2.12 Center Manifold Theory

In Section 2.8 we presented the Hartman–Grobman Theorem, which showed that, in a neighborhood of a hyperbolic critical point $\mathbf{x}_0 \in E$, the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

is topologically conjugate to the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2}$$

with $A = D\mathbf{f}(\mathbf{x}_0)$, in a neighborhood of the origin. The Hartman–Grobman Theorem therefore completely solves the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point of a nonlinear system. In the last section, we gave some results for determining the stability and qualitative behavior in a neighborhood of a nonhyperbolic critical point of the nonlinear system (1) with $\mathbf{x} \in \mathbf{R}^2$ where $\det A = 0$ but $A \not\equiv 0$. In this section, we present the Local Center Manifold Theorem, which generalizes Theorem 1 of the previous section to higher dimensions and shows that the qualitative behavior in a neighborhood of a nonhyperbolic critical point $\mathbf{x}_0$ of the nonlinear system (1) with $\mathbf{x} \in \mathbf{R}^n$ is determined by its behavior on the center manifold near $\mathbf{x}_0$. Since the center manifold is generally of smaller dimension than the system (1), this simplifies the problem of determining the stability and qualitative behavior of the flow near a nonhyperbolic critical point of (1). Of course, we still must determine the qualitative behavior of the flow on the center manifold near the hyperbolic critical point. If the dimension of the center manifold $W^c(\mathbf{x}_0)$ is one, this is trivial; and if the dimension of $W^c(\mathbf{x}_0)$ is two and a linear term is present in the differential equation determining the flow on $W^c(\mathbf{x}_0)$, then Theorems 2 and 3 in the previous section or the method in Section 2.9 can be used to determine the flow on $W^c(\mathbf{x}_0)$. The remaining cases must be treated as they appear; however, in the next section we will present a method for simplifying the nonlinear part of the system of differential equations that determines the flow on the center manifold.

Let us begin as we did in the proof of the Stable Manifold Theorem in Section 2.7 by noting that if $\mathbf{f} \in C^1(E)$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, then the system (1) can be written in the form of equation (6) in Section 2.7 where, in this case, the matrix $A = D\mathbf{f}(\mathbf{0}) = \operatorname{diag}[C, P, Q]$ and the square matrix C has c eigenvalues with zero real parts, the square matrix P has s eigenvalues with negative real parts, and the square matrix Q has u eigenvalues with positive real parts; i.e., the system (1) can be written in diagonal form

$$\begin{aligned}
\dot{\mathbf{x}} &= C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})\\
\dot{\mathbf{y}} &= P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y}, \mathbf{z})\\
\dot{\mathbf{z}} &= Q\mathbf{z} + \mathbf{H}(\mathbf{x}, \mathbf{y}, \mathbf{z}),
\end{aligned} \tag{3}$$

where $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{R}^c \times \mathbf{R}^s \times \mathbf{R}^u$, $\mathbf{F}(\mathbf{0}) = \mathbf{G}(\mathbf{0}) = \mathbf{H}(\mathbf{0}) = \mathbf{0}$, and $D\mathbf{F}(\mathbf{0}) = D\mathbf{G}(\mathbf{0}) = D\mathbf{H}(\mathbf{0}) = O$.

We first shall present the theory for the case when $u = 0$ and treat the general case at the end of this section. In the case when $u = 0$, it follows from the center manifold theorem in Section 2.7 that for $(\mathbf{F}, \mathbf{G}) \in C^r(E)$ with $r \geq 1$, there exists an s-dimensional invariant stable manifold $W^s(\mathbf{0})$ tangent to the stable subspace E^s of (1) at $\mathbf{0}$ and there exists a c-dimensional invariant center manifold $W^c(\mathbf{0})$ tangent to the center subspace E^c of (1) at $\mathbf{0}$. It follows that the local center manifold of (3) at $\mathbf{0}$,

$$W^c_{\text{loc}}(\mathbf{0}) = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^c \times \mathbf{R}^s \mid \mathbf{y} = \mathbf{h}(\mathbf{x}) \text{ for } |\mathbf{x}| < \delta\} \tag{4}$$

for some $\delta > 0$, where $\mathbf{h} \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, and $D\mathbf{h}(\mathbf{0}) = O$ since $W^c(\mathbf{0})$ is tangent to the center subspace $E^c = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^c \times \mathbf{R}^s \mid \mathbf{y} = \mathbf{0}\}$ at the origin. This result is part of the Local Center Manifold Theorem, stated below, which is proved by Carr in [Ca].

