MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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COURSE OVERVIEW

First half (*Oliver Maclaren*) [15 lectures]

1. *Introduction and basic concepts* [3 lectures]
   PDEs, basic definitions. Modelling the diffusion (heat) equation and boundary conditions. Introduction to separation of variables.

2. *Expansions in orthogonal functions: Fourier series* [4 lectures]
   Orthogonality of functions/sets of functions and series expansions. Real trigonometric series. Convergence and sketching Fourier series. Complex Fourier series. Use in separation of variables.

3. *Sturm-Liouville eigenvalue problems* [4 lectures]
   Eigenvalue problems for function spaces: eigenvalues, eigenfunctions, Sturm-Liouville problems. Existence and orthogonality of solutions, eigenfunction expansions.

COURSE OVERVIEW

4. *Separation of variables revisited* [3 lectures]
   Separation of variables for the wave equation and Laplace's equation, in several geometries.

5. *Wave Equation* [1 lecture]
   D'Alembert's solution.

LECTURE 1

TOPICS

Basic concepts and origins

Notation

Classification

WHAT ARE PARTIAL DIFFERENTIAL EQUATIONS (PDES)?

A differential equation (DE) is an equation relating a function and its derivatives.

A *partial* differential equation (PDE) is, unsurprisingly, a differential equation that contains *partial* derivatives. A PDE for a function $u(x, y)$ which depends on x and y looks like

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0$$

where F is some function, x, y are the *independent* variables and u is called the *dependent* variable. Note $u_x := \frac{\partial u}{\partial x}$ etc.

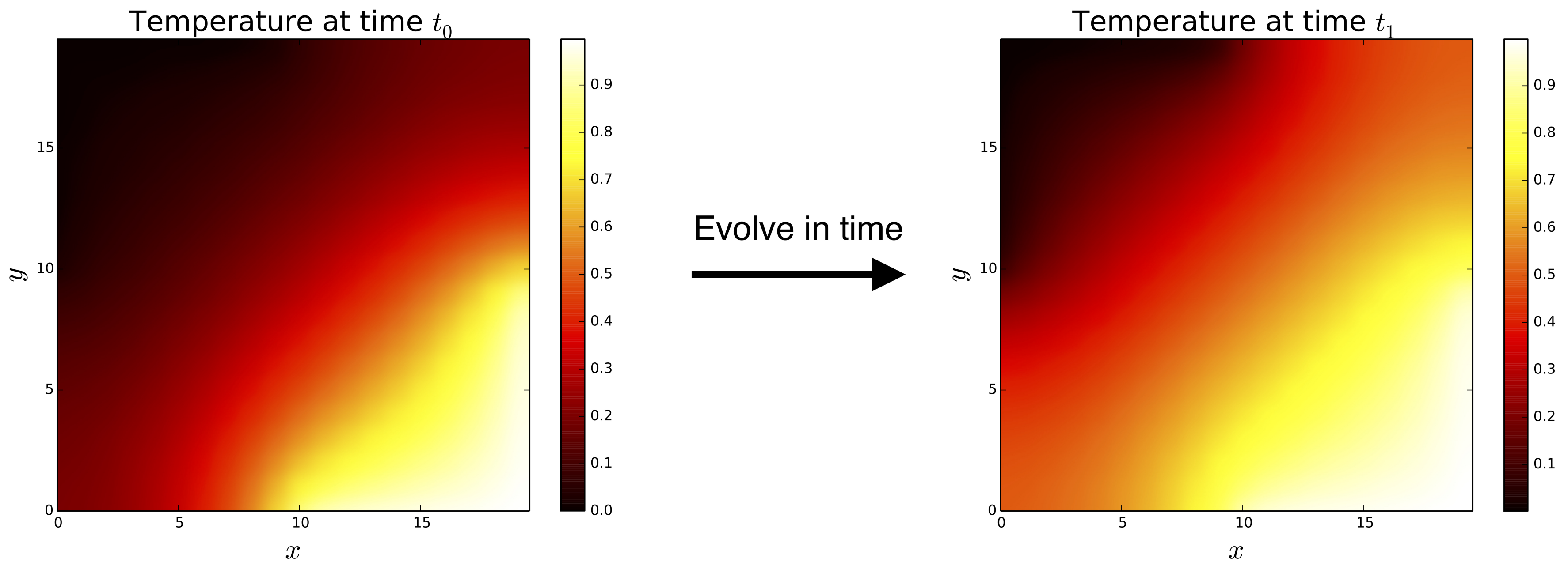
A function or function(s) *$u(x, t, \ldots)$ satisfying* a PDE

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$

is called a *solution** and, in contrast to ordinary differential equations (ODEs), it clearly depends on *more than one independent variable*.

