambda_1 = i, \lambda_2 = -i$	\rightarrow	non-hyp.

} need
center
manifold
theory

More formal: Hartman-Grobman theorem

"If $x=0, x \in \mathbb{R}^n$, is a hyperbolic fixed point of $\dot{x} = f(x)$, then there is a continuous 1-1 map { - homeomorphism } on some neighbourhood of x_0 that maps orbits of the nonlinear flow to those of the linear flow & which preserves the direction of time"

i.e. the linear & nonlinear systems are locally, topologically equivalent near a hyperbolic fixed point:



topologically conjugate:
 f
 $X \xrightarrow{f} X$
 $h \downarrow$
 $Y \xrightarrow{g} Y$
 $h \circ f = g \circ h$
 $f = h^{-1} \circ g \circ h$

$y = x + \text{nonlinear terms}$
 → h is a (weakly) nonlinear change of coordinates* } just a 'stretched'
 → same number of fixed points } version of
 → fixed points have same stability etc } 'same thing'
 → qualitatively equivalent

* Idea goes back to Poincaré: 'normal form theory'

'linearization theorem' / h extends this

Bifurcations? (Preview of next section of course)

- We have looked at stability of solutions to a fixed system

- a related idea is stability of the systems/equations themselves
- } important since:
'all models are wrong'!
- \hookrightarrow "Structural stability"

eg $\dot{x} = f(x)$

$$\downarrow$$

$$\dot{x} = f(x) + \epsilon g(x)$$

↑ small epsilon
small change to equations themselves

Q: do these systems have 'similar' properties:
 → eg same number of fixed points with same stability?

} similar to asking if linear system is 'like' nonlinear system!
 → ignoring higher order terms

Not surprisingly, the answer is:

- yes if near hyperbolic fixed point
- no if near non-hyperbolic fixed point

→ see later.

More Examples

$$\begin{cases} \dot{x} = y(2x-y) \\ \dot{y} = x^2 - y \end{cases}$$

From: Exam 2016 Q5.

- Find fixed points
- Linearise
- Determine if origin is hyperbolic
- Determine if origin is stable

• Fixed Points

'Divide & conquer'

- $\dot{x} = 0$: $\Rightarrow y(2x-y) = 0$
 $\Rightarrow y=0$ OR $y=2x$

- $y=0$ & $\dot{y}=0$:

$$x^2 = 0 \Rightarrow x = 0$$

so: $(0,0)$

- $y=2x$ & $\dot{y}=0$:

$$x^2 - 2x = 0 \Rightarrow x(x-2) = 0$$

$$\Rightarrow x=0$$
 OR $x=2$

so: $(0,0)$ (already have)
 OR $(2,4)$

Solutions: $\boxed{\{(0,0), (2,4)\}}$ (Fixed points)

Linearisation: arbitrary point (x, y)

$$\dot{x} = y(2x - y) = 2xy - y^2 = f_1(x, y)$$

$$\dot{y} = x^2 - y = f_2(x, y)$$

want Df , where $Df_{ij} = \frac{\partial f_i}{\partial x_j}$
row col.

$$Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 2y \\ 2x & -1 \end{bmatrix} \quad \text{(valid for any } (x, y))$$

At origin $(x, y) = (0, 0)$ (recall: is a fixed point)

$$Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\lambda = 0, -1 \Rightarrow \text{non-hyperbolic!}$$

\Rightarrow marginal/unknown stability
