giving $\alpha_1^2 = -2G_\mu/G_{xx}$, or $\alpha_1 = \pm\sqrt{-2G_\mu/G_{xx}}$, which we already knew. At order $\mu^{\frac{3}{2}}$ we find that we need to include cubic terms from (8.19) to obtain

$$G_{\mu x}\alpha_1 + G_{xx}\alpha_1\alpha_2 + \tfrac{1}{6}G_{xxx}\alpha_1^3 = 0 \tag{8.24}$$

and so

$$\alpha_2 = -\frac{1}{G_{xx}}(G_{\mu x} + \tfrac{1}{6}G_{xxx}\alpha_1^2) = \frac{1}{3G_{xx}^2}\left(G_\mu G_{xxx} - 3G_{\mu x}G_{xx}\right). \tag{8.25}$$

Hence, provided both $G_\mu/G_{xx} < 0$ (and, in particular, $G_\mu \neq 0$ and $G_{xx} \neq 0$) we obtain a pair of stationary points if $\mu > 0$ at

$$x \sim \pm\sqrt{-\frac{2G_\mu}{G_{xx}}}\mu^{\frac{1}{2}} + \frac{1}{3G_{xx}^2}\left(G_\mu G_{xxx} - 3G_{\mu x}G_{xx}\right)\mu + O(\mu^{\frac{3}{2}}). \tag{8.26}$$

In principle we could continue to find α_3, α_4 and so on. Note that if $G_\mu/G_{xx} > 0$ then we would find a pair of solutions if $\mu < 0$ which could be represented as an asymptotic series in powers of $(-\mu)^{\frac{1}{2}}$.

Finally we should investigate the stability of these stationary points (for small $|\mu|$). This is determined by the sign of the Jacobian matrix which, for one-dimensional systems, is simply the function $A_1 + 2A_2x + \ldots$ evaluated at the stationary points. From this it is clear that the dominant term is A_2x (which is of order $|\mu|^{\frac{1}{2}}$) and so the positive (resp. negative) stationary point is stable if $G_{xx} < 0$ (resp. $G_{xx} > 0$) whilst the stationary point in $x < 0$ (resp. $x > 0$) is unstable.

This argument shows (informally) that if $x = 0$ is a non-hyperbolic stationary point of the family of differential equations $\dot{x} = G(x, \mu)$ on the real line when $\mu = 0$ then provided $G_\mu(0,0)$ and $G_{xx}(0,0)$ are non-zero a curve of stationary states bifurcates from $(x, \mu) = (0, 0)$, tangential to the x-axis. The direction of the bifurcation is determined by the sign of (G_μ/G_{xx}) and the stability of the bifurcation solutions is determined by the sign of G_{xx}. This set of results can be expressed as the following theorem.

(8.2) THEOREM (SADDLENODE BIFURCATION)

Suppose that $\dot{x} = G(x, \mu)$ with $G(0,0) = G_x(0,0) = 0$. Then provided

$$G_\mu(0,0) \neq 0 \quad \textit{and} \quad G_{xx}(0,0) \neq 0$$

there is a continuous curve of stationary points in a neighbourhood of $(x, \mu) = (0,0)$ which is tangent to $\mu = 0$ at $(0,0)$. If $G_\mu G_{xx} < 0$ (resp. $G_\mu G_{xx} > 0$) there are no stationary points near $(0,0)$ if $\mu < 0$ (resp. $\mu > 0$) whilst for each value of $\mu > 0$ (resp. $\mu < 0$) in some sufficiently small neighbourhood of $\mu = 0$ there are two stationary points near $x = 0$. For $\mu \neq 0$ both stationary points are hyperbolic and the upper one is stable and the lower unstable if $G_{xx} < 0$. The stability properties are reversed if $G_{xx} > 0$.

This is an example of a bifurcation theorem (this bifurcation is called the saddlenode bifurcation). Typically such theorems state that provided some genericity conditions hold (in this case the non-vanishing of two partial derivatives of the equations at the bifurcation point) then, locally, certain changes in the flow will arise. In Section 8.7 we will outline a different approach to such changes, but the next few sections will be concerned with what happens if one or other of the genericity conditions for the saddlenode bifurcation fails to be satisfied.

Example 8.3 — Saddlenode (& TC)

Consider the equations

$$\dot{x} = G(x, \mu) = x^3 + x^2 - (2 + \mu)x + \mu.$$

We want to locate bifurcations of stationary points for this system, so we begin by looking for solutions to $x^3 + x^2 - (2 + \mu)x + \mu = 0$. This is made easier by noticing that the cubic factorizes, so

$$\boxed{(x - 1)(x^2 + 2x - \mu) = 0.}$$

Hence $x = 1$ is a stationary point for all values of μ whilst the quadratic terms give real solutions

$$\boxed{x_\pm = -1 \pm \sqrt{1 + \mu}}$$

provided $\mu > -1$. This is strong evidence that there is a saddlenode bifurcation from $x = -1$ when $\mu = -1$. To confirm this we note that

$$G_x(x, \mu) = 3x^2 + 2x - (2 + \mu), \quad G_\mu(x, \mu) = -x + 1$$

$$\text{and} \quad G_{xx}(x, \mu) = 6x + 2$$

and so, evaluating these quantities at the bifurcation value $(x, \mu) = (-1, -1)$ we find

