# 1 A Set of Axioms for the Real-Number System

## 1.1 The field axioms

**Axiom 1.1 (Commutative laws)** $x + y = y + x, \quad xy = yx.$

**Axiom 1.2 (Associative laws)** $x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z.$

**Axiom 1.3 (Distributive law)** $x(y + z) = xy + xz.$

**Axiom 1.4 (Existence of identity elements)** *There exist two distinct real numbers, which we denote by 0 and 1, such that for every real $x$ we have $x + 0 = x$ and $1 \cdot x = x$.*

**Axiom 1.5 (Existence of identity elements)** *For every real number $x$ there is a real number $y$ such that $x + y = 0$.*

**Axiom 1.6 (Existence of reciprocals)** *For every real number $x \neq 0$ there is a real number $y$ such that $xy = 1$.*

**Theorem 1.1 (Cancellation law for addition)** *if $a + b = a + c$, then $b = c$ (In particular, this shows that the number 0 of Axiom 1.4 is unique.)*

**Theorem 1.2 (Possibility of subtraction)** *Given $a$ and $b$, there is exactly one $x$ such that $a+x = b$. This $x$ is denoted by $b - a$. In particular, $0 - a$ is written simply $-a$ and is called the negative $a$.*

**Theorem 1.3** $b - a = b + (-a).$

**Theorem 1.4** $-(-a) = a.$

**Theorem 1.5** $a(b - c) = ab - ac.$

**Theorem 1.6** $0 \cdot a = a \cdot 0 = 0$

**Theorem 1.7 (Cancellation law for multiplication)** *if $ab = ac$ and $a \neq 0$. then $b = c$. (In particular, this shows that the number 1 of Axiom 1.4 is unique.)*

**Theorem 1.8 (Possibility of division)** *Given $a$ and $b$ with $a \neq 0$, there is exactly one $x$ such that $ax = b$. This $x$ is denoted by $b/a$ or $\frac{b}{a}$ and is called the quotient of $b$ and $a$. In particular, $1/a$ is also written $a^{-1}$ and is called the reciprocal of $a$.*

**Theorem 1.9** *If $a \neq 0$, then $b/a = b \cdot a^{-1}$.*

**Theorem 1.10** *If $a \neq 0$, then $(a^{-1})^{-1} = a$.*

**Theorem 1.11** *If $ab = 0$, then $a = 0$ of $b = 0$.*

**Theorem 1.12** $(-a)b = -(ab)$ *and* $(-a)(-b) = ab$.

**Theorem 1.13** $(a/b) + (c/d) = (ad + bc)/(bd)$ *if* $b \neq 0$ *and* $d \neq 0$.

**Theorem 1.14** $(a/b)(c/d) = (ac)/(bd)$ *if* $b \neq 0$ *and* $d \neq 0$.

**Theorem 1.15** $(a/b)/(c/d) = (ad)/(bc)$ *if* $b \neq 0$, $c \neq 0$ *and* $d \neq 0$.

## 1.2 The order axioms

**Axiom 1.7** *If $x$ and $y$ are in $\mathbb{R}^+$, so are $x + y$ and $xy$.*

**Axiom 1.8** *For every real $x \neq 0$, either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$, but not both.*

**Axiom 1.9** $0 \notin \mathbb{R}^+$.

Now we can define the symbols $<$, $>$, $\leq$, and $\geq$, called, respectively, *less than*, *greater than*, *less than or equal to*, and *greater than or equal to*, as follows:

- $x < y$ means that $y - x$ is positive;
- $x > y$ means that $x < y$;
- $x \leq y$ means that either $x < y$ or $x = y$;
- $y \geq x$ means that $x \leq y$.

**Theorem 1.16 (Trichotomy law)** *For arbitrary real numbers $a$ and $b$, exactly one of the three relations $a < b$, $b < a$, $a = b$ holds.*

**Theorem 1.17 (Transitive law)** *If $a < b$ and $b < c$, then $a < c$.*

**Theorem 1.18** *If $a < b$, then $a + c < b + c$.*

**Theorem 1.19** *If $a < b$ and $c > 0$, then $ac < bc$.*

**Theorem 1.20** *If $a \neq 0$, then $a^2 > 0$.*

**Theorem 1.21** $1 > 0$.

**Theorem 1.22** *If $a < b$ and $c < 0$, then $ac > bc$.*

**Theorem 1.23** *If $a < b$, then $-a > -b$. In particular, if $a < 0$, then $-a > 0$.*

**Theorem 1.24** *If $ab > 0$, then both $a$ and $b$ are positive or both are negative.*

**Theorem 1.25** *If $a < c$ and $b < d$, then $a + b < c + d$.*

## 1.3 Integers and rational numbers

**Definition 1.1 (Inductive set)** *A set of real numbers is called an inductive set if it has the following two properties:*
*(a) the number 1 is in the set.*
*(b) For every $x$ in the set, the number $x + 1$ is also in the set.*

