Demonstrating the Coverage of a (1 – a)100% Confidence Interval For a Given Parameter

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Question of Interest:

Confidence intervals are extremely important in statistical inference when estimating population parameters with a specified level of uncertainty. The coverage probability, which represents the percentage of times the interval does, in fact, contain the true population parameter in repeated sampling, is a core part of confidence intervals. A thorough understanding of the coverage probability and subsequent examination of it are essential for ensuring the accuracy and dependability of different statistical inference methods and processes.

Imagine a situation where estimating a population parameter with a confidence interval, like the population mean, denoted by "μ," is of importance. Evaluating the coverage of a (1-α)100% confidence interval—where α represents the likelihood of a Type I error taking place—would be the primary goal in such a situation. In order to do this, one must determine the percentage of times the confidence interval accurately represents the true population mean across a large number of randomly selected population samples.

The purpose of this research was to model a scenario of decision-making in which the population is sampled several times, confidence intervals are calculated, and the coverage of the true population mean is evaluated. By following these procedures, important information regarding the behavior of confidence intervals and their ability to yield accurate population parameter estimates in the face of uncertainty will be discovered.

Chosen Population Parameter:

The aim of the project is to "demonstrate the coverage of a (1 – a)100% confidence interval for a given parameter," so it was chosen that the population parameter that would be worked on in this simulation is the population mean, μ. The reason behind selecting the population mean as the chosen parameter is that it is a core parameter in statistics that is frequently utilized to reflect and encapsulate the central tendency of a collection of data. By concentrating and working on the population mean, how to create and simulate confidence intervals and then evaluate their coverage probability, which provides very valuable information regarding the validity of statistical inference techniques, would be clearly demonstrated.

Objective of the Simulation:

The purpose of this simulation is to find the confidence interval for a particular population parameter, in this case, the population mean. To accomplish this goal, random sampling from a population, calculating confidence intervals from these particular samples, and figuring out if the true population parameter lies inside the interval are the necessary steps. The ultimate goal of completing this sequence of actions is to obtain a better understanding of how well the generated confidence intervals characterize the underlying population parameter and how well the constructed confidence intervals capture the true population parameter.

Design of the Experiment:

The design of the experiment includes the following parts:

1. **<u>Sampling:</u>**

Random samples are generated from a population and are assumed to be normally distributed mainly for demonstration purposes. Therefore, the function, "sample_generation," works to generate random sample data that follows the normal distribution.

2. **<u>Calculation of the Confidence Interval:</u>**

The confidence intervals are calculated for each sample, using the sample mean, standard deviation, and the desired confidence interval. Therefore, the function, "CI_calculation," works to calculate the confidence interval for a given sample data and confidence interval.

3. **<u>Checking the Coverage:</u>**

The coverage of the confidence intervals is assessed by checking if the true population parameter, μ, does or does not fall somewhere within the confidence interval. Therefore, the function, "check_coverage," works to check if the true parameter, in this case, μ, does or does not fall within the confidence interval.

4. **<u>Simulation:</u> (Calling The Three Functions Above)**

The simulation works to iterate over a declared number of simulations where it goes over the three previous steps as it will generate sample data, calculate the confidence intervals, and also check the coverage.

Explaining the Code:

As previously stated above, the R code consists of three functions which are the "sample_generation," "CI_calculation," and the "check_coverage" functions.

Function 1:

sample_generation(n)

Its Purpose:

The purpose of this first function is to generate random sample data from a specified distribution. This particular sample data is utilized to simulate the process of drawing random samples from a population, which is a necessary process for constructing confidence intervals and assessing their coverage.

Steps:

1. Input Parameter(s): This function takes only one input parameter which is **'n,'** which represents the sample size to be generated
2. Generating the Sample Data: This function uses the built-in R function, rnorm, to generate **'n'** random observations from a normal distribution with a mean equal to zero and a standard deviation equal to one.
3. Output: The output is a vector of **'n'** random observations

Explanation:

For the sake of demonstration, the mean and standard deviation should be equal to zero and one, respectively. Owing to the Central Limit Theorem and its well-known characteristics, the normal distribution is a widely accepted assumption in many statistical processes.

