Engsci 711

Assignment 2

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Due: Wednesday 8th June (or before; office 223)

Question 1

The purpose of this question is to understand how we can get to bifurcation theory via centre manifold theory using the idea of an 'extended' centre manifold.

Really, everything works just the same as in normal centre manifold theory, *we just 'upgrade' the parameter to a (still trivial!) state variable.*

Don't worry, I'll guide you through!

Consider the system of equations

$$\dot{x} = y - x - x^2,$$
$$\dot{y} = \mu x - y - y^2$$

- Find a value $\mu = \mu_c$, for which the origin $(0, 0)$ is non-hyperbolic.
- Linearise the system about the origin $(0, 0)$ and with μ fixed at μ_c.
- Determine a linear change of coordinates $(x, y) \to (u, v)$ that puts the linearised system into diagonal form. Hint: one of the tutorials (and handouts!) shows how to do this.
- Define a new parameter $\lambda = \mu - \mu_c$ which is zero at the non-hyperbolic point. Write the *full, nonlinear* system in terms of your u, v above and your new parameter λ.

Now comes the key - yet simple - step.

- 'Upgrade' the parameter λ to a state variable. This means we take the u, v equations and add the trivial equation $\dot{\lambda} = 0$.

Note that, since we are in diagonal form this corresponds to adding another *centre* variable (eigenvalue = 0). In this case it is 'super slow' since *both* linear and nonlinear parts are zero (the other centre variable will have 'zero-eigenvalue' linear dynamics but non-trivial higher-order dynamics, so can be thought of as 'slowly varying').

You should now have a system of the form

$$\dot{u} = ...$$
$$\dot{v} = ...$$
$$\dot{\lambda} = 0$$

Note that, while for the λ-as-parameter system a term like λu is linear, when λ is considered as a state variable a term like this is considered *nonlinear*!

We should now have a diagonalised system where the (extended) centre manifold component is two-dimensional (check you understand why). Suppose u and λ are your centre manifold variables. Our centre manifold will be tangent to the (λ, u) plane at $(u, v, \lambda) = (0, 0, 0)$.

We can now proceed as normal in centre manifold theory

- Assume that the two-dimensional centre manifold is described by a restriction of three-dimensional (u, v, λ) space by one constraint $v = h(\lambda, u)$, and that h can be approximated using a two-variable Taylor series expansion. This takes the form

$$v = a + b\lambda + cu + d\lambda^2 + e\lambda u + fu^2$$

where a, b, c, d, e, f are constants.

- What are a, b, c? You should be able to write these down instantly.
- Now use the usual procedure for finding the other coefficients. That is, use

$$\dot{v} = \frac{\partial h}{\partial \lambda}\dot{\lambda} + \frac{\partial h}{\partial u}\dot{u}$$

and substitute in what you know about $\dot{v}, \dot{\lambda}, \dot{u}$.

- Equate coefficients to determine d, e, f.
- Now, use the Reduction Principle to determine the dynamics on the extended (u, λ) centre manifold. That is, substitute your expression into the u equation (the λ equation remains trivial). Your answer should consist of writing down two differential equations.

Note: we have effectively obtained a one-dimensional bifurcation problem (as expected)! To see, note that since the λ dynamics are trivial, we can effectively downgrade λ back to a control parameter. That is, we fix it to different values and solve the u equation for each of these.

This can be considered as either a u vs λ phase-portrait OR a u vs λ bifurcation diagram (particularly when we just plot the equilibria of u). The point of 'upgrading' it temporarily was to derive the bifurcation diagram from the centre manifold phase portrait. Interestingly, reducing it back to a parameter can be thought of as an *additional* centre manifold reduction with λ as the (super) slow variable.

So we can now carry out the last step.

- Draw a bifurcation plot/(u, λ) phase portrait. What sort of bifurcation is this?

Question 2

Carry out a bifurcation analysis for the systems

$$(a) \quad \dot{x} = x(\mu - x - x^2)$$
$$(b) \quad \dot{x} = x(\mu - x^2)$$

Remember to say what type of bifurcation occurs for each. Note - you do not need to use extended centre manifold theory for this!

Question 3

Consider the system (from Tutorial 3)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

- Find the curves in (μ, a) space at which you expect a Hopf bifurcation to occur.
- Fix a value of a and then verify, by plotting various phase portraits with XPPAut (or Matlab etc), that a Hopf bifurcation does occur for your predicted μ values.
- Use XPPAut to determine whether the Hopf bifurcation is subcritical (appearing/disappearing periodic orbit is unstable), supercritical (appearing/disappearing periodic orbit is stable) or degenerate (periodic solution appears at bifurcation but disappears for all other parameter values).