Linear Algebra Problem Solving Strategies

- **Gauss-Jordan Elimination**

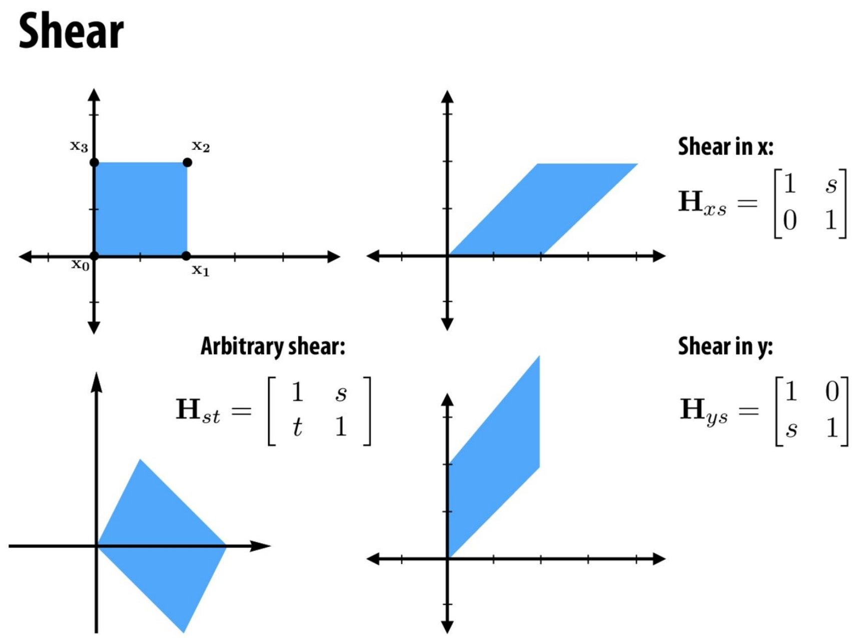
  1. *Given: a matrix A, find its kernel, or given a system $A\vec{x} = \vec{b}$, find all possible $\vec{x}$'s, or given some vectors $v_1 ... v_n$ determine whether a vector $\vec{b}$ can be represented as a linear combination of the vectors*
  2. Either start row-reducing, or augment A with $\vec{b}$.
  3. Find the reduced row echelon form of the matrix, by following these three steps:
     - Add or subtract scalar multiples of rows to one another.
     - Divide rows by any scalar $k \in R$.
     - Switch the position of any row with another.
  4. Usually, the solutions given are either the kernel of the matrix A, or the $\ker(A)$ + 1 unique solution to the given system $A\vec{x} = \vec{b}$.

- **Sets as Subspaces**

  1. *Given: a set or a set relation, determine whether the set of vectors is a subspace of R^n*
  2. Failsafe: make sure that the $\vec{0}$ is within the set; if not, the set is not a subspace.
  3. Confirm set preserves vector addition: using two arbitrary vectors $\vec{v}_1, \vec{v}_2$ in the set, confirm that $\vec{v}_1 + \vec{v}_2$ is also in the set.
  4. Confirm set preserves scalar multiplication: using arbitrary set element $\vec{v}$ and scalar $k \in R$, confirm that $K\vec{v}$ the given set.

- **Matrix Functions**

  1. *Given: given a matrix A with dimensions $m \times n$, determine what it does to it's image*
  2. One by one, multiply A by $\vec{e}_1 ... \vec{e}_m$ to see what the matrix does to a basis of its domain.
  3. Usually, a given matrix can be any combination of the following, with each property holding a particular characteristic:
     - Rotation, where dimensions remain the same but the basis of the domain is rotated by an angle θ counterclockwise.
     - Scaling, where vectors are scaled up or down by a particular scalar.
     - Projection, where all vectors are projected onto a particular subspace of the domain.
     - Reflection, which can be undone by multiplying the reflected domain-basis vectors by A once again ($A^2 = I_n$).
     - A vertical or horizontal shear, which do the following:



- **Orthogonal Projection onto Line L**

  1. *Given: a vector/line with n components, find the matrix of its projection*
  2. Find the vector on the line L, which is oftentimes given.
  3. Find the vectors perpendicular to the line L by multiplying by a general vector $\vec{x}$, and re-wiriting the result as a linear combination of $n-1$ vectors.
  4. Use all of the found vectors as columns for a matrix S.
  5. Find the $\mathfrak{B}$-matrix of the projection, which is just what the projection does to the columns of S. Let this matrix be called C.
  6. The matrix of the projection is the matrix $A = SCS^{-1}$.

