ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*) *Oliver Maclaren*
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LECTURE 12

Formalisation, implications and applications of centre manifold theory

- Centre manifold theorem
- Reduction/emergence principle
- More complicated examples
  - Two-dimensional centre manifolds
  - One-dimensional centre manifolds in three-dimensional systems

MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.

CENTRE MANIFOLD THEOREM

Consider $\dot{x} = f(x)$ having a *non-hyperbolic fixed point* at $x = 0$, where $x \in \mathbb{R}^n$.

Assume that there are n^+ eigenvalues (counting repeated cases) with Re $\lambda > 0$, n^0 eigenvalues with Re $\lambda = 0$, and n^- eigenvalues with Re $\lambda < 0$.

CENTRE MANIFOLD THEOREM

Then there is a locally defined smooth n^0-dimensional invariant manifold $W^c_{loc}(0)$ that is tangent to the (linear) centre eigenspace E^c.

Moreover, there is a neighborhood U of $x_0 = 0$, such that if $\phi(x,t) \in U$ for all $t \geq 0$ (≤ 0) then $\phi(x,t) \to W^c_{loc}(0)$ for $t \to \infty$ ($t \to -\infty$).

RECALL: INTERPRETATION

The price of the decoupling reduction is that the *centre manifold* subsystem is (typically) *nonlinear*. On the other hand, the *stable/unstable* subsytems will be *fast* (disappearing when $t \to \infty$) and the *centre manifold dynamics* will be *slow* and hence *emerge* as $t \to \infty$.

For example, a stable non-hyperbolic nonlinear system will, instead of decaying to a unique fixed point as in a hyperbolic system, *rapidly decay to* a centre/slow manifold and then *travel along this at a slower rate*.

Q: what happens in the linearised non-hyperbolic system?

INTERPRETATION

A system is *exponentially attracted to* (or repelled from if we allow for positive eigenvalues) *the centre manifold*.

We can *formalise* this a bit better in terms of a *reduction* near a non-hyperbolic fixed point to a *decoupled* system of *three subsystems* with eigenvalues that have real part positive, negative and zero , respectively.

This generalises the idea of decoupling a system near a hyperbolic fixed point into a linear, real part positive subsystem and a linear, real part negative subsystem.

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

If $(x,y,z) \in E^c \times E^s \times E^u$ are *coordinates in terms of the system's eigenbasis (diagonalised/Normal form)* representation then we *first* write our system as

$$\dot{u} = Au + g(u,v)$$
$$\dot{v} = Bv + h(u,v)$$

where $u \in \mathbb{R}^{n^0}$ are our centre manifold variables and $v \in \mathbb{R}^{n^+ + n^-}$ are our (locally) exponentially growing/decaying solutions (lumped together for convenience).

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

Note:

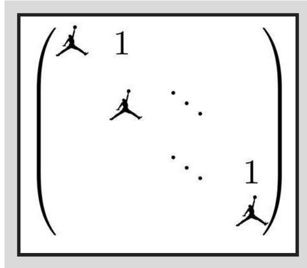
- The matrices A and B have *eigenvalues with zero and non-zero real-part* respectively, (and A is just zero if there are no imaginary parts), while
- The *functions g and h represent the higher-order* (at least quadratic) terms (since A and B represent the linear dynamics) and we will assume they have Taylor expansions (which clearly start from quadratic order)

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

> The key point is to write the equations so that they are *linearly decoupled* according to the *sign of the real part of the eigenvalues*

This corresponds to putting the *linear part* of the system in (Jordan) *'normal form'* (think: diagonal).

INTERLUDE I: TRANSFORMING TO JORDAN NORMAL FORM



...see handouts...

Key is *diagonalising* the *linear part* or getting *as close as possible*.

INTERLUDE II - FULL OR PARTIAL DIAGONALISATION?

