ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*…and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*)
[~17-18 lectures/tutorials]

1. *Basic concepts, stability and linearisation* [4 lectures/tutorials]
   Basic concepts and some formal definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest: fixed points, periodic orbits etc. Overview of basic analysis procedure including linearisation, connecting stability of nonlinear systems and stability of linearised systems. Computer-based analysis.
2. *Phase plane analysis and geometry of hyperbolic systems* [5 lectures/tutorials]
   Analysis of two-dimensional linear and nonlinear systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds) and decoupling for general linear and nonlinear hyperbolic systems. Connecting geometry of nonlinear and linearised hyperbolic systems.

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]
   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations for parameter dependent systems. Bifurcation diagrams.
4. *Centre manifold theory and putting it all together* [4 lectures/tutorials]
   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.
5. *Bonus topic?* [1 lectures if time]
   Introduction to mathematics of chaos…

LECTURE 1 - ‘BIG PICTURE’

Why? *What problem are we trying to solve?*

- Complex models (and the phenomena they represent) are *difficult to understand*
- Even ‘simple’ models can be difficult to understand

LECTURE 1 - COMPLEX MODELS

- No *closed-form* solutions
- *Brute-force* simulation doesn't necessarily help us *understand* our model (and the phenomenon we are modelling)
- All models are *wrong* (Box)
- Better to be *approximately right* than exactly wrong (Tukey)

EXAMPLES

- Lorenz System
- Car and Trailer
- Emergent dynamics
  - Ecological catastrophe
  - Training deep neural nets

EXAMPLE - LORENZ SYSTEM

Can we *understand* this three-dimensional, three-parameter ODE?

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

EXAMPLE - LORENZ SYSTEM

'Time' simulation.

See 'lorenz.ode'

HIDDEN COMPLEXITY AND HIDDEN SIMPLICITY

Despite having a small number of parameters and variables, the Lorenz system *exhibits complex behaviour* (even simple 1-D discrete maps can also exhibit such complexity)

However, by looking at it from *a different point of view* we can get *some* understanding of this system.

THE QUALITATIVE POINT OF VIEW

Rather than finding explicit, exact solutions, our point of view here will be

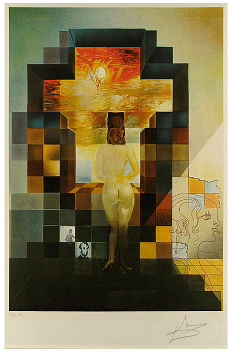
- The *'qualitative'* analysis of ODEs (and 'dynamical systems' in general). We step back from the full detail to enable a better overall picture
- Essentially a *geometric* perspective
- Possibly a *different way of thinking* than you are used to
- Both *hand calculation and computer-based methods* will be used; the goals are the same though

HIDDEN SIMPLICITY AND HIDDEN COMPLEXITY

Let's plot a so-called *phase-portrait*

See 'lorenz.ode'

THE QUALITATIVE POINT OF VIEW



What can we say just by looking at 'pictures' in various ways?

WAYS OF THINKING

- Part of this module is about 'cookbook' methods to add to your toolbox
- Part of it is to give you an introduction to the underlying mathematical/geometric viewpoint

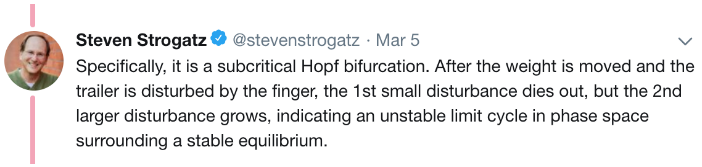
I'll *try* to balance practical analysis methods with formal definitions and mathematical ideas!

- Hands-on experience is important: make sure to do *tutorials*

HOW YOU SHOULD BE THINKING BY THE END OF THE COURSE

Car and trailer video https://bit.ly/2dJOXvw

Steven Strogatz comment:



SO...WHAT IS A 'DYNAMICAL SYSTEM'?

