MATHS 361 PARTIAL DIFFERENTIAL EQUATIONS

Oliver Maclaren

oliver.maclaren@auckland.ac.nz

HOUSEKEEPING

~~LEARNING GOALS~~ WHAT DO I NEED TO KNOW FOR THE EXAM?

I will put up a summary sheet for each sub-topic (set of 3-4 lectures)

PER UNIT WHAT? (FROM LECTURE 2)

The quantities in PDEs are usually measured per unit length/volume/time etc. In our typical 1-D case

- $u(x, t)$ is a *stored* amount of substance *per unit length**
- $j(x, t)$ is a *transported* amount of substance *per unit time* (*flux*)
- $f(x, t)$ is a *source/added amount* of substance *per unit length, per unit time*

* Or per unit volume, mass etc in general, depending on the specific problem. This is usually called a *density*, e.g. mass density, energy density etc

PER UNIT WHAT? (LECTURE 2)

Decide on the appropriate the units/dimensions of the terms in the following balance (valid for small Δx)

$$u_t(x,t)A\Delta x = j(x,t)A - j(x+\Delta x)A + f(x,t)A\Delta x$$

RECAP OF BC/IC (FROM LECTURE 2)

See end of Lecture 2 slides - motivation, types, example simulation.

LECTURE 3

Introduction to separation of variables

Steady state solutions

LET'S SOLVE AN EQUATION BY HAND!

...using *separation of variables*

BASIC IDEA - GUESS AND CHECK!

Ansatz?

- An ansatz is a guess (hopefully educated!) about the *form* of a solution. We use it to constrain our search for exact or approximate solutions.
- Once we find a candidate solution of this form we can then check whether it does in fact satisfy the original problem.
- We will also need to satisfy the *boundary and initial conditions!*

SEPARATION OF VARIABLES

The basic ansatz for separation of variables is to guess* that we can split a function like $u(x, t)$ into

$$u(x, t) \stackrel{?}{=} X(x)T(t)$$

We then substitute this into our equation and see what happens!

* An intuitive motivation for why it might be a reasonable idea/how to interpret it: for probability distrbutions in two variables we can write $P(x, y) = P(x|y)P(y)$ in general. For *independent* probabilities then we can further write this as $P(x, y) = P(x)P(y)$.

HEAT EQUATION

Let's write the heat equation as

$$u_t = Du_{xx}$$

for $0 < x < 1$, where here u is temperature (not energy!) and use

BC

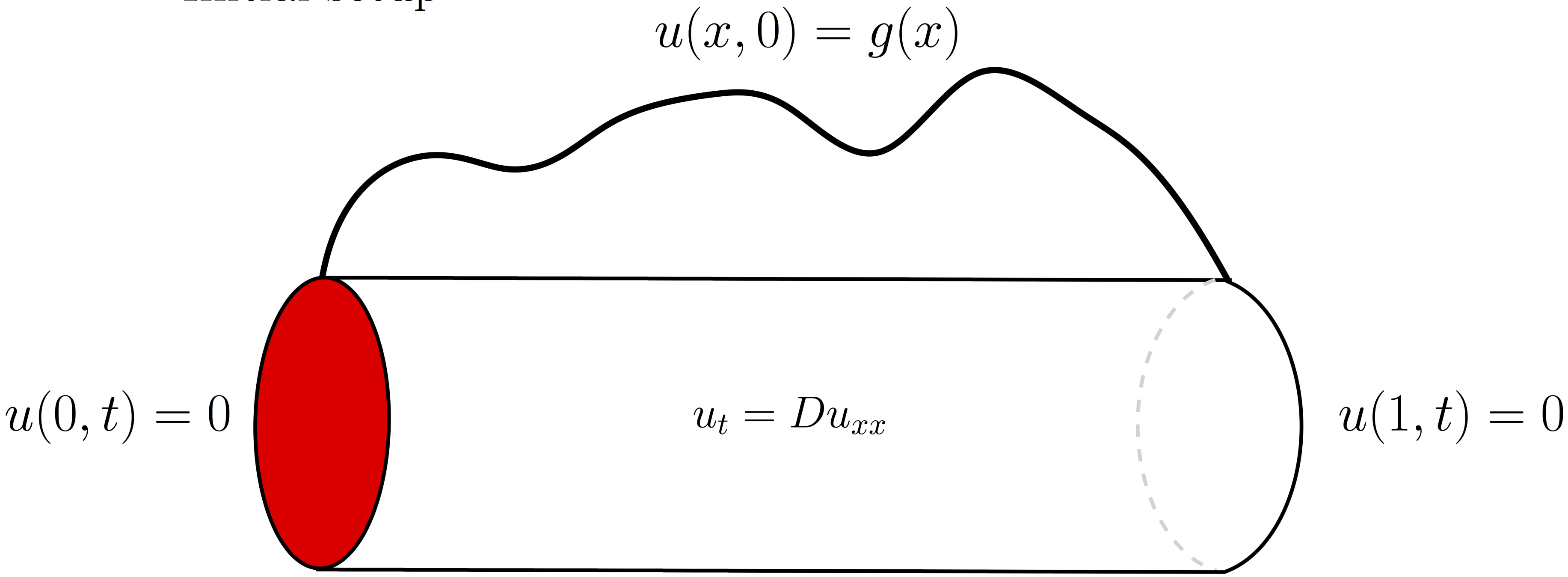
- $u(0, t) = 0 \quad \text{for} \;\; 0 < t < \infty$
- $u(1, t) = 0 \quad \text{for} \;\; 0 < t < \infty$

