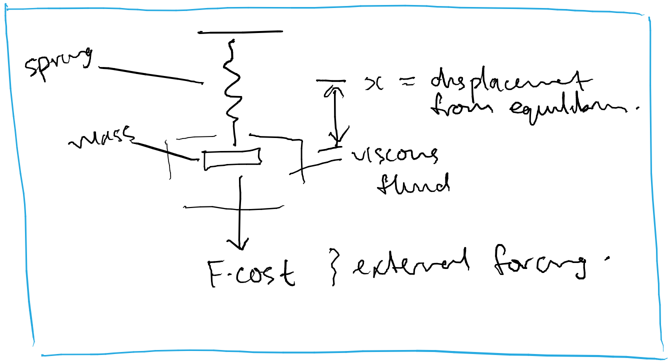
Engsci 711 L2 Examples.

State space

- see Q6b 2016 Exam in last handout.
- Strogatz (1994 Ed) section 1.2: Harmonic Osc.

Given

$$\underbrace{m\frac{d^2x}{dt^2}}_{\text{mass}\times\text{accel.}} + \underbrace{b\frac{dx}{dt}}_{\text{damping force}} + \underbrace{kx}_{\text{spring force}} = \underbrace{F\cdot\cos t}_{\text{external force}}$$



Q: what is the 'state' / 'state space'?

A: Rewrite as system of first-order ODEs

Trick: $x_1 = x,\ x_2 = \dot{x}$ } one less than highest derivative

$$\Rightarrow \quad \dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = \frac{1}{m}\left[-b\dot{x} - kx + F\cos t\right]$$

$$= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{F}{m}\cos t = f(x_1, x_2, t)$$

$\Rightarrow$

we have

$$\dot{x}_1 = x_2 = f_1(x_1, x_2) \checkmark$$

$$\dot{x}_2 = \frac{1}{m}\left[-bx_2 - kx_1 + F\cos t\right] = f_2(x_1, x_2, t)$$

we want to elim. this too.

into $x_3 = t$

$\Rightarrow \dot{x}_3 = 1$

(allows us to convert dynamics to 'static' geometric picture.)

so:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \frac{1}{m}\left[-bx_2 - x_1 + F\cos x_3\right] \\ 1 \end{Bmatrix}$$

$\Rightarrow$ 3-dimensional state space

- state vars are x_1, x_2, x_3
- evolution rule is $\frac{d}{dt}(\text{state})$

Note: x_1 : displacement
x_2 : velocity
x_3 : time

Solutions, integral curves, orbits, vector fields

Based on Wiggins 2003 example 0.0.1.

Consider $\dot{u} = v$ $\quad (u,v) \in \mathbb{R}^1 \times \mathbb{R}^1$

$\dot{v} = -u$

1. Find the
   a) - solution with $(u,v) = (1,0)$ at $t=0$
   b) - integral curve with $(u,v) = (1,0)$ at $t=0$
   c) - orbit that includes $(u,v) = (1,0)$

   & sketch them graphically

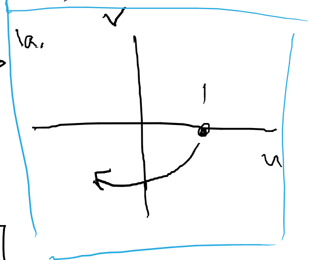
2. Sketch the vector field in the u-v plane.

1a). The solution is a function $t \mapsto (u(t), v(t))$

Here it is

$(u(t), v(t)) = (\cos t, -\sin t)$.

pic



[verify: $\dot{u} = -\sin t$, $\dot{v} = -\cos t$

$\Rightarrow \dot{u} = v$ ✓, $\dot{v} = -u$ ✓

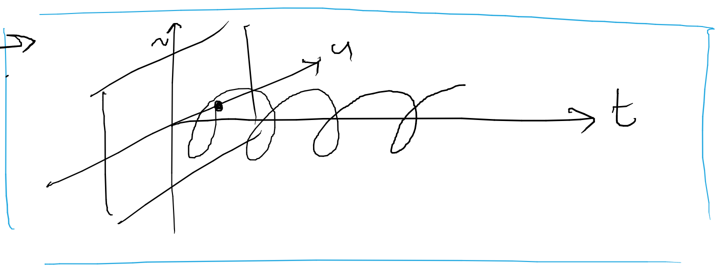
& $\cos 0 = 1$, $\sin 0 = 0$ ✓]

[$\varphi((1,0), t)$] in terms of flow fnct.

