Engsci 711

Tutorial 2: Full phase plane analysis

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Overview

The purpose of this tutorial is to give you some practice analysing (mainly) two-dimensional systems by hand from woah to go.

Tips and tricks

Analysis procedure

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).
- Sketch/compute the *phase portrait*.

Tips for phase portrait sketching

- Start drawing locally near individual fixed points
- Check for any obvious invariant axes, lines/curves
- Draw nullclines and any other helpful/obvious flow directions (e.g. trapping regions)
- Think about whether periodic orbits might exist anywhere
- Think about various ways the local flows might connect up or be extended more globally.

Calculating manifolds

Consider a system such as $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$. We usually use our information (give or take a swap of x, y variables etc) to construct series expansions for stable/unstable manifolds as follows

- Assume the manifold can be described by a *functional relationship* such as $y = h(x)$ (or $x = g(y)$).

- Substitute this functional relationship into our x and y equations to give e.g. $\dot{x} = f_1(x, h(x))$ and $\dot{y} = f_2(x, h(x))$.

- Use the functional relation again, along with the chain rule for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dh}{dx}\dot{x}$.

- Use the above relationships along with an assumed power series expansion such as $h(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain two polynomial expressions in x (say) for $\dot{y}$ involving the unknown coefficients of the power series. Equate powers of x to determine the coefficients.

- You will need to use the information that the stable/unstable manifold passes through the fixed point and is tangent to the linearised stable/unstable manifold to determine the first two terms of the series. These will not be zero in general (but should be known)!

Phase plane analysis

1. Find the fixed points of the following equations, determine their stability and sketch their phase portraits

$$\begin{aligned}
&(a)\quad \dot{x} = -2x - y, \quad \dot{y} = x + x^3 \\
&(b)\quad \dot{x} = x + y - 2x^2, \quad \dot{y} = -2x + y + 3y^2 \\
&(c)\quad \ddot{x} + \sin x = 0 \\
&(d)\quad \dot{x} = x(1 - x - y), \quad \dot{y} = y(2 - x - y) \\
&(e)\quad \dot{x} = -x + 4y, \quad \dot{y} = -x - y^3
\end{aligned}$$

For the last case, prove or disprove the existence of periodic solutions.

2. Kermack and McKendrick (1927) were among the first to mathematical model epidemics, including relating their models to real data. They proposed the following simple model:

$$\begin{aligned}
\dot{x} &= -kxy \\
\dot{y} &= kxy - ly \\
\dot{z} &= ly
\end{aligned}$$

where x, y, z represent healthy, sick and dead people respectively.

- Give an interpretation of the terms of the equations in terms of interacting populations and their kinetics.
- Explain why you can analyse this system in the $x - y$ phase *plane* (rather than in three-dimensions).
- Find and classify all the fixed points.
- Sketch the nullclines and some vector field directions.
- Find a conserved quantity (hint: consider dy/dx and try to integrate it).
- Sketch a phase portrait using the information found so far.
- What do you expect to happen as $t \to \infty$.
- Let (x_0, y_0) be the initial condition. An *epidemic* is said to occur if $y(t)$ increases initially. Under what conditions does an epidemic occur?

3. Here are two simple 'inverse problems': rather than giving you equations to determine the behaviour of, here you need to construct appropriate equations given desired solution behaviour.

a) Write down a system of differential equations

$$\begin{aligned}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{aligned}$$

such that it has two and only two fixed points: one at $(x, y) = (2, 1)$, and one at $(x, y) = (-2, 1)$.

b) Write down a system of differential equations

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

such that both the x and y axes are invariant under the flow, some orbits tend to the origin as $t \to \infty$ and all other orbits have an x-component tending to $+\infty$ as $t \to \infty$.

For this last case, verify your system has these properties by plotting it using XPP. Include the nullclines (XPP can draw these for you!).

4. (Exam 2018) Consider the system:

$$\dot{x} = x^2 - y - 1$$
$$\dot{y} = (x - 2)y$$

where $x, y \in \mathbb{R}$.

a) Find and classify all three of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields *on* the nullclines in your sketch.

c) The nullclines above separate the phase-plane into eight regions, each of which has one of four different qualitative flow directions: $\boxed{\nearrow}$, $\boxed{\searrow}$, $\boxed{\swarrow}$, $\boxed{\nwarrow}$. Add boxed arrows such as these to your sketch to indicate the direction field in each region of phase space (you can use the information you determined in the previous part of this question without further justification).

Approximating stable/unstable manifolds

Short

1. Complete the example from the lecture, i.e. find the stable manifold $W^s_{loc}(0)$ for

$$\dot{x} = x$$
$$\dot{y} = -y + x^2$$

2. Consider

$$\dot{x} = 2x + y^2$$
$$\dot{y} = -y$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ to all orders.

Long

(Warning: somewhat long and tedious - but worth attempting. Can also use symbolic math in Matlab/Python/Wolfram alpha to help...).

3. Consider the system

$$x' = -2x - 3y - x^2$$
$$y' = x + 2y + xy - 3y^2$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ up to (i.e. including) cubic order.

Periodic orbits, trapping regions etc.

1. Work through Strogatz (1994) Examples 7.3.2 and 8.3.1 on constructing trapping regions (see the Lecture 7 handout).
2. See if you can determine whether the following systems have any periodic orbits.

a)

$$\dot{x} = y$$
$$\dot{y} = (x^2 + 1)y - x^5$$

b)

$$\dot{x} = y$$
$$\dot{y} = y^2 + x^2 + 1$$

c)

$$\dot{x} = 1 + x^2 + y^2$$
$$\dot{y} = (x - 1)^2 + 4$$

3. (Warning - relatively challenging/involved question). Consider the system

$$x' = x - y - x(x^2 + 2y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

- Re-write the system in polar coordinates (r, θ) where $x = r\cos\theta, y = r\sin(\theta)$. Hint: use the identities $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.
- Determine a region bounded by two circles (i.e. annulus shaped), each of which are centred at the origin, such that the flow is radially outward on the inner circle and radially inward on the outer circle.
- Show that there is a periodic orbit in this region, i.e. for $r_{inner} \leq \sqrt{x^2 + y^2} \leq r_{outer}$.
- Use XPP (or Matlab or Python etc) to plot the periodic orbit (and some neighbouring orbits) in the phase plane.

4. Look up the definition of Liapunov/Lyapunov functions (e.g. in Strogatz 1994 Section 7.2). When available, these give a general method for proving global stability of fixed points. The basic idea is to find 'energy-like' functions that are monotonically decreasing along trajectories.

- Explain how these could be used as another method to rule out the existence of periodic orbits.