MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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NEXT MODULE

2. *Expansions in orthogonal functions: Fourier series* [4 lectures]

Orthogonality of functions/sets of functions and series expansions. Real trigonometric series. Convergence and sketching Fourier series. Complex Fourier series. Use in separation of variables.

RECALL

We found an *infinite number of solutions* to the heat equation of the form

$$u_n(x,t) = A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$
$$n = 1, 2, \ldots$$

And, since we had a *linear* equation, our general solution was constructed as a sum of these *fundamental solutions* (or 'modes'), i.e.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$

where...

RECALL

...the A_n describe *how much* each fundamental solution contributes to the solution of our *particular* problem and are *determined by the initial conditions*.

We then used our IC to determine the the constraint on A_n values:

$$u(x, t = 0) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n sin(n\pi x) = g(x)$$

For a known IC - or 'initial stimulus' $g(x)$.

RECALL

Determining the A_n from an expression like

$$\sum_{n=1}^{\infty} A_n sin(n\pi x) = g(x)$$

requires us to learn some new mathematics - orthogonal functions and Fourier series...this module!

LECTURE 4 BASIC CONCEPTS FOR FOURIER SERIES

Orthogonality of functions

Series expansions and classical/trigonometric Fourier series

Convergence theorem

EASIER THAN WE THOUGHT

Determining the A_n from an expression like

$$\sum_{n=1}^{\infty} A_n sin(n\pi x) = A_1 sin(\pi x) + A_2 sin(2\pi x) + \ldots = g(x)$$

is *actually pretty easy*!

ORTHOGONALITY

It turns out that the following *orthogonality* property holds:

$$\int_0^1 sin(m\pi x)sin(n\pi x)dx = \begin{cases} 0, & \text{if } m \neq n \\ 1/2, & \text{if } m = n \end{cases}$$

PROOF

Use

$$sin(mx)sin(nx) = \frac{1}{2}[cos((m - n)x) - cos((m + n)x)]$$

CALCULATION

We can use orthogonality to get

$$A_m = 2\int_0^1 g(x)sin(m\pi x)$$

Which we can calculate (in principle) and substitute back into

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$

Done!

EXAMPLE

What would our A_m look like if

$$g(x) = \frac{1}{3} sin(3\pi x) + 7 sin(5\pi x)$$

?

WHERE DID THIS COME FROM?

We have previously used an analogy with finite-dimensional vector spaces for thinking about differential equations and differential operators.

It turns out that many sets of functions often do in fact form *infinite-dimensional vector spaces*. Some examples*:

- $C_p[a, b]$ the set of all real-valued piecewise-continuous functions defined on $[a, b]$
- $L^2[a, b]$ the of all real-valued function defined on $[a, b]$ that are square-integrable

$*$: In contrast to finite-dimensional problems the appropriate choice of function space is usually problem-specific. Note we define $f + g = f(x) + g(x)$ and $\alpha f = \alpha f(x)$ where the LHS is the vector operation and the RHS defines it in turns of ordinary function calculations at a point.

WHERE DID THIS COME FROM?

As in linear algebra, we can introduce extra structure into our 'function space'. Here we have introduced an *inner product**

$$< f, g > = \int_a^b f(x)g(x)dx$$

We see that our orthogonality property of sin functions *works just like orthogonality of vectors*, i.e.

$$< f, g > = 0 \equiv f \text{ and } g \text{ are orthogonal}$$

$*$: Check that this satisfies the definition of an inner product! We can also introduce an extra weight function $w(x)$ into this definition.

ORTHOGONAL FOURIER BASIS

More generally the set

$$\{1, cos\frac{\pi x}{l}, cos\frac{2\pi x}{l}, \ldots, sin\frac{\pi x}{l}, sin\frac{2\pi x}{l}, \ldots, \}$$

is an orthogonal set of functions* on $[-l, l]$.

We want to understand how to represent different types of functions using expansions in this orthogonal set of functions and when to expect it to work.

$*$: An orthogonal set of functions/vectors is one where each function/vector is orthogonal to all of the others, except itself.

INFINITE SERIES EXPANSIONS - REMEMBER TAYLOR?

You should already be familiar with the idea of expanding a function in other infinite series such as the Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $a_n = f^{(n)}(x_0)/n!$

The Taylor series is typically good for local approximations (i.e. near x_0). It is *not* an orthogonal expansion btw.

