Engsci 711 L7 Bifurcation theory Examples

Key points:

- dependence on problem parameters
- non-hyperbolic fixed points

- non-hyperbolic FP are sensitive to variations in parameters etc
- 'all the action' is here
  - slow modes dominate in long term (see centre manifold theory).
  - but these slow modes may undergo bifurcations

Consider the algebraic equation

$x^2 - a = 0$, $x \in \mathbb{R}$, a is a parameter

solutions $\begin{cases} \text{none if } a < 0 \\ \text{one if } a = 0 \\ \text{two if } a > 0 \end{cases}$

ie the 'key features' (number of solutions) depend on a

$\Rightarrow$

Expanded state space (x, μ):

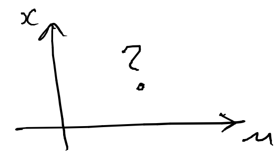
$\dot{x} = f(x, \mu)$

$\dot{\mu} = 0 \Rightarrow \mu = \mu_0$ for some (free) μ_0.

$\dot{\mu} = 0$ 'very slow' ie 'frozen'

actual value is arbitrary $\rightarrow$ can vary 'externally' $\rightarrow$ 'problem parameter'

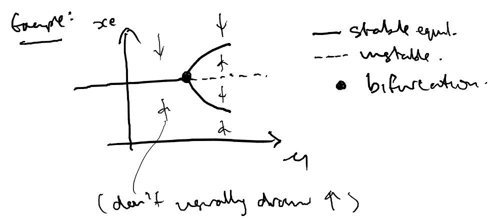
'phase portrait'



But $\mu = \mu_0$ is 'fixed' for each case.

$\Rightarrow$ Plot long term features of x vs μ_0 values.

eg $x_e \leftarrow$ equilibria.



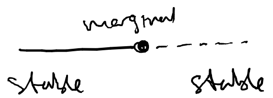
Bifurcation diagram.

$\dot{x} = f(x;\mu)$, $x \in \mathbb{R}$

$Df(x_e) = \lambda = 0$ } non-hyperbolic F.P.

< 0 stable, hyperbolic

> 0 unstable, hyperbolic.

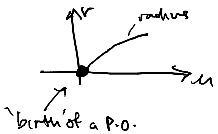


change in stability

change in number of solutions.*

also: Periodic orbits ?

→ plot radius of orbit etc



* Terminology: 'supercritical' : new solutions are stable

'subcritical' : new solutions are unstable.

Key bifurcations

'Normal forms' → 'standard' forms. [reversed forms etc same

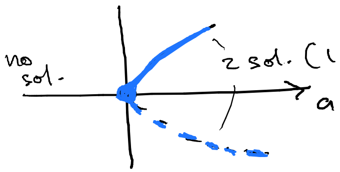
eg   same]

Saddle-node [turning point, fold]:

— stable
--- unstable

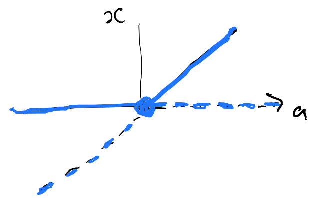
$\dot{x} = a - x^2$

'net stability' is constant.

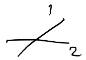


Transcritical

$\dot{x} = a \cdot x(1 - \frac{b}{a}x)$ $= ax - bx^2$ (logistic).



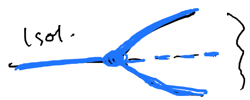
2 sol. swap stability.



Q: net stability?

Pitchfork

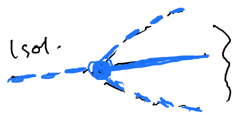
$\dot{x} = ax(1 - \frac{b}{a}x^2) = ax - bx^3$



} 3 sol

supercritical (branches stable)

OR



} 3 sol

subcritical (branches unstable).

+ Hopf (see later → birth of P.O.)

Details

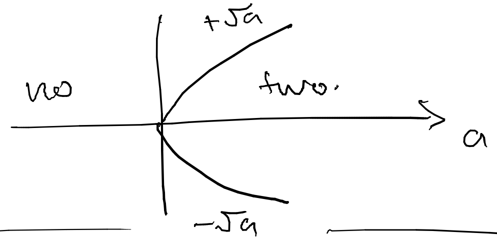
SN. $\dot{x} = a - x^2$, $x \in \mathbb{R}$

1. Fixed points $\dot{x} = 0$

$\Rightarrow x^2 = a \Rightarrow \begin{cases} a < 0 & \text{no sol.} \\ a = 0 & \text{one sol.} \\ a > 0 & \text{two sol.} \end{cases}$

$x = \pm\sqrt{a}$.

