Decision-Making & Modelling Under Uncertainty (DMU)

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[10 lectures/tutorials]

- Decision-making under uncertainty [5/10]
  - Basic concepts
  - Risk, probability, utility
  - Statistical: extended setup
    - formulation & empirical risk approx.
    - minimax & Bayes
  - Tutorial sheet
- Modelling under uncertainty { models of risk & intervention } [5/10]
  - probability, graphical models, & independence
  - causal interpretations of graphical models
  - stochastic process models (esp. Markov)
  - simulation & estimation tools
  - Tutorial sheet

Lecture 2: Decisions under risk

Let's consider decision making under $\boxed{\text{risk}}$ ie known probabilities over 'states of nature'

We need to learn/recap basic ideas of $\boxed{\text{probability}}$ & $\boxed{\text{expectation}}$

& then consider how we go from $\boxed{\text{utility}}$ to $\boxed{\text{expected utility}}$

I will assume some familiarity with basic probability tho' will recap here + provide more material later.

- eg summary sheet?

Probability

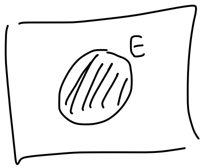
What is probability?

- Multiple interpretations ('meaning') of same formal/mathematical theory eg
  - classical (ratios of cases)
  - frequentist (limit of relative freq.)
  - propensity (tendency to produce obs. freq.)
  - subjective (degree of belief)

  → see eg Resnik (1987) or Gillies (2012), Hacking (2006)

- Multiple (equivalent) formalisations too! Eg
  - Kolmogorov (sets)
  - Boole, Peirce, Bernstein, Keynes, Cox (propositions)
  - Ramsey, De Finetti (bets)
  - Huygens (expectation)

Event: ⟷ Proposition (statement)



$x \in E$ } "event E occurs"

stats | philosophy

Probability: standard setup (Kolmogorov)

1. Sample space S (or Ω) of all possible 'elementary outcomes' s (ie $s \in S$)
2. A collection of events, where each event E is a subset $E \subseteq S$ of the sample space
3. A probability function $\mathbb{P}$ (or 'measure') that assigns a 'probability' $\mathbb{P}(E)$ to each event $E \subseteq S$.

where:
- $0 \leq \mathbb{P}(E) \leq 1$ for all $E \subseteq S$
- $\mathbb{P}(S) = 1$
- $\mathbb{P}(\emptyset) = 0$ ($\emptyset$ is empty set)
- if $A, B \subseteq S$ are mutually exclusive, ie $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Note: $\mathbb{P}(E) \longleftrightarrow \mathbb{P}(s \in E)$
$\longleftrightarrow \mathbb{P}$('event E occurs')

→ see any textbook on prob. / supplement / later in course for more.

Example

- toss coin twice

$S = \{HH, HT, TH, TT\}$

$E =$ "first coin is heads"

$= \{HH, HT\} \subseteq S.$

→ Fair coin, independent tosses:

$\underbrace{\mathbb{P}(\{s\})}_{\text{event so subset}} = 0.25$ for all $s \in S$

- conditional prob, for $\mathbb{P}(B) > 0$: (see later)

$$\mathbb{P}(A|B) = \underbrace{\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}_{\text{definition}} = \underbrace{\frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}}_{\text{Bayes' theorem}},$$

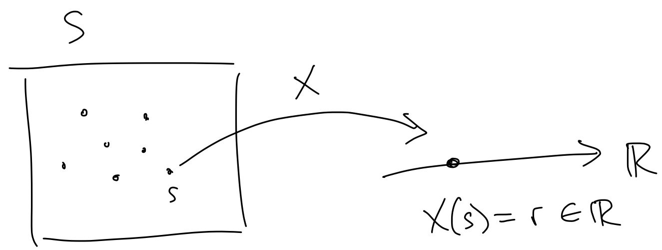
(from def & $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$)

Random variables: assign numbers to outcomes & hence define events.

A random variable is actually a function from outcomes to numbers

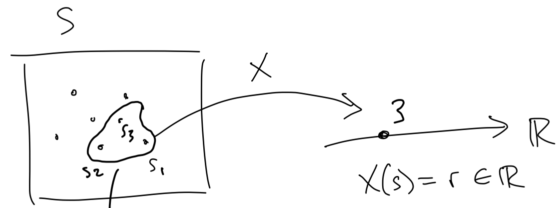
$$\boxed{X : S \to \mathbb{R}}$$

ie:



Particular values of random variables define events

$E_r = \{s \mid X(s) = r\} = X^{-1}[\{r\}]$



all same value of $X(s)$: $\{s_1, s_2, s_3\}$

$= \{s \mid X(s) = 3\}$

Expectation of random variables

For random variables X, X_1, X_2, expectation $\mathbb{E}$ satisfies

1. If $X > 0$, $\mathbb{E}[X] > 0$
2. If c is a constant, $\mathbb{E}[cX] = c\mathbb{E}[X]$
3. $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$
4. $\mathbb{E}(1) = 1$

→ ie $\mathbb{E}$ is a linear operator on RVs.

Probability via expectation: for event A,

$$\boxed{\mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A]}$$

where $\mathbb{1}_A$ is the indicator RV for A:

$$\mathbb{1}_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \notin A \end{cases}$$

→ can develop prob. theory from expectation!

Example:

$S = \{HH, HT, TH, TT\}$ as before

$X(\{s\}) =$ "number of heads in $s \in S$"

$$\underbrace{X \text{ values}} = \{2, \underbrace{1, 1}_{\text{same}}, 0\}$$

So
$$\mathbb{P}(X=2) = \mathbb{P}(\{HH\})$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{H,T, TH\})$$
$$\underbrace{\mathbb{P}(X=0)}_{\text{RV form}} = \underbrace{\mathbb{P}(\{TT\})}_{\text{event form}}$$

For discrete random vars : $\mathbb{E}$ = weighted sum

$$\mathbb{P}(\{s_i\}) = p_i \quad , i = 1, \ldots, N$$

$$\mathbb{E}[X(s)] = \sum_{i=1}^{N} p_i \cdot X(s_i)$$

Sample probabilities (sample approx. of $\mathbb{E}$)

$$p_i = f_i = \text{observed frequency}$$

$$f_i = \frac{\#\text{ times outcome } i \text{ occurred}}{\#\text{ 'trials'}}$$

For distinct outcomes:

$$f_i = \frac{1}{N} \quad \text{for all } i$$

$$\& \quad \mathbb{E}[X] \approx \sum_{i=1}^{N} \frac{1}{N} \cdot X(s_i)$$

$$= \frac{1}{N} \sum_i X(s_i)$$

$$= \text{sample average}$$

(can just count repeated values multiple times, each with weight $\frac{1}{N}$)

So...

Utility revisited: expected utility rule ?

Idea:

From :

Preferences over outcomes
<->
Utility of outcomes

So far

To :

Preferences over 'lotteries' over outcomes
<->
Utility of 'lotteries' = Expected utility of outcomes under 'lottery'

Lotteries (or 'prospects'): consist of

- collection of outcomes (or 'prizes')
- probability for each outcome

eg $L = \begin{cases} \text{win} & \text{lose} & \text{draw} \\ 0.6 & 0.3 & 0.1 \end{cases}$

We can start from two (& one*) outcome lotteries & build up (see later)

$\boxed{\text{Define}}$ $L(p, x, y)$ as 'lottery where you recieve outcome (prize) x with probability p & outcome y with prob. $1-p$.'

