MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

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~~NEXT~~ LAST MODULE

4. *Separation of variables revisited, waves.* [~~3 lectures~~ 4 lectures]

   Derivation, separation of variables and *travelling waves (D'Alembert's solution)* for the wave equation. Derivation and seperation of variables for Laplace’s equation, in several geometries.

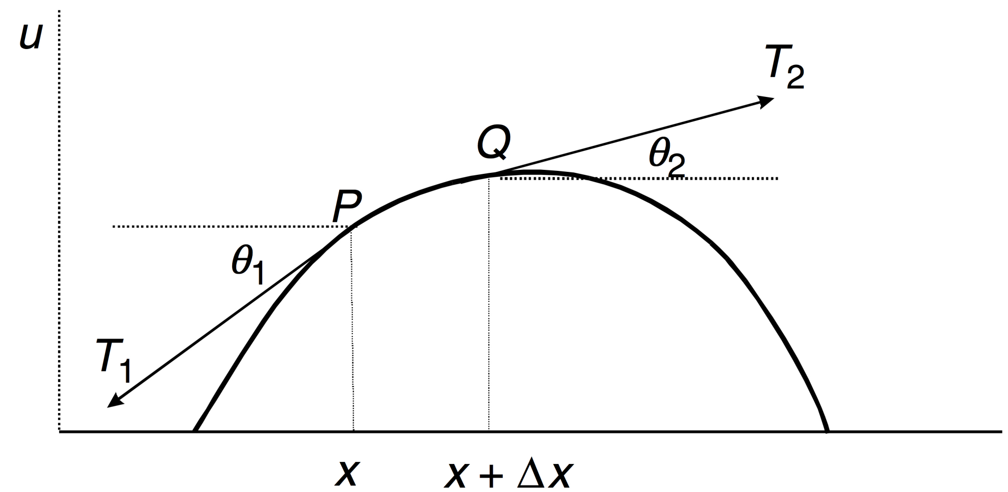
LECTURE 12

The wave equation:

- derivation
- separation of variables

DERIVATION OF THE WAVE EQUATION

Consider a (vertically) vibrating 'string' at time t:



Let's derive an equation!

A 'PLUCKED' STRING PROBLEM

Consider the problem

$$\text{PDE: } \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

$$\text{IC: } u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

$$\text{BC: } u(0,t) = 0, \qquad u(L,t) = 0.$$

Note the *two initial conditions* now. Why?

Let's solve using separation of variables (first for arbitrary $f(x)$)!

SEPARATION OF VARIABLES

Solve the following problem (for arbitrary $f(x)$) using separation of variables:

$$\text{PDE: } \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

$$\text{IC: } u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

$$\text{BC: } u(0,t) = 0, \qquad u(L,t) = 0.$$

A 'PLUCKED' STRING PROBLEM

What $f(x)$ shall we use? Suppose we pull the string up a distance h in the middle and then release it. Model as:

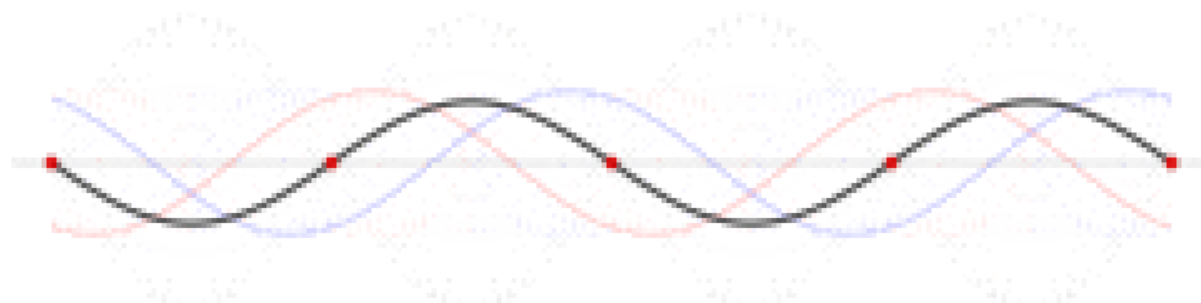
$$f(x) = \begin{cases} \frac{2h}{L}x, & 0 \leq x \leq L/2 \\ \frac{2h}{L}(L-x), & L/2 \leq x \leq L \end{cases}$$

Let's complete the solution for this $f(x)$ using our knowledge of Fourier series/orthogonal functions

A HINT OF MODES, STANDING WAVES AND TRAVELLING WAVES

Our solution is a superposition of distinct (i.e. one for each n) *modes* of vibration (with both a spatial frequency and a temporal frequency).

Each mode forms a *standing wave* which appears to vibrate 'in place'.



from https://en.wikipedia.org/wiki/Standing_wave

NEXT TIME!

We will discuss these and their relation to *travelling waves* in the next lecture.

HOMEWORK

Go over the derivation and solution from today

Plot the solution to our problem using MuPad. What do you notice?

Read about 'standing waves', 'normal modes' and 'travelling waves' (in Tang or online)