- **Matrix Inverse**

  1. *Given: a matrix A with dimensions $n \times n$, find its inverse*
  2. If easy to do so, find the determinant of A to make sure it is in fact invertible (if det(A) $\neq 0$).
  3. Augment A with the identity matrix I_n
  4. Use Gauss Jordan Elimination until you achieve the rref(A) on the right side of the augmentation, which should just be the identity matrix I_n. The matrix on the left side is A^{-1}

- **Matrix Image**

  1. *Given: a matrix A, find a basis of its image*
  2. Find rref(A).
  3. The columns of rref(A) with leading 1's correspond to the columns of A that form a basis of im(A).
  4. If given the size of the kernel, confirm that the size of the image upholds the Rank-Nullity Theorem.

- **$\mathfrak{B}$ Matrices**

  1. *Given: a basis of vectors $\vec{v}_1 ... \vec{v}_n$, and an $n \times n$ matrix A, find the $\mathfrak{B}$ matrix of A*
  2. Multiply A by each of the basis vectors $\vec{v}_1 ... \vec{v}_n$. Label the resultant vectors $\vec{g}_1 ... \vec{g}_n$.
  3. Express $\vec{g}_1 ... \vec{g}_n$ as a linear combination of the vectors $\vec{v}_1 ... \vec{v}_n$. Those linear combination coefficients become the $\mathfrak{B}$-matrix of A.

- **Orthogonal Projection of $\vec{x}$ onto V**

  1. *Given: a vector $\vec{x}$ and a vector space V*
  2. Find an orthonormal basis for V
  3. Use formula to calculate projection:

  $$\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2 + ... + (\vec{x} \cdot \vec{u}_n)\vec{u}_n$$

- **The Gram-Schmidt Process**

  1. *Given: orthogonal vectors $v_1 ... v_n$*
  2. Find u_1: Scale v_1 by multiplying it by $\frac{1}{||v_1||}$
  3. Find v_2^*: $v_2 - (u_1 \cdot v_2)u_1$. Pay attention to the fact that we are using u_1 in caluclations, NOT v_1.
  4. Find u_2: Scale v_2^* by multiplying it by $\frac{1}{||v_2^*||}$
  5. Generalization: Find v_n^* using the following formula:

  $$v_n^* = v_n - (u_1 \cdot v_n)u_1 - ... - (u_{n-1} \cdot v_n)u_{n-1}$$

  6. Generalization: Find u_n: Scale v_n^* by multiplying it by $\frac{1}{||v_n^*||}$

- **Least Square Solutions**

  1. *Given: system of equations*
  2. Rewrite system of equations as matrices in the form $A\vec{x} = \vec{b}$
  3. Since the system is inconsistent, find the least squares solution by using the formula $A^T A\vec{x}^* = A^T\vec{b}$
  4. Find the residual sum of squares by using the equation $||b - A\vec{x}^*||^2$

- **Determinant, Trace, and Triangularity**
  1. *Given: $n \times n$ matrix A*
  2. If A is a 2×2 matrix, use formula to find determinant: $ad - bc$
  3. If A is larger, expand along any row or column of A alternating sign between minors.
  4. For any A, $\text{tr}(A)$ is just the sum of the values on the diagonal.
  5. If the values in A are all zeros above the diagonal it's called lower triangular. If the values in A are all zeros below the diagonal, it's called upper triangular. The determinant of these matrices is the product of the diagonal entries.

- **Eigenvalue Search**
  1. *Given: $n \times n$ matrix A and a relationship A observes, tasked with finding the eigenvalues/ characteristic polynomial of A*
  2. Pick arbitrary eigenvalue λ; we know that $A\vec{v} = \lambda\vec{v}$.
  3. Substitute this fact into the behavior A, observes.
  4. Solve the equation, factoring out $\vec{v}$ to find the characteristic polynomial or possible eigenvalues.