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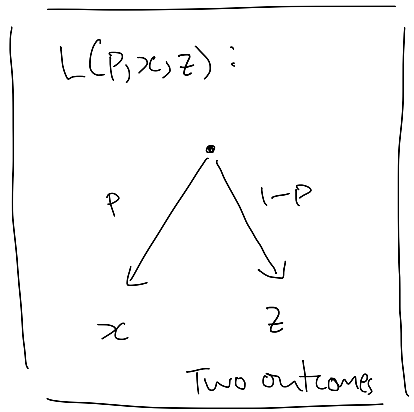
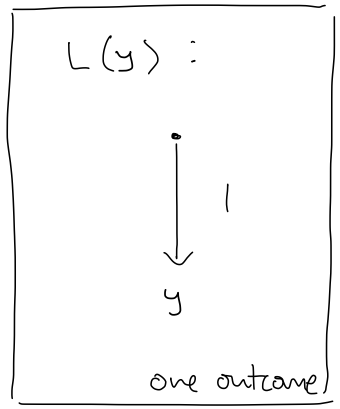
* If $p = 1$ then you are guaranteed to recieve x & we can write $L(x)$

→ each outcome can be treated as a 'degenerate' lottery

Example lotteries:





Question: How do we value (assign utilities to) lotteries?

→ Assume have utilities over outcomes:

$$u(x) > u(y) > u(z)$$

Eg x = trip to Wellington

y = trip to Christchurch

z = trip to Dunedin

(assume $x \succ y \succ z$)

Sketch/Motivation

→ Step one: define value of guaranteed outcome (special lottery)

Define: $\boxed{u(L(y)) = u(y)}$

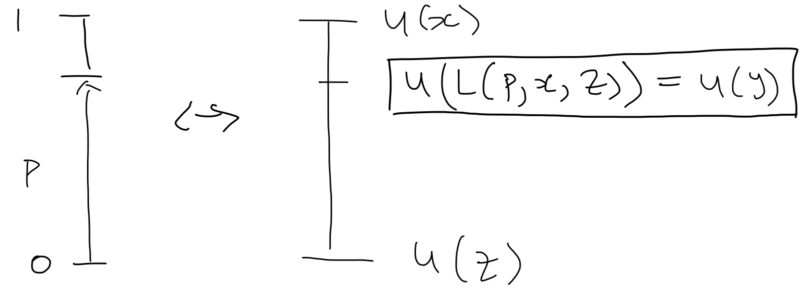
→ Step two: given y, for what probability p you would be indifferent between (prefer neither) $L(p, x, z)$ & $L(y)$?

Key: von Neumann & Morgenstern (VNM) assumed that for any x, y, z, $u(x) > u(y) > u(z)$, there is always a lottery with prob. p such that $L(p, x, y)$ & $L(y)$ are valued the same.

VNM axioms define interval scale [see Resnik for axioms/derivation]

- relative differences in values matter
  - not just order
- unique up to positive linear transformation
  - [see Resnik]

visualise as:



ie $\boxed{\text{probability} \leftrightarrow \text{relative degree of preference}}$

Then:

$$
\begin{aligned}
u(L(p, x, z)) &= u(y) \quad \text{[indifferent]} \\
&= u(z) + p \cdot [u(x) - u(z)] \quad \text{[interval location]} \\
&= p \cdot u(x) + (1-p)\, u(z) \\
&= \mathbb{E}[u(\text{outcome})] = \text{expected utility of outcomes under } L(p, x, z).
\end{aligned}
$$

ie

$$\boxed{u(L(p,x,z)) = \mathbb{E}[u(\text{outcome})]} \text{ [under lottery]}$$

$\Rightarrow$ Utility of lottery = expected utility of outcomes of lottery — Take home!

Von Neumann & Morgenstern

$\rightarrow$ proved that if people's preferences between lotteries satisfy a particular set of axioms then the utility of a lottery is given by the expected utility of the outcomes under that lottery

$\rightarrow$ called the 'expected utility theorem'

(see Resnik 1987 for details)

General lotteries : same idea

$L(\mathbb{P}, Z)$

probability distribution over outcome set

outcome set

$\underset{\text{outcome}}{z} \in \underset{\text{set}}{Z}$

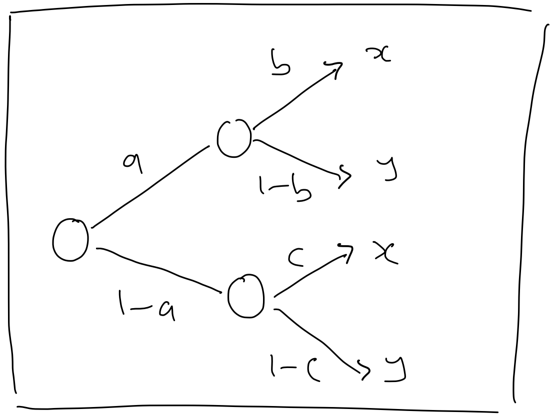
$$\Rightarrow \boxed{U(L(\mathbb{P}, Z)) = \mathbb{E}_{\mathbb{P}}[U(z)]}$$

(under VNM axioms)

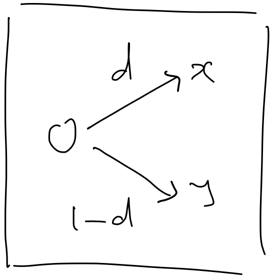
Compound lotteries

According VNM theory we can evaluate compound lotteries (lotteries where the outcomes are lotteries etc) according to probability trees

Eg $L_1 = L(a, L(b, x, y), L(c, x, y))$:



is equivalent to $L_2 = L(d, x, y)$:



where $\boxed{d = ab + (1-a)c}$

ie $\underline{U(L_1) = U(L_2)}$

Application to decision theory

- Each decision defines a lottery
- We value decisions by valuing the lotteries:

	s_1	s_2	
d_1	$U(d_1, s_1)$, $\mathbb{P}(s_1; d_1)$	$U(d_1, s_2)$, $\mathbb{P}(s_2; d_1)$	lottery 1. $U(d_1) = U(\text{lott.}1) = \mathbb{E}[U(d_1, S)]$
d_2	$U(d_2, s_1)$, $\mathbb{P}(s_1; d_2)$	$U(d_2, s_2)$, $\mathbb{P}(s_2; d_2)$	lottery 2 $U(d_2) = U(\text{lott.}2) = \mathbb{E}[U(d_2, S)]$

By choosing the highest expected utility decision we are choosing the lottery we prefer

→ a 'representation' theorem (preferences first, utility second)

Value (utility) of a decision problem

→ value of a decision = expected utility of lottery

→ value of set of decisions (decision problem)

= max value of available decisions

= value of maximum utility decision

= value of maximum expected utility lottery

$$\boxed{U(\{d_1, d_2, ..\}) = \max_{d_i \in \{d_1, d_2, ..\}} U(d_i)}$$

Examples / Problems

A) Suppose we have three outcomes x, y, z with
$u(x) = 1 \quad u(y) = c, \quad u(z) = 0$
for some $0 < c < 1$.

Suppose you are indifferent between a lottery $L(p, x, z)$ & y for given p.
→ determine c, ie $u(y)$, in terms of p.

B). Calculate the expected utility of the compound lottery:

$$L(0.6, L(0.5, 0, 4), 1)$$

C). Solve problem 1 in Resnik 4-1 (attached)

Answers (A & B).