In practice, we only need to *split the system into the centre and non-centre systems.* That is, we only need to put the system in the *decoupled block form*

$$\begin{pmatrix} C & \dots \\ \dots & F \end{pmatrix}$$

where the '...' are zero, C is a *diagonalised/Jordan normal form block* for the centre variables, and *F corresponds to the non-centre (fast) subsystem but is not necessarily diagonalised/in Jordan normal form.*

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

We then assume (justified by the CMT, and just like for the stable/unstable manifold case) that we can *locally represent the centre manifold by a smooth curve*, i.e.

$$W^c_{loc} = \{\ (u, v) \mid v = V(u)\}$$

(this may be a vector equation if v and/or u are multi-dimensional quantites!)

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

As described previously, the basic idea is to substitute this relationship into the *chain rule* applied *along the manifold*:

$$\frac{dv}{dt}(u, V(u)) = \frac{dv}{du}(u)\frac{du}{dt}(u, V(u))$$

from which to find $V(u)$.

Let's call this the *'manifold equation'*. We usually solve it by assuming a *power series solution*, as discussed previously (and justified by the CMT).

CENTRE MANIFOLD THEOREM - REDUCTION PRINCIPLE

Putting all this together leads to the following *Reduction (or decoupling) Principle*:

Near a non-hyperbolic fixed point our system (written in its eigenbasis/diagonalised form) is *locally* topologically equivalent to the system

$$\begin{aligned}\dot{u} &= Au + g(u, V(u)) \\ \dot{v} &= Bv\end{aligned}$$

where $V(u)$ is the expression for the centre manifold (found from the procedure on the previous slide).

CENTRE MANIFOLD THEOREM - EMERGENCE PRINCIPLE

Note that these reduced, local dynamics are now *uncoupled* and the dynamics in v are *linear, 'fast' and essentially 'trivial'!*

This allows us to justify (in particular when the linear fast dynamics are stable) using the following *emergent, long-time approximate model of the full system*

$$\dot{u} = Au + g(u, V(u))$$

i.e. we can *just focus on the centre manifold dynamics*.

EngSci 711 L12 Centre manifold theory cont'd.

- Linear decoupling into fast/slow
- Reduction/emergence principle for nonlinear systems
- Example with two-dimensional centre manifold.

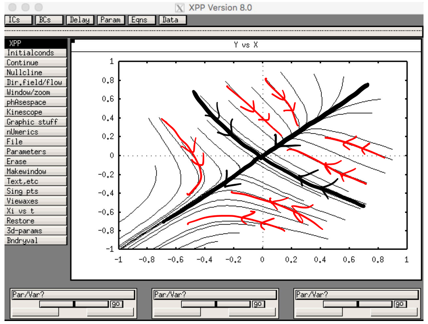
Example Questions

- None from exams!
- See tutorial & assignment

Motivating examples (re: yesterday)

② $$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$

XPP



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③ $$\dot{x}_1 = x_1 y - x_1 x_2^2$$
$$\dot{x}_2 = x_2 y - x_2 x_1^2$$
$$\dot{y} = -y + x_1^2 + x_2^2$$

Consider

$$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$

Let's consider near $(0,0)$.

1. FP? $\dot{x}(0,0) = 0$, $\dot{y}(0,0) = 0$

2. $Df(x,y) = \begin{pmatrix} -1-2x & 1 \\ 1 & -1-2y \end{pmatrix}$

3. $Df(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ (note: columns are linearly dependent here).

$tr = -2$, $det = 0$

Characteristic eq^n

$\lambda^2 + 2\lambda = 0 \Rightarrow \lambda = 0, -2$

$\Rightarrow$ Clearly there is a fast, stable direction & a slow direction

Problem: which is fast, which is slow? Neither x nor y are 'purely fast' or 'purely slow'

Fast or slow?