- **Diagonalization**
  1. *Given: 2×2 matrix A*
  2. Find the eigenvalues of A by finding $\ker(A - \lambda I_2)$
  3. If A has one eigenvalue λ_1, check the eigenspace of λ_1. If $\ker(A-\lambda_1 I_2)$ IS NOT dimension 2, the matrix is not diagonalizable. Otherwise, you are left with two eigenvectors: $\vec{v}_1$ and $\vec{v}_2$
  4. If A has two eigenvalues λ_1 and λ_2, it's assumed that A is diagonalizable. For each eigenvalue, find its eigenspace by finding a basis for $\ker(A - \lambda_n I_2)$.
  5. Overview of possible scenarios: 1 eigenvalue with eigenspace dimension 2; 2 real eigenvalues, each with eigenspace dimension 1; 2 complex conjugate eigenvalues with complex conjugate eigenspaces, each with dimension 1.
  6. Rewrite A as SDS^{-1}, where S is an invertible matrix with columns $v_1 ... v_2$, and where D is a matrix with the eigenvalues, in order, across the diagonal.
  7. Additionally, you might have to find S^{-1}. For non-orthogonally diagonalizable matrices, switch the a and d entries in the matrix, negate the b and c entries, and the multiply the resultant matrix by $\frac{1}{det(S)}$.
  8. If the matrix is **orthogonally diagonalizable**, then a few extra steps/pieces of information are important:
     - S should be an orthogonal matrix. To check if a matrix is orthogonally diagonalizable, dot A's eigenspaces together. If the dot product is zero, you can normalize the eigenspaces to get an orthogonal S.
     - S^{-1} is just S^T.
     - In problems involving a 3×3 matrix, there will be two eigenspaces which must be orthogonal to orthogonally diagonalize them. This is usually done by carrying out the Gram-Schmidt process on the 3 vectors: 2 from a plane and 1 from the eigenspace perpendicular to it.

- **Discrete Dynamical System**

  1. *Given: system of* 2 *equations; tasked with finding* $\vec{x}(t+1)$
  2. Find A, a 2×2 matrix whose entries are the coefficients of the system of equations. **Each coefficient is equal to how a variable changes from** t **to** $t+1$.
  3. Express $\vec{x}(t) = A^t \vec{x}(0)$. There are two ways to continue beyond this point:
     - Method A: Find the eigenvalues and eigenvectors of A. Express $\vec{x}(0)$ as a linear combination of the eigenvectors. The resultant form should look like:

       $$\vec{x}(t) = c_1 \lambda_1^t \vec{v}_1 + ... + c_n \lambda_n^t \vec{v}_n$$

     - Method B: Diagonalize A. The resultant form should look like:

       $$\vec{x}(t) = S D^t S^{-1} \vec{x}(0)$$

  4. If the eigenvalues of A are complex, rewrite them using polar form.

- **Continuous Dynamical System**

  1. *Given: system of* 2 *differential equations; tasked with finding* $\vec{x}(t)$
  2. Find A, a 2×2 matrix whose entries are the coefficients of the system of equations. **Each coefficient reflects a rate of change**.
  3. Express $\vec{x}(t) = e^{At}\vec{x}(0)$. There are two ways to go beyond this point:
     - Method A: Find the eigenvalues and eigenvectors of A. Express $\vec{x}(0)$ as a linear combination of the eigenvectors. The resultant form should look like:

       $$\vec{x}(t) = b_1 e^{\lambda_1 t}\vec{v}_1 + ... + b_n e^{\lambda_n t}\vec{v}_n$$

     - Method B: Diagonalize A. The resultant form should look like:

       $$\vec{x}(t) = Se^{Dt}S^{-1}\vec{x}(0)$$

  4. If the eigenvalues of A are complex, use Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$.

- **Converting Complex Numbers to Polar Form**

  1. *Given: complex numbers* $a \pm bi$
  2. We need to rewrite these numbers in the form $re^{i\theta}$.
  3. To do so, we first find r, which is just $\sqrt{a^2 + b^2}$
  4. Then, find θ by looking at the first two quadrants for $a + bi$ and using knowledge of trigonometric functions. We now have all the necessary components.
  5. Knowing how to rewrite $a + bi$, we can rewrite its complex conjugate too!

- **Phase Portraits**

  1. *Given: general solution to a system of differential equations*
  2. Plot the eigenvectors of the general solution.
  3. For each eigenspace, orient it inward if its corresponding eigenvalue is negative, and outward if its corresponding eigenvalue is positive.
  4. Look at the behavior of the system when $t \to \infty$ and when $t \to -\infty$:
     - If both eigenvalues are positive, $-\infty$ should be movement towards the origin, and ∞ would be movement away from the origin.
     - If both eigenvalues are negative, $-\infty$ should be movement away from the origin, and ∞ would be movement towards the origin.
     - If eigenvalues have different signs, the phase portrait's trajectories will look like the trajectories of a rational function, with eigenspaces serving as "asymptotes".
  5. If dealing with complex roots, trajectories will be cyclical. They will shrink inward if the real component of the eigenvalues is negative; spiral outward if the real component of the eigenvalues is positive; and cycle periodically if the real component of the eigenvalues is equal to 1.
  6. If dealing with a zero eigenvalue, then regardless of the other eigenvalue, the equilibrium **points** will fall along the zero eigenspace.

