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Improved Adiabatic Elimination in Laser Equations.

G. L. OPPO
Department of Chemistry, University of Toronto
Toronto, Ontario M5S 1A1, *Canada*

A. POLITI
Istituto Nazionale di Ottica - Largo E. Fermi 6, 50125 *Firenze, Italy*

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Abstract. – An adiabatic-elimination approach based on centre manifold theory is presented, allowing to determine a two-dimensional flow which describes in a unified way different classes of lasers, which range from CO_2 to raser ones. The model displays, for the expected parameter values, Haken second threshold.

The adiabatic-elimination (AE) procedure is used [1] to simplify the equations of motion close to bifurcation points, where the slowing-down of a few variables let the fast ones relax to their instantaneous equilibrium positions. This procedure is particularly successful in those physical systems where some variables undergo fast-relaxation processes. A particular care has to be taken, however, in these cases, first suitably rescaling the variables, and then examining their effective fastness. LUGIATO and co-workers have recently suggested [2] a series of conditions to be fulfilled in order to guarantee the correctness of AE, and showed some examples where it cannot be applied.

In this letter, following the guideline of the centre manifold theory [3], we develop a more general approach, and apply it to the homogeneously broadened resonant laser. The main idea is to look for a global expression of the invariant surface where the slow motion develops, thus overcoming the explicit dependence on the co-ordinates of the standard method. In this way we will be able to handle general cases where no component of the vector field vanishes on the manifold.

The centre manifold theory has been already successfully applied to improve the understanding both of deterministic and stochastic processes: in the former cases greatly simplifying the investigation of bifurcation unfolding [3], in the latter ones yielding more accurate results [1, 4]. In particular, it is to be mentioned a local application to the Lorenz system [5] (isomorphic to the Maxwell-Bloch equations [6]) around one of the two asymmetric fixed points (corresponding to the lasing state).

Here, while referring to the same equations of ref. [5], we perform a global analysis of lasers characterized by small values of

$$\mu \equiv \sqrt{\frac{\gamma_{\parallel}(\gamma_{\perp}+k)}{\gamma_{\perp} k}}\,, \tag{1}$$

where $\gamma_{\parallel}$, $\gamma_{\perp}$, and k are the damping constants of population Δ, polarization P, and electric field E, respectively. Such a class, which goes from CO_2 lasers (good cavity limit) to rasers (bad cavity limit), will be shown to display an adiabatic following between field and polarization variables. The resulting two-dimensional model describes a quasi-conservative dynamics in a Toda potential, and, for the expected parameter values, a Hopf bifurcation (Haken second threshold [7]).

Let us begin with the Maxwell-Bloch equations in the resonant case:

$$\begin{cases} \dot{P} = -\gamma_{\perp}(P - E\Delta)\,, \\ \dot{E} = -k(E-P)\,, \\ \dot{\Delta} = -\gamma_{\parallel}(\Delta - \Delta_0 + EP)\,, \end{cases} \tag{2}$$

where Δ_0 is the pump parameter, $\Delta_0 = 1$ corresponding to the first laser threshold. It is well known that when $\gamma_{\perp}$ is the largest decay constant, *i.e.* $\tilde{\gamma} \equiv \gamma_{\perp}/k \gg 1$, the polarization can be adiabatically eliminated yielding the so-called rate equations. On the other side, when $\tilde{\gamma} \ll 1$ (raser), the electric field can be eliminated yielding a different two-dimensional flow.

