ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

ORGANISATIONAL

- Assignment 2 (from Mike) due in class on Wednesday 27th April
- Further reading list: see Canvas
- Lectures vs tutorials
- Python (PyDSTool) vs Matlab (MatCont) vs XPPAut

LECTURE 1 - 'BIG PICTURE'

- The 'qualitative' point of view of ODEs (and dynamical systems in general)
- Terminology, basic concepts and examples: state/phase space, solutions, integral curves, flow functions, orbits and vector fields.

QUALITATIVE VIEW OF ODES

We will mainly focus on *ODEs* here (maybe a little bit on discrete systems/maps)

We will take a *dynamical systems* perspective however.

This means we will try to understand the *'qualitative', 'geometric' and 'robust'* features of ODEs rather than obtaining 'exact' analytical solutions (all models are wrong!)

Numerical simulation will be an important complement to help us understand 'emergent' stucture.

WHAT IS A 'DYNAMICAL SYSTEM'?

Informally, a *dynamical system* is a mathematical model of a *process which evolve in time*.

There are *three key ingredients*: a set or interval of *'times'*, possible *'states'* of a 'system' at any given time and an *'evolution rule'* or law governing how the system transititions between these states.

WHAT IS A 'DYNAMICAL SYSTEM'?

Examples are everywhere - ODEs, PDEs, difference equations/maps, stochastic processes even iterative computer algorithms and constructive mathematical proofs.

The *'dynamic' point of view* complements the (also important) *'static' point of view* - e.g. limiting processes vs the actual limits.

ORDINARY DIFFERENTIAL EQUATIONS

We will first look at ODE systems in the form

$$\dot{x} = f(x, t; \mu)$$

where $x \in \mathbb{R}^n$ is a vector of *state variables*, $t \in \mathbb{R}$ is the *independent variable* (usually time), $\mu \in \mathbb{R}^m$ is a vector of *problem parameters*.

(For now we often suppress the dependence on problem parameters - but see bifurcation theory!)

ORDINARY DIFFERENTIAL EQUATIONS

We have one equation in f for each entry in the state vector x e.g.

$$x = (x_1, x_2, \ldots)^T$$

$$f = (f_1, f_2, \ldots)^T$$

If there is no dependence on t then we say the system is *autonomous* (we can always add a new dependent variable to track time dependence). We will focus on these in this course.

EXAMPLE

A differential equation example.

STATE/PHASE SPACE

In practice the *'state'* is defined by *'contains everything we need to know to get from the current state to the next state'*. E.g. position and momentum for classical mechanics.

The *state space/phase space* is...the 'space' of all states - usually (embedded in) $\mathbb{R}^n$, for real-valued differential equations.

(More general phase spaces include the circle, torus etc, depending on the 'problem structure')

EXAMPLE

State space example.

SOLUTIONS AND INTEGRAL CURVES

A *solution* x_s (or trajectory) is a *function* assigning a state vector to each time in a given time interval and which satisfies the ODE, i.e. $x_s : T \subset \mathbb{R} \to \mathbb{R}^n$, where

$$\dot{x_s}(t) = f(x_s(t)).$$

An *integral curve* is the *graph* of a solution, including the time dimension, i.e. the *set of points*

$$\{(x, t) \mid t \in T \text{ and } x(t) \text{ defines a solution}\}$$

FLOW FUNCTIONS

The *flow function* ϕ is a convenient way to combine our description of *solutions and their dependence on initial conditions*.

We write

$$\phi(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

where for fixed x_0, $\phi(x_0,t)$ gives the solution to the differential equation at time t which starts from an initial value (at $t=0$) equal to x_0.

FLOW FUNCTIONS

So, for any t such that $\phi(x, 0) = x$ we have

$$\frac{d}{dt}\phi(x,t) = f(\phi(x,t))$$

and

$$\phi(x, t+s) = \phi(\phi(x,t), s) = \phi(\phi(x,s), t) = \phi(x, s+t)$$

(for any 'allowed' t, s). We talk about 'flows' when we want to emphasise the dependence on initial conditions (rather than just time)

EXISTENCE AND UNIQUENESS?

We won't go into this, but for a sufficiently smooth system written in our standard form and given appropriate initial conditions *there exists a unique solution*.

Upshot: the solution curves/trajectories (for autonomous systems) *do not intersect in phase space*. We always know exactly where to go next!

ORBITS

Orbits are the geometric objects in the phase plane that are generated by solutions/flows.

In terms of the flow function an orbit beginning at x_0 can be described by $\{\phi(x_0, t) \mid t \in T\}$.

Usually we take $T = \mathbb{R}$ and hence consider all solutions passing through x_0 (and both forwards and backwards in time *if* invertible!).

These can be labelled with a 'time direction' but are otherwise somewhat 'static' (geometric) objects.

INVARIANT SETS

A set of points in the phase space M is called *invariant under the flow* if for all $x \in M$ we have

$$\phi(x, t) \in M$$

for all t. That is, every point in M leads to another point in M - once in, we never leave!

EXAMPLE

A comparison of a solution, integral curve and orbit for a simple example.

VECTOR FIELD

The solutions/orbits are *tangent to the 'velocity vector'* $(\dot{x}_1, \dot{x}_2, \ldots)^T$ defined by the ODE $\dot{x} = f(x)$ at each point in the state space (and at each time).

We often call $f(x)$ the *vector field* of the ODE. The *direction* (relative velocity of components) can be determined by dividing through by one (non-zero) component i.e.

$$\frac{\dot{x}_k}{\dot{x}_1} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, \ldots, x_n)}{f_1(x_1, \ldots, x_n)}$$

(Now how this relates the 'static' and 'dynamic' objects).

FEATURES OF INTEREST

We will look at (and define!) various *'interesting features'* of our equations in phase space e.g.

- Stationary/fixed/equilibrium points
- Periodic orbits

and try to analyse their properties such as *stability* under different types of 'perturbations' - both 'within' a model (*solution* stability) and 'externally' (*structural* stability) to a model.

PHASE PORTRAITS

We will summarise these key features in *phase portraits* of a given system.

A phase portrait is a 'picture' of the phase space in which we further *partition it according to orbit/solution 'types' or behaviour* in different regions.

Example.

HOMEWORK

- Look into PyDSTool, MatCont and XPPAut (Google them!)

Reading for fun (see Canvas):

- The dynamical systems approach to differential equations by M.W. Hirsch
- Linear vs. nonlinear and infinite vs finite: an interpretation of chaos by V. Protopopescu

Reading for study:

- see Canvas for some background references.