ENGSCI 711

QUALITATIVE ANALYSIS OF DIFFERENTIAL EQUATIONS

(*...and other dynamical systems*)

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MODULE OVERVIEW

Qualitative analysis of differential equations (*Oliver Maclaren*) [~14 lectures/tutorials]

1. *Basic concepts* [3 lectures/tutorials]

   Basic concepts and (boring) definitions: state/phase space, solutions, integral curves, flows, orbits and vector fields. Key qualitative features of interest. Overview of basic analysis procedures. Computer-based analysis.

2. *Phase plane analysis, stability, linearisation and classification* [5 lectures/tutorials]

   General linear systems. Linearisation of nonlinear systems. Analysis of two-dimensional systems - stability and classification of fixed points, periodic orbits. Geometry (invariant manifolds).

MODULE OVERVIEW

3. *Introduction to bifurcation theory* [3 lectures/tutorials]

   Hyperbolic vs non-hyperbolic systems and structural instability. Various types of bifurcations. Bifurcation diagrams.

4. *Centre manifold theory and putting it all together*
   [3 lectures/tutorials]

   Geometry of non-hyperbolic systems. In particular: centre manifold theorem and reduction/emergence principle. Applications: asymptotic stability of non-hyperbolic systems, justification of bifurcation theory using the geometric perspective, fast/slow systems.

LECTURE 10

Implications and applications of centre manifold theory

- Reduction/emergence principle
- Extension to systems with parameters (*assignment/tutorial* material only - not on exam)
  - Application to bifurcation theory
  - Application to fast/slow systems (geometric singular perturbation theory)

RECALL: CENTRE MANIFOLD THEOREM

Consider $\dot{x} = f(x)$ having a *non-hyperbolic fixed point* at $x = 0$, where $x \in \mathbb{R}^n$.

Assume that there are n^+ eigenvalues (counting repeated cases) with Re $\lambda > 0$, n^0 eigenvalues with Re $\lambda = 0$, and n^- eigenvalues with Re $\lambda < 0$.

RECALL: CENTRE MANIFOLD THEOREM

Then there is a locally defined smooth n^0-dimensional invariant manifold $W^c_{loc}(0)$ that is tangent to the (linear) centre eigenspace E^c.

Moreover, there is a neighborhood U of $x_0 = 0$, such that if $\phi(x, t) \in U$ for all $t \geq 0$ (≤ 0) then $\phi(x, t) \to W^c_{loc}(0)$ for $t \to \infty$ ($t \to -\infty$).

RECALL: INTERPRETATION

A system is *exponentially attracted to* (or repelled from if we allow for positive eigenvalues) *the centre manifold*.

We can *formalise* this a bit better in terms of a *reduction* near a non-hyperbolic fixed point to a *decoupled* system of *three subsystems* with eigenvalues that have real part positive, negative and zero , respectively.

This generalises the idea of decoupling a system near a hyperbolic fixed point into a linear, real part positive subsystem and a linear, real part negative subsystem.

RECALL: INTERPRETATION

The price of the decoupling reduction is that the *centre manifold* subsystem is (typically) *nonlinear*. On the other hand, the *stable/unstable* subsytems will be *fast* (disappearing when $t \to \infty$) and the *centre manifold dynamics* will be *slow* and hence *emerge* as $t \to \infty$.

For example, a stable non-hyperbolic nonlinear system will, instead of decaying to a unique fixed point as in a hyperbolic system, *rapidly decay to* a centre/slow manifold and then *travel along this at a slower rate*.

Q: what happens in the linearised non-hyperbolic system?

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

If $(x, y, z) \in E^c \times E^s \times E^u$ are *coordinates in terms of the system's eigenbasis (diagonalised/Normal form)* representation then we *first* write our system as

$$\begin{aligned} \dot{u} &= Au + g(u, v) \\ \dot{v} &= Bv + h(u, v) \end{aligned}$$

where $u \in \mathbb{R}^{n^0}$ are our centre manifold variables and $v \in \mathbb{R}^{n^+ + n^-}$ are our (locally) exponentially growing/decaying solutions (lumped together for convenience).