* There are in fact various *types* of solution - e.g. *classical*, *weak* and *distributional* - depending on what we mean by things like 'satisfy'. Or, to put it another way, what *function space* we consider and how we *interpret* the operations on these functions. As an analogy, think about what it means to satisfy $x = \sqrt{-1}$. The answer clearly depends on whether we think of x as real or complex. We will cover weak solutions and distribution theory in the second-half of the course.

WHAT DOES THIS LOOK LIKE?



Spatiotemporal temperature field $u(x, y, t)$

WHAT SORT OF PDE WOULD MODEL THIS?

The *diffusion* (heat) equation in *one space dimension** and *one time dimension*, with constant diffusion constant is given by

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

This can describe heat conduction, diffusion of a chemical species or the evolution of a probability density.

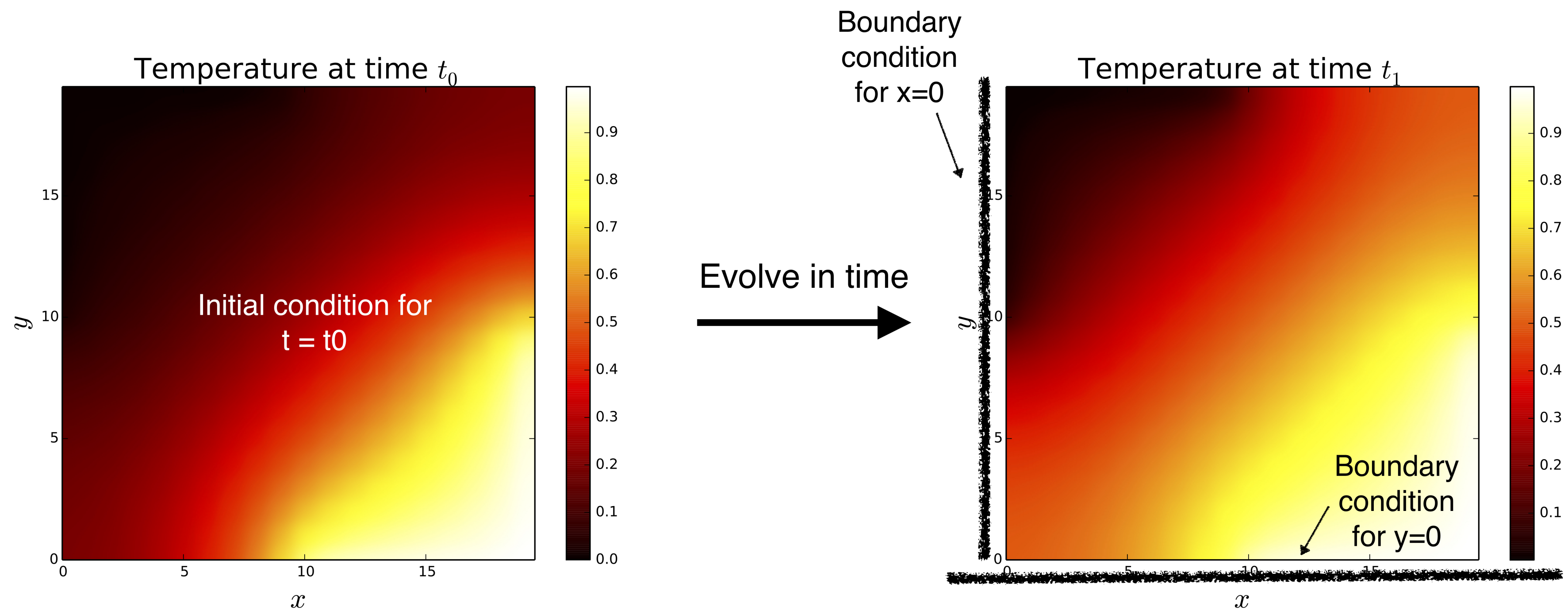
Later, we will look at how we can *derive* this equation from *physical* principles for particular problems, as well as how we can *solve* this and similar equations.

* For simplicity

WELL-POSED?