- **Phase Plane Analysis**

  1. *Given: system of nonlinear differential equations*
  2. Find the $\frac{dx}{dt}$ nullclines (vertical dashes).
  3. Find the $\frac{dy}{dt}$ nullclines (horizontal dashes).
  4. Mark equilibrium points.
  5. Orient the regions of the diagram.
  6. Orient the nullclines based on the regions' orientations.
  7. Determine whether the equilibrium points are asymptotically stable. For some points, observation isn't enough to make a decision, so we construct a Jacobian for the point at (a,b):
     - The Jacobian J has the form $\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$.
     - Find the eigenvalues of J.
     - If the real components of both eigenvalues are negative, the point is asymptotically stable.
     - If at least one real component of the eigenvalues is positive, the point is asymptotically unstable.
     - Otherwise, the Jacobian is inconclusive.

- **Linear Homogeneous Second Order Differential Equations**

  1. *Given: a linear second order differential equation of the form* $f'' + bf' + cf$
  2. Replace the f in the equation with λ and you get the characteristic polynomial.
  3. Find the roots of the characteristic polynomial, we'll name them α and β.
  4. If α and β are real, the general solution to this differential equation is:
  $$f(t) = c_1 e^{\alpha t} + c_2 e^{\beta t}$$
  5. If the roots have non-real components, $p \pm iq$, the general solution to this differential equation is:
  $$f(t) = a_1 e^{pt} \cos(qt) + a_2 e^{pt} \sin(qt)$$

- **Find Values for Diagonalizable Matrix**

  1. *Given:* $n \times n$ *matrix* A*, with a variable 'a' as one of the entries*
  2. Find the characteristic polynomial of the matrix.
  3. Find the eigenvalues of the matrix, (oftentimes) using the quadratic formula.
  4. Find possible values a such that the algebraic multiplicity of an eigenvalue is ≥ 2. All values a besides these produce diagonalizable A's.
  5. Plug in the remaining values, find the $\ker(A - \lambda I_n)$ for the repeated eigenvalues λ, to see whether they produce diagonalizable matrices.

- **Linear Spaces**

  1. *Given: a set S, task is to determine whether it is a linear space*
  2. Check failsafe: Does the zero element fit the description of the set of functions?
  3. Check if S is closed under addition: Using 2 "things" $x, y \in S$, is $x + y \in S$?
  4. Check if S is closed under scalar multiplication: Using 1 "thing" $x \in S$ and scalar k, is $kx \in S$?

- **Linear Transformations**

  1. *Given: transformation T, decide whether it's a linear transformation*
  2. Failsafe: Check if plugging in the zero element produces a zero element.
  3. Is T closed under addition? Is $T(f + g) = T(f) + T(g)$ for all $f, g \in V$?
  4. Is T closed under scalar multiplication? Is $T(kf) = kT(f)$ for all $f \in V$ and all real k?

- **Linear Transformation Kernel & Image**

  1. *Given: linear transformation T, find its kernel and its image*
  2. Find a basis of the kernel by testing what is necessary for the entries of the output to equal zero. This sometimes involves constructing an augmented matrix using information that would produce a zero element as output.
  3. The image of T is the set of all possible outputs of T. Find a spanning set of the possible outputs (this includes redundancies).
  4. Use the dimension of the domain and Rank-Nullity Theorem to determine the proper size of the image.
  5. Either by inspection or by looking at the kernel, find a minimal basis for the image.

- **Linear Differential Equations of Order n**

  1. *Given: linear differential equation of order n*
  2. There are n linearly independent solutions to this equation.
  3. Create the characteristic polynomial of the equation by replacing the f^n's with λ^n's. Find the eigenvalues of the characteristic polynomial.
  4. If the eigenvalues are:
     - If $(\lambda - m)$ is part of the characteristic polynomial, then $c_1 e^{mt}$ is part of the solution.
     - If $(\lambda^2 - m)$ is part of the characteristic polynomial with solutions $\alpha + \beta i$, then $c_1 e^{\alpha t} \cos \beta t + c_1 e^{\alpha t} \cos \beta t$ is part of the solution.
     - If $(\lambda - m)^p$ is part of the characteristic polynomial, then $c_1 e^{mt} + c_2 t e^{mt} + ... + c_{p-1} t^{p-1} e^{mt}$ is part of the solution.
  5. If the linear differential equation is **not** set equal to zero, then another solution needs to be found. To do that, you will be provided with a general solution, usually of the form $A \cos qt + B \sin qt$.
  6. Plug in the general solution you have been given into the linear differential equation. Collect terms and solve the system of differential equations to find A and B.