Generating random samples from a normal distribution will replicate the sampling procedure, which is necessary to understand how confidence intervals behave under repeated sampling.

<u>Function 2:</u>

CI_calculation(sample_data, alpha)

<u>Its Purpose:</u>

The following function has the goal of determining the $(1 - \alpha)100\%$ confidence interval of the sample mean. With respect to the sample data, this interval offers a range of values that the true population mean is probably going to fall somewhere within.

<u>Steps:</u>

1. Input Parameter(s): This function takes two input parameters which are **'sample_data,'** which represents a vector of sample observations, as well as **'alpha,'** which represents the significance level where in this case it was chosen to be equal to 0.05 for a 95% confidence interval.
2. Calculating Sample Statistics:
   a. **'n'** represents the sample size, calculated as the length of the sample_data vector.
   b. **'sample_mean'**: represents the mean of the sample data
   c. **'sample_sd'**: represents the standard deviation of the sample data
3. Finding the Critical Value: **'z'** represents the critical value from the standard normal distribution which was calculated using the built-in R function, qnorm, for a two-tailed test. *In this case, it was **z = qnorm(1 - alpha/2)**.*
4. Finding the Margin of Error (MOE): The formula z * (sample_sd / sqrt(n)) is used to compute the margin of error, which is the critical value times the standard error of the mean.
5. Putting Together the Confidence Interval (CI): The lower and upper bounds of the confidence interval are calculated using the two equations sample_mean - margin_of_error and sample_mean + margin_of_error, respectively.
6. Output: The output of this function is a list which includes the lower and upper bounds of the confidence interval (defined as lower_bound and upper_bound)

<u>Explanation:</u>

The range of values provided by the confidence interval is thought to include the true population mean, with a given degree of confidence of 95%.

The interval width is modified by the margin of error, which takes into consideration the variability in the sample data.

Additionally, since the Central Limit Theorem (CLT) does guarantee that the sampling distribution of the mean approximates a normal distribution, using the built-in R function 'qnorm,' is appropriate when the sample size is said to be large enough.

<u>Function 3:</u> check_coverage(confidence_interval, true_parameter)

<u>Its Purpose:</u>

The purpose of this final function is to determine whether or not the true population parameter, where in this case it is the mean μ, falls within the calculated confidence interval. This is necessary in order to evaluate the probability of the confidence interval.

<u>Steps:</u>

1. Input Parameter(s): This function also takes two input parameters which are **'confidence_interval,'** a list that includes the **'lower_bound'** and **'upper_bound'** of the confidence interval, and **'true_parameter'** which is the true value of the population parameter, again, in this case, it is μ.
2. Checking The Coverage: This function sees if the true parameter (true_parameter) is greater than or equal to the lower limit (lower_bound) and if it is also less than or equal to the upper limit (upper_bound). Simply put, this function checks if the true parameter falls within the calculated confidence interval.
3. Output: The output of this function is a boolean as it returns either a 'TRUE' or 'FALSE' depending on whether the true parameter does or does not fall within the confidence interval. If the true parameter does indeed fall within the confidence interval it returns 'true' and if it does not fall in the interval, then the output would be 'false.'

<u>Explanation:</u>

One of the most important steps in assessing how well confidence intervals work is the coverage check. We can evaluate how frequently the intervals accurately capture the parameter across numerous samples by checking to see if the true population parameter lies inside the interval. The coverage probability is the fraction of intervals over a large number of simulations that contain the real value. To indicate the reliability of the interval estimation approach, this probability should preferably be near the nominal confidence level (e.g., 95% for a 95% confidence interval).

Simulation: **(Calling The Three Functions Above)**

In the simulation, numerous random samples are created, confidence intervals are computed for each sample, and the true population parameter is compared to these intervals to see if it falls inside them. We can estimate the coverage probability and assess the effectiveness of the confidence intervals by repeatedly going through this process.

Example:

1. Setting the parameters: The number of simulations is defined as 'number_of_simulations,' the sample size is defined as 'sample_size,' and the confidence interval is defined as 'confidence_level.'
2. Running the code: For each one of the simulations, a random sample will be generated using the sample_generation() function, the confidence interval will be calculated using the CI_calculation() function, and finally, the interval will be checked if it does or does not cover the true parameter using the check_coverage() function.
3. Determine the percentage of intervals that contain the true parameter in order to compute the coverage probability.

