SECTION B: INVERSE PROBLEMS

Question 3 (15 marks)

Consider the standard Tikhonov form of the regularised least squares problem

$$\min_{\boldsymbol{x}} \ \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \lambda\|\boldsymbol{x}\|_2^2$$

where $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}, \lambda \in \mathbb{R}$ and $\boldsymbol{A}$ is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix.

i) By differentiating the objective function with respect to $\boldsymbol{x}$, derive the following corresponding normal equations that characterise solutions to the optimisation problem above:

$$(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A} + \boldsymbol{\lambda I})\boldsymbol{x} = \boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{y}.$$

[5 marks]

ii) Show how to derive an *iterative solution method* to the above normal equations. Note: you do NOT need to give conditions under which your scheme converges.

[5 marks]

iii) Suppose that you decided to set the explicit Tikhonov regularisation parameter $\boldsymbol{\lambda} \in \mathbb{R}$ to zero in your problem. Explain how you could *still* obtain a regularised solution just using an iterative scheme.

[5 marks]

Question 4 (10 marks)

i) Consider the general reduced form of the SVD of a matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{U_r \Sigma_r V_r^T},$$

where $\boldsymbol{r}$ is the rank of $\boldsymbol{A}$.

a. Express the pseudoinverse $\boldsymbol{A}^+$ in terms of the SVD component matrices given above.

[1 mark]

b. Use your answer to (a) to derive an expression for the *model (parameter) resolution operator* $\boldsymbol{R_m}$ in terms of the SVD component matrices.

[2 marks]

c. Use your answer to (a) to derive an expression for the *data resolution operator* $\boldsymbol{R_d}$ in terms of the SVD component matrices.

[2 marks]

ii) Briefly explain how to obtain regularised solutions to a linear inverse problem by using the SVD. Include a brief intuitive explanation of *why* this works.

[5 marks]