ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*)

Oliver Maclaren
oliver.maclaren@auckland.ac.nz

LECTURE 6

- Nonlinear stable and unstable manifolds continued
- Piecing together local phase portraits into 'plausible' global pictures

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

2. *Phase plane analysis and geometry of hyperbolic systems*
   [5 lectures/tutorials]

   Analysis of two-dimensional linear and nonlinear systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling for linear and nonlinear hyperbolic systems. Connecting geometry of nonlinear and linearised hyperbolic systems.

RECALL: STABLE MANIFOLD (LOCAL)

Given some neighbourhood U of a stationary point x, the *local stable manifold* on U for a nonlinear system $W^s_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to \infty, \phi(y,t) \in U \text{ for all } t \geq ($$

RECALL: UNSTABLE MANIFOLD (LOCAL)

Similarly, given some neighbourhood U of a stationary point x, the *local unstable manifold* on U for a nonlinear system $W^u_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to -\infty, \phi(y,t) \in U \text{ for all } t \leq$$

STABLE MANIFOLD THEOREM

What's the *connection between these linear and nonlinear stable/unstable manifolds?* We have the following theorem (for local manifolds).

Suppose the origin is a *hyperbolic fixed point* for $\dot{x} = f(x)$ in $\mathbb{R}^n$ and that $E^s(0)$ and $E^u(0)$ are the stable and unstable manifolds of the linearised system $\dot{x} = Df(0)x$.

Then…

STABLE MANIFOLD THEOREM

…there exist *local stable and unstable manifolds* $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the *same dimension* as $E^s(0)$ and $E^u(0)$, respectively, and which are (respectively) *tangent* to E^s and E^u at the origin.

These manifolds are equally smooth/unsmooth as the original function f.

CALCULATING THE MANIFOLDS - THE 'MANIFOLD EQUATION'

The basic idea is to substitute the defining equation $y = U(x)$ or $x = V(y)$ into the governing equations and use the *chain rule* applied *along the manifold*:

$$\frac{dy}{dt}(x, U(x)) = \frac{dy}{dx}(x)\frac{dx}{dt}(x, U(x))$$

from which to find $U(x)$.

Let's call this the *'manifold equation'*. We usually solve it locally by assuming a *power series solution* - justified by the SMT!

POWER SERIES EXPANSIONS IN TWO-DIMENSIONAL SYSTEMS

Even on small neighbourhoods of our fixed points, *our manifolds are no-longer straight lines* (or hyperplanes etc in higher dims) as in the linear case - they are *curves* (or surfaces in higher-dimensions).

We can, however, use the information from the previous theorem to (try to) *compute local expressions for these curves*.

POWER SERIES: STEPS

- *Assume* the manifold can be described by $y = U(x)$ (or the other way around).
- *Substitute* $y = U(x)$ into our x and y equations to give $\dot{x} = f_1(x, U(x))$ and $\dot{y} = f_2(x, U(x))$.
- Use $y = U(x)$ *again*, along with the *chain rule* for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dU}{dx}\dot{x}$.

POWER SERIES EXPANSIONS FOR ONE-DIMENSIONAL MANIFOLDS

Assume a stable/unstable manifold of interest can be described by a curve $y = U(x)$ (or $x = V(y)$).

We can try to approximate this by a *local series expansion* of the form

$$y = U(x) = \sum_{n=0}^{\infty} a_n x^n$$

POWER SERIES: STEPS

- Use the above relationships along with an assumed *power series* expansion such as $U(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain *two expressions* in x for $\dot{y}$
- *Equate* powers of x to determine the unknown coefficients.
- Make sure to use the fact that the stable/unstable manifold *passes through* the fixed point and *is tangent* to the linearised stable/unstable manifold to determine the *first two terms* of the series.

GLOBAL PHASE PORTRAITS

So far we have mainly looked at the *local geometry*. We can now imagine *'gluing together' local pictures* near fixed points to get a better *'global' picture* of our system.

