ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds. Bifurcation diagrams.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

LECTURE 6

- Introduction to bifurcation theory

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

We now return to systems of the form

$$\dot{x} = f(x; \mu)$$

where $x \in \mathbb{R}^n$ is the usual vector of state variables but we have explicitly included $\mu \in \mathbb{R}^m$, a vector of *problem parameters*.

Intuitively, parameters may be thought of as extra *'slowly-varying' state variables* and our dynamical system for a fixed parameter value as a *'projection'* of a 'background' larger system onto a smaller state space.

PROBLEM PARAMETERS IN DYNAMICAL SYSTEMS

This means we are *neglecting some 'processes' or model components and 'summarising' their effects in problem-specific parameters.*

In a sense, bifurcation theory is about what happens *when this 'projection' of a higher-dimensional system fails to be reliable* - our choice of parameter value matters crucially.

These 'system' or 'structural' (c.f. solution) instabilities are called *bifurcations*.

RECALL: BIFURCATIONS AND STRUCTURAL INSTABILITY

When do these occur?

Fixed points for which the local linearisation has a *zero eigenvalue* are called *non-hyperbolic*.

In these cases *linear stability analysis fails to hold* for the nonlinear system and we get *structural instabilities and hence bifurcations*

E.g. the number of stationary points or periodic orbits (and/or their stability) may change.

THE CURSE OF INSTABILITY VS THE CURSE OF DIMENSIONALITY?

See Kuehn 'The curse of instability' (http://arxiv.org/abs/1505.04334)

Curse? (Structural) instabilities cause an *increase in dimensionality*, substantially raise the analytical difficulty and are a strong indicator for multiscale dynamical complexity.

THE CURSE OF INSTABILITY VS THE CURSE OF DIMENSIONALITY?

Cure? Separate your system/data/model into *regimes with and without (structural) instabilities* in the underlying process. So-called “universal” or “generic” dynamical principles are always based upon the absence or presence of certain instability classes.

ONE DIMENSIONAL, ONE PARAMETER SYSTEMS

Even in large systems we *often only have bifurcations occuring at for a small number of parameters at a time* - e.g. only one eigenvalue crossing the imaginary axis.

Centre manifold theory (which we will come back to) provides a way of analysing the reduced dynamics near a bifurcation.

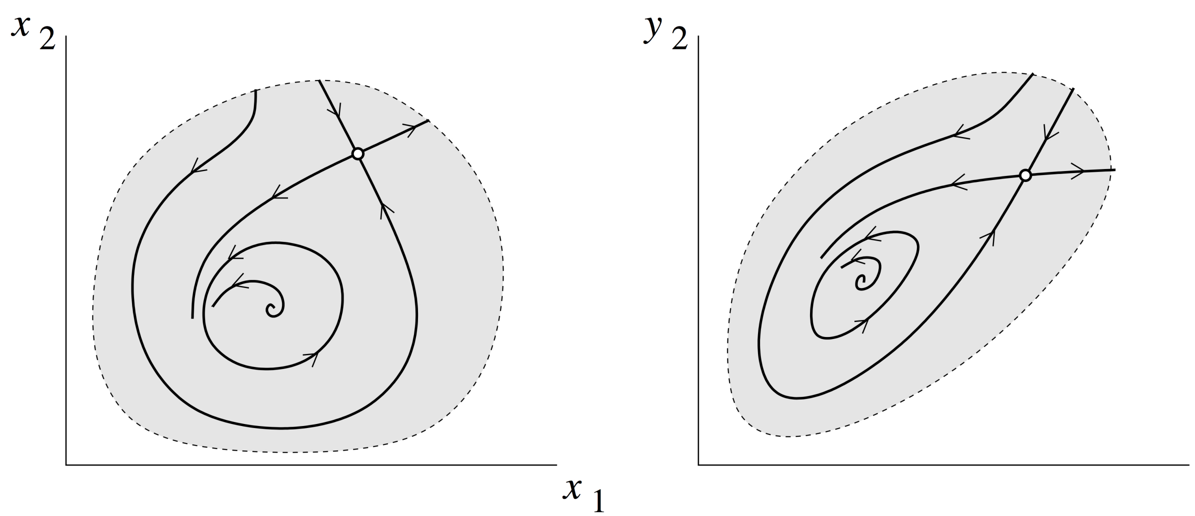
Because of this *we will focus on one-state variable, one-parameter systems*, i.e. $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$, which are frequently encountered during bifurcations in larger systems

systems.

TOPOLOGICAL EQUIVALANCE: A PICTURE

If a system is *structurally stable/unstable* then the phase portrait doesn't/does change qualitatively.

We'll (probably) come back to a more formal definition of topological/qualitative equivalance. For now, a picture:



Q: is this a picture of a structurally stable or unstable system.

LOCAL VS GLOBAL?

Note: we can analyse changes (bifurcations) in either or both *local* and *global* qualitative features of the phase portrait.

We will mainly focus on local bifurcations.

BIFURCATION DIAGRAMS

A bifurcation diagram shows how some *property of interest* of a system - e.g. location of an equilibrium point - *depends on a system parameter* (or parameters).

The best way to get a feel is to look at some examples, so let's do that!

EXAMPLES

(See e.g. chapters 3 and 8 of Strogatz)

- saddle-node/turning point bifurcation
- transcritical bifurcation
- pitchfork bifurcation
- Hopf bifurcation

(I'll work through some examples here and give you a fuller summary later)