Maths 361 Partial Differential Equations - supplement to Lecture 6

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(email me any typos!)

Working for complex Fourier series lecture examples

Derivation from real Fourier series

Goal

Our goal is to derive

$$\text{FS } f = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

for $n \in \mathbb{Z}$

using the definition of the real Fourier series. Note that the c_n and exponentials in this expression are complex but *the series has a real-valued sum if f is real valued.*

Definitions

The real Fourier series if of course

$$\text{FS } f := a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})] \tag{1}$$

where

$$a_0 := \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$a_n := \frac{1}{l}\int_{-l}^{l} f(x)cos(\frac{n\pi x}{l})dx$$

$$b_n := \frac{1}{l}\int_{-l}^{l} f(x)sin(\frac{n\pi x}{l})dx \tag{2}$$

$n = 1, 2, ...$

We know that

$$e^{iy} = cos(y) + isin(y)$$
$$e^{-iy} = cos(y) - isin(y)$$

$$cos(y) = \frac{e^{iy} + e^{-iy}}{2}$$
$$sin(y) = \frac{e^{iy} - e^{-iy}}{2i} \tag{3}$$

We will be substituting these identities in the Fourier series.

Derivation: the infinite sum as a limit of the finite sums

First note that we can bring a_0 back into the sum in (1)

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

$$= \sum_{n=0}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

Q: what's $cos(0)$? What's $sin(0)$?.

Now, the infinite sum is the limit of finite sums:

$$\text{FS } f = \lim_{N\to\infty}\sum_{n=0}^{N}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

Derivation: using the Euler trig/exponential identities

Let's substitute in our complex exponentials from (3)

$$\text{FS } f = \lim_{N\to\infty} \sum_{n=0}^{N} [a_n(e^{in\pi x/l} + e^{-in\pi x/l})/2 + b_n(e^{in\pi x/l} - e^{-in\pi x/l})/2i]$$

Recalling $1/i = -i$ (multiply top and bottom by i), and rearranging terms gives

$$\text{FS } f = \lim_{N\to\infty} \sum_{n=0}^{N} [\frac{(a_n - ib_n)}{2} e^{in\pi x/l} + \frac{(a_n + ib_n)}{2} e^{-in\pi x/l}]$$

We can consider the sums separately for each exponential:

$$\text{FS } f = \lim_{N\to\infty} \{\sum_{n=0}^{N} [\frac{(a_n - ib_n)}{2} e^{in\pi x/l}] + \sum_{n=0}^{N} [\frac{(a_n + ib_n)}{2} e^{-in\pi x/l}]\}$$

Derivation: manipulating the dummy variable for the second sum

Let's consider the second term

$$\sum_{n=0}^{N} [\frac{(a_n + ib_n)}{2} e^{-in\pi x/l}] \tag{4}$$

Here n is just a dummy variable. Let's re-write the sum using $n' = -n$, noting

$$\sum_{n=0}^{N} E(n) \equiv \sum_{(-n)=0}^{-N} E(n) \equiv \sum_{(n')=0}^{-N} E(-n')$$

for summing over any expression $E(n)$. We will drop the 'dash' form the dummy variable from now.

This means (4) can be re-written as

$$\sum_{n=0}^{-N} [\frac{(a_{-n} + ib_{-n})}{2} e^{in\pi x/l}]$$

or

$$\sum_{n=-N}^{0} [\frac{(a_{-n} + ib_{-n})}{2} e^{in\pi x/l}]$$

since the order of summation doesn't matter.

Derivation: combining results

We can write

$$\text{FS } f = \lim_{N\to\infty} \{ \sum_{n=0}^{N} [\frac{(a_n - ib_n)}{2} e^{in\pi x/l}] + \sum_{n=-N}^{0} [\frac{(a_{-n} + ib_{-n})}{2} e^{in\pi x/l})] \}$$

as the combined sum

$$\text{FS } f = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{in\pi x/l}$$

where

$$c_n = \begin{cases} (a_n - ib_n)/2, & \text{if } n > 0 \\ (a_{-n} + ib_{-n})/2, & \text{if } n < 0 \\ (a_0 - ib_0)/2 + (a_{-0} - ib_{-0})/2 = a_0, & \text{if } n = 0 \end{cases}$$

It turns out that we can write this seemingly awkward 'case' expression as simply

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} dx$$

Let's look at how.