IC

- $u(x, 0) = g(x) \quad \text{for} \;\; 0 \le x \le 1$ and $g(x)$ known.

SCHEMATIC OF FORMULATION

Initial setup



SEPARATION ASSUMPTION

Substituting our guess we get

$$\frac{dT(t)}{dt}X(x) = \frac{k}{C}\frac{d^2X(x)}{dx^2}T(t)$$

Note that we now have *ordinary* derivatives for X and T.

HEAT EQUATION

Rearranging we get

$$\frac{dT(t)}{dt}\frac{1}{T(t)}\frac{1}{D} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$$

i.e.

$$F(t) = G(x)$$

But how can a function of x be equal to a function of t *for all* x, t?

HEAT EQUATION

We deduce that both must be *constant** functions (and equal to the same constant), giving

$$\frac{dT(t)}{dt}\frac{1}{T(t)}\frac{1}{D} = \lambda$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda$$

Thus we get *two separate ODEs to solve* instead of one PDE.

* Exercise - try to give a more mathematical proof/argument for this.

LINEAR ODE FACTS

You should (aim to) know how to write down the solution to these homogeneous *ODEs* without thinking

- $u' + au = 0$
- $u'' - k^2 u = 0$
- $u'' + \omega^2 u = 0$

COMPLEX EXPONENTIALS, TRIGONOMETRIC AND HYPERBOLIC FUNCTION FACTS

...know these too (and what their graphs look like):

$$e^{iy} = cos(y) + isin(y)$$
$$e^{-iy} = cos(y) - isin(y)$$

$$cos(y) = (e^{iy} + e^{-iy})/2$$
$$sin(y) = (e^{iy} - e^{-iy})/2i$$

$$cosh(y) = (e^{y} + e^{-y})/2$$
$$sinh(y) = (e^{y} - e^{-y})/2i$$

TIME SOLUTION

$$\frac{dT(t)}{dt}\frac{1}{T(t)}\frac{1}{D} = \lambda \implies T(t) = Ae^{\lambda D t}$$

for A an arbitary constant

TIME SOLUTION

We want either

- *Decay in time* (c.f. blow-up), i.e. $\lambda < 0$ (physically, we take $D > 0$), or
- *Steady state/time-independent* solutions, i.e. $\lambda = 0$

So use $\lambda = -\alpha^2 \leq 0$ (α is real) and

$$T(t) = \begin{cases} Ae^{-\alpha^2 Dt}, & \text{if } \alpha \neq 0 \\ B, & \text{if } \alpha = 0 \end{cases}$$

For A, B arbitrary constants.

SPACE SOLUTION

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda = -\alpha^2$$

$$\Longrightarrow$$

$$X(x) = \begin{cases} Asin(\alpha x) + Bcos(\alpha x), & \text{if } \alpha \neq 0 \\ C_0 + C_1 x, & \text{if } \alpha = 0 \end{cases}$$

for A, B, C_0, C_1 arbitrary constants* .

* Note - These are different from the previous slide - I'm just re-using letters due to laziness. Note I'm also trying to avoid using D for the diffusion coefficient and for a different, arbitrary, constant by introducing C_0 and C_1 !

COMBINED SOLUTION

For this equation

- $\alpha \neq 0$ corresponds to transient decay solutions $u^{tr}(x, t)$ (we rule out $\alpha > 0$, i.e. 'blow-up' solutions)
- $\alpha = 0$ corresponds to steady-state (time-independent) solutions $u^{ss}(x, t) = u^{ss}(x)$

Since we have a *linear* equation the sum of these is also a solution so we write

$$u(x, t) = u^{tr}(x, t) + u^{ss}(x, t)$$

where...