This is generally a good idea but is *not guaranteed to give a unique or correct answer!*

EngSci 711 L06 Geometry cont'd: Local & global

- Nonlinear stable & unstable manifolds (L05)
- 'Piecing together' local phase portraits (2D systems) to make global pictures
  - └ plausible guesses + warnings

Examples

- Manifolds: see Lecture 05 examples
- Phase portraits:

Exam style: (2017)

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

Recall:

Analysis of

$$\boxed{\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y + x^2\end{aligned}}$$

(4.2 in Glendinning)

Key steps

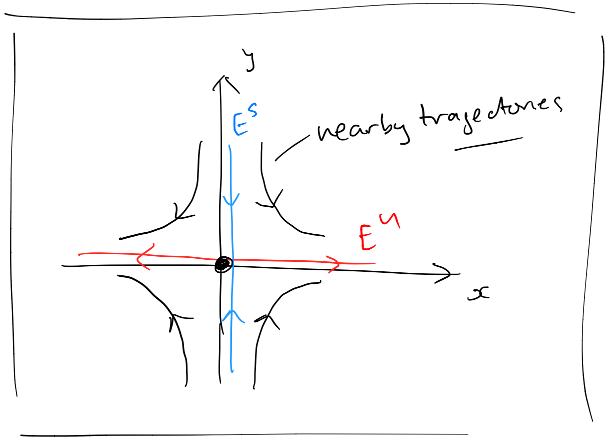
- Find fixed points
- Linearise near fixed points
- Classify fixed points
- Sketch a 'local picture' of flow near fixed points
- Build up a more 'global' picture
  - └ extend/join flows
  - └ look for other 'global' objects (like periodic orbits etc)

need to understand the local 'geometry'
- linearised
- nonlinear

↓ link

Recall:

Local, linear picture:



$E^u = \{(x,y) \mid y=0\}$ (implicit)

$= \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$ (explicit)

$E^s = \{(x,y) \mid x=0\}$

$= \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

→ want nonlinear versions

'manifolds'

Recall:

Manifolds – 'nonlinear subspaces'

- Basic idea:

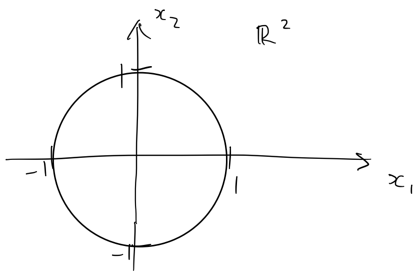
  ↳ subset W of $\mathbb{R}^n$ (here), instead of 'E' for linear

  defined by m (possibly) $\boxed{\text{nonlinear constraints}}$ } easier to generalise than 'span'

- Example:

  $\{x \in \mathbb{R}^2 \mid \underbrace{x_1^2 + x_2^2 = 1}_{\text{nonlinear constraint}}\}$

  → defines a circle!



→ a linear subspace is just a special kind of manifold:

- generated by linear constraints

→ manifolds can 'curve' & 'close', but locally look linear

Recall: nonlinear $\boxed{\text{stable/unstable}}$ manifolds

- Intuition:
  - stable as an $\boxed{\text{invariant}}$ set near origin / FP such that: $\boxed{\text{points} \to \text{FP}}$ as $\boxed{t \to \infty}$

  - unstable: $\boxed{\text{invariant}}$ set near origin / FP such that: $\boxed{\text{points} \to \text{FP}}$ as $\boxed{t \to -\infty}$

└ trick: unstable as $t \to \infty$ is stable as $t \to -\infty$!

Local vs global

- These are local definitions
- But, as invariant manifolds, can trace out / piece together / extend to global objects
- see videos for Lorenz (UoA math dep)

---

we'll focus on local for now.

Recall:

Connection between linear & nonlinear

- Hartman-Grobman
  - → existence & stability of FP for hyperbolic:

$\boxed{\text{Linear FP} \longleftrightarrow \text{nonlinear FP}}$

stable manifold theorem

→ existence & tangency of manifolds for hyperbolic:

$\boxed{\begin{array}{ccc} \text{Linear geometry} & \longleftrightarrow & \text{nonlinear geometry} \\ \text{(stable/unstable subspaces)} & & \text{(stable/unstable manifolds)} \end{array}}$

---

ie

$$\left.\begin{array}{l} E^s \to W^s \\ E^u \to W^u \end{array}\right\} \text{for hyperbolic use above} \Big\} \text{today}$$

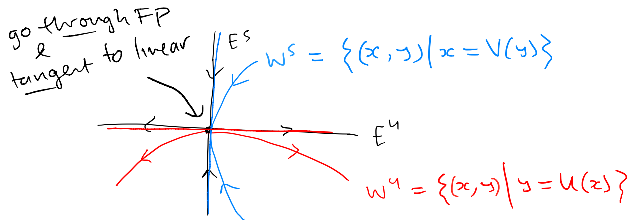
$$E^c \to \,? \quad \Big\} \text{need 'centre manifold theorem'} \Big\} \text{later}$$

- Linear is usually $\boxed{\text{easier}}$
  - → we can use this theorem to base nonlinear calc. on linear manifolds & determine 'corrections'

'Manifold Equation'

- Idea: exists either $y = U(x)$ or $x = V(y)$ } local (or multivariable surface if higher dim)

expressions for manifolds which
- satisfy ODE & are
- tangent to associated linear manifold at FP.