A. $U(y) = U(L(p, x, z))$

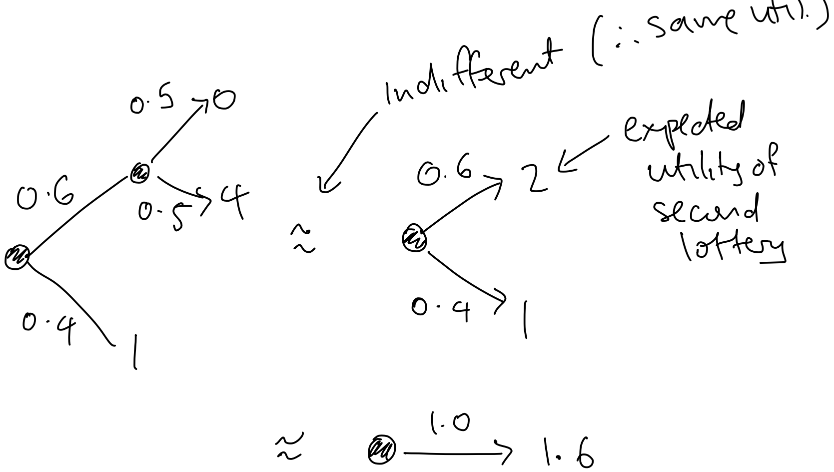
$= \mathbb{E}_L(U)$ } expected utility of outcomes under lottery

$= pU(x) + (1-p)U(z)$

$= p$

$\Rightarrow c = p$!

B.



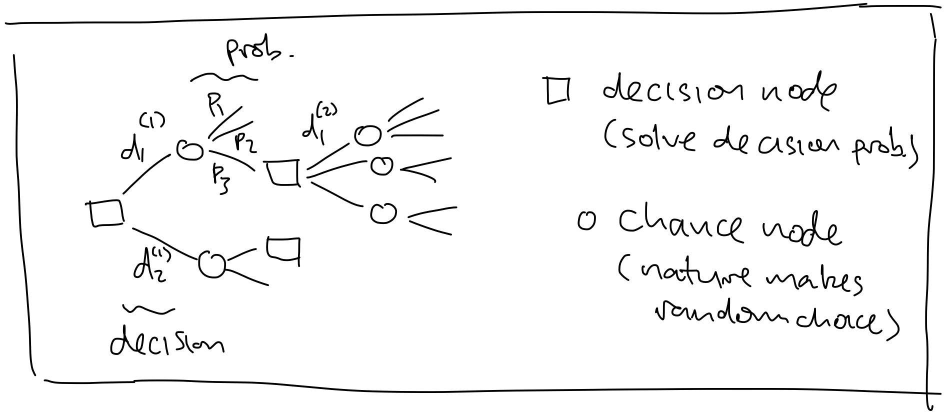
$\Rightarrow U(\text{comp. lottery}) = 1.6$

Multistage decisions under risk (see Dynamic Prog. module)

Averaging out & folding back (Raiffa 1968)

(= Dynamic programming, Bellman)

Decision trees (Lecture 1 handout):



→ Decompose single complex decision into multiple stages:

↳ equivalent but often more convenient/natural than single decision form

↳ single decision = policy/strategy = vector of simpler decisions

ie $\boxed{d = (d^{(1)}, d^{(2)}, d^{(3)}, \dots)}$

→ Alternate between decision problems & chance outcomes

→ Can use expected utility + optimality principle to 'solve' decision tree (see later module)

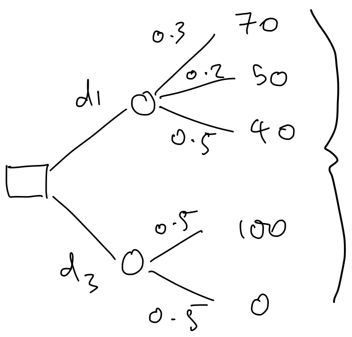
Brief idea:

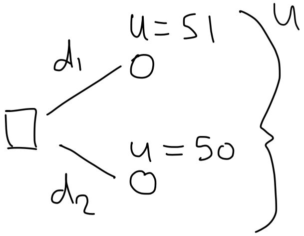
Lottery → utility of lottery = expected utility of outcomes

Decision point → max utility (value of decision problem = value of best decision)

key!

Solve backwards from outcomes (can value):





$u(\{d_1, d_2\}) = 51$
$= u(d_1)$

□ 51

$\boxed{\text{etc}}$

Note: result/outcome of decision often lottery + immediate payoff/cost ie lottery is pay \$50 + get chance p at \$100. Can include before or after chance payoff since:

$\mathbb{E}[\text{random} + \text{fixed}] = \mathbb{E}[\text{random}] + \text{fixed}$

Chapter 4
DECISIONS UNDER RISK: UTILITY

4-1. Interval Utility Scales

Utilities are just as critical to our account of decisions under risk as probabilities since the rule of maximizing expected utility operates on them. But what are utilities? And what do utility scales measure? In discussing decisions under ignorance I hinted at answers to these questions. But such allusions will not suffice for a full and proper understanding of decisions under risk. So let us begin with a more thorough and systematic examination of the concept of utility.

The first point we should observe is that ordinal utility scales do not suffice for making decisions under risk. Tables 4-1 and 4-2 illustrate why this is so. The

4-1

A_1	6 ¼	1 ¾
A_2	5 ¼	2 ¾

expected utilities of A_1 and A_2 are 9/4 and 11/4, respectively, and so A_2 would be picked. But if we transform table 4-1 ordinally to table 4-2 by simply raising

4-2

A_1	20 ¼	1 ¾
A_2	5 ¼	2 ¾

the utility number 6 to 20, the expected utilities are now 23/4 and 11/4, which results in A_1 being picked. Thus two scales that are ordinal transformations of each other might fail to be equivalent with respect to decisions under risk.

This stands to reason anyway. Ordinal scales represent only the relative standings of the outcomes; they tell us what is ranked first and second, above and below, but no more. In a decision under risk it is often not enough to know that you prefer one outcome to another; you might also need to know whether you prefer an outcome *enough* to take the risks involved in obtaining it. This is reflected in our disposition to require a much greater return on an investment of $1,000,000 than on one of $10—even when the probabilities of losing the investment are the same.

Happily, interval scales are all we require for decisions under risk. In addition to recording an agent's ranking of outcomes, we need measure only the relative lengths of the "preference intervals" between them. To understand what is at stake, suppose I have represented my preferences for cola (C), ice cream (I), apples (A), and popcorn (P) on the following line.

least preferred			most preferred
P	A	C	I

If I now form a scale by assigning numbers to the line, using an ordinal scale would require me only to use numbers in ascending order as I go from left to right. But if I use an interval scale, I must be sure that the *relative* lengths of the intervals on the line are reflected as well. Thus I could not assign 0 to P, 1 to A, 2 to C, and 3 to I, for this would falsely equate the interval between, say P and A with that between C and I. More generally, if items x, y, z, and w are assigned utility numbers $u(x)$, $u(y)$, $u(z)$, and $u(w)$ on an interval scale, these numbers must satisfy the following conditions:

a. xPy if and only if $u(x) > u(y)$.

b. xIy if and only if $u(x) = u(y)$.

the preference interval between x and y is greater than or equal to that between z and w if and only if $|u(x) - u(y)| \geq |u(z) - u(w)|$.

More than one assignment of numbers will satisfy these two conditions but every assignment that does is a positive linear transformation of every other one that does. To illustrate this point, suppose I assign numbers to my preferences as indicated here.

least preferred			most preferred
1	2	7	10
P	A	C	I

Then I have properly represented both the ordinal and interval information. But I could have used other numbers, such as the next set.

least preferred			most preferred
15	25	75	105
P	A	C	I

These are obtained from the first set by a positive linear transformation. (What is it?)