Derivation: expressions for c_n

Consider $n > 0$. Using our usual expressions for a_n and b_n we get

$$c_n = (a_n - ib_n)/2 = \frac{1}{2l} \int_{-l}^{l} f(x)[cos(\frac{n\pi x}{l}) - isin(\frac{n\pi x}{l})]dx$$

$$= \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} dx$$

Now consider $n < 0$.

$$c_n = (a_{-n} + ib_{-n})/2 = \frac{1}{2l} \int_{-l}^{l} f(x)[cos(\frac{-n\pi x}{l}) + isin(\frac{-n\pi x}{l})]dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)[cos(\frac{n\pi x}{l}) - isin(\frac{n\pi x}{l})]dx$$

Since *cos* is an even function and *sin* is an odd function. This also gives

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

The $n = 0$ case is simply

$$c_0 = a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)cos(0)dx = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^0 dx$$

which is also consistent.

Note then that, for $n \neq 0$

$$c_n = (a_n - ib_n)/2 = (a_{-n} + ib_{-n})/2$$

and for $n = 0$

$$c_0 = (a_0 - ib_0)/2 + (a_{-0} + ib_{-0})/2 = a_0/2 + a_0/2 = a_0$$

Summary

We have hence shown that

$$\text{FS } f = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

with

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

for $n \in \mathbb{Z}$ follows from the definition of the real Fourier series.

Example solution and comparison to results from real case

Our goal is to compute the *complex* Fourier series of the square-wave from last lecture and then compare it to the results from the *real* Fourier series.

Our square-wave was defined by

$$f(x) = \begin{cases} 2, & \text{on } (-\pi, 0] \\ 0, & \text{on } (0, \pi] \end{cases}$$

and

$$f(x + 2\pi) = f(x)$$

Definitions

We will need the Euler identity:

$$e^{i\pi} = -1$$

Note that $e^{-i\pi} = 1/-1 = -1$, $e^{2i\pi} = e^{i\pi}e^{i\pi} = (-1)(-1) = 1$ etc. So

$$e^{-in\pi} = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

Solution

We need to compute

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

Since f is zero on $(0, \pi]$ we will focus on $[-\pi, 0)$.

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx = \frac{1}{2\pi}[\int_{-\pi}^{0} 2e^{-inx}dx + \int_{0}^{\pi} 0dx]$$

$$= -\frac{1}{2\pi}\int_{0}^{-\pi} 2e^{-inx}dx$$

(swapping end-points of integration).

If $n \neq 0$ then

$$= -\frac{1}{\pi}\ \frac{e^{-inx}}{-in}\Big|_{x=0}^{x=-\pi}$$

$$= \frac{1}{\pi}\ \frac{e^{-inx}}{in}\Big|_{x=0}^{x=-\pi}$$

$$= -\frac{i}{n\pi}\ e^{-inx}\Big|_{x=0}^{x=-\pi}$$

$$= -\frac{i}{n\pi}[e^{in\pi} - 1] = \begin{cases} \frac{2i}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

If $n = 0$ then the integral is simply

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{0} 2dx = 1$$

Solution summary

So we get

$$\text{FS } f = 1 + \sum_{n=-\infty, n \text{ is odd}}^{\infty} \frac{2i}{n\pi} e^{inx}$$

Note that the c_n are complex. Let's see how this relates to the real case.

Comparison to real case

We've just found

$$\text{FS } f = 1 + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2i}{n\pi} e^{inx}$$

Previously we found

$$\text{FS } f = 1 + \sum_{n=1, n \text{ odd}}^{\infty} \frac{-4}{n\pi} sin(nx)$$

Are these consistent?

We have shown above that

$$c_n = (a_n - ib_n)/2 = (a_{-n} + ib_{-n})/2$$

for $n \neq 0$. Let's consider this case (clearly the $n = 0$ term agrees).

First note $a_n = 0$ since the *cosine* terms are zero. This means we have

$$b_n = -2c_n/i = -2\frac{2i}{ni\pi} = -\frac{4}{n\pi}$$

which is consistent with our results from the real series. For fun, let's consider the the other expression, i.e.

$$c_n = (a_{-n} + ib_{-n})/2$$

which implies

$$b_{-n} = 2c_n/i \Leftrightarrow b_n = 2c_{-n}/i = 2\frac{2i}{(-n)i\pi} = \frac{-4}{n\pi}$$

which is also consistent.