We'll see a theorem soon which assumes we have identified the slow/fast variables, at least linearly

└ actually just need to separate the slow variables from the rest

We then write:

non-slow = f(slow)

linearisation:

$$\begin{bmatrix} [\text{slow}] & 0 \\ 0 & [\text{other}] \end{bmatrix}$$

→ To identify these, we'll use a linear change of coordinates to put the linear part in block diagonal form

↳ in particular, Jordan normal form, at least the centre part.

↳ this be be used to transform the full nonlinear system

↳ Here we just use a direct, naive approach. More sophisticated methods can be found in refs.

Fast or slow? Linear separation

The idea is simple: use our eigendirections as our new coordinate axes

→ this clearly separates the fast (E^s or E^u) dynamics from the slow (or at least centre) dynamics

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The procedure is discussed in more detail in the Linear Algebra supplement

(End of LOS, attached here again too)

Here I'll show via an example →

Continuing with

$$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$

where
$$A = Df(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ with } \lambda_s = -2,\ \lambda_c = 0.$$

First we need:

1. Eigenvectors

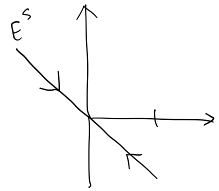
$$(A - \lambda I)u = 0$$

$\lambda_s = -2$ (stable)

$$\begin{pmatrix} -1+2 & 1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow u_1 + u_2 = 0$ (both equations give this)

$\Rightarrow$ choose $u_1 = 1, u_2 = -1$ say $\Rightarrow \boxed{e_s = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$



$E^s = \{(x,y) \mid x = -y\}$

→

$\lambda_c = 0$

$$(A - \lambda I)u = 0$$

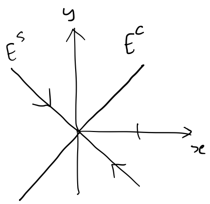
$$\iff$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad u_1 - u_2 = 0 \quad$ ie $u_1 = u_2$ (both).

choose eg $u_1 = 1, u_2 = 1 \Rightarrow \boxed{e_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

So :



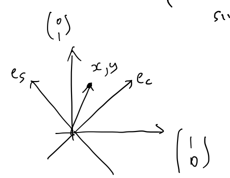
$E^s = \{(x,y) \mid x = -y\}$

$E^c = \{(x,y) \mid y = x\}$

2. Change coord from $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ to $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \{e_c, e_s\}$

Use $Q = \begin{pmatrix} \vdots & \vdots \\ e_c & e_s \\ \vdots & \vdots \end{pmatrix}$

(here: expect simple rotation)



& $\begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} u \\ v \end{pmatrix}$

ie $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underset{\substack{\uparrow \\ \text{slow var}}}{u}\begin{pmatrix} e_c \end{pmatrix} + \underset{\substack{\uparrow \\ \text{fast var}}}{v}\begin{pmatrix} e_s \end{pmatrix}$ } same point, different basis.

Gives

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \vdots & \vdots \\ e_c & e_s \\ \vdots & \vdots \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

Here, $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$Q^{-1} = \frac{1}{\det Q} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}Q$$

basically: symmetric, orthogonal modulo a scale factor $\to$ rotation

So : $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ & $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

ie $\boxed{\begin{aligned} u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \end{aligned}}$ & $\boxed{\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}}$

$\Rightarrow \boxed{\begin{aligned} \dot{u} &= \tfrac{1}{2}(\dot{x}+\dot{y}) \\ \dot{v} &= \tfrac{1}{2}(\dot{x}-\dot{y}) \end{aligned}}$ where $\boxed{\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2 \end{aligned}}$

$$\dot{u} = \frac{1}{2}(\dot{x} + \dot{y}) =$$

$$= -\frac{1}{2}(x^2 + y^2) = -\underline{(u^2 + v^2)}$$

$$\dot{v} = \frac{1}{2}(\dot{x} - \dot{y}) = \frac{1}{2}\left(2y - 2x - (x^2 - y^2)\right)$$

$$= y - x - \frac{1}{2}(x^2 - y^2)$$

$$= -2v - 2uv$$

$$= \underline{-2v(1+u)}$$

So ...