Theorem 1 (The Local Center Manifold Theorem). *Let $\mathbf{f} \in C^r(E)$, where E is an open subset of $\mathbf{R}^n$ containing the origin and $r \geq 1$. Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $D\mathbf{f}(\mathbf{0})$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$. The system (1) then can be written in diagonal form*

$$\begin{aligned}
\dot{\mathbf{x}} &= C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y})\\
\dot{\mathbf{y}} &= P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y}),
\end{aligned}$$

where $(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^c \times \mathbf{R}^s$, C is a square matrix with c eigenvalues having zero real parts, P is a square matrix with s eigenvalues with negative real parts, and $\mathbf{F}(\mathbf{0}) = \mathbf{G}(\mathbf{0}) = \mathbf{0}$, $D\mathbf{F}(\mathbf{0}) = D\mathbf{G}(\mathbf{0}) = O$; furthermore, there exists a $\delta > 0$ and a function $\mathbf{h} \in C^r(N_\delta(\mathbf{0}))$ that defines the local center manifold (4) and satisfies

$$D\mathbf{h}(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - P\mathbf{h}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0 \tag{5}$$

for $|\mathbf{x}| < \delta$; and the flow on the center manifold $W^c(\mathbf{0})$ is defined by the system of differential equations

$$\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x})) \tag{6}$$

for all $\mathbf{x} \in \mathbf{R}^c$ with $|\mathbf{x}| < \delta$.

Equation (5) for the function $\mathbf{h}(\mathbf{x})$ follows from the fact that the center manifold $W^c(\mathbf{0})$ is invariant under the flow defined by the system (1) by substituting $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ from the above differential equations in Theorem 1 into the equation

$$\dot{\mathbf{y}} = D\mathbf{h}(\mathbf{x})\dot{\mathbf{x}},$$

which follows from the chain rule applied to the equation $\mathbf{y} = \mathbf{h}(\mathbf{x})$ defining the center manifold. Even though equation (5) is a quasilinear partial differential equation for the components of $\mathbf{h}(\mathbf{x})$, which is difficult if not impossible to solve for $\mathbf{h}(\mathbf{x})$, it gives us a method for approximating the function $\mathbf{h}(\mathbf{x})$ to any degree of accuracy that we wish, provided that the integer r in Theorem 1 is sufficiently large. This is accomplished by substituting the series expansions for the components of $\mathbf{h}(\mathbf{x})$ into equation (5); cf. Theorem 2.1.3 in [Wi-II]. This is illustrated in the following examples, which also show that it is necessary to approximate the shape of the local center manifold $W^c_{\text{loc}}(\mathbf{0})$ in order to correctly determine the flow on $W^c(\mathbf{0})$ near the origin. Before presenting these examples, we note that for c = 1 and $s = 1$, the Local Center Manifold Theorem given above is the same as Theorem 1 in the previous section (if we let $t \to -t$ in equation (2) in Section 2.11). Thus, Theorem 1 above is a generalization of Theorem 1 in Section 2.11 to higher dimensions. Also, in the case when c = s = 1, as in Theorem 1 in Section 2.11, it is only necessary to solve the algebraic equation determined by setting the last two terms in equation (5) equal to zero in order to determine the correct flow on $W^c(\mathbf{0})$.

It should be noted that while there may be many different functions $\mathbf{h}(\mathbf{x})$ which determine different center manifolds for (3), the flows on the various center manifolds are determined by (6) and they are all topologically equivalent in a neighborhood of the origin. Furthermore, for analytic systems, if the Taylor series for the function $\mathbf{h}(\mathbf{x})$ converges in a neighborhood of the origin, then the analytic center manifold $\mathbf{y} = \mathbf{h}(\mathbf{x})$ is unique; however, not all analytic (or polynomial) systems have an analytic center manifold. Cf. Problem 4.