Informally, a *dynamical system* is a mathematical model of a *process which evolve in time*.

There are *three key ingredients*: a set or interval of *'times'*, possible *'states'* of a 'system' at any given time and an *'evolution rule'* or law governing how the system transititions between these states.

SO...WHAT IS A 'DYNAMICAL SYSTEM'?

Examples are everywhere - ODEs, PDEs, difference equations/maps, stochastic processes even iterative computer algorithms and constructive mathematical proofs.

Our aim is to relate the *'dynamic' point of view* to the *'static' point of view* by looking for *invariant and/or limiting features in state space*

TERMINOLOGY: STATE/PHASE SPACE

In practice the *'state'* is defined by *'contains everything we need to know to get from the current state to the next state'*. E.g. position and momentum for classical mechanics.

The *state space/phase space* is…the 'space' of all states - usually (embedded in) $\mathbb{R}^n$, for real-valued differential equations.

(More general phase spaces include the circle, torus etc, depending on the 'problem structure')

ORDINARY DIFFERENTIAL EQUATIONS

We have one equation in f for each entry in the state vector x e.g.

$$x = (x_1, x_2, \ldots)^T$$

$$f = (f_1, f_2, \ldots)^T$$

If there is no dependence on t then we say the system is *autonomous* (we can always add a new dependent variable to track time dependence). We will focus on these in this course.

ORDINARY DIFFERENTIAL EQUATIONS

Our main focus is on a very familiar type of dynamical system - systems of ordinary differential equations of the form:

$$\dot{x} = f(x, t; \mu)$$

where $x \in \mathbb{R}^n$ is a vector of *state variables*, $t \in \mathbb{R}$ is the *independent variable* (usually time), $\mu \in \mathbb{R}^m$ is a vector of *problem parameters*.

(For now we often suppress the dependence on problem parameters - but see bifurcation theory!)

EXAMINABLE EXPECTATIONS

"But what do you expect me to be able to do for the exam?"

Let's look at the exam from last year!

FURTHER READING/WATCHING

- Strogatz course on YouTube! https://goo.gl/Kqus6G
- Handout on helpful books/references

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Qualitative analysis of differential equations

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Some references specifically from the 'dynamical systems' perspective

Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering by Steven H. Strogatz.

- Great book, easy to read, lots of examples. A little on the wordy side. A math prof. once complained 'it's great but it doesn't mention the centre manifold theorem'. I didn't care at the time, though I have more sympathy for this complaint now. Often recommended as a supplement in more 'advanced' courses to give the 'big picture', also used in many introductory courses. We will mainly work at this sort of level, but introduce some material and definitions from the other books below.
- Two copies of the 1994 edition (the one I have!) are available in the main library. The 2005 Edition is available as online ebook through the library (you should be able to find it - if not, ask me).

Simulating, analyzing, and animating dynamical systems : a guide to XPPAUT for researchers and students by Bard Ermentrout.

- Bard wrote XPPAut, an interface to the AUTO and XPP dynamical systems software. Also known for his contributions to mathematical biology. Easy to read, great way to get some hands-on experience doing examples. If you wanted to use XPPAut for a real problem then this is a great place to start. My PhD supervisor used it to make many figures in his book 'Mathematical Physiology' and they don't look terrible.
- Available from general library and to read online through the library access to SIAM.

Stability, instability, and chaos : an introduction to the theory of nonlinear differential equations by Paul Glendinning

- Nice coverage, all the key theorems. For the more mathematically-inclined but still quite 'applied'. In the 'Cambridge Texts in Applied Mathematics' series. I used this when I took Maths 761 at Auckland, it really grew on me.
- Multiple copies available from main library, engineering library and short loan (since used for Maths 761 course?).

Introduction to applied nonlinear dynamical systems and chaos by Stephen Wiggins.

- Multiple copies/editions in both general and engineering library. Also available for free download through library access to Springer book. Another solid book, similar level to Glendinning. Probably more comprehensive.

