Engsci 711 Tut 3 Worked Example

Multiple co-dim-one bifurcations.

$$\boxed{\dot{x} = (x-1)(x^2+2ax-\mu) = f(x;\mu)}$$

$\boxed{a>0 \quad x, \mu \in \mathbb{R}}$

FP

$(x-1)(x^2+2ax-\mu) = 0$

$\Rightarrow \boxed{x=1}$ &/or $\boxed{x^2+2ax-\mu=0}$

always sol^n.

$\Leftrightarrow x = -a \pm \sqrt{a^2+\mu}$

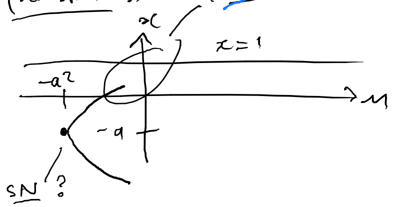
sol^n for $a^2+\mu \geq 0$

ie $\mu \geq -a^2$

(1 for $\mu=-a^2$, 2 for $\mu > -a^2$).

when $\mu = -a^2$
$x = -a$

Pic (no stability) intersect? tangent?.



Key trick:

$\boxed{\begin{array}{l}\bullet \text{ convenient points} \\ + \\ \bullet \text{ stability only changes at bif.} \\ \quad \hookrightarrow \text{continuity of stability along branches.}\end{array}}$

Bif. at intersection?

[Either]: first check for intersection / OR first check for change in stability

Both: [equivalent to other]

$\boxed{1.}$ stability then intersection ← $\boxed{\text{stability then intersect}}$

$f = x^3 + 2ax^2 - \mu x$
$\quad - x^2 - 2ax + \mu$

$Df = 3x^2 + (4a-2)x - (\mu + 2a)$

at $\boxed{x=1}$

$Df(x=1) = 3 + 4a - 2 - \mu - 2a$
$= 1 + 2a - \mu$
< 0 for $\boxed{\mu > 1+2a}$ $\boxed{\text{stable}}$
> 0 for $\boxed{\mu < 1+2a}$ $\boxed{\text{unstable}}$

↳ Now,

verify intersects other sol. at change in stability

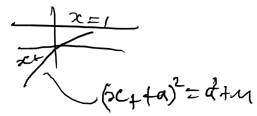
$(x+a)^2 = a^2+\mu$ } eqn of the branch of other sol

$x^2 + 2a + \cancel{a^2} = \cancel{a^2} + \mu$

$x^2 = \mu - 2a$

when $\mu = 1+2a$ } change in stab.

$x^2 = 1 \Rightarrow \boxed{x=1}$ ✓ intersects other

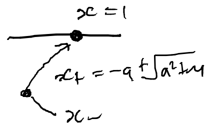


Other approach (can use either)
— doesn't need both!

---

2. Intersection then stability.

$$\left[\begin{array}{ll} (x_+ + a)^2 = a^2 + \mu & ① \\ x = 1 & ② \end{array}\right]$$



← OR $\boxed{\text{intersect then stability}}$

$\Rightarrow (1+a)^2 = a^2+\mu \Rightarrow 1+2a+a^2 = a^2+\mu$
$\Rightarrow \boxed{\mu = 1+2a}$ (same!) ← $\boxed{\text{intersection point}}$

↳ verify change in stab.

check: $Df = 3x^2 + (4a-2)x - (\mu+2a)$

at $x=1$ & $\mu = 1+2a$
$\Rightarrow \boxed{Df = 0}$ as expected (bif.).

But still need stability either side....
→ no free lunch

---

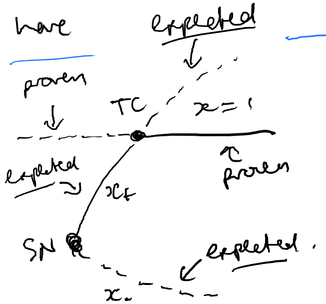
use convenient points + continuity of stability

↳ $\mu < 1+2a$: $\mu = -2a$; $x=1$, $Df = 3+4a-2 = 1+4a > 0 \Rightarrow$ unstable $\mu < 1+2a$
(convenient)

↳ $\mu > 1+2a$: $\mu = 2a+2$, $x=1$, $Df = 3 + \cancel{4a} - 2 - \cancel{4a} - 2 = -1 < 0 \Rightarrow$ stable $\mu > 1+2a$
(convenient) (same).