→ Let's calculate our manifolds & the flows on these! (Semi-quantitative):

- Substitute in expression & use chain rule

1. subs: $\dot{x} = f_1(x,y) = f_1(x, U(x))$
   $\dot{y} = f_2(x,y) = f_2(x, U(x))$

2. chain: (along $y = U(x)$) $\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} \Leftrightarrow \boxed{\dot{y} = \dot{x} \cdot \frac{dy}{dx}}$

Gives the 'Manifold Equation'

$$\boxed{\dot{y}(x) = \dot{x}(x) \cdot \frac{dy}{dx}(x)}$$

Solution is $\boxed{y = U(x)}$

Power Series Sol^n

We only really want 'local' sol^n to a given order

→ use a power series expansion

Example cont'd: unstable W^u for

$$\dot{x} = x$$
$$\dot{y} = -y + x^2$$

$$y = U(x) = a_0 + a_1 x + a_2 x^2 + \dots \quad (1)$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + \dots \quad (2).$$

$$\dot{x}(x) = f_1(x, y(x)) = x \quad (3)$$

$$\dot{y}(x) = f_2(x, y(x)) = -(a_0 + a_1 x + \dots) + x^2$$
$$= -a_0 - a_1 x + (1 - a_2)x^2 + \dots \quad (4)$$

Gives: $\dot{y} = \frac{dy}{dx} \cdot \dot{x} = \left[a_1 + 2a_2 x + \dots\right] x$ (manifold eq^n)

$$= -a_0 - a_1 x + (1 - a_2) x^2$$

Cont'd.

So

$$a_1 x + 2a_2 x^2 + \dots = -a_0 - a_1 x + (1-a_2)x^2$$

solving:

1. use
   a) pass through origin
   b) tangent to E^u at origin } distinguishes E^u & E^s
   c) equate coefficients for rest.

a & b) (From SM Theorem)

$$y = a_0 + a_1 x + \dots$$

$$y(0) = 0 \Rightarrow \boxed{a_0 = 0}$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + \dots$$

$$\frac{dy}{dx}(0) = 0 \Rightarrow \boxed{a_1 = 0}$$

tip: do this straight away!

c).

$$2a_2 x^2 + \dots = (1-a_2)x^2$$

$$\Rightarrow 2a_2 = 1 - a_2$$

$$\Rightarrow \boxed{a_2 = \frac{1}{3}}$$

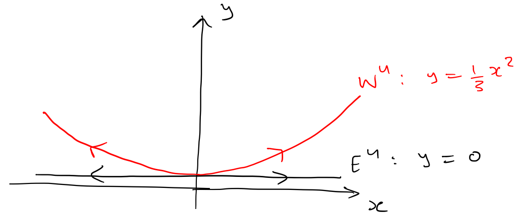
So we have:

$y = \frac{1}{3}x^2$ ie,

Unstable manifold:

$$\boxed{W^u = \{(x,y) \mid y = \tfrac{1}{3}x^2\}}$$

So:



Note:

$$\dot{x}(u(x)) = x \quad \begin{cases} < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0 \end{cases}$$

$$\dot{y}(u(x)) = -y(x) + x^2 = -\frac{1}{3}x^2 + x^2 = \frac{2}{3}x^2 > 0$$

flow on unstable

(recall is invariant satisfies ODE $\rightarrow$ E^u doesn't in general)

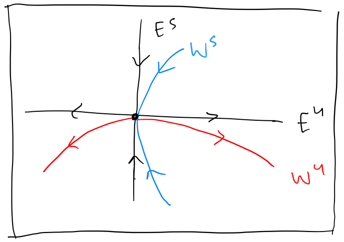
Exercise: use $x = V(y)$ & find stable manifold!