Example 1. Consider the following system with c = s = 1:

$$\dot{x} = x^2 y - x^5$$
$$\dot{y} = -y + x^2.$$

In this case, we have $C = O$, $P = [-1]$, $F(x,y) = x^2y - x^5$, and $G(x,y) = x^2$. We substitute the expansions

$$h(x) = ax^2 + bx^3 + 0(x^4) \quad \text{and} \quad Dh(x) = 2ax + 3bx^2 + 0(x^3)$$

into equation (5) to obtain

$$(2ax + 3bx^2 + \cdots)(ax^4 + bx^5 + \cdots - x^5) + ax^2 + bx^3 + \cdots - x^2 = 0.$$

Setting the coefficients of like powers of x equal to zero yields $a - 1 = 0$, $b = 0$, c = 0, …. Thus,

$$h(x) = x^2 + 0(x^5).$$

Substituting this result into equation (6) then yields

$$\dot{x} = x^4 + 0(x^5)$$

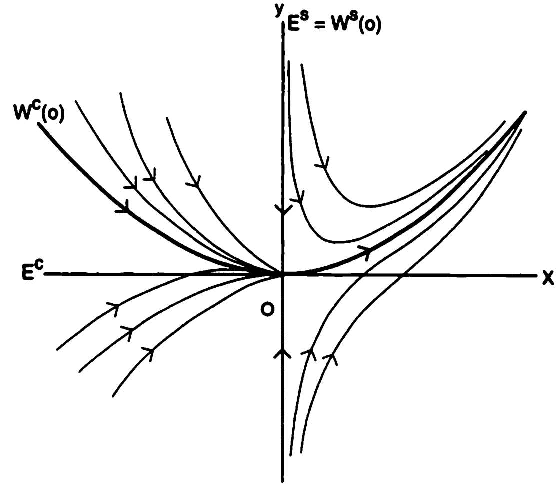

Figure 1. The phase portrait for the system in Example 1.

on the center manifold $W^c(\mathbf{0})$ near the origin. This implies that the local phase portrait is given by Figure 1. We see that the origin is a saddle-node and that it is unstable. However, if we were to use the center subspace approximation for the local center manifold, i.e., if we were to set $y = 0$ in the first differential equation in this example, we would obtain

$$\dot{x} = -x^5$$

and arrive at the incorrect conclusion that the origin is a stable node for the system in this example.

The idea in Theorem 1 in the previous section now becomes apparent in light of the Local Center Manifold Theorem; i.e., when the flow on the center manifold has the form

$$\dot{x} = a_m x^m + \cdots$$

near the origin, then for m even (as in Example 1 above) equation (2) in Section 2.11 has a saddle-node at the origin, and for m odd we get a topological saddle or a node at the origin, depending on whether the sign of a_m is the same as or the opposite of the sign of $\dot{y}$ near the origin.

Example 2. Consider the following system with $c = 2$ and $s = 1$:

$$\begin{aligned}\dot{x}_1 &= x_1y - x_1x_2^2\\ \dot{x}_2 &= x_2y - x_2x_1^2\\ \dot{y} &= -y + x_1^2 + x_2^2.\end{aligned}$$

In this example, we have $C = O$, $P = [-1]$,

$$\mathbf{F}(\mathbf{x}, y) = \begin{pmatrix} x_1y - x_1x_2^2 \\ x_2y - x_2x_1^2 \end{pmatrix} \quad \text{and} \quad G(\mathbf{x}, y) = x_1^2 + x_2^2.$$

We substitute the expansions

$$h(\mathbf{x}) = ax_1^2 + bx_1x_2 + cx_2^2 + 0(|\mathbf{x}|^3)$$

and

$$Dh(\mathbf{x}) = [2ax_1 + bx_2, bx_1 + 2cx_2] + 0(|\mathbf{x}|^2)$$

into equation (5) to obtain

$$\begin{aligned}(2ax_1 + bx_2)[x_1(ax_1^2 + bx_1x_2 + cx_2^2) - x_1x_2^2]&\\ + (bx_1 + 2cx_2)[x_2(ax_1^2 + bx_1x_2 + cx_2^2) - x_2x_1^2]&\\ + (ax_1^2 + bx_1x_2 + cx_2^2) - (x_1^2 + x_2^2) + 0(|\mathbf{x}|^3) &= 0.\end{aligned}$$

