Engsci 711

Tutorial 4: Centre manifold theory

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Overview

The purpose of this tutorial is to get some experience with carrying out centre manifold analysis. The most relevant parts are the basic centre manifold reductions (no parameters etc) - see e.g. the 'Exam' questions.

Basic centre manifold reduction problems

Problem 1

Consider the worked example from Lecture 11 (Kuznetsov, 2004, Example 5.1)

$$\begin{aligned}\dot{x} &= xy + x^3\\ \dot{y} &= -y - 2x^2\end{aligned}$$

- Confirm the origin $(0, 0)$ is an equilibrium point
- Find the Jacobian derivative Df and evaluate it at the origin
- Find the eigenvalues and associated linear subspaces E^s, E^u, E^c.
- Is the system expressed relative to the eigenbasis? That is, are the eigen-directions parallel to the x and y axis?
- If they are, proceed. If they aren't, define a linear transformation converting x, y to new variables u, v such that u, v are coordinates in the eigenbasis (hint: here we are fine; for other cases, you might need to change coordinates).
- Assume that (relative to the eigendirections) the centre manifold W^c_{loc} can be expressed as a curve expressing the non-centre coordinate(s) in terms of the centre coordinate(s), $y = V(x)$.
- Assume this has a series expansion and use the same process as outlined in the previous tutorial and the lectures to determine the terms of the series for the centre manifold.
- Use this expression for W^c_{loc} and the Reduction/Emergence Principle from the lectures to determine the (approximate) dynamics on the centre manifold (i.e. substitute your expression for $y(x)$ into the $\dot{x}$ equation!).
- Are these dynamics stable or unstable? Are the dynamics of the full system stable or unstable? Sketch the local dynamics near the origin.
- What are the relative rates of the dynamics on the centre and the stable/unstable manifolds near the origin (look at the equations you have!).
- Instead of substituting W^c_{loc} into the $\dot{x}$ equation, try substituting the equation defining E^c into the $\dot{x}$ equation. What do you notice?

Problem 2 (Exam Qs)

Carry out the same process for (2016 Exam)

$$\begin{aligned}\dot{x} &= y(2x - y)\\ \dot{y} &= x^2 - y\end{aligned}$$

(2017 Exam)

$$\begin{aligned}\dot{x} &= 2xy + x^3\\ \dot{y} &= -y - x^2\end{aligned}$$

and (2018 Exam)

$$\begin{aligned}\dot{x} &= -x + 2xy + y^3\\ \dot{y} &= -2xy - y^3\end{aligned}$$

Problem 3

Carry out the same process for the example from Lecture 12

$$\dot{x} = y - x - x^2$$
$$\dot{y} = x - y - y^2$$

Problem 4

Carry out the same process for

$$\dot{x} = y - 3x^2 + xy$$
$$\dot{y} = -3y + y^2 + x^2$$

Two-dimensional centre manifold reduction

Consider the system

$$\dot{x_1} = x_1 y - x_1 x_2^2$$
$$\dot{x_2} = x_2 y - x_2 x_1^2$$
$$\dot{y} = -y + x_1^2 + x_2^2$$

where $x_1, x_2, y \in \mathbb{R}$.

- Verify that the origin is a fixed point of this system.
- Find the Jacobian derivative - first as a function of x_1, x_2 and y and then evaluated at the origin $(0, 0, 0)$.
- Find the eigenvalues of the linearisation about the origin and - if they exist - the associated stable, unstable and centre eigenspaces, E^s, E^u and E^c respectively.
- Use a power series expansion to calculate an expression for the centre manifold $W^c_{loc}(0, 0, 0)$ that is correct up to and including quadratic order.

Note: If your centre manifold is two-dimensional then you need to expand in two variables. An example of expanding an arbitrary two-variable function to quadratic order is

$$f(u, v) \approx a + bu + cv + du^2 + euv + fv^2$$

where $a, b, ..., f$ are constants. Note that a cross term like uv in the above is considered quadratic order.

- Use your expression for the centre manifold to determine the dynamics on the centre manifold, again correct up to and including quadratic order, and thus determine whether these dynamics are (asymptotically) stable or unstable.
- Would you have gotten the same answer for this particular example if you used the linear centre manifold $E^c(0, 0, 0)$? Justify your answer.