$\dot{u} = -(u^2 + v^2)$ ← slow

$\dot{v} = -2v - 2uv$ ← fast

ie

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} \boxed{0} & 0 \\ 0 & \boxed{-2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}_{\text{Linear decoupled}} + \underbrace{\begin{pmatrix} -(u^2+v^2) \\ -2uv \end{pmatrix}}_{\text{nonlinear}}$$

$$\boxed{\begin{aligned} u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \end{aligned}}$$

$$\boxed{\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}}$$

$$\boxed{\begin{aligned} &x^2 - y^2 \\ &= \\ &\cancel{u^2} + 2uv + \cancel{v^2} \\ &- \\ &(\cancel{u^2} - 2uv + \cancel{v^2}) \\ &= \\ &4uv \end{aligned}}$$

$$\boxed{\begin{aligned} \dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2 \end{aligned}}$$

$$\boxed{\begin{aligned} &x^2 + y^2 = \\ &u^2 + \cancel{2uv} + v^2 \\ &+ \\ &u^2 - \cancel{2uv} + v^2 \\ &= \\ &2(u^2 + v^2) \end{aligned}}$$

Now we have:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} C & \cdots \\ \vdots & F \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

We can use the same procedure as yesterday to

> Reduce our model to the centre manifold & hence identify the emergent (long-time) dynamics

This is justified by the following theorems/principles

> **Centre manifold theorem**: a nonlinear manifold exists & is tangent to E^c
>
> **Reduction principle**: centre manifold 'acts like' a manifold of fixed points: dynamics are locally exponentially attracted/repelled from it, & it is 'linearly frozen'
>
> **Emergence principle**: when the 'fast' variables are stable, they exponentially decay 'away' & the slower dynamics 'emerge' & are confined to/near the centre manifold

(see slides for formal statements)

Doing this we get:

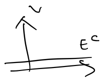
$\dot u = -(u^2+v^2)$ ← slow

$\dot v = -2v - 2uv$ ← fast

so

$v = f(u)$ (fast = f(slow))

$= \cancel{a_0} + \cancel{a_1} u + a_2u^2 + a_3u^3 + \dots$



convenient coord!

$\frac{dv}{du} = 2a_2 u + 3a_3u^2 + \dots$

$\dot v = -2(a_2u^2+a_3u^3+\dots) - 2u(a_2u^2 + \cancel{\dots})$

$= -2a_2u^2 - (2a_3+2a_2)u^3 + \dots$ (1)

$= \frac{dv}{du}\dot u = \left[2a_2u + 3a_3\cancel{u^2}+\dots\right]\left[-(u^2+\cancel{v^2})\right]$

$= -2a_2u^3$ (2)

equate powers in (1) = (2):

$(u^2): -2a_2 = 0 \Rightarrow a_2 = 0$

$(u^3): -(2a_3 + 2\cancel{a_2}^{0}) = -2\cancel{a_2}^{0}$

$\Rightarrow a_3 = 0$

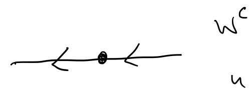
$W^c = E^c$ in this particular case (not in general)

So $v = 0$, ie $W^c = \{(u,v) \mid v=0\}$ →

Motion on centre (slow) manifold:

$\dot u(u,v)\big|_{W^c} = -u^2$: $\{ < 0$ always

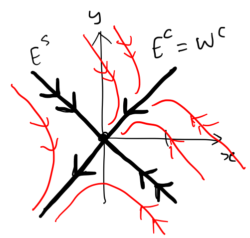
Note:



asymptotically unstable! (careful)

original coord:

XPP:



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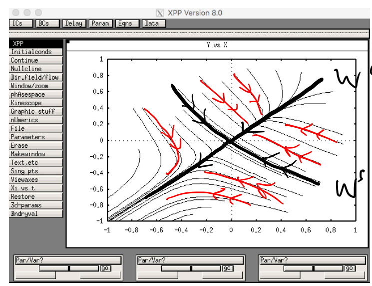


$W^c = E^c$ (here).