$\Rightarrow$ Turns out $W^s = \{(x,y) \mid x = 0\}$

ie $W^s = E^s$ in this case.

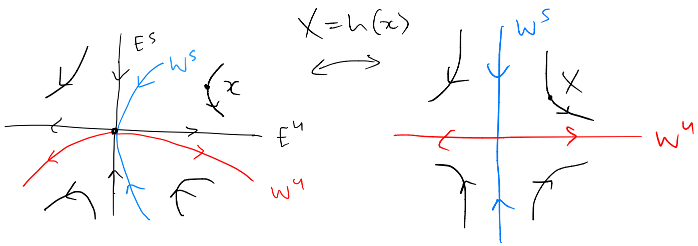
Side Note: Linear vs nonlinear

Consider:



HG & SMT:

Local nonlinear change of coord:



→ Nonlinear manifolds of original system are linear manifolds of a nonlinear transformation of the original system!

(still need nonlinear change of coord)

→ we will mainly consider W^s, W^u etc as nonlinear manifolds of the original system

[see Roberts 2015 for more on coord. transforms & 'normal forms']

Think global, act local?

→ we've been doing analysis local to a given fixed point

→ we might have many:

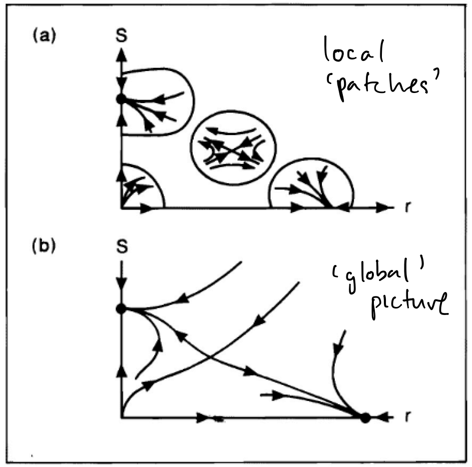
what about the 'big picture'?

'Global' picture for nonlinear?

'Typical' steps as follows: (in x-y plane here)

- Find fixed points $(x^*, y^*)_i$ $i = 1, 2, \ldots$
- Find linearisation at arbitrary point $Df(x,y)$
- Evaluate Df at each fixed point: $Df(x^*, y^*)$
  - find eigenvalues & classify
  - find eigenvectors
- sketch local pictures around each (x^*, y^*)
- Combine into 'plausible' global picture

Eg (Fig 5.14 Glendinning) [interacting pops]



guess

Fig. 5.14 (a) Local patches; (b) global phase portrait.

What could go wrong??

Cautionary Examples [can skip if you want]

Point: complications to naive approach (even tho' you should try it!)

- Glendinning

  Example 5.6

  Figures 5.14

  5.15

  5.16

- Perko

  Example 2 Chap 2.7 — Fig 2 & Fig 3.

$\longrightarrow$

Non-uniqueness!

Figures 5.15 & 5.16 from Glendinning:

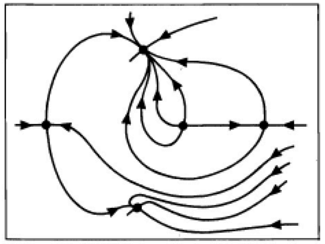


Fig. 5.15 One of the consistent non-symmetric configurations.

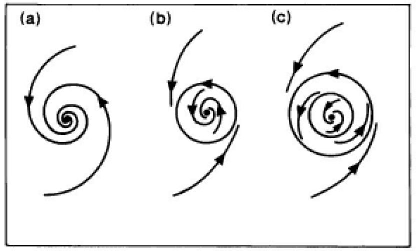


Fig. 5.16 Local behaviour and periodic orbits.

→ Other possible ways to connect the same local patches together

→ How many periodic orbits? orbits in orbits in ....

Need 'global' methods too!

→ tools exist

→ many open problems in general!

Another example (Perko 2.7, Example 2).