Nonlinear systems by P. G. Drazin.

- Good broad coverage of both difference and differential equations. Has particularly good material on bifurcation theory. For the more mathematically-inclined. A bit less systematic than Glendinning but lots of good extra stuff in there. Also in the 'Cambridge Texts in Applied Mathematics' series. I used this when I TA'd a third-year applied mathematics course in the Oxford mathematics department.
- A copy in the main library and a copy in the engineering library.

Differential equations and dynamical systems by Lawrence Perko

- Good material on centre manifold theory, including simple examples!

- Available online through the main library (can download through Springer) and multiple hard copies in the general library.

Model Emergent Dynamics in Complex Systems by A. J. Roberts

- An attempt to present undergraduate mathematical modelling from the point of view of centre manifold theory and nonlinear coordinate transformations
- The theory is viewed as a way to derive emergent, simple models from more complex models (rather than just as a tool for analysing given models)
- Lots of interesting material and examples. Quite idiosyncratic presentation.
- Available online through the main library.

Nonlinear systems by Khalil

- Two copies, one in engineering one in general library. Focused on engineering applications but is a graduate-level text. Accessible material on centre manifold theory.

Nonlinear oscillations, dynamical systems, and bifurcations of vector fields by John Guckenheimer and Philip Holmes.

- Multiple copies available in general library and short loan (probably for Maths 761?). The classic, still really good, written by 'legends' of the field. A little more advanced that the others. John Guckenheimer often visits/works with people in the Auckland math department, who are really strong on this stuff. If you like this area and wanna continue (e.g. do a Masters/PhD) then I can point you to them.

Elements of applied bifurcation theory by Y. A. Kuznetsov.

- Multiple copies in both engineering and main library. Similar level to Guckenheimer and Holmes but a surprisingly easy read for a fairly advanced book. Good material on 'practical' bifurcation analysis e.g. numerical methods. Yuri wrote (with others) 'MatCont' which is Matlab-based software for bifurcation analysis.

Other references

Almost all books on 'Advanced Engineering Mathematics' or (often 'Introduction to') 'Applied Mathematics' will have good condensed material on qualitative analysis (esp. phase plane analysis) of differential equations. They often also have some brief material on bifurcation theory.

- In the 'Advanced Engineering Mathematics' category I quite like Greenberg
- In the 'Applied Mathematics' category I quite like Logan.

Both are available in the engineering library.

These sorts of references might often be easier to quickly learn the basics from than the more comprehensive books exclusively focused on dynamical systems.

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Qualitative Analysis of ODEs (& Other Dynamical Systems)

Today:
- overview
- state space

Plan: 4 Sections

1. Basic overview/terminology/linearisation
2. Phase Plane & Geometry of Hyperbolic syst.
3. Bifurcation Theory
4. Centre Manifold Theory
   - + intro to chaos?

Slides:
- Big Picture
- Technical Definitions
  - for the mathematicians
  - moral obligation to show you!
- Tidier

Handwritten & Scanned:
- Messier .... BUT:
- What you need to be able to do & how!

Challenges

- Complex models eg 100s – 1000s of parameters, equations etc; nonlinear
- We can simulate for one set of parameters, then another etc etc
  - Brute force quantitative simulation approach

BUT (can be):
- inefficient/infeasible?
- lacking in insight?
  - why do we get answer we get?
  - under what conditions etc?
  - how could we change the behaviour?

nonlinearity makes difficult (not just 'sum of parts')

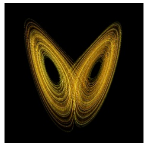
→ we want more qualitative, approximate methods to compliment eg direct simulation

→ we use a geometrical & topological perspective

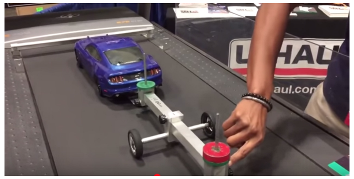
→ apply to non-linear systems

Some Examples : qualitative analysis of complex behaviour

Weather prediction { geometry of Lorenz system



Car + trailer dynamics { Stability depends on mass location



Deep Learning, Scale separation & 'emergent' dynamics { dynamics become slower & simpler in final stages.