$W^s \neq E^s$

Two-dimensional centre manifold

③
$$\left.\begin{aligned}\dot{x}_1 &= x_1 y - x_1 x_2^2\\ \dot{x}_2 &= x_2 y - x_2 x_1^2\end{aligned}\right\}\ \text{slow } x_1, x_2$$
$$\left.\dot{y} = -y + x_1^2 + x_2^2\ \right\}\ \text{fast.}$$

Same idea: fast = f(slow).

We get (near (0,0,0)) (exercise: fill in details)

$$\begin{array}{c} \\ \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{y}\end{pmatrix} = \end{array}
\begin{array}{c}\text{slow}\\ \left(\begin{array}{cc|c} 0 & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & \boxed{-1}\end{array}\right)\\ \text{fast}\end{array}
\begin{array}{c} \\ \begin{pmatrix}x_1\\ x_2\\ y\end{pmatrix} + \begin{pmatrix}x_1 y - x_1 x_2^2\\ x_2 y - x_2 x_1^2\\ x_1^2 + x_2^2\end{pmatrix}\end{array}$$

Use same but 2D expansion/chain in slow vars

Fast = f(slow) $y = f(x_1, x_2)$

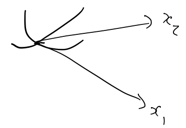
$$\boxed{y = a + b x_1 + c x_2 + d x_1^2 + e x_1 x_2 + f x_2^2}$$

&

chain
$$\boxed{\dot{y} = \frac{\partial y}{\partial x_1}\dot{x}_1 + \frac{\partial y}{\partial x_2}\dot{x}_2}$$

&

Tangency (to x_1, x_2 plane).
$$\frac{\partial y}{\partial x_1}(0) = \frac{\partial y}{\partial x_2}(0) = 0 \Rightarrow \boxed{b = c = 0}$$
$$y(0) = 0 \Rightarrow \boxed{a = 0}$$



So $y = a x_1^2 + b x_1 x_2 + c x_2^2$ (reusing a,b,c names)

Also
$$\frac{\partial y}{\partial x_1} = 2a x_1 + b x_2, \quad \frac{\partial y}{\partial x_2} = b x_1 + 2c x_2$$

$$\dot{x}_1(x_1, x_2) = x_1 \cdot (O(\|\ \|^2)) = O(\|\ \|^3)$$
$$\dot{x}_2(x_1, x_2) = x_2 \cdot (O(\|\ \|^2)) = O(\|\ \|^3)$$
$$\Rightarrow \dot{y} = \frac{\partial y}{\partial x_1}\dot{x}_1 + \frac{\partial y}{\partial x_2}\dot{x}_2 = 0 + O(\|\ \|^3)$$

So just need:
$$\begin{aligned}\dot{y} &= -y + x_1^2 + x_2^2\\ &= -(a x_1^2 + b x_1 x_2 + c x_2^2) - x_1^2 + x_2^2\\ &= 0\end{aligned}$$
$$\Rightarrow a = 1, b = 0, c = 1$$

& $y = x_1^2 + x_2^2$ (name QSS!)