$$G(-1, -1) = G_x(-1, -1) = 0, \quad G_\mu(-1, -1) = 2$$

$$\text{and} \quad G_{xx}(-1,-1) = -4.$$

Saddle node

Hence there is indeed a saddlenode bifurcation from $(-1,-1)$ and since $G_\mu G_{xx} < 0$ the pair of solutions bifurcates into $\mu > -1$, as we knew already from the form of the solutions $x_\pm$. Furthermore, since $G_{xx} < 0$ x_+ is stable and x_- is unstable for $\mu+1 > 0$ sufficiently small. This is all we can determine from the local analysis, but note that if $\mu = 3$, $x_+ = 1$ and so the stationary point x_+ coincides with the stationary point at $x = 1$. Evaluating G_x at $(x, \mu) = (1, 3)$ we find that this stationary point is non-hyperbolic. Is there a further bifurcation? The answer to this rhetorical question is, of course, yes, but since $G_\mu(1,3) = 0$ it cannot be a saddlenode bifurcation. Figure 8.4 shows what happens here. The full analysis of this second bifurcation is left until the next section.

Remark: This example shows that a much more sensible way of determining the nature and existence of saddlenode bifurcations is simply to look at the equations for a stationary point and find places where a pair of zeros disappear (typically this will be when some discriminant vanishes). In effect, this is precisely what we have done above, but rather than work with a particular example we have used the general form of the (local) Taylor series of the system.

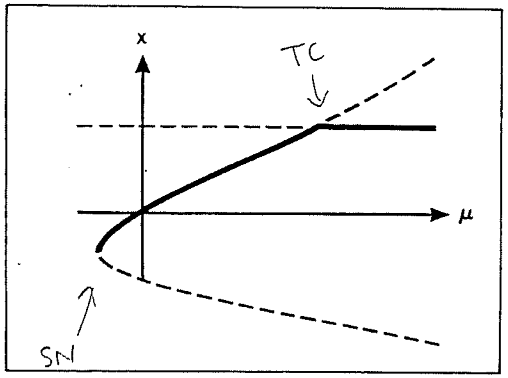

Fig. 8.4 Bifurcation diagram for Example 8.3.

8.4 The transcritical bifurcation

The transcritical bifurcation is another bifurcation which occurs if $\dim E^c(0) = 1$ for a non-hyperbolic stationary point at the origin, but in this case $G_\mu(0,0) = 0$, and so the genericity conditions for the saddle-node bifurcation do not apply.

Example 8.4

Consider the differential equation

$$\dot{x} = \mu x - x^2.$$

Looking for stationary points (solutions of $\dot{x} = 0$) we find that there are two solutions, one at $x = 0$ and another at $x = \mu$. So for $\mu < 0$ and for $\mu > 0$ there are two stationary points and if $\mu = 0$ there is only one stationary point (at $x = 0$). The (1×1) Jacobian matrix of the equation is $\mu - 2x$, which is zero if $x = \mu = 0$, so the stationary point for $\mu = 0$ is not hyperbolic. From the Jacobian we can also see that the stationary point at $x = 0$ is stable if $\mu < 0$ and unstable if $\mu > 0$, whilst the stationary point at $x = \mu$ is unstable if $\mu < 0$ and stable if $\mu > 0$. The bifurcation diagram is shown in Figure 8.5. This bifurcation is usually referred to as a transcritical bifurcation, but (for obvious reasons) it is sometimes called an exchange of stability.

The important feature of this example is that the origin, $x = 0$, is constrained to be a stationary point both before and after the bifurcation

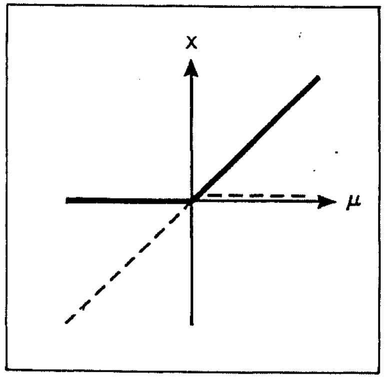

Fig. 8.5 Bifurcation diagram for Example 8.4.

(at $\mu = 0$) and it is easy to verify that if $G(x,\mu) = \mu x - x^2$ then $G(0,0) = G_x(0,0) = G_\mu(0,0) = 0$ and $G_{xx}(0,0) = 2$. This suggests that the first attempt to generalize this example should be by considering differential equations on the centre manifold,

$$\dot{x} = G(x,\mu) = \tfrac{1}{2}(G_{xx}x^2 + 2G_{\mu x}\mu x + G_{\mu\mu}\mu^2) + O(3), \qquad (8.27)$$

i.e. assuming that the first derivative of G with respect to each variable vanishes. As in the previous section we attempt to find the leading order expressions for stationary points of this system near $(x,\mu) = (0,0)$ by solving the quadratic equation obtained by setting the right hand side of the $\dot{x}$ equation to zero and ignoring terms of order x^3 and higher. This gives

$$x \sim \frac{-G_{\mu x}\mu \pm \mu\sqrt{G_{\mu x}^2 - G_{xx}G_{\mu\mu}}}{G_{xx}} \qquad (8.28)$$

provided $G_{xx} \neq 0$. Set $\Delta^2 = G_{\mu x}^2 - G_{xx}G_{\mu\mu}$, then provided $\Delta^2 > 0$ there are two curves of stationary points near $(0,0)$ given by

$$x \sim -\frac{G_{\mu x} \pm \Delta}{G_{xx}}\mu + O(\mu^2). \qquad (8.29)$$

These two curves intersect transversely at $(x,\mu) = (0,0)$. As with the saddlenode bifurcation the stability of these stationary points is easy to determine: the upper branch of solutions is stable if $G_{xx} < 0$ (with the lower branch being unstable) and stability properties are reversed if $G_{xx} > 0$ (see Fig. 8.6).