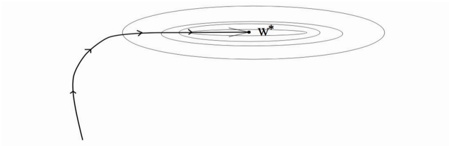


Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum $\mathbf{w}^*$ and it is a good approximation of the minimum eigenvalue direction of the Hessian.

Examples

- Lorenz system

ODE system

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

in XPPAut :

```
init x=-7.5  y=-3.6  z=30
par r=27  s=10  b=2.66666
x'=s*(y-x)
y'=x*(r-z)-y
z'=x*y-b*z
```

↳ see lab !

how? see later!

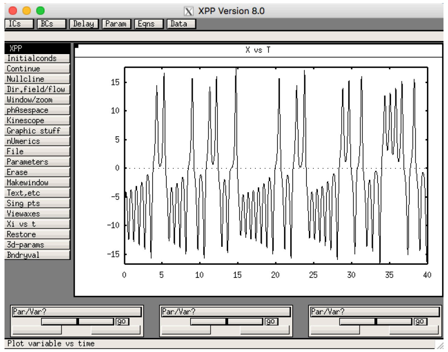
- simplified model of atmospheric convection (also arise as model of lasers, chemical reactions etc)
- Exhibits chaotic solutions ! (even though 'simplified')

→

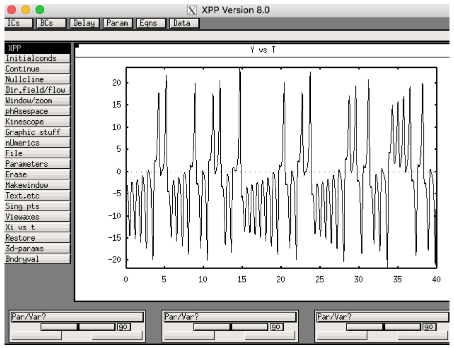
'Brute force' / direct time simulation

→ plot x, y, z vs time for fixed set of parameters

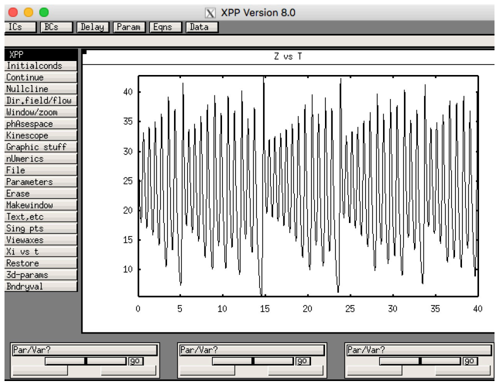
x vs t



y vs t



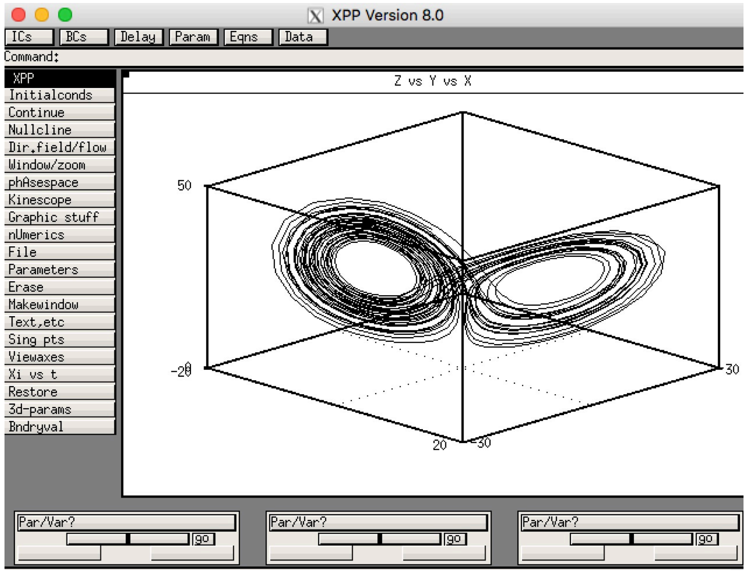
z vs t



→ what's going on?