Since this is an identity for all x_1, x_2 with $|\mathbf{x}| < \delta$, we obtain $a = 1$, $b = 0$, $c = 1, \ldots$. Thus,

$$h(x_1, x_2) = x_1^2 + x_2^2 + 0(|\mathbf{x}|^3).$$

Substituting this result into equation (6) then yields

$$\begin{aligned}\dot{x}_1 &= x_1^3 + 0(|\mathbf{x}|^4)\\ \dot{x}_2 &= x_2^3 + 0(|\mathbf{x}|^4)\end{aligned}$$

on the center manifold $W^c(\mathbf{0})$ near the origin. Since $r\dot{r} = x_1^4 + x_2^4 + O(|\mathbf{x}|^5) > 0$ for $0 < r < \delta$, this implies that the local phase portrait near the origin is given as in Figure 2. We see that the origin is a type of topological saddle that is unstable. However, if we were to use the center subspace approximation for the local center manifold, i.e., if we were to set $y = 0$ in the first two differential equations in this example, we would obtain

$$\begin{aligned}\dot{x}_1 &= -x_1x_2^2\\ \dot{x}_2 &= -x_2x_1^2\end{aligned}$$

and arrive at the incorrect conclusion that the origin is a stable nonisolated critical point for the system in this example.

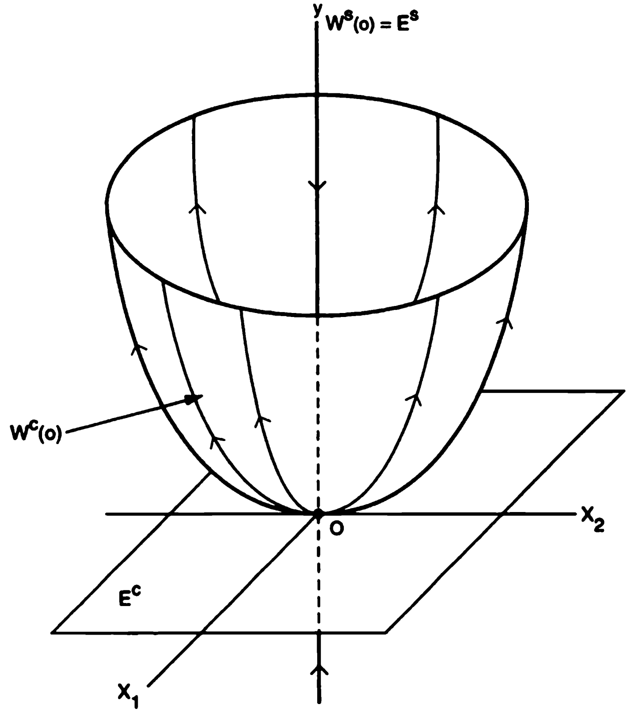

Figure 2. The phase portrait for the system in Example 2.

Example 3. For our last example, we consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + y\\ \dot{x}_2 &= y + x_1^2\\ \dot{y} &= -y + x_2^2 + x_1y.\end{aligned}$$

The linear part of this system can be reduced to Jordan form by the matrix of (generalized) eigenvectors

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This yields the following system in diagonal form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1^2 + (x_2 - y)^2 + x_1 y$$
$$\dot{y} = -y + (x_2 - y)^2 + x_1 y$$

with $c = 2$, $s = 1$, $P = [-1]$,

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}, y) = \begin{pmatrix} 0 \\ x_1^2 + (x_2 - y)^2 + x_1 y \end{pmatrix},$$

and

$$G(\mathbf{x}, y) = (x_2 - y)^2 + x_1 y.$$

Let us substitute the expansions for $h(\mathbf{x})$ and $Dh(\mathbf{x})$ in Example 2 into equation (5) to obtain

$$(2ax_1 + bx_2)x_2 + (bx_1 + 2cx_2)[x_1^2 + (x_2 - y)^2 + x_1 y] + (ax_1^2 + bx_1x_2 + cx_2^2) - (x_2 - ax_1^2 - bx_1x_2 - cx_2^2)^2 - x_1(ax_1^2 + bx_1x_2 + cx_2^2) + 0(|\mathbf{x}|^3) = 0.$$

Since this is an identity for all x_1, x_2 with $|\mathbf{x}| < \delta$, we obtain $a = 0$, $b = 0$, $c = 1, \ldots$, i.e.,

$$h(\mathbf{x}) = x_2^2 + 0(|\mathbf{x}|^3).$$

Substituting this result into equation (6) then yields

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1^2 + x_2^2 + O(|\mathbf{x}|^3)$$

on the center manifold $W^c(\mathbf{0})$ near the origin.

