Engsci 711

Tutorial 2: Full phase plane analysis. Calculating manifolds.

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Overview

The purpose of this tutorial is to give you some practice analysing (mainly) two-dimensional systems by hand from woah to go.

Tips and tricks

Analysis procedure

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* (and/or the *trace* and *determinant*) of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).
- Sketch/compute the *phase portrait*.

We will often do the above without calculating any nonlinear corrections for the stable/unstable manifolds, or without even calculating the linear manifolds, but you should also be able to do this if asked. Later we'll look at calculating centre manifolds too.

Tips for phase portrait sketching

- Start drawing locally near individual fixed points
- Check for any obvious invariant axes, lines/curves
- Draw nullclines and any other helpful/obvious flow directions (e.g. trapping regions)
- Think about whether periodic orbits might exist anywhere
- Think about various ways the local flows might connect up or be extended more globally.

Calculating manifolds

Consider a system such as $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$. We usually use our information (give or take a swap of x, y variables etc) to construct series expansions for stable/unstable manifolds as follows

- Assume the manifold can be described by a *functional relationship* such as $y = h(x)$ (or $x = g(y)$).

- Substitute this functional relationship into our x and y equations to give e.g. $\dot{x} = f_1(x, h(x))$ and $\dot{y} = f_2(x, h(x))$.

- Use the functional relation again, along with the chain rule for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dh}{dx}\dot{x}$.

- Use the above relationships along with an assumed power series expansion such as $h(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain two polynomial expressions in x (say) for $\dot{y}$ involving the unknown coefficients of the power series. Equate powers of x to determine the coefficients.

- You will need to use the information that the stable/unstable manifold passes through the fixed point and is tangent to the linearised stable/unstable manifold to determine the first two terms of the series. These will not be zero in general (but should be known)!

Phase plane analysis

1. Find the fixed points of the following equations, determine their stability and sketch their phase portraits

$$\begin{aligned}
&(a)\quad \dot{x} = -3x - 2y, \quad \dot{y} = x + x^3 \\
&(b)\quad \dot{x} = x + y - 2x^2, \quad \dot{y} = -2x + y + 3y^2 \\
&(c)\quad \ddot{x} + \sin x = 0 \\
&(d)\quad \dot{x} = x(1 - x - y), \quad \dot{y} = y(2 - x - y) \\
&(e)\quad \dot{x} = -x + 4y, \quad \dot{y} = -x - y^3
\end{aligned}$$

For the first case, repeat the analysis for when $\dot{x} = -2x - y$ (refer to the Strogatz material in Lecture 4 and the linear algebra handout from Lecture 5 if you get stuck).

For the second case you can just analyse near the origin as the second fixed point is a bit ugly.

For the last case, prove or disprove the existence of periodic solutions.

2. (Exam 2018) Consider the system:

$$\begin{aligned}
\dot{x} &= x^2 - y - 1 \\
\dot{y} &= (x - 2)y
\end{aligned}$$

where $x, y \in \mathbb{R}$.

a) Find and classify all three of the equilibria of the system. You do not need to draw any pictures (yet) or find any eigenvectors.

b) Write down the equations for the x— and y—nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields *on* the nullclines in your sketch.

c) The nullclines above separate the phase-plane into eight regions, each of which has one of four different qualitative flow directions: $\boxed{\nearrow}$, $\boxed{\searrow}$, $\boxed{\swarrow}$, $\boxed{\nwarrow}$. Add boxed arrows such as these to your sketch to indicate the direction field in each region of phase space (you can use the information you determined in the previous part of this question without further justification).

(Note: there are very similar questions in the 2016, 2017 and 2019 exams if you want more example questions!)

3. Here are two simple 'inverse problems': rather than giving you equations to determine the behaviour of, here you need to construct appropriate equations given desired solution behaviour.

a) Write down a system of differential equations

$$\begin{aligned}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{aligned}$$

such that it has two and only two fixed points: one at $(x, y) = (2, 1)$, and one at $(x, y) = (-2, 1)$.

b) Write down a system of differential equations

$$\begin{aligned}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{aligned}$$

such that both the x and y axes are invariant under the flow, some orbits tend to the origin as $t \to \infty$ and all other orbits have an x-component tending to $+\infty$ as $t \to \infty$.

For this last case, verify your system has these properties by plotting it using XPP. Include the nullclines (XPP can draw these for you!). Warning: Here you *may* need to consider non-hyperbolic systems!

Approximating stable/unstable manifolds

Short

1. Complete the example from the lecture, i.e. find the stable manifold $W^s_{loc}(0)$ for

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y + x^2\end{aligned}$$

2. Consider

$$\begin{aligned}\dot{x} &= 2x + y^2 \\ \dot{y} &= -y\end{aligned}$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ to all orders.