Taking a deep breath, it is now possible to give the conditions for a transcritical bifurcation to occur at $(x,\mu) = (0,0)$ for the system $\dot{x} = G(x,\mu)$.

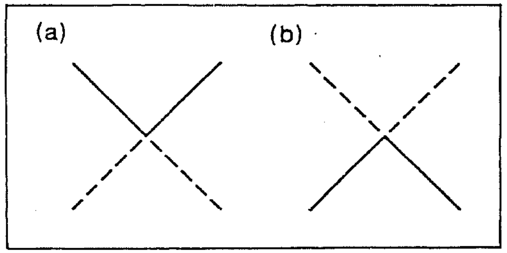

Fig. 8.6 Transcritical bifurcations (x against μ): (a) $G_{xx} < 0$; (b) $G_{xx} > 0$.

(8.3) THEOREM (TRANSCRITICAL BIFURCATION)

Suppose $\dot{x} = G(x,\mu)$ and $G(0,0) = G_x(0,0) = 0$. Then if

$$G_\mu = 0, \quad G_{xx} \neq 0 \quad \text{and} \quad G_{\mu x}^2 - G_{xx}G_{\mu\mu} > 0$$

there are two curves of stationary points in a neighbourhood of $(x,\mu) = (0,0)$. These curves intersect transversely at $(0,0)$ and for each $\mu \neq 0$ sufficiently small there are two hyperbolic stationary points near $x = 0$. The upper stationary point is stable (resp. unstable) and the lower stationary point is unstable (resp. stable) if $G_{xx} < 0$ (resp. $G_{xx} > 0$).

Example 8.5 8.3 con'td

Consider Example 8.3:

$$\dot{x} = G(x,\mu) = x^3 + x^2 - (2+\mu)x + \mu.$$

We have already shown that there is a saddlenode bifurcation when $\mu = -1$ and that if $\mu = 3$ there is a non-hyperbolic stationary point at $x = 1$ with $G_\mu(1,3) = 0$. From the expression in Example 3 we see that $G_{xx}(1,3) = 8$ and also

$$G_{\mu x}(x,\mu) = -1 \quad \text{and} \quad G_{\mu\mu} = 0.$$

Hence $G_{xx}(1,3) \neq 0$ and $G_{\mu x}^2 - G_{xx}G_{\mu\mu} = 1 > 0$, so there is a transcritical bifurcation for $\mu = 3$ (this should be obvious from the sketch in Fig. 8.4). Recall that the two stationary points involved in this bifurcation are $x = 1$ and $x_+ = -1 + \sqrt{1+\mu}$ and so $x_+ < 1$ if $\mu < 3$ and $x_+ > 1$ if $\mu > 3$. Since $G_{xx} > 0$ this implies that x_+ is stable and $x = 1$ is unstable if $\mu < 3$ and the stability properties are exchanged if $\mu > 3$. This completes the full analysis of the bifurcations in this example.

8.5 The pitchfork bifurcation

The final example of bifurcations involving only stationary points of a flow is often found in systems which are invariant under the transformation $x \to -x$. It occurs if both G_μ and G_{xx} vanish at the origin (together with G and G_x, the standard conditions for existence of a bifurcation which we assume all along).

Example 8.6

Consider the differential equation

$$\dot{x} = \mu x - x^3.$$

The origin, $x = 0$, is always a fixed point, and the Jacobian is $\mu - 3x^2$. Thus $x = 0$ is non-hyperbolic when $\mu = 0$ and is stable if $\mu < 0$ and unstable if $\mu > 0$. Provided $\mu > 0$ there are two other stationary points: $x_\pm = \pm\sqrt{\mu}$, both of which are stable. This situation is shown schematically in Figure 8.7, which also explains the name *pitchfork*.

The analytic approach to this bifurcation is similar to the saddlenode bifurcation, in that the only thing we need to worry about is the correct asymptotic expansion for the stationary points. Recall that we are considering bifurcations on the centre manifold of the system

$$\dot{x} = G(x, \mu)$$

where $G(0,0) = G_x(0,0) = 0$, so that $x = 0$ is a non-hyperbolic stationary point when $\mu = 0$, together with the extra conditions $G_\mu(0,0) = G_{xx}(0,0) = 0$. Hence the Taylor expansion of G in a neighbourhood of $(x,\mu) = (0,0)$ can be written as

$$\begin{aligned}\dot{x} = G(x,\mu) &= \tfrac{1}{2}(2G_{\mu x}\mu x + G_{\mu\mu}\mu^2) \\ &+ \tfrac{1}{3!}(G_{xxx}x^3 + 3G_{xx\mu}x^2\mu + 3G_{x\mu\mu}x\mu^2 + G_{\mu\mu\mu}\mu^3) \\ &+ O((|x| + |\mu|)^4). \end{aligned} \tag{8.30}$$