COMBINED SOLUTION

...since $u(x, t) = T(t)X(x)$...we get (after merging arbitrary constants)...

$$u^{tr}(x, t) = e^{-\alpha^2 Dt}[Asin(\alpha x) + Bcos(\alpha x)]$$

$$u^{ss}(x, t) = C_0 + C_1 x = u^{ss}(x)$$

where A, B, C_0, C_1 and α are arbitrary, and are to be *determined from IC/BC*.

** verify that this satisfies the PDE by direct substitution!*

STEADY-STATE AND BOUNDARY CONDITIONS

We will choose the *steady-state solution to satisfy the boundary conditions* for *all time* since the transient eventually $\rightarrow 0$. Here we need

$$u^{ss}(x=0,t) = u^{ss}(x=0) = C_0 = 0$$
$$u^{ss}(x=1,t) = u^{ss}(x=1) = C_1 = 0$$

So we just have the *trivial steady-state solution* and we will hence focus on the *transient* solutions. In the case of *non-homogeneous BC (in particular) the steady-state solution will be chosen to match these* (if possible).

A NOTE ON STANDARD FORM FOR SEPARATION OF VARIABLES

In general, separation of variables works when we can write the problem in *standard form* where

- The *PDE is linear and homogeneous*
- The *boundary conditions are linear and homogeneous*, i.e.

$$\alpha u(x = x_a, t) + \beta u_x(x = x_a, t) = 0$$

for arbitrary constants α, β

CONVERTING TO STANDARD FORM

For *some* problems we can *remove non-homogeneous terms* by using a *transient + steady-state (or similar) split* like we considered.

One problem is that *removing* non-homogeneous *BCs* often *creates* a non-homogeneous RHS in the new *PDE* (and *vice-versa*)

Later we will look at a number of specific examples, and also investigate the *general solvability* of (linear) equations + BC/IC in more detail...

BOUNDARY CONDITIONS

...But for now back to our problem. Substitute our candidate *transient* solutions into the BC to get

$$u(x=0,t) = 0 = e^{-\alpha^2 Dt}[Asin(0) + Bcos(0)] \implies B = 0$$

$$u(x=1,t) = 0 = e^{-\alpha^2 Dt}[Asin(\alpha) + \cancel{B}\, cos(\alpha)]$$

$$= e^{-\alpha^2 Dt} Asin(\alpha)$$

$$\implies Asin(\alpha) = 0$$

$$\implies \alpha = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

(for $A, \alpha \neq 0$)

INFINITE SERIES

We have found an *infinite number of solutions* of the form

$$u_n(x, t) = A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$
$$n = 1, 2, \ldots$$

Since we have a *linear* equation, our general solution will be constructed as a sum of these *fundamental solutions* (or 'modes'), i.e.

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$

where...

INITIAL CONDITIONS

...the A_n describe *how much* each fundamental solution contributes to the solution of our *particular* problem and are *determined by the initial conditions*.

We require

$$u(x, t = 0) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n sin(n\pi x) = g(x)$$

For a known IC - or 'initial stimulus' $g(x)$.

INITIAL CONDITIONS

Determining the A_n from an expression like

$$\sum_{n=1}^{\infty} A_n sin(n\pi x) = g(x)$$

requires us to learn some new mathematics - orthogonal functions and Fourier series...next module!

HOMEWORK

Follow the same steps for heat equation problem with the new BC:

A

$$u(0,t) = u_1 \quad \text{for} \;\; 0 < t < \infty$$
$$u(1,t) = u_2 \quad \text{for} \;\; 0 < t < \infty$$

for known constants u_1, u_2

B

$$u_x(0,t) = 0 \quad \text{for} \;\; 0 < t < \infty$$
$$u_x(1,t) = 0 \quad \text{for} \;\; 0 < t < \infty$$

...you should find that the general solutions are

$$A.\ u(x,t) = u_1 + (u_2 - u_1)x + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 Dt} sin(n\pi x)$$

$$B.\ u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 Dt} cos(n\pi x)$$

Q: What does the expression relating the A_n to $g(x)$ look like in each case?

Hints: - Split into a non-trivial steady-state solution satisfying the boundary conditions and a decaying transient part. Solve the resulting problem for the transient part using the same steps. *Combine* both solutions and *then* match the IC.