Two ordinal scales count as equivalent if and only if they can be obtained

from each other by means of order-preserving (ordinal) transformations. Two interval scales will count as equivalent if and only if they can be obtained from each other by means of positive linear transformations.

Another type of scale with which you are familiar is ratio scales. We use these for measuring lengths (in yards, feet, meters) or weights (in pounds, grams, ounces). The scales all share two important features: First, they all have natural zero points (no length, no speed, no weight), and second, the scales are used to represent the ratio of a thing measured to some standard unit of measurement. (Something 10 yards long bears the ratio of 10 to 1 to a standard yardstick; the latter can be laid off ten times against the former.) In converting from one ratio scale to another we multiply by a positive constant. (Thus, to obtain inches from feet we multiply by 12.) This turns out to be the equivalence condition for ratio scales: Two ratio scales are equivalent to each other if and only if they may be obtained from each other by multiplying by positive constants. This is a special case of a positive linear transformation; thus ratio scales are a tighter kind of interval scale.

One way to appreciate the difference between ratio and interval scales is to think of changing scales in terms of changing the labels on our measuring instruments. If we had a measuring rod 9 feet long, labeled in feet, and relabeled it in yards, we would need fewer marks on the stick. If we labeled it in inches we would need more marks. But in either case the zero point would remain the same. This is not necessarily so with interval scales. If we had a thermometer marked in degrees Fahrenheit and changed it to degrees Celsius, we would use fewer marks (between the freezing and boiling points of water there are 100 Celsius units in contrast to 180 Fahrenheit units), and we would also shift the zero point upward. Because there is no fixed zero point on our temperature scales, we must be quite careful when we say that something is twice as hot as something else. Suppose that at noon yesterday, the outdoor temperature was at the freezing point of water and that today at noon it measured 64 degrees on the Fahrenheit scale. Was it twice as hot today at noon as it was yesterday? On the Fahrenheit scale, yes, but not on the Celsius scale. Suppose, for contrast, that I am driving at 60 miles per hour and you are driving at 30. Now convert our speeds to kilometers per hour. You will see that I would still be driving twice as fast as you.

Since we will use interval scales rather than ratio scales to represent preference intervals, we cannot assume that arithmetic operations that we perform freely on speeds, weights, or distances make sense for utilities. I will return to this point later.

We have already seen that decisions under risk require more than ordinal scales. Will interval scales suffice? Or must we move on to ratio scales? No, we need not, for *any two scales that are positive linear transformations of each other will produce the same ranking of acts in a decision table* and, thus, will yield the same decisions. In short, interval scales suffice for decisions under risk. Let us now prove this.

Let table 4-3 represent any decision table and any two acts in it. I will call the acts A_i and A_j, and for convenience I will write them next to each other, but in fact they might have many rows between them or above and below them.

4-3

A_i	u_1 p_1	u_2 p_2	. . .	u_n p_n
A_j	v_1 q_1	v_2 q_2	. . .	v_n q_n

The u's and v's are utility numbers and the p's and q's are probabilities. Since the states may be dependent on the acts, we cannot assume that the p's and q's are equal. However, we can assume that the probabilities across for each row sum to 1. The expected utilities for A_i and A_j are given by the formulas

$$EU(A_i) = u_1p_1 + u_2p_2 + \ldots + u_np_n$$
$$EU(A_j) = v_1q_1 + v_2q_2 + \ldots + v_nq_n.$$

Now a positive linear transformation of the scale used in table 4-3 would cause each u and each v to be replaced, respectively, by $au + b$ and $av + b$ (with $a > 0$). Thus after the equation the formulas for the expected utilities would be

$$EU_{\text{new}}(A_i) = (au_1 + b)p_1 + (au_2 + b)p_2 + \ldots + (au_n + b)p_n$$
$$EU_{\text{new}}(A_j) = (av_1 + b)q_1 + (av_2 + b)q_2 + \ldots + (av_n + b)q_n.$$

If we multiply through by the p's and q's and then gather at the end all the terms that contain no u's or v's we obtain

$$EU_{\text{new}}(A_i) = [au_1p_1 + au_2p_2 + \ldots + au_np_n] + [bp_1 + bp_2 + \ldots + bp_n]$$
$$EU_{\text{new}}(A_j) = [av_1q_1 + av_2q_2 + \ldots + av_nq_n] + [bq_1 + bq_2 + \ldots + bq_n].$$

If we now factor out the a's the expressions remaining in the left-hand brackets are the old expected utilities. On the other hand, if we factor out the b's the expressions remaining in the right-hand brackets are p's and q's that sum to 1. Thus our new utilities are given by this pair of equations:

$$EU_{\text{new}}(A_i) = aEU(A_i) + b$$
$$EU_{\text{new}}(A_j) = aEU(A_j) + b.$$

Now since $a > 0$, $aEU(A_i)$ is greater than (less than, or equal to) $aEU(A_j)$ just in case $EU(A_i)$ stands in the same relation to $EU(A_j)$. Furthermore, this relation is preserved if we add b to $aEU(A_i)$ and $aEU(A_j)$. In other words, $EU_{\text{new}}(A_i)$ and $EU_{\text{new}}(A_j)$ will stand in the same order as $EU(A_i)$ and $EU(A_j)$. This means that using expected utilities to rank the two acts A_i and A_j will yield the same results whether we use the original scale or the positive linear transformation of it. But our reasoning has been entirely general, so the same conclusion holds for all expected utility rankings of any acts using these two scales. In short, they are equivalent with respect to decision making under risk.

PROBLEMS

1. Suppose you can bet on one of two horses – Ace or Jack – in a match race. If Ace wins you are paid \$5; if he loses you must pay \$2 to the track. If you bet on Jack and he loses, you pay the track \$10. You judge each horse to be as likely to win as the other. Assuming you make your decisions on the basis of expected monetary values, how much would a winning bet on Jack have to pay before you would be willing to risk \$10?
2. Suppose the interval scale u may be transformed into u' by means of the transformation

   $$u' = au + b \qquad (a > 0).$$

   Give the transformation that converts u' back into u.
3. Suppose the u' of the last problem can be transformed into u'' by means of the transformation

   $$u'' = cu' + d \qquad (c > 0).$$

   Give the transformation for converting u into u''.
4. Suppose s and s' are equivalent ratio scales. Show that if $s(x) = 2s(y)$, then $s'(x) = 2s'(y)$.
5. Suppose you have a table for a decision under risk in which the probabilities are independent of the acts. Show that if you transform your utility numbers by adding the number b_i to each utility in column i (and assume that the numbers used in different columns are not necessarily the same), the new table will yield the same ordering of the acts.

expected regret

4-2. Monetary Values vs. Utilities

A popular and often convenient method for determining how strongly a person prefers something is to find out how much money he or she will pay for it. As a general rule people pay more for what they want more; so a monetary scale can be expected to be at least an ordinal scale. But it often works as an interval scale too – at least over a limited range of items. A rough test of this is the agent's being indifferent between the same increase (or reduction) in prices over a range of prices, since the intervals remain the same though the prices change. Thus if I sense no difference between \$5 increases (e.g., from \$100 to \$105) for prices between \$100 and \$200, it is likely that a monetary scale can adequately function as an interval scale for my preferences for items in that price range. Within this range it would make sense for me to make decisions under risk on the basis of expected monetary values (EMVs).

Since we are so used to valuing things in terms of money – we even price intangibles, such as our own labor and time or a beautiful sunset, as well as necessities, such as food and clothing – it is no surprise that EMVs are often used as a basis for decisions under risk. My earlier insurance and car purchase examples typify this approach. Perhaps this is the easiest and most appropriate method for making business decisions, for here the profit motive is paramount.

