More generally, the equation

$$\boxed{\langle L[u], v\rangle = \langle u, L^*[v]\rangle} \tag{61}$$

is used to *define* the **Hermitian conjugate** (or **adjoint**) L^* of the operator L, relative to whatever inner product is chosen. Let us illustrate.

EXAMPLE 5. Find the Hermitian conjugate of the operator consisting of the differential operator

$$L = \frac{d^2}{dx^2} + \frac{d}{dx} + 1 \tag{62a}$$

on the domain $\mathcal{D}$ of real-valued functions defined and having continuous second derivatives on $[0, \pi]$ and satisfying the homogeneous initial conditions

$$u(0) = 0, \qquad u'(0) = 0, \tag{62b}$$

subject to the inner product definition

$$\langle u, v\rangle = \int_0^\pi u(x)v(x)\, dx, \tag{62c}$$

say. Begin with the left-hand side of (61),

$$\langle L[u], v\rangle = \int_0^\pi (u'' + u' + u)v\, dx. \tag{63}$$

Integrating the $u''v$ term by parts twice, the $u'v$ term once, leaving the uv term intact, and using (62b), gives

$$\begin{aligned} \langle L[u], v\rangle &= (u'v - uv' + uv)\Big|_0^\pi + \int_0^\pi u(v'' - v' + v)\, dx \\ &= [u'(\pi) + u(\pi)]v(\pi) - [u(\pi)]v'(\pi) + \langle u, L^*[v]\rangle, \end{aligned} \tag{64}$$

where, from the $v'' - v' + v$ in the integral, we can infer that

$$L^* = \frac{d^2}{dx^2} - \frac{d}{dx} + 1. \tag{65a}$$

To obtain the boundary conditions associated with L^* we see, by comparing (64) with (61), that we need the boundary terms in (64) to drop out. Whereas $u(0) = 0$ and $u'(0) = 0$, the bracketed quantities $u'(\pi) + u(\pi)$ and $u(\pi)$ are not prescribed, so we must have both

$$v(\pi) = 0, \qquad v'(\pi) = 0. \tag{65b}$$

Thus, the Hermitian conjugate operator is the differential operator (65a) on the domain $\mathcal{D}^*$ of real-valued functions defined and having continuous second derivatives on $[0, \pi]$ and satisfying the conditions (65b). ▮

If the operator and its Hermitian conjugate (or adjoint) are identical, then we say that it is **Hermitian** (or **self-adjoint**). Thus, the operator in Example 5 is not