ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~16-17 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Stability, linearisation and classification. Phase plane analysis* [5-6 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [4 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*

   [4 lectures/tutorials]

   Putting everything together - asymptotic stability, structural stability and bifurcation using the geometric perspective. In particular: the centre manifold theorem, reduction principle and approximately decoupling non-hyperbolic systems.

LECTURE 2 - 'BIG PICTURE P.II'

- More (boring) terminology: definitions, key features of interest, analysis types
- How these fit together

i.e. a quick tour of the *dynamical systems dictionary*

BASIC ANALYSIS PROCEDURE

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).
- Sketch/compute the *phase portrait*.

BASIC ANALYSIS PROCEDURE

...we can then go on to

- find more 'global' features such as *periodic orbits*
- analyse *bifurcations* (loss of stability)
- use *perturbation methods* to construct approximate solutions

etc.

TERMINOLOGY

We've used terms like *equilibria, stability, Jacobian, eigenvalues, bifurcations* etc. What do these mean?

- The next section covers linearisation, stability etc in detail. Then bifurcations.
- Today, though, we'll introduce a few more basic 'geometric' concepts/features of interest
- A preview our analysis procedure can be found in the worked exam Qs from last year.

TERMINOLOGY: STATE/PHASE SPACE

In practice the *'state'* is defined by *'contains everything we need to know to get from the current state to the next state'*. E.g. position and momentum for classical mechanics.

The *state space/phase space* is...the 'space' of all states - usually (embedded in) $\mathbb{R}^n$, for real-valued differential equations.

(More general phase spaces include the circle, torus etc, depending on the 'problem structure')

EXAMPLE

State space example.

TERMINOLOGY: SOLUTIONS AND INTEGRAL CURVES

A *solution* x_s (or trajectory) is a *function* assigning a state vector to each time in a given time interval and which satisfies the ODE, i.e. $x_s : T \subset \mathbb{R} \to \mathbb{R}^n$, where $\dot{x}_s(t) = f(x_s(t))$.

An *integral curve* is the *graph* of a solution, including the time dimension, i.e. the *set of points*

$$\{(x, t) \mid t \in T \text{ and } x(t) \text{ defines a solution}\}$$

TERMINOLOGY: FLOW FUNCTIONS

The *flow function* ϕ is a convenient way to combine our description of *solutions and their dependence on initial conditions*.

We write $\phi(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

where for fixed x_0, $\phi(x_0, t)$ gives the solution to the differential equation at time t which starts from an initial value (at $t = 0$) equal to x_0.

TERMINOLOGY: FLOW FUNCTIONS

So, for any t such that $\phi(x, 0) = x$ we have

$$\frac{d}{dt}\phi(x, t) = f(\phi(x, t))$$

and

$$\phi(x, t + s) = \phi(\phi(x, t), s) = \phi(\phi(x, s), t) = \phi(x, s + t)$$

(for any 'allowed' t, s). We talk about 'flows' when we want to emphasise the dependence on initial conditions (rather than just time)

TERMINOLOGY: ORBITS

Orbits are the geometric objects in the phase plane that are generated by solutions/flows.

In terms of the flow function an orbit beginning at x_0 can be described by $\{\phi(x_0, t) \mid t \in T\}$.

Usually we take $T = \mathbb{R}$ and hence consider all solutions passing through x_0 (and both forwards and backwards in time *if* invertible!).

These can be labelled with a 'time direction' but are otherwise somewhat 'static' (geometric) objects.

TERMINOLOGY: VECTOR FIELD

The solutions/orbits are *tangent to the 'velocity vector'* $(\dot{x}_1, \dot{x}_2, ...)^T$ defined by the ODE $\dot{x} = f(x)$ at each point in the state space (and at each time).

We often call $f(x)$ the *vector field* of the ODE. The *direction* (relative velocity of components) can be determined by dividing through by one (non-zero) component i.e.

$$\frac{\dot{x}_k}{\dot{x}_1} = \frac{dx_k}{dx_1} = \frac{f_k(x_1, ..., x_n)}{f_1(x_1, ..., x_n)}$$

(Now how this relates the 'static' and 'dynamic' objects).

EXAMPLE

A comparison of a solution, integral curve and orbit for a simple example.

Example of vector field.

EXISTENCE AND UNIQUENESS?

We won't go into this, but for a sufficiently smooth system written in our standard form and given appropriate initial conditions *there exists a unique solution*.

Upshot: the solution curves/trajectories (for autonomous systems) *do not intersect in phase space*. We always know exactly where to go next!

TERMINOLOGY: PHASE PORTRAITS

We will summarise these key features in *phase portraits* of a given system.

A phase portrait is a 'picture' of the phase space in which we further *partition it according to orbit/solution 'types' or behaviour* in different regions.

FEATURES OF INTEREST

We will look at (and define!) various *'interesting features'* of our equations in phase space e.g.

- Stationary/fixed/equilibrium points
- Periodic orbits

and try to analyse their properties such as *stability* under different types of 'perturbations' - both 'within' a model (*solution* stability) and 'externally' (*structural* stability) to a model.

INTERESTING FEATURES - EQUILIBRIA

A point x_e is an *equilibrium solution/fixed point/stationary point* iff

$$\phi(x_e, t) = x_e$$

for all t. Equivalently it is a *zero of the vector field* (RHS)

$$f(x_e) = 0$$

MORE FEATURES - PERIODIC POINTS AND PERIODIC ORBITS

A point x_e is a *periodic point* with least period T iff

$$\phi(x_e, t + T) = \phi(x_e, t)$$

for all t and $\phi(x_e, t + s) \neq \phi(x_e, t)$ for $0 < s < T$.

If x_e is a periodic point then the orbit

$$\{\phi(x_e, t) \mid t \in \mathbb{R}\}$$

is a *periodic orbit* passing through x_e.

MORE FEATURES: INVARIANT SETS

A set of points in the phase space M is called *invariant under the flow* if for all $x \in M$ we have

$$\phi(x, t) \in M$$

for all t. That is, every point in M leads to another point in M - once in, we never leave!

MORE FEATURES - LIMIT SETS

Other useful definitions include the following (invariant!) sets:

The *ω-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in forward time*.

Formally it consists of elements y such that there exists a sequence (t_n) with $t_n \to \infty$ and $\phi(x, t_n) \to y$ as $n \to \infty$.

MORE FEATURES - LIMIT SETS

The *α-limit set* of a point $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all points y to which the flow from x *tends to in backwards time* (exercise: write down the formal definition!)

Note that the points in these sets *don't have to lie on the orbits* through x - they are *limit* points for a reason!

HOMEWORK

- Downloading and running XPPAut - see Canvas
- Strogatz videos
- Read through the preview worked exam Qs