→

Motion on CM:

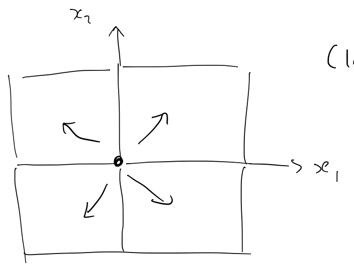
$$\dot{x}_1\big|_{W^c} = x_1(x_1^2+x_2^2) - x_1x_2^2 = x_1^3$$

$$\dot{x}_2\big|_{W^c} = x_2(x_1^2+x_2^2) - x_2x_1^2 = x_2^3$$

$$\dot{x}_1 \begin{cases} >0 & \text{for } x_1>0 \\ <0 & \text{for } x_1<0 \end{cases}$$

& same for $\dot{x}_2$

Unstable:

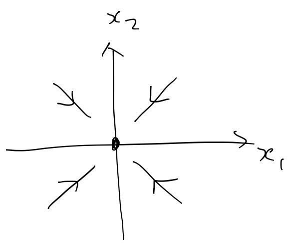


(looking down)

Compare to (incorrect) linear:

$$\dot{x}_1\big|_{E^c} = -x_1x_2^2 \qquad (E^c: y=0)$$

$$\dot{x}_2\big|_{E^c} = -x_2x_1^2$$



Engsci 711

Linear Algebra supplement

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Note: you don't need to know this material for the exam (I don't think!), other than basic stuff used in lectures.

Other the other hand, some *might* be useful for assignments and for e.g. understanding the full range of possible behaviour in linear systems.

Eigenvalues, eigenvectors and all that

Here we recall some basic facts from linear algebra. Some will be stated in full generality, i.e. for $\mathbb{R}^n$, some will be stated for $\mathbb{R}^2$ or $\mathbb{R}^3$. This is because I'm lazy. For more detail see e.g. Glendinning or Kuznetsov (or more generally, any book on linear algebra - I recommend the ones by Strang).

Fact

An $n \times n$ matrix always has n eigenvalues, some of these may be repeated however. The number of times an eigenvalue is repeated is called the multiplicity of that eigenvalue (e.g. the number of times it is a repeated root of the characteristic equation).

Fact

The trace of $n \times n$ matrix is equal to the sum of the n eigenvalues. The determinant is equal to the product of the n eigenvalues.

Fact

If an $n \times n$ matrix is upper or lower triangular, then the eigenvalues are just the diagonal elements. Similarly for a purely diagonal matrix.

Fact

If the eigenvalues of a matrix A are all distinct then the associated eigenvectors are linearly independent (but not necessarily orthogonal). Moreover these are unique up to a scale factor. These eigenvectors can be used to diagonalise A (see below).

Fact

Square matrices with distinct eigenvalues are always diagonalisable. Some matrices with non-distinct eigenvalues are diagonalisable, however - a simple example is the identity matrix.

Fact

Diagonalisability and invertibility are distinct concepts: e.g. you can diagonalise some non-invertible matrices. You can also invert some non-diagonalisable matrices.

Fact

Square matrices are invertible if and only if all eigenvalues are non-zero. Furthermore, all eigenvalues are non-zero if and only if the matrix has a non-zero determinant, if and only if it has linearly independent columns, and if and only if it has linearly independent rows.

Fact

Given a repeated eigenvalue you can sometimes find multiple corresponding standard eigenvectors in the usual way. You just get multiple independent solutions to $(A - \lambda I)x = 0$. This is not guaranteed in general, however (see next result for a condition).

Implication for two-dimensional differential equations If the eigenvalue of a two-dimensional (planar) system is repeated and it *happens to have distinct standard eigenvectors*, then *every* vector in the plane is an eigenvector.

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5). This is called a 'star node' (see figure below). These are on the margin between being a spiral and being a standard node. So they are *sort of* structurally unstable cases in the sense of being boundary cases. On the other hand the actual *existence and stability* of the fixed point is *unchanged* (even though it may change from a spiral to a node, for example, both will have the *same stability*) so they do *still count as hyperbolic fixed points* (another way to see this is to note that a stable spiral and a stable node are topologically equivalent, i.e. can be deformed into one another).

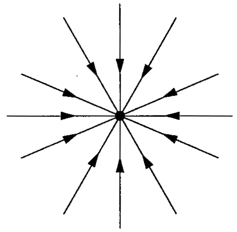

Figure 1: Star node - repeated eigenvalues but distinct eigenvectors. The eigenspace is thus the whole plane.