Theorem 3 in Section 2.11 then implies that the origin is a cusp for this system. The phase portrait for the system in this example is therefore topologically equivalent to the phase portrait in Figure 3.

As on pp. 203–204 in [Wi-II], the above results can be generalized to the case when the dimension of the unstable manifold $u \neq 0$ in the system (3). In that case, the local center manifold is given by

$$W^c_{\text{loc}}(\mathbf{0}) = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{R}^c \times \mathbf{R}^s \times \mathbf{R}^u \mid \mathbf{y} = \mathbf{h}_1(\mathbf{x}) \text{ and } \mathbf{z} = \mathbf{h}_2(\mathbf{x}) \text{ for } |\mathbf{x}| < \delta\}$$

for some $\delta > 0$, where $\mathbf{h}_1 \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h}_2 \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h}_1(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}_2(\mathbf{0}) = \mathbf{0}$, $D\mathbf{h}_1(\mathbf{0}) = 0$, and $D\mathbf{h}_2(\mathbf{0}) = 0$ since $W^c(\mathbf{0})$ is tangent to the center subspace $E^c = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{R}^c \times \mathbf{R}^s \times \mathbf{R}^u \mid \mathbf{y} = \mathbf{z} = \mathbf{0}\}$ at the origin. The functions $\mathbf{h}_1(\mathbf{x})$ and $\mathbf{h}_2(\mathbf{x})$ can be approximated to any desired degree of accuracy (provided that r is sufficiently large) by substituting their power series expansions into the following equations:

$$D\mathbf{h}_1(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x}))] - P\mathbf{h}_1(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x})) = 0$$
$$D\mathbf{h}_2(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x}))] - Q\mathbf{h}_2(\mathbf{x}) - \mathbf{H}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x})) = 0. \quad (7)$$

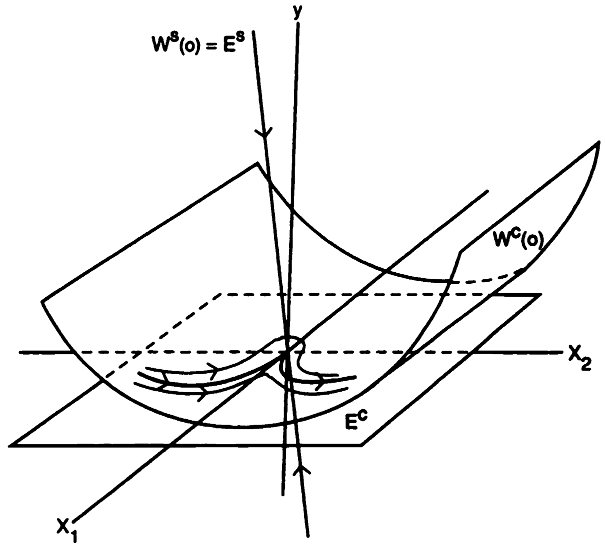

Figure 3. The phase portrait for the system in Example 3.

The next theorem, proved by Carr in [Ca], is analogous to the Hartman–Grobman Theorem except that, in order to determine completely the qualitative behavior of the flow near a nonhyperbolic critical point, one must be able to determine the qualitative behavior of the flow on the center manifold, which is determined by the first system of differential equations in the following theorem. The nonlinear part of that system, i.e., the function $\mathbf{F}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x}))$, can be simplified using the normal form theory in the next section.

Theorem 2. *Let E be an open subset of $\mathbf{R}^n$ containing the origin, and let $\mathbf{f} \in C^1(E)$; suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that the $n \times n$ matrix $D\mathbf{f}(\mathbf{0}) =$ diag$[C, P, Q]$, where the square matrix C has c eigenvalues with zero real parts, the square matrix P has s eigenvalues with negative real parts, and the square matrix Q has u eigenvalues with positive real parts. Then there exists C^1 functions $\mathbf{h}_1(\mathbf{x})$ and $\mathbf{h}_2(\mathbf{x})$ satisfying* (7) *in a neighborhood of the origin such that the nonlinear system* (1)*, which can be written in the form* (3)*, is topologically conjugate to the C^1 system*

$$\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}_1(\mathbf{x}), \mathbf{h}_2(\mathbf{x}))$$
$$\dot{\mathbf{y}} = P\mathbf{y}$$
$$\dot{\mathbf{z}} = Q\mathbf{z}$$

for $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{R}^c \times \mathbf{R}^s \times \mathbf{R}^u$ in a neighborhood of the origin.

