Selected solutions

1 a).

$$\dot{x} = -2x - y \quad (1)$$
$$\dot{y} = x + x^3 \quad (2)$$

(slightly unusual case!)

FP. $(1) = 0 \Rightarrow y = -2x$ ← x nullcline

$(2) = 0 \Rightarrow x(1+x^2) = 0$ ← y nullcline.

$(2) = 0$: $x = 0$ or $x = \pm i$ ✗

⇒ assume real only

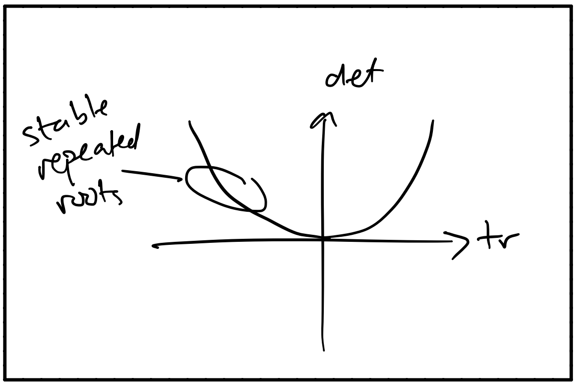
$x = 0$ & $(1) = 0$

$y = 0$

so $\{(0,0)\}$ is only fixed point.

Df. $$Df(x,y) = \begin{pmatrix} -2 & -1 \\ 1+3x^2 & 0 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$$



tr $= -2$

det $= 1$

$\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda+1)(\lambda+1) = 0$

$\Rightarrow \lambda = -1$, repeated! ∴ < deg node / star node } But $Re(\lambda) < 0$, so both stable

Poss: 2 indep. eigenvector ⇒ star (whole plane eig.)

1 indep. eig. vector, 1 indep. generalised eig. vector

→ try!

Cont'd

$$Df(0,0) = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} = A$$

$\lambda = -1.$

$$A - \lambda I = \begin{pmatrix} -2+1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

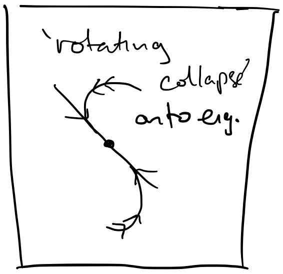
$$(A-\lambda I)u = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$u_1 + u_2 = 0$ (only one LI eqn).

$\Rightarrow u_1 = -u_2$

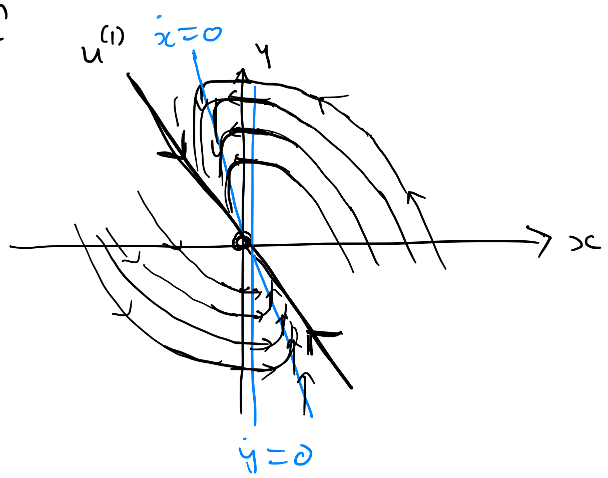
eg $u^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

recall:



- No other LI eigenvector $\Rightarrow$ deg. node. } expect:
- We can draw without finding generalised eigenvector
  $\rightarrow$ do this first!

Sketch



$x = 0$

$\Rightarrow \dot{y} = 0$

$\dot{x} = -y$

$y = -2x$

$\Rightarrow \dot{x} = 0$

$\dot{y} = x(1+x^2)$

$= -\frac{1}{2}y(1+4y^2)$

$y = 0$

$\dot{x} = -2x$

$\dot{y} = x(1+x^2)$

Q: Generalised eigenvector? $\rightarrow$ see linear algebra handout (later)

Idea: solve $(A-\lambda I)^2 u = 0$ ← note

$$(A-\lambda I)^2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 u = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

} I don't really expect you to do!

$\{u^{(1)}, u^{(2)}\}$ form basis.

Stable/unstable

Sketch

$\to$ see class for full

$$\dot{x} = x$$
$$\dot{y} = -y + x^2$$

FP : $(0,0)$

$$Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda_1 = 1, \lambda_2 = -1$$

&

$\lambda_1 = 1$

$$\Rightarrow u^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

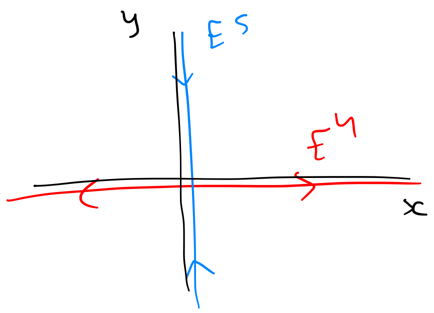
$\lambda_2 = -1$

$$\Rightarrow u^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

verify!

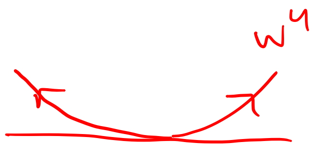
Linear

$$E^u = \{(x,y) \mid y = 0\}$$
$$E^s = \{(x,y) \mid x = 0\}$$



Nonlinear

W^u : $\{(x,y) \mid y = \frac{1}{3}x^2\}$ (see class)



W^s. $x = g(y) = a_0 + a_1 y + a_2 y^2 + \dots$

$a_0 = 0$, $a_1 = 0$ } tangent at $(0,0)$

① $\dot{x} = x = a_2 y^2 + \dots$ $\boxed{\&}$ $\dot{y} = -y + x^2 = -y + \dots$ } note: neglect terms!

② $\dot{x} = \frac{dx}{dy}\dot{y} = (2a_2 y + \dots)(-y) = -2a_2 y^2 + \dots$

$= a_2 y^2$

$\Rightarrow a_2 = 0$! $\Rightarrow x = 0 + \dots$

$$\Rightarrow \boxed{W^s = E^s = \{(x,y) \mid x = 0\}}$$