Consider

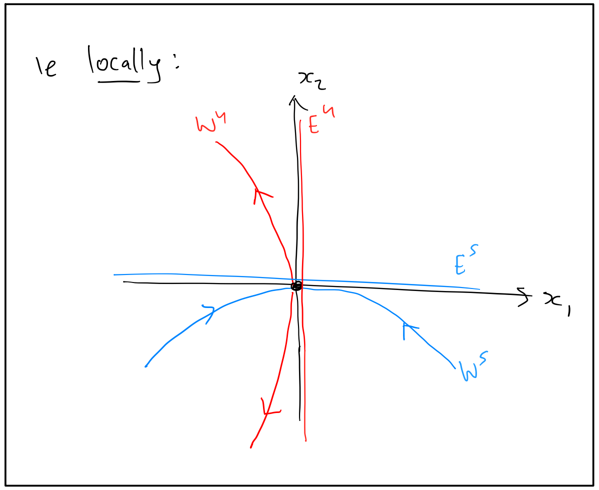
$$\dot{x}_1 = -x_1 - x_2^2$$
$$\dot{x}_2 = x_2 + x_1^2$$

} similar to example from L5

Get (exercise!) local stable/unstable manifolds:

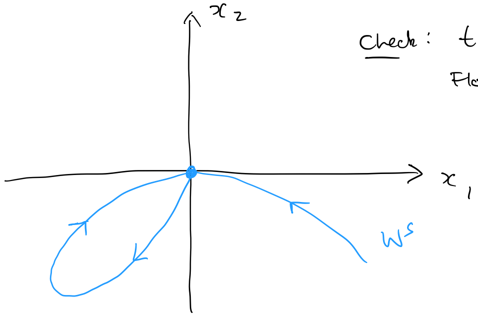
$$W^s_{loc} = \left\{ (x_1, x_2) \;\middle|\; x_2 = -\frac{x_1^2}{3} + O(x_1^5) \right\}$$

$$W^u_{loc} = \left\{ (x_1, x_2) \;\middle|\; x_1 = -\frac{x_2^2}{3} + O(x_2^5) \right\}$$



Q: global? Extended manifolds?

Global stable manifold (don't expect you to calculate, unless XPP!)



Check: $t \to \infty$

Flow $\to$ fixed point.

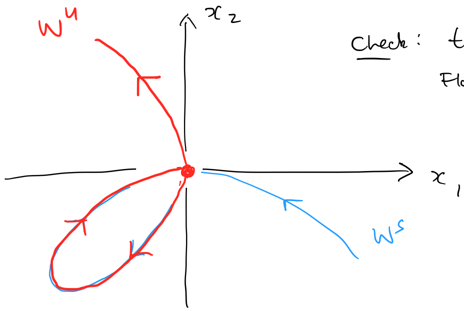
$\to$ loop back to same fixed point

$\to$ 'Homoclinic loop/cycle/orbit'.

} more common in conservative/reversible systems

(as are centres & heteroclinic).

Global unstable manifold



Check: $t \to -\infty$

Flow $\to$ fixed point.

$\to$ same loop!

$\to$ 'Homoclinic loop/cycle/orbit'.

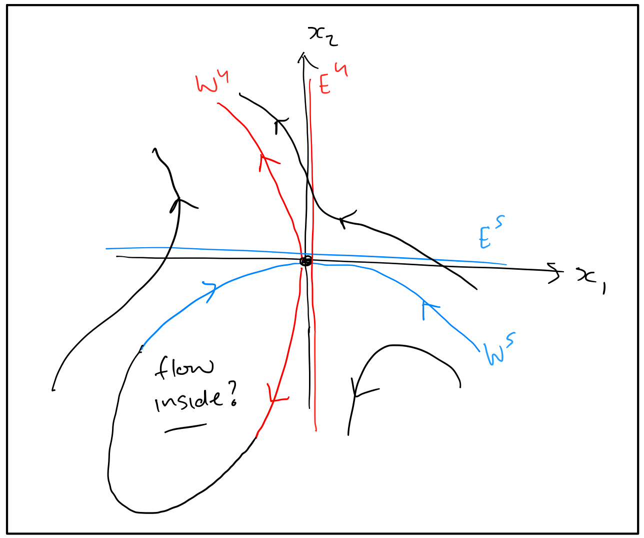
} $t \to \infty$, $t \to -\infty$, both $\to$ Fixed point

$\llcorner$ part of both stable & unstable global manifolds.

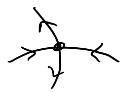
The global versions actually 'join up'

In this case ('Branched manifolds')

ie 'funky' (to use a technical term)



$\to$ need 'global' methods to detect.

(locally just )

Note: Homoclinic orbit is not periodic orbit!

why?

$\to$ More global tips & tricks next lecture...

Strogatz

Hyperbolic Fixed Points, Topological Equivalence, and Structural Stability

If $\mathrm{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\mathrm{Re}(\lambda) \neq 0$.

