ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~15 lectures]

1. *Basic concepts* [3 lectures]

   Basic concepts and definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [4 lectures]

   Two-dimensional systems. Linearisation of nonlinear systems. Linear systems - stability and classification of fixed points. Periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Geometry of bifurcations - invariant manifolds. Bifurcation diagrams.

4. *Introduction to fast-slow systems and singular perturbation problems* [4 lectures]

   Canonical fast-slow examples and importance. Key geometric concepts and perturbation theory.

LECTURE 5

- Finish: perturbation expansion example for nonlinear stable/unstable manifolds
- Periodic orbits in the plane: nonexistence, existence
- Periodic orbits in higher dimensions: discrete maps as a tool
- Note: Assignment!

RECALL: STABLE MANIFOLD (LOCAL)

Given some neighbourhood U of a stationary point x, the *local stable manifold* on U for a nonlinear system $W^s_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y,t) \to x \text{ as } t \to \infty, \phi(y,t) \in U \text{ for all } t \geq 0\}$$

RECALL: UNSTABLE MANIFOLD (LOCAL)

Similarly, given some neighbourhood U of a stationary point x, the *local unstable manifold* on U for a nonlinear system $W^u_{loc}(x)$ is defined by

$$\{y \in U \mid \phi(y, t) \to x \text{ as } t \to -\infty, \phi(y, t) \in U \text{ for all } t \leq 0\}$$

RECALL: STABLE MANIFOLD THEOREM

Suppose the origin is a *hyperbolic fixed point* for $\dot{x} = f(x)$ in $\mathbb{R}^n$ and that $E^s(0)$ and $E^u(0)$ are the stable and unstable manifolds of the linearised system $\dot{x} = Df(0)x$, then

$\boxed{\text{...there exist local stable and unstable manifolds } W^s_{loc}(0) \text{ and } W^u_{loc}(0) \text{ of the same dimension as } E^s(0) \text{ and } E^u(0), \text{ respectively, and which are (respectively) tangent to } E^s \text{ and } E^u \text{ at the origin.}}$

These manifolds are equally smooth/unsmooth as the original function f.

FINISH: POWER SERIES EXPANSIONS FOR ONE-DIMENSIONAL MANIFOLDS

Example 4.2 from Glendinning.

Assume a stable/unstable manifold of interest can be described by a curve $x = g(y)$ (or $y = h(x)$).

We can try to approximate this by a *local series expansion* of the form

$$g(y) = \sum_{n=0}^{\infty} a_n y^n$$

PERIOIC ORBITS IN THE PLANE

A key question for a given nonlinear ODE system is *whether it admits closed curve solutions* - i.e. *periodic orbits* (oscillations).

How can you *rule then out?* How can you *rule them 'in'?*

These results *typically apply only in the plane* - 'no chaos in the plane'. All bets off (well, almost) for higher dimensions and/or discrete (non-smooth) systems!

NONLINEAR PLANAR SYSTEM

Note that here we are looking at *planar nonlinear systems*

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$

RECALL: PERIOIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t + T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t + s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

RULING OUT PERIODIC ORBITS

The *(Poincare) index/winding number* is a (topological) invariant of closed curves in the plane.

We won't go into it (see p. 126-129 Glendinning, p. 174-180 Strogatz (1994) if interested) but note that it can be used to show (among other things)

Inside any closed orbit there *must be at least one fixed point*.

Example 6.8.5 Strogatz (1994).

RULING OUT PERIODIC ORBITS

Recall the *divergence theorem* (with a weight g):

Suppose Γ is a simple (doesn't cross itself) closed curve with outward normal n enclosing a region R and f and g are continuously differentiable functions $\mathbb{R}^2 \to \mathbb{R}^2$ then

$$\int_\Gamma g(n \cdot f)dl = \int\int_R \nabla \cdot (gf)dxdy$$

where $gf := g(x, y)f(x, y)$.

RULING OUT PERIODIC ORBITS

This can be used to formulate *Dulac's criterion* (see Glendinning 5.6):

If there exists a g (refer previous slide) such that $\nabla \cdot (gf)$ is continuous and has one sign throughout a simply connected domain D then there are *no closed orbits lying entirely in D.*

If we take $g = 1$ then this is often called the *divergence test*.

Example (Glendinning 5.9).

RULING IN PERIODIC ORBITS - THE POINCARE-BENDIXSON THEOREM

The *Poincare-Bendixson theorem* allows one to establish the *existence of a periodic orbit*. It also establishes that there is *'no chaos in the plane'*.

Let D be a closed and bounded domain in the plane and suppose there are *no stationary solutions in D*. Then, if the orbit $\phi(x_0, t)$ *begins in/enters D and does not leave D for all time*, then the *orbit is either closed or spirals toward a closed orbit* as $t \to \infty$

THE POINCARE-BENDIXSON THEOREM - TRAPPING REGIONS

The standard trick to finding an appropriate region is to construct a *trapping region* R - a closed connected subset such that the vector field *points 'inwards' everywhere on the boundary*.

This implies (proof not shown!) that *all orbits are confined* to R (i.e. once in don't leave).

If we can construct an R *without a fixed point inside* then there exists a closed (i.e. periodic) orbit. See Strogatz (1994, 7.3).

PERIODIC ORBITS IN HIGHER DIMENSIONS

Periodic orbits in high dimensions can be (very!) complicated.

We will hence try to *introduce a 'simpler' object to study* which can help us *understand periodic orbits in quite general/complicated systems*.

This leads to our first encounter with another type of dynamical system - *discrete maps*.

Here these will arise as a *tool to study (e.g. periodic orbits in) ODEs*; note that they can arise as interesting models in their own right.

RETURN MAPS

Given a nonlinear ODE system $\dot{x} = f(x)$.

A *Poincare section* is P_Σ, is a transverse section of the trajectories of an ODE system, which is nowhere tangential to any trajectory.

We can label the points in (time) order of their intersection with P_Σ, giving $x_0, x_1, \ldots x_n, \ldots$.

Picture.

RETURN MAPS

The Poincare section defines a (discrete!) *Poincare/return map*

$$x_{n+1} = F(x_n)$$

There is a corresponding *theory of stability/instability/bifurcation for discrete maps*. This can be used to *deduce properties* of e.g. equilibria and periodic orbits in the *original ODE system*.

We will look at doing this with XPP in a later tutorial/assignment.

RETURN MAPS

Picture and intuition.