It is both surprising and disquieting that a large number of nonbusiness de-

$1,000,000 – but never $1. Thus how can we connect this figure with a cash value for the bet? Why – assuming money is all that counts – would it be rational for you to sell your ticket for $2? One is tempted to answer in terms of averages or long runs: If there were many people in your situation, their winnings would average $1; if this happened to you year after year, your winnings would average $1. But this will not work for the case at hand, since, by hypothesis, you alone have a free ticket and there will be only one lottery. With a one-shot decision there is nothing to average; so we have still failed to connect EMVs with cash values.

PROBLEMS

1. Given a utility scale u, we can formulate a disutility scale, $d(u)$, by multiplying each entry on the u-scale by -1. The expected disutility of an act is calculated in the same way as its expected utility except that every utility is replaced by its corresponding disutility. Reformulate the rule for maximizing expected utilities as a rule involving expected disutilities.
2. Show that the expected disutility of an act is equal to -1 times its expected utility.
3. Show that using a disutility scale and the rule you formulated in problem 1 yields the same rankings of acts as maximizing expected utilities does.
4. The St. Petersburg game is played as follows. There is one player and a "bank." The bank tosses a fair coin once. If it comes up heads, the player is paid $2; otherwise the coin is tossed again with the player being paid $4 if it lands heads. The game continues in this way with the bank continuing to double the amount set. The game stops when the coin lands heads.

   Consider a modified version of this game. The coin will be tossed no more than two times. If heads comes up on neither toss, the player is paid nothing. What is the EMV of this game?

   Suppose the coin will be tossed no more than n times. What is the EMV of the game?

   Explain why an EMVer should be willing to pay any amount to play the unrestricted St. Petersburg game.
5. Consider the following answer to the one-shot lottery objection to EMVs: True, there is only one lottery and only one person has a free ticket. But in a hypothetical case in which there were many such persons or many lotteries, we would find that the average winnings would be $1. Let us identify the cash value of the ticket with the average winnings in such hypothetical cases. It follows immediately that the cash value equals the EMV.

   Do you think this approach is an adequate solution to the problem of relating EMVs to cash values?

4-3. Von Neumann-Morgenstern Utility Theory

John Von Neumann, a mathematician, and Oskar Morgenstern, an economist, developed an approach to utility that avoids the objections we raised to EMVs.

Although Ramsey's approach to utility antedates theirs, today theirs is better known and more entrenched among social scientists. I present it here because it separates utility from probability, whereas Ramsey's approach generates utility and subjective probability functions simultaneously.

Von Neumann and Morgenstern base their theory on the idea of measuring the strength of a person's preference for a thing by the risks he or she is willing to take to receive it. To illustrate that idea, suppose that we know that you prefer a trip to Washington to one to New York to one to Los Angeles. We still do not know how *much more* you prefer going to Washington to going to New York, but we can measure that by asking you the following question: Suppose you were offered a choice between a trip to New York and a lottery that will give you a trip to Washington if you "win" and one to Los Angeles if you "lose." How great a chance of winning would you need to have in order to be indifferent between these two choices? Presumably, if you prefer New York quite a bit more than Los Angeles, you will demand a high chance at Washington before giving up a guaranteed trip to New York. On the other hand, if you only slightly prefer New York to Los Angeles, a small chance will suffice. Let us suppose that you reply that you would need a 75% chance at Washington – no more and no less. Then, according to Von Neumann and Morgenstern, we should conclude that the New York trip occurs 3/4 of the way between Washington and Los Angeles on your scale.

Another way of representing this is to think of you as supplying a ranking not only of the three trips but also of a lottery (or gamble) involving the best and worst trips. You must be indifferent between this lottery and the middle-ranked trip. Suppose we let the expression

$$L(a, x, y)$$

stand for the lottery that gives you a chance equal to a at the prize x and a chance equal to $1 - a$ at the prize y. Then your ranking can be represented as

Washington
New York, $L(3/4$, Washington, Los Angeles$)$
Los Angeles.

We can use this to construct a utility scale for these alternatives by assigning one number to Los Angeles, a greater one to Washington, and the number 3/4 of the way between them to New York. Using a 0 to 1 scale, we would assign 3/4 to New York – but any other scale obtained from this by a positive linear transformation will do as well.

Notice that since the New York trip and the lottery are indifferent they are ranked together on your scale. Thus the utility of the lottery itself is 3/4 on a 0 to 1 scale. But since on that scale the utilities of its two "prizes" (the trips) are 0 and 1, the expected utility of the lottery is also 3/4. We seem to have forged a link between utilities and expected utilities. Indeed, Von Neumann and Morgenstern showed that if an agent ranks lotteries in the manner of our example, their utilities will equal their expected utilities. Let us also note that the Von

Neumann-Morgenstern approach can be applied to any kind of item—whether or not we can sensibly set a price for it—and that it yields an agent's personal utilities rather than monetary values generated by the marketplace. This permits us to avoid our previous problems with monetary values and EMVs.

(You might have noticed the resemblance between the Von Neumann-Morgenstern approach and our earlier approach to subjective probability, where we measured degrees of belief by the amount of a valued quantity the agent was willing to stake. Ramsey's trick consisted in using these two insights together without generating the obvious circle of defining probability in terms of utility and utility in turn in terms of probability.)

The Von Neumann-Morgenstern approach to utility places much stronger demands on agents' abilities to fix their preferences than do our previous conditions of rationality. Not only must agents be able to order the outcomes relevant to their decision problems, they must also be able to order all lotteries involving these outcomes, all compound lotteries involving those initial lotteries, all lotteries compounded from those lotteries, and so on. Furthermore, this ordering of lotteries and outcomes (I will start calling these *prizes*) is subject to constraints in addition to the ordering condition. Put in brief and rough form, these are: (1) Agents must evaluate compound lotteries in agreement with the probability calculus (reduction-of-compound-lotteries condition); (2) given three alternatives A, B, C with B ranked between A and C, agents must be indifferent between B and $\boxed{\text{some}}$ lottery yielding A and C as prizes (continuity condition); (3) given two other lotteries agents will prefer the one giving the better "first" prize—if everything else is equal (better-prizes condition); (4) given two otherwise identical lotteries, agents will prefer the one that gives them the best chance at the "first" prize (better-chances condition). If agents can satisfy these four conditions plus the ordering condition of chapter 2, we can construct an interval utility function u with the following properties:

(1) $u(x) > u(y)$ if and only if xPy (P: 'prefer')
(2) $u(x) = u(y)$ if and only if xIy (I: 'indifferent')
(3) $u[L(a, x, y)] = au(x) + (1-a)u(y)$
(4) Any u' also satisfying (1)–(3) is a positive linear transformation of u.

You should recognize (1) and (2) from our discussion of decisions under ignorance (chapter 2). They imply that u is at least an ordinal utility function. But (3) is new. It states that the utility of a lottery is equal to its expected utility. We can also express this by saying that u has the *expected utility property*. The entire result given by (1)–(4) is known as the *expected utility theorem*. Let us now turn to a rigorous proof of it.

First we must specify lotteries more precisely than we have. The agent is concerned with determining the utilities for some set of outcomes, alternatives, or prizes. Let us call these *basic prizes*. Let us also assume that the number of basic prizes is a finite number greater than 1 and that the agent is not indifferent between all of them. Since we can assume that the agent has ranked the prizes, some will be ranked at the top and others at the bottom. For future reference,

let us select a top-ranked prize and label it "B" (for best). Let us also select a bottom-ranked prize and label it "W" (for worst).

