$$\dot{x} = (x-1)(x^2+2ax-\mu) = f(x;\mu) \qquad (1)$$

(1) $a > 0$
$x, \mu \in \mathbb{R}$.

(Based on Glendinning 8.3 & 8.5)

FP $f(x;\mu) = 0$

(1) $\boxed{x = 1}$ & /or (2) $x^2 + 2ax - \mu = 0$

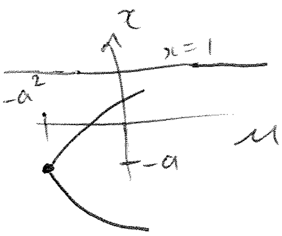
always sol^n

(2) ~~[illegible]~~

$x^2 + 2ax - \mu = 0$

$\Leftrightarrow$

$x = -a \pm \sqrt{a^2+\mu}$



require $x \in \mathbb{R}$

so $a^2 + \mu \geq 0$

ie $\mu \geq -a^2$

If $\mu = -a^2$
$x = -a$ < 0 } one sol^n

If $\mu < -a^2$
$\Rightarrow$ no sol^n

If $\mu > -a^2$
$x_+ = -a + \sqrt{a^2+\mu}$
$x_- = -a - \sqrt{a^2+\mu}$ } two sol^n

imply saddle node / turning point bifurcation.

$$f(x;\mu) = x^3 + 2ax^2 - \mu x - x^2 - 2ax + \mu$$

$$Df(x;\mu) = 3x^2 + (4a-2)x - (\mu + 2a)$$

But first $\rightarrow$

Q: do sol^{n}s intersect?

$(\mu = -a^2)$; $x = -a < 0 < 1$ → not this

$(\mu > -a^2)$; $x_- = -a - \sqrt{a^2+\mu} < -a$ → not this

$(\mu > -a^2)$; $x_+ = -a + \sqrt{a^2+\mu}$ → maybe.

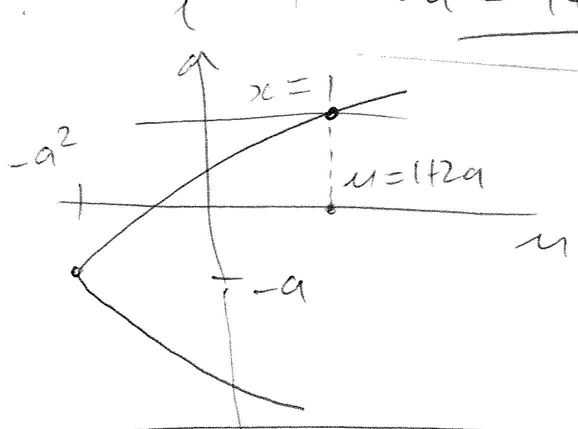
try $x_+ = 1 = -a + \sqrt{a^2+\mu}$

$(1+a)^2 = a^2 + \mu$

$1 + 2a + \cancel{a^2} = \cancel{a^2} + \mu$

$\mu = 1 + 2a$. } intersection here.

so far.



Now, stability.

$Df(x, \mu)$ at $x = -a$, $\mu = -a^2$

$= 3a^2 + (4a-2)(-a) - (-a^2 + 2a)$

$= 3a^2 - 4a^2 + 2a + a^2 - 2a$

$= 0$ (as expected → bifurcation at non-hyperbolic point).

How about x_+ & x_-? (+ good for getting intuition)