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:

1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

A specific model that incorporates these assumptions is

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where

$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Now consider the four fixed points in turn:

$(0,0)$: Then $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node*. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

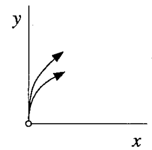


Figure 6.4.1

$(0,2)$: Then $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since

Strogatz

the matrix is triangular. Hence the fixed point is a *stable node.* Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1, -2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0, 2)$.

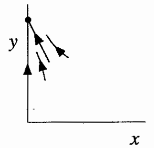


Figure 6.4.2

$(3,0)$: Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node.* The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3, -1)$, as shown in Figure 6.4.3.

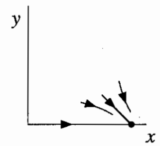


Figure 6.4.3

$(1,1)$: Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.

Hence this is a *saddle point.* As you can check, the phase portrait near $(1,1)$ is as shown in Figure 6.4.4.

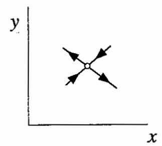


Figure 6.4.4

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x} = 0$ when $x = 0$, and $\dot{y} = 0$ when $y = 0$.

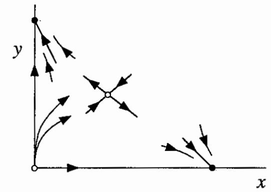


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.

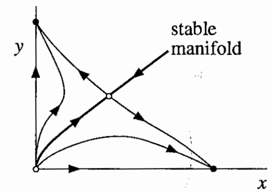


Figure 6.4.6

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

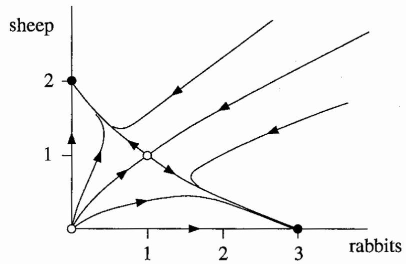


Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist. See Pianka (1981) for a biological discussion, and

Strogatz

Pielou (1969), Edelstein–Keshet (1988), or Murray (1989) for additional references and analysis.

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.

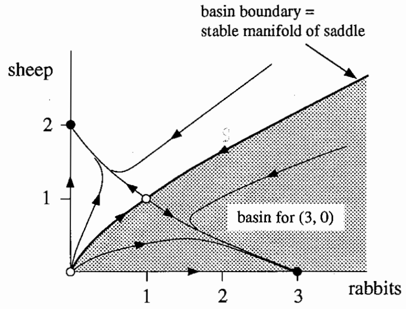


Figure 6.4.8

Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

6.5 Conservative Systems

Newton's law $F = ma$ is the source of many important second-order systems. For example, consider a particle of mass m moving along the x-axis, subject to a nonlinear force $F(x)$. Then the equation of motion is

$$m\ddot{x} = F(x).$$

Notice that we are assuming that F is independent of both $\dot{x}$ and t; hence there is no damping or friction of any kind, and there is no time-dependent driving force.

Under these assumptions, we can show that *energy is conserved*, as follows. Let $V(x)$ denote the ***potential energy***, defined by $F(x) = -dV/dx$. Then

$$m\ddot{x} + \frac{dV}{dx} = 0. \qquad (1)$$

at $(0, y_0)$ and so all orbits in a neighbourhood of the origin are periodic and the origin is a nonlinear centre. The existence of either a symmetry or a conserved energy-type function is frequently the only way of showing that a centre remains a centre under nonlinear perturbation.

Example 5.5

Suppose

$$\dot{x} = -y + x(x^2 + y^2)\sin(\log\sqrt{(x^2 + y^2)})$$

$$\dot{y} = x + y(x^2 + y^2)\sin(\log\sqrt{(x^2 + y^2)})$$

or

$$\dot{r} = r^3 \sin(\log r), \quad \dot{\theta} = 1.$$

This has periodic orbits whenever $\sin(\log r) = 0$, i.e. whenever $\log r = \pm n\pi$. Hence there is an infinite sequence of isolated periodic orbits with radii $r_n = e^{-n\pi}$, $n = 1, 2, 3, \ldots$ which accumulate on the origin.

