Lecture 9: Intro to Centre Manifold Theory.

Examples.

$$\boxed{\begin{aligned}\dot{x} &= xy + x^3 \\ \dot{y} &= -y - 2x^2\end{aligned}}$$

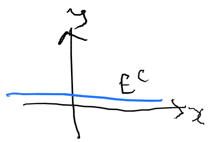
- $(x,y) = (0,0)$ is $\boxed{\text{FP}}$ (focus on this one).
- $\boxed{Df} = \begin{bmatrix} y+3x^2 & x \\ -4x & -1 \end{bmatrix}$
- $\boxed{Df(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

$\lambda_1 = 0$, $\lambda_2 = -1$

centre, stable

- Eigenspace? Same as normal.

$\boxed{\lambda_1 = 0}$ $\begin{bmatrix} 0-0 & 0 \\ 0 & -1-0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$\Rightarrow -u_2 = 0 \Rightarrow u_2 = 0$, u_1 free.

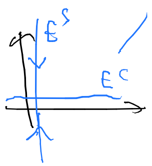
$\Rightarrow e_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow$ ie $y = 0$

So $\boxed{E^c = \{(x,y) \mid y = 0\}}$

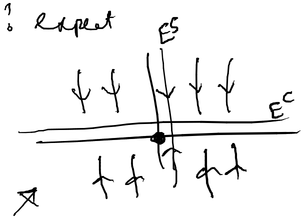
$\boxed{\lambda_2 = -1}$ $\Rightarrow u_1 = 0$, u_2 free

$e_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ie $x = 0$

$\boxed{E^s = \{(x,y) \mid x = 0\}}$



Trajectories? Expect

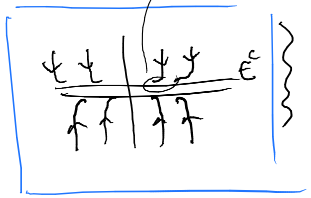


fast attraction to E^c

linear picture

– what happens $\boxed{\text{near } E^c}$?

→ need to $\boxed{\text{zoom in}}$

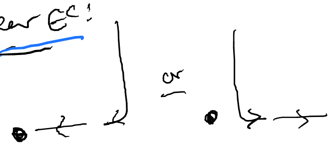


fast attraction to E^c then $\boxed{\text{slow motion on } E^c}$

↑

need $\boxed{\text{nonlinear analysis.}}$

Near E^c:



? eg asympt. stable or unstable?

Need W^c.

→ Same procedure as for W^s, W^u (but some caveats)

$\boxed{\text{Steps}}$

- Assume $\boxed{y = h(x)}$ $\longrightarrow E^c$

(expect tangent)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\left.\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \end{aligned}\right\} \text{ tangent to } E^c.$$

$$\Rightarrow y = a_2 x^2 + a_3 x^3 + \dots$$

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① $\dot{y} = \frac{dy}{dx}\dot{x} = \left[2a_2 x + 3a_3 x^2 + \dots\right] x \cdot \left[a_2 x^2 + a_3 x^3 + x^2\right]$

② $\dot{y} = -\left(a_2 x^2 + a_3 x^3 + \dots\right) - 2x^2$

Let's do to cubic order

①: $= 0 + O(x^4)$ $\rightarrow$ easy! $\nearrow$ W^c often easier than W^s/W^u!

②: $= -(a_2+2)x^2 - a_3 x^3 + \dots$

$\Rightarrow$ $\boxed{\begin{aligned} a_2 &= -2 \\ a_3 &= 0 \end{aligned}}$

$\Rightarrow$ $\boxed{y = -2x^2}$

Note: this order could have been obtained by just assuming $\dot{y} = 0$!
$\rightarrow$ quasi-steady state
[y fast, x slow] *

higher order requires full calc. though...

so $\boxed{W^c = \{(x,y) \mid y = -2x^2\}}$

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* Note: $\dot{x}$ = quadratic & higher $\leftarrow$ slow near $x,y = 0,0$ (c $\boxed{x \cdot y = \text{smaller than } y}$)

$\dot{y}$ = linear & higher
$\uparrow$
fast near $x,y = 0,0$.