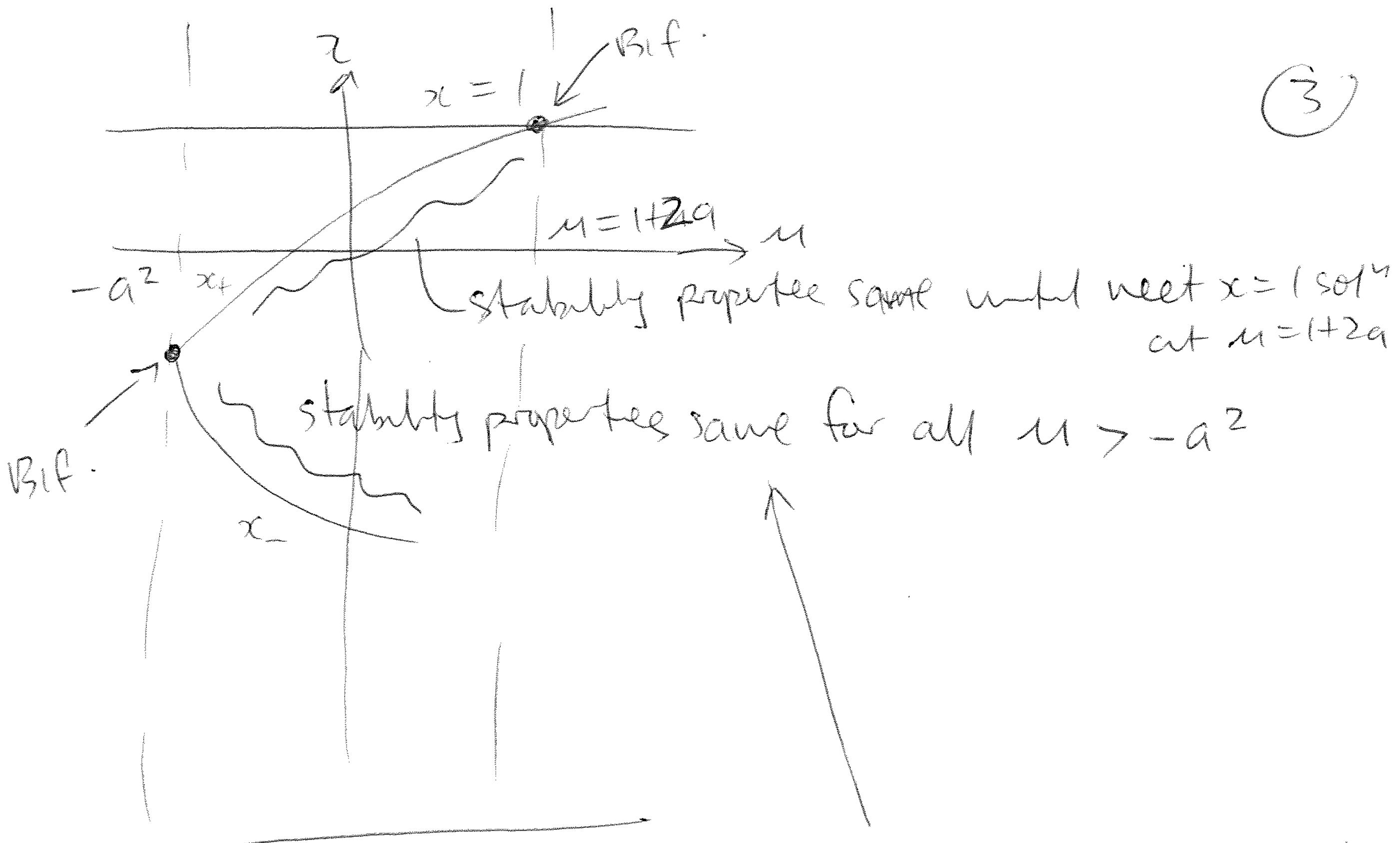
→ get 'clever' → sol^n only change stability at bif.

or

(if can diff.) → (easier to automate, harder to remember) { (use SN theorem)

→ require FP meet or appear / disappear

→ none until meet $x = 1$.



