MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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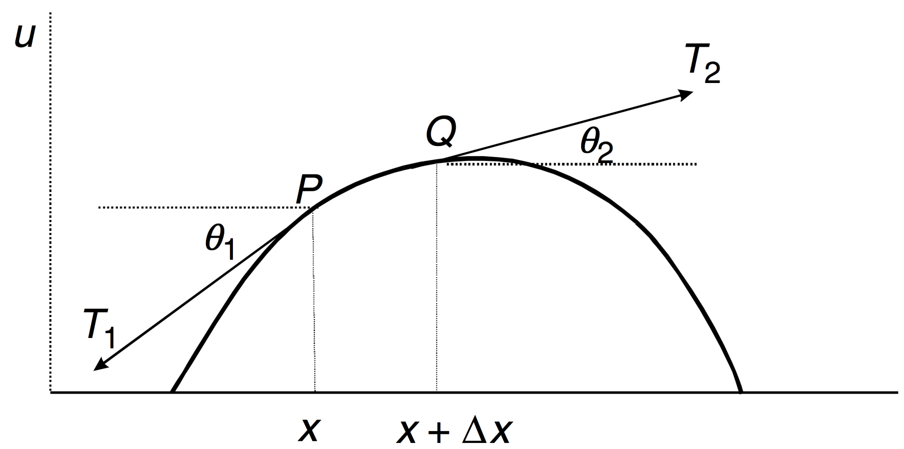
LECTURE 13

The wave equation:

- Standing waves and travelling waves
- D'Alembert's solution

RECALL: DERIVATION OF THE WAVE EQUATION

We derived an equation to represent a (vertically) vibrating 'string'



finding...

RECALL: DERIVATION OF THE WAVE EQUATION

...The *wave Equation*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

which is a *hyperbolic* equation

RECALL: 'PLUCKED' STRING PROBLEM

We considered the problem

$$\text{PDE: } \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

$$\text{IC: } u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

$$\text{BC: } u(0,t) = 0, \qquad u(L,t) = 0.$$

$$f(x) = \begin{cases} \frac{2h}{L}x, & 0 \le x \le L/2 \\ \frac{2h}{L}(L-x), & L/2 \le x \le L \end{cases}$$

SOLUTION

Recall: solution by separation of variables (arbitrary $f(x)$)

$$u(x,t) = \sum_{n=1}^{\infty} C_n sin(\frac{n\pi x}{L}) cos(\frac{n\pi at}{L})$$

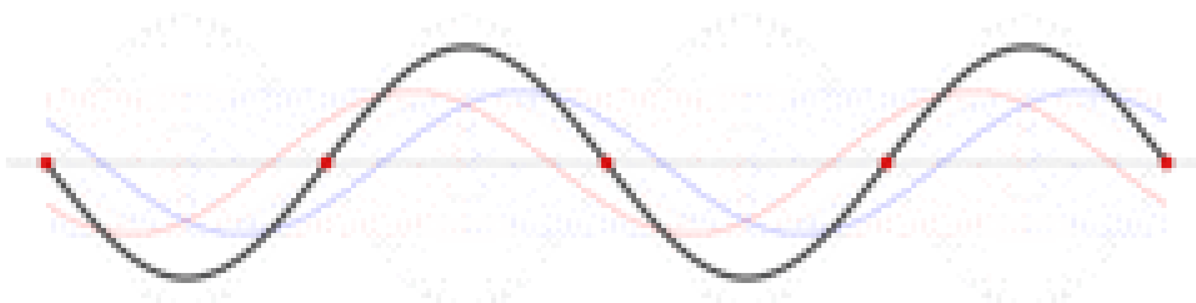
$$f(x) = \sum_{n=1}^{\infty} C_n sin(\frac{n\pi x}{L})$$

Full solution to plucked string problem using MuPad.

MODES, STANDING WAVES AND TRAVELLING WAVES

We see that our solution is a superposition of distinct (i.e. one for each n) *modes* of vibration (with both a spatial frequency and a temporal frequency).

Each mode forms a *standing wave* which appears to vibrate 'in place'.



from https://en.wikipedia.org/wiki/Standing_wave

TRAVELLING WAVE INTERPRETATION

We can see that *a standing wave can be thought of as the superposition of two travelling waves*, each travelling in the *opposite* direction.

In this case we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \left[sin(\frac{n\pi}{L}(x-at)) + sin(\frac{n\pi}{L}(x+at)) \right]$$

which (here) sums to the *closed form*

$$u(x,t) = \frac{1}{2}\left[f_{odd,per}(x-at) + f_{odd,per}(x+at) \right]$$

D'ALEMBERT'S SOLUTION

This is actually the more general phenomenon - there is (in contrast to the heat equation) a *general solution to the wave equation* and it has the form of *two travelling waves moving in opposite directions at the same speed* i.e.

$$u(x, t) = F(x - at) + G(x + at)$$

where F and G are *arbitrary functions*

D'ALEMBERT'S SOLUTION

Back to our MuPad example.

D'ALEMBERT'S SOLUTION: VERIFICATION AND PROOF

We can verify that $u(x, t) = F(x - at) + G(x + at)$ is a solution by substitution.

We can prove that this is the general solution by a clever change of variables and direct integration (See Section 5.1.6 of Tang).

D'ALEMBERT'S SOLUTION: EXAMPLE

Let's consider solving:

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty.$$

D'ALEMBERT'S SOLUTION: EXAMPLE

We find $u(x, t) = F(x - at) + G(x + at)$ becomes (in terms of our IC)

$$u(x, t) = \frac{1}{2}[f(x - at) + f(x + at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi$$

(See also Section 5.1.6 of Tang)

PROPERTIES OF THE WAVE EQUATION

Some of the notable features of the wave equation (apparent in the x-t plane) include

- Characteristics (with)
- Finite speeds of propagation (c.f. heat equation) (and determining)
- Domains of influence/dependence

HOMEWORK

Assignment 2!

Read the derivation of the general solution and use for specific examples (5.1.6 of Tang)

Start preparing for the test!