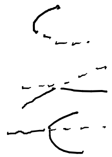
Lecture 8 Examples: Hopf & more complicated co-dimension-one examples

'co-dim': how many params do I need to get bif.

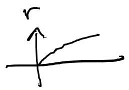
co-dim-one:

all
- saddlenode
- transcritical
- pitchfork



&

- $\boxed{\text{Hopf}}$.



} this lecture.

Hopf: need two-dimensional system

but still co-dim one.

Basic idea: fixed point changes stability & periodic orbit created or destroyed

Hopf-bifurcation theorem → too hard...

in practice: find possible, verify directly

Lessonally numerically.

Example. (simple enough to do analytically)

$$\boxed{\begin{aligned} \dot{x} &= -\omega y + x(\mu - (x^2+y^2)) \\ \dot{y} &= \omega x + y(\mu - (x^2+y^2)) \end{aligned}}$$

$\boxed{\omega > 0}$

1. FP

$-\omega y + x(\mu - (x^2+y^2)) = 0$ ①

$\omega x + y(\mu - (x^2+y^2)) = 0$ ②.

$\boxed{x=0, y=0\ ?} \Rightarrow 0 + 0\cdot(\ ) = 0$ ✓①

$0 + 0\cdot(\ ) = 0$ ✓②

$\boxed{\text{Any others ?}}$

Nope, since:

$-\omega y + x(\mu - c(x,y)) = 0 \rightarrow y = \frac{x(\mu - c(x,y))}{\omega}$

$\omega x + y(\mu - c(x,y)) = 0$

$\omega x + \frac{x}{\omega}(\mu - c)(\mu - c) = 0$

$\Rightarrow x[\omega + \frac{1}{\omega}(\mu - c)^2] = 0$

$x \neq 0 \Rightarrow \omega + \frac{1}{\omega}(\mu - c)^2 = 0$

but $\omega + \frac{1}{\omega}(\mu - c)^2 > 0$ (sum & prod. of positive quantities).

$\Rightarrow$ no other soln.

2. $Df = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

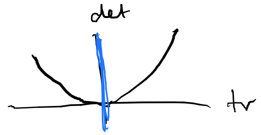
$tr = 2\mu$

$det = \mu^2 + \omega$

$\lambda^2 - tr.\lambda + det = 0$

$\lambda^2 - 2\mu\lambda + \omega$

pure im?



$\left.\begin{aligned} tr &= 0 \\ det &> 0 \end{aligned}\right\} \mu = 0$

$\Rightarrow \boxed{\lambda = \pm\sqrt{\omega}\, i}$

when $\mu = 0$.

(note: $Re(\lambda) = 0$)

(so expect when: $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$)

→ we have a possible Hopf bifurcation.

In this case we can solve analytically & verify

→ in gen., use simulations.

Here: Polar coord. & complex numbers.

$x = r\cos\theta$

$y = r\sin\theta$

$x+iy = re^{i\theta}$

[Note: if I want you to do, I'll prob. guide]

$$\frac{d(re^{i\theta})}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

$$= \omega(-y+ix) + (\mu - x^2 - y^2)(x+iy)$$

so

$$\left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)\cancel{e^{i\theta}} = \omega i r\cancel{e^{i\theta}} + (\mu - r^2)re^{i\theta}$$

⇒ equate real & imaginary....

$$\boxed{\begin{aligned}\frac{dr}{dt} &= r(\mu - r^2)\\ \frac{d\theta}{dt} &= \omega\end{aligned}}$$

(other ways of doing coord trans. too — see text.)

→

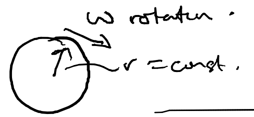
Key point:

$$\boxed{\begin{aligned}\dot r &= r(\mu - r^2)\\ \dot\theta &= \omega\end{aligned}}$$

$r=0, \omega=0$ is also FP.

Note: if $r^2 = \mu$ then $\dot r = 0$ & $\dot\theta = \omega$

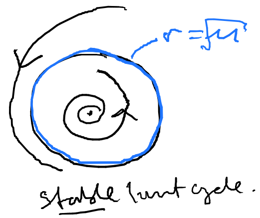
→ circle!



Case: $\mu > 0$

if $r^2 < \mu$: $\dot r > 0$

$r^2 > \mu$: $\dot r < 0$



Case $\mu < 0$

⇒ no P.O. since $r \neq \sqrt{\mu}$ doesn't exist

Case: $\mu = 0$

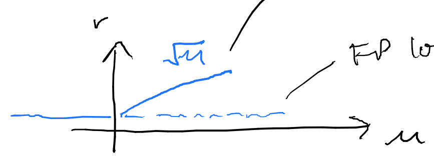
→ bifurcation

→ pure osc. 'at' origin.

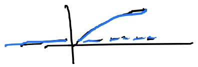
Here.

a stable P.O. is born and grows at rate $\sqrt{\mu}$.

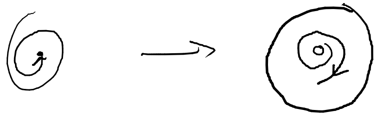
FP loses stability.



Supercritical



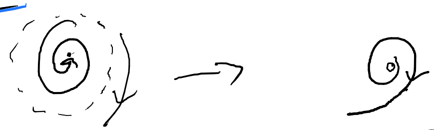
$\boxed{\text{an unstable FP creates a P.O.}}$



stable P.O. is born

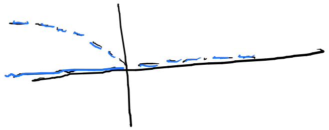
$\mu < 0$ $\qquad\qquad$ $\mu > 0$

Subcritical



$\mu < 0$ $\qquad\qquad$ $\mu > 0$

an $\boxed{\text{unstable P.O. destabilises a FP.}}$



Exam 2016: (exercise)

$$\ddot{x} + \mu\dot{x} + (x - x^3) = 0$$

i) rewrite as system of first-order eqns.

ii) det. any value where Hopf could occur.

iii) sketch generic Hopf bif. diagram.