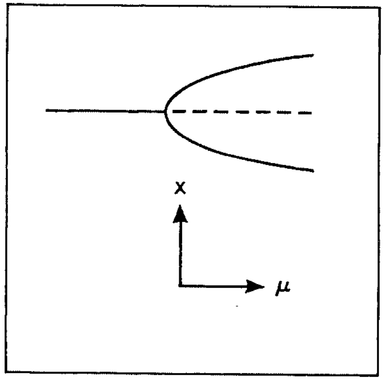

Fig. 8.7 Bifurcation diagram for Example 8.6.

In the two previous sections there has been only one scaling of x which leads to a natural balance in setting the right hand side of the $\dot{x}$ equation to zero: $x \sim O(|\mu|^{\frac{1}{2}})$ for the saddlenode bifurcation and $x \sim O(|\mu|)$ for the transcritical bifurcation. Here, however, both of these possibilities lead to coherent asymptotic expansions for curves of stationary points. If we set $x \sim O(|\mu|^{\frac{1}{2}})$ then we get

$$G_{\mu x}\mu x + \tfrac{1}{3!}G_{xxx}x^3 \approx 0 \tag{8.31}$$

at leading order, whilst if we set $x \sim O(|\mu|)$ we find

$$2G_{\mu x}\mu x + G_{\mu\mu}\mu^2 \approx 0 \tag{8.32}$$

at leading order. This suggests that we need to investigate both possibilities in order to determine the nature of stationary points near $(x,\mu) = (0,0)$. The latter scaling of x is clearly easier, so pose the solution

$$x \sim \sum_{n\geq 1} \alpha_n \mu^n \tag{8.33}$$

for stationary points. Substituting into the right hand side of (8.30) gives

$$\alpha_1 = -\frac{G_{\mu\mu}}{2G_{\mu x}} \tag{8.34}$$

at order μ^2 provided $G_{\mu x} \neq 0$. At order μ^3 the equation is

$$G_{\mu x}\alpha_2 + \frac{1}{6}(G_{xxx}\alpha_1^3 + 3G_{xx\mu}\alpha_1^2 + 3G_{x\mu\mu}\alpha_1 + G_{\mu\mu\mu}) = 0 \tag{8.35}$$

which, provided $G_{\mu x} \neq 0$ can easily be used to find α_2. Hence there is a curve of stationary points through $(x,\mu) = (0,0)$ of the form

$$x \sim -\frac{G_{\mu\mu}}{2G_{\mu x}}\mu + O(\mu^2) \tag{8.36}$$

if $G_{\mu x} \neq 0$. To investigate the other possibility, (8.31), we pose solutions of the form

$$x \sim \sum_{n\geq 1} \beta_n \mu^{\frac{n}{2}} \tag{8.37}$$

for $\mu > 0$ (we will consider the case $\mu < 0$ when we have understood this case). This implies that

$$x^2 \sim \beta_1^2\mu + 2\beta_1\beta_2\mu^{\frac{3}{2}} + O(\mu^2) \tag{8.38}$$

and

$$x^3 \sim \beta_1^3\mu^{\frac{3}{2}} + 3\beta_1^2\beta_2\mu^2 + O(\mu^{\frac{5}{2}}). \tag{8.39}$$

Substituting into (8.30) and setting $\dot{x} = 0$ we obtain, at order $\mu^{\frac{3}{2}}$,

$$G_{\mu x}\beta_1 + \frac{1}{6}G_{xxx}\beta_1^3 = 0 \tag{8.40}$$

and so, provided $G_{\mu x}G_{xxx} < 0$, we obtain the solution

$$\beta_1 = \pm\sqrt{-\frac{6G_{\mu x}}{G_{xxx}}}. \tag{8.41}$$

Now, at order μ^2,

$$G_{\mu x}\beta_2 + \tfrac{1}{2}G_{\mu\mu} + \tfrac{1}{3!}(3G_{xxx}\beta_1^2\beta_2 + 3G_{xx\mu}\beta_1^2) + \tfrac{1}{4!}G_{xxxx}\beta_1^4 = 0 \tag{8.42}$$

or

$$\beta_2 = \frac{1}{4G_{\mu x}G_{xxx}^2}\left(G_{\mu\mu}G_{xxx}^2 - 6G_{\mu x}G_{xx\mu}G_{xxx} + 3G_{xxxx}G_{\mu x}^2\right). \tag{8.43}$$

We could continue to obtain higher order terms in the asymptotic expansion, but the moral should be clear. Provided $G_{\mu x}G_{xxx} < 0$ a pair of stationary points bifurcates into $\mu > 0$ from $\mu = 0$ in a manner analogous to the curve of stationary points in the saddlenode bifurcation. We leave it as an exercise to show that these stationary points are stable if $G_{xxx} < 0$ and unstable if $G_{xxx} > 0$. If $G_{\mu x}G_{xxx} > 0$ then a similar curve bifurcates into $\mu < 0$ (this can be obtained by trying an asymptotic expansion in $(-\mu)^{\frac{1}{2}}$). We are now in a position to state the main result suggested by this discussion.