I will now introduce compound lotteries by the following rule of construction. (In mathematical logic this is called an inductive definition.)

Rule for Constructing Lotteries:
1. Every basic prize is a lottery.
2. If L_1 and L_2 are lotteries, so is $L(a, L_1, L_2)$, where $0 \le a \le 1$.
3. Something is a lottery if and only if it can be constructed according to conditions 1 and 2.

Thus lotteries consist of basic prizes, simple lotteries involving basic prizes, further lotteries involving basic prizes or simple lotteries, ad infinitum. This means that B and W are lotteries, $L(1/2, B, W)$ and $L(3/4, W, B)$ are, and so are $L(2/3, B, L(1/2, B, W))$ and $L(1, L(0, B, W), W)$

PROBLEMS

1. Why can we not assume that there is a single best prize?
2. Why have I assumed that there is more than one basic prize? That the agent is not indifferent between all the prizes? If these assumptions were not true, would the expected utility theorem be false?
3. What chance at B does each of the following lotteries give?
   a. $L(1, B, W)$
   b. $L(1/2, L(1, W, B), L(1/2, B, W))$
   c. $L(a, B, B)$
4. Why should an agent be indifferent between $L(a, B, W)$ and $L(1-a, W, B)$?
5. Show how to construct one of our lotteries that is equivalent to the lottery with *three* prizes, A, B, and C, that offers a 50% chance of yielding A and 25% chances of yielding B and C.

Now that we have a precise characterization of lotteries, let us turn to precise formulations of the "rationality" conditions the agent must satisfy. The first of these is the familiar *ordering condition*, applied this time not just to basic prizes but also to all lotteries. This means that conditions O1–O8 (discussed in chapter 2) apply to all lotteries. An immediate consequence of this is that we can partition the lotteries into ranks so as to rank together lotteries between which the agent is indifferent while placing each lottery below those the agent prefers to it.

The next condition is called the *continuity condition* because one of its consequences is that the ordering of the lotteries is continuous. It is formulated as follows:

> For any lotteries x, y, and z, if xPy and yPz, then there is some real number a such that $0 \le a \le 1$ and $y\ I\ L(a, x, z)$.

In less formal terms this says that if the agent ranks y between x and z, there is some lottery with x and z as prizes that the agent ranks along with y.

The *better-prizes condition* is next. Intuitively, it says that other things being equal, the agent prefers one lottery to another just in case the former involves better prizes. Put formally:

For any lotteries x, y, and z and any number a ($0 \leq a \leq 1$), xPy if and only if $L(a, z, x)\ P\ L(a, z, y)$ and $L(a, x, z)\ P\ L(a, y, z)$.

There is also the *better-chances condition*, which says roughly that, other things being equal, the agent prefers one lottery to another just in case the former gives a better chance at the better prize. Put in precise terms:

For any lotteries x and y and any numbers a and b (both between 0 and 1, inclusively), if xPy, then $a > b$ just in case $L(a, x, y)\ P\ L(b, x, y)$.

The final condition is called the *reduction-of-compound-lotteries condition* and requires the agent to evaluate compound lotteries in accordance with the probability calculus. To be exact, it goes:

For any lotteries x and y and any numbers a, b, c, d (again between 0 and 1 inclusively), if $d = ab + (1 - a)c$, then $L(a, L(b, x,y), L(c, x, y))\ I\ L(d, x, y)$.

To get a better grip on this condition, let us introduce the *lottery tree* notation (figure 4-1), which is similar to the decision tree notation.

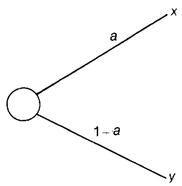


Figure 4-1

This diagram represents the simple lottery that yields the prize x with a chance of a and the prize y with a chance of $1 - a$. Compound lotteries can be represented by iterating this sort of construction, as figure 4-2 illustrates. This is a two-stage lottery whose final prizes are x and y. What are the chances of getting x in this lottery? There is a chance of a at getting into lottery 2 from lottery 1 and a chance of b of getting x. In other words, there is an ab chance of getting x through lottery 2. Similarly, there is a $(1 - a)c$ chance of getting x through lottery 3. Since these are the only routes to x and they are mutually exclusive, the chances for x are $ab + (1 - a)c$. The same type of reasoning shows that the chances for y are $a(1 - b) + (1 - a)(1 - c)$. If we set $d = ab + (1 - a)c$, a little algebra will show that $1 - d = a(1 - b) + (1 - a)c$. The reduction-of-compound-lotteries condition simply tells us that the agent must be indifferent between the compound lottery given earlier and the next simple one (figure 4-3). Notice, by the way, that the condition is improperly named, since the agent is indifferent

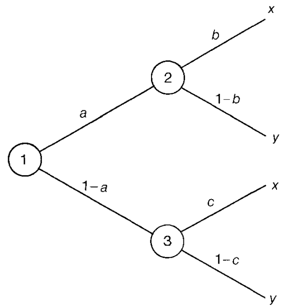


Figure 4-2

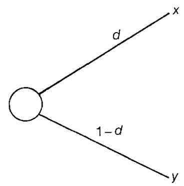


Figure 4-3

to both reductions and *expansions* of lotteries so long as they accord with the probability calculus.

Let us turn now to the proof of the expected utility theorem. I will divide the proof into two parts. The first part will establish that, given that the agent satisfies the ordering, continuity, better-prizes, better-chances, and reduction-of-compound-lotteries conditions, there is a utility function, u, satisfying the expected utility property that represents his preferences. This is called the *existence* part of the proof, since it proves that there exists a utility function having the characteristics given in the theorem. The second part of the proof is called the *uniqueness* part, because it establishes that the utility function constructed in the first part is unique up to positive linear transformations; that is, it is an interval utility function.

Directing ourselves now to the proof of the existence of u, recall that we have already established that there are at least two basic prizes B and W, where the agent prefers B to W and regards B as at least as good as any basic prize and every basic prize as at least as good as W. Since all lotteries ultimately pay in basic prizes and are evaluated by the agent in terms of the probability calculus, there is no lottery ranked above B or below W. (See exercises 4-8 in the next problems section.) Accordingly, fixing the top of our utility scale at 1 and the bottom at 0, we stipulate that

$u(B) = 1$ and $u(x) = 1$ for all lotteries x indifferent to B,
$u(W) = 0$ and $u(x) = 0$ for all lotteries x indifferent to W.

Having taken care of the lotteries at the extremes we must now define u for those in between. So let x be any lottery for which

BPx and xPW

holds. Applying the continuity axiom to this case, we can conclude that there is a number a, where $0 \leq a \leq 1$, such that

$x\ I\ L(a, B, W)$.

If there is just one such a, we will be justified in stipulating that $u(x) = a$. So let us assume that $a' \neq a$ and $x\ I\ L(a', x, y)$ and derive a contradiction. Since a and a' are assumed to be distinct, one is less than the other. Suppose $a < a'$. Then by the better-chances condition,

$L(a', B, W)\ P\ L(a, B, W)$,

but this contradicts the ordering condition since both lotteries are indifferent to x. A similiar contradiction follows from the alternative that $a' < a$. Having derived a contradiction from either alternative, we may conclude that $a = a'$.

We are now justified in stipulating that

$u(x) = a$,

where a is the number for which $x\ I\ L(a, B, W)$. Note that by substituting equals for equals we obtain

$(*)\ x\ I\ L(u(x), B, W)$;

that is, the agent is indifferent between x and the lottery that gives a $u(x)$ chance at B and a $1 - u(x)$ chance at W.