Back to example

So.... we have



so far

Need to consider

- x_+ change in stability at TC
- x_+ & x_- stability at SN.

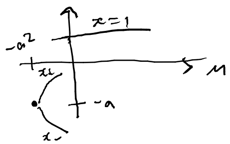
Approach: SN first.
Then x_+ change at TC

SN $x = -a \pm \sqrt{a^2+\mu}$, $a > 0$.

$Df(x,\mu) = 3x^2 + (4a-2)x - (\mu+2a)$

→ need to constrain to upper & lower

Key trick: $\boxed{\text{Convenient points + continuity of stability away from bif. point}}$ ← only change stability at bif.



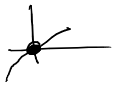
$\boxed{\text{upper}}$ branch: passes through $x=0$ ← $\boxed{\text{convenient}}$

$x_+ = 0 = -a + \sqrt{a^2+\mu}$

$a = \sqrt{a^2+\mu}$

$a^2 = a^2+\mu$

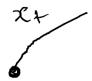
$\Rightarrow \mu = 0$



$\Rightarrow$ check stability at $\boxed{\begin{matrix} x_+=0 \\ \mu=0 \end{matrix}}$ (convenient)

$\Rightarrow \boxed{Df}(x_+=0, \mu=0)$

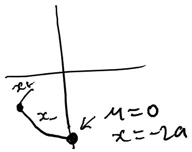
$= -2a < 0 \Rightarrow \boxed{\text{stable}}$



$\boxed{\text{Lower}}$ branch: set $\mu = 0$ ← $\boxed{\text{convenient}}$

$x_- = -a - \sqrt{a^2+0}$

$\Rightarrow x_- = -a - a = -2a$



check stability at $\boxed{\begin{matrix} x_- = -2a \\ \mu = 0 \end{matrix}}$ (convenient)

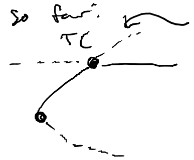
$\boxed{Df}(-2a, 0) = 3\times 4a^2 + (4a-2)(-2a) - \cancel{\mu}^{\,0} - 2a$

$= 12a^2 - 8a^2 + 4a - 2a$

$= 4a^2 + 2a > 0$ $\boxed{\text{unstable}}$

$\Rightarrow$

$\boxed{\text{TC}}$?



So far: TC ← to show $\left\}\begin{array}{l} \text{need } \mu > 1+2a \text{ or } x > 1 \\ \& \quad x_+ = -a + \sqrt{a^2+\mu} \\ \text{easiest: } x > 1 \end{array}\right.$

$\boxed{Df(x,\mu) = 3x^2 + (4a-2)x - (\mu+2a)}$

$(x_+ + a)^2 = a^2 + \mu$ on upper

$\Rightarrow \mu = x_+^2 + 2a$

set $\boxed{x_+ = 2}$ $\quad$ } $\boxed{\text{easiest}}$ here to choose x & solve for μ.

$\Rightarrow \boxed{\mu = 4+2a}$

$Df(x_+=2, \mu=4+2a)$

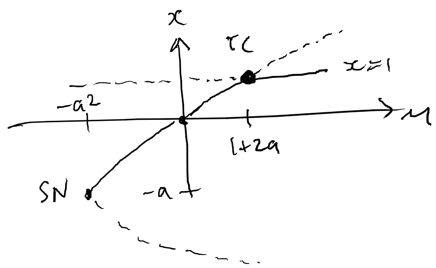
$= 3\times 4 + (4a-2)\times 2 - 4 - 4a$

$= 12 + 8a - 4 - 4 - 4a$

$= 4 + 4a > 0 \Rightarrow \boxed{\text{unstable}} \Rightarrow$ as expected

---

Summary:



..... more worked examples to come .....