Key Techniques / Defs, → Adjointness

Integration by parts.

$$\boxed{\int_{x_1}^{x_2} f'g = -\int_{x_1}^{x_2} fg' + gf\Big]_{x_1}^{x_2}}$$

remove ' from f → put on g

Inner product

$$\boxed{\langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \phi\psi \, dx}$$

self-adjoint ~ 'symmetric'

Adjoint defⁿ → 'normal' linear op. (such u & v, transpose / adjoint L).

$$\boxed{\langle v, Lu \rangle} = \langle L^* v, u \rangle = \boxed{\langle u, L^* v \rangle}$$

$$\left[\text{cf } \boxed{\underline{v^T L u}} \overset{\text{scaler}}{=} (v^T L u)^T = \boxed{u^T L^T v} \right].$$

– like transpose

Adjoint of $\boxed{\text{Diff}}$ → slightly diff → BC!

$$\langle v, Lu \rangle = \int (Lu) v = \int \Big(\overset{\text{def}^n}{\quad} \Big) v$$

→ integrate by parts. (remove deriv. from u, put on v → ignore BC).

get
Lagrange Identity

$$\boxed{\int (Lu) v - \int u (L^* v) = \underset{\uparrow}{J(u,v)} \Big|_a^b}$$

Conjunct of u, v.

General 2^{nd} order diff op.

$$\boxed{L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)}$$

Use Lagrange (via int. by parts)

$$\boxed{L^* v = (a_2 v)'' - (a_1 v)' + a_0 v}$$

$$= (a_2' v + a_2 v')' - (a_1' v + a_1 v') + a_0 v$$

$$= (a_2'' v + a_2' v' + a_2' v' + a_2 v'')$$

$$- a_1' v - a_1 v' + a_0 v$$

$$= a_2 v'' + (2a_2' - a_1) v' + (a_2'' - a_1' + a_0) v$$

$$\therefore \boxed{L^* = a_2 \frac{d^2}{dx^2} + (2a_2' - a_1)\frac{d}{dx} + (a_2'' - a_1' + a_0)}$$

or

$$\boxed{L^* = b_2 \frac{d^2}{dx^2} + b_1 \frac{d}{dx} + b_0} \quad (b\text{'s } f(x))$$

where $\boxed{b_2 = a_2,\ b_1 = 2a_2' - a_1,\ b_0 = (a_2'' - a_1' + a_0)}$

& $J(u,v) = a_2(vu' - uv') + uv(a_1 - a_2')$

$\boxed{\text{Self-adjoint conditions}}$: $a_2 = a_2$ ✓, $2a_2' - a_1 = a_1 \Rightarrow a_2' = a_1$

& $a_2'' = a_1'$, sat. by $\boxed{a_2' = a_1}$ if C^∞.

Sub into $\boxed{\text{gen form}}$

get $a_2 \frac{d^2}{dx^2} + a_2' \frac{d}{dx} + a_0 = \boxed{\frac{d}{dx}\left(a_2 \frac{d}{dx}\right) + a_0}$ $\boxed{\text{self-adjoint ODE.}}$