So far we have simply established the existence of a function u that assigns a number to each lottery. We must also show that this is an (interval) utility function that satisfies the expected utility property.

Let us first show that, for all lotteries x and y,

(1) xPy if and only if $u(x) > u(y)$.

By the better chances condition we have

(a) $L(u(x), B, W)\ P\ L(u(y), B, W)$ if and only if $u(x) > u(y)$.

By $(*)$, above, we have

(b) $x\ I\ L(u(x), B, W)$ and $y\ I\ L(u(y), B, W)$.

Using the ordering condition, we can easily prove

(c) for all lotteries x, y, z, and w, if xIy and zIw, then xPz if and only if yPw.

This together with (a) and (b) immediately yields (1).

It is now easy to prove

(2) xIy if and only if $u(x) = u(y)$, for all lotteries x and y.

For if xIy and $u(x) > u(y)$, then, by (1) xPy—a contradiction; so if xIy, then not $u(x) > u(y)$. Similarly, if xIy, then it is false that $u(y) > u(x)$. Thus if xIy, $u(x) = u(y)$. On the other hand, if $u(x) = u(y)$, we can have neither xPy nor yPx without contradicting (1) and the ordering condition. This means that if $u(x) = u(y)$, then xIy, since the ordering condition implies that either xIy, xPy, or yPx.

The rest of this part of our proof will be concerned with showing that for all lotteries x and y,

(3) $u(L(a, x, y)) = au(x) + (1 - a)u(y)$.

To prove this, however, it will be convenient for us to first prove that the following condition follows from the others.

Substitution-of-Lotteries Condition: If $x\ I\ L(a, y, z)$, then both
(a) $L(c, x, v)\ I\ L(c, L(a, y, z), v)$
(b) $L(c, v, x)\ I\ L(c, v, L(a, y, z))$.

This condition states that if the agent is indifferent between a prize (lottery) x and some lottery $L(a, y, z)$, the agent is also indifferent to substituting the lottery $L(a, y, z)$ for x as a prize in another lottery.

Turning to the derivation of the substitution condition, let us abbreviate "$L(a, y, z)$" by "L". Now assume that xIL. We want to show that (a) and (b) hold. I will derive (a) and leave (b) as an exercise. By the ordering condition, $L(c, x, v)$ is indifferent to $L(c, L, v)$ or one is preferred to the other. If $L(c, x, v)$ is preferred to $L(c, L, v)$, then by the better-prizes condition, xPL. But that contradicts xIL. Similarly, if $L(b, L, w)$ is preferred to $L(b, x, w)$, then LPx—again contradicting xIL. So the only alternative left is that the two lotteries are indifferent. This establishes (a).

With the substitution condition in hand we can now turn to the proof of (3). Let us use "L" this time to abbreviate "$L(a, x, y)$." By $(*)$ we have

i) $L\ I\ L(u(L), B, W)$
ii) $x\ I\ L(u(x), B, W)$
iii) $y\ I\ L(u(y), B, W)$.

Thus by substituting $L(u(x), B, W)$ for x and $L(u(y), B, W)$ for y in the lottery L and applying the substitution condition, we obtain

$L\ I\ L(a, L(u(x), B, W), L(u(y), B, W))$.

We can reduce the compound lottery on the right to a simple lottery, thus obtaining

$L(a, L(u(x), B, W), L(u(y), B, W))\ I\ L(d, B, W),$

where $d = au(x) + (1-a)u(y)$. But then by the ordering condition we must have

$L\ I\ L(d, B, W),$

which together with (i) yields

$L(u(L), B, W)\ I\ L(d, B, W).$

If $u(L) > d$ or $d > u(L)$, we would contradict the better-chances condition. So $u(L) = d$. But this is just an abbreviated form of (3).

This completes the existence part of the expected utility theorem.

PROBLEMS

1. Using the existence part of the expected utility theorem show
   a. $L(1, x, y)\ \ I\ x$
   b. $L(0, x, y)\ \ I\ y$
   c. $L(a, x, y)\ \ I\ L(1-a, y, x)$
   d. $L(a, x, x)\ \ I\ x$
2. Derive part (b) of the substitution-of-lotteries condition.
3. Using just the ordering condition, prove:

   If xIy and zIw, then xPz if and only if yPw.
4. In this and the following exercises *do not* appeal to the expected utility theorem. Instead reason directly from the rationality conditions of the theorem.
   a. There is no number a or basic prize x distinct from B for which $L(a, x, B)\ P\ L(a, B, B)$ or $L(a, B, x)\ P\ L(a, B, B)$.
   b. There is no number a or basic prizes x and y distinct from B for which $L(a, x, y)\ P\ L(a, B, B)$.
5. Define the degree of a lottery as follows:

   All basic prizes are of degree zero.

   Let n be the maximum of the degrees of L_1 and L_2, then the degree of $L(a, L_1, L_2)$ is equal to $n+1$.

   Suppose no lottery of degree less than n is preferred to B. Show that

   There is no number a and lottery L of degree less than n for which $L(a, B, L)\ P\ L(a, B, B)$ or $L(a, L, B)\ P\ L(a, B, B)$.

   There is no number a and no lotteries L_1 and L_2 of degree less than n for which $L(a, L_1, L_2)\ P\ L(a, B, B)$.

   It follows from this and the previous exercise that no lottery of degree greater than 0 is preferred to $L(a, B, B)$ for any number a.
6. Show that for no number a, $L(a, B, B)\ P\ B$. Hint: Apply the conclusion of exercise 5 to $L(a, L(a, B, B), L(a, B, B))$.
7. Show that for no number a, $B\ P\ L(a, B, B)$.

   Exercises 5–7 establish that no lottery is preferred to B. We can similarly show that no lottery is less preferred than W.
8. Show that if BPx and xPW, then $L(a, x, x)\ I\ x$ for any number a.

Let us turn now to the uniqueness part of the proof. Now our task is to show that if u' is a utility function defined for the same preferences as u that satisfies

(1) $u'(x) > u'(y)$ if and only if xPy,
(2) $u'(x) = u'(y)$ if and only if xIy,
(3) $u'[L(a, x, y)] = au'(x) + (1-a)u'(y)$,

then there are numbers c and d with $c > 0$ such that

$u'(x) = cu(x) + d.$

Since the two functions u and u' give rise to utility scales for the same preference ordering, we can picture the situation as follows (figure 4-4). (We

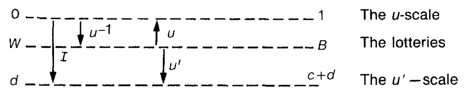


Figure 4-4

know that the end points of the u'-scale will be d and $c+d$ if our proof is correct, since

$cu(W) + d = c0 + d = d,$
$cu(B) + d = c1 + d = c + d.$)

The function u assigns utilities on the u-scale, the function u' assigns them on the u'-scale, and the function I converts assignments on the u-scale into assignments on the u'-scale. Given a number e on the u-scale, the function I first selects a lottery L for which $u(L) = e$ [this is $u^{-1}(e)$], then I applies the function u' to L to find the number $f\ [= u'(L)]$. In short, we have

(a) $I(e) = u'[u^{-1}(e)] = f.$

Now let k and m be any numbers on the u-scale. Note that for any number a, such that $0 \leq a \leq 1$, the number $ak + (1-a)m$ is between k and m or one of them. So the number $ak + (1-a)m$ is also on the u-scale. Substituting this number in (a), we obtain

(b) $I[ak + (1-a)]m = u'\{u^{-1}[ak + (1-a)m]\}.$

But $u^{-1}[ak + (1-a)m]$ is a lottery whose utility on the u-scale is $ak + (1-a)m$. Since k and m are also on the u-scale, they are utilities of some lotteries x and y; that is, $u(x) = k$ and $u(y) = m$. But then by (3), the expected utility condition, we have

(c) $u[L(a, x, y)] = au(x) + (1-a)u(y) = ak + (1-a)m.$

From which it follows that

(d) $I[ak + (1-a)m] = u'[L(a, x, y)].$

Since u' also satisfies the expected utility condition we have

(e) $u'[L(a, x, y)] = au'(x) + (1-a)u'(y)$,

and since $u(x) = k$ and $u(y) = m$ we must have

(f) $I(k) = u'(x)$ and $I(m) = u'(y)$.