Here we will show that both cases are nothing, but special examples of the same, more general, AE procedure. To begin with, we rescale the time variable t to the dimensionless quantity

$$\tau \equiv \frac{\gamma_{\perp} k}{\gamma_{\perp}+k}\, t\,, \tag{3}$$

and introduce a new variable $w = (\Delta - 1)/\mu$. Equations (2) then read as

$$\begin{cases} \dot{P} = -(1+\tilde{\gamma})(P - E - \mu w E)\,, \\ \dot{E} = -(1+\tilde{\gamma}^{-1})(E-P)\,, \\ \dot{w} = \mu\ (D - \mu w - EP)\,, \end{cases} \tag{4}$$

where $D = \Delta_0 - 1$. The smallness of μ seems to suggest the existence of two rapidly contracting directions in the plane (P, E). However, the eigenvalues of the first and second of equations (4) are, for $\mu = 0$,

$$\lambda_1 = 0\,, \qquad \lambda_2 = -(1+\tilde{\gamma})^2/\tilde{\gamma}\,, \tag{5}$$

indicating the existence of only one stable direction. By introducing two new variables,

$$\begin{cases} R \equiv (P-E)/(1+\tilde{\gamma})\,, \\ z \equiv (P + \tilde{\gamma} E)/(1+\tilde{\gamma})\,, \end{cases} \tag{6}$$

along the respective eigendirection, eqs. (4) are rewritten as

$$\begin{cases} \dot{R} = (-(1+\tilde{\gamma})^2/\tilde{\gamma})R + \mu w(z-R) , \\ \dot{z} = \mu w(z-R) , \\ \dot{w} = \mu(D - \mu w - (z-R)(z+\tilde{\gamma}R)) . \end{cases} \tag{7}$$

For $\mu = 0$, the R variable is not coupled to z and w, hence $R = 0$ defines the zeroth-order approximation of the invariant surface. Then, going back to eq. (6), where the centre manifold is defined in terms of physical variables, we notice it can be written as $E = P$, *i.e.* a small value of μ is a sufficient condition to guarantee the adiabatic following between field and polarization variables. Substituting $R = 0$ in the second and third of eqs. (7) yields

$$\dot{E} = \mu w E , \qquad \dot{w} = \mu(D - E^2 - \mu w) , \tag{8}$$

which, with the introduction of $s = \ln E$, are shown to be equivalent to a Toda oscillator with a small linear damping term. Therefore, three well-separated time scales exist in the motion described by eqs. (2): on the shortest one ($O(1)$), laser variables approach the invariant manifold; on the middle one ($O(1/\mu)$), a conservative motion is followed in a Toda potential; on the longest one ($O(1/\mu^2)$), the system relaxes toward the equilibrium value.

Hence, first-order corrections to centre manifold $R = 0$ turn out to be structurally relevant in so far as they guarantee an exact computation, up to order μ, of damping terms. Before determining such corrections, we notice that eqs. (7) are well suited for an application of the centre manifold theory. Indeed, their structure is of the type

$$\begin{cases} \dot{\boldsymbol{x}} = B\boldsymbol{x} + \mu f_x(x, y) , \\ \dot{\boldsymbol{y}} = A\boldsymbol{y} + \mu f_y(x, y) , \end{cases} \tag{9}$$

whit $\boldsymbol{x} = R$, $\boldsymbol{y} = (z, w)$ and A, B are linear operators, the eigenvalues of A having zero real parts. The centre manifold theory [3] guarantees the existence of an invariant manifold $R(z, w)$ of order μ for z, w values $O(1)$. Its expression is easily evaluated after substitution of eqs. (7) in the time derivative of $R(z, w)$

$$\dot{R} = \frac{\partial R}{\partial z}\dot{z} + \frac{\partial R}{\partial w}\dot{w} . \tag{10}$$

Retaining only first-order terms in μ, we obtain

$$R = \frac{\tilde{\gamma}\mu w z}{(1+\tilde{\gamma})^2} . \tag{11}$$

The subsequent substitution of eq. (11) in eqs. (7) yields, neglecting μ^2 terms and rescaling time variable by a factor μ, the general two-dimensional flow

$$\begin{cases} \dot{z} = wz\left(1 - \dfrac{\mu w\tilde{\gamma}}{(1+\tilde{\gamma})^2}\right) , \\ w = D - \mu w - z^2\left(1 + \dfrac{\mu w\tilde{\gamma}}{(1+\tilde{\gamma})^2}(\tilde{\gamma} - 1)\right) , \end{cases} \tag{12}$$

which describes the laser dynamics under the only assumption of small μ. In studying the stability properties of eqs. (12), we prefer, instead of referring to the abstract variable z, to go back to the more physical one $E = z - R$, and introduce again $s = \ln E$.