Output(s) and Interpretations:

The coverage probability, which indicates the percentage of simulations in which the true population parameter, μ, does fall inside the computed confidence ranges, is the output of the code. A higher probability would suggest that the confidence interval accurately represents the true parameter value, indicating the validity of the process used to estimate the confidence interval.

Output: `Coverage Probability: 0.949`

1. **Having a High Coverage Probability**

With a coverage probability of 0.949, or roughly 0.95, we can see that the confidence intervals correctly captured the underlying population parameter in 95% of the simulations. The confidence interval estimate method is, therefore, very good at delivering precise information and predictions about the population mean, as evidenced by its exceptionally high coverage probability.

2. **How Reliable Is The Inference?**

Once more, it is reasonable to state that the computed confidence intervals reliably contain the population mean, with a coverage probability of nearly 0.95. *"We are 95% confident that the population mean falls within this interval."* This suggests that the technique used to assemble the intervals is, in fact, very dependable.

3. **How Precise Is The Estimation?**

The estimate process produces accurate confidence intervals, as indicated by the slight deviation of the coverage probability from the desired confidence level. It takes a great deal of precision and accuracy to be able to use statistical inference to make accurate conclusions.

The simulation was run five more times to try to observe any variance in the outputs but there was hardly any, which can be observed directly below. Below is a list of other outputs to show how the outputs, which represent the coverage probability, only slightly deviate from each other and are all, more or less the same value:

```
Coverage Probability: 0.952
Coverage Probability: 0.955
Coverage Probability: 0.957
Coverage Probability: 0.956
Coverage Probability: 0.945
```

Therefore, the obvious consistency of the above findings shows that the coverage probability of the confidence interval always stays extremely near the 95% confidence level, and is almost equal to it. All five of the above output values, just like the original output of 0.949, were consistently within the above probability, even though they did vary by a small amount on either side. This provides additional support for the reliability of the confidence interval estimate technique of the study. The conclusion that confidence intervals calculated in this method provide highly reliable estimates of the population mean across multiple different samples is reinforced by the modest differences seen, which fall within the two desired bounds. All things considered, these results highlight the dependability of confidence intervals as a key instrument in statistical inference, guaranteeing that they retain their expected coverage probability even in the face of recurrent sampling circumstances.

4. **<u>Reliability of the Confidence Intervals</u>**

Coverage probabilities being consistently approximately equal to 0.95 indicated that if similar samples were extracted from the population and confidence intervals were put together through the same procedure time and time again, it is reasonable to expect that the true population mean to fall in about 95% of all of the cases. This particular level of confidence is usually classified as and deemed to be adequate to proceed with making important assumptions and decisions.

5. **<u>Potential Considerations for the Future</u>**

While it is highly encouraging and adequate that the coverage probability is nearly at the targeted degree of confidence, it is equally critical to consider the potential influences that may have contributed to the current direction of the coverage. A few potential contributing variables could include the sample size, the underlying distribution of the population, or even the sequence of actions taken to create the confidence intervals. Further investigation into these potential contributors may provide extremely useful insights into how to improve the accuracy and consistency of the essential inference procedures.

Summary and Conclusion:

In conclusion, this simulation had set the achievable objective of investigating the coverage probability of a $(1 - \alpha)100\%$ confidence interval for the population mean, μ. It has been shown that the performance of confidence intervals under repeated sampling by generating several random samples from a normally distributed population, computing confidence intervals for each one of the samples, and then finally determining if these intervals do or do not include the true population mean.

The findings of the simulation show that the empirical coverage probability roughly matches the nominal confidence level, indicating that the confidence intervals that were created are capable of accurately capturing the true population mean, specifically 95% of the time which is what is expected from a 95% confidence interval. This supports the validity of confidence intervals as a statistical inference tool, giving mathematicians and researchers the necessary assurance that they can accurately estimate population parameters and come out with reliable findings. This kind of understanding of confidence interval behavior is essential for guaranteeing the validity of statistical results and for making well-informed judgments based on sample data. Overall, this simulation emphasizes how genuinely crucial thorough statistical techniques are in order to draw trustworthy and insightful conclusions from data.