'Shift' perspective

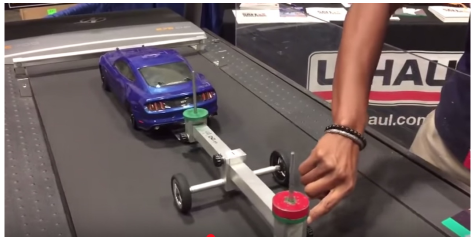
- plot in so-called phase/state space
- focus on the 'geometry'/topology of the problem
- where are trajectories attracted to?



- a clearer structure 'emerges' as time goes on
- the dynamics are attracted to the 'strange' butterfly-like shape

Examples: Stability of a trailer [Bifurcation theory]

Video: see link in slides.



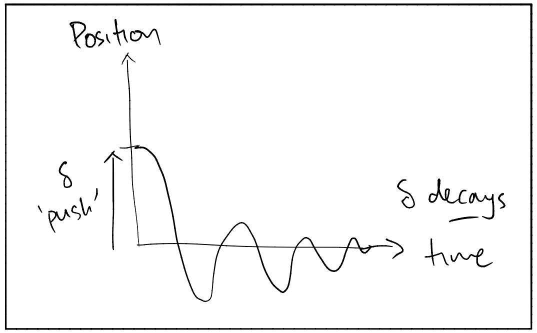
change mass distribution

↓

| unstable |: avoid!

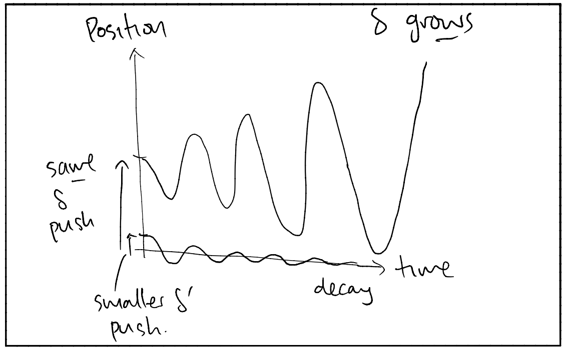
Steven Strogatz @stevenstrogatz · Mar 5
Specifically, it is a subcritical Hopf bifurcation. After the weight is moved and the trailer is disturbed by the finger, the 1st small disturbance dies out, but the 2nd larger disturbance grows, indicating an unstable limit cycle in phase space surrounding a stable equilibrium.

Examples: Stability of a trailer [Bifurcation theory]



mass closer to car

Change in stability as 'control parameter' varied.



mass further away from car

"Bifurcation"

→ Change in qualitative behaviour

| stable → unstable |

as an external control parameter is varied

Examples: Reduction & Emergence near centre manifold

Reduction of complex models/dynamics
↓ ↓
Emergence of simple models/dynamics

↓ longer time scale

Eg

- How might we get a 'simplified' model like the Lorenz system from a more complicated model (eg Navier-Stokes)?

- Turns out this model can actually be obtained using centre manifold reduction theory

  ↳ we will cover this theory towards the end of this part!

- Again, focus on the geometry/topology of the long term dynamics

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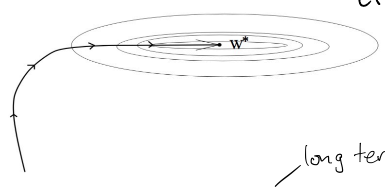
Examples:

[Emergence of 'simpler' dynamics near 'centre manifolds']

Deep Learning!