Long

(Warning: somewhat long and tedious - but worth attempting. Can also use symbolic math in Matlab/Python/Wolfram alpha to help…).

3. Consider the system

$$\begin{aligned}x' &= -2x - 3y - x^2 \\ y' &= x + 2y + xy - 3y^2\end{aligned}$$

- Find and classify the equilibria.
- Find the power series expansions for $W^u_{loc}(0), W^s_{loc}(0)$ up to (i.e. including) cubic order.

Periodic orbits, trapping regions etc.

1. Work through Strogatz (1994) Examples 7.3.2 and 8.3.1 on constructing trapping regions (see the Lecture 7 handout).
2. See if you can determine whether the following systems have any periodic orbits.

a)

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= (x^2 + 1)y - x^5\end{aligned}$$

b)

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= y^2 + x^2 + 1\end{aligned}$$

c)

$$\begin{aligned}\dot{x} &= 1 + x^2 + y^2 \\ \dot{y} &= (x-1)^2 + 4\end{aligned}$$

3. (Warning - relatively challenging/involved question). Consider the system

$$\begin{aligned}x' &= x - y - x(x^2 + 2y^2) \\ y' &= x + y - y(x^2 + y^2)\end{aligned}$$

- Re-write the system in polar coordinates (r, θ) where $x = r\cos\theta, y = r\sin(\theta)$. Hint: use the identities $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.

- Determine a region bounded by two circles (i.e. annulus shaped), each of which are centred at the origin, such that the flow is radially outward on the inner circle and radially inward on the outer circle.
- Show that there is a periodic orbit in this region, i.e. for $r_{inner} \leq \sqrt{x^2+y^2} \leq r_{outer}$.
- Use XPP (or Matlab or Python etc) to plot the periodic orbit (and some neighbouring orbits) in the phase plane.

4. Look up the definition of Liapunov/Lyapunov functions (e.g. in Strogatz 1994 Section 7.2). When available, these give a general method for proving global stability of fixed points. The basic idea is to find 'energy-like' functions that are monotonically decreasing along trajectories.

- Explain how these could be used as another method to rule out the existence of periodic orbits.

Applications and extensions

Interacting populations - background

Many interesting nonlinear models arise from applications involving two or more interacting populations or species, e.g. animals/people or chemical species. These models can often be derived using a combination of:

- Balance/conservation/interaction 'structural' equations with 'placeholder' fluxes or forces
- 'Constitutive' or rate models to give specific expressions for the fluxes or forces.

A simple example would be combining conservation of mass and the 'law' of mass action to get equations for chemical kinetics. For example, given the reaction $R : A + B \to C$, we start with mass conservation:

$$\begin{aligned}\frac{dA}{dt} &= -J\\ \frac{dB}{dt} &= -J\\ \frac{dC}{dt} &= +J\end{aligned}$$

where J is the 'flux' of the reaction R. Each 'step' dt of R reduces A and B by Jdt and increases C by Jdt. Then we model the 'law' for J via mass action kinetics to give $J = kAB$ and

$$\begin{aligned}\frac{dA}{dt} &= -kAB\\ \frac{dB}{dt} &= -kAB\\ \frac{dC}{dt} &= kAB\end{aligned}$$

This sort of reasoning can be extended to many types of interacting systems, including those for which 'conservation' laws are not obvious but we can still think about differential equations and rate laws capturing the key 'interactions'. Common examples include the *predator-prey* model (y eats x):

$$\begin{aligned}\dot{x} &= -kxy\\ \dot{y} &= kxy,\end{aligned}$$

the *competitive species* model (x and y have negative, and potentially asymmetric, influences on each other):

$$\begin{aligned}\dot{x} &= -k_1xy\\ \dot{y} &= -k_2xy,\end{aligned}$$

and the *symbiotic species* model (x and y have positive, and potentially asymmetric, influences on each other):

$$\begin{aligned}\dot{x} &= k_1 xy\\ \dot{y} &= k_2 xy.\end{aligned}$$

You can often think of the above interactions in terms of generalised 'reactions', e.g. $X + Y \rightarrow 2Y$ for 'y eats x to grow'. Or 'y converts x to more y'.

When putting together multiple influences, another common model component is pure growth, e.g. $\dot{x} = kx$. Logistic growth is given by e.g.

$$\dot{x} = k_1 x\left(1 - \frac{1}{k_2}x\right) = k_1 x - \frac{k_1}{k_2}x^2$$

with 'carrying capacity' $x_c = k_2$. This growth model can hence be thought of as being made up of a 'pure' growth component and a negative self interaction component (e.g. due to overcrowding).

Further reading on interacting population models can be found in the Strogatz and Glendinning handouts at the end of Lecture 6.