The integral curve is given by the set

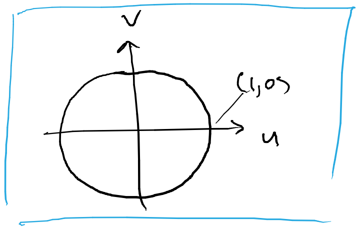
$\{(u,v,t) \mid u(t) = \cos t, v(t) = -\sin t \;\; \forall t\}$

pic



1c) Orbit is the set of points in u,v space generated by all traj. passing through $(1,0)$.

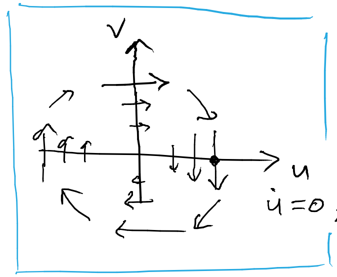
convenient rep: $\{(u,v) \mid u^2 + v^2 = 1\}$



2. vector field: think local tangent arrows in u-v plane.

Note $\frac{\dot{v}}{\dot{u}} = \frac{dv}{du}$ = slope of v vs u.

$= -\frac{u}{v}$ here (for $v \neq 0$)



plug in points in general

[calc $\dot{u}$ & $\dot{v}$ at various points.]

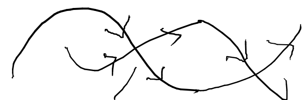
$\dot{u} = 0$, $\dot{v} = -1$ at $(u,v = 0,1)$

Existence & Uniqueness.

"Always know where to go next"

$\Rightarrow$ no intersecting / crossing trajectories

NO:



can't keep going

[unless stop fixed point]

Phase Portrait

→ Plot interesting features, trajectories etc in phase space.

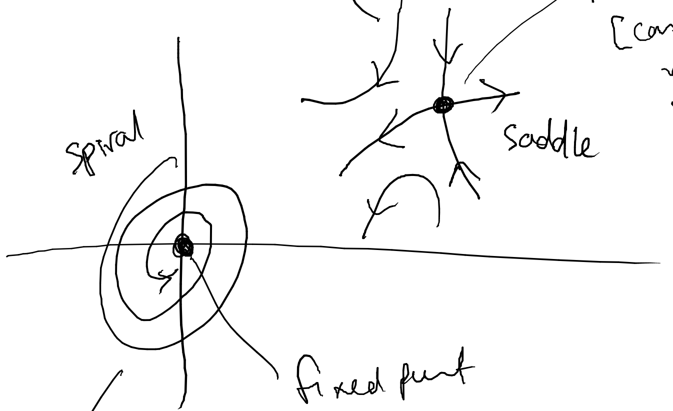
→ will often be different regions of interesting behaviour

unstable diverge away from FP.

fixed point [can't cross, must stop / approach inf. slowly]

Saddle

spiral



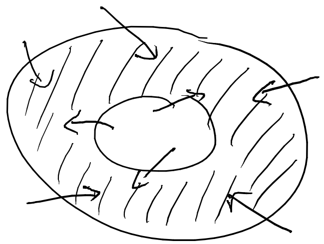
fixed point

↳ if start exactly there then don't move

stable decay towards FP

Invariant & Limit Sets.

Invariant set : trapping region (see eg Strogatz 1994, Fig 7-3.2)



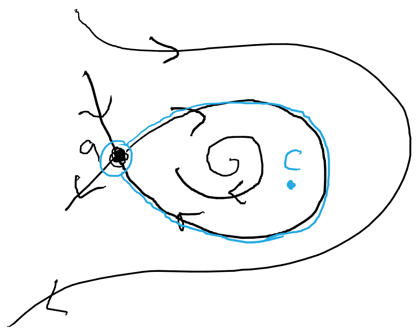
flow is always into region

Invariant set : {Fixed point}

$$\phi(x^*, t) = x^* \quad \forall t.$$

so $\{x^*\}$ is invariant.

ω-limit set (forward limit set)



$\omega(C)$?

- flow approaches a arb. closely.
- also the orbit connecting a to itself ('homoclinic orbit')

⇒ {points indicated in blue},

{a + homoclinic orbit}.