Engsci 711

Tutorial 3: Bifurcation theory

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

Overview

The main purpose of this tutorial is to get some experience carrying out bifurcation analysis of various one- or two-dimensional systems. These include one-dimensional/co-dimension one problems and two-dimensional/co-dimension one problems.

I put in some one-dimensional/co-dimension two problems and a two-dimensional/co-dimension two problem for fun. *These are not examinable: I only expect you to be able to handle co-dimension one problems in the exam.*

Furthermore, *a number of these problems are longer/more involved than I expect you to be able to do under exam conditions* - some past exam questions are indicated and give you a feel for what I expect. Doing more interesting problems is, however, good for the soul and/or developing your cultural sophistication (or something like that, I guess).

One of the key tricks in many problems is, when in doubt

- Find some *convenient points* on fixed point branches (points on branches of fixed points where you can easily evaluate their stability)

- Use the fact that branches of fixed points *only change stability at bifurcations*, so the *whole branch* (between bifurcation points) has the same stability as your convenient point.

This trick of using 'continuity of stability' is related to the numerical strategy of *parameter continuation* in bifurcation analysis.

One-dimensional, co-dimension one problems

Problem 1

Draw the bifurcation diagrams of the following equations

$$\begin{aligned}
(a)\quad & \dot{x} = \mu - x^2 \\
(b)\quad & \dot{x} = \mu x - x^2 \\
(c)\quad & \dot{x} = \mu x - x^3 \\
(d)\quad & \dot{x} = \mu x + x^3 - x^5
\end{aligned}$$

What types of bifurcations occur in each of the above?

Problem 2

Suppose the equations from Problem 1 arose from potential functions $V(x)$, i.e.

$$\dot{x} = -\frac{dV}{dx}$$

- Sketch the forms of the potentials in each case (a) - (d), and how they depend on the values of μ.
- The last case, i.e. (d), is an example of a *first-order phase transition* occuring at the critical value of μ. A common example is the freezing of water into ice. Do some further research on the phase transition between water and ice and how the mathematical analysis above corresponds to the physical phenomenon observed.

Problem 3

Find all the bifurcations and sketch the bifurcation diagrams for the following systems

$$\begin{aligned}
(a)\quad & \dot{x} = \mu + x - \ln(1+x) \\
(b)\quad & \dot{x} = \mu x - \ln(1+x) \\
(c)\quad & \dot{x} = x(\mu - e^x) \\
(d)\quad & \dot{x} = \mu x - \sinh(x) \\
(e)\quad & \dot{x} = x + \frac{\mu x}{1+x^2}
\end{aligned}$$

What kind of bifurcations do each correspond to?

Problem 4 (Multiple co-dimension-one bifurcations in one system)

Find all bifurcations and sketch the bifurcation diagram for the following system

$$\dot{x} = (x-1)(x^2 + 2ax - \mu)$$

where $x \in \mathbb{R}$, $a > 0$ is a fixed positive parameter and $\mu \in \mathbb{R}$ is a controllable parameter. (Hint: see lecture notes!).

Problem 5 (Exam 2016)

Determine the equilibria and their stability, and hence find all bifurcations, in the following system

$$\dot{x} = (\lambda - b)x - ax^3$$

where $x \in \mathbb{R}$, $a, b > 0$ are fixed positive parameters and $\lambda \in \mathbb{R}$ is a controllable parameter. Sketch the bifurcation diagram.

(Hint: it may help to consider the cases $\lambda < b$, $\lambda = b$, and $\lambda > b$ separately.)

Problem 6 (Exam 2017)

Consider the equation

$$\dot{u} = (u-2)(\lambda - u^2)$$

where $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a controllable parameter.

- Determine the equilibria and their stability as λ varies.
- Sketch the bifurcation diagram showing how the equilibria vary with λ. What types of bifurcations occur?

Problem 7 (Exam 2018)

Consider the equation

$$\dot{x} = x(\mu - 3 - x^2)$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a parameter that can vary.

- Determine the equilibria and their stability as μ varies.
- Sketch the bifurcation diagram showing how the equilibria vary with μ. What type(s) of bifurcations occur?

Next consider the equation

$$\dot{x} = \mu x - \ln(1 + x)$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a parameter that can vary.

