Engsci 711

Tutorial 2: Full phase plane analysis

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Overview

The purpose of this tutorial is to give you some practice analysing (mainly) two-dimensional systems by hand from woah to go. There is also a computer-based exercise for constructing Poincare sections and looking at the three-dimensional Lorenz equations.

Tips and tricks

Analysis procedure

Given a nonlinear system $\dot{x} = f(x)$, the usual first steps we'll follow in this course are

- Find all the *equilibria* x_e by solving $f(x) = 0$.
- Find the *linearisation* $\dot{u} = Df(x_e)u$ where Df is the Jacobian matrix associated with f and $u = x - x_e$.
- Determine all the *eigenvalues* of Df at the equilibrium points and hence the local stability of the equilibria.
- *Classify* each equillibrium (eg. as a saddle, node, etc).

- Sketch/compute the *phase portrait*.

Tips for phase portrait sketching

- Start drawing locally near individual fixed points
- Check for any obvious invariant axes, lines/curves
- Draw nullclines and any other helpful/obvious flow directions (e.g. trapping regions)
- Think about whether periodic orbits might exist anywhere
- Think about various ways the local flows might connect up or be extended more globally.

Calculating manifolds

Consider a system such as $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$. We usually use our information (give or take a swap of x, y variables etc) to construct series expansions for stable/unstable manifolds as follows

- Assume the manifold can be described by a functional relationship such as $y = h(x)$.
- Substitute $y = h(x)$ into our x and y equations to give $\dot{x} = f_1(x, h(x))$ and $\dot{y} = f_2(x, h(x))$.
- Use the functional relation again, along with the chain rule for our y (say) equation $\dot{y} = f_2(x, y)$, to relate $\dot{x}$ and $\dot{y}$ giving (e.g.) $\dot{y} = \frac{dh}{dx}\dot{x}$.
- Use the above relationships along with an assumed power series expansion such as $h(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ to obtain two polynomial expressions in x (say) for $\dot{y}$ involving the unknown coefficients of the power series. Equate powers of x to determine the coefficients.
- You will need to use the information that the stable/unstable manifold passes through the fixed point and is tangent to the linearised stable/unstable manifold to determine the first two terms of the series. These will not be zero in general (but should be known)!

Exercises

Find the equilibria, classify them and sketch the phase portraits for the following.

a)

$$\begin{aligned} \dot{x} &= x(1 - x - y) \\ \dot{y} &= y(2 - x - y) \end{aligned}$$

b)

$$\ddot{x} + xx^3 = 0$$

c)

$$\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3 \end{aligned}$$

For this last case see if you can prove/disprove the existence of periodic solutions.

Periodic orbits

Determine whether the following systems have any periodic orbits.

a)

$$\dot{x} = y$$
$$\dot{y} = (x^2 + 1)y - x^5$$

b)

$$\dot{x} = y$$
$$\dot{y} = y^2 + x^2 + 1$$

c)

$$\dot{x} = 1 + x^2 + y^2$$
$$\dot{y} = (x - 1)^2 + 4$$

Approximating stable/unstable manifolds

Consider

$$\dot{x} = 2x + y$$
$$\dot{y} = -y + x^2$$

- Find the equilibria.
- Find the linearisation about $(0, 0)$.
- Find $E^u(0, 0)$ and $E^s(0, 0)$
- Construct power series approximations for $W^u_{loc}(0, 0)$ and $W^s_{loc}(0, 0)$.