ENGSCI 213: MATHEMATICAL MODELLING 2SE

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MODULE OVERVIEW

Markov Processes (*Oliver Maclaren*) [~~6 lectures~~ 5 lectures]

1. *Basic concepts* [2 lectures]

   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions. Diagrams of Markov chains.

2. *Properties of Markov chains* [2 lectures]

   Accessibile, recurrent, transient states. Communication of states. Stationary/invariant distributions and limiting/equilibrium distributions. Estimating transition matrices. Expected value calculations using invariant/limiting distributions.

3. *Applications of Markov chains* [~~2 lectures~~ 1 lecture]

   Introduction to MCMC. We can use the other lecture as revision/or play around with MCMC some more.

LECTURE 4

Recap of some properties/examples of Markov chains

Using invariant/limiting distributions to calculate expected values

Estimating transition matrices

RECALL - PROPERTIES

We looked at properties at the level of

States

- Accessibile, recurrent, transient, absorbing states.
- Communication of states; classes, trapping sets

and

Distributions

- Stationary/invariant distributions.
- Limiting/equilibrium distributions.

RECALL - WHAT DO WE WANT TO KNOW?

- What are the possible dynamics?
  - Which *states* can we get to?
  - Which *states* attract' or 'trap' the process etc?
  - Which *distributions* does the process tend to/get 'trapped' in?
- What does this look like *graphically*?
- What does this look like in terms of *matrix calculations*?

WHAT ELSE DO WE WANT? EXPECTED VALUES

If we reach a stationary (invariant) or equilibrium (limiting) distribution we can *use this like an ordinary probability distribution*, e.g. to *calculate expected values* of random variables following this distribution!

We'll see this shortly. First a quick recap of some examples.

RECALL: COMMUNICATION AND ACCESSIBILITIY OF STATES

We say that a state *i communicates with j* if $p_{ij}(n) > 0$ for some n, and write $i \rightarrow j$.

(Also: i *reaches* j, j is *accessible* from i)

If $i \rightarrow j$ and $j \rightarrow i$ then we say that *i and j communicate*, and write $i \leftrightarrow j$.

This is easiest to see in a *state transition diagram*.

RECALL: COMMUNICATION AND ACCESSIBILITIY OF STATES

Question: does a state communicate with itself even if it has no arrows pointing to itself?

STATIONARY (INVARIANT) DISTRIBUTIONS.

- Given a candidate invariant distribution, easy to *check by matrix multiplication*.
- *Find by solving an eigenvalue problem*: $\pi\mathbb{P} = \lambda\pi$ with $\lambda = 1$.

Question: how to I solve this eigenvalue problem?

EXPECTED VALUE CALCULATIONS

Suppose we are interested in a function $f(X_n)$ of the state X_n of our Markov process.

This might *represent e.g. the 'value' or 'score'* etc for *being in state* $X_n = j$ for each possible state value.

Since X_n varies stochastically, so will $f(X_n)$, though it is known exactly for each given X_n value. Hence it has the *same distribution as* X_n.

EXPECTED VALUE CALCULATIONS - STATIONARY/LIMITING DISTRIBUTION

Suppose our Markov chain 'settles down' to a unique stationary/limiting* distribution π

Then it can be shown

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n) \to \mathbb{E}_\pi(f)$$

where

$$\mathbb{E}_\pi(f) := \sum_j f(j)\pi_j$$

* Note again that we won't distinguish these here - assume both exist and are equal.

EXPECTED VALUE CALCULATIONS - STATIONARY/LIMITING DISTRIBUTION

Upshot - *suppose we have a 'value' measure $V(j)$ of being in each state j stored, say, in the (row) vector v* and a stationary (or equilibrium) distribution π.

Then the *'long-term' value* - i.e. expected value - for the process is *given by the expected value of v under the distribution π* and is calculated by the *inner product*

$$\pi v^T \text{ or, equivalently, } v\pi^T$$

(note how the transpose is used for the inner product of row vectors here, c.f. column vectors)

EXPECTED VALUE CALCULATIONS - STATIONARY/LIMITING DISTRIBUTION

Example.

ESTIMATING TRANSITION MATRICES FROM DATA

A *realisation* of a Markov chain simply consists of a sequence of values of the state variable.

E.g. if $\mathbb{X} = \{0, 1, 2\}$ then a realisation might be 0, 1, 1, 2, 1, 1, 2, 2, 0,...

Given a finite sequence of such observations *how might we estimate the single-step transition matrix?*

Answer - *count*, just the way you would expect! (BTW - it turns out this is the M.L.E.)

ESTIMATING TRANSITION MATRICES FROM DATA

Given a state space written as $\mathbb{X} = \{1, 2, \ldots, N\}$ and a *matrix of counts* $\mathbb{C}$ for each transition - e.g. $C_{01} = 5$ means 5 transitions from state 0 to state 1, we *estimate the transition matrix by*

$$\mathbb{P}_{ij} = \frac{\mathbb{C}_{ij}}{\sum_{j=1}^{N} \mathbb{C}_{ij}}$$

Note the normalisation. Example.