Engsci 741

Problem Set 2:

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Problem

Write down as many formulations of the linear regularised least squares problem that you can (I can think of at least 3-5).

Explain the interpretation of any relevant regularisation parameters.

Sketch trade-off curves for some of these and how solutions vary with the associated regularisation parameter.

Which of these formulations are applicable to both linear and nonlinear problems? Show how to express nonlinear problems for these formulations.

Problem

Consider the standard Tikhonov form of the regularised least squares problem

$$\min_x ||Ax - y||^2 + \lambda ||x||^2$$

- Rewrite this in the form of a new 'augmented' but 'standard' least squares problem.
- Explain how the previous procedure enables us to convert e.g. an underdetermined problem into an overdetermined problem.
- Justify the linear independence of the columns of the matrix in the resulting overdetermined problem.
- Explain why this enables us to 'solve' the resulting problem and derive an explicit expression for the 'Tikhonov inverse'. You may state/use the normal equations without proof.
- Describe the solutions corresponding to the two limits $\lambda \to 0$ and $\lambda \to \infty$.

Problem

Describe a method for choosing the regularisation parameter λ in Tikhonov regularisation.

Problem

Suppose you wanted to obtain solutions that emphasise 'smoothness' instead of small norm $||x||$. Show how to formulate an optimisation problem to do this. Define and give expressions for any new operators you introduce.

Problem

Next suppose that you wanted to obtain solutions that preserve *sharp* transitions or features in a signal or image.

- Formulate an appropriate regularised optimisation problem to do this.
- Briefly explain why your formulation is appropriate for this task.
- Sketch typical solutions that you might obtain from this approach compared to an approach emphasising smooth solutions.

Problem

Contrast the problem that must be solved to obtain eigevalues/eigenvectors of a matrix A to that which must be solved to obtain singular values/singular vectors.

Problem

Write the collection of solutions to the above problem in matrix form and hence derived the singular value decomposition of a matrix A. Interpret this geometrically.

Write down both the full and reduced forms of the SVD.

Problem

Give an explicit form of the generalised inverse of A in terms of its SVD component matrices. Verify that it is a left/right inverse when it should be.

Problem

Consider the 'inverse problem' of polynomial regression for data $\{(x_i, y_i)\}, i = 1, .., m$. Suppose you have a model with four observations (i.e. $m = 4$), and three parameters.

- Write down the general expressions for the *model resolution* operator and the *data resolution* operator, respectively. State which spaces they map between.
- Which of these do you expect to be (near) identity and which not? Why?
- Give a simple example of U_r, V_r compatible with the above.
- Verify your answers for the case where the forward mapping and its (reduced) matrices of left and right singular vectors are given by:

$$A = \begin{bmatrix} 1. & 1. & -0.5 \\ 1. & 3. & -4.5 \\ 1. & 5. & -12.5 \\ 1. & 13. & -84.5 \end{bmatrix}, U_r = \begin{bmatrix} -0.00769051 & 0.29107872 & 0.89685522 \\ -0.05693158 & 0.58920415 & 0.1049587 \\ -0.15174676 & 0.73868339 & -0.42646699 \\ -0.98674848 & -0.14986157 & 0.05253846 \end{bmatrix}, V_r = \begin{bmatrix} -0.01388544 & 0.35601275 & 0.93437793 \\ -0.15886432 & 0.92181289 & -0.3535861 \\ 0.98720278 & 0.15334901 & -0.04375794 \end{bmatrix}$$

respectively. (It might be tidier to do this in Python/Matlab and just recompute U_r, V_r e.g. via np.linalg.svd(A, full_matrices=False)!). Plot the left/right singular vectors in each case.

Problem

What would you expect the singular spectrum of the forward operator appearing in a typical ill-posed problem to look like?

Problem

Explain how to use truncated SVD to obtain regularised solutions.

Problem

Reformulate the least squares normal equations

$$A^T A x = A^T y$$

as a fixed point problem and show how to derive a simple iterative solution scheme. Hint: if you get stuck, use a 'theory of everything' or two.

Problem

In reference to the problem above, explain the concept of 'iterative regularisation' (and/or 'semi-convergence') and how you might use it to obtain a 'good' solution to an ill-posed inverse problem. Make sure to describe what you'd expect/want the solutions to look like a) at the very begininning of iteration, b) in the 'middle' phase of iteration, and c) in the very last phase of iteration.