Dynamics & stability

$E^c : \{(x,y) \mid y = 0\}$

$W^c : \{(x,y) \mid y = -2x^2\}$

Consider slow var:

$\boxed{\dot{x} = xy + x^3}$

opposite conclusions

$\boxed{\dot{x}\big|_{E^c} = x^3}$ $\Rightarrow$ $\boxed{\text{unstable}}$ $\left(\begin{aligned} x > 0 &\Rightarrow \dot{x} > 0 \\ x < 0 &\Rightarrow \dot{x} < 0 \end{aligned}\right)$

restricted to E^c

vs

$\boxed{\text{opposite}}$ conclusions.

$\boxed{\dot{x}\big|_{W^c} = x \cdot (-2x^2) + x^3}$

restricted to W^c

$= -x^3$ $\Rightarrow$ $\boxed{\text{stable}}$ $\left(\begin{aligned} x > 0, \dot{x} < 0 \\ x < 0, \dot{x} > 0 \end{aligned}\right)$

$\Rightarrow$ $\boxed{\text{linear stability/analysis is unreliable here} \\ \text{Lneed } W^c \text{ for } \boxed{\text{nonhyperbolic}}}$

Simulation example.

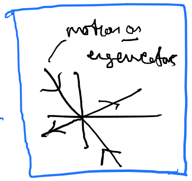
see $\boxed{\text{XPPAut}}$ .ode file on Canvas.

Coordinate transformation Example.

The system: $\boxed{\begin{aligned}\dot{x} &= y - x - x^2 \\ \dot{y} &= x - y - y^2\end{aligned}}$ } note: linearly coupled*

has eigenvectors that aren't parallel to the x–y axes.

→ We want to use the eigenvectors as our new coord system.



→ this gives a linearly separated/diagonal matrix for distinct eigenvalues & an upper triangular matrix in general. } 'Jordan normal form'

→ this makes it easier to identify slow/fast components, & carry out analysis in general.

↳ eg manifolds are tangent to eigenvectors, so in eigencoords our power series are simplified

↳ our reduction principle will assume we put in this form

Working (see Linear Alg. handout next lecture for full details)

$$Df(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

* linearly coupled ∵ off-diagonals non zero.

$tr = -2$, $det = 0$

$$\boxed{\lambda^2 - tr.\lambda + det = 0}$$

$\Leftrightarrow \lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\Rightarrow \boxed{\lambda_1 = 0,\ \lambda_2 = -2}$ ⇒

$\lambda = 0$

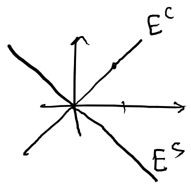
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -u_1 + u_2 = 0$

& $u_1 - u_2 = 0$

$\Rightarrow u_1 = u_2$

$$\boxed{e^c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$



$\lambda = -2$

$$\begin{pmatrix} -1-(-2) & 1 \\ 1 & -1-(-2) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$u_1 + u_2 = 0$
$u_1 + u_2 = 0$ } same

set $u_1 = 1 \Rightarrow u_2 = -1$

$$\boxed{e^s = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

[expresses x & y as linear comb. of eigenvectors]

To change to $\boxed{e^s, e^c \text{ coord}}$ use (handout to come)

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \vdots & \vdots \\ e^c & e^s \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}$$

← use eigenvectors as columns of transformation matrix.

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$\Rightarrow x = u + v$
$\quad y = u - v$ } coord transform

ie $u = \dfrac{x+y}{2}$, $v = \dfrac{x-y}{2}$ } inverse coord transf.

$$\boxed{\dot{u} = \tfrac{1}{2}[\dot{x} + \dot{y}],\ \dot{v} = \tfrac{1}{2}[\dot{x} - \dot{y}]}$$

$\Rightarrow \dot{u} = -(u^2 + v^2)$, $\dot{v} = -2v(1+u)$ } $\boxed{\text{linearly decoupled}}$

exercise: verify

$$\boxed{Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}}$$

in u,v coord.

[cf $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ before]