$$\ddot{s} = D - \exp[2s] - \mu\dot{s}\left[1 + \frac{2D\tilde{\gamma}}{(1+\tilde{\gamma})^2} + \frac{\tilde{\gamma}(\tilde{\gamma}-3)}{(1+\tilde{\gamma})^2}\exp[2s]\right]. \tag{13}$$

The general relevance of Toda potential in laser systems, then clearly stands out from its independence of laser type (strength of $\tilde{\gamma}$). The dependence is, indeed, entirely contained in the $\dot{s}$ term, which changes from $(1 + \exp[2s])$ for $\tilde{\gamma} \gg 1$, as already found in ref. [8], to a constant in the opposite case $\tilde{\gamma} \ll 1$. The further analysis for generic $\tilde{\gamma}$ values shows the existence of a Hopf bifurcation: the coefficient of $\dot{s}$ can, indeed, change sign depending on the field intensity $\exp[2s]$. More precisely, substituting the equilibrium value $\exp[2s] = D$ in the $\dot{s}$ term, we see that it vanishes for

$$D = \frac{(1+\tilde{\gamma})^2}{\tilde{\gamma}(1-\tilde{\gamma})}, \tag{14}$$

a value in perfect agreement with Haken second threshold [7]. Finally, we investigated the nature (direct-reverse) of Hopf bifurcation. This is easily done by introducing the pseudo-energy $H = (\dot{s}^2 + \exp[2s])/2 - Ds$ and evaluating its variation ΔH over an oscillation period T. Recalling that, up to order μ, $\ddot{s} = D - \exp[2s]$, we have

$$\Delta H = \mu\int_0^T \dot{s}^2\left[1 + \frac{\tilde{\gamma}D(\tilde{\gamma}-1)}{(1+\tilde{\gamma})^2}\right]\mathrm{d}t. \tag{15}$$

Since the expression in square brackets coincides with the $\dot{s}$ term of eq. (13), once $\exp[2s]$ has been substituted with D, a global stability analysis yields the same results as the local one. Hence, if higher-order terms in μ are needed to decide on the bifurcation unfolding, it is at least possible to conclude that a small pump variation is sufficient to lead to an overall instability of the phase space. Obviously, such divergence does not reflect a similar phenomenon in Maxwell-Bloch equations: for $\exp[2s] > 1/\mu$ first-order approximation of centre manifold fails and a more detailed analysis is required for very large field values.

However, the relevance of eq. (13) remains in so far as it provides a more accurate model for instance for CO_2 lasers with nonirrelevant modification to losses terms for realistic physical parameters ($3 \lesssim \tilde{\gamma} \lesssim 10$).

In conclusion, we have presented a constructing way to perform the AE even in the limit cases $\gamma_\parallel \ll k \ll \gamma_\perp$, $\gamma_\parallel \ll \gamma_\perp \ll k$, where the standard approach fails as shown in ref. [2].

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REFERENCES

[1] H. HAKEN: *Advanced Synergetics* (Springer, Berlin, 1983).
[2] L. A. LUGIATO, P. MANDEL and L. M. NARDUCCI: *Phys. Rev. A*, **29**, 1438 (1984).
[3] J. CARR: *Applications of Center Manifold Theory* (Springer, Berlin, 1981).
[4] A. FERNANDEZ: *Phys. Rev. A*, **32**, 3070 (1985).
[5] M. MIARI: *Physica D*, to be published.
[6] H. HAKEN: *Phys. Lett. A*, **53**, 77 (1975).
[7] H. HAKEN: *Z. Phys.*, **190**, 327 (1966); H. RISKEN, C. SCHMID and W. WEIDLICH: **194**, 337 (1966).
[8] G. L. OPPO and A. POLITI: *Z. Phys. B*, **59**, 111 (1985).