Maths 361: Lecture 13:
Separation of variables solution for wave equation
Travelling waves and a hint of D'Alembert's solution

Initial displacement:

```
f:=x->piecewise([x<L/2,2*h*x/L ],[x>=L/2, 2*h*(L-x)/L])
```

$$x \to \text{piecewise}\left(\left[x < \frac{L}{2}, \frac{2\,h\,x}{L}\right], \left[\frac{L}{2} \le x, \frac{2\,h\,(L-x)}{L}\right]\right)$$

Take L=1, a=1, h=1:

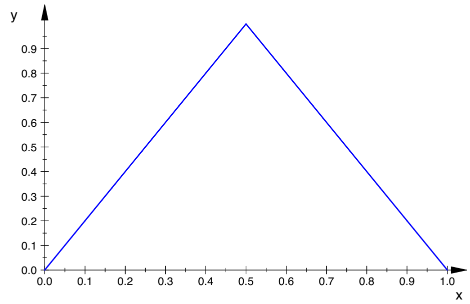
```
L:=1;a:=1;h:=1;
```

$$1$$

$$1$$

$$1$$

```
plot(f(x),x=0..1)
```



In our calculations we assume that n is a non-negative integer and we need to put this information into MuPAD:

```
assume(n,Type::NonNegInt);
```

The formula for the Fourier coefficients for f are:

```
cn:=(2/L)*simplify(int(f(x)*sin((n)*PI*x),x=0..L))
```

$$\frac{2\,(-1)^{n/2}\,(2\,(-1)^{n}\,\mathrm{i} - 2\,\mathrm{i})}{n^2\,\pi^2}$$

It is useful to write c_n as a function c of an integer. Check that you understand the following:

```
c:=m->subs(cn,n=m);
```

$$m \to \text{subs}(\text{cn}, n = m)$$

Let's define a MuPAD function U(x,t,N) for the sum of
the first N terms of the solution:

```
U:=(x,t,N)->sum(c(n)*sin((n)*PI*x)*cos(((n)*PI)*t),n=1..N)
```

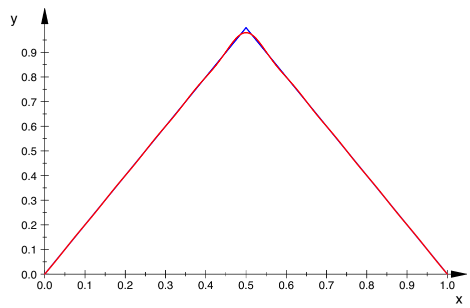
$$(x, t, N) \to \sum_{n=1}^{N} c(n)\,\sin(n\,\pi\,x)\,\cos((\pi\,n)\,t)$$

Of course the exact solution requires N=infinity but we just take N large.

Here's a plot of the initial condition and its approximation. Try making N larger.

```
N:=20;
plot(f(x), U(x,0,N),x=0..1)
```

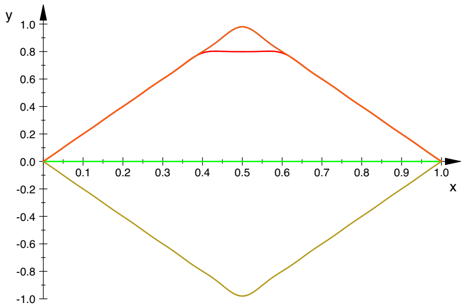
$$20$$



Next, let's set a value for N and use it to plot u for various values of t

```
N:=20;
plot(U(x,0,N), U(x,0.1,N), U(x,0.5,N),U(x,1,N), U(x,2,N),x=0..1)
```