(8.4) THEOREM (PITCHFORK BIFURCATION)

Suppose $\dot{x} = G(x,\mu)$ and $G(0,0) = G_x(0,0) = 0$. Then if

$$G_\mu(0,0) = G_{xx}(0,0) = 0, \quad G_{\mu x}(0,0) \neq 0 \text{ and } G_{xxx}(0,0) \neq 0$$

there exist two curves of stationary points in a neighbourhood of $(x,\mu) = (0,0)$. One of these passes through $(0,0)$ transverse to the axis $\mu = 0$ whilst the other is tangential to $\mu = 0$ at $(0,0)$. If $G_{\mu x}G_{xxx} < 0$ then for each μ with $|\mu|$ sufficiently small there is one stationary point near $x = 0$ if $\mu < 0$ which is stable if $G_{xxx} < 0$ and unstable if $G_{xxx} > 0$, and if $\mu > 0$ there exist three stationary points near $x = 0$. Of these the outer pair are stable (resp. unstable) and the inner stationary point is unstable (resp. stable) if $G_{xxx} < 0$ (resp. $G_{xxx} > 0$).

If $G_{\mu x}G_{xxx} < 0$, then there exist three stationary points near $x = 0$ if $\mu < 0$ (the outer pair are stable and the inner one is unstable if $G_{xxx} < 0$) and one stationary point near $x = 0$ if $\mu > 0$ (stable if $G_{xxx} < 0$). Stability properties are reversed if $G_{xxx} > 0$.

The bifurcation is said to be *supercritical* if the bifurcating pair of stationary points is stable, otherwise the bifurcation is *subcritical*. These various possibilities are sketched in Figure 8.8.

8.6 An example

(long way)

In this section we shall go through the calculation of the centre manifold and determination of the type of bifurcation by projecting the flow (locally) onto the extended centre manifold in full detail. For our example we take the two-dimensional differential equation

$$\dot{x} = (1+\mu)x - 4y + x^2 - 2xy$$
$$\dot{y} = 2x - 4\mu y - y^2 - x^2$$

which has a stationary point at the origin for all real values of μ. The Jacobian matrix evaluated at $(0,0)$ is

$$\begin{pmatrix} (1+\mu) & -4 \\ 2 & -4\mu \end{pmatrix}$$

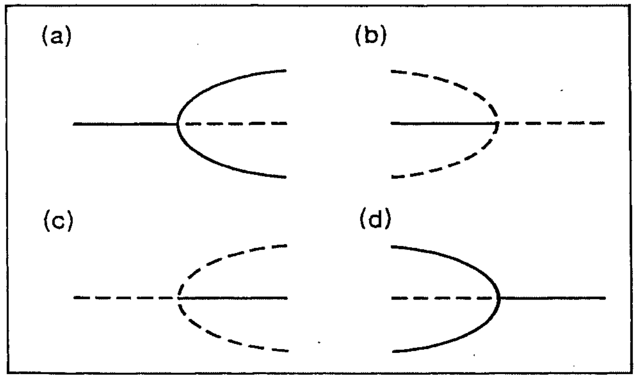

Fig. 8.8 Pitchfork bifurcations: (a) supercritical, $G_{\mu x} > 0$, $G_{xxx} < 0$; (b) subcritical, $G_{\mu x} > 0$, $G_{xxx} > 0$; (c) subcritical, $G_{\mu x} < 0$, $G_{xxx} > 0$; (d) supercritical, $G_{\mu x} < 0$, $G_{xxx} < 0$.

with characteristic equation

$$s^2 - (1-3\mu)s - 4\mu(1+\mu) + 8 = 0.$$

The origin is non-hyperbolic for values of μ at which $s = 0$ or $s = i\omega$ is a solution of the characteristic equation. Setting $s = 0$ we see that the origin has a simple zero eigenvalue if $-4\mu(1+\mu)+8 = 0$, i.e. if $\mu = -2$ or $\mu = 1$. Similarly, setting $s = i\omega$ we find that the origin is non-hyperbolic if $\mu = \frac{1}{3}$ with a pair of purely imaginary eigenvalues. We will ignore this case until the next section and concentrate on the case $\mu = 1$: what sort of bifurcation occurs as μ passes through one? We begin the calculation by changing coordinates so that the linear part of the flow at the origin when $\mu = 1$ is in canonical form. Setting $\mu = 1$ in the Jacobian matrix we find that the linear part of the equation at the origin is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so the linear flow has eigenvalues 0 (as we knew already) and -2. The corresponding eigenvectors are $e_0 = (2,1)^T$ and $e_{-2} = (1,1)^T$. Now let P be the matrix whose columns are the eigenvectors,

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} P = P \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence if we set $\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ we obtain the linear part of the equation for u and v as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Furthermore, $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and so

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u+v \\ u+v \end{pmatrix}.$$

Going back to the original equations and rewriting them in terms of the new variables u and v using these equations gives ($\dot{u} = \dot{x}-\dot{y}$, $\dot{v} = -\dot{x}+2\dot{y}$)

$$\dot{u} = (-1+\mu)(6u+5v) - 3u^2 - 4uv - v^2$$
$$\dot{v} = 10(1-\mu)u + (7-9\mu)v - 10u^2 - 10uv - 3v^2.$$