Fact

Given an $n \times n$ symmetric real matrix $A = A^T$ (or *Hermitian* when dealing with complex matrices) then there are n standard eigenvectors and these are distinct and linearly independent. Furthermore, the eigenvalues are all real, though not necessarily distinct.

Fact

For a non-symmetric $n \times n$ matrix with repeated eigenvalues we can always find an orthogonal basis of *generalised eigenvectors* (sometimes called principle vectors). A generalised eigenvector x of rank m is defined as a non-zero solution to

$$(A - \lambda I)^m x = 0$$

where $m \geq 1$ and

$$(A - \lambda I)^n x \neq 0$$

for $n < m$. That is, it satisfies the usual equation, except $(A - \lambda I)$ is raised to the power of m. The second part of the definition just makes sure x doesn't satisfy the equations of lower rank and so the rank well-defined (i.e. it is the smallest power m satisfying the above equation).

Fact

A general *block diagonal* form of a matrix is:

$$\begin{pmatrix} A_1 & ... & ... \\ ... & A_2 & ... \\ ... & ... & A_3 \end{pmatrix}$$

where each A_i can be any size and internal structure, and any '...' entries are taken to be zero. When this represents a system of linear differential equations, this form is sufficient to *decouple* the system into *self-contained subsystems that can be solved separately.*

Fact

Generalised eigenvectors can be used to put A into Jordan normal form (see below). For square matrices this gives a particular type of *block diagonal* structure. It is a *generalisation of matrix diagonalisation, and is always possible*. It can be thought of as 'as close as possible to diagonal'. This is a preview of the Jordan normal form:

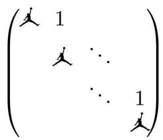

Figure 2: Jordan blocked (stolen from the internet...).

The following illustrates the structure in more detail, including the form of the *Jordan blocks*:

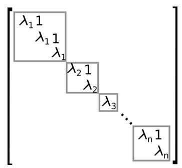

Figure 3: Jordan normal form with Jordan blocks illustrated (stolen from Wikipedia...). The off-diagonal ones occur when the corresponding eigenvalue is repeated, and the size of the block to the number of times the eigenvalue is repeated.

Fact

If x is a generalised eigenvector of rank m then $(A - \lambda I)^n x$ for $n < m$ is a generalised eigenvector of A of rank $m - n$. For example, if $m = 2$ and $n = 1$ then $(A - \lambda I)x$ is a generalised eigenvector of A.

Fact

From the above we see that, given a generalised eigenvectors of A of rank m, we can form a so-called chain of generalised eigenvectors $\{x_m, x_{m-1}, ...x_1\}$ of the form $x_m, (A - \lambda I)x_m, ..., (A - \lambda I)^{m-1}x_m$.

As an example, if we have a 2x2 system and a rank 2 eigenvector x_2, we form the *lower* rank eigenvector x_1 via

$$x_1 = (A - \lambda I)x_2$$

Implication for two-dimensional differential equations Suppose the eigenvalue of a two-dimensional (planar) system is repeated and there is *only one standard eigenvector* and hence one generalised eigenvector. As you might remember, in such cases we mutliply our solutions by t to get a linearly independent basis for our general solution.

Similarly, it can be shown (see e.g. Glendinning) that if e_1 is the standard eigenvector and e_2 is the generalised eigenvector, then the solution can be considered a linear supposition of exponential motions on e_1 and $te_1 + e_2$. Note that as $t \to \infty$, the vector $te_1 + e_2$ points more and more towards e_1, as the first term dominates the second term. Thus the motion on $te_1 + e_2$ - and, as time increases, the full motion - *collapses into motion approximately along the single, standard eigenvector* e_1 (note though: the collapse onto the standard eigenvector is not *exponentially* quick as it would be with a true slow manifold).