- **The Gram-Schmidt Process & Inner Product Spaces**

  1. *Given: Space W spanned by functions $f_1, f_2, ...$ and inner product rule for some functions $f, g \in V$*
  2. To find g_1, normalize f_1 by dividing by its norm, or simply, find $\frac{f_1}{||f_1||}$.
  3. To find g_2, find $f_2^{\perp}$, or $f_2 - \langle g_1, f_2 \rangle$, and then divide that by its norm.
  4. Repeat until you get an orthonormal basis!
  5. To find a projection of some function h onto the subspace W, use the projection formula:

  $$\text{proj}_W h = \langle f, g_1 \rangle g_1 + \langle f, g_2 \rangle g_2 + ...$$

- **Fourier Analysis**

  1. *Given: some function f, find the Fourier coefficient of f, used to find the* $\text{proj}_{T_n} f$
  2. $\text{proj}_{T_n} f$ has the form $a_0 \cdot \frac{1}{\sqrt{2}} + b_1 \sin x + c_1 \cos x + ... + b_n \sin nx + c_n \cos nx$
  3. To find a_0, find the inner product $\langle \frac{1}{\sqrt{2}}, f(x) \rangle$.
  4. To find b_k, find the inner product $\langle \sin kx, f(x) \rangle$.
  5. To find c_k, find the inner product $\langle \cos kx, f(x) \rangle$.
  6. Generally, keep these things in mind for arbitrary functions $f, g \in V$:
     - If the two functions are even, their product is an even function.
     - If the two functions are odd, their product is an even function.
     - If one of the functions is even and the other odd, their product is an odd function.
     - The inner product of an odd function equals 0.
     - The inner product of functions whose product is even can be rewritten as:

     $$\frac{2}{\pi} \int_0^{\pi} f(x) \cdot g(x) dx$$

     - Integration by substitution is oftentimes necessary, here's a reminder:

     $$\int f(g(x)) g'(x)\, dx = \int f(u)\, du = F(u) + c$$

     where $u = g(x)$ and $du = g'(x)dx$

     - Integration by parts is oftentimes necessary, here's a reminder:

     $$\int u\, dv = uv - \int v\, du$$

  7. To find the Fourier Sine Series of a function f, you only need to find the b_k coefficients, which equal $\frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin kx dx$

- **Verify Solutions to the Heat Equation**
  1. *Given: some function $f(t,x)$, determine whether f is a solution to the heat equation*
  2. Check that the f meets the boundary conditions, which are usually given.
  3. Check that $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$ for some constant μ.
- **Solving the Heat Equation**
  1. *Given: a heat equation f with some boundary conditions and initial condition $f(0,x)$*
  2. The general solution to the Heat Equation is $f(t,x) = \sum_{k=1}^{\infty} P_k e^{-\mu k^2 t} \sin kx$
  3. We plug in 0 for t in this general solution, and then set that equal to our initial condition $f(0,x)$.
  4. To find the P_k terms, we will often only need to find the Fourier Sine Series of the initial condition, if not given the initial condition in terms of the sine function already.
  5. After finding the P_k terms, we just plug them back into the general solution to find our solution.
- **Solving the Wave Equation**
  1. *Given: a wave equation f with some boundary conditions and initial conditions $f(0,x)$ and $\frac{\partial f}{\partial t}(0,x)$*
  2. The general solution to the Wave Equation is $f(t,x) = \sum_{k=1}^{\infty} (P_k \cos kct + Q_k \sin kct) \sin kx$
  3. We plug in 0 for t in this general solution, and then set that equal to our initial condition $f(0,x)$.
  4. To find the P_k terms, we will often only need to find the Fourier Sine Series of the initial condition if the initial condition wasn't in terms of the sine function already.
  5. After finding the P_k terms, we find the partial derivative of our general solution and equate that to $\frac{\partial f}{\partial t}(0,x)$ to find the Q_k terms.
  6. After finding the P_k and Q_k terms, we just plug them back into the general solution to find our solution.