2. Initial diagram without stability



3. Stability.

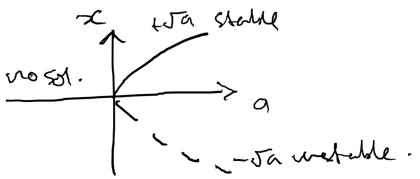
$$Df = -2x \quad \begin{aligned} &< 0 \text{ if } x > 0 \\ &> 0 \text{ if } x < 0 \\ &= 0 \text{ if } x = 0 \end{aligned}$$

$\Rightarrow$ upper sol^n $(x>0)$ is stable $(\lambda < 0)$

lower sol^n $(x<0)$ is unstable $(\lambda > 0)$

bif. when $x = 0$.

So



Details

TC. $\dot{x} = ax - bx^2$, $x \in \mathbb{R}$; assume $b > 0$ fixed, a can be varied

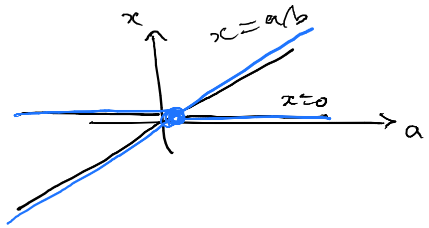
1. FP: $ax\left(1 - \frac{b}{a}x\right) = 0$

$x = 0$ &/or $x = \frac{a}{b}$

→ always exist

↳ look for change in stability

2. Diagram without stability vs a

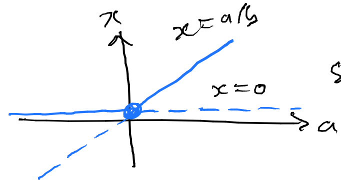


3. Stability $Df = a - 2bx$

$x=0$: $Df(0) = a \quad \begin{aligned} &> 0 \text{ if } a > 0 \text{ (unstable)} \\ &< 0 \text{ if } a < 0 \text{ (stable)} \end{aligned}$

$x = a/b$: $Df(x = \frac{a}{b}) = a - 2b\frac{a}{b} = -a \quad \left.\begin{aligned} &> 0 \text{ if } a < 0 \\ &< 0 \text{ if } a > 0 \end{aligned}\right\}$ opposite

So



switch stability

Details

Pitchfork.

$\dot{x} = ax - bx^3$, $x \in \mathbb{R}$, assume $b > 0$;
a can be varied

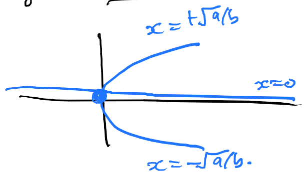
1. FP $\dot{x} = 0 \Rightarrow ax - bx^3 = 0$

$$ax\left(1 - \frac{b}{a}x^2\right) = 0$$

$x = 0$ &/OR $x^2 = a/b$.

always exists — only if $a > 0$ (assuming $b > 0$)

2. Diagram without stability



3. Stability

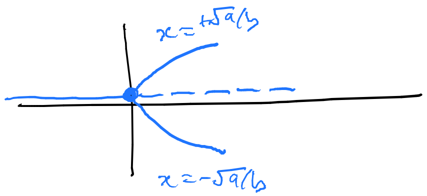
$Df = a - 3bx^2$

$\boxed{x=0}$ $Df(0) = a$ $\quad > 0$ if $a > 0$; < 0, $a < 0$

$\boxed{x^2 = \frac{a}{b}}$ $Df\left(x^2 = \frac{a}{b}\right) = a - 3b \cdot \frac{a}{b} = -2a$

< 0 if $a > 0$
> 0 if $a < 0$ ← not possible (no solⁿ).

So



More complicated (exercises) — good practice for exam!

① $\dot{x} = x + \frac{\mu x}{1 + x^2}$, $x \in \mathbb{R}$

② $\dot{x} = (\lambda - b)x - ax^3$, $x \in \mathbb{R}$

a, b fixed
λ parameter that can vary (2016 Exam).

③ $\dot{x} = (x-1)(x^2 + 2ax - \mu)$

$a > 0$
$x, \mu \in \mathbb{R}$

[Note: multiple bifurcations, one system.].