From Le Cun et. al. "Efficient BackProp" in 'Neural Networks: 'Tricks of the trade' (book)



long term

Fig. 1.5. Convergence of the flow. During the final stage of learning the average flow is approximately one dimensional towards the minimum $\mathbf{w}^*$ and it is a good approximation of the minimum eigenvalue direction of the Hessian.

reduction

centre manifold

⇒ 'long term reduction to the centre manifold'!

CBMM Memo No. 073 **January 30, 2018**

Theory of Deep Learning III: the non-overfitting puzzle

} explaining deep learning/non-over fitting using qualitative theory of ODEs, incl. centre manifolds.

by

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with
X. Boix†, J. Hidary††, H. Mhaskar◇,
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⊤CSAIL, MIT
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◇Claremont Graduate University

Abstract: A main puzzle of deep networks revolves around the apparent absence of overfitting intended as robustness of the expected error against overparametrization, despite the large capacity demonstrated by zero training error on randomly labeled data.

In this note, we show that the dynamics associated to gradient descent minimization of nonlinear networks is topologically equivalent, near the asymptotically stable minima of the empirical error, to a gradient system in a quadratic potential with a degenerate (for square loss) or almost degenerate (for logistic or crossentropy loss) Hessian. The proposition depends on the qualitative theory of dynamical systems and is supported by numerical results. The result extends to deep nonlinear networks two key properties of gradient descent for linear networks, that have been recently recognized (1) to provide a form of implicit regularization:

qualitative theory of dynamical systems

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. Thus the first step is to linearize the system, which means considering the Jacobian of F or equivalently the Hessian of L at W^*, that is

$$\mathbf{H}_{ij} = \frac{\partial^2 L}{\partial w_i \partial w_j} \tag{2}$$

.

We obtain

$$\dot{W} = -HW, \tag{3}$$

where the matrix H, which has only real eigenvalues (since it is symmetric), defines in our case (by hypothesis we do not consider unstable critical points) two main subspaces:

- the stable subspace spanned by eigenvectors corresponding to negative eigenvalues
- the center subspace corresponding to zero eigenvalues.

centre manifold theory

The center manifold existence theorem (16) states then that if F has r derivatives (as in the case of deep polynomial networks) then at every equilibrium W^* there is a $\mathbf{C}^r$ stable manifold and a $\mathbf{C}^{r-1}$ center manifold which is sometimes called *slow manifold*. The center manifold emergence theorem says that there is a neighborhood of W^* such that all solutions from the neighborhood tend exponentially fast to a solution in the center manifold. In general, properties of the solutions in the center manifold depends on the nonlinear parts of F. We assume that the center manifold is not unstable in our case, reflecting empirical results in training networks. Of course, the dynamics associated with the center manifold may be non trivial, especially in the case of SGD. We simply assume for now that perturbations of it will not grow and trigger instability.

A simpler example (illustrates same idea as previous 'deep learning' slide)

Consider the system:

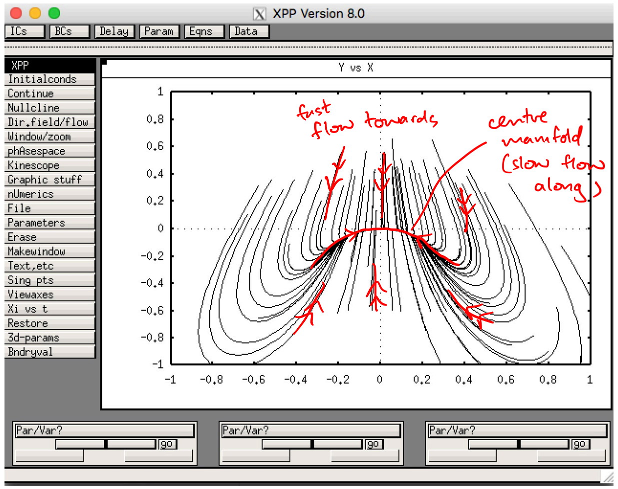
$$\left.\begin{aligned}\frac{dx}{dt} &= xy + x^3,\\ \frac{dy}{dt} &= -y - 2x^2\end{aligned}\right\} \text{nonlinear}$$

(Turns out the usual linear analysis doesn't work — we'll look at when & why this happens)

Emergent, attracting 'manifold' — curved surface/subspace

→ fast motion to manifold

→ slow motion along manifold



Qualitative View?