Since we are interested in the bifurcation at $\mu = 1$ we define a new parameter ν by $\mu = 1+\nu$ so that the bifurcation occurs when $\nu = 0$ and we are able to treat $|\nu|$ as small in our local investigation. Substituting for μ and adding the extra equation $\dot{\nu} = 0$ we obtain the extended system

$$\dot{u} = \nu(6u+5v) - 3u^2 - 4uv - v^2$$
$$\dot{v} = -10\nu u - (2+9\nu)v - 10u^2 - 10uv - 3v^2$$
$$\dot{\nu} = 0.$$

Note that since we are now treating ν in the same way as u and v, the first equation has no linear terms, the only linear term in the second equation is $-2v$ and the third equation has no terms at all! The linear centre manifold is thus $E^c(0) = \{(u,v,\nu)|v = 0\}$ and the linear stable manifold is $E^s(0) = \{(u,v,\nu)|u = \nu = 0\}$. To find an approximation to the nonlinear centre manifold we pose the solution

$$v = h(u,\nu) \text{ with } \frac{\partial h}{\partial u}(0,0) = \frac{\partial h}{\partial \nu}(0,0) = 0$$

i.e. $h(u,\nu) = au^2 + bu\nu + c\nu^2 + \dots$. Now, from the $\dot{v}$ equation we have

$$\dot{v} = -10u\nu - (2+9\nu)h(u,\nu) - 10u^2 - 10uh(u,\nu) - 3[h(u,\nu)]^2$$

but since, on the centre manifold, $v = h(u,v)$ we also have that

$$\dot{v} = \dot{u}\frac{\partial h}{\partial u} + \dot{\nu}\frac{\partial h}{\partial \nu} = \frac{\partial h}{\partial u}(\nu(6u+5h(u,\nu)) - 3u^2 - 4uh(u,\nu) - [h(u,\nu)]^2).$$

Substituting the series expansion of h into these two expressions and equating powers of u^2 gives $a = -5$, powers of $u\nu$ give $b = -5$ and powers of ν^2 give $c = 0$. Hence the centre manifold is given by

$$v = -5u(u+\nu) + \text{ cubic terms.}$$

Finally we substitute this expression back into the equation of $\dot{u}$ to obtain the projection of the motion on the centre manifold onto the u axis:

$$\dot{u} = 6\nu u - 3u^2 + \text{ cubic terms}$$

(note that the expression for the centre manifold does not contribute any quadratic terms in this case, so we didn't really need to calculate the centre manifold). Hence on the centre manifold $\dot{u} = G(u,\nu)$ with

$$G(0,0) = G_u(0,0) = G_\nu(0,0) = 0,\ G_{uu} = -6,\ G_{u\nu} = 6 \text{ and } G_{\nu\nu} = 0.$$

This implies that the conditions for a transcritical bifurcation are satisfied: the origin is stable in $\nu < 0$ and there is a separate branch of unstable stationary points, in $\nu > 0$ this other branch becomes stable and the origin becomes unstable. When $\nu = 0$ the equation on the centre manifold is approximately $\dot{u} = -3u^2$ and so the origin is stable if approached from $u > 0$ and unstable if approached from $u < 0$. These results are illustrated schematically in Figure 8.9.

8.7 The Implicit Function Theorem

In the previous three sections we have discussed various bifurcations by finding stationary points as asymptotic series in the parameter μ. In order to prove such results rigorously we really need to invoke a result from analysis called the Implicit Function Theorem, which states that solutions to equations exist provided certain conditions hold. We have already used the Implicit Function Theorem (implicitly) when discussing the persistence of hyperbolic stationary points under perturbations of differential equations in Chapter 4. In this section we will give a formal statement of the theorem and indicate how it can be used to put the preceding analysis on a rigorous footing.

(8.5) IMPLICIT FUNCTION THEOREM

Suppose that $F : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ *is a continuously differentiable function of the variables* $(y_1, \dots, y_n) \in \mathbf{R}^n$ *and* $z \in \mathbf{R}$, *and that* $F(0,0) = 0$. *If the Jacobian matrix* $DF(0,0)$ *(where* $(DF)_{ij} = \frac{\partial F_i}{\partial y_j}$*) is invertible then*

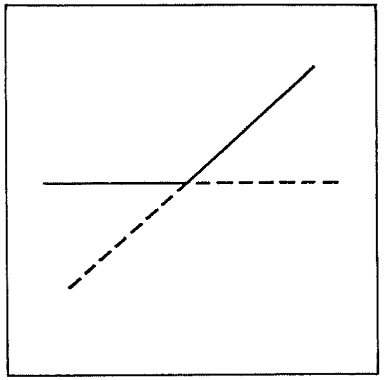

Fig. 8.9

there exists $\epsilon > 0$ *and a smooth curve*

$$y_i = Y_i(z), \quad i = 1, \dots, n,$$

which is the unique solution of $F(y_1, \dots, y_n, z) = 0$ *in* $|z| < \epsilon$, $|y| < \epsilon$.