5.3 Rabbits and sheep

Models from population dynamics provide a rich source of nonlinear behaviour. These models, introduced by Lotka and Volterra, represent the most simple models of nonlinear interaction between species. We shall consider a grassy island with two species of animal, which may be in competition for the island's resources (rabbits and sheep) or one may prey on the other (wolves and sheep). For simplicity we assume that these two populations are large, so that the number of individuals of a population can be taken to be a real (positive!) number, with the understanding that if the number of sheep, $s(t)$, is small then $s(t)$ is interpreted as the number of sheep divided by some large number, so $s = 1$ might represent a population of 100,000 sheep.

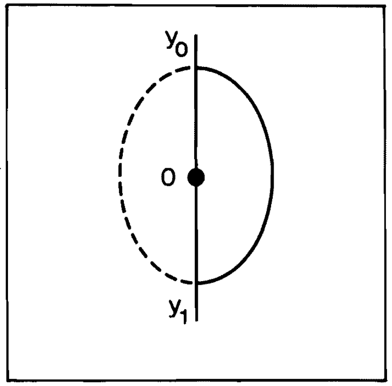


Fig. 5.13 A nonlinear centre by symmetry.

Suppose that there is a grassy island supporting populations of two species, x and y. If the populations are large then it is reasonable to let the normalized populations be continuous function of time. We propose a simple model of the change in population of the form

$$\dot{x} = x(A + a_1 x + b_1 y) \tag{5.37a}$$

$$\dot{y} = y(B + b_2 x + a_2 y) \tag{5.37b}$$

where A, B, a_i and b_i are constants. These equations can be interpreted as the rate of change of the population equals the present population multiplied by (the birth rate − the death rate). Consider the x equation when $y = 0$: $\dot{x} = x(A + a_1 x)$. The coefficient a_1 describes the interaction of the species with itself and is negative since the island is finite and so large populations suffer from overcrowding. On the other hand $A > 0$ if the species eats grass (so the population increases if the initial population is small) and $A \le 0$ if the species preys on the second species (since if $y = 0$ there is no available food and the population dies of starvation). Finally, the coefficient b_1 describes the effect of species y on species x. If $b_1 > 0$ then this term increases the rate of population growth, for example if x feeds upon y, whilst if $b_1 < 0$ this term decreases the population growth of x, for example if x and y compete for the same resources. Similar interpretations hold for B, a_2 and b_2. This means that there are four classes of population models, depending on the signs of b_1 and b_2: if $b_i > 0$, $i = 1, 2$, both populations benefit each other (a symbiotic relationship), if $b_i < 0$, $i = 1, 2$, both populations inhibit each other (competitive species), whilst if $b_1 < 0$ and $b_2 > 0$ we have a predator-prey model where x is the predator, and if $b_1 > 0$ and $b_2 < 0$ the situation is the same but y is the predator. These are two species models, and of course they can be generalized to N species with

Glendinning

populations $x_i(t)$ which satisfy

$$\dot{x}_i = x_i \left(A_i + \sum_{k=0}^{N} a_{ik} x_k \right) \tag{5.38}$$

for $i = 1, 2, ..., N$ with $a_{ii} < 0$ and the signs of the remaining coefficients depend upon the relationships between the various species.

These models provide examples which can be treated using the fairly basic knowledge about the nature of stationary points that has already been established. The strategy is first, to locate the stationary points, then determine their type and find the relevant eigenvectors of the linearization about each stationary point and finally to join together this local information into a convincing global phase diagram.

(5.1) EXERCISE

Show that the x- and y-axes are invariant for these population models. Why should this be a necessary feature of a population model?

We shall illustrate this technique by considering an example with two competitive species: rabbits, r, and sheep, s.

Example 5.6

Consider the model

$$\dot{r} = r(3 - r - 2s), \quad \dot{s} = s(2 - r - s)$$

with $r, s \geq 0$.

Step 1. To find the stationary points we need to solve

$$r(3 - r - 2s) = 0 \text{ and } s(2 - r - s) = 0$$

in the positive quadrant. This is a straightforward process (solving a pair of simultaneous equations) and gives four stationary points at (r, s) equal to

$$(0,0) \quad (0,2) \quad (3,0) \quad \text{and} \quad (1,1).$$

Step 2. To determine the type of each stationary point we need the Jacobian matrix $Df(r, s)$. Differentiating the defining equations we get

$$Df(r,s) = \begin{pmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - r - 2s \end{pmatrix}.$$

Step 3. We now need to evaluate the eigenvalues and eigenvectors of the Jacobian matrix at each stationary point. If the stationary point is a node it is important to determine the eigenvector to which trajectories are tangential as they approach or leave a neighbourhood of the stationary point (cf. Fig. 5.1).