Extended centre manifold reduction problems

Extended lecture manifold reduction - guided example (see also Lecture 13!)

The purpose of this question is to understand how we can get to bifurcation theory via centre manifold theory using the idea of an 'extended' centre manifold. This is the same example as 'covered' in Lecture 13, repeated here for convenience. Really, everything works just the same as in normal centre manifold theory, *we just 'upgrade' the parameter to a (still trivial!) state variable.*

Consider the system of equations

$$\dot{x} = y - x - x^2,$$
$$\dot{y} = \lambda x - y - y^2$$

- Find a value $\lambda = \lambda_c$, for which the origin $(0,0)$ is non-hyperbolic.
- Linearise the system about the origin $(0,0)$ and with λ fixed at λ_c.
- Determine a linear change of coordinates $(x,y) \to (u,v)$ that puts the linearised system into diagonal form (for $\lambda = \lambda_c$).
- Define a new parameter $\mu = \lambda - \lambda_c$ which is zero at the non-hyperbolic point. Write the *full, nonlinear* system in terms of your u, v above and your new parameter μ.

Now comes the key - yet simple - step.

- 'Upgrade' the parameter μ to a state variable. This means we take the u, v equations and add the trivial equation $\dot{\mu} = 0$.

Note that, since we are in diagonal form this corresponds to adding another *centre* variable (eigenvalue $= 0$). In this case it is 'super slow' since *both* linear and nonlinear parts are zero (the other centre variable will have 'zero-eigenvalue' linear dynamics but non-trivial higher-order dynamics, so can be thought of as 'slowly varying').

You should now have a system of the form

$$\begin{aligned} \dot{u} &= ... \\ \dot{v} &= ... \\ \dot{\mu} &= 0 \end{aligned}$$

Note that, while for the μ-as-parameter system a term like μu is linear, when μ is considered as a state variable a term like this is considered *nonlinear*!

We should now have a diagonalised system where the (extended) centre manifold component is two-dimensional (check you understand why). Suppose u and μ are your centre manifold variables. Our centre manifold will be tangent to the (μ, u) plane at $(u, v, \mu) = (0, 0, 0)$.

We can now proceed as normal in centre manifold theory

- Assume that the two-dimensional centre manifold is described by a restriction of three-dimensional (u, v, μ) space by one constraint $v = h(\mu, u)$, and that h can be approximated using a two-variable Taylor series expansion. This takes the form

$$v = a + b\mu + cu + d\mu^2 + e\mu u + fu^2$$

where a, b, c, d, e, f are constants.

- What are a, b, c? You should be able to write these down instantly.
- Now use the usual procedure for finding the other coefficients. That is, use

$$\dot{v} = \frac{\partial h}{\partial \mu}\dot{\mu} + \frac{\partial h}{\partial u}\dot{u}$$

and substitute in what you know about $\dot{v}, \dot{\mu}, \dot{u}$.

- Equate coefficients to determine d, e, f.
- Now, use the Reduction/Emergence Principle to determine the dynamics on the extended (u, μ) centre manifold. That is, substitute your expression into the u equation (the μ equation remains trivial). Your answer should consist of writing down two differential equations.

Note: we have effectively obtained a one-dimensional bifurcation problem (as expected)! To see, note that since the μ dynamics are trivial, we can effectively downgrade μ back to a control parameter. That is, we fix it to different values and solve the u equation for each of these.

This can be considered as either a u vs μ phase-portrait OR a u vs μ bifurcation diagram (particularly when we just plot the equilibria of u). The point of 'upgrading' it temporarily was to derive the bifurcation diagram from the centre manifold phase portrait. Interestingly, we can carry out a further simplification because the parameter dynamics are trivial since μ is a (super) slow variable.

So we can now carry out the last step.

- Draw a bifurcation plot/(u, μ) phase portrait. What sort of bifurcation is this?

Extended centre manifold reduction - your turn!

Carry out an extended centre manifold reduction/analysis for the system

$$\begin{aligned}\dot{x} &= \mu x - xy,\\ \dot{y} &= -y + x^2\end{aligned}$$

- Carry out a bifurcation analysis of the resulting system.