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

Note

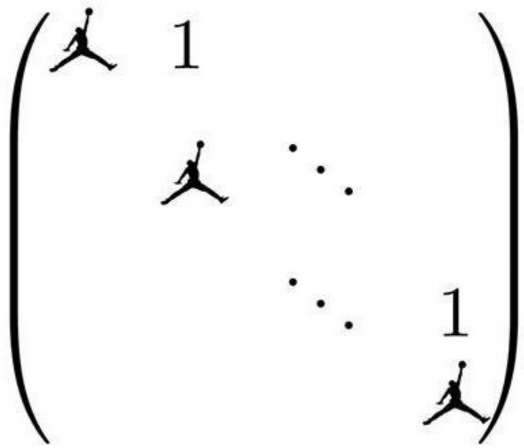
- The matrices A and B have *eigenvalues with zero and non-zero real-part* respectively, (and A is just zero if there are no imaginary parts), while
- The *functions g and h represent the higher-order* (at least quadratic) terms (since A and B represent the linear dynamics) and we will assume they have Taylor expansions (which clearly start from quadratic order)

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

The key point is to write the equations so that they are *linearly decoupled* according to the *sign of the real part of the eigenvalues*

This corresponds to putting the *linear part* of the system in (Jordan) *'normal form'* (think: diagonal).

INTERLUDE: TRANSFORMING TO JORDAN NORMAL FORM



...see handouts...

Key is *diagonalising* the *linear part* or getting *as close as possible*.

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

We then assume (justified by the CMT, and just like for the stable/unstable manifold case) that we can *locally represent the centre manifold by a smooth curve*, i.e.

$$W^c_{loc} = \{ (u, v) \mid v = V(u) \}$$

(this may be a vector equation if v and/or u are multi-dimensional quantites!)

CENTRE MANIFOLD THEOREM - REDUCTION PROCEDURE

As described previously, the basic idea is to substitute this relationship into the *chain rule* applied *along the manifold*:

$$\frac{dv}{dt}(u, V(u)) = \frac{dv}{du}(u)\frac{du}{dt}(u, V(u))$$

from which to find $V(u)$.

Let's call this the *'manifold equation'*. We usually solve it by assuming a *power series solution*, as discussed previously (and justified by the CMT).

CENTRE MANIFOLD THEOREM - REDUCTION PRINCIPLE

Putting all this together leads to the following *Reduction (or decoupling) Principle*:

Near a non-hyperbolic fixed point our system (written in its eigenbasis/diagonalised form) is *locally* topologically equivalent to the system

$$\begin{aligned} \dot{u} &= Au + g(u, V(u)) \\ \dot{v} &= Bv \end{aligned}$$

where $V(u)$ is the expression for the centre manifold (found from the procedure on the previous slide).

CENTRE MANIFOLD THEOREM - EMERGENCE PRINCIPLE

Note that these reduced, local dynamics are now *uncoupled* and the dynamics in v are *linear, 'fast' and essentially 'trivial'!*

This allows us to justify (in particular when the linear fast dynamics are stable) using the following *emergent, long-time approximate model of the full system*

$$\dot{u} = Au + g(u, V(u))$$

i.e. we can *just focus on the centre manifold dynamics*.

EXAMPLE CONTINUED

Exercise: complete the example (find the centre manifold and the flow on this).

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

Note: the material on *extended* centre manifolds that *follows next is not on the exam*, but may be in the assignment to some extent.

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

We surely want to consider systems where *some eigenvalues are much smaller* than the others but *not exactly zero*.

We also want to analyse the *dynamics in systems with parameter-dependent bifurcations*.

Both of these cases can handled by constructing an *extended* centre manifold which *includes the parameter(s)* of interest.

EXTENDED CENTRE MANIFOLD - EXTENSION TO PARAMETERS

The key trick is simple: treat the *parameter* of interest as a *(super slow!) centre state variable*. i.e. rewrite a system like

$$\dot{x} = f(x;\mu)$$

as

$$\begin{aligned}\dot{x} &= f(x,\mu)\\ \dot{\mu} &= 0\end{aligned}$$

where μ is *now a state variable*. Note: this means that in the second system terms like e.g. μx in f are now considered *nonlinear!*

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

How is this relevant to *bifurcation* theory? Suppose we have a *non-hyperbolic fixed point* $x = 0$ when $\mu = 0$. From Wiggins (2003):

> *...the center manifold exists for all μ in a sufficiently small neighborhood of $\mu = 0$... [but as we know] it is possible for solutions to be created or destroyed by perturbing nonhyperbolic fixed points...*

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

> *...Thus, since the invariant center manifold exists in a sufficiently small neighborhood in both x and μ of $(x, \mu) = (0, 0)$, all bifurcating solutions will be contained in the lower dimensional [extended] center manifold.*

i.e.

'...all the action is on the centre manifold...'

EXTENDED CENTRE MANIFOLD - APPLICATION TO BIFURCATION THEORY

The easiest way to understand this is via an example.

Transcritical bifucation example: *bifurcation analysis the long way* (using centre manifold theory).

CENTRE MANIFOLD THEOREM - APPLICATION TO SINGULAR PERTURBATION PROBLEMS

See assignment?