At $(0,0)$, $Df(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$ with eigenvectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $(0,0)$ is an unstable node and trajectories leave tangential to the e_2 eigenvector, since this corresponds to the eigenvalue of smallest modulus.

At $(0,2)$, $Df(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$ with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ and corresponding eigenvectors $e_1 = (1,-2)^T$ and $e_2 = (0,1)^T$. Hence $(0,2)$ is a stable node and trajectories tend to $(0,2)$ tangential to the e_1 axis.

At $(3,0)$, $Df(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ with eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$ and corresponding eigenvectors $e_1 = (1,0)^T$ and $e_2 = (3,-1)^T$. Hence $(3,0)$ is another stable node and trajectories tend to $(3,0)$ tangential to the e_2 axis.

At $(1,1)$, $Df(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ with eigenvalues $\lambda_\pm = -1 \pm \sqrt{2}$ and eigenvectors $e_\pm = (1, \mp\frac{1}{\sqrt{2}})$. Hence $(1,1)$ is a saddle, with linear stable manifold in the direction of e_- and linear unstable manifold in the direction of e_+.

Step 4. Guess the global phase portrait from the local analysis. This is a little like joining up the dots in puzzle books, and gives a picture like the one shown in Figure 5.14: the stable manifold of $(1,1)$ divides the positive quadrant into two regions, everything below this curve tends to the stationary point at $(3,0)$ and everything above tends to $(0,2)$. Thus for this choice of the parameters we find that one of the species always dies out, but that the question as to which one dies is determined by the initial populations. Furthermore, very small changes of the initial condition near the stable manifold of the saddle lead to radically different asymptotic steady states.

This section has given a simple way of getting some idea of the behaviour of simple nonlinear models using the linearization results near stationary points. It is easy to implement, but has some drawbacks. The linearization determines the flow in a neighbourhood of the stationary

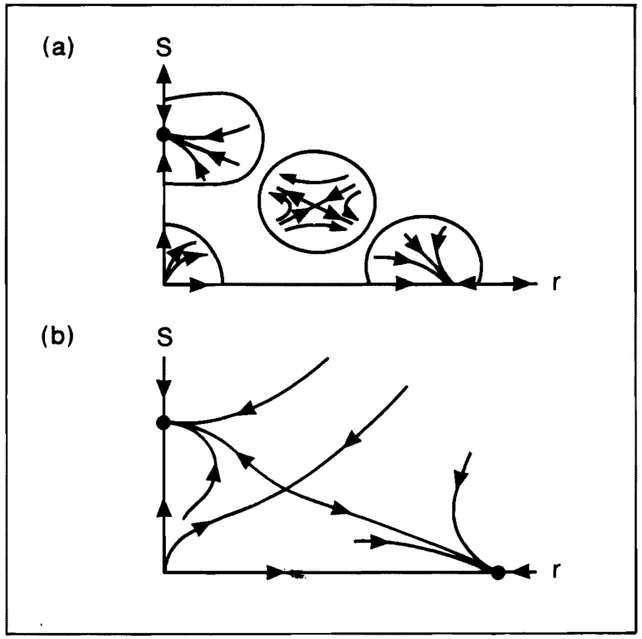


Fig. 5.14 (a) Local patches; (b) global phase portrait.

points, and away from the stationary points trajectories are locally parallel. When joining together patches there are often few topologically distinct solutions if this is done in such a way that no new stationary points are introduced. Indeed, if trajectories come in from infinity *and all stationary points are hyperbolic and there are no periodic orbits* then there must be five or more stationary points before any choice becomes possible. Figure 5.15 shows a flow for which more than one choice of patching the linear neighbourhoods together is possible. There is a symmetric solution (not drawn) and two asymmetric solutions, one of which is illustrated. Can you find other topologically distinct phase portraits for this configuration of stationary points? [Hint: the behaviour of the stable and unstable manifolds of the saddles determines the flow.]

The problems due to periodic orbits are illustrated in Figure 5.16. Suppose the flow has a stable focus, so in a neighbourhood of this stationary point trajectories are spiralling into the point. Then we cannot guarantee that there are no periodic orbits as shown in either the second or third sketch of Figure 5.16. This suggests that we need to do more