- Write down the condition, in terms of the two functions $f_1(x;\mu) = \mu x$ and $f_2(x) = \ln(1 + x)$, that defines fixed point solutions to the above equation.
- Assuming $x \geq 0$, use a series of sketches of the functions $f_1(x;\mu) = \mu x$ and $f_2(x) = \ln(1 + x)$ to indicate how you would expect the number of fixed points of the above equation to vary with μ. You only need to indicate qualitative behaviour here, e.g. you do NOT need to determine the value of μ for which a bifurcation could occur.
- Based on the above purely graphical reasoning, what sort of bifurcation would you expect to be able to produce by varying μ?

Two-dimensional, co-dimension one (Hopf bifurcation) problems

Problem 1

Consider the system

$$\dot{x} = -y + \mu x + xy^2$$
$$\dot{y} = x + \mu y - x^2$$

- For what parameter values do you expect a Hopf bifurcation to occur at the origin?
- Check the stability and type of the fixed point at the origin on either side of the bifurcation.
- Use XPPAut to explore the neighbourhood this bifurcation and determine whether the Hopf bifurcation is subcritical, supercritical or degenerate (periodic solution appears at bifurcation but disappears for all other parameter values).
- Verify analytically that the Hopf bifurcation is non-degenerate, i.e. that it has a *non-zero crossing speed* as the parameter varies.

Problem 2

Carry out the same steps as in the previous problem for the systems

$$\dot{x} = y + \mu x$$
$$\dot{y} = -x + \mu y - x^2 y$$

and (Predator-prey model)

$$\dot{x} = x(x(1 - x) - y)$$
$$\dot{y} = y(x - a)$$

where in the latter case $x, y, a \geq 0$ and x, y represent, respectively, the prey population, predatory population

- In this latter case, given an interpretation to the terms in the model, e.g. reproduction, ‘predation’ (predator eating prey) etc. What do you think the a parameter represents?

Problem 3

(More difficult?). Carry out the same steps as above for the predator-prey model

$$\dot{x} = x(b - x - \frac{y}{1+x})$$
$$\dot{y} = y(\frac{x}{1+x} - ay)$$

One-dimensional, co-dimension two problems

Problem 1

Consider the following 'imperfect' bifurcation equations (see Strogatz 3.6 or Drazin 2.3) - i.e. standard bifurcation equations with 'external control' parameter μ plus an additional (uncontrolled, small) 'imperfection parameter' δ.

$$(a) \quad \dot{x} = \mu - x^2 + \delta$$
$$(b) \quad \dot{x} = \mu x - x^2 + \delta$$
$$(c) \quad \dot{x} = \mu x - x^3 + \delta$$

- Which bifurcations do these correspond to for $\delta = 0$.
- For each of $\delta >, =, < 0$, plot the usual bifurcation diagram vs μ.
- Which of the $\delta = 0$ bifurcation diagrams are 'structurally stable' (preserved) under small variations of δ?
- Summarise the bifurcation behaviour of each by drawing a picture in the (μ, δ) plane indicating lines/curves separating regions with different qualitative phase-space properties. Indicate in each region of the parameter space what the behaviour in the phase-space looks like (e.g. number of fixed points, stability etc).

Note: the above examples illustrate that bifurcation diagrams can themselves be structurally unstable! Just as with the usual situation of a structurally unstable vector field/ODE system, however, our classification depends in general on the *type* of perturbation considered. E.g. centres can be structurally stable when we *restrict ourselves to conservative perturbations* of the governing equations.

In the above you would have hopefully found that the saddle-node bifurcation is 'generic' (structurally stable) under the given perturbation, while the transcritical and pitchfork were not. However, the transcritical bifurcation *is* generic when we require that *there is always at least one fixed point*. Similarly, the pitchfork bifurcation *is* generic, when we require the *existence of a pair of fixed points after bifurcation*. These can be related to ideas about symmetries that the physical system must satisfy.

Two-dimensional, co-dimension two problems

For the brave. The Bogdanov-Takens bifurcation. See e.g. the scholarpedia article by Guckenheimer and Kuznetsov at http://www.scholarpedia.org/article/Bogdanov-Takens_bifurcation or Kuznetsov's book (Elements of Applied Bifurcation Theory - see Further Reading).

Consider the system

$$\dot{x} = \lambda - \mu x + y^2 + xy$$
$$\dot{y} = x$$

- See if you can find a saddle-node bifurcation
- See if you can find a (possible) Hopf-bifurcation
- Draw a bifurcation picture in the two-dimensional parameter space (λ, μ). That is, try to find where in the (λ, μ) plane (i.e. curves/lines) there are bifurcations and indicate in each parameter region what the corresponding typical behaviour in the phase-plane looks like.