What do we need to make our problem 'well-posed'*

* solution exists, is unique, is stable to small perturbations in the problem formulation - Hadamard



Spatiotemporal temperature field $u(x, y, t)$ with *initial* and *boundary* conditions

INITIAL AND BOUNDARY CONDITIONS

A necessary condition for obtaining a *well-posed problem* for a PDE is specifying both

- *Initial conditions*

e.g. $u(x, y, t = 0) = f(x, y)$

and

- *Boundary conditions*

e.g. $u(x = 0, y, t) = g(y, t)$ and $u(x, y = 0, t) = h(x, t)^*$

* We also need boundary conditions at the other ends!

WHERE DO PDES COME FROM?

Historically, PDEs came from the study of surfaces in *geometry* and the formulation of problems in *mechanics*.

Nowadays a very large range of phenomena can be modelled and studied using PDEs:

...fluid dynamics, solid mechanics, electricity and magnetism, optics, heat flow, quantum mechanics, biological signalling and pattern formation, stochastic motion of interacting particles...

Exercise: Google some!

PDEs continue to be important in the modern study of and interaction between physics, geometry and analysis.

We will mainly focus on the modelling and solution of PDEs related to simple physical (and linear) phenomena.

This course is only a small taste of PDEs!

Other revelant courses include: MATHS 333, 340, 362, 363, 763, 769, 786, 731, 762...

HOW DO YOU SOLVE A PDE?

There is no (known) universal theory for the solution of an arbitrary PDE...

(The theory of linear PDEs is well-established; nonlinear PDEs are a different story)

HOW DO YOU SOLVE A PDE?

Methods covered in first half of course

Method	Motivations
Separation of variables	Reduce a PDE in n variables to n ODEs
Eigenfunction expansion	Find solution as an infinite sum of eigenfunctions by solving an associated eigenproblem
Numerical methods [covered in more detail in other courses]	Convert into system of discrete/symbolic equations, solve on computer

HOW DO YOU SOLVE A PDE?

Transformations (see later)

Method	**Motivations**
Integral transforms	Reduce number of independent vars; work in an easier problem domain
Change of independent/dependent variables	Change PDE into ODE or easier PDE, PDE in new unknown
Symmetry/group theory analysis (e.g. Lie, Renormalization)	Identify exact/approximate symmetries to reduce/clarify problem structure

HOW DO YOU SOLVE A PDE?

Solve a related problem (see later)

Method	Motivations
Perturbation methods	Change nonlinear problem into sequence on linear problems
Greens functions	Find solution to overal response as sum of response to 'simple impulses'
Integral equations	Convert PDE to an integral equation (often allows wider class of solutions)
Calculus of variations	Reformulate as extremising an integral. Also basis for FEM numerical method

WHAT ARE THE DIFFERENT KINDS OF PDES?

We can classify a *general* PDE according to

- Order (highest partial derivative)
- Number of independent variables
- Linearity* (linear or nonlinear)
    - Unknown function and its derivatives only appear in linear combination with coefficients depending only on the independent variables

We can give a nicer discussion of linearity by considering *operator notation*.

* A *quasi*-linear system is linear in the highest-ordered derivative, where the coefficient may depend in an arbitary manner on the lower-order terms.

EXAMPLE CLASSIFICATIONS

$$u_t - ku_{xx} = 0$$

with k constant.

$$uu_t + cu_x + 2txu - sin(tx) = 0$$

with c constant.

OPERATOR NOTATION

A general PDE for $u(x, y, \ldots)$ can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$

We can rearrange this and group the terms involving u and its derivatives on one side and the terms only involving the independent variables $x, y, \ldots$ on the other, giving

$$G(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = f(x, y, \ldots)$$

OPERATOR NOTATION

$G(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots)$ can be considered to be generated from u via a *differential operator* acting on u,

$$L(u(x, y, \ldots)) := G(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = f(x, y, \ldots)$$

Suppressing the arguments, this gives*

$$Lu = f$$

where, as mentioned, L is called a *differential operator*, which *operates on* u. (An *operator* is a function which takes a function and returns another function.)

* This should remind you of matrix equations in finite dimensional spaces, though here we are in *infinite dimensions* since $u(x)$ can be considered an infinite-dimensional vector - see later.

FORCING

Here f is often referred to as the 'forcing' function

We say that the equation $Lu = f$ is

- *homogeneous* if $f = 0$ and
- *non-homogeneous* otherwise.

