ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: centre manifold theorem and reduction principle.

LECTURE 8

More examples of *co-dimension-one* bifurcations:

- Hopf bifurcation
- Systems with multiple co-dimension-one bifurcations

BIFURCATIONS - 'CO-DIMENSION'

For our purposes, we define the *co-dimension* of a bifurcation type as the *minimum number of parameters* we need to vary *to get this type of bifurcation*.

Note: our original system may be higher-dimensional, but

1. the bifurcation typically occurs in lower dimensions and
2. is determined by a small number of parameters (low co-dimension) - e.g. one eigenvalue crosses the imaginary axis (the real part changes sign, hence stability).

RECALL: BIFURCATIONS

We have seen three *co-dimension one* bifurcations:

- saddle-node/turning point/fold bifurcation
- transcritical bifurcation
- pitchfork bifurcation

There is one more - which is *also co-dimension one* but is slightly more complicated - the *Hopf bifurcation*.

These four give all the possible co-dimension one bifurcations.

HOPF BIFURCATION

The Hopf bifurcation occurs when a *pair of complex conjugate eigenvalues cross the imaginary axis together*.

In contrast to before, we now have a non-zero imaginary component and hence have to deal with oscillatory components.

(Thus we need the *system* to have at least dimension two, though the *co-dimension* is still one.)

HOPF BIFURCATION

A Hopf bifurcation (for our purposes) is characterised by *a change in stability of a fixed point, along with the appearance or the disappearance of a periodic orbit at this fixed point*.

HOPF BIFURCATION THEOREM (OR NOT)

There is a *Hopf bifurcation theorem* (See e.g. Glendinning) giving *conditions under which periodic solutions are created/destroyed* as a pair of complex eigenvalues pass through the imaginary axis (and the associated fixed point changes stability)

Unfortunately it is a bit tricky/ugly to verify the conditions for creation/destruction.

HOPF BIFURCATION THEOREM (OR NOT)

Instead we typically a) *find where a pair of complex eigenvalues become purely imaginary* and b) *directly verify that a periodic solution was created/destroyed* (as we pass through the bifurcation) via simulation (or analytical solution in simple cases).

HOPF BIFURCATION - CO-DIMENSION AGAIN

The Hopf bifurcation occurs in *two-dimensional systems* (or on a two-dimensional reduced/centre manifold of a larger system) BUT

The Hopf bifurcation essentially only *depends on varying one parameter, hence the co-dimension is one*.

HOPF BIFURCATION - CANONICAL EXAMPLE

$$\dot{x} = -\omega y + x(\mu - (x^2 + y^2))$$
$$\dot{y} = \omega x + y(\mu - (x^2 + y^2))$$

HOPF BIFURCATION ANALYSIS

Steps

- *Verify* we have a pair of complex conjugate eigenvalues crossing the imaginary axis and an associated change in stability of the fixed point.
- *Verify* (in this example by direct solution, in general via numerical methods) that a periodic orbit exists on one side of the bifurcation.

Note: If a direct solution is possible, then it is typically easiest to construct/verify in polar coordinates.

HOPF BIFURCATION - CROSSING SPEED

As seen, *a necessary condition for a Hopf bifurcation is finding a pair of complex conjugate eigenvalues crossing the imaginary axis*.

We can also sometimes *verify that this crossing occurs with a non-zero speed* (another necessary condition)

So, we have eigenvalues $\lambda = \pm i\omega$ at critical parameter value $\mu = \mu_c$.

and (for a non-degenerate bifurcation) $\frac{\partial \lambda^r}{\partial \mu} \neq 0$ at $\mu = \mu_c$.

where λ^r is the real part of the eigenvalue.

HOPF BIFURCATION KEY FEATURES

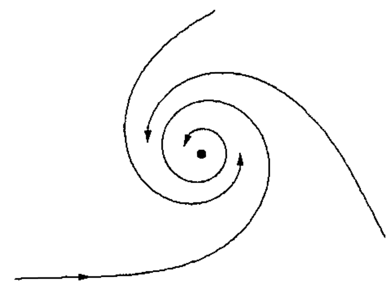
- We call the bifurcation *supercritical* if the emerging/disappearing periodic orbit is stable.
- If it is unstable, we call it *subcritical*[*]
- The *radius* of the limit cycles grow/shrink continuously from/to zero and proportional to $\sqrt{\mu - \mu_c}$ near μ_c
- The *frequency* of the limit cycle is approximately Im λ, evaluated at $\mu = \mu_c$

[*] Similar terminology is used for other bifurcations (e.g. supercritical pitchfork - stable FP are born).

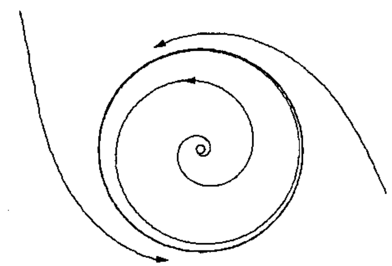
HOPF BIFURCATION KEY FEATURES

- The periodic orbit and the fixed point have *opposite stability for the parameter values that they both exist*
- I.e. *supercritical*: stable PO, unstable FP; *subcritical*: unstable PO, stable FP
- Note that it is also *difficult to manually check the theorem conditions* for whether the Hopf bifurcation is 'supercritical' or 'subcritical' (or degenerate).
- Again, it is easier to do it by simulation, direct construction, direct checking etc on a *case-by-case basis*.

HOPF BIFURCATION PICTURES

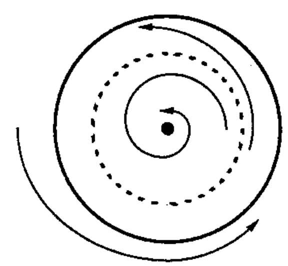


$\mu < 0$

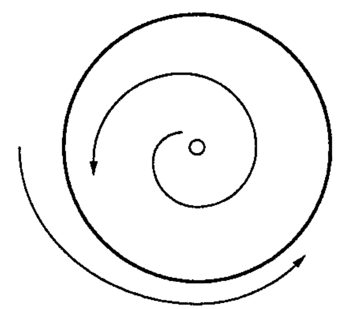


$\mu > 0$

Supercritical



$\mu < 0$



$\mu > 0$

Subcritical

ANOTHER HOPF BIFURCATION EXAMPLE

Exam 2016: Exercise (will go over in tutorial).

MULTIPLE ONE-PARAMETER BIFURCATIONS

Returning to our general four types of co-dimension one bifurcations, can we get *multiple one-parameter bifurcations in a single system?* Of course!

Exam 2016.

TWO PARAMETERS - TUTORIAL/ASSIGNMENT ONLY (NOT EXAM)

What happens when we have bifurcations *depending on more than one parameter?*

Tutorial (and maybe assignment...)