All this theorem really says is that if we know a solution to an equation and the Jacobian matrix of the equation is invertible (i.e. has non-zero determinant) at that point, then there are solutions at nearby values of the parameter which are continuously connected to the known solution. A proof of this result can be found in any good book on analysis, but essentially all we are saying is that if $F(0,0) = 0$ then

$$F(y,z) = DF(0,0)y + F_z(0,0)z + \dots$$

and so if $DF(0,0)$ is invertible then solutions to $F(y,z) = 0$ are given approximately (for small $|y|$ and $|z|$) by

$$y \sim [DF(0,0)]^{-1} F_z(0,0)z.$$

This is, of course, precisely the sort of manipulation that we have been doing in the last few sections, all the Implicit Function Theorem says is that this procedure is justified.

To illustrate the use of the Implicit Function Theorem we consider the transcritical bifurcation (the saddlenode bifurcation is trivial because the condition $G_\mu \neq 0$ means that the theorem can be applied directly with $y = \mu$ and $z = x$). So we have

$$\dot{x} = G(x, \mu)$$

with $G(0,0) = G_x(0,0) = G_\mu(0,0) = 0$. Stationary points are given by the equation $G(x,\mu) = 0$ and for small $|x|$ and $|\mu|$

$$2G(x,\mu) = G_{xx}x^2 + 2G_{x\mu}\mu x + G_{\mu\mu}\mu^2 + O((|x| + |\mu|)^2).$$

Since both G_x and G_μ vanish at $(x,\mu) = (0,0)$ we cannot apply the Implicit Function Theorem to $G(x,\mu)$ to prove the existence of continuous families of solutions. To treat this case, define a new variable ω by $\omega\mu = x$ and consider

$$g(\omega, \mu) = G(\omega\mu, \mu).$$

Zeros of g correspond to zeros of G and

$$g_\mu(\omega,\mu) = \omega G_x(\omega\mu, \mu) + G_\mu(\omega\mu, \mu),$$
$$g_\omega(\omega,\mu) = \mu G_x(\omega\mu, \mu).$$

Thus both partial derivatives of g are zero if $\mu = 0$. To apply the Implicit Function Theorem define

$$H(\omega,\mu) = \begin{cases} \dfrac{2g(\omega,\mu)}{\mu^2} & \text{if } \mu \neq 0 \\ g_{\mu\mu}(\omega,\mu) & \text{if } \mu = 0 \end{cases}$$

so H is smooth and $H(\omega,0) = 0$ if $g_{\mu\mu}(\omega,0) = 0$. Now,

$$g_{\mu\mu}(\omega,\mu) = \omega^2 G_{xx}(\omega\mu,\mu) + 2\omega G_{\mu x}(\omega\mu,\mu) + G_{\mu\mu}(\omega\mu,\mu)$$

and so $H(\omega,0) = 0$ if $\omega = \omega_\pm$, where

$$\omega_\pm = \frac{-G_{x\mu} \pm \sqrt{G_{x\mu}^2 - G_{xx}G_{\mu\mu}}}{G_{xx}}$$

provided

$$\begin{cases} D = G_{\mu x}^2 - G_{xx}G_{\mu\mu} > 0 \\ G_{xx} \neq 0 \end{cases}.$$

Furthermore,

$$\frac{\partial H}{\partial \omega}(\omega_\pm, 0) = 2G_{xx}\omega_\pm + 2G_{x\mu} = \pm 2\sqrt{D}.$$

Thus, provided $D > 0$ we can apply the Implicit Function Theorem (with $y = \omega$ and $z = \mu$) at $(\omega_+, 0)$ and $(\omega_-, 0)$ to obtain two curves of solutions to $H(\omega,\mu) = 0$ in the form

$$\omega = W_\pm(\mu), \quad W_\pm(0) = \omega_\pm,$$

and hence two branches of stationary points (solutions to $G(x,\mu) = 0$):

$$x = \mu W_\pm(\mu).$$

These two curves intersect at $(x,\mu) = (0,0)$ and the stability results of Section 8.4 are straightforward to verify.

We leave it as an exercise for the interested reader to apply the Implicit Function Theorem to the pitchfork bifurcation.

8.8 The Hopf bifurcation

All the bifurcations discussed so far have involved motion on a one-dimensional centre manifold on which stationary points can be created or destroyed as parameters vary. The Hopf bifurcation is several orders of magnitude harder to analyse since it involves a non-hyperbolic stationary point with linearized eigenvalues $\pm i\omega$, and thus a two-dimensional centre manifold, and the bifurcating solutions are periodic rather than stationary. All this implies that the algebra involved becomes significantly harder, although it is not very different in any conceptual sense. In order to make the results comprehensible this section is split into several parts. First we look at a simple example which illustrates the bifurcation and then go on to state the theorem and rehearse the steps of the proof without going into any detail. Then we look at another example suggested by the sketch of the proof and finally go through the gory details. In fact, the essential part of the proof is relatively straightforward as it only needs the type of manipulations that were used in the proof of Poincaré's Linearization Theorem. The major headache is caused by a need to relate the results back to the partial derivatives of the two-dimensional differential equation on the centre manifold. This manipulation, although straightforward, is lengthy and unexciting and is left until the end of the section.

