ENGSCI 213: MATHEMATICAL MODELLING 2SE

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MODULE OVERVIEW

Markov Processes (*Oliver Maclaren*) [6 lectures]

1. *Basic concepts* [2 lectures]
   Motivation and key questions. Definitions - state space, index set, Markov property, Markov processes, Markov chains, homogeneous Markov chains. Transition probabilities and matrices. Chapman-Kolmogorov equation and n-step matrices. Initial and marginal distributions.

2. *Properties of Markov chains* [2 lectures]
   Diagrams of Markov chains. Accessibile, recurrent, transient states. Communication of states. Stationary/invariant distributions and limiting/equilibrium distributions.

3. *Applications of Markov chains* [2 lectures]
   Modelling with Markov chains. Value calculations. Possible examples: random walks, branching processes, a hint of MCMC.

LECTURE 1

- Stochastic processes, state space, index set.
- Markov property, Markov processes, Markov chains.
- Homogeneous Markov chains
- Transition probabilities and transition matrices.
- Examples.

RECALL: SEQUENCES OF RANDOM VARIABLES AND TRIALS

We can often think of the result from an *overall* experiment to consist of an ordered sequence of sub-experiments - or *trials* - where *each trial has some relation to the previous*.

If the outcome of trial n, for each $n = 1, 2, \ldots N$, can be described by a RV X_n, then a collection

$$\{X_n \mid n = 1, 2, \ldots N\}$$

defines a *random experiment given by sequence of random variables* (or a discrete-time stochastic/random process).

DYNAMIC VS STATIC

You might imagine this as switching back and forward between thinking of the 'overall, static' outcome of the full experiment and the 'step-by-step, dynamic' outcomes of each trial.

Here we will focus on *processes where the outcome of the next trial depends on the outcome of the current trial* - compare with the independent Bernouilli trials!

These processes are the basic building blocks of stochastic process theory.

RECALL: STOCHASTIC PROCESSES

First, recall that a *stochastic process* is an indexed collection of random variables:

$$\{X_t \mid t \in T\}$$

Each of the RVs X_t takes values in the same set $\mathbb{X}$ called the *state space*.

- *This can be discrete or continuous*.

The RVs are 'indexed' by $t \in T$, called the *index set* (think: time/stage/trial number).

- *This can also be discrete or continuous*.

STOCHASTIC PROCESSES

A *realisation* of a stochastic process is a particular value for the *whole sequence* (i.e. 'overall experiment').

Examples.

- IID trials
- Weather
- Stock prices

MARKOV PROCESSES AND THE MARKOV PROPERTY

A *Markov process* is a stochastic process for which *the future only depends on the current state* and not the rest of the past.

This is called the *Markov property*.

MARKOV PROCESSES AND THE MARKOV CHAINS

Terminology varies, but *we will call a discrete-valued, discrete-time Markov process a Markov chain*.

> We will only consider (what we have called) Markov chains in this module.

We often use n rather than t for the index variable (think 'stage' vs 'time'). Also note that T is sometimes called the *horizon*

MARKOV PROPERTY FOR MARKOV CHAINS

The *Markov property is easy to formulate for Markov chains*.

We say that the discrete-state, discrete-time process $\{X_t \mid n \in T\}$ *satisfies the Markov property* - i.e. is a Markov chain - if

$$P(X_n = x \mid X_0, X_1, \ldots, X_{n-1}) = P(X_n = x \mid X_{n-1})$$

for all $n \in T$ and $x \in \mathbb{X}$.

Example.

HOMOGENEOUS MARKOV CHAINS

The key properties of a process are the *probabilities of jumping from one state to another*.

A Markov chain is *homogeneous* if these probabilities depend only on the state and not on the time. i.e.

$$P(X_{n+1} = j \mid X_n = i)$$

does not depend on n, i.e.

$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i)$ etc.

HOMOGENEOUS MARKOV CHAINS

We will (mainly?) only consider homogeneous Markov chains in this course (unless explicitly stated).

TRANSITION PROBABILITIES AND TRANSITION MATRIX

For homogeneous Markov chains the *transition probabilities* are defined by

$$p_{ij} := P(X_{n+1} = j \mid X_n = i)$$

The matrix $\mathbb{P}$ with (i, j)th element $\mathbb{P}_{ij}$ equal to p_{ij} is called the *transition matrix*.

Note: i is 'from' and j is 'to'.

EXAMPLES

Random walk

Weather

HOMEWORK

What were the key definitions introduced today?

Try simulating a Markov process! (I'll put up an example on Canvas/consider in the tutorial).