- emergent (long time) simple (reduced) descriptions of complex systems
- approximate, qualitative & geometric approach to complement detailed simulation methods
- stability, instability & emergence for nonlinear systems

(Dali)

→ what you see depends on the scale

→ a qualitatively different picture 'emerges' on the coarser scale.

Dynamical System?

→ something that 'evolves in time' from 'state to state'

ingredients:
- sequence of 'times'
- possible 'states'
- evolution/update rule

→ This can be quite abstract!

- eg 'time' can be any natural ordering

Examples: ODEs
PDEs
⋮
Computer programs
Constructive proofs
⋮

Dynamic ⟷ static:

dyn. ↓ static

> 'When things change, look for what stays the same & what "limits" the process tends to'

static ↓ dyn.

> 'Given some (implicitly defined) "static" object, think about how to define a procedure or sequence that can construct/converge to this object'

State & State space/Phase space

- Informally, the system 'state' is defined as 'everything you need to know to get from the current state to the next state using the update rule' } enough to define a first order in time, deterministic evolution rule
  → a bit circular!
  → best understood via examples & practice
- the state space/phase space is the set of all possible states

  eg $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$

Examples:
- In classical mechanics, knowing the position & momentum at time t is enough to tell you the position & momentum at time $t+dt$, using Newton's law
- In quantum mechanics you need to know the wave function
  → although this defines a probability distribution over observables, the wave function itself evolves deterministically (state ≠ observable)
  ↳ (Schrodinger eqn.)

ODEs

We'll focus on systems of ODEs in the form

$$\underset{\substack{\uparrow \\ \text{first order} \\ \text{in time}}}{\dot{x}} = f(\underset{\substack{\uparrow \\ \text{state} \\ \text{vector, } x \in \mathbb{R}^n}}{x}, \underset{\text{time} \in \mathbb{R}}{t}\,;\ \underset{\substack{\text{problem} \\ \text{parameters}}}{\mu})$$

ie

$$\underset{\substack{\uparrow \\ \text{vector}}}{\dot{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(\overbrace{x_1, x_2, \ldots}^{x \text{ vector}}) \\ f_2(\ldots) \\ \vdots \\ f_n(\ldots) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \underset{\substack{\uparrow \\ \text{vector}}}{f}$$

- parameter?
  - L held fixed during evolution
  - L eg rate constant, diffusion constant etc
  - L later, we'll sometimes imagine as super slow state vars, ie $\dot{\mu} = 0 \Rightarrow \mu = \text{const.}$
  - L for now we'll most ignore explicit dep. on param.

- time dependence?
  - L when f independent of t, called autonomous system
  - L can always intro $x_{n+1} = t$ & $\dot{x}_{n+1} = 1$

    to convert n dim non-autonomous to $n+1$ dim autonomous system

{ → $\boxed{\text{hence we will assume autonomous!}}$ }

ODE example

→ converting to state-space form (should be familiar!)

Q.

(b) Consider the second-order equation

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is a system parameter.

(i). Re-write the above equation as system of two first-order equations.

Exam 2016.

} ie state-space form.

A.

Let $\boxed{\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} = \dot{x}_1 \end{aligned}}$ define new state vars: x & x derivatives up to one less than highest derivative in ODE.