Interacting populations - examples

1. Consider the following system

$$\begin{aligned}\dot{x} &= x(4 - x - 2y)\\ \dot{y} &= y(2 - x - y)\end{aligned}$$

- What sort of real-world system might this set of ODEs be used as a model for? Hence give an interpretation for the terms on the RHS of the equations (e.g. as growth, symbiotic interaction, competitive interaction terms etc). Hint: see Lecture 6 handouts.
- Find the fixed points and determine their stability.
- Determine the eigenvectors of the fixed points.
- Write down the equations for the $x-$ and $y-$nullclines. Sketch these in the phase plane. Include the equilibria you found above and the direction fields *on* the nullclines in your sketch.
- Complete the sketch above by adding some possible compatible trajectories.
- Use Dulac's criterion to disprove the existence of periodic solutions in the positive x, y quadrant.

Another common application is modelling the spread of disease in populations. The next question considers this.

2. Kermack and McKendrick (1927) were among the first to mathematically model epidemics, and applied their models to real data such as observations from the Bombay plague epidemic of 1905–1906. In addition to more complicated models, they proposed the following simple model:

$$\begin{aligned}\dot{S} &= -rSI\\ \dot{I} &= rSI - aI\\ \dot{R} &= aI\end{aligned}$$

where $r, a > 0$ are rate parameters, and $S, I, R \geq 0$ represent susceptible, infected, and recovered (or 'removed') populations of people, respectively. Hence this model is often called an SIR model. This model assumes that recovered people cannot be reinfected in contrast to, for example, an SIS (susceptible-infected-susceptible) model.

Typical initial conditions are $S(0) = S_0 > 0,\ I(0) = I_0 > 0,\ R(0) = 0.$

- Give an interpretation of the terms in the differential equation system in terms of interacting populations and their kinetics. Interpret r and a.
- This model in fact has a whole *line* of non-hyperbolic equlibria for $I = 0$. Verify this. We haven't analysed this type of system yet, but we can still make some progress for now as follows.

- Show that the total population size, $N = S+I+R$, is constant. This (and the fact that the first two equations are independent of R) implies that we can analyse the key features of the system as a *two-dimensional* system in the $S-I$ plane. Further show that $S(t)+I(t) < S_0 + I_0$ for all $t > 0$.
- Let (S_0, I_0) be the initial condition in the reduced $S-I$ system. An *epidemic* is (formally) said to occur if $I(t)$ increases initially. By considering $\frac{dI}{dt}(t=0)$, determine a threshold S_0 value for an epidemic occur. Call this ρ. This is known as the *relative removal rate*.
- Use the above to define a parameter $\mathcal{R}_0$ such that $\mathcal{R}_0 > 1$ implies an epidemic will occur. This is called the *basic reproduction number*. It can be interpreted as 'the average number of secondary cases caused by a primary case in an entirely susceptible population'. The definition/derivation of this in more complicated models can be...more complicated.
- Determine the nullclines and the flow on these. Sketch these and the qualitative flow directions of the vector field. Include a line representing all possible initial conditions for which $S_0 + I_0 = N =$ constant, and representative trajectories for such initial conditions.
- Besides the total number of people infected during the course of an epidemic, another important measure of how severe an epidemic will be is $I_{\text{max}} = \max_t\{I(t)\}$, the *maximum* number of infected people *at any time* in the epidemic. If this is too high then health care systems may become overburdened. We can determine this in three steps.
  - First, use $\frac{dI}{dt}$ to determine the S value for which the number of infected people I is maximum. Write this in terms of ρ.
  - Next, consider $\frac{dI}{dS}$ and integrate this to determine a relationship between S, I, S_0, I_0 and ρ. You should use the initial conditions to determine the constant of integration.
  - Combine the above two relationships to obtain an expression for I_{max} in terms of S_0, I_0 and ρ.
- Reducing I_{max} through interventions, even if the total number of people infected during the whole epidemic is roughly the same, is sometimes called *flattening the curve* (though interventions tend to *both* reduce the peak and the total number infected). What strategies does the above suggest for flattening the curve, i.e. reducing I_{max}?
- It can be shown (see e.g. Murray 2002 'Mathematical Biology' vol. I) that the total number of people infected during an epidemic, I_{tot}, is given (for this model) by

$$I_{\text{tot}} = N - S(\infty) = I_0 + S_0 - S(\infty)$$

where we have assumed $R(0) = 0$, and where the final number of susceptibles, $S(\infty)$, is implicitly given by the solution to

$$S(\infty) = S_0 \exp\left(-\frac{N - S(\infty)}{\rho}\right) = S_0 \exp\left(-\frac{I_0 + S_0 - S(\infty)}{\rho}\right)$$

- Write some code to plot I_{max} and I_{tot} as a function of ρ, and/or as a function of the individual rate parameters r, a. What do you notice?