Extended centre manifold reduction - your turn again!

Carry out an extended centre manifold reduction/analysis for the system (Glendinning Example 8.6)

$$\begin{aligned}\dot{x} &= (1+\mu)x - 4y + x^2 - 2xy,\\ \dot{y} &= 2x - 4\mu y - y^2 - x^2\end{aligned}$$

- Carry out a bifurcation analysis of the resulting system.

Slow-fast systems

Here we give a brief preview of how to tackle singular perturbation problems via centre manifold theory (based on Carr Example 3). Note that we only get as far as a *local* analysis for small initial conditions here, but that it is possible to construct a more global approximation, again using centre manifold theory, as discussed in the books by Carr and by Roberts (see course references).

Consider the system in 'singular perturbation' form:

$$\begin{aligned}\dot{x} &= -x + (x+c)y\\ \epsilon\dot{y} &= x - (x+1)y\end{aligned}$$

Singular perturbation theory is one way to justify the procedure of a) setting $\epsilon = 0$ to give a degenerate differential-algebraic system and b) using this to construct an approximate solution to the original full system. In particular this procedure gives

$$\begin{aligned}y &\approx \frac{x}{x+1},\\ \dot{x} &\approx -x + (x+c)\frac{x}{x+1}\end{aligned}$$

as the first approximation. Centre manifold theory works similarly, but usually begins from a nonsingular form - i.e. no ϵ terms multiplying derivatives and hence $\epsilon = 0$ still gives a full (but simplified) system of ODES, i.e. not a differential-algebraic system.

- Put the system in the correct form for centre manifold theory by first re-scaling time as $t = \epsilon\tau$ and then writing it as a *linearly decoupled, extended system* with the small parameter as a state variable. Use y' etc to represent derivatives with respect to τ, and you should use a linear change of variables to decouple the system. The result should be:

$$\begin{aligned}x' &= \epsilon f(x,z),\\ \epsilon' &= 0,\\ z' &= -z + x^2 - xz + \epsilon f(x,z)\end{aligned}$$

where $f(x,z) = -x + (x+c)(x-z)$ and $z = x - y$. Identify the fast and slow variables from the above.

- Show that an approximate centre manifold is given by

$$z = x^2 - (1-c)\epsilon x.$$

- Hence show that the centre manifold dynamics are, on the *original* time scale, approximately given by

$$\dot{x} = (c-1)(x - x^2)$$

.

- Compare this result to that obtained from solving the degenerate system when using the singular perturbation form and $\epsilon = 0$. In particular, compare these results *near the origin* $x = 0$. Do the two procedures agree?
- Look up the book by Carr or the book by Roberts (see course references) for how to construct an approximation that is global in x and y (or $z = x - y$), though still local in ϵ, using centre manifold theory. This will give $y \approx \frac{x}{x+1}$ for the centre manifold, valid for x not just near the origin, as well as high-order corrections. This works by basing analysis on a whole *nonlinear manifold of equilibria* instead of just the one at the origin.
- To see the hints of how the global analysis might work, note that $\{(x, \epsilon, z) | \epsilon = 0, z = \frac{x^2}{x+1}\}$ defines a whole *manifold* of equilibria for the extended system. Show that $z = \frac{x^2}{x+1}$ is equivalent to $y = \frac{x}{x+1}$.

Miscellaneous further reading

- The articles (see Canvas) by Oppo and Politi on lasers, adiabatic approximations and centre manifold theory.
- Suslov, S. A., & Roberts, A. J. (1998). Similarity, attraction and initial conditions in an example of nonlinear diffusion. ANZIAM Journal, 40, 1-26.

This above article by Suslov and Roberts gives a preview of how centre manifold theory connects to 'intermediate asymptotics'. This latter topic is (sometimes) covered in the other part of this course by Richard. Roberts' book is in the recommended reading handout and has a lot of nice material on centre manifolds and normal form coordinate changes.

- 'Critical slow down' and ecological disasters: https://www.quantamagazine.org/critical-slowing-warns-of-looming-disasters-20151118/
- Centre manifolds and deep learning: Poggio, et al. (2018). Theory of deep learning III: the non-overfitting puzzle. CBMM memo 073.