The Hopf Bifurcation Theorem has a slightly curious history. It is named after Hopf, who gave the first proof in $\mathbf{R}^n$ in 1942, but had been proved by Andronov and Leontovich in the late 1930s using techniques due to Poincaré and Bendixson. Indeed, Poincaré makes it clear in *Méthodes nouvelles de la mécanique céleste, Vol. 1* (1892) that he was aware of the result, but finds it too trivial to bother to write down! Some authors refer to the theorem as the Poincaré–Andronov–Hopf bifurcation, or simply the oscillatory bifurcation. The proof presented here is sketched in Wiggins (1991) and owes a great deal to the work of Hassard and Wan (1978), who seem to have been the first authors to use the complex notation adopted below.

Example 8.7

Consider the differential equation

$$\dot{x} = \mu x - \omega y - (x^2 + y^2)x$$
$$\dot{y} = \omega x + \mu y - (x^2 + y^2)y.$$

A little linearization about the origin $(x,y) = (0,0)$ shows that the origin is a stable focus if $\mu < 0$ and an unstable focus if $\mu > 0$ (the eigenvalues of the Jacobian matrix at the origin are $\mu \pm i\omega$). Hence the origin is non-hyperbolic, with linearized eigenvalues $\pm i\omega$, when $\mu = 0$. From the experience of the previous sections we expect some sort of bifurcation to occur when $\mu = 0$. In order to discover what happens it is easiest to change into polar coordinates. This gives

$$\dot{r} = \mu r - r^3, \quad \dot{\theta} = \omega.$$

From the $\dot{\theta}$ equation it is immediately clear that provided $\omega \neq 0$ the only stationary point is at the origin, $r = 0$, so there is no bifurcating stationary solution. However, $\dot{r} = 0$ if $r = 0$ or $\mu = r^2$. Hence if $\mu > 0$ there is a periodic orbit at $r = \sqrt{\mu}$. Plotting $\dot{r}$ against $\mu r - r^3$ we see that this periodic orbit is stable. This gives the picture illustrated in Figure 8.10: a stable periodic orbit with radius $\sqrt{\mu}$ bifurcates from the origin from $\mu = 0$ into $\mu > 0$.

This is an example of a supercritical Hopf bifurcation (supercritical because the bifurcating periodic orbit is stable). The general result is stated below, but first note that the linearized differential equation at the bifurcation point is $\dot{x} = Lx$ where $L = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ and so L is invertible. Hence, by the Implicit Function Theorem (see Chapter 4 and Section 8.7), there is a stationary point of the flow near the origin for all parameter values close enough to the bifurcation value. This implies that without loss of generality we can consider differential equations for which the origin is always a stationary point for parameter values near zero (which will be taken to be the bifurcation value) since we can always arrange for this to be the case by a simple shift of the origin.

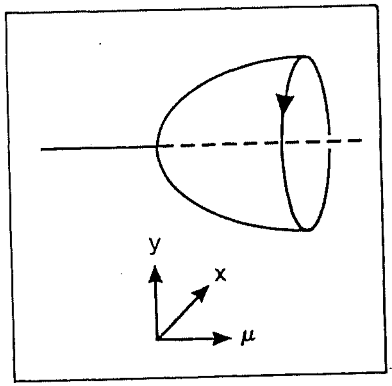

Fig. 8.10 Bifurcation diagram for Example 8.7.

(8.6) THEOREM (HOPF BIFURCATION THEOREM)

Suppose that

$$\dot{x} = f(x, y, \mu), \quad \dot{y} = g(x, y, \mu)$$

with $f(0,0,\mu) = g(0,0,\mu) = 0$ *and that the Jacobian matrix* $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$ *evaluated at the origin when* $\mu = 0$ *is*

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

for some $\omega \neq 0$. *Then if*

$$f_{\mu x} + g_{\mu y} \neq 0$$

and

$$a \neq 0$$

where a *is a constant defined below, a curve of periodic solutions bifurcates from the origin into* $\mu < 0$ *if* $a(f_{\mu x} + g_{\mu y}) > 0$ *or* $\mu > 0$ *if* $a(f_{\mu x} + g_{\mu y}) < 0$. *The origin is stable for* $\mu > 0$ *(resp.* $\mu < 0$*) and unstable for* $\mu < 0$ *(resp.* $\mu > 0$*) if* $f_{\mu x} + g_{\mu y} < 0$ *(resp.* > 0*) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of* $\mu = 0$ *for which the periodic solutions exist. The amplitude of the periodic orbits grows like* $|\mu|^{\frac{1}{2}}$ *whilst their periods tend to* $\frac{2\pi}{|\omega|}$ *as* $|\mu|$ *tends to zero. The bifurcation is supercritical if the bifurcating periodic orbits are stable, otherwise it is subcritical.*

The genericity condition, $a \neq 0$, is the usual sort of condition for a bifurcation, involving the partial derivatives of the vector field $(f(x,y,\mu), g(x,y,\mu))$ evaluated at $(0,0,0)$. More explicitly

$$\begin{aligned} a = {} & \tfrac{1}{16}(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) \\ & + \tfrac{1}{16\omega}\left(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right). \end{aligned} \quad (8.44)$$

This horrible expression will be the cause of not a little pain at the end of this section. However, the most important part of the theorem, the existence of bifurcating periodic solutions, is not too hard to prove and the pain referred to above comes from the need to do lots of nasty algebraic manipulation rather than any sophisticated conceptual trickery. As promised we begin with an outline of the strategy of the proof.