So... use 'convenient points' (+ 'continuity of properties')

x_- : use $\underline{u=0}$. } whole branch same stability if $u > -a^2$

so $x_- = -a - \sqrt{a^2+u}$ & $u = 0$

$$\Rightarrow x_-\Big|_{\underline{u=0}} = -a - a = -2a.$$

$$\begin{aligned} Df(x=-2a, u=0) &= 3 \times (-2a)^2 + (4a-2)(-2a) - 2a \\ &= 12a^2 - 8a^2 + 4a - 2a \\ &= 4a^2 + 2a \\ &= 2(2a^2+1) > 0 \Rightarrow \underline{\text{unstable}}. \end{aligned}$$

→

Expect x^+ stable for $\mu > -a^2$ & $\mu < 1+2a$ (4)

Prove: set $\underline{\mu = 0}$

so $x_+\big|_{\mu=0} = -a + a = 0$

so $Df(x=0, \mu=0)$

$= -2a < 0 \rightarrow \underline{\text{stable}}.$

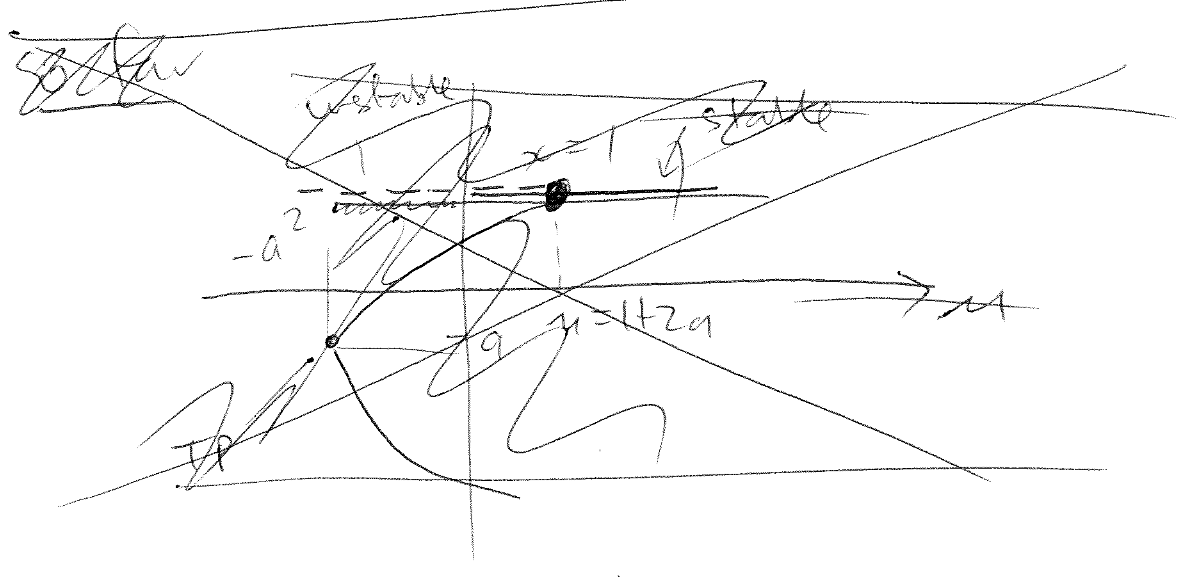
Now consider $x = 1$ for $\begin{cases} \mu > 1+2a \\ \quad < 1+2a \text{ resp.} \end{cases}$