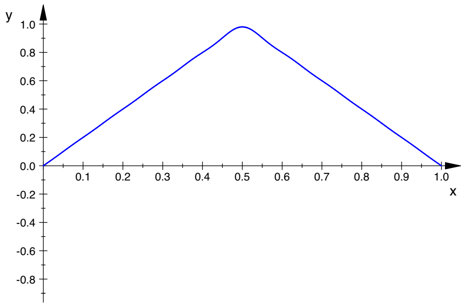
$$20$$



We can animate the solution from t=0 to 1 by including t=0..1.
This will take a few minutes to run, so be patient.
When it is finished, click on it and animation controls will appear.
Note the different kinds of reflections that occur at the boundaries x=0, x=1.

```
N:=20;
plot(U(x,t,N),x=0..1, t=0..2)
```

$$20$$



First normal mode:

```
u:=(x,t)->sin(PI*x/L)*cos(PI*a*t/L);
```

$$(x, t) \to \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi a t}{L}\right)$$

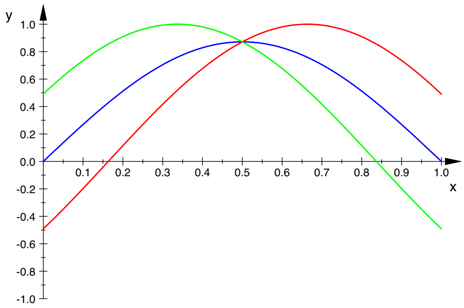
Travelling wave version

```
utrav1:=(x,t)->sin(PI*(x-a*t)/L);
utrav2:=(x,t)->sin(PI*(x+a*t)/L)
```

$$(x, t) \to \sin\left(\frac{\pi (x - a t)}{L}\right)$$

$$(x, t) \to \sin\left(\frac{\pi (x + a t)}{L}\right)$$

```
plot(u(x,t),utrav1(x,t),utrav2(x,t),x=0..1, t=0..2)
```



second normal mode

```
u:=(x,t)->sin(2*PI*x/L)*cos(2*PI*a*t/L);
```

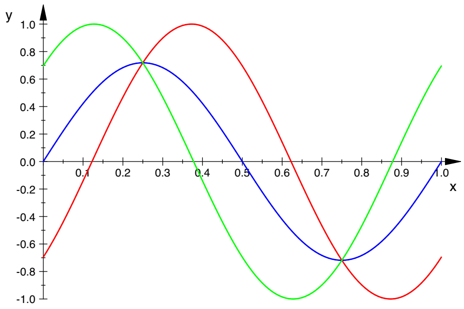
$$(x, t) \to \sin\left(\frac{2 \pi x}{L}\right) \cos\left(\frac{2 \pi a t}{L}\right)$$

```
utrav1:=(x,t)->sin(2*PI*(x-a*t)/L);
utrav2:=(x,t)->sin(2*PI*(x+a*t)/L)
```

$$(x, t) \to \sin\left(\frac{2 \pi (x - a t)}{L}\right)$$

$$(x,\ t) \to \sin\left(\frac{2\ \pi\ (x + a\ t)}{L}\right)$$

```
plot(u(x,t),utrav1(x,t),utrav2(x,t),x=0..1, t=0..2)
```



Try some other initial conditions by entering your own functions.

To repeat the computations for a different initial condition, enter the following and then go back up and re-enter the MuPAD commands above (by clicking on them) starting from the command plot(f(x),x=0..1).

Another initial function - good for illustrating D'Alemberts

```
f:=x->piecewise([x<L/3,0 ],[x>=L/3 and x<L/2,6*h*(x-L/3)/L], [x>=L/2 and x<2*L/3,6*h*(2*L/3-x)/L],[x>2*L/3,0])
```

$$x \to \text{piecewise}\left(\left[x < \frac{L}{3},\ 0\right],\ \left[\frac{L}{3} \le x \wedge x < \frac{L}{2},\ \frac{6\ h\ \left(x - \frac{L}{3}\right)}{L}\right],\ \left[\frac{L}{2} \le x \wedge x < \frac{2\ L}{3},\ \frac{6\ h\ \left(\frac{2\ L}{3} - x\right)}{L}\right],\ \left[\frac{2\ L}{3} < x,\ 0\right]\right)$$

Things to notice:
1. The initial displacement becomes two waves of half the initial size moving in opposite directions.
2. Unlike the heat equation, solutions don't become smoother than the initial displacement.
3. Reflections are occuring at the boundaries.