Again, these cases are on the margin of being a standard node and being a spiral. The solutions thus have a sort of *spiralling collapse onto the standard eigenvector.*

Example: See end of Lecture 4 Handout (Strogatz Example 5.2.5; see also figure below). Note: you can get the directions of motion (the 'spiralling' direction) by guessing a few interesting points and plugging them into the equations to find the flow direction.

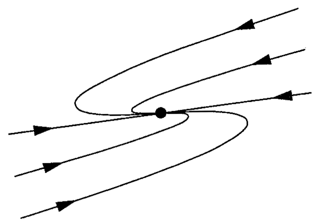

Figure 4: Degenerate node - repeated eigenvalues and only one standard eigenvector. As $t \to \infty$, the contribution of the generalised eigenvector disappears and the motion collapses to the single standard eigenvector.

Change of coordinates, diagonalisation and the Jordan normal form

Change of coordinates: eigenvector system

If we use eigenvectors in our coordinate system we get nice looking matrices.

Suppose you have a vector x whose components are (x_1, x_2) in a given basis. Here we will assume the basis is given by $\{(1,0)^T, (0,1)^T\}$ i.e. the standard basis for $\mathbb{R}^2$.

Then, given a new desired basis (coordinate axes) $\{a, b\}$ with coordinates specified relative to the original basis (standard $\mathbb{R}^2$ basis) (a_1, a_2) and (b_1, b_2), for a and b respectively, we can find the coordinates of x relative to the new coordinate axes, u_1, u_2, using

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

i.e. the columns of the transformation matrix are the vectors of the desired new coordinate system (as expressed relative to the original basis).

To motivate this, consider the invariant vector equation

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + u_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which expresses the *same* vector x relative to two different bases : $\{(1,0)^T, (0,1)^T\}$ and $\{(a_1, a_2)^T, (b_1, b_2)^T\}$, respectively.

So, to change to coordinates u_1, u_2 relative to an eigenbasis with eigenvectors e^s and e^u say, use

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1^s & e_1^u \\ e_2^s & e_2^u \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where e_1^s, e_2^s are the components of e^s relative to the original coordinate system (etc).

This gives two equations to solve for x_1, x_2 in terms of u_1, u_2 - we can then e.g. derive a new differential equation in u variables by replacing the x variables (in the differential equations for x) by u variables.

Easiest way to check you understand this is to try it! See handwritten examples from Lectures 11 and 12. The next two facts elaborate on this.

Fact

Suppose we are given a matrix A. In the case of distinct, real eigenvalues, we can define a change of coordinates from old variables x to new variables y as above, i.e.

$$x = Py$$

where the columns of P are distinct eigenvectors and P is invertible. In this case the matrix

$$\Lambda = P^{-1}AP$$

is diagonal (and hence the diagonal terms are the eigenvalues).

Fact

In the above case, a linear differential equation $\dot{x} = Ax$ gives

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy$$

i.e. the diagonalised linear system

$$\dot{y} = \Lambda y$$

The motion is thus a linear superposition of flows along with eigendirections, with rates given by the eigenvalues. This can be extended to complex eigenvalues, but I'm too lazy to write this out.

Fact

In the case of non-distinct real eigenvalues, we either still have distinct normal eigenvectors - in which case we proceed as normal - or we instead use our generalised eigenvectors in the transformation. This leads to the *Jordan normal form* pictured previously. In the 2x2 case we have

$$e_1 = (A - \lambda I)e_2$$

and so

$$Ae_2 = \lambda e_2 + e_1$$

whereas, obviously,

$$Ae_1 = \lambda e_1$$

which means

$$P^{-1}AP = [e_1, e_2]^{-1}A[e_1, e_2] = [e_1, e_2]^{-1}[\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2]^{-1}[e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

giving our Jordan normal form. Again, we can extend to the complex but I won't here.