DATA SCIENCE

The combination of RHS values for the

- (boundary conditions,
- initial conditions,
- differential equation)

are sometimes called the problem data.

OPERATORS AND PROBLEM DATA: SOLVABILITY

- Knowing how the particular combinations of problem data + action of the differential operator fit together is important for understanding the solvability of a given problem.
- It will often help take the definition of a given differential operator L to consist of both* the action of the differential operator and the boundary/initial data

This will all (hopefully) become clearer later - just a heads up for now!

* Think about the definition of a function actually requires three things: a rule/mapping as well as a domain of things to map and a codomain to map into.

EXAMPLE

The diffusion equation with a spatially-dependent source $\phi(x)$,

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + \phi(x)$$

Can be written in the form $Lu = f$ using*

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)u = \phi$$

where here $L = \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}$ and $f = \phi$.

* For general nonlinear operators we won't be able to factorise L so nicely. We have to be content with describing L via its action, e.g. something like $L : u \rightarrow u^2 u''$.

LINEARITY AND LINEAR OPERATORS

Thinking of L as an operator allows us to classify it as a *linear* or *non-linear operator* in the usual, abstract way:

L is a *linear* operator if

- $L(au) = aL(u)$
- $L(u + v) = L(u) + L(v)$

for a constant a and for any two solutions u and v

Otherwise, L is *non-linear*.

EXAMPLE CLASSIFICATIONS REVISITED

$$Lu = u_t - ku_{xx} = 0$$

with k constant.

$$L(u + w) =?$$
$$L(au) =?$$

$$Lu = uu_t + cu_x + 2txu = sin(tx) = f$$

with c constant.

$$L(u + w) =?$$
$$L(au) =?$$

A NOTE ON NOTATION FOR NONLINEAR OPERATORS

Note that the second case is *nonlinear*.

For general nonlinear operators we *won't be able to factorise L so nicely as we do for linear operators*.

We have to be content with describing L via its *action on u*, e.g. something like $L : u \to u^2 u_{tt}$ (say).

Question: What is $L(u + v)$ in the above case?

COEFFICIENTS OF LINEAR PDES

We can further classify *linear* PDEs, written

$$L(u) = \ldots$$

$$a(x_1, x_2, \ldots)u + \sum b_i(x_1, x_2, \ldots)u_{x_i} + \sum c_{ij}(x_1, x_2, \ldots)u_x$$

$$\ldots = f(x_1, x_2, \ldots)$$

In terms of *kinds of coefficients*

- constant/non-constant $a, b_i, c_{ij}, \ldots$

(SOME) TYPES OF LINEAR PDES

We can further (further) classify *linear, second-order* PDEs according to one of three '*basic types*'

- *Parabolic* (diffusion-like)
- *Hyperbolic* (wave-like)
- *Elliptic* (equilibrium)

CANONICAL SECOND-ORDER LINEAR PDES

Parabolic (*heat/diffusion equation*)

$$\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

Hyperbolic (*wave equation*)

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

Elliptic (*Laplace equation*)

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

FURTHER CLASSIFICATION OF LINEAR PDES II

Here we will just consider the formal 'type' classification of *second-order, linear* PDEs in *two variables* written in the *standard form*

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F, G are coefficients that are functions of x, y in general*.

* This classification will hence apply *at a given point* (x, y): a given PDE may have *different types at different locations* if the coefficients change signs etc.

FURTHER CLASSIFICATION OF LINEAR PDES II

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

is, at a given x, y point,

- *Parabolic* if $B^2 - 4AC = 0$
- *Hyperbolic* if $B^2 - 4AC > 0$
- *Elliptic* if $B^2 - 4AC < 0$

~~SUMMARY~~ 10 LIFE LESSONS FROM DIFFERENTIAL EQUATIONS

From http://www.johndcook.com/blog/2015/07/23/life-lessons-from-differential-equations/

1. Some problems simply have no solution.
2. Some problems have no simple solution.
3. Some problems have many solutions.
4. Determining that a solution exists may be half the work of finding it.
5. Solutions that work well locally may blow up when extended too far. ...

…

6. Boundary conditions are the hard part.
7. Something that starts out as a good solution may become a very bad solution.
8. You can fool yourself by constructing a solution where one doesn’t exist.
9. Expand your possibilities to find a solution, then reduce them to see how good the solution is.
10. You can sometimes do what sounds impossible by reframing your problem.