So $\underbrace{\dot{x}_2 = \ddot{x}}_{\substack{\text{highest order} \\ \text{derivative}}} = \underbrace{-\mu\dot{x} - (x - x^3)}_{\text{rest of ODE}}$

$\Rightarrow \dot{x}_2 = \underbrace{-\mu x_2 - (x_1 - x_1^3)}_{\text{in terms of new vars}}$

& $\dot{x}_1 = x_2$ by definition (just relates derivatives/definitions)

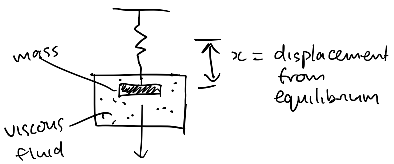
$$\Rightarrow \underset{\substack{\uparrow \\ \text{system of first order}}}{\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 - (x_1 - x_1^3) \end{aligned}}} \begin{aligned} &\overset{\checkmark}{=} f_1(x_1, x_2) \\ &\overset{\checkmark}{=} f_2(x_1, x_2) \end{aligned}$$

Another example

Given

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F\cdot\cos t$$ } Newton's second law

$\underbrace{m\frac{d^2x}{dt^2}}_{\text{mass}\times\text{accel}} + \underbrace{b\frac{dx}{dt}}_{\text{damping force}} + \underbrace{kx}_{\text{spring force}} = \underbrace{F\cdot\cos t}_{\text{external force}}$



$F\cdot\cos t$ } external forcing

state space?

Let $x_1 = x$, $x_2 = \dot{x}$

$$\Rightarrow \dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left[-b\dot{x} - kx + F\cdot\cos t\right]$$

$$= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m}\cos t = \underbrace{f(x_1, x_2, t)}_{\text{non-auton.}}$$

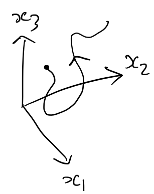
Next, Let $x_3 = t$

$$\Rightarrow \dot{x}_3 = 1$$

So $$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{cases} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m}\cos t \\ 1 \end{cases}$$

ie $\frac{d}{dt}(\text{states}) = \text{evolution rule}$

3D state space:



'But what do I need to be able to do for the exam?'

→

Exam Qs 2017 — Not too bad!
(Most did very well)

SECTION B

Question 4 (16 marks)

Consider the system

$$\dot{x} = 2xy + x^3$$
$$\dot{y} = -y - x^2$$

where $x, y \in \mathbb{R}$.

(a) Verify that the origin is a fixed point of this system.

(1 mark)

— solve RHS = 0

(b) Find the Jacobian derivative - first as a function of x and y and then evaluated at the origin $(0, 0)$.

(2 marks)

— linearise

(c) Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively. Sketch the eigenspaces in the (x, y) plane. You do not need to show any nearby trajectories.

(3 marks)

— calc eigenvalues of a matrix

(d) Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0, 0)$ that is correct up to and including cubic order.

(8 marks)

(e) Use the previous expression to determine the dynamics on the centre manifold, again correct up to and including cubic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.

(2 marks)

— use a power series approx & sub into centre manifold equation (new)

Question 5 (16 marks)

Consider the equation

$$\dot{u} = (u - 2)(\lambda - u^2)$$

where $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a parameter that can vary.

(a) Determine the equilibria and their stability as λ varies.

(11 marks)

— solve RHS = 0 as function of λ

(b) Sketch the bifurcation diagram showing how the equilibria vary with λ. What types of bifurcations occur?

(5 marks)

— draw a picture & label

Question 6 (18 marks)

Consider the system

$$\dot{x} = x^2 + y^2 - 2$$
$$\dot{y} = x - 1$$

where $x, y \in \mathbb{R}$.

(a) Find and classify all of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

(6 marks)

— solve RHS = 0

(b) Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields on the nullclines in your sketch.

(10 marks)

— solve RHS = 0 one at a time

(c) Add some possible compatible trajectories, including compatible local behaviour near the equilibria, to your diagram. You do not need to do any further explicit calculation (e.g. you do not need to find any eigenvectors) - a qualitative sketch is enough.

(2 marks)

— draw some pictures.