$Df(x=1; \mu) = 3 + 4a - 2 - \mu - 2a$

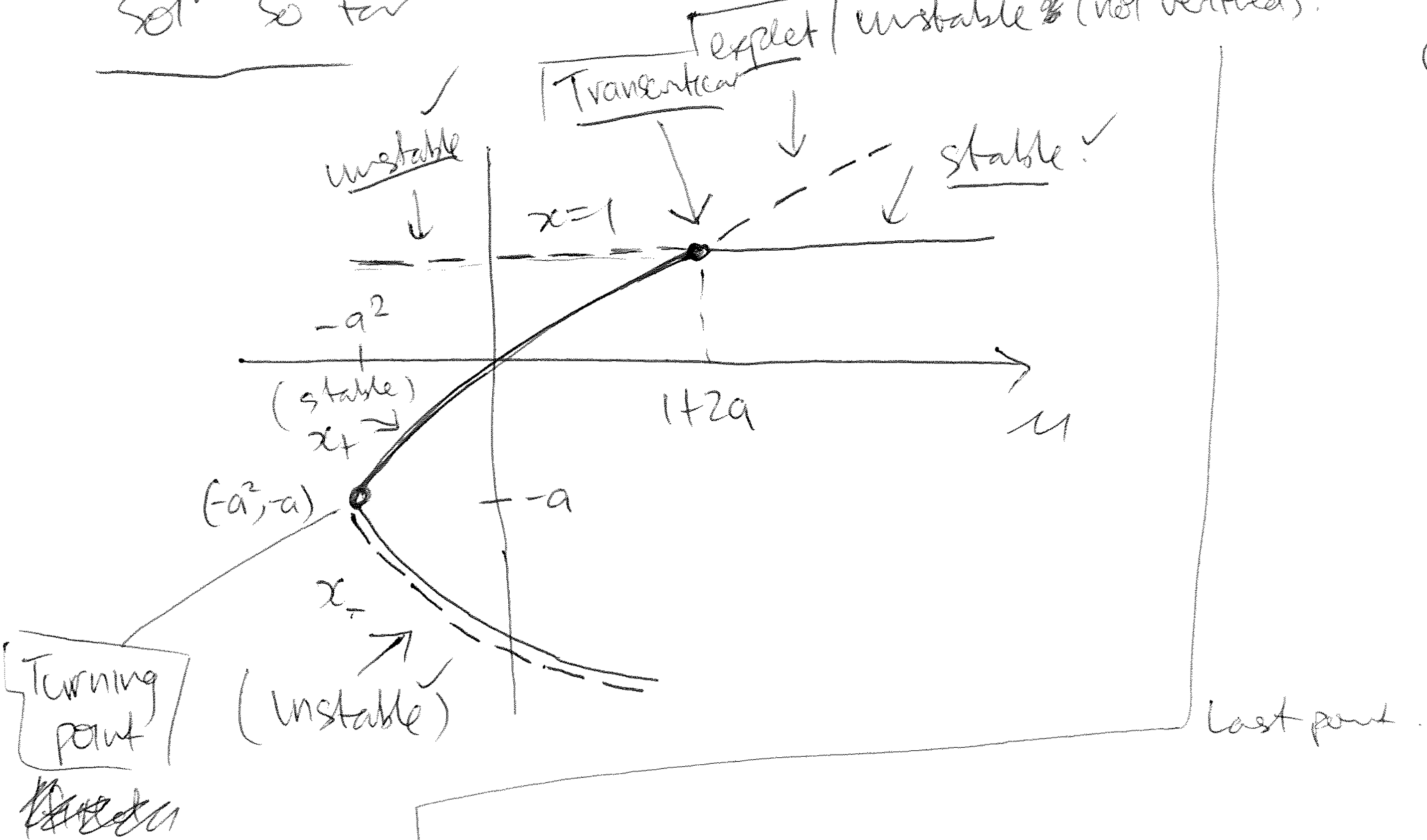
$= 1 + 2a - \mu$

> 0 for $\mu < 1+2a$ (unstable)

< 0 for $\mu > 0$ (stable)



Sol^n so far



stability of $\boxed{x^+ \text{ for } \mu > 1+2a\ ?}$

any point **after** intersection.

Try 'clever' eg $x^+ = 2$ $\quad (\underline{>1})$

If in doubt try convenient points + 'properties preserved until bifurcation'

so $\mu\big|_{x^+} = x_+^2 + 2a x_+$ (remember, have (x,μ) curve).

$= 4 + 4a$

$= 4(1+a)$.

(check consistent with $\mu > 1+2a$:

$4 + 4a \overset{?}{>} 1 + 2a$

$2a > -3$ ✓ since $a > 0$.)

so $\partial f\,(x=2, \mu = 4+4a)$

$= 3 \times 4 + (4a-2) \times 2 - (4a+4+2a)$

$= 12 + \underline{6a} - 4 - \underline{4a} - \underline{2a} - 4$

$= 4 > 0 \Rightarrow$ unstable.

so verified x_+ is unstable on $\mu > 1+2a$ branch