Engsci 213 Continuous Probability Models - Problems (v. 1)

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Continuous probability models

Let's consider some simple problems involving continuous probability models.

The first part will be problems to do by hand, the second part is about using R to do probability calculations and simulation. (You can use whatever language you want but you will use R in the next (Data Analysis) module, so may as well start getting used to it!)

You will find the R Exercises online on Canvas (since you have a laptop!)

Note:
There are lots of extra worked problems and resources on canvas. E.g. in the past course book. I also put up a free (GNU) 'Introduction to Probability and Statistics Using R' by Kerns, too. It has lots of R code as well as example hand calculations.

There are also plenty of R tutorials etc on the internet. See the notebook I put up from lectures for some starting points (or just Google).

Basic problems

Consider

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

- Sketch the graph of this function.
- Is it a valid cumulative distribution function? Justify your answer.
- What is the probability density function? Does this make sense?

Suppose we had

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 2 & 0 \leq x \leq 0.5 \\ 0 & x > 0.5 \end{cases}$$

- Sketch the function.
- Is this a valid probability density function? Justify your answer.

Uniform distribution

Suppose we were manufactoring rulers (what's a ruler??!) on a machine which generates rulers with length

$$L \sim U(1.4, 6.7)$$

where we are measuring (with…a…ruler) in metres.

- What is the probability that a randomly selected ruler has length between 1.6 and 2.4 metres?
- What is the probability that a randomly selected ruler has length of exactly 2.3 metres?
- What is the probability that a randomly selected ruler has length greater than or equal to 6.7 metres?

Exponential distribution

Emails again

You are tracking email arrivals to you email account. Emails arrive in your inbox one-by-one according to a Poisson process with rate $\lambda = 20$ emails per hour.

Note - this is the same Poisson model as in the discrete model tutorial sheet but now we will be looking at 'waiting times' instead of 'number of events'.

- What distribution represents the length of time (in minutes) you expect to wait between emails? Write down the distribution (density and/or cumulative) and sketch a graph. [If you refer to the notes make sure you use the updated ones!]
- What is the probability of waiting less than a minute for an email? Evaluate it if you have a calculator, otherwise write out the expression to evaluate.
- What is the probability of waiting at least three minutes for an email?
- What is the probability of waiting at least six minutes given you have waited three already? Hint - try calculating this both a) the short way and b) the long way
- What is the probability of waiting at least 30 minutes for an email?

Mean and variance

Derive the expressions for the mean and variance of the Exponential distribution from first principles. Hint - you will need to integrate by parts!

Normal distribution

[If you refer to the notes make sure you use the most recent typo-corrected slides!]

- Write down the expression for the density function of the Normal distribution.
- Do the same for the cumulative distribution ;-)
- Redo the derivation of the standardised Normal distribution from an arbitrary Normal distribution using your knowledge of the behaviour of the Normal distribution under linear transformations (i.e. the effect of multiplying a normally-distributed variable by a constant, or adding a constant to it)

The length of human pregnancies from conception to birth varies according to a distribution that is approximately normal with mean 264 days and standard deviation 16 days. We model this as

$$X \sim N(\mu = 264, \sigma = 16)$$

- What is the probability that a pregnancy lasts londer than 287 days?

Hint: First write down an expression in terms of the original distribution. Then transform this to an expression in terms of the standardised distribution. Then look up the answer in the table attached at the end of this sheet (or use a computer!).

Central limit theorem

Triangular distribution

Consider the 'triangular distribution' with density function

$$f_X(x) = 2x, \quad 0 < x < 1$$

(and zero otherwise).

- Find the expected value and variance

Now, let $S_n = X_1 + X_2 + ... + X_n$ where X_i are all independent, identically-distributions 'triangular' random variables.

- To what distribution to you expect this sum to converge as n gets large?
- What are the mean and variance of this distribution?

Standard Normal Probabilities

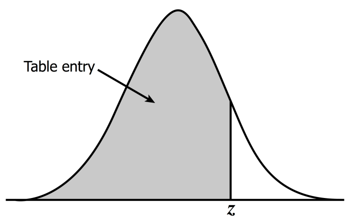

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Figure 1: Table for standardised Normal distribution