ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*
   [4 lectures/tutorials]

   Geometry of non-hyperbolic systems. In particular: centre manifold theorem and reduction principle. Applications: asymptotic stability of non-hyperbolic systems, understanding of bifurcation theory using the geometric perspective, fast/slow systems.

LECTURE 9

- Intro. to *centre manifold theory* (geometry of non-hyperbolic systems)
- Application to *asymptotic stability* of non-hyperbolic systems
- Preview of *reduction principle*

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

Most of the first part of the course dealt with *hyperbolic* fixed points.

We could decide if our system was asymptotically stable just by looking at the *linearisation*.

This is *not the case for non-hyperbolic systems* - linearisation is too 'rough' to handle these sensitive cases: here we *need to look at the higher-order terms*.

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

Bifurcation theory gave us a first taste of analysing non-hyperbolic systems. This was essentially a 'static' analysis - no dynamics.

Now we look in more detail at the *geometry of non-hyperbolic systems* - we extend our stable/unstable manifold analysis to include a *centre manifold*.

This allows us to analyse the *dynamics* near non-hyperbolic fixed points.

CENTRE MANIFOLD THEORY: BASIC MOTIVATION

In particular, we can use *centre manifold theory* to:

- Help *reduce* complex dynamic models to 'emergent' simpler, approximate dynamic models
- Gain a deeper understanding of *bifurcation theory*
- Analyse *fast/slow systems* ('geometric' view of singular perturbation theory)

'all the good stuff happens on the centre manifold'

CENTRE SUBSPACE? SLOW SUBSPACE?

The centre *subspace* (linear manifold) $E^c(0)$ is just the eigenspace corresponding to the *eigenvalues with real part zero*.

This works the *same way* as for $E^s(0)$ and $E^u(0)$.

If the eigenvalues are *exactly zero* - i.e. the *imaginary part is also zero* - then we call the centre subspace a *slow subspace*.

CENTRE MANIFOLD? SLOW MANIFOLD?

The centre/slow *manifold* $W^c(0)$ is just the *nonlinear correction* to the linear subspace $E^c(0)$.

Again, this is *just like* how $W^s(0)$ and $W^u(0)$ correct $E^s(0)$ and $E^u(0)$.

(The key difference is how we can further use $W^c(0)$ to do interesting things)

EXAMPLE (KUZNETSOV EXAMPLE 5.1)

Consider the system:

$$\frac{dx}{dt} = xy + x^3,$$
$$\frac{dy}{dt} = -y - 2x^2$$

Let's first do the usual *linear analysis*.

CENTRE MANIFOLD? SLOW MANIFOLD?

Now we want to calculate the *nonlinear correction*.

We will see that, *in contrast to hyperbolic fixed points*, these corrections can be very important - e.g. in determining *asymptotic stability*.

CENTRE MANIFOLD CALCULATION: BASIC PROCEDURE

The *basic procedure* is (essentially) *the same* as for the stable/unstable manifolds:

- *Assume* functional relationship $y = g(x)$ or $x = h(y)$
- *Substitute* in
- Use *chain rule*
- *Equate coefficients* of a power series
- Use *tangency* to $E^c(0)$

(we will see later why the centre manifold has some very useful additional properties though, and some caveats to look out for).

EXAMPLE (KUZNETSOV EXAMPLE 5.1) CONTINUED

$$\frac{dx}{dt} = xy + x^3,$$
$$\frac{dy}{dt} = -y - 2x^2$$

ASYMPTOTIC STABILITY OF A NON-HYPERBOLIC FIXED POINT

Note:

Using our linear eigenspace to determine the asymptotic stability our our non-hyperbolic fixed point gives the incorrect answer.

We need to calculate the *nonlinear* centre manifold to *correctly determine asymptotic stability* (note: we still need justify this somewhat!).

OTHER NOTES

Somewhat in contrast to the stable/unstable manifold cases, for the centre manifold theory we *usually first carefully separate out the fast linear and slow linear dynamics* before calculating the nonlinear centre manifold.

This allows us to *just focus on the emergent centre manifold dynamics*

We will first look at the theorem, however, before previewing transforming to the linearly separated 'normal form' (which is most important for the next lecture).

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Consider $\dot{x} = f(x)$ having a *non-hyperbolic fixed point* at $x = 0$, where $x \in \mathbb{R}^n$.

Assume that there are n^+ eigenvalues (counting repeated cases) with Re $\lambda > 0$, n^0 eigenvalues with Re $\lambda = 0$, and n^- eigenvalues with Re $\lambda < 0$.

CENTRE MANIFOLD THEOREM (FOLLOWING KUZNETSOZ)

Then there is a locally defined smooth n^0-dimensional invariant manifold $W^c_{loc}(0)$ that is tangent to the (linear) centre eigenspace E^c.

Moreover, there is a neighborhood U of $x_0 = 0$, such that if $\phi(x,t) \in U$ for all $t \geq 0$ (≤ 0) then $\phi(x,t) \to W^c_{loc}(0)$ for $t \to \infty$ ($t \to -\infty$).

CENTRE MANIFOLD - UNIQUENESS?

The centre manifold is unique to all orders of its Taylor expansion.

That is, center manifolds are *not quite unique but differ only by exponentially small functions* of the distance from the fixed point (think: 'faster scales').

CENTRE MANIFOLD THEOREM - WHY/WHAT?

The solutions on the centre *eigenspace* are *'frozen'* - neither growing nor decaying. The solutions on the centre *manifold* are *slowly varying*.

We can thus think of the eigenvalue = 0 case as defining the *linearised steady-state* behaviour of the full system.

The linearised dynamics are 'infinitely slow' flows relative to the *exponential behaviour on the other eigenspaces*.

The *nonlinear* dynamics can *vary slowly* (e.g. 'quasi-steady states'). This is *often our 'emergent' timescale of interest!*

CENTRE MANIFOLD THEOREM - RESTRICTION AND REDUCTION

Usually we are interested in equilibria where $n^+ = 0$, i.e. where *all eigenvalues are negative or zero*.

Thus the dynamics are *exponentially attracted to the centre manifold*. (see simulation example).

This justifies our use of the *restriction to the centre manifold* for determining asymptotic stability.

It also naturally leads to a *reduction principle* based on centre manifold theory (next lecture).

EXAMPLE (KUZNETSOV EXAMPLE 5.1) - SIMULATION

$$\frac{dx}{dt} = xy + x^3,$$
$$\frac{dy}{dt} = -y - 2x^2$$

CENTRE MANIFOLD - COORDINATE TRANSFORMATIONS

Our *reduction principle* (next lecture), which is based on the centre manifold theorem, will assume that we have *transformed coordinates to put our system into linearly-decoupled* (i.e. diagonalised/upper triangular) form.

This can be done by using our (generalised) *eigenvectors as our new coordinate axes*.

A brief preview of this coordinate transformation procedure can be found in the handout (details to come).