Engsci 213 Markov Chain Problems - Set 2

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Topics

So far we've covered the main foundations - Parts 2 of 3 - of Markov chains:

1. *Basic concepts* [2 lectures]
   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions. Diagrams of Markov chains.
2. *Properties of Markov chains* [2 lectures]
   Accessibile, recurrent, transient states. Communication of states. Stationary/invariant distributions and limiting/equilibrium distributions. Estimating transition matrices. Expected value calculations using invariant/limiting distributions.

The remaining lectures will just cover some examples of modelling with/applying Markov chains. You might be expected to outline or fill-in some steps of some of this material on modelling/applying Markov chains but the main material to be examined is covered in Lectures 1-4.

So...let's do some more examples covering this core material!

States: accessibility, communication, trapping and absorbing

Problem one

Re-consider the note-passing example from the previous tutorial with the following transition matrix for the stochastic dynamics of note passing:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.75 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.0 & 0.0 \\ 0.25 & 0.0 & 0.25 & ? & 0.0 & 0.25 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & ? \end{bmatrix}$$

- Fill in the two missing '?' values in the transition matrix

- Draw a state transition diagram (using your choice of labels for the state space)
- Find any trapping sets or absorbing states
- Determine the classes of the Markov process

Problem two

Consider a Markov process with state space $\mathbb{X} = \{1, 2, 3, 4\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0.0 & 0.0 \\ 0.25 & 0.75 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.0 & 0.25 & ? \end{bmatrix}$$

- Fill in the missing entry
- Draw a state-transition diagram
- Determine any trapping sets and absorbing states
- Determine the classes
- Which states does state 4 communicate with? Which states communicate with state 4?

Distributions: stationary/invariant and limiting/equilibrium

Problem one

Given the following transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

- Find an invariant distribution by solving the eigenvalue problem (i.e. find the eigenvector)
- Verify that the distribution is in fact invariant
- Describe the difference between an invariant and a limiting distribution.
- Suppose the distribution you found above is also a limiting distribution. Write down the form the limiting transition matrix $\mathbf{P}^n, n \to \infty$ takes.

Problem two

Given the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Find a stationary distribution (note - I originally said that I would only ask you to find 2D invariant distributions but a) this case is nice and simple and b) since we don't have to solve for the eigenvalues things are generally quite simple. I therefore reserve the right to ask you to find invariant distributions for 3D cases too!).
- Calculate $\mathbf{P}^2$, $\mathbf{P}^3$ and $\mathbf{P}^4$.
- Is the stationary distribution you found also a limiting distribution?

Problem three

Given the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & ? \end{bmatrix}$$

- Fill in the missing entry
- Find the stationary distribution
- Describe how you might check whether this is also a limiting distribution

Expected value calculations and inferring transition matrices from observations

Problem one

Consider a Markov chain with state space $\mathbb{X} = \{0, 1, 2, 3, 4\}$. Suppose you decide on a 'value' function $V = 2j + j^2$ for each state $j \in \mathbb{X}$.

- Calculate the expected value of V under the distribution $\pi = (1/12, 2/12, 3/12, 5/12, 1/12)$.
- Calculate the variance of V under π.

Problem two

Consider the sequence of observations taking values in $\{1, 2, 3\}$

$$1, 2, 1, 1, 1, 2, 3, 1, 1, 2, 1, 3, 1, 1, 3, 2$$

- Assuming the underlying process is a Markov chain, write down an expression that you could use to estimate the transition matrix
- Estimate the transition matrix

Problem three

Suppose you observed a process assumed to be Markov and found the following matrix of counts

$$\mathbf{P} = \begin{bmatrix} 412 & 54 & 165 \\ 54 & 19 & 25 \\ 165 & 24 & ? \end{bmatrix}$$

- Is it possible to determine the missing entry? Why/why not?
- Given the missing entry is 143, determine the associated transition matrix
- If the data from this question are observations of the same process as in the previous question, which matrix would you prefer to use to estimate the true process? You can justify this informally.
- Use a calculator or computer to compare the entries of the two transition matrices (this one and the previous one). Do they give similar answers?

For the above question you should decide on some reasonable criteria for comparing the two matrices (this is called specifying a matrix *norm* $|\ |$ and measuring the distance $|\mathbf{P} - \mathbf{P'}|$ for two matrices $\mathbf{P}$ and $\mathbf{P'}$).

Bonus problems

Problem one

Consider a Markov process $X_0, X_1, ...$ with state space $\mathbb{X} = \{0, 1, 2\}$ and a transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0.0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$$

- Assuming $\mu_0 = (0.3, 0.4, 0.3)$, find the joint probabilities $P(X_0 = 0, X_1 = 1, X_2 = 2)$ and $P(X_0 = 0, X_1 = 1, X_2 = 1)$. Hint: use the Markov property.