**Definition 1.2 (Positive integers)** *A real number is called a positive integer if it belongs to every inductive set.*

## 1.4 Upper bound of a set, maximum element, least upper bound (supremum)

**Definition 1.3 (Least upper bound)** *A number $B$ is called a least upper bound of a nonempty set $S$ if $B$ has the following two properties:*
*(a) $B$ is an upper bound for $S$.*
*(b) No number less than $B$ is an upper bound for $S$.*

**Theorem 1.26** *Two different numbers cannot be least upper bounds for the same set.*

## 1.5 The least-upper-bound axiom (completeness axiom)

**Axiom 1.10** *Every nonempty set $S$ of real numbers which is bounded above has a supremum; that is, there is a real number $B$ such that $B = \sup S$.*

**Theorem 1.27** *Every nonempty set $S$ that is bounded below has a greatest lower bound; that is, there is a real number $L$ such that $S = \inf S$.*

## 1.6 The Archimedean property of the real-number system

**Theorem 1.28** *The set $\mathbb{P}$ of positive integers $1, 2, 3, ...$ is unbounded above.*

**Theorem 1.29** *For every real $x$ there exists a positive integer $n$ such that $n > x$.*

**Theorem 1.30** *If $x > 0$ and if $y$ is an arbitrary real number, there exists a positive integer $n$ such that $nx > y$.*

**Theorem 1.31** *If three real numbers $a$, $x$, and $y$ satisfy the inequalities*

$$a \leq x \leq a + \frac{y}{n}$$

*for every integer $n \geq 1$, then $x = a$.*

### 1.6.1 Fundamental properties of the supremum and infimum

**Theorem 1.32** *Let $h$ be given positive number and let $S$ be a set of real numbers.*
*(a) If $S$ has a supremum, then for some $x$ in $S$ we have*

$$x > \sup S - h.$$

*(b) If $S$ has an infimum, then for some $x$ in $S$ we have*

$$x < \inf S + h.$$

**Theorem 1.33 (Additive property)** *Given nonempty subsets $A$ and $B$ of $\mathbb{R}$, let $C$ denote the set*

$$C = \{a + b \mid a \in A,\ b \in B\}.$$

*(a) if each of $A$ and $B$ has a supremum, then $C$ has a supremum, and*

$$\sup C = \sup A + \sup B.$$

*(b) if each of $A$ and $B$ has an infimum, then $C$ has an infimum, and*

$$\inf C = \inf A + \inf B.$$

**Theorem 1.34** *Given two nonempty subsets $S$ and $T$ of $\mathbb{R}$ such that*

$$s \leq t$$

*for every $s$ in $S$ and every $t$ in $T$. Then $S$ has a supremum, and $T$ has an infimum, and they satisfy the inequality*

$$\sup S \leq \inf T$$

## 1.7 Existence of square roots of nonnegative real numbers

**Theorem 1.35** *Every nonnegative real number $a$ has a unique nonnegative square root.*

Note: if $a \geq 0$, we denote its nonnegative square root by $a^{1/2}$ or by $\sqrt{a}$. if $a > 0$. the negative square root is $-a^{1/2}$ or $-\sqrt{a}$.

# 2 Mathematical Induction, Summation Notation, and Related Topics

## 2.1 The principle of mathematical induction

*Method of proof by induction.* Let $A(n)$ be an assertion involving an integer $n$. We conclude that $A(n)$ is true for every $n \geq n_1$ if we can perform the following two steps:

(a) Prove that $A(n_1)$ is true.

(b) Let $k$ be an arbitrary but fixed integer $\geq n_1$. Assume that $A(k)$ is true and prove that $A(k+1)$ is also true.

In actual practice $n_1$ is usually 1. The logical justification for this method of proof is the following theorem about real numbers.

**Theorem 2.1 (Principle of mathematical induction)** *Let $S$ be a set of positive integers which has the following two properties:*

*(a) The number $1$ is in the set $S$.*

*(b) If an integer $k$ is in $S$, then so is $k+1$.*

*Then every positive integer is in the set $S$.*

## 2.2 The well-ordering principle

**Theorem 2.2 (Well-ordering principle)** *Every nonempty set of positive integers contains a smallest member.*