ENGSCI 213: MATHEMATICAL MODELLING 2SE

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CURRENT (& LAST) PROBABILITY TOPIC

3. *Continuous probability models* [3 lectures]

   Continuous random variables and continuous probability distributions. Probability density functions and cumulative distribution functions. (Expectation and variance). Uniform, Exponential and Normal distributions.

LECTURE 9

- Recap: Exponential distribution
- The Normal (Gaussian) distribution
- The Central Limit Theorem

RECAP: EXPONENTIAL DISTRIBUTION

The *Exponential distribution* can be related to the *Poisson process* - rather than count the number of events in a fixed time interval it describes

The length of time between events - i.e. the 'waiting time' - in a Poisson process

EXPONENTIAL DISTRIBUTION

The exponential distribution has *one parameter*, λ (equals the rate in a Poisson process), which must be positive.

We write

$$\boxed{X \sim Exponential(\lambda), \text{ or } X \sim Exp(\lambda).}$$

RECAP: EXPONENTIAL DISTRIBUTION

If $X \sim \mathrm{Exp}(\lambda)$, then

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

&

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-\lambda x} & \text{otherwise} \end{cases}$$

PLOTTING IN R

Use `dexp()` and `pexp()`. See R Markdown notebook on Canvas.

RECAP: MEAN AND VARIANCE OF EXPONENTIAL DISTRIBUTION

If $X \sim \mathrm{Exp}(\lambda)$, then

$$E(X) = \frac{1}{\lambda}$$

and

$$Var(X) = \frac{1}{\lambda^2}$$

RECAP: MEMORYLESSNESS OF THE EXPONENTIAL DISTRIBUTION

If $X \sim Exp(\lambda)$ then for any $s, t \geq 0$

$$P(X > (s + t) \mid X > t) = P(X > s)$$

So the process is *'memoryless'*

EXAMPLE

Yet another example for the Exponential distribution

THE NORMAL (GAUSSIAN) DISTRIBUTION

The Normal (or Gaussian) distribution is the familiar *'bell-shaped' distribution*. It has *two parameters*, the *mean*, μ, and the *variance*, σ^2.

We write $X \sim Normal(\mu, \sigma^2)$, *or*
$X \sim N(\mu, \sigma^2)$

PROBABILITY DENSITY FUNCTION

The *probability density function* is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for $-\infty < x < \infty$ and $-\infty < \mu < \infty, \sigma^2 > 0$

CUMULATIVE DISTRIBUTION FUNCTION

There is *no closed form for the cumulative distribution function* of the Normal distribution.

If $X \sim \mathrm{Normal}(\mu, \sigma^2)$, then $F_X(x)$ can only be calculated by computer.

PLOTTING IN R

Use `dnorm()` and `pnorm()` (note: expect SD not VAR!) See R Markdown notebook on Canvas.

MEAN AND VARIANCE OF THE NORMAL DISTRIBUTION

If $X \sim \text{Normal}(\mu, \sigma^2)$, then

$$E(X) = \mu, \quad \text{Var}(X) = \sigma^2$$

Note: The Normal distribution is *probably the most used of all the statistical distributions*, mainly due to the *Central Limit Theorem* (see end of these slides). The theorem implies that the Normal distribution can approximate many natural phenomena.

TRANSFORMATION PROPERTIES

If $X \sim \text{Normal}(\mu, \sigma^2)$ then for constants a, b we get

$$aX + b \sim \text{Normal}(a\mu + b, a^2\sigma^2)$$

In particular this leads to...

STANDARD NORMAL DISTRIBUTION

If $X \sim \mathrm{Normal}(\mu, \sigma^2)$ then

$$\frac{X - \mu}{\sigma} \sim \mathrm{Normal}(0, 1)$$

This is the *'standard form'* that many algorithms/tables of values expect.

EXAMPLE

THE CENTRAL LIMIT THEOREM

The *Central Limit Theorem* states:

If $X_1, X_2, \ldots X_n$ are a *large* set (n 'big') of *independent, identically-distributed* random variables, each with mean μ and variance σ^2 (but are *otherwise arbitrary*), then their *sum* is approximately distributed as

$$X_1 + X_2 + \ldots + X_n \sim \text{approx. Normal}(n\mu, n\sigma^2)$$

EXAMPLES

$$\boxed{Bin(n, p) \to Normal(np, np(1 - p))}$$

for $n \to \infty$ and p fixed.

$$\boxed{Poi(\lambda) \to Normal(\lambda, \lambda)}$$

for λ large.

R PLOTS

See R Markdown notebook on Canvas.