PROBLEM SET 12

1. Consider the system in Example 3 in Section 2.7:
$$\dot{x} = x^2$$
$$\dot{y} = -y.$$
By substituting the expansion
$$h(x) = ax^2 + bx^3 + \cdots$$
into equation (5) show that the analytic center manifold for this system is defined by the function $h(x) \equiv 0$ for all $x \in \mathbf{R}$; i.e., show that the analytic center manifold $W^c(\mathbf{0}) = E^c$ for this system. Also, show that for each c $\in \mathbf{R}$, the function
$$h(x, c) = \begin{cases} 0 & \text{for} \quad x \geq 0 \\ ce^{1/x} & \text{for} \quad x < 0 \end{cases}$$
defines a C^∞ center manifold for this system, i.e., show that it satisfies equation (5). Also, graph $h(x, c)$ for various c $\in \mathbf{R}$.

2. Use Theorem 1 to determine the qualitative behavior near the nonhyperbolic critical point at the origin for the system
$$\dot{x} = y$$
$$\dot{y} = -y + \alpha x^2 + xy$$
for $\alpha \neq 0$ and for $\alpha = 0$; i.e., follow the procedure in Example 1 after diagonalizing the system as in Example 3.

3. Same thing as in Problem 2 for the system
$$\dot{x} = xy$$
$$\dot{y} = -y - x^2.$$

4. Same thing as in Problem 2 for the system
$$\dot{x} = -x^3$$
$$\dot{y} = -y + x^2.$$
Also, show that this system has no analytic center manifold, i.e., show that if $h(x) = a_2 x^2 + a_3 x^3 + \cdots$, then it follows from (5) that $a_2 = 1, a_{2k+1} = 0$ and $a_{n+2} = na_n$ for n even; i.e., the Taylor series for $h(x)$ diverges for $x \neq 0$.

5. (a) Use Theorem 1 to find the approximation (6) for the flow on the local center manifold for the system
$$\dot{x}_1 = -x_2 + x_1 y$$
$$\dot{x}_2 = x_1 + x_2 y$$
$$\dot{y} = -y - x_1^2 - x_2^2 + y^2$$
and sketch the phase portrait for this system near the origin.
Hint: Convert the approximation (6) to polar coordinates and show that the origin is a stable focus of (6) in the x_1, x_2 plane.

(b) Same thing for the system
$$\dot{x}_1 = x_1 y - x_1 x_2^2$$
$$\dot{x}_2 = -2x_1^2 x_2^2 - x_1^4 + y^2$$
$$\dot{y} = -y + x_1^2 + x_2^2.$$
Hint: Show that the approximation (6) has a saddle-node in the x_1, x_2 plane.

(c) Same thing for the system
$$\dot{x}_1 = x_1^2 y - x_1^2 x_2^2$$
$$\dot{x}_2 = -2x_1^2 x_2^2 + y^2$$
$$\dot{y} = -y + x_1^2 + x_2^2.$$
Hint: Show that the approximation (6) has a nonhyperbolic critical point at the origin with two hyperbolic sectors in the x_1, x_2 plane.

6. Use Theorem 1 to find the approximation (6) for the flow on the local center manifold for the system
$$\dot{x} = ax^2 + bxy + cy^2$$
$$\dot{y} = -y + dx^2 + exy + fy^2,$$
and show that if $a \neq 0$, then the origin is a saddle-node. What type of critical point is at the origin if $a = 0$ and $bd \neq 0$? What if $a = b = 0$ and $cd \neq 0$? What if $a = d = 0$?

7. Use Theorem 1 to find the approximation (6) for the flow on the local center manifold for the system
$$\dot{x}_1 = -x_2^3 - x_1^3$$
$$\dot{x}_2 = x_1 y - x_2^3$$
$$\dot{y} = -y + x_1^2.$$
What type of critical point is at the origin of this system?

2.13 Normal Form Theory

The Hartman–Grobman Theorem shows us that in a neighborhood of a hyperbolic critical point, the qualitative behavior of a nonlinear system
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$