NOTE ON NOTATION

We can also write

$$\text{TS } f|_{x=x0} = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

to denote the RHS series *without implying that it equals* (converges to) the LHS $f(x)$ on the previous slide.

SERIES EXPANSIONS - THE TRIGONOMETRIC (CLASSICAL FOURIER) SERIES

As noted, our sin expansion was a special case of the trigonometric - or classical Fourier - series which will involve expansions of the form

$$a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

for a given domain 'length' parameter l.

We will find that the series represents functions more 'globally' over the domain $[-l, l]$ (or over the whole domain for periodic functions).

FORMAL DEFINITION

Given a function $f : [-l, l] \to \mathbb{R}$, the trigonometric (classical Fourier) series* is defined as

$$\text{FS} f = a_0 + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{l}) + b_n sin(\frac{n\pi x}{l})]$$

where ...

$*$: We will see that a *Generalised Fourier series* includes expansions in other orthogonal sets of functions besides trigonometric functions.

FORMAL DEFINTION

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x)cos(\frac{n\pi x}{l})dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x)sin(\frac{n\pi x}{l})dx$$

$n = 1, 2, \ldots$ are called the *Fourier coefficients*.

CONVERGENCE

Note that we *haven't said anything about convergence here.*

The following convergence theorem will be of interest to us.

CONVERGENCE THEOREM

Let f be a periodic function with fundamental period $2l$ such that f and f' are both piecewise continuous (i.e. f is piecewise smooth) on $[-l, l]$.

Then the Fourier series FS f of f converges to

- $f(x)$ at each point x at which f is continuous, and to
- the mean value $(f(x^+) + f(x^-))/2$ at every point x at which f is discontinuous, where $f(x^+)$ and $f(x^-)$ are the right- and left-hand (i.e. one-sided) limits, respectively.

SOME MORE DEFINITIONS

To understand the previous theorem we need some *definitions and facts*. I will list them here - make sure you are OK with all of these.

We will also get a feel for the theorem through doing *examples*.

CONTINUITY DEFINITIONS

A function is *piecewise continuous* on $[a, b]$ if it is continuous except at a finite number of points in $[a, b]$, where it has simple jump discontinuities.

f has a *simple jump discontinuity* at $x = c$ if both one-sided limits of $f(x)$ exist and are finite at c.

A function f is called *piecewise smooth* if both f and f' are piecewise continuous on $[a, b]$.

LIMIT DEFINITIONS

The *right-hand limit* $f(x^+)$ is defined by

$$f(x^+) = \lim_{h \to 0^+} f(x+h)$$

i.e. h approaches zero through positive values.

The left-hand limit is defined analogously. These are also called *one-sided limits*.

PERIODIC, ODD, EVEN DEFINITIONS

A function f is

- *periodic* with period T if
$$f(x+T) = f(x)$$
- *even* if
$$f(-x) = f(x)$$
- *odd* if
$$f(-x) = -f(x)$$

The smallest T for which a function is periodic is called its *fundamental period*.

PROPERTIES OF EVEN AND ODD FUNCTIONS

even + even = even

even x even = even

odd + odd = odd

odd x odd = even

even x odd = odd

[Exercise: prove from definitions]

MORE PROPERTIES OF EVEN AND ODD FUNCTIONS

$$\int_{-a}^{a} f(x) = 2 \int_{0}^{a} f(x)$$

if f is *even*

and

$$\int_{-a}^{a} f(x) = 0$$

if f is *odd*

DECOMPOSITION OF AN ARBITRARY FUNCTION IN EVEN AND ODD PARTS

An arbitrary function f may be written in terms of an *even part* and an *odd part* using

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$\equiv f_e(x) + f_o(x)$$

HOMEWORK

Calulate the following integrals

$$\int_{-l}^{l} sin(\frac{\pi x}{l})dx$$

$$\int_{-l}^{l} cos(\frac{n\pi x}{l})dx, \ \ n = 0, 1, 2, \dots$$

Go over the list of definitions at the end of this lecture

Go over integration by parts

EXTRA HOMEWORK

Look up the definition of a vector space

Look up the definition of an inner product vector space and orthogonality

Check that our inner product satisfys the required properties of an inner product