Putting these in (e) and (d) we obtain

(g) $I[ak + (1-a)m] = aI(k) + (1-a)I(m)$.

With (g) in hand (which tells us that I mimicks the expected utility property) we can complete the proof. Since each number k on the u-scale is $u(x)$ for some lottery x, we have

(h) $I[u(x)] = u'(x)$.

But by simple algebra

(i) $u(x) = u(x)1 + [1 - u(x)]0$

Thus by (g), (h), and (j)

(j) $$\begin{aligned} u'(x) &= I[u(x)] = I\{u(x)1 + [1-u(x)]0\} \\ &= u(x)I(1) + [1-u(x)]I(0) \\ &= u(x)[I(1) - I(0)] + I(0). \end{aligned}$$

Thus by setting

(k) $c = I(1) - I(0)$ and $d = I(0)$

and substituting in (j), we have

$$u'(x) = cu(x) + d.$$

To finish our proof we need only show that $c > 0$. That is left as an exercise.

PROBLEMS

1. Prove that c as defined in (k) above is greater than zero.
2. Prove that given any number k on the u-scale, there is some lottery x for which $u(x) = k$.
3. Show how to transform a 0 to 1 scale into a 1 to 100 scale using a positive linear transformation. Similarly, show how to transform a -5 to $+5$ scale into a 0 to 1 scale.
4. If we measure an agent's preferences on a Von Neumann-Morgenstern utility scale, does it make sense to say that the agent prefers a given prize twice as much as another?

4-3a. Some Comments and Qualifications on the Expected Utility Theorem

Now that we have concluded the proof of the expected utility theorem, let us reflect on what it has accomplished for us. The theorem is a *representation theorem*; that is, it shows that a certain nonnumerical structure can be represented numerically. Specifically, it tells us that if an agent's preferences have a sufficiently rich structure, that structure can be represented numerically by means

of an interval utility function having the expected utility property. We proved the theorem by assigning numbers to each prize and lottery and then verifying that the resulting numerical scale had the desired properties. However, if the agent's preferences had failed to satisfy any one of the conditions of the theorem, then our construction would have failed to have the desired properties. For example, without the continuity condition we could not be assured of a numerical assignment for each lottery or prize, and without the reduction-of-compound-lotteries condition we could not have established the expected utility property. In a sense, then, the theorem merely takes information already present in facts about the agent's preferences and reformulates it in more convenient numerical terms. It is essential to keep this in mind when applying the theorem and discussing its philosophical ramifications.

How might we apply the theorem? Recall that we needed an interval utility scale for use with the rule of maximizing expected utility. Monetary scales proved unsatisfactory, because monetary values sometimes part company with our true preferences and because EMVs cannot ground our one-time decisions.

By contrast, utility scales do assign "true values" in the sense that utilities march along with an agent's preferences. Furthermore, each act in a decision under risk is itself a lottery involving one or more of the outcomes of the decision. Thus we can expect our agent to rank all the acts open to him along with all the prizes and lotteries. When he applies the rule of maximizing expected utility, he chooses an act whose expected utility is maximal among his options. But the utility of that act, since it is a lottery, is equal to its expected utility. Thus the agent chooses an act whose utility is highest. If there were an act he preferred to that one, it would have a higher utility. Hence in picking this act, the agent is simply taking his most preferred option. This is true even in the case of a one-shot decision. So we now know what justifies the use of expected utilities in making decisions under risk—in particular one-time decisions. It is this: In choosing an act whose expected utility is maximal an agent is simply doing what he wants to do!

Closer reflection on these facts about the theorem may cause you to wonder how it can have any use at all. For the theorem can be applied only to those agents with a sufficiently rich preference structure; and if they have such a structure, they will not need utility theory—because they will already prefer what it would advise them to prefer.

Still, decision theory can be useful to us mortals. Although the agents of the theorem are ideal and hypothetical beings, we can use them as guides for our own decision making. For example, although we may find that (unlike the ideal agents) we must calculate the expected utility of an act before we can rank it, this still does not prevent us from ranking one act above another if its expected utility is higher. We also can try to bring our preferences into conformity with the conditions of the expected utility theorem. In practice we might construct our personal utility functions by setting utilities for some reasonably small number of alternatives and then obtain a tentative utility function from these points by extrapolation and curve fitting. This tentative function can be modified by checking its fit with additional alternatives and a new function can be projected from the results, and so on, until we obtain a function satisfactory for our current purposes.

Furthermore, utility functions are useful even for ideally rational agents—for the same reason that arabic numerals are preferable to roman numerals. Utility functions facilitate the manipulation of information concerning preferences—even if the manipulator is an ideally rational being.

But, whether they are our own or those of ideally rational beings, we must approach such manipulations with caution. Suppose an agent assigns a utility of 2 to having a dish of ice cream. Can we conclude that the agent will assign a utility of 4 to having two dishes? No, utility is not an additive quantity; that is, there is no general way of combining prizes with the result that the utility of the combination equals the sum of the utilities of the components. As a result, it does not make sense to add, subtract, multiply, or divide utilities. In particular, we have no license to conclude that two dishes of ice cream will be worth twice the utility of one to our agent. If eating the second dish would violate his diet, then having two dishes might even be worth less to him than having one.

It would also be fallacious to conclude, for example, that something assigned a utility of 2 on a given scale is twice as preferable to something assigned a 1 on the same scale. For suppose that the original scale is a 1 to 10 scale. If we transform it to a 1 to 91 scale by the permissible transformation of multiplying every number by 10 and then subtracting 9, the item originally assigned 1 will continue to be assigned 1 but the one assigned 2 will be assigned 11. Thus its being assigned twice the utility on the first scale is simply an artifice of the scale and not a scale-invariant property of the agent's preferences.

As a general rule, we must be cautious about projecting properties of utility numbers onto agents preferences. A utility scale is only a numerical representation of the latter. Consequently, agents have no preferences *because* of the properties of their utility scales; rather their utility scales have some of their properties because the agents have the preferences they have.

PROBLEMS

1. Suppose that yesterday the highest temperature in New York was 40 degrees Fahrenheit whereas in Miami it was 80 degrees Fahrenheit. Would it be correct to say that Miami was twice as hot as New York?
2. To graph utility against money, we represent amounts of money on the x-axis and utilities for amounts of money on the y-axis and draw utility graphs in the usual way. One utility graph for an EMVer is the straight line given by the equation $y = x$. This graphs the function $u(x) = x$. All the other utility functions of the EMVer are positive linear transformations of this one. Describe their graphs.
3. Suppose you have an aversion to monetary risks; that is, you prefer having an amount of money for certain to having a lottery whose EMV is that